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Philippe G. LeFloch · Jacques Smulevici

Weakly regular T^2-symmetric spacetimes. The global geometry of future Cauchy developments

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Abstract. We provide a geometric well-posedness theory for the Einstein equations within the class of weakly regular vacuum spacetimes with T^2-symmetry, as defined in the present paper, and we investigate their global causal structure. Our assumptions allow us to give a meaning to the Einstein equations under weak regularity as well as to solve the initial value problem under the assumed symmetry. First, introducing a frame adapted to the symmetry and identifying certain cancellation properties taking place in the standard expressions of the connection and the curvature, we formulate the initial value problem for the Einstein field equations under the proposed weak regularity assumptions. Second, considering the Cauchy development of any weakly regular initial data set and denoting by R the area of the orbits of symmetry, we establish the existence of a global foliation by the level sets of R such that R grows to infinity in the future direction. Our weak regularity assumptions only require that R is Lipschitz continuous while the metric coefficients describing the initial geometry of the symmetry orbits are in the Sobolev space H^1 and the remaining coefficients have even weaker regularity.

Keywords. Einstein equations, T^2-symmetry, vacuum spacetime, weakly regular, energy space, global geometry

Contents

P. G. LeFloch: Laboratoire Jacques-Louis Lions, Centre National de la Recherche Scientifique, Université Pierre et Marie Curie, 4 Place Jussieu, 75252 Paris, France; e-mail: contact@philippelefloch.org

J. Smulevici: Département de Mathématiques, Faculté des Sciences d'Orsay, Université Paris-Sud, 91405 Orsay, France; e-mail: jacques.smulevici@math.u-psud.fr

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1. Introduction

This is the first of a series of papers [25, 26] devoted to *weakly regular* vacuum spacetimes of general relativity satisfying Einstein's field equations (that is, the Ricci-flat condition) under certain symmetry assumptions. One of the main difficulties we overcome here is to determine the natural weak regularity conditions that are required to deal with the Einstein equations under the assumed symmetry. Within this framework, for any initial data set with weak regularity we determine the *global geometric structure* of the associated development. Our symmetry assumption is that the initial data are defined on a manifold diffeomorphic to the 3-dimensional torus T^3 and are invariant under the action of the Lie group T^2. This requirement characterizes the so-called *T^2-symmetric spacetimes on T^3* with possibly non-vanishing twist constants, one of the simplest classes which allow one to study the propagation of gravitational waves. A large literature is available on T^2-symmetric vacuum spacetimes when *sufficiently high regularity* on the initial data is assumed. Let us especially refer to Moncrief [29], Chruściel [8], Berger, Chruściel, Isenberg, and Moncrief [3], and Isenberg and Weaver [17]. (For further references, cf. [37].)

The present paper is motivated by earlier work by LeFloch and Stewart [27, 28] (see also [1]) and LeFloch and Rendall [23], which treats a special case of T^2-symmetric spacetimes (namely Gowdy-symmetric spacetimes) but also includes the coupling with matter and thus covers the Einstein–Euler equations; see also [4, 13, 14]. Therein, it was recognized that, due to the formation of shock waves in the fluid and by virtue of the

Einstein equations, only weak regularity of the geometry can be allowed. It is also of physical importance to include impulsive gravitational waves, and therefore these papers provide us with a strong motivation for the present work. In addition, we recall that Christodoulou's proof [7] of the weak and strong cosmic censorship conjectures for spherically symmetric Einstein-scalar field spacetimes also relied on the introduction of a class of spacetimes with weak regularity.

Let us recall briefly the formulation of the initial value problem in general relativity (in the vacuum case). An initial data set for the vacuum Einstein equations is a triple (Σ, h, K) such that (Σ, h) is a 3-dimensional Riemannian manifold, K is a symmetric 2-tensor field defined on Σ, and satisfying the so-called Einstein constraint equations

$$R^{(3)} - |K|^2 + (\mathrm{Tr}(K))^2 = 0, \tag{1.1}$$

$$\nabla^{(3)j} K_{ij} - \nabla^{(3)}_i \mathrm{Tr}(K) = 0, \tag{1.2}$$

in which the covariant derivative $\nabla^{(3)}$ and the scalar curvature $R^{(3)}$ are computed from the Riemannian metric h. Then a solution to the initial value problem associated with the initial data set (Σ, h, K), by definition, is a $(3+1)$-dimensional Lorentzian manifold $(\mathcal{M}, g)$ satisfying the vacuum Einstein equations

$$R_{\mu\nu} = 0, \tag{1.3}$$

together with an embedding $\phi : \Sigma \to \mathcal{M}$ such that $\phi(\Sigma)$ is a Cauchy surface of $(\mathcal{M}, g)$ and the pull-back of its first and second fundamental forms coincides with h and K, respectively.

Recall that the existence of a unique (up to diffeomorphism) maximal globally hyperbolic solution $(\mathcal{M}, g)$, or *maximal Cauchy development*, was established in pioneering work by Choquet-Bruhat [11] and Choquet-Bruhat and Geroch [6]. The local existence theorem given by Hughes, Kato, and Marsden [15] requires that the initial data (h, K) belong to the Sobolev space $H^s_{\mathrm{loc}}(\Sigma) \times H^{s-1}_{\mathrm{loc}}(\Sigma)$ for some $s > 5/2$. The current state of the art is provided by Klainerman and Rodnianski [18] (see also [35]) and requires asymptotically flat initial data with $s > 2$ only. Moreover, beginning with [19], there has been considerable work on the L^2 curvature conjecture, which asserts that the Einstein equations are well-posed for initial data sets having curvature with finite local L^2 norm and second fundamental form whose covariant derivatives have finite local L^2 norms [20].

In this paper, we restrict attention to the class of T^2-symmetric spacetimes, while our regularity assumptions go far below those which can be covered without symmetry. For the precise definitions and concepts presented in this introduction, we refer to Section 2, and here we only provide an overview of the theory established in the present work. These results were first announced in [24].

As far as the initial data set is concerned, our weak regularity conditions can be summarized as follows. First of all, we assume that the area R of the orbits of T^2-symmetry is Lipschitz continuous, and we observe that additional regularity of the function R (namely, admitting integrable second-order derivatives) is implied by Einstein's constraint equations. The remaining components of the data set prescribed on the initial slice Σ either represent the geometry of the T^2-orbits and are assumed to belong to the Sobolev space

$H^1(\Sigma)$, or represent its orthogonal complement and have even lower regularity. A weak regularity property is also imposed on the second fundamental form. (See again Section 2.)

These regularity assumptions are weaker than those needed to define, in the weak sense, the Levi-Civita connection or the curvature of the metric without symmetry assumptions, as recognized by LeFloch and Mardare [22]. Hence, our first task is to exploit the symmetry assumptions to provide alternative definitions for these objects, and in particular to reformulate equations (1.1)–(1.3) under our regularity assumptions.

Theorem 1.1 (Weak formulation of the Einstein equations for weakly regular spacetimes). *If (Σ, h, K) is a weakly regular T^2-symmetric triple, then Einstein's constraint equations* (1.1)–(1.2) *can be reformulated in a weak sense. Similarly, if $(\mathcal{M}, g)$ is a weakly regular T^2-symmetric Lorentzian manifold, then Einstein's* (*constraint and evolution*) *field equations* (1.3) *can be reformulated in a weak sense.*

We refer to Section 2 for the terminology and the proof of this theorem. In particular, we introduce therein a set of *frames adapted to the symmetry*, which are not smooth under our regularity assumptions. However, in any such frame, we uncover certain *cancellation properties* within the standard expressions of the Riemann and Ricci curvatures, and these properties allow us to introduce an alternative (but equivalent if the metric has enough regularity) definition of the Riemann and Ricci curvatures, by suppressing certain (otherwise ill-defined) terms.

Our second main result establishes the *existence of a weakly regular, future Cauchy development* of any given initial data set, and provides detailed information about the *global geometric structure* of the constructed spacetime. In particular, we establish that these weakly regular developments may be covered by a global foliation whose spacelike leaves coincide with the level sets of the area function R, as stated now.

Theorem 1.2 (Existence theory for the Einstein equations of weakly regular spacetimes). *Given any, nonflat, weakly regular T^2-symmetric initial data set (Σ, h, K) with topology T^3 and with orbits of symmetry having initially constant area denoted by $R_0 > 0$, there exists a weakly regular, vacuum spacetime with T^2-symmetry on T^3, say $(\mathcal{M}, g)$, which is a future development of (Σ, h, K), is maximal among all weakly regular T^2-symmetric developments, and admits a unique global foliation by the level sets of the area $R \in [R_0, \infty)$.*

The proof of this theorem will rely on the material developed in Sections 3 to 6 and be finally provided at the end of Section 7. We emphasize that the restriction that the initial slice has constant area is not an essential assumption and is made only for convenience of presentation. Also, the past direction can also be covered by our technique and (with some additional estimates) we could also extend to weakly regular metrics the argument by Isenberg and Weaver [17] which shows that (except for flat Kasner spacetimes) the area R approaches zero in past directions.

In addition, since we derive estimates on the difference of two solutions (cf. Section 6.5), we also establish *uniqueness* of solutions for the reduced partial differential equations, both in areal and conformal coordinates (as defined below). This result can be easily upgraded to a more geometric uniqueness theorem,[1] within the class of weakly regular T^2-symmetric developments of a given initial data set. The definition of the class of spacetimes under consideration here includes, in particular, a natural replacement for the standard condition of global hyperbolicity. See Section 8 for further details.

Moreover, based on the norms to be introduced in Section 6.5, we establish continuous dependence on initial data (see, for instance, Proposition 6.10). Hence, our results provide a fully satisfactory well-posedness theory. Observe in passing that on any compact time interval our weak solutions can be uniformly approximated by smooth solutions. Due to the weak regularity conditions, many key estimates derived in [29, 3] for smooth T^2-symmetric solutions no longer hold for our larger class of spacetimes, especially the estimates involving second-order derivatives of the metric coefficients. The new estimates and compactness arguments of this paper rely, in particular, on identifying a certain null structure of the Einstein equations within this symmetry class which enables us to control the quadratic nonlinearities in the equations.

Interestingly, our analysis relies on two different sets of coordinates, the so-called conformal and areal coordinate systems. One difficulty arises from the fact that these coordinate systems must be constructed together with the solution to the Einstein equations, and therefore also enjoy weak regularity only. We show that the coordinate charts are nonetheless at least C^1 compatible and we establish that the coordinate transformation preserves our regularity conditions (cf. Section 5.2). On the one hand, conformal coordinates enable us to establish compactness properties and to study the local well-posedness of solutions. Indeed, the equations become semilinear, whereas the constraints and certain nonlinear terms take a more involved form. On the other hand, in areal coordinates, the evolution equations admit a monotone energy-like functional and the constraint equations degenerate to a first-order system. We take advantage of this to control the long-time behavior of solutions and analyze the global structure of the spacetimes.

To establish our existence result for the Einstein equations, we also need to investigate the constraints imposed on the initial data. We propose here a novel *regularization scheme* that allows us to approximate any weakly regular initial data set by a sequence of smooth initial data sets, while preserving the Einstein constraints. In addition, we establish the *existence of weakly regular initial data sets,* in which each metric coefficient has just the assumed regularity.

An outline of this paper is as follows. In Section 2, we define the class of weakly regular initial data and spacetimes of interest, and we provide a fully geometric reformulation of the Einstein constraint and evolution equations; this analysis leads us to a proof of Theorem 1.1. Then, in Section 3, we rely on our symmetry and weak regularity assumptions and introduce certain (admissible, conformal, areal) coordinates adapted to the symmetry. In Section 4, we express the weak form of the Einstein equations as a system of partial differential equations whose (generalized) solutions are understood in the sense of distributions. Section 5 contains several preliminary results, and in particular

[1] This uniqueness statement is naturally not as general as the one in Choquet-Bruhat and Geroch [6], since it holds within the class of T^2-symmetric solutions only.

includes a discussion of the regularization of initial data sets. Section 6 is concerned with local existence and compactness arguments and takes advantage of the (null) structure of the Einstein equations under the assumed symmetry. In Section 7 we analyze the global geometry of the constructed spacetime and complete the proof of Theorem 1.2. Finally, in Section 8, we state and establish a uniqueness theorem for the constructed solutions.

2. Geometric formulation

2.1. *Weakly regular T^2-symmetric Riemannian manifolds*

All topological manifolds[2] under consideration are of class C^∞, that is, are defined by local charts such that the overlap maps are differentiable of any order. On the other hand, metric structures under consideration have *low regularity,* specified in the course of our analysis. Throughout the paper, we use standard notation for Lebesgue and Sobolev spaces such as L^1, H^2, etc.

Observe first that the Lie derivative $\mathcal{L}_Z h$ of a measurable and locally integrable 2-tensor h (on a differentiable manifold) is defined in the weak sense, for any C^1 vector fields X, T, Z, by

$$(\mathcal{L}_X h)(T, Z) := X(h(T, Z)) - h(\mathcal{L}_X T, Z) - h(T, \mathcal{L}_X Z), \tag{2.1}$$

where the last two terms are (classically defined as) locally integrable functions (that is, in L^1_{loc}), but the first one is defined in a weak sense only. We begin with several standard properties of manifolds admitting a torus action, and refer to [8] and [21, Chap. 9] for a proof of the following lemma.

Lemma 2.1. *Let Σ be a smooth (connected, orientable) 3-manifold and assume that Σ admits a smooth effective action G of $T^2 = U(1) \times U(1)$ on Σ such that Σ has no fixed point under G. Then Σ is diffeomorphic to T^3 and the action is unique up to an automorphism of $U(1) \times U(1)$ and a diffeomorphism of T^3. Moreover, there exist two smooth, linearly independent (in particular nonvanishing) commuting vector fields X and Y on Σ that are generators of G.*

Note that, conversely, given two smooth, linearly independent commuting vector fields X and Y with closed orbits, the flows of X and Y define an effective T^2-action with *no fixed point on Σ.*

Definition 2.2. Under the assumption of the above lemma, one says that a function ψ in L^1_{loc} is **invariant** by the action G if $(\psi \circ G)(p, g) = \psi(p)$ for all $(p, g) \in \Sigma \times T^2$.

This definition can be extended to tensor fields in the usual way and, for instance, a (0, 2)-tensor field S_{ab} in L^1_{loc} is invariant by G if $G^*(\cdot, g)[S \circ G(p, g)] = S(p)$ for all (p, g) in $\Sigma \times T^2$, where $G^*(\cdot, g)$ is the pull-back associated with the map $G(\cdot, g) : \Sigma \to \Sigma$, defined for all $g \in T^2$.

[2] All manifolds in this paper are assumed to be Hausdorff, orientable, connected, and paracompact.

Lemma 2.3. *Under the assumptions of Lemma 2.1, if an L^1_{loc} tensor field S is invariant by G, then $\mathcal{L}_X S = \mathcal{L}_Y S = 0$ in the weak sense. Conversely, if $\mathcal{L}_X S = \mathcal{L}_Y S = 0$ in the weak sense, then S is invariant by G.*

Proof. The proof is elementary and we sketch it only for completeness. Since Σ is diffeomorphic to T^3, there exist periodic coordinates (x, y, z) defined on Σ such that $X = \partial_x$ and $Y = \partial_y$ are generators of G. Let us assume for simplicity that S is a scalar function. Let ϕ_t be the flow of the field X, hence $(S \circ \phi_t)(x, y, z) = S(x + t, y, z) = S(x, y, z)$. If ψ is any smooth 3-form field, then we have

$$\int_\Sigma \big(S\psi - \phi_t^*((S\psi) \circ \phi_t)\big) = 0.$$

By dividing by t and letting $t \to 0$, it follows that

$$\int_\Sigma S\mathcal{L}_X \psi = 0,$$

so that $\mathcal{L}_X S = 0$ in the weak sense. This also clearly shows the converse statement. □

In view of the above lemma, we can state the symmetry property in terms of either the group action or the generators X, Y. From now on, this fact will be used without further reference to the above lemma. We can now introduce the following definition.

Definition 2.4. A **weakly regular T^2-symmetric Riemannian manifold** (Σ, h) is a compact, C^∞ differentiable 3-manifold Σ endowed with a tensor field h, enjoying the following properties:

1. **Riemannian structure.** The field h is a Riemannian metric in L^∞.
2. **Symmetry.** The Riemannian manifold (Σ, h) is invariant under the action of the Lie group T^2 generated by two (smooth, linearly independent, commuting) Killing fields X, Y (with closed orbits) satisfying, therefore, in particular
$$\mathcal{L}_X h = 0, \qquad \mathcal{L}_Y h = 0, \tag{2.2}$$
understood in the weak sense (2.1).
3. **Regularity of the orbits.** The functions $h(X, X)$, $h(X, Y)$, and $h(Y, Y)$ belong to the Sobolev space $H^1(\Sigma)$, and the area $\overline{R}$ of the orbits of symmetry defined by
$$\overline{R}^2 := h(X, X)h(Y, Y) - h(X, Y)^2, \tag{2.3}$$
which then lies in $W^{1,1}(\Sigma)$, actually belongs to $W^{1,\infty}(\Sigma)$.
4. **Regularity of the orthogonal complement.** There exists a (smooth) vector field Θ defined on Σ such that (X, Y, Θ) forms a frame of commuting vector fields ($\mathcal{L}_X \Theta = \mathcal{L}_Y \Theta = 0$) for which, by introducing the (nonsmooth!) vector field
$$Z := \Theta + aX + bY, \qquad Z \in \{X, Y\}^\perp, \tag{2.4}$$
for some real functions a, b, the regularity $h(Z, Z) \in W^{1,1}(\Sigma)$ holds with[3]
$$\inf_\Sigma h(Z, Z) > 0. \tag{2.5}$$

[3] In fact, the lower bound (2.5) is a consequence of the assumed regularity and symmetry (stated explicitly for clarity of presentation), since $W^{1,1}$ regularity and symmetry imply continuity.

In the context of Definition 2.4, we refer to the triple (X, Y, Z) as an **adapted frame** on Σ. Observe that, by Lemma 2.1, the existence of a T^2 action leaving no point of Σ fixed implies that Σ is diffeomeorphic to the 3-torus T^3, which guarantees the existence of a vector field Θ commuting with (X, Y).

We emphasize that the above definition is fully geometric, as it is easily checked that it does not depend on the specific choice of Killing fields within the generators of the T^2-symmetry. We emphasize that *no regularity is required* on the derivatives of the "cross-terms" $h(X, \Theta)$ and $h(Y, \Theta)$. On the other hand, since h is a Riemannian metric, the definition (2.3) yields a positive function $\overline{R}^2$, and since $\overline{R}$ is a continuous function defined on a compact set,

$$\min_{\Sigma} \overline{R} > 0. \tag{2.6}$$

The strict positivity conditions (2.5) and (2.6) ensure that the isomorphism $^\sharp$ (and the isomorphism $^\flat$, respectively) which transforms covectors into vectors (and vice versa, resp.) is a multiplicative operator with L^∞ coefficients. Moreover, (2.6), the symmetry assumptions and the H^1 regularity of h on the orbits imply that the inverse metric components h^{XX}, h^{XY} and h^{YY} all have H^1 regularity.

To fully describe the class of initial data sets of interest, we need to consider Riemannian manifolds endowed with a 2-covariant tensor field which will later stand for the second fundamental form describing the extrinsic geometry of the initial slice.

Definition 2.5. A **weakly regular T^2-symmetric triple** (Σ, h, K) is a weakly regular T^2-symmetric Riemannian manifold (Σ, h) with an adapted frame (X, Y, Z) satisfying the following conditions:

1. **Regularity.** K is a symmetric 2-tensor field on Σ such that

$$K(Z, Z) \in L^1(\Sigma) \tag{2.7}$$

and, for all $U, V \in \{X, Y, Z\}^2$ with $(U, V) \neq (Z, Z)$,

$$K(U, V) \in L^2(\Sigma). \tag{2.8}$$

2. **Symmetry.** The field K is invariant under the action of the Lie group T^2 generated by (X, Y):

$$\mathcal{L}_X K = \mathcal{L}_Y K = 0, \tag{2.9}$$

understood in the weak sense (2.1).
3. **Additional regularity.** The trace of K on the orbits of symmetry is bounded:

$$\mathrm{Tr}^{(2)}(K) := h^{XX} K(X, X) + 2h^{XY} K(X, Y) + h^{YY} K(Y, Y) \in L^\infty(\Sigma), \tag{2.10}$$

where each product involves an H^1 function and an L^2 function.

As far as solutions to the Einstein equations (which are not assumed yet) are concerned, the additional regularity in (2.10) corresponds to a Lipschitz continuous bound on the time derivative of the area R of the orbits of symmetry, and therefore is a natural regularity condition in view of the assumption $\overline{R} \in W^{1,\infty}$ made in Definition 1.1. Note also that we could have assumed a lower regularity of $K(X, Z)$ and $K(X, Y)$, namely that they only belong to L^1; however, this is unnecessary since Einstein's momentum constraints will eventually imply that these components lie even in L^∞. Indeed, importantly, in Section 2.3 we shall show that the weak regularity described in the above two definitions is suitable to deal with the constraints associated with Einstein's field equations.

2.2. *Weakly regular T^2-symmetric Lorentzian manifolds*

We now introduce the class of spacetimes of interest.

Definition 2.6. An L^∞ **Lorentzian structure** is a (3+1)-dimensional manifold $\mathcal{M}$ (possibly with boundary) endowed with a Lorentzian metric g in $L^\infty_{\text{loc}}(\mathcal{M})$ whose volume form is L^∞_{loc} and bounded below.

Definition 2.7. Let $(\mathcal{M}, g)$ be an L^∞ Lorentzian structure such that $\mathcal{M} = I \times T^3$ and assume that $(\mathcal{M}, g)$ is invariant under an effective action of the Lie group T^2 with no point of $\mathcal{M}$ being fixed by the action. One says that $(\mathcal{M}, g)$ admits a $(3 + 1)$**-decomposition adapted to the symmetry** if the following conditions are satisfied:

1. There exist global coordinates (t, θ, x, y) adapted to the product decomposition of $\mathcal{M}$, with $t \in I$ and (θ, x, y) periodic coordinates on T^3, such that $X = \partial/\partial x$ and $Y = \partial/\partial y$ are generators of the symmetry group.
2. There exists a family of scalars $n(t)$ and Riemannian metrics $h(t)$ defined on each level set of t and belonging to L^∞, uniformly in t on any compact subset of I, such that, in the coordinates (t, θ, x, y), the metric takes the form
$$g = -n^2(t)dt^2 + h_{ij}(t)d\zeta^i \otimes d\zeta^j$$
with $(d\zeta^i) = (dx, dy, dz)$ (so that the so-called shift vector[4] vanishes in these coordinates).

As usual, $n(t)$ is referred to as the **lapse function.** Since $(\mathcal{M}, g)$ is invariant by the group action, we have
$$\mathcal{L}_X(n^2) = \mathcal{L}_Y(n^2) = 0 \tag{2.11}$$
and
$$\mathcal{L}_X h = \mathcal{L}_Y h = 0 \tag{2.12}$$
in the weak sense. We shall denote by Σ_t the level sets of t, i.e. the hypersurfaces $\{t\} \times T^3$. Note that since the function t is invariant by the group action, there is a natural induced action on each Σ_t.

[4] We are restricting attention to zero shift, since the areal and conformal coordinates (constructed later) in the weakly regular case—which are known to always exist for smooth T^2-symmetric spacetimes on T^3—enjoy this property.

Definition 2.8. Let $(\mathcal{M}, g)$ be an L^∞ Lorentzian structure with $\mathcal{M} = I \times T^3$. Assume that $(\mathcal{M}, g)$ is invariant under an effective action of the Lie group T^2 such that no point of $\mathcal{M}$ is fixed by the action and that $(\mathcal{M}, g)$ admits a (3 + 1)-decomposition adapted to the symmetry, as in Definition 2.7. Let (T, Θ, X, Y) be the induced basis associated to the global coordinates (t, θ, x, y) of Definition 2.7

One says that $(\mathcal{M}, g)$ is a **weakly regular T^2-symmetric Lorentzian manifold** with spatial topology T^3 if the following regularity properties hold:

1. **Timelike regularity.** The field $\mathcal{L}_T h$ belongs to $L^1(\Sigma_t)$, uniformly in t in any compact subset of I.
2. **Spacelike regularity.** For each t (and uniformly on any compact subset of I), the triple $(\Sigma_t, h(t), K(t))$ with

$$K(t) := -\frac{1}{2n(t)} (\mathcal{L}_T h)(t) \tag{2.13}$$

   is a weakly regular T^2-symmetric triple in the sense of Definition 2.5, with the group action being the induced action on Σ_t. The implied constants are bounded on each compact subset of I.
3. **Conformal metric regularity.** Finally, the vector field

$$Z := \Theta + aX + bY, \qquad Z \in \{T, X, Y\}^\perp, \tag{2.14}$$

   satisfies

$$\rho^2 := \frac{h(Z, Z)}{n^2} = -\frac{g(Z, Z)}{g(T, T)} \in W^{1,\infty}(\mathcal{M}). \tag{2.15}$$

Spacetimes satisfying the above definition will indeed be constructed in the present work by solving the initial value problem for the Einstein equations from initial data sets satisfying Definition 2.5. The solutions will in fact have more regularity and be actually continuous in time (in certain topologies in space). We will first construct one specific foliation along which the regularity conditions in the definition are satisfied, and next deduce the same regularity along general foliations. Note also that the function $\rho^2 = -g(Z, Z)/g(T, T)$ in (2.15) determines the conformal quotient metric and the wave operator relevant later in this paper when dealing with the evolution part of the Einstein equations.

A frame such as (T, X, Y, Z) will be referred to as an *adapted frame* for a weakly regular T^2-symmetric Lorentzian manifold. The restriction of (X, Y, Z) to a surface Σ_t provides an adapted frame for a weakly regular T^2-symmetric triple on Σ_t. From the definition of Z and the regularity of $\mathcal{L}_T h$, it follows that

$$\mathcal{L}_T Z = T(a)X + T(b)Y,$$

where $T(a)$ and $T(b)$ are in $L^1(\Sigma_t)$ (uniformly in t on any compact time interval). Moreover, from the definition (2.13) and for all $e_i, e_j \in \{X, Y\}$, we have

$$K(e_i, e_j) = -\frac{1}{2n} T(h(e_i, e_j)), \qquad K(Z, Z) = -\frac{1}{2n} T(h(Z, Z)),$$
$$K(Z, e_i) = K(e_i, Z) = \frac{1}{2n} h(e_i, \mathcal{L}_Z T),$$

which, by our definition, are L^1 or L^2 functions on each slice.

We conclude this section by introducing a notion of second fundamental form associated with the orbits of symmetry (of any T^2-symmetric weakly regular manifold). Observe that, due to our low regularity assumptions, a *family* of second fundamental forms *defined almost everywhere* only can be introduced here.

Definition 2.9. Let $(\mathcal{M}, g)$ be a weakly regular T^2-symmetric Lorentzian manifold with adapted frame (T, X, Y, Z). Then the **weak version of the second fundamental form in the Z-direction** associated with the orbits of symmetry is defined (almost everywhere only) as the tensor field

$$\chi_{ij} := -\tfrac{1}{2}(h(t)(Z,Z))^{-1/2}\gamma_i^a\gamma_j^b Z(h(t)_{ab}) \qquad \text{almost everywhere in } \mathcal{M},$$

where γ_i^a is the projector on the space generated by X, Y, with $i, j = X, Y, Z$ and $a, b = X, Y$. Similarly, the **weak version of the second fundamental form in the T-direction** associated with the orbits of symmetry is defined (almost everywhere only) as the tensor field

$$\kappa_{ij} := -\tfrac{1}{2}(h(t)(T,T))^{-1/2}\gamma_i^a\gamma_j^b T(h(t)_{ab}) \qquad \text{almost everywhere in } \mathcal{M}.$$

Recall that, by definition, $\gamma_i^a = h_i^a - h(Z,Z)^{-1}Z^a Z_i$. Thus, in view of Definition 2.4, the components of γ_i^a in the frame (X, Y, Z) are in $L^\infty(\Sigma_t)$, and it follows from the H^1 regularity of h_{ab} that χ belongs to $L^2(\Sigma_t)$ uniformly in the time variable on any compact time interval. The same is true for the components of κ, so

$$\chi_{ij}, \kappa_{ij} \in L^2(\Sigma_t) \qquad \text{locally uniformly in } t. \tag{2.16}$$

We can now state without proof the following elementary result.

Lemma 2.10 (Normal derivative of the area element). *If $(\mathcal{M}, g)$ is a weakly regular Lorentzian manifold with adapted frame (T, X, Y, Z), then $\mathrm{Tr}^{(2)}(\chi)$ is determined by the (normalized) Z-derivative of the area element:*

$$\mathrm{Tr}^{(2)}(\chi) := h^{ab}\chi_{ab} = (h(t)(Z,Z))^{-1/2}Z(\ln R) \qquad \textit{almost everywhere in } \mathcal{M}. \tag{2.17}$$

Similarly, $\mathrm{Tr}^{(2)}(\kappa)$ is determined by the (normalized) T-derivative of the area element:

$$\mathrm{Tr}^{(2)}(\kappa) := h^{ab}\kappa_{ab} = -\frac{1}{n}T(\ln R) \qquad \textit{almost everywhere in } \mathcal{M}. \tag{2.18}$$

Similarly, we have the following.

Lemma 2.11 (Normal derivative of the volume element). *Let $(\mathcal{M}, g)$ be a weakly regular T^2-symmetric Lorentzian manifold and let $K(t)$ be the second fundamental form associated with the slice of constant time t, as in Definition 2.8. Then the trace $\mathrm{Tr}(K) := h^{ij}K_{ij}$ of the second fundamental form is determined by the time derivative of the determinant $h := \det h_{ij}$,*

$$\mathrm{Tr}(K) := -\frac{1}{n}T(\ln\sqrt{h}) \qquad \textit{almost everywhere in } \mathcal{M}. \tag{2.19}$$

2.3. *Weak version of Einstein's constraint equations*

Christoffel symbols

The standard definition of the Christoffel symbols involves certain nonlinear terms that *cannot be defined*—even as distributions—under the weak regularity conditions introduced in Sections 2.1 and 2.2 above. A fortiori, it is unclear whether any component of the curvature could be well-defined. In fact, for general manifolds, the minimal regularity assumption for the curvature to make sense as a tensor distribution is known to be $H^1 \cap L^\infty$ (cf. [22]). In the present paper, we assume a *weaker regularity* for *certain components* of the metric and need to take advantage of the symmetry of the spacetimes under consideration. We will *reformulate* Einstein's constraint and evolution equations so that, for weakly regular T^2-symmetric spacetimes, all of the geometric objects of interest are well-defined in a suitably weak sense, and our definitions reduce to the classical ones when sufficient regularity is assumed.

First of all, we emphasize that a geometric standpoint based on an adapted frame, as we propose in this work, is required. Indeed, under the conditions stated in Proposition 2.12, below, one *cannot define* the Christoffel symbol $\Gamma^\Theta_{\Theta\Theta}$, as this would involve products of the form $h^{i\Theta}\Theta(h_{\Theta b})$ (with $b = X, Y$), which cannot be defined in the weak sense when $h_{\Theta b} \in L^\infty(\Sigma)$. This is why we introduce a (nonsmooth) adapted frame (X, Y, Z) (as defined in (2.4) or (2.14)) where the problematic terms vanish by construction since Z is orthogonal to X, Y.

A preliminary remark is in order. The vector field Z introduced is *not smooth* so that it does not apply to *general* functions of class L^1, but yet can be applied to T^2*-symmetric* functions, by defining

$$Z(f) := \Theta(f) \quad \text{for } T^2\text{-symmetric } f \in L^1(\Sigma),$$

where the right-hand side involves the C^∞ vector field Θ of the frame (X, Y, Θ), as in Definition 2.4. In the following, this observation will be used without further notice.

Proposition 2.12 (Definition and regularity of the Christoffel symbols in an adapted frame). *Let (Σ, h) be a weakly regular T^2-symmetric Riemannian manifold with adapted frame (X, Y, Z) and, for all $i = X, Y, Z$ and $a, b = X, Y$, consider the formal expressions Γ^i_{jk} defined by*

$$\begin{aligned}
&\Gamma^c_{ab} := 0, \quad \Gamma^Z_{ab} := -\tfrac{1}{2} h^{ZZ} Z(h_{ab}), \quad \Gamma^Z_{aZ} = \Gamma^Z_{Za} := 0,\\
&\Gamma^b_{aZ} = \Gamma^b_{Za} := \tfrac{1}{2}\left(h^{bX} Z(h_{aX}) + h^{bY} Z(h_{aY})\right),\\
&\Gamma^X_{ZZ} = \Gamma^Y_{ZZ} := 0, \quad \Gamma^Z_{ZZ} := \tfrac{1}{2} h^{ZZ} Z(h_{ZZ}).
\end{aligned}$$

Then, for $(j, k) \neq (Z, Z)$, the symbols Γ^i_{jk} are well-defined as functions in $L^2(\Sigma)$, while Γ^Z_{ZZ} is well-defined as a function in $L^1(\Sigma)$ and, in addition,

$$\Gamma^a_{aZ} \in L^\infty(\Sigma). \tag{2.20}$$

Moreover, if (Σ, h) is sufficiently regular, then these functions Γ^i_{jk} coincide with the standard Christoffel symbols (in the frame X, Y, Z) associated with the metric h.

Proof. We observe that the given expressions do make sense and have the claimed regularity, as follows immediately from Definition 2.4. The additional regularity of Γ^a_{aZ} is a direct consequence of (2.10). On the other hand, when the data are sufficiently regular and since X, Y, Z commute, the Christoffel symbols can be computed in a standard way. For $i = X, Y, Z$ and $a, b = X, Y$ and by using the symmetry properties of h and the orthogonality condition $Z \in \{X, Y\}^\perp$, we find (a comma indicating differentiation)

$$\begin{aligned}
\Gamma^i_{ab} &= \tfrac{1}{2} h^{ij}(h_{aj,b} + h_{jb,a} - h_{ab,j}) = -\tfrac{1}{2} h^{iZ} Z(h_{ab}),\\
\Gamma^Z_{aZ} &= \tfrac{1}{2} h^{Zj}(h_{aj,Z} + h_{jZ,a} - h_{aZ,j}) = 0,\\
\Gamma^b_{aZ} &= \tfrac{1}{2} h^{bj}(h_{aj,Z} + h_{jZ,a} - h_{aZ,j}) = \tfrac{1}{2}(h^{bX} Z(h_{aX}) + h^{bY} Z(h_{aY})),\\
\Gamma^a_{ZZ} &= \tfrac{1}{2} h^{aj}(2h_{jZ,Z} - h_{ZZ,j}) = 0,\\
\Gamma^Z_{ZZ} &= \tfrac{1}{2} h^{Zj}(2h_{Zj,Z} - h_{ZZ,j}) = \tfrac{1}{2} h^{ZZ} Z(h_{ZZ}).
\end{aligned} \tag{2.21}$$

Hence, when the data are sufficiently regular, Γ^i_{jk} do coincide with the standard Christoffel symbols. □

Although this is not needed in the rest of this paper, it is possible to introduce a suitable notion of connection whose components in the adapted frame (X, Y, Z) are the coefficients we have just constructed. This is done in the Appendix, where we also prove that standard results such as the uniqueness of the Levi-Civita connection can be extended to our setting.

It follows from Proposition 2.12 that, for weakly regular T^2-symmetric Riemannian manifolds, the main obstacle to defining the curvature tensor in a weak sense (in the frame X, Y, Z) comes from the component Γ^Z_{ZZ} *which is only in* $L^1(\Sigma)$, and therefore cannot be multiplied by Christoffel coefficients—which are in $L^2(\Sigma)$ or $L^1(\Sigma)$. Fortunately, as we check below, for *sufficiently regular* T^2-symmetric spacetimes and within the expression of the curvature, the formal products involving such coefficients *cancel out.* This suggests *redefining* the Ricci scalar by a *new formula* taking this cancellation into account, as we now explain.

Weak version of the Hamiltonian constraint

To write down a weak form of the Ricci scalar, denoted by $R^{(3)}$, we need first to *redefine the component* $R^{(3)}_{ZZ}$ of the Ricci tensor as follows. The definition will be fully justified below in the proof of Proposition 2.16, where terms of the form $\pm\Gamma^Z_{ZZ}\Gamma^Z_{ZZ}$ will be checked to cancel out and, for that reason, do not arise in the definition.

Definition 2.13. Let (Σ, h) be a weakly regular T^2-symmetric Riemannian manifold with adapted frame (X, Y, Z). The **weak version of the Ricci curvature in the direction** (Z, Z) is defined as

$$R^{(3)}_{ZZ} := -Z(\Gamma^a{}_{aZ}) + \Gamma^a{}_{aZ}\Gamma^Z_{ZZ} - \Gamma^a_{bZ}\Gamma^b_{aZ}, \tag{2.22}$$

where the first term of the right-hand side is defined in the weak sense only, and the other terms are products of the type $L^\infty L^1$ or $L^2 L^2$.

Based on the above definition, we can now formulate the Hamiltonian constraint in the weak sense. First, we observe that, in view of the Gauss equation and the flatness of the orbits of symmetry,

$$R^{(3)} = 2h(Z,Z)^{-1} R^{(3)}_{ZZ} + |\chi|^2 - (\mathrm{Tr}^{(2)}(\chi))^2 \quad \textit{for sufficiently regular metrics} \tag{2.23}$$

(with χ given in Definition 2.9). However, in our setting, the Ricci curvature term $R^{(3)}_{ZZ}$ of (2.22) is defined in the weak sense only, and it *does not make sense* to multiply it by the factor $h(Z,Z)^{-1}$. This motivates introducing the following *normalized* version.

Definition 2.14. Let (Σ, h) be a weakly regular T^2-symmetric Riemannian manifold with adapted frame (X, Y, Z). Then the **weak version of the normalized scalar curvature** of (Σ, h) is defined as

$$R^{(3)}_{\mathrm{norm}} := 2R^{(3)}_{ZZ} + h(Z,Z)\big(|\chi|^2 - (\mathrm{Tr}^{(2)}(\chi))^2\big),$$

where χ is the weak version of the second fundamental form in the Z-direction. In addition, a weakly regular T^2-symmetric triple (Σ, h, K) is said to satisfy the **weak version of the Hamiltonian constraint** if

$$R^{(3)}_{\mathrm{norm}} + h(Z,Z)\big((\mathrm{Tr}^{(2)}(K))^2 + 2\,\mathrm{Tr}^{(2)}(K)K^Z_Z - K_{ab}K^{ab} - 2K_{aZ}K^{aZ}\big) = 0. \tag{2.24}$$

Remark 2.15. The weak form of the Hamiltonian constraint is independent of the specific choice of adapted frame (X, Y, Z). Indeed, the orthogonal complement to the orbits is one-dimensional, and thus the vector field Z is uniquely determined up to multiplication by a C^∞ function, which does not change the set of solutions to (2.24). This fact can be checked as follows. Write $Z = \theta + aX + bY \in \{X, Y\}^\perp$ and $Z = \theta' + a'X + b'Y \in \{X, Y\}^\perp$ for some other field θ'. Since $\{X, Y\}^\perp$ is a one-dimensional vector space, there exists a scalar field φ such that $Z' = \varphi Z$. Furthermore, one can decompose θ in the basis θ', X, Y and write $\theta = \gamma\theta' + \alpha X + \beta Y$ where α, β, γ are C^∞ and $\gamma \neq 0$ since θ', X, Y is also a basis. Then an elementary calculation shows that $\varphi = 1/\gamma$, which is thus C^∞ (despite the coefficients a, b, a', b' being only weakly regular).

Proposition 2.16 (Equivalence to the classical definition). *Let (Σ, h, K) be a weakly regular T^2-symmetric triple. If h, K are sufficiently regular, then (Σ, h, K) satisfies the weak version of the Hamiltonian constraint equation (in the sense of Definition 2.14) if and only if it satisfies the constraint equation (1.1) in the classical sense.*

Proof. In view of (2.23), and since, if K has sufficient regularity,

$$(\mathrm{Tr}^{(2)}(K))^2 + 2\,\mathrm{Tr}^{(2)}(K)K^Z_Z - K_{ab}K^{ab} - 2K_{aZ}K^{aZ} = (\mathrm{Tr}(K))^2 - |K|^2,$$

the result follows if, assuming now *sufficient regularity,* we can prove that the classical definition for $R^{(3)}_{ZZ}$ coincides with the one adopted in Definition 2.9. Namely, computing $R^{(3)}_{ZZ}$ in the classical sense from the trace of the Riemann curvature, we find $R^{(3)}_{ZZ} = \Omega_1 + \Omega_2$ with (a comma indicating differentiation, as mentioned earlier)

$$\Omega_1 := \Gamma^i_{ZZ,i} - \Gamma^i_{iZ,Z}, \qquad \Omega_2 := \Gamma^j_{ji}\Gamma^i_{ZZ} - \Gamma^j_{Zi}\Gamma^i_{jZ}.$$

On the one hand, since $\Gamma^a_{ZZ,a} = 0$ we have

$$\Omega_1 = \Gamma^a_{ZZ,a} + \Gamma^Z_{ZZ,Z} - \Gamma^Z_{ZZ,Z} - \Gamma^a_{aZ,Z} = -Z(\Gamma^a_{aZ}),$$

where we have cancelled out the terms $\pm\Gamma^Z_{ZZ,Z}$. On the other hand, we have

$$\begin{aligned}\Omega_2 &= \Gamma^Z_{ZZ}\Gamma^Z_{ZZ} + \Gamma^a_{aZ}\Gamma^Z_{ZZ} + \Gamma^Z_{Za}\Gamma^a_{ZZ} + \Gamma^a_{ab}\Gamma^b_{ZZ} \\ &\quad - \Gamma^Z_{ZZ}\Gamma^Z_{ZZ} - \Gamma^Z_{Zb}\Gamma^b_{ZZ} - \Gamma^a_{ZZ}\Gamma^Z_{aZ} - \Gamma^a_{Zb}\Gamma^b_{aZ} \\ &= \Gamma^a_{aZ}\Gamma^Z_{ZZ} - \Gamma^a_{Zb}\Gamma^b_{aZ},\end{aligned}$$

where we have used $\Gamma^Z_{aZ} = \Gamma^a_{ZZ} = 0$ and cancelled out the products $\pm\Gamma^Z_{ZZ}\Gamma^Z_{ZZ}$. This leads us to (2.22), as claimed. □

Weak version of the momentum constraints

Next, we introduce the following definition.

Definition 2.17. A weakly regular T^2-symmetric triple (Σ, h, K) is said to satisfy the **weak version of the momentum constraints** if the equations

$$\begin{aligned} Z(\mathrm{Tr}^{(2)} K) - h(Z,Z)^{1/2}\,\mathrm{Tr}^{(2)}(\chi)K^Z_Z - \Gamma^a_{Zb}K^b_a &= 0, \\ Z(h(Z,Z)^{1/2}K^Z_a) - \Gamma^b_{bZ}K^Z_a &= 0, \qquad a = X, Y, \end{aligned} \tag{2.25}$$

hold in the weak sense, with $\mathrm{Tr}^{(2)}(K) = \mathrm{Tr}(K) - K^Z_Z$.

Observe that the second set of equations in (2.25) has been *weighted* by the scalar $h(Z,Z)^{1/2}$—in order for it to be well-defined in a weak sense, while the first equation has a different homogeneity in Z.

Proposition 2.18 (Equivalence to the classical definition). *Let (Σ, h, K) be a weakly regular T^2-symmetric triple. If h, K are sufficiently regular, then (Σ, h, K) satisfies the weak version of the momentum constraint equations (in the sense of Definition 2.17) if and only if it satisfies the constraint equations (1.2) in the classical sense.*

Proof. Assuming sufficient regularity and that the momentum constraint equations hold in the classical sense, i.e.

$$\nabla^{(3)j}K_{ij} - \nabla^{(3)}_i\,\mathrm{Tr}(K) = 0,$$

we begin by computing $\nabla^{(3)j}K_{Zj}$ in an adapted frame:

$$\begin{aligned}\nabla^{(3)j}K_{Zj} &= K^Z_{Z,Z} + K^a_{Z,a} - \Gamma^i_{jZ}K^j_i + \Gamma^j_{ji}K^i_Z \\ &= K^Z_{Z,Z} - \Gamma^Z_{ZZ}K^Z_Z - \Gamma^a_{ZZ}K^Z_a - \Gamma^Z_{aZ}K^a_Z - \Gamma^a_{bZ}K^b_a \\ &\quad + \Gamma^Z_{ZZ}K^Z_Z + \Gamma^Z_{Za}K^a_Z + \Gamma^b_{bZ}K^Z_Z + \Gamma^b_{ba}K^a_Z \\ &= K^Z_{Z,Z} - \Gamma^a_{bZ}K^b_a + \Gamma^b_{bZ}K^Z_Z,\end{aligned}$$

where we used $\Gamma^Z_{aZ} = \Gamma^a_{ZZ} = 0$ and *cancelled* (potentially problematic) terms $\pm\Gamma^Z_{ZZ}$. Since $h(Z,Z)^{1/2}\,\mathrm{Tr}(\chi) = -\Gamma^b_{bZ}$, the momentum constraint equation in the Z-direction is equivalent to the first equation in (2.25). For the remaining two momentum constraint equations, we write

$$\begin{aligned}
\nabla^{(3)}_j K^j_a &= K^j_{a,j} - \Gamma^i_{aj} K^j_i + \Gamma^j_{ji} K^i_a \\
&= Z(K^Z_a) - \Gamma^Z_{aZ} K^Z_Z - \sum_{(i,j)\neq(Z,Z)} \Gamma^i_{aj} K^j_i + \Gamma^Z_{ZZ} K^Z_a + \sum_{(i,j)\neq(Z,Z)} \Gamma^j_{ji} K^i_a \\
&= Z(K^Z_a) + \Gamma^Z_{ZZ} K^Z_a + \sum_{(i,j)\neq(Z,Z)} (\Gamma^j_{ji} K^i_a - \Gamma^i_{aj} K^j_i).
\end{aligned}$$

Recalling that the term involving Γ^Z_{ZZ} is not well-defined for weakly regular spacetimes, we multiply the above equations by $h(Z,Z)^{1/2}$ and expand the Christoffel symbol of the second term, in order to get

$$\begin{aligned}
h(Z,Z)^{1/2}\nabla^{(3)}_j K^j_a = h(Z,Z)^{1/2}\big(Z(K^Z_a) + \tfrac{1}{2} h(Z,Z)^{-1} Z(h(Z,Z)) K^Z_a\big) \\
+ h(Z,Z)^{1/2} \sum_{(i,j)\neq(Z,Z)} (\Gamma^j_{ji} K^i_a - \Gamma^i_{aj} K^j_i).
\end{aligned}$$

Combining the first two terms on the right-hand side yields the second set of equations in (2.25), as expected. We conclude that (1.2) and (2.25) are equivalent for sufficiently regular data. □

2.4. Weak version of Einstein's evolution equations

We are now in a position to discuss the Einstein equations. As before, we need first to examine the regularity of the Christoffel symbols, now associated with a spacetime metric.

Proposition 2.19 (Definition and regularity of the Christoffel symbols in an adapted frame). *Let* $(\mathcal{M}, g)$ *be a weakly regular* T^2*-symmetric Lorentzian manifold with adapted frame* (T, X, Y, Z)*, spacelike slices* Σ_t *with* $t \in I$*, and second fundamental form* K*, as introduced in Definition 2.8, and for all* $a, b = X, Y$ *define*

$$\begin{aligned}
&\Gamma^T_{ab} := -\frac{1}{n} K_{ab}, && \Gamma^T_{ZZ} := -\frac{1}{n} K(Z,Z), \\
&\Gamma^T_{TZ} = \Gamma^T_{ZT} := \frac{1}{n} Z(n), && \Gamma^T_{Za} = \Gamma^T_{aZ} := -\frac{1}{n} K(\cdot, Z)_a, \\
&\Gamma^T_{Ta} = \Gamma^T_{aT} := 0, && \Gamma^Z_{TZ} = \Gamma^Z_{ZT} := -g^{ZZ} n K(Z,Z), \\
&\Gamma^Z_{aT} = \Gamma^Z_{Ta} := -n g^{ZZ} K(\cdot, Z)_a, && \Gamma^a_{Tb} = \Gamma^a_{bT} := -g^{ac} n K_{bc}, \\
&\Gamma^a_{TZ} = \Gamma^a_{ZT} := n K(\cdot, Z)^a, && \Gamma^a_{TT} := 0, \\
&\Gamma^T_{TT} := T(n) n^{-1}, && \Gamma^Z_{TT} := n g^{ZZ} Z(n).
\end{aligned}$$

In addition, define Γ^i_{kj} *for* $i, j, k = X, Y, Z$ *as in Proposition* 2.12*, but with* h *replaced by* $h(t)$*. Then these functions are well-defined, have the regularity*

$$\Gamma^T_{ZZ}, \Gamma^Z_{TT}, \Gamma^T_{TT} \in L^1(\Sigma_t), \qquad \Gamma^T_{ab} \in L^2(\Sigma_t), \qquad \Gamma^T_{Zi}, \Gamma^Z_{iT}, \Gamma^i_{ZT} \in L^\infty(\Sigma_t), \tag{2.26}$$

uniformly in the time variable in any compact subset of I*. Furthermore, the following linear combinations of Christoffel symbols are better behaved:*

$$\Gamma^a_{Ta} = \Gamma^a_{aT} \in L^\infty(\Sigma_t), \tag{2.27}$$

$$\Gamma^T_{TT} - \Gamma^Z_{ZT} \in L^\infty(\Sigma_t), \qquad \Gamma^Z_{ZZ} - \Gamma^T_{ZT} \in L^\infty(\Sigma_t). \tag{2.28}$$

Finally, if $(\mathcal{M}, g)$ *is sufficiently regular, then the above definition coincides with the standard definition of the Christoffel symbols.*

Proof. In the frame $(T, X, Y, Z) = (e_0, e_1, e_2, e_3)$ (which is not induced by coordinates), and provided the data are sufficiently regular, we have the classical definition:

$$\Gamma^\alpha_{\beta\gamma} = \frac{1}{2} g^{\alpha\delta}(g_{\beta\delta,\gamma} + g_{\gamma\delta,\beta} - g_{\beta\gamma,\delta} + c_{\delta\beta\gamma} + c_{\delta\gamma\beta} + c_{\beta\gamma\delta}),$$
$$c_{\beta\gamma\delta} := [e_\beta, e_\gamma]_\delta = g_{\delta\rho}[e_\beta, e_\gamma]^\rho.$$

Except for (2.27) and (2.28), the regularity properties stated in the proposition follow immediately from our definitions. Then (2.27) is an immediate consequence of Lemma 2.11 and of the assumptions on $\mathrm{Tr}^{(2)}(K)$. To derive (2.28), we use the regularity assumption on $\rho = n^{-2} g_{ZZ}$ to obtain

$$\Gamma^T_{TT} - \Gamma^Z_{ZT} = \frac{1}{2n^2}\big(T(n^2) - T(g_{ZZ}) n^2 g^{ZZ}\big) = \frac{1}{2n^2} g_{ZZ} T(\rho^{-2})$$

and

$$\Gamma^Z_{ZZ} - \Gamma^T_{ZT} = \frac{1}{2} g^{ZZ} Z(g_{ZZ}) - \frac{1}{n} Z(n^2) = \frac{n^2}{2} g^{ZZ} Z\left(\frac{g_{ZZ}}{n^2}\right) = \frac{n^2}{2} g^{ZZ} Z(\rho^2).$$

It remains to check that the above definition agrees with the standard one when the spacetime is *sufficiently regular,* which we now assume. We recall that $[e_1, e_2] = [e_2, e_3] = 0$, and from the definition of T^2-symmetric spacetime, we also have $[e_0, e_2] = [e_0, e_3] = 0$. Indeed,

$$g([e_0, e_a], e_0) = 2e_a(g(e_0, e_0)) = 0$$

by using the symmetry assumptions, and

$$g([e_0, e_a], e_i) = -g(e_0, [e_a, e_i]) = 0$$

by the orthogonality of e_0, e_1, and the commutation property of e_a, e_i.

Moreover, still under the assumption that the spacetime is *sufficiently regular*, it follows from the definition of Z that $[Z, T] = fX + gY$ for some functions f and g, and in particular $[Z, T]$ is orthogonal to both Z and T. We now compute the Christoffel symbols as follows. For $i, j = 1, 2, 3$, we obtain

$$g(T, \nabla_i e_j) = g_{TT} \Gamma^T_{ij} = g(nN, \nabla_i e_j) = nK_{ij},$$

where N is the timelike unit normal to Σ_t, thus $\Gamma^T_{ij} = -\frac{1}{n} K_{ij}$.

For Γ^T_{TZ}, Γ^T_{TT}, Γ^T_{Ta} and Γ^Z_{TT}, one derives the desired formulas directly from (2.4), for instance

$$\Gamma^T_{TZ} = \frac{1}{2} g^{TT} (g_{TZ,T} + g_{TT,Z} - g_{TZ,T} + c_{TTZ} + c_{TZT} + c_{ZTT}) = Z(n)/n.$$

On the other hand, for $i = 1, 2, 3$ we find

$$g(e_1, \nabla_T e_i) = g_{ZZ} \Gamma^Z_{Ti} = g(e_1, \nabla_i T) = -g(\nabla_i e_1, T) = -ng(\nabla_i e_1, N) = -nK_{iZ}.$$

For Γ^a_{Tb}, we get

$$g(e_c, \nabla_T e_b) = g_{ca} \Gamma^a_{Tb} = g(e_c, \nabla_b T) = -g(\nabla_b e_c, T) = -nK_{bc},$$

while, for Γ^a_{TZ},

$$g(e_c, \nabla_T Z) = g_{ca} \Gamma^a_{TZ} = -g(\nabla_T e_c, Z) = -g(\nabla_c T, Z) = g(T, \nabla_c Z) = nK_{cZ},$$

and finally, for Γ^a_{ZT},

$$g(e_c, \nabla_Z T) = g_{ca} \Gamma^a_{ZT} = -g(\nabla_Z e_c, T) = -nK_{Zc}.$$ □

Finally, we introduce a weak version of Einstein's evolution equations. Given a (3 + 1)-splitting of Einstein equations and provided the constraint equations are satisfied on each slice, the evolution equations are equivalent to $R_{ij} = 0$ (cf. [5, Sec. VI-3.1]). Hence, since we have already derived the constraint equations in a weak form in the previous section, we can now restrict attention to the components R_{ij} of the Ricci curvature.

Definition 2.20. When $(\mathcal{M}, g)$ is a weakly regular T^2-symmetric Lorentzian manifold, the **weak version of the component** R_{ZZ} of the Ricci tensor is defined as

$$\begin{aligned} R_{ZZ} := {} & T(\Gamma^T_{ZZ}) - Z(\Gamma^T_{TZ}) - Z(\Gamma^a_{aZ}) - \Gamma^a_{Zb}\Gamma^b_{aZ} + \Gamma^a_{aT}\Gamma^T_{ZZ} + \Gamma^a_{aZ}\Gamma^Z_{ZZ} \\ & + 2\Gamma^T_{Za}\Gamma^a_{ZT} + \Gamma^T_{ZZ}(\Gamma^T_{TT} - \Gamma^Z_{ZT}) + \Gamma^T_{TZ}(\Gamma^Z_{ZZ} - \Gamma^T_{ZT}), \end{aligned} \tag{2.29}$$

in which the first three terms of the right-hand side are derivatives of L^p functions on each slice, while the remaining terms belong to L^1 on each slice. (Observe that the last two terms make sense, thanks to (2.28).)

Definition 2.21. When $(\mathcal{M}, g)$ is a weakly regular T^2-symmetric Lorentzian manifold, the **weak version of the components** R_{Zd} of the Ricci tensor is defined as

$$R_{Zd} := T(\Gamma^T_{dZ}) + \Gamma^T_{dZ}(\Gamma^T_{TT} - \Gamma^Z_{ZT}) + \Gamma^a_{aT}\Gamma^T_{dZ}.$$

Finally, the components R_{cd}, $c, d = X, Y$, need to be suitably weighted by the norm of the vector field Z, in order to be well-defined as distributions.

Definition 2.22. When $(\mathcal{M}, g)$ is a weakly regular T^2-symmetric Lorentzian manifold, the **weak version of the normalized components** R_{cd}^{norm} of the Ricci tensor is defined as

$$\begin{aligned} R_{cd}^{\text{norm}} := T(ng(Z,Z)^{1/2}\Gamma_{dc}^{T}) + Z(ng(Z,Z)^{1/2}\Gamma_{dc}^{Z}) \\ + ng(Z,Z)^{1/2}\big(\Gamma_{aT}^{a}\Gamma_{dc}^{T} + \Gamma_{aZ}^{a}\Gamma_{dc}^{Z} - \Gamma_{dZ}^{T}\Gamma_{Tc}^{Z} - \Gamma_{dT}^{Z}\Gamma_{Zc}^{T} \\ - \Gamma_{da}^{T}\Gamma_{Tc}^{a} - \Gamma_{dT}^{a}\Gamma_{ac}^{T} - \Gamma_{da}^{Z}\Gamma_{Zc}^{a} - \Gamma_{dZ}^{a}\Gamma_{ac}^{Z}\big). \end{aligned}$$

Definition 2.23. A weakly regular T^2-symmetric Lorentzian manifold $(\mathcal{M}, g)$ is said to satisfy the **weak version of Einstein's evolution**[5] **equations** if

$$R_{ZZ} = 0, \quad R_{Zd} = 0, \quad R_{cd}^{\text{norm}} = 0, \quad c, d = X, Y, \tag{2.30}$$

in the weak sense introduced in Definitions 2.20 to 2.22.

Again, we have an equivalence result establishing the link with the classical definition.

Proposition 2.24 (Equivalence to the classical definition). *For any sufficiently regular T^2-symmetric spacetime $(\mathcal{M}, g)$, the weak version (2.30) of the Einstein evolution equations is satisfied if and only if the Ricci flatness condition*

$$\text{Ric}(e_i, e_j) = 0, \quad e_i, e_j \in \{X, Y, Z\},$$

holds, where Ric *denotes the Ricci tensor of g defined in the classical sense.*

Before providing a proof of this result, we summarize our conclusions in this section:

Theorem 2.25 (Weak formulation of the Einstein equations). *If (Σ, h, K) is a weakly regular T^2-symmetric triple, then Einstein's constraint equations (1.1)–(1.2) make sense in the weak form (2.24)–(2.25). Similarly, if $(\mathcal{M}, g)$ is a weakly regular T^2-symmetric Lorentzian manifold, Einstein's evolution equations (1.3) make sense in the weak form (2.30). Furthermore, the new geometric objects introduced in Definitions 2.13–2.23 coincide with the classical ones when sufficiently high regularity is assumed.*

We have thus restated and established Theorem 1.1 (presented earlier in the introduction). We refer to a weakly regular T^2-symmetric triple satisfying the weak version of the Hamiltonian and momentum constraint equations as a **weakly regular T^2-symmetric initial data set.** Analogously, we refer to a weakly regular T^2-symmetric Lorentzian manifold satisfying the weak version of the Einstein constraint and evolution equations as a **weakly regular T^2-symmetric vacuum spacetime.**

Proof of Proposition 2.24. We assume that g is smooth and we will show that the distributions defined by $\text{Ric}(e_i, e_j)$ for $e_i, e_j = X, Y, Z$ agree with the ones introduced in

[5] Of course, we shall later consider only spacetimes which are solutions of both the evolution and constraint equations.

(2.20)–(2.22) (where an additional weight must be introduced for $\mathrm{Ric}(e_c, e_d)$). Abusing notation, we denote the Christoffel symbols defined in the classical sense by $\Gamma^{\alpha}_{\beta\delta}$. With $c^f_{aZ} = [e_c, e_d]^f$, we find

$$\mathrm{Ric}(Z, Z) = R^{\alpha}_{Z\alpha Z} = \Gamma^{\alpha}_{ZZ,\alpha} - \Gamma^{\alpha}_{\alpha Z,Z} + \Gamma^{\alpha}_{\alpha\beta}\Gamma^{\beta}_{ZZ} - \Gamma^{\alpha}_{Z\beta}\Gamma^{\beta}_{YZ} - c^{\beta}_{\alpha Z}\Gamma^{\alpha}_{Z\beta}. \tag{2.31}$$

We expand the right-hand side of (2.31) by focusing our attention on the terms which, for a T^2-symmetric solution having only weak regularity, are a priori not well-defined:

$$\begin{aligned}\mathrm{Ric}(Z, Z) = {} & Z(\Gamma^Z_{ZZ}) + T(\Gamma^T_{ZZ}) - (Z(\Gamma^Z_{ZZ}) + Z(\Gamma^T_{TZ}) + Z(\Gamma^a_{aZ}))\\ & + \Gamma^T_{TT}\Gamma^T_{ZZ} + \Gamma^Z_{ZZ}\Gamma^Z_{ZZ} + \Gamma^T_{TZ}\Gamma^Z_{ZZ} + \Gamma^Z_{ZT}\Gamma^T_{ZZ}\\ & + \Gamma^T_{ta}\Gamma^a_{ZZ} + \Gamma^a_{at}\Gamma^T_{ZZ} + \Gamma^Z_{Za}\Gamma^a_{ZZ} + \Gamma^a_{aZ}\Gamma^Z_{ZZ}\\ & + \Gamma^a_{ab}\Gamma^b_{ZZ} - (\Gamma^T_{ZT}\Gamma^T_{TZ} + \Gamma^Z_{ZZ}\Gamma^Z_{ZZ} + \Gamma^T_{ZZ}\Gamma^Z_{TZ} + \Gamma^Z_{ZT}\Gamma^T_{ZZ})\\ & - (\Gamma^T_{Za}\Gamma^a_{TZ} + \Gamma^a_{ZT}\Gamma^T_{aZ} + \Gamma^Z_{Za}\Gamma^a_{ZZ} + \Gamma^a_{ZZ}\Gamma^Z_{aZ}) - \Gamma^a_{Zb}\Gamma^b_{aZ} - c^b_{TZ}\Gamma^T_{Zb}.\end{aligned}$$

To handle the latter term, we observe that the only non-vanishing commutator is $[T, Z]$, and it is orthogonal to both Z, T. Note that this last term can be rewritten in terms of the connection coefficients since

$$[T, Z]^b = \Gamma^b_{TZ} - \Gamma^b_{ZT}.$$

Taking into account the cancellations in $Z(\Gamma^Z_{ZZ})$, $\Gamma^Z_{ZZ}\Gamma^Z_{ZZ}$ and $\Gamma^Z_{ZT}\Gamma^T_{ZZ}$, as well as the antisymmetry of Γ^a_{ZT} and the fact that $\Gamma^T_{ta} = \Gamma^Z_{Za} = 0$, we obtain

$$\begin{aligned}\mathrm{Ric}(Z, Z) = {} & T(\Gamma^T_{ZZ}) - Z(\Gamma^T_{TZ}) - Z(\Gamma^a_{aZ}) + (\Gamma^T_{TT}\Gamma^T_{ZZ} + \Gamma^T_{TZ}\Gamma^Z_{ZZ})\\ & + (\Gamma^a_{at}\Gamma^T_{ZZ} + \Gamma^a_{aZ}\Gamma^Z_{ZZ}) + \Gamma^a_{ab}\Gamma^b_{ZZ}\\ & - (\Gamma^T_{ZT}\Gamma^T_{TZ} + \Gamma^T_{ZZ}\Gamma^Z_{TZ}) - \Gamma^a_{Zb}\Gamma^b_{aZ} + 2\Gamma^T_{Za}\Gamma^a_{ZT},\end{aligned}$$

and the expression for $\mathrm{Ric}(T, T)$ then follows by factoring out Γ^Z_{ZT} and Γ^Z_{TT}.

For $\mathrm{Ric}(Z, e_d)$, we proceed similarly and obtain

$$\begin{aligned}\mathrm{Ric}(Z, e_d) & = \Gamma^{\alpha}_{dZ,\alpha} - \Gamma^{\alpha}_{\alpha Z,d} + \Gamma^{\alpha}_{\alpha\beta}\Gamma^{\beta}_{dZ} - \Gamma^{\alpha}_{d\beta}\Gamma^{\beta}_{\alpha Z} - c^{\beta}_{\alpha d}\Gamma^{\alpha}_{Z\beta}\\ & = T(\Gamma^T_{dZ}) + Z(\Gamma^Z_{Zd}) + (\Gamma^T_{TT}\Gamma^T_{dZ} + \Gamma^Z_{ZZ}\Gamma^Z_{dZ} + \Gamma^T_{TZ}\Gamma^Z_{dZ} + \Gamma^Z_{ZT}\Gamma^T_{dZ})\\ & \quad + \Gamma^T_{ta}\Gamma^a_{dZ} + \Gamma^a_{at}\Gamma^T_{dZ} + \Gamma^Z_{Za}\Gamma^a_{dZ} + \Gamma^a_{aZ}\Gamma^Z_{dZ}\\ & \quad + \Gamma^a_{ab}\Gamma^b_{dZ} - (\Gamma^T_{dt}\Gamma^T_{TZ} + \Gamma^Z_{dZ}\Gamma^Z_{ZZ} + \Gamma^T_{dZ}\Gamma^Z_{TZ} + \Gamma^Z_{dT}\Gamma^T_{ZZ})\\ & \quad - (\Gamma^T_{da}\Gamma^a_{TZ} + \Gamma^a_{dT}\Gamma^T_{aZ} + \Gamma^Z_{da}\Gamma^a_{ZZ} + \Gamma^a_{dZ}\Gamma^Z_{aZ}) - \Gamma^a_{db}\Gamma^b_{aZ}.\end{aligned}$$

Next, using $\Gamma^a_{ZZ} = \Gamma^Z_{Za} = \Gamma^T_{ta} = \Gamma^a_{bc} = 0$ and the fact that X, Y commute with Z, T, we obtain

$$\begin{aligned}\mathrm{Ric}(Z, e_d) = {} & T(\Gamma^T_{dZ}) + \Gamma^T_{TT}\Gamma^T_{dZ} + \Gamma^Z_{ZT}\Gamma^T_{dZ} + \Gamma^a_{aT}\Gamma^T_{dZ}\\ & - (\Gamma^T_{dZ}\Gamma^Z_{TZ} + \Gamma^Z_{dT}\Gamma^T_{ZZ}) - (\Gamma^T_{da}\Gamma^a_{TZ} + \Gamma^a_{dT}\Gamma^T_{aZ}).\end{aligned}$$

We also note the cancellation in $\Gamma_{ZT}^{Z}\Gamma_{dZ}^{T}$ as well as the identities

$$\Gamma_{TT}^{T}\Gamma_{dZ}^{T} - \Gamma_{dT}^{z}\Gamma_{ZZ}^{T} = \Gamma_{Zd}^{T}(\Gamma_{TT}^{T} - \Gamma_{ZT}^{Z}), \qquad -\Gamma_{da}^{T}\Gamma_{TZ}^{a} - \Gamma_{dT}^{a}\Gamma_{aZ}^{T} = 0,$$

and arrive at the desired formula

$$\mathrm{Ric}(Z, e_d) = T(\Gamma_{dZ}^{T}) + \Gamma_{Zd}^{T}(\Gamma_{TT}^{T} - \Gamma_{ZT}^{Z}) + \Gamma_{aT}^{a}\Gamma_{dZ}^{T}.$$

Next, for $\mathrm{Ric}(e_c, e_d)$, we have

$$\begin{aligned}
\mathrm{Ric}(e_c, e_d) &= \Gamma_{dc,\alpha}^{\alpha} - \Gamma_{\alpha c,d}^{\alpha} + \Gamma_{\alpha\beta}^{\alpha}\Gamma_{dc}^{\beta} - \Gamma_{d\beta}^{\alpha}\Gamma_{\alpha c}^{\beta} - c_{\alpha d}^{\beta}\Gamma_{c\beta}^{\alpha} \\
&= T(\Gamma_{dc}^{T}) + Z(\Gamma_{dc}^{z}) + (\Gamma_{TT}^{T}\Gamma_{dc}^{T} + \Gamma_{ZZ}^{Z}\Gamma_{dc}^{Z}) + (\Gamma_{TZ}^{T}\Gamma_{dc}^{Z} + \Gamma_{ZT}^{Z}\Gamma_{dc}^{T}) \\
&\quad + (\Gamma_{Ta}^{T}\Gamma_{dc}^{a} + \Gamma_{Za}^{Z}\Gamma_{dc}^{a}) + (\Gamma_{aT}^{a}\Gamma_{dc}^{T} + \Gamma_{aZ}^{a}\Gamma_{dc}^{Z}) + \Gamma_{ab}^{a}\Gamma_{dc}^{b} - (\Gamma_{dT}^{T}\Gamma_{Tc}^{T} + \Gamma_{dz}^{z}\Gamma_{Zc}^{Z}) \\
&\quad - (\Gamma_{dZ}^{T}\Gamma_{Tc}^{z} + \Gamma_{dT}^{Z}\Gamma_{Zc}^{T}) - (\Gamma_{da}^{T}\Gamma_{Tc}^{a} + \Gamma_{dT}^{a}\Gamma_{ac}^{T}) - (\Gamma_{da}^{Z}\Gamma_{Zc}^{a} + \Gamma_{dZ}^{a}\Gamma_{ac}^{Z}) - \Gamma_{da}^{b}\Gamma_{bc}^{a}
\end{aligned}$$

and, using $\Gamma_{Za}^{Z} = \Gamma_{Ta}^{T} = \Gamma_{bc}^{a} = 0$,

$$\begin{aligned}
\mathrm{Ric}(e_c, e_d) = T(\Gamma_{dc}^{T}) + Z(\Gamma_{dc}^{z}) &+ (\Gamma_{TT}^{T}\Gamma_{dc}^{T} + \Gamma_{ZZ}^{Z}\Gamma_{dc}^{Z}) + (\Gamma_{TZ}^{T}\Gamma_{dc}^{Z} + \Gamma_{ZT}^{Z}\Gamma_{dc}^{T}) \\
&+ (\Gamma_{at}^{a}\Gamma_{dc}^{T} + \Gamma_{aZ}^{a}\Gamma_{dc}^{Z}) - (\Gamma_{dz}^{T}\Gamma_{Tc}^{Z} + \Gamma_{dT}^{z}\Gamma_{Zc}^{T}) \\
&- (\Gamma_{da}^{T}\Gamma_{Tc}^{a} + \Gamma_{dT}^{a}\Gamma_{ac}^{T}) - (\Gamma_{da}^{Z}\Gamma_{Zc}^{a} + \Gamma_{dZ}^{a}\Gamma_{ac}^{Z}).
\end{aligned}$$

The first six terms of the right-hand side above can be rewritten as

$$\begin{aligned}
T(\Gamma_{dc}^{T}) + \Gamma_{TT}^{T}\Gamma_{dc}^{T} + \Gamma_{dc}^{T}\Gamma_{ZT}^{Z} &= T(\Gamma_{dc}^{T}) + \Gamma_{dc}^{T}\frac{T(n)}{n} + \Gamma_{dc}^{T}g(Z, Z)^{-1/2}T(g(Z, Z)^{1/2}) \\
&= n^{-1}g(Z, Z)^{-1/2}T(ng(Z, Z)^{1/2}\Gamma_{dc}^{T}),
\end{aligned}$$

and similarly

$$\begin{aligned}
Z(\Gamma_{dc}^{Z}) + \Gamma_{ZZ}^{Z}\Gamma_{dc}^{Z} + \Gamma_{dc}^{Z}\Gamma_{TZ}^{T} &= Z(\Gamma_{dc}^{z}) + \Gamma_{dc}^{Z}\frac{Z(n)}{n} + \Gamma_{dc}^{Z}g(Z, Z)^{-1/2}Z(g(Z, Z)^{1/2}) \\
&= n^{-1}g(Z, Z)^{-1/2}Z(ng(Z, Z)^{1/2}\Gamma_{dc}^{Z}).
\end{aligned}$$

This suggests introducing a weight in $\mathrm{Ric}(e_c, e_d)$, that is, $ng(Z, Z)^{1/2}$, which leads us to the desired expression:

$$\begin{aligned}
ng(Z, Z)^{1/2}\,\mathrm{Ric}(e_c, e_d) = T(ng(Z, Z)^{1/2}\Gamma_{dc}^{T}) &+ Z(ng(Z, Z)^{1/2}\Gamma_{dc}^{Z}) \\
&+ ng(Z, Z)^{1/2}\big(\Gamma_{aT}^{a}\Gamma_{dc}^{T} + \Gamma_{aZ}^{a}\Gamma_{dc}^{Z} - (\Gamma_{dZ}^{T}\Gamma_{Tc}^{Z} + \Gamma_{dT}^{Z}\Gamma_{Zc}^{T}) \\
&\quad - (\Gamma_{da}^{T}\Gamma_{Tc}^{a} + \Gamma_{dT}^{a}\Gamma_{ac}^{T}) - (\Gamma_{da}^{Z}\Gamma_{Zc}^{a} + \Gamma_{dZ}^{a}\Gamma_{ac}^{Z})\big). \qquad \square
\end{aligned}$$

2.5. Twist coefficients

We end this section with an important property of the so-called **twist coefficients** associated with two Killing fields X, Y. In the smooth case, they are defined by

$$C_X := \mathcal{E}_{\alpha\beta\gamma\delta} X^\alpha Y^\beta \nabla^\gamma X^\delta, \quad C_Y := \mathcal{E}_{\alpha\beta\gamma\delta} Y^\alpha Y^\beta \nabla^\gamma X^\delta,$$

where $\mathcal{E}_{\alpha\beta\gamma\delta}$ is the volume form of $(\mathcal{M}, g)$. On the other hand, under our weak regularity assumptions, we can rely on an adapted frame (T, X, Y, Z) and set

$$C_X := \mathcal{E}_{\alpha\beta\gamma\delta} X^\alpha Y^\beta g^{\rho\gamma} \Gamma^\delta_{X\rho}, \quad C_Y := \mathcal{E}_{\alpha\beta\gamma\delta} Y^\alpha Y^\beta g^{\rho\gamma} \Gamma^\delta_{Y\rho},$$

where each term is evaluated in the frame (T, X, Y, Z) and so the expresssions above are well-defined since they involve products of L^∞ by L^1 functions, or L^2 by L^2 functions.

We recall that for all *sufficiently regular* spacetimes, it is well-known that the vacuum Einstein equations imply that the twists are constant [8]. We check now that this property is preserved at our level of (weak) regularity.

Proposition 2.26 (Constant twist property). *The twist coefficients of any weakly regular T^2-symmetric spacetime are constants. Furthermore, one can always choose the Killing fields X, Y in such a way that one of them vanishes identically.*

Proof. It follows from the antisymmetry of the volume form that

$$C_X = \mathcal{E}_{XYTZ} g^{TT} \Gamma^Z_{TX} + \mathcal{E}_{XYZT} g^{ZZ} \Gamma^T_{ZX}.$$

Moreover, in view of the relation $\Gamma^Z_{TX} = n^2 g^{ZZ} \Gamma^T_{ZX}$, we have

$$C_X = 2\mathcal{E}_{XYZT} g^{ZZ} \Gamma^T_{ZX} = -2\frac{R}{\rho} \Gamma^T_{ZX}.$$

It follows immediately from one of the Hamiltonian constraint equations and the evolution equation $R_{ZX} = 0$ that C_X is a constant. The same holds for C_Y, and moreover one of the twists can be made to vanish by introducing a suitable linear combination of the Killing vectors, say

$$X' = aX + bY, \quad Y' = cX + dY, \quad ad - bc = 1, \tag{2.32}$$

where the last restriction on a, b, c, d ensures that the transformation preserves the periodicity property. Then the conclusion follows easily from

$$C_{X'} = \mathcal{E}_{\alpha\beta\gamma\delta} X'^\alpha Y'^\beta \nabla^\gamma X'^\delta = (ad - bc)(aC_X + bC_Y). \qquad \square$$

3. Weakly regular metrics in admissible coordinates

3.1. Weakly regular Riemannian manifolds in admissible coordinates

In this section, we introduce several choices of coordinates, in which we will later (cf. Section 4) express the weak version of the Einstein equations in a form amenable to techniques of analysis for nonlinear partial differential equations. We determine here the regularity of the metric coefficients that is implied by the geometric regularity assumptions made in the previous section. From now on, functions invariant by the action of the Killing fields are identified with functions defined on the circle S^1 (and, later in this section, also depending on a time variable).

If (X, Y, Z) is an adapted frame, with Z being the orthogonal projection on $\{X, Y\}^\perp$ of some vector field Θ (commuting with X,Y), a system of coordinates (x, y, θ) such that (X, Y, Θ) is the basis of vector fields induced by (x, y, θ) is said to be adapted to the symmetry or *admissible*.

Lemma 3.1 (Weakly regular T^2-symmetric metrics in admissible coordinates). *Let (Σ, h) be a weakly regular T^2-symmetric Riemannian manifold and (x, y, θ) be coordinates adapted to the symmetry. Then the metric h takes the form*

$$h = \frac{e^{2\overline{\nu}-2\overline{P}}}{\overline{R}} d\theta^2 + e^{2\overline{P}}\overline{R}\big(dx + \overline{A}dy + (\overline{G} + \overline{A}\,\overline{H})d\theta\big)^2 + e^{-2\overline{P}}\overline{R}(dy + \overline{H}d\theta)^2, \quad (3.1)$$

where the coefficients $\overline{R}$, $\overline{P}$, $\overline{A}$, $\overline{\nu}$, $\overline{G}$, $\overline{H}$ depend on the variable $\theta \in S^1$ only, and satisfy

$$\overline{P}, \overline{A} \in H^1(S^1), \qquad \overline{\nu} \in W^{1,1}(S^1), \qquad \overline{G}, \overline{H} \in L^\infty(S^1),$$

while the area function $\overline{R}$ (already defined in (2.3)) satisfies $\overline{R} \in W^{1,\infty}(S^1)$ and is bounded above and below by positive constants.

Proof. We rely here on the conditions introduced in Definition 2.4. Clearly, any metric can be expressed in the form (3.1), provided one defines $\overline{\nu}$, $\overline{P}$, $\overline{A}$, $\overline{G}$ and $\overline{H}$ by $h(X, X) =: e^{2\overline{P}}\overline{R}$, $h(X, Y) =: \overline{R}e^{2\overline{P}}\overline{A}, \ldots$ Since the metric is T^2-symmetric, all coefficients are independent of the variables (x, y). By our assumption (2.3), we have $\overline{R} \in W^{1,\infty}(\Sigma)$ and, after identifying $\overline{R}$ with a function on S^1, it follows that $\overline{R} \in W^{1,\infty}(S^1)$. Since $e^{2\overline{P}}\overline{R} = h(X, X) \in H^1(\Sigma)$, we obtain $e^{2\overline{P}} \in H^1(\Sigma)$, and after identifying $e^{2\overline{P}}$ with a function of $\theta \in S^1$, it follows that $e^{2\overline{P}} \in H^1(S^1)$. In particular, $\overline{P}$ (defined almost everywhere) admits a Hölder continuous representative. Since $e^{2\overline{P}} \in C^0(S^1)$ is positive and defined on the compact set S^1, its inverse $e^{-2\overline{P}}$ also belongs to the space $L^\infty(S^1)$. From this, it also follows that $\overline{P}_\theta = \frac{1}{2}e^{-2\overline{P}}(e^{2\overline{P}})_\theta$ belongs to $L^2(S^1)$, and we conclude that $\overline{P} \in H^1(S^1)$. A completely similar argument applies to $\overline{A}$ and shows that $\overline{A} \in H^1(S^1)$.

For $\overline{G}$ and $\overline{H}$, we have $h(X, Z) = e^{2\overline{P}}\overline{R}(\overline{G} + \overline{A}\,\overline{H}) \in L^\infty(S^1)$ and thus $\overline{P} \in C^0(S^1)$ and $\overline{R} \in C^0(S^1)$. So, we find

$$\overline{G} + \overline{A}\,\overline{H} \in L^\infty(S^1). \quad (3.2)$$

On the other hand, from the assumptions on $h(Y, Z)$, we also know that

$$h(Y, Z) = \overline{R}e^{2\overline{P}}\overline{A}(\overline{G} + \overline{A}\,\overline{H}) + e^{-2\overline{P}}\overline{R}\,\overline{H} \in L^\infty(S^1),$$

in which the first term is in $L^\infty(S^1)$ by (3.2), and so we have $e^{-2\overline{P}}\overline{R}\,\overline{H} \in L^\infty(S^1)$. Moreover, using the lower bound on $\overline{R}$, the function $e^{2\overline{P}}/\overline{R}$ belongs to $L^\infty(S^1)$ and thus $\overline{H} \in L^\infty(S^1)$. From (3.2), it then follows that $\overline{G} \in L^\infty$. Finally, by observing that $h(Z, Z)$ provides a control of $e^{2\overline{\nu}}$, similar arguments show that $\overline{\nu}$ belongs to $W^{1,1}(S^1)$. □

Relying on Definition 2.5, we now introduce a decomposition of the tensor field K and specify the regularity of each component. The proof of the following statement is omitted.

Lemma 3.2 (Decomposition of weakly regular tensor fields K). *Let (Σ, h, K) be a weakly regular T^2-symmetric triple in admissible coordinates (3.1). Then there exist functions $\underset{0}{\overline{P}}$, $\underset{0}{\overline{A}}$, $\underset{0}{\overline{G}}$, $\underset{0}{\overline{H}}$, $\underset{0}{\overline{R}}$, $\underset{0}{\overline{\nu}}$ and a symmetric 2-tensor $\underset{0}{h_{ab}}$ such that, in an adapted frame (X, Y, Z), the components of K read*

$$\begin{aligned}
K_{ab} &= \tfrac{1}{2}h(Z,Z)^{-1/2}\underset{0}{h_{ab}},\\
K(X,Z) &= \tfrac{1}{2}e^{-\overline{\nu}+3\overline{P}}(\underset{0}{\overline{G}} + \overline{A}\,\underset{0}{\overline{H}}),\\
K(Y,Z) &= \tfrac{1}{2}e^{-\nu-\overline{P}}\overline{R}^2\underset{0}{\overline{H}} + \overline{A}K(X,Z) = \tfrac{1}{2}e^{-\overline{\nu}+\overline{P}}\big(\overline{R}^2e^{-2\overline{P}}\underset{0}{\overline{H}} + \overline{A}e^{2\overline{P}}(\underset{0}{\overline{G}} + \overline{A}\,\underset{0}{\overline{H}})\big),\\
\mathrm{Tr}^{(2)}(K) &= e^{-\overline{\nu}+\overline{P}}\underset{0}{\overline{R}}\,\overline{R}^{-1} = e^{-\overline{\nu}+\overline{P}}\overline{R}^{-1/2}\underset{0}{\overline{R}},\\
K_{ZZ} &= e^{\overline{\nu}-\overline{P}}\overline{R}^{-1/2}\big(\underset{0}{\overline{\nu}} - \underset{0}{\overline{P}} - \underset{0}{\overline{R}}(2\overline{R})^{-1}\big),
\end{aligned}$$

with

$$\begin{aligned}
\underset{0}{h_{ab}} &= \overline{R}^{-1}\underset{0}{\overline{R}}h_{ab} + \overline{R}\big(e^{2P}2\underset{0}{\overline{P}}(dx + \overline{A}dy)^2 - 2\underset{0}{\overline{P}}e^{-2\overline{P}}dy^2\big)\\
&\quad + \overline{R}e^{2\overline{P}}(2\underset{0}{\overline{A}}dxdy + 2\overline{A}\,\underset{0}{\overline{A}}dy^2),
\end{aligned}$$

and the following regularity properties hold:

$$\underset{0}{\overline{P}}, \underset{0}{\overline{A}}, \underset{0}{\overline{G}}, \underset{0}{\overline{H}}, \underset{0}{h_{ab}} \in L^2(S^1), \qquad \underset{0}{\overline{R}} \in W^{1,\infty}(S^1), \qquad \underset{0}{\overline{\nu}} \in L^1(S^1).$$

3.2. Weakly regular Lorentzian manifolds in admissible coordinates

If $(\mathcal{M}, g)$ is a weakly regular T^2-symmetric Lorentzian manifold, a system of global smooth coordinates (t', x', y', θ') on $\mathcal{M}$ such that the metric when expressed in these coordinates takes the form and the regularity of Definitions 2.7 and 2.8 will be called **admissible coordinates adapted to the symmetry** or simply **admissible coordinates**. From now on, (t, x, y, θ) will denote an arbitrary system of admissible coordinates. We shall also refer to the expression (3.3) below as the **metric in admissible coordinates.**

In the context of Definition 2.8, applying Lemma 3.1 to each slice of the foliation, one has the following result.

Lemma 3.3 (Weakly regular (3 + 1)-metrics in admissible coordinates). *Let $(\mathcal{M}, g)$ be a weakly regular T^2-symmetric spacetime and (t, x, y, θ) be admissible coordinates adapted to the symmetry. Then the spacetime metric g takes the form*

$$g = -n^2 dt^2 + \frac{e^{2\nu-2P}}{R} d\theta^2 + e^{2P} R\big(dx + A dy + (G + AH)d\theta\big)^2 + e^{-2P} R(dy + H d\theta)^2 \tag{3.3}$$

with coefficients P, A, ν, G, H depending only on $t \in I$ and $\theta \in S^1$, satisfying

$$P, A \in L^\infty_{\text{loc}}(I, H^1(S^1)), \qquad \nu \in L^\infty_{\text{loc}}(I, W^{1,1}(S^1)), \qquad G, H \in L^\infty_{\text{loc}}(I, L^\infty(S^1)).$$

and such that the area function R (defined in (2.3)) satisfies $R \in W^{1,\infty}(I \times S^1)$ and is bounded above and below by positive constants.

From now on, a subscript (like t and θ) denotes a partial derivative, possibly understood in the weak sense. The regularity assumed on the second fundamental form implies some regularity on the time derivative of the metric coefficients.

Lemma 3.4 (Timelike regularity in admissible coordinates). *Let $(\mathcal{M}, g)$ be a weakly regular T^2-symmetric spacetime and (t, x, y, θ) be admissible coordinates adapted to the symmetry, with $t \in I$. Then the metric coefficients in (3.3) enjoy the following regularity in time:*

$$P_t, A_t \in L^\infty_{\text{loc}}(I, L^2(S^1)), \qquad R_t \in L^\infty_{\text{loc}}(I, L^\infty(S^1)),$$
$$\nu_t \in L^\infty_{\text{loc}}(I, L^1(S^1)), \qquad G_t, H_t \in L^\infty_{\text{loc}}(I, L^2(S^1)).$$

Proof. In view of Definition 2.8, the components of K satisfy

$$L^1(S^1) \ni 2nK(Z, Z) = \big(e^{-2P} H^2 R + Re^{2P}(G + AH)^2 + e^{2\nu-2P} R^{-1}\big)_t,$$

while all other components belong to $L^2(S^1)$:

$$\begin{aligned}
&2nK(Z, X) = \big(e^{2P} R(G + AH)\big)_t,\\
&2nK(Z, Y) = \big(e^{2P} RA(G + AH) + e^{-2P} RH\big)_t,\\
&2nK(X, X) = (e^{2P} R)_t, \qquad 2nK(X, Y) = (e^{2P} RA)_t,\\
&2nK(Y, Y) = (e^{2P} RA^2 + Re^{-2P})_t
\end{aligned}$$

with, moreover,

$$\begin{aligned}
L^\infty(S^1) \ni \mathrm{Tr}^{(2)}(K) = {} & e^{2P} R^{-1} K(Y, Y) - 2Ae^{2P} R^{-1} K(X, Y)\\
& + (e^{2P} A^2 R^{-1} + 1)K(X, X).
\end{aligned}$$

We first use the conditions $(e^{2P} R)_t \in L^2(S^1)$ and $(e^{2P} RA)_t \in L^2(S^1)$ and deduce that $P_t, A_t \in L^2(S^1)$. Then the condition on $\mathrm{Tr}^{(2)}(K)$ implies that $R_t \in L^\infty(S^1)$. We then deduce a control on the functions G_t, H_t, and finally the condition $2nK(Z, Z) \in L^1(S^1)$ yields $\nu_t \in L^1(S^1)$. □

3.3. *Conformal coordinates for weakly regular metrics*

A well-known problem in general relativity and, more generally, in geometric analysis is to exploit the gauge freedom at our disposal to simplify the analysis. This typically means choosing a coordinate system or a frame well-adapted to the problem. Here, it will turn out that we need to make two different gauges, i.e. we will use two different choices of admissible coordinates, specifically the so-called conformal and areal coordinate systems. We begin by proving the existence of conformal coordinates.

Lemma 3.5 (Existence of conformal coordinates). *Let $(\mathcal{M}, g)$ be a weakly regular T^2-symmetric spacetime and (t, x, y, θ) be admissible coordinates adapted to the symmetry, with $t \in I$ and $x, y, \theta \in S^1$. Also assume that the function ρ introduced in (2.15) belongs*[6] *to the space $W^{2,1}$. Then there exist functions $\tau, \xi : \mathcal{M} \to \mathbb{R}$ such that:*

1. *In the coordinates (t, x, y, θ), the functions τ, ξ depend on (t, θ) only, and belong to $W^{1,\infty}_{\text{loc}}(I \times S^1)$.*
2. *The functions τ, ξ, x, y determine a global chart on $\mathcal{M}$ and hence define a smooth differential structure on $\mathcal{M}$. Moreover, the charts (τ, ξ, x, y) and (t, θ, x, y) are $W^{1,\infty}$ compatible at least (but need not be C^∞ compatible).*
3. *In the coordinate system (τ, ξ, x, y), the metric takes the form*

$$g = \frac{e^{2\nu - 2P}}{R}(-d\tau^2 + d\xi^2) + e^{2P} R\big(dx + A dy + (G + AH)d\xi\big)^2 + e^{-2P} R(dy + H d\xi)^2, \tag{3.4}$$

   where the coefficients ν, P, A, R, G, H depend on $\tau \in J$ and $\xi \in S^1$ only, where J is an interval.
4. *The hypersurface $t = t_0$ coincides with a level set of τ.*

In fact, the coefficients of the metric, when expressed as functions of (τ, ξ), will also enjoy the same regularity properties as those presented in Lemmas 3.3 and 3.4, provided the weak version of the Einstein equations holds true. This fact will be checked later in Section 5.

Proof of Lemma 3.5. We restrict attention to the quotient metric

$$\widehat{g} = \frac{e^{2\nu - 2P}}{R}(-\rho^2 dt^2 + d\theta^2),$$

and establish the existence of functions τ, ξ such that

$$\widehat{g} = \frac{e^{2\hat{\nu} - 2P}}{R}(-d\tau^2 + d\xi^2)$$

(the relation between ν and $\hat{\nu}$ being specified below). We are going to construct null coordinates $u, v : \mathcal{M} \to \mathbb{R}$ enjoying the following properties:

[6] This higher regularity will indeed be established within our proof of existence.

1. The functions u, v depend on (t, θ) only, and belong to $W^{1,\infty}(I \times S^1)$.
2. The following equations hold:

$$u_t + \rho u_\theta = 0, \qquad v_t - \rho v_\theta = 0. \tag{3.5}$$

3. The following periodicity conditions hold:

$$u(t, \theta + 2\pi) = u(t, \theta) - 2\pi, \qquad v(t, \theta + 2\pi) = v(t, \theta) + 2\pi. \tag{3.6}$$

4. The map $I \times S^1 \ni (t, \theta) \mapsto (u, v)$ is a $W^{1,\infty}$ diffeomorphism onto its image.

Once this is established, one easily checks that the functions

$$\xi := \frac{v - u}{2}, \qquad \tau := \frac{v + u}{2}$$

satisfy the desired requirements, and that the functions ν and $\hat{\nu}$ (in the expressions of the metric) are related by writing $du = u_t dt + u_\theta d\theta$ and $dv = v_t dt + v_\theta d\theta$, which lead to

$$e^{2\nu} = e^{2\hat{\nu}} u_\theta v_\theta.$$

Actually, the equations (3.5) are linear transport equations and are easily solved by the method of characteristics. First of all, setting $I = [t_1, t_2]$, we can choose the initial data $t - \theta$ and $t + \theta$ for the functions u, v at time t_1, that is,

$$u(t_1, \cdot) := t_1 - \theta, \qquad v(t_1, \cdot) := t_1 + \theta.$$

Then we consider the characteristic equations

$$\frac{d\overline{\theta}}{dt} = \pm\rho(t, \overline{\theta}(t))$$

with initial condition $\overline{\theta}(t_1, \theta) = \pm\theta$, and we denote by $\overline{\theta}_\pm = \overline{\theta}_\pm(t, \theta)$ the corresponding solutions. Since $\rho \in W^{2,\infty}(I \times S^1)$ by assumption,[7] from a standard theorem on ordinary differential equations it follows that $\overline{\theta}_\pm \in W^{1,\infty}(I \times S^1)$, and

$$\overline{\theta}_{\pm,\theta}(t, \theta) = \exp\left(\int_{t_1}^{t} \frac{\rho_\theta}{\rho}(t', \overline{\theta}_\pm(t', \theta))\, dt'\right) \in L^\infty_{\text{loc}}$$

never vanishes. Thus, the maps $I \times \mathbb{R} \ni (t, \theta) \mapsto (t, \theta_\pm) \in I \times \mathbb{R}$ are $W^{1,\infty}$-diffeomorphisms. Since the solutions are unique and the data are periodic, we obtain $\overline{\theta}_\pm(t, \theta + 2\pi) = \overline{\theta}_\pm(t, \theta) \pm 2\pi$. Finally, we arrive at the desired conclusion by defining the functions u, v by

$$u(t, \theta) := u_1(t, \overline{\theta}_+(t, \theta)), \qquad v(t, \theta) := v_1(t, \overline{\theta}_-(t, \theta)).$$ □

[7] The higher regularity on ρ is used in order to ensure that the functions $\overline{\theta}_\pm$ are Lipschitz continuous, as required.

3.4. Areal coordinates for weakly regular metrics

We will also use a time function coinciding with the area of the orbits of symmetry. In such coordinates, the area function is obviously of class C^∞, while the metric coefficient a introduced below in (3.7) has weak regularity.

Later we will justify this choice and show (cf. Proposition 5.1 below) that the gradient of the area function R is timelike so that the area can be used as a time coordinate. In the so-called areal coordinates, the metric takes the form

$$\begin{aligned} g = {} & e^{2(\eta-U)}(-dR^2 + a^{-2}d\theta^2) + e^{2U}\big(dx + Ady + (G + AH)d\theta\big)^2 \\ & + e^{-2U}R^2(dy + Hd\theta)^2, \end{aligned} \tag{3.7}$$

where U, A, η, a, G, H are functions of R and $\theta \in S^1$. The variable R describes some interval $[R_0, R_1)$ and the variables x, y, θ describe S^1. As in the conformal case, we will prove in Section 5 that areal coordinates are admissible if the weak version of the Einstein equations holds and that, in particular, the regularity in Lemmas 3.3 and 3.4 holds in areal coordinates, as now stated.

Lemma 3.6 (Weak regularity in areal coordinates). *Let $(\mathcal{M}, g)$ be a weakly regular T^2-symmetric spacetime and suppose that the area function has a timelike gradient ∇R. Assume that the areal coordinates (R, x, y, θ) are admissible and that the metric takes the form (3.7) where all functions depend only on R, θ with $R \in I \subset (0, \infty)$ (an interval) Then the following regularity properties hold:*

$$\begin{aligned} & U_R, A_R, U_\theta, A_\theta \in L^\infty_{loc}(I, L^2(S^1)), \qquad \eta_R, \eta_\theta, G, H \in L^\infty_{loc}(I, L^1(S^1)), \\ & a \in L^\infty_{loc}(I, W^{1,\infty}(S^1)). \end{aligned} \tag{3.8}$$

4. Field equations in admissible coordinates

4.1. Constraint equations in admissible coordinates

In this section, we derive the Einstein equations in admissible coordinates from the geometric formulation of the equations presented in the previous sections. To begin with, we consider the constraint equations.

Lemma 4.1 (Weak version of the constraint equations in admissible coordinates). *Let (Σ, h, K) be a weakly regular T^2-symmetric triple and consider the metric in admissible coordinates as described in Lemmas 3.1 and 3.2. Then the weak version of the constraint equations defined in (2.24)–(2.25) is equivalent to the following four equations:*

$$\begin{aligned} & \overline{R}_{\theta\theta} + \frac{1}{4\overline{R}}(\overline{R}_\theta^2 + \underset{0}{\overline{R}}^2) - \overline{R}_\theta(\overline{\nu}_\theta - \overline{P}_\theta) - \underset{0}{\overline{R}}(\underset{0}{\overline{\nu}} - \underset{0}{\overline{P}}) + \overline{R}(\overline{P}_\theta^2 + \underset{0}{\overline{P}}^2) \\ & \quad + \tfrac{1}{4}\overline{R}(\overline{A}_\theta^2 + \underset{0}{\overline{A}}^2)e^{4\overline{P}} + \tfrac{1}{4}e^{-2\overline{\nu}+4\overline{P}}\overline{R}^2(\underset{0}{\overline{G}} + \overline{A}\,\underset{0}{\overline{H}})^2 + \tfrac{1}{4}e^{-2\overline{\nu}}\overline{R}^3\underset{0}{\overline{H}}^2 = 0, \end{aligned} \tag{4.1}$$

$$(\underset{0}{\overline{R}})_\theta - (\overline{\nu}_\theta - \overline{P}_\theta)\underset{0}{\overline{R}} - (\underset{0}{\overline{\nu}} - \underset{0}{\overline{P}})\overline{R}_\theta + \frac{1}{2\overline{R}}\overline{R}_\theta \underset{0}{\overline{R}} + \overline{R}(2\underset{0}{\overline{P}}\,\overline{P}_\theta + \tfrac{1}{2}\underset{0}{\overline{A}}\,\overline{A}_\theta e^{4\overline{P}}) = 0, \tag{4.2}$$

$$\big(\overline{R}e^{4\overline{U}-2\overline{\nu}}(\underset{0}{\overline{G}} + \overline{A}\,\underset{0}{\overline{H}})\big)_\theta = 0, \tag{4.3}$$

$$\big(\overline{R}^3 e^{-2\overline{\nu}}\underset{0}{\overline{H}} + \overline{A}\,\overline{R}e^{4\overline{U}-2\overline{\nu}}(\underset{0}{\overline{G}} + \overline{A}\,\underset{0}{\overline{H}})\big)_\theta = 0, \tag{4.4}$$

in which $\underset{0}{\overline{G}}, \underset{0}{\overline{H}}, \underset{0}{\overline{A}}, \underset{0}{\overline{U}}, \underset{0}{\overline{\nu}}$, *and* $\underset{0}{\overline{R}}$ *were introduced in Lemma 3.2, and the equations above hold in the weak sense.*

Since the term $\overline{R}_{\theta\theta}$ is the only one containing second-order derivatives, if one evaluates the constraint equations above on a hypersurface of *constant* area R, then no second-order derivative of the metric arises in the constraints; doing so suppresses the elliptic nature of these equations and is the key reason why the analysis of T^2-symmetric spacetimes is natural in areal coordinates.

Proof of Lemma 4.1. We consider first the Hamiltonian equation (2.24) and compute the normalized scalar curvature $R^{(3)}_{\text{norm}}$ in terms of the metric coefficients (introduced in Lemmas 3.1 and 3.2):

$$\begin{aligned} R^{(3)}_{\text{norm}} &= 2R^{(3)}_{ZZ} + h(Z,Z)(|\chi|^2 - \operatorname{Tr}(\chi)^2) \\ &= -2Z(\Gamma^a_{aZ}) + 2\Gamma^a_{aZ}\Gamma^Z_{ZZ} - 2\Gamma^a_{bZ}\Gamma^b_{aZ} + h(Z,Z)(|\chi|^2 - \operatorname{Tr}(\chi)^2). \end{aligned}$$

Observe then that

$$\chi_{ab} = g\big(h(Z,Z)^{-1/2}Z, \nabla_{e_a}e_b\big) = h(Z,Z)^{1/2}\Gamma^Z_{ab} = -\tfrac{1}{2}(h^{ZZ})^{1/2}Z(h_{ab}),$$

thus

$$\begin{aligned} R^{(3)}_{\text{norm}} = {} & -2Z(\Gamma^a_{aZ}) + 2\Gamma^a_{aZ}\Gamma^Z_{ZZ} - 2\tfrac{1}{2}h^{ac}Z(h_{cb})\tfrac{1}{2}h^{bd}Z(h_{da}) \\ & + h(Z,Z)\big(\tfrac{1}{2}(h^{ZZ})^{1/2}Z(h_{ab})\tfrac{1}{2}(h^{ZZ})^{1/2}Z(h_{cd})h^{ac}h^{bd} \\ & \qquad - \big(\tfrac{1}{2}(h^{ZZ})^{1/2}Z(h_{ab})h^{ab}\big)^2\big). \end{aligned}$$

Hence, we obtain

$$R^{(3)}_{\text{norm}} = -2Z(\Gamma^a_{aZ}) + 2\Gamma^a_{aZ}\Gamma^Z_{ZZ} - \tfrac{1}{4}Z(h_{cb})Z(h_{da})h^{ac}h^{bd} - \tfrac{1}{4}(Z(h_{ab})h^{ab})^2.$$

Using the identity

$$\tfrac{1}{2}h^{ab}Z(h_{ab}) = Z(\ln R) = \Gamma^a_{aZ} = -h(Z,Z)^{1/2}\operatorname{Tr}(\chi),$$

where $R^2 = \det(h_{ab})$, we find

$$\begin{aligned} R^{(3)}_{\text{norm}} &= -2Z(Z(\ln R)) + 2Z(\ln R)\left(-\frac{R_\theta}{2R} + \nu_\theta - P_\theta\right) - (Z(\ln R))^2 \\ &\quad - \tfrac{1}{4}Z(h_{cb})Z(h_{ad})h^{ac}h^{bd} \\ &= -2Z\left(\frac{R_\theta}{R}\right) + 2\frac{R_\theta}{R}(\nu_\theta - P_\theta) - 2\left(\frac{R_\theta}{R}\right)^2 - \tfrac{1}{4}Z(h_{cb})Z(h_{da})h^{ac}h^{bd}. \end{aligned} \tag{4.5}$$

Thus, we need to evaluate $\frac{1}{4}Z(h_{cb})Z(h_{da})h^{ac}h^{bd}$.

To this end, by decomposing h_{ab} in the form $h_{ab} = R\aleph_{ab}$ with $\det(\aleph_{ab}) = 1$, we obtain

$$\begin{aligned}
-\tfrac{1}{4}Z(h_{cb})Z(h_{da})h^{ac}h^{bd} &= -\tfrac{1}{4}Z(R\aleph)(R\aleph)^{-1}Z(R\aleph)(R\aleph)^{-1} \\
&= -\tfrac{1}{4}Z(\aleph)\aleph^{-1}Z(\aleph)\aleph^{-1} - \tfrac{1}{2}Z(\aleph)\frac{R_\theta}{R}\aleph^{-1} - \tfrac{1}{4}2\left(\frac{R_\theta}{R}\right)^2 \\
&= -\tfrac{1}{4}Z(\aleph)\aleph^{-1}Z(\aleph)\aleph^{-1} - \tfrac{1}{2}\left(\frac{R_\theta}{R}\right)^2,
\end{aligned}$$

where we have used $\mathrm{Tr}(Z(\aleph)\aleph^{-1}) = 0$ (since $\aleph$ has constant determinant). Therefore,

$$\aleph = \begin{pmatrix} e^{2P} & Ae^{2P} \\ Ae^{2P} & A^2e^{2P} + e^{-2P} \end{pmatrix},$$

and a straightforward computation gives

$$-\tfrac{1}{4}Z(\aleph)\aleph^{-1}Z(\aleph)\aleph^{-1} = -2P_\theta^2 - \tfrac{1}{2}A_\theta^2e^{4P},$$

from which it follows that

$$R^{(3)}_{\text{norm}} = -2Z\left(\frac{R_\theta}{R}\right) + 2\frac{R_\theta}{R}(\nu_\theta - P_\theta) - \tfrac{5}{2}\left(\frac{R_\theta}{R}\right)^2 - 2P_\theta^2 - \tfrac{1}{2}A_\theta^2e^{4P}. \tag{4.6}$$

To complete the derivation of the Hamiltonian constraint equations in admissible coordinates, it remains to determine the contribution of the tensor K.

Note that

$$\begin{aligned}
&h(Z,Z)((\mathrm{Tr}(K))^2 - |K|^2) \\
&\quad = \frac{R_0^2}{R^2} + (K_{ZZ})^2h^{ZZ} + 2K_{ZZ}(h^{ZZ})^{1/2}\frac{1}{\overline{R}}\underset{0}{\overline{R}} \\
&\qquad - (K_{ZZ})^2h^{ZZ} - 2h(Z,Z)K_{Za}K^{Za} - h(Z,Z)K_{ab}K^{ab} \\
&\quad = \frac{R_0^2}{R^2} + 2K_{ZZ}(h^{ZZ})^{1/2}\frac{1}{\overline{R}}\underset{0}{\overline{R}} - 2h(Z,Z)K_{Za}K^{Za} - h(Z,Z)K_{ab}K^{ab},
\end{aligned} \tag{4.7}$$

and, as before, we define $\underset{0}{\aleph}_{ab}$ by

$$\underset{0}{h}_{ab} = \frac{1}{\overline{R}}\underset{0}{\overline{R}}h_{ab} + \overline{R}\underset{0}{\aleph}_{ab},$$

so that the trace of $\underset{0}{\aleph}_{ab}$ vanishes: $\underset{0}{\aleph}_{ab}h^{ab} = 0$, which follows from the definition of $\underset{0}{\overline{R}}$. One then has

$$\begin{aligned}
-h(Z,Z)K_{ab}K^{ab} &= -\frac{1}{2}\left(\frac{1}{\overline{R}}\underset{0}{\overline{R}}\right)^2 - \frac{\overline{R}^2}{4}\underset{0}{\aleph}_{ab}\underset{0}{\aleph}_{cd}h^{ad}h^{bd} \\
&= -\frac{1}{2}\left(\frac{1}{\overline{R}}\underset{0}{\overline{R}}^2\right)^2 - 2\underset{0}{\overline{P}}^2 - \tfrac{1}{2}\underset{0}{\overline{A}}^2e^{4P},
\end{aligned}$$

and moreover

$$2K_{ZZ}(h^{ZZ})^{1/2}\frac{1}{\overline{R}}\underset{0}{\overline{R}} = 2\frac{1}{\overline{R}}\underset{0}{\overline{R}}\left(\underset{0}{\overline{\nu}} - \underset{0}{\overline{P}} - \frac{1}{2\overline{R}}\underset{0}{\overline{R}}\right).$$

We now consider the last term on the right-hand side of (4.7). From the definition of $\underset{0}{\overline{G}}$ and $\underset{0}{\overline{H}}$, it follows that

$$
\begin{aligned}
K_{Za}K^{Za} &= K_{Za}K_{Zb}h^{ab} = K_{ZX}^2 h^{XX} + K_{ZY}h^{YY} + 2K_{ZX}K_{ZY}h^{XY} \\
&= \tfrac{1}{4}e^{-2\overline{\nu}+4\overline{P}}R(\underset{0}{\overline{G}} + \overline{A}\,\underset{0}{\overline{H}})^2 + \tfrac{1}{4}e^{-2\nu}\overline{R}^2\underset{0}{\overline{H}}^2.
\end{aligned}
$$

Collecting all the terms computed above, we have established that the Hamiltonian constraint equation reads

$$
\begin{aligned}
&-2\left(\frac{\overline{R}_\theta}{\overline{R}}\right)_\theta + 2\frac{\overline{R}_\theta}{R}(\overline{\nu}_\theta - \overline{P}_\theta) - \tfrac{5}{2}\left(\frac{\overline{R}_\theta}{R}\right)^2 - \frac{1}{2\overline{R}^2}\underset{0}{\overline{R}}^2 + \frac{2}{\overline{R}}\underset{0}{\overline{R}}(\underset{0}{\overline{\nu}} - \underset{0}{\overline{P}}) \\
&\quad - 2(\overline{P}_\theta^2 + \underset{0}{\overline{P}}^2) - \tfrac{1}{2}(\overline{A}_\theta^2 + \underset{0}{\overline{A}}^2)e^{4\overline{P}} - \tfrac{1}{2}e^{-2\overline{\nu}+4\overline{P}}\overline{R}(\underset{0}{\overline{G}} + \overline{A}\,\underset{0}{\overline{H}})^2 - \tfrac{1}{2}e^{-2\overline{\nu}}\overline{R}^2\underset{0}{\overline{H}}^2 = 0.
\end{aligned}
$$

From a straightforward density argument it then follows that this equation is equivalent to (4.1). On the other hand, the twist equations (4.4) are obtained easily by observing that the geometric formulation is equivalent to $Z(h(Z,Z)^{1/2}\overline{R}^{-1}K_a^Z) = 0$, and then using the decomposition of K.

We now consider the last momentum constraint equation (4.2). For this, we compute all the terms appearing in the first equation of (2.25) one by one. For the first term we have

$$
\begin{aligned}
-Z(\mathrm{Tr}^2(K)) &= -Z(e^{-\overline{\nu}+\overline{U}}\underset{0}{\overline{R}}\,\overline{R}^{-1}) \\
&= -Z(e^{-\overline{\nu}}\underset{0}{\overline{R}})e^{U}\overline{R}^{-1} + \underset{0}{\overline{R}}(\overline{R})^{-2}\overline{R}_\theta e^{-\overline{\nu}+\overline{U}} - \underset{0}{\overline{R}}\,\overline{R}^{-1}(-\overline{\nu}_\theta - \overline{U}_\theta)e^{-\overline{\nu}+\overline{U}} \\
&= -Z(e^{-\overline{\nu}}\underset{0}{\overline{R}}\,\overline{R}^{-1/2})e^{\overline{P}} - Z(\overline{P})e^{-\overline{\nu}+\overline{P}}\underset{0}{\overline{R}}\,\overline{R}^{-1/2}.
\end{aligned}
$$

For the second term we find

$$
-h(Z,Z)^{1/2}\,\mathrm{Tr}(\chi)K_Z^Z = -\overline{R}_\theta\overline{R}^{-1}K_Z^Z = \overline{R}_\theta R^{-1/2}e^{-\overline{\nu}+\overline{P}}\big(\underset{0}{\overline{\nu}} - \underset{0}{\overline{P}} - 2\underset{0}{\overline{R}}(2\overline{R})^{-1}\big)
$$

and, for the last term,

$$
-\Gamma_{Zb}^a K_a^b = -\tfrac{1}{2}h^{ac}Z(h_{bc})h^{bd}K_{ad} = -\tfrac{1}{4}e^{-\overline{\nu}+\overline{U}}h^{ac}h^{bd}Z(h_{bc})\underset{0}{h_{bd}}.
$$

Using the fact that the traces of $\underset{0}{\aleph_{ab}}$ and $Z(\aleph)$ vanish, we obtain

$$
-\Gamma_{Zb}^a K_a^b = -\tfrac{1}{2}e^{-\nu+P}\overline{R}^{-3/2}\underset{0}{\overline{R}}\,\overline{R}_\theta - \tfrac{1}{4}e^{-\nu+P}\overline{R}^{1/2}h^{bd}h^{ac}Z(\aleph_{bc})\underset{0}{\aleph_{ad}},
$$

Finally, in view of

$$
-\tfrac{1}{4}e^{-\nu+P}\overline{R}^{1/2}h^{bd}h^{ac}Z(\aleph_{bc})\underset{0}{\aleph_{ad}} = -R^{1/2}e^{-\overline{\nu}+\overline{P}}\big(2\underset{0}{\overline{P}}\,\overline{P}_\theta + \tfrac{1}{2}\underset{0}{\overline{A}}\,\overline{A}_\theta e^{4\overline{P}}\big),
$$

the last momentum constraint equation follows by collecting all the terms. □

4.2. Evolution equations in admissible coordinates

In this section, we rely on the geometric formulation introduced earlier and derive the Einstein equations in admissible coordinates.

Proposition 4.2 (Weak version of the evolution equations in admissible coordinates). *Let $(\mathcal{M}, g)$ be a weakly regular T^2-symmetric spacetime with admissible coordinates (t, x, y, θ). Then $(\mathcal{M}, g)$ satisfies the weak formulation (2.30) of the Einstein equations if and only if $(\Sigma_t, h(t))$ satisfies the constraint equations on each slice and the following equations are satisfied:*

$$\begin{aligned} 0 = T(\rho\nu_t) - Z(\rho^{-1}\nu_\theta) - \rho\left(P_t - \frac{R_t}{2R}\right)^2 + \rho^{-1}\left(P_\theta - \frac{R_\theta}{2R}\right)^2 \\ - \frac{e^{4P}}{4}(\rho A_t^2 - \rho^{-1}A_\theta^2) + \frac{3}{4R^4}\rho^{-1}e^{2\nu}K^2, \end{aligned} \tag{4.8}$$

$$\begin{aligned} 0 = \left(\rho\left(P_t + \frac{R_t}{2R}\right)\right)_t - \left(\rho^{-1}\left(P_\theta + \frac{R_\theta}{2R}\right)\right)_\theta - \rho\frac{R_t U_t}{R} \\ + \rho^{-1}\frac{R_\theta U_\theta}{R} - \frac{\rho}{2}e^{4P}A_t^2 - \frac{\rho^{-1}}{2}e^{4P}A_\theta^2, \end{aligned} \tag{4.9}$$

$$\begin{aligned} 0 = (\rho A_t)_t - (\rho^{-1}A_t)_t - \rho\frac{R_t A_t}{R} - \rho^{-1}\frac{R_\theta A_\theta}{R} \\ - 4\left(\rho^{-1}A_\theta\left(P_\theta + \frac{R_\theta}{2R}\right) - \rho A_t\left(P_t + \frac{R_t}{2R}\right)\right), \end{aligned} \tag{4.10}$$

$$0 = (\rho R_t)_t - (\rho^{-1}R_\theta)_\theta - \frac{1}{2R^3}\rho^{-1}e^{2\nu}K^2, \tag{4.11}$$

$$0 = (\rho R^2 e^{-2\nu}H_t)_t, \quad 0 = \left(\rho R^2 e^{-2\nu+4P}(G_t + AH_t)\right)_t. \tag{4.12}$$

Proof. The equations (4.12) are easily obtained from $R_{Za} = 0$, as in Proposition 2.26. We now consider the equations $R_{cd} = 0$ which read

$$\begin{aligned} 0 =& T(ng(Z,Z)^{1/2}\Gamma^T_{dc}) + Z(ng(Z,Z)^{1/2}\Gamma^Z_{dc}) + ng(Z,Z)^{1/2}(\Gamma^a_{at}\Gamma^T_{dc} + \Gamma^a_{aZ}\Gamma^Z_{dc} \\ &- \Gamma^T_{dZ}\Gamma^Z_{tc}) + ng(Z,Z)^{1/2}(-\Gamma^Z_{dt}\Gamma^T_{Zc} - \Gamma^T_{da}\Gamma^a_{tc} - \Gamma^a_{dt}\Gamma^T_{ac} - \Gamma^Z_{da}\Gamma^a_{Zc} - \Gamma^a_{dZ}\Gamma^Z_{ac}). \end{aligned}$$

First, we note the following identities:

$$\begin{aligned} & ng(Z,Z)^{1/2} = \rho n^2, \quad \Gamma^T_{dc} = \frac{1}{2n^2}g_{dc,t}, \quad T(ng(Z,Z)^{1/2}\Gamma^T_{dc}) = \tfrac{1}{2}T(\rho g_{dc,t}), \\ & Z(ng(Z,Z)^{1/2}\Gamma^Z_{dc}) = -\tfrac{1}{2}Z(\rho^{-1}g_{dc,\theta}), \quad \Gamma^a_{at}\Gamma^T_{dc} = \frac{R_t}{R}\frac{1}{2n^2}g_{dc,t}, \\ & \Gamma^a_{aZ}\Gamma^Z_{dc} = -\frac{R_\theta}{R}\frac{1}{2}g^{Z,Z}g_{dc,\theta}, \quad -2\Gamma^T_{dZ}\Gamma^Z_{tc} = -\frac{1}{2}\frac{K_d K_c}{R^2}, \\ & \Gamma^T_{da}\Gamma^a_{tc} = \frac{1}{4n^2}g_{da,t}g_{bc,t}g^{ab}, \quad \Gamma^Z_{da}\Gamma^a_{Zc} = -\frac{1}{4\rho^2 n^2}g_{da,\theta}g_{bc,\theta}g^{ab}, \end{aligned}$$

where $K_d = K$ if $d = y$ and 0 otherwise. To investigate the last two expressions in more detail, we set $g_{ab} =: R\aleph_{ab}$. Then we have

$$g_{da,t}g_{bc,t}g^{ab} = 2R_t\aleph_{dc,t} + \frac{R_t^2}{R}\aleph_{dc} + R\aleph_{da,t}\aleph_{bc,t}\aleph^{ab}$$
$$= 2\frac{R_t}{R}g_{dc,t} - \frac{R_t^2}{R^2}g_{dc} + R\aleph_{da,t}\aleph_{bc,t}\aleph^{ab}.$$

Now we compute, for $d = c = x$,

$$\begin{aligned}\aleph_{ax,t}\aleph_{bx,t}\aleph^{ab} &= (2P_te^{2P})^2(e^{-2P} + A^2e^{2P}) + 2(-Ae^{2P})(2P_te^{2P})(2P_tAe^{2P} + A_te^{2P})\\ &\quad + (2P_tAe^{2P} + A_te^{2P})^2e^{2P}\\ &= e^{2P}(4P_t^2 + A_t^2e^{4P}),\end{aligned}$$

for $d = c = y$,

$$\begin{aligned}\aleph_{ay,t}\aleph_{by,t}\aleph^{ab} &= (A_te^{2P} + 2P_tAe^{2P})^2(e^{-2P} + A^2e^{2P})\\ &\quad + 2(-Ae^{2P})(-2P_te^{-2P} + 2AA_te^{2P} + A^22P_te^{2P})(A_te^{2P} + 2P_tAe^{2P})\\ &\quad + (-2P_te^{-2P} + 2AA_te^{2P} + A^22P_te^{2P})^2e^{2P}\\ &= (4P_t^2 + A_t^2e^{4P})(e^{-2P} + A^2e^{2P}),\end{aligned}$$

and for $d = x$ and $c = y$,

$$\begin{aligned}\aleph_{ax,t}\aleph_{by,t}\aleph^{ab} &= (2P_te^{2P})(A_te^{2P} + 2P_tAe^{2P})(e^{-2P} + A^2e^{2P})\\ &\quad + (-Ae^{2P})(2P - te^{2P})(-2P_te^{-2P} + 2AA_te^{2P} + 2P_tA^2e^{2P})\\ &\quad + (-Ae^{2P})(A_te^{2P} + 2P_tA^2e^{2P})^2\\ &\quad + e^{2P}(A_te^{2P} + 2P_tAe^{2P})(-2P_te^{-2P} + 2AA_te^{2P} + 2P_tA^2e^{2P})\\ &= 4P_t^2Ae^{2P} + AA_t^2e^{6P} = Ae^{2P}(4P_t^2 + A_t^2e^{4P}).\end{aligned}$$

Similar expressions are valid for $\aleph_{ac,\theta}\aleph_{bd,\theta}\aleph^{ab}$ by replacing the t-derivatives by θ-derivatives.

Putting everything together, we obtain for $d = c = x$

$$\begin{aligned}0 &= \tfrac{1}{2}T(\rho g_{xx,t}) - \tfrac{1}{2}Z(\rho^{-1}g_{xx,\theta}) - \frac{\rho}{2}\frac{R_t}{R}g_{xx,t} + \frac{\rho^{-1}}{2}\frac{R_\theta}{R}g_{xx,\theta}\\ &\quad - \frac{\rho}{2}\left(2\frac{R_t}{R}g_{xx,t} - \frac{R_t^2}{R^2}g_{xx} + (4P_t^2 + A_t^2e^{4P})g_{xx}\right)\\ &\quad + \frac{\rho^{-1}}{2}\left(2\frac{R_\theta}{R}g_{xx,\theta} - \frac{R_\theta^2}{R^2}g_{xx} + (4P_\theta^2 + A_\theta^2e^{4P})g_{xx}\right).\end{aligned}$$

Finally, substituting $g_{xx} = P_t + \frac{R_t}{2R}$, one easily obtains (4.9). The wave equation (4.10) for A is derived similarly.

To derive equation (4.11), we note that, for sufficiently regular solutions,

$$\begin{aligned} &g^{cd}\big(\tfrac{1}{2}T(\rho g_{cd,t}) - \tfrac{1}{2}Z(\rho^{-1}g_{cd,\theta})\big)\\ &= T\left(\rho\frac{R_t}{R}\right) - Z\left(\rho^{-1}\frac{R_\theta}{R}\right) - \tfrac{1}{2}\rho g^{ad}g^{cd}g_{ab,t}g_{cd,t} + \tfrac{1}{2}\rho^{-1}g^{ad}g^{cd}g_{ab,\theta}g_{cd,\theta}, \end{aligned}$$

from which (4.11) follows. Finally, a straightforward density argument shows that (4.11) remains true under our regularity assumptions. We now consider the equation $R_{ZZ} = 0$ and, in view of the definition, we have

$$\begin{aligned} R_{ZZ} = {} & T(\Gamma^T_{ZZ}) - Z(\Gamma^T_{TZ}) - Z(\Gamma^a_{aZ}) - \Gamma^a_{Zb}\Gamma^b_{az} + \Gamma^a_{aZ}\Gamma^Z_{ZZ} + \Gamma^a_{at}\Gamma^T_{ZZ} + 2\Gamma^T_{Za}\Gamma^a_{ZT}\\ & + \Gamma^T_{ZZ}(\Gamma^T_{TT} - \Gamma^Z_{ZT}) + \Gamma^T_{TZ}(\Gamma^Z_{ZZ} - \Gamma^T_{ZT}). \end{aligned}$$

We evaluate successively each of the terms above and obtain

$$\begin{aligned} &T(\Gamma^T_{ZZ}) = T\left(-\frac{1}{n}K(Z,Z)\right) = T\left(\frac{1}{2n^2}g_{ZZ,t}\right), \qquad -Z(\Gamma^T_{TZ}) = -Z\left(\frac{1}{n}Z(n)\right),\\ &-Z(\Gamma^a_{aZ}) = -Z(Z(\ln R)d), \qquad \Gamma^a_{at}\Gamma^T_{ZZ} = T(\ln R)\frac{1}{2n^2}g_{ZZ,t}. \end{aligned}$$

The algebraic expressions of the products

$$\Gamma^a_{Zb}\Gamma^b_{aZ}, \qquad \Gamma^a_{aZ}\Gamma^Z_{ZZ}, \qquad \Gamma^T_{Za}\Gamma^a_{ZT}$$

have already been computed in terms of the metric functions (for the derivation of the constraint equations):

$$\begin{aligned} &-\Gamma^a_{Zb}\Gamma^b_{aZ} = -\tfrac{1}{4}Z(h_{cb})Z(h_{ad})h^{ac}h^{bd} = -2P_\theta^2 - \tfrac{1}{2}A_\theta^2 e^{4P},\\ &\Gamma^a_{aZ}\Gamma^Z_{ZZ} = Z(\ln R)\left(-\frac{R_\theta}{2R} + \nu_\theta - P_\theta\right),\\ &\Gamma^T_{Za}\Gamma^a_{ZT} = K^a_Z K_{aZ} = \rho^2\big(\tfrac{1}{4}e^{-2\nu+4P}R(G_t + AH_t)^2 + \tfrac{1}{4}e^{-2\nu}R^2H_t^2\big) = \tfrac{1}{4}R^2e^{-2\nu}K^2, \end{aligned}$$

where K denotes the only non-vanishing twist constant. The last two terms in the definition of R_{ZZ} give

$$\begin{aligned} \Gamma^T_{ZZ}(\Gamma^T_{TT} - \Gamma^Z_{ZT}) &= \frac{1}{4n^4}g_{ZZ,t}g^{ZZ}T(\rho^{-2}),\\ \Gamma^T_{TZ}(\Gamma^Z_{ZZ} - \Gamma^T_{ZT}) &= \frac{n}{2}g^{ZZ}Z(n)Z(\rho^2). \end{aligned}$$

Adding all the terms together, we obtain the equation

$$\begin{aligned} 0 = {} & T\left(\rho^2\left(\nu_t - P_t - \frac{R_t}{2R}\right)\right) - \rho\rho_t\left(\nu_t - P_t - \frac{R_t}{2R}\right) - Z\left(\nu_\theta - P_\theta - \frac{R_\theta}{2R}\right)\\ & + \frac{\rho_\theta}{\rho}\left(\nu_\theta - P_\theta - \frac{R_\theta}{2R}\right) - Z\left(\frac{R_\theta}{R}\right) + \frac{R_t}{R}\rho^2\left(\nu_t - P_t - \frac{R_t}{2R}\right)\\ & + \frac{R_\theta}{R}\left(\nu_\theta - P_\theta - \frac{R_\theta}{2R}\right) - 2P_\theta^2 - \tfrac{1}{2}A_\theta^2e^{4P} + \tfrac{1}{4}R^2e^{-2\nu}K^2. \end{aligned}$$

Equation (4.8) then follows by using (4.9) as well as (4.1) to eliminate all second-order derivatives of P, R. □

4.3. Field equations in conformal coordinates

Applying Proposition 4.2 to the special case of conformal coordinates, we obtain the following result.

Proposition 4.3 (Weak version of the field equations in conformal coordinates). *Let $(\mathcal{M}, g)$ be a weakly-regular T^2-symmetric spacetime and let (τ, ξ, x, y) be a system of conformal admissible coordinates for $(\mathcal{M}, g)$ in which the metric*[8] *takes the following form:*

$$g = e^{2(\nu-U)}(-d\tau^2 + d\xi^2) + e^{2U}\big(dx + A dy + (G + AH)d\xi\big)^2 + e^{-2U} R^2 (dy + H d\xi)^2. \tag{4.13}$$

Then the weak version (2.30) of the Einstein equations is equivalent to the following system of evolution and constraint equations:

1. *Four constraint equations:*

$$0 = U_\tau^2 + U_\xi^2 + \frac{e^{4U}}{4R^2}(A_\tau^2 + A_\xi^2) + \frac{R_{\xi\xi}}{R} - \frac{\nu_\tau R_\tau}{R} - \frac{\nu_\xi R_\xi}{R} + \frac{e^{2\nu}}{4R^4}K^2, \tag{4.14}$$

$$0 = 2U_\tau U_\xi + \frac{e^{4U}}{2R^2}A_\tau A_\xi + \frac{R_{\xi\tau}}{R} - \frac{\nu_\xi R_\tau}{R} - \frac{\nu_\tau R_\xi}{R}, \tag{4.15}$$

$$K_\xi = 0, \qquad K_\tau = 0. \tag{4.16}$$

2. *Four evolution equations:*

$$U_{\tau\tau} - U_{\xi\xi} = \frac{R_\xi U_\xi}{R} - \frac{R_\tau U_\tau}{R} + \frac{e^{4U}}{2R^2}(A_\tau^2 - A_\xi^2), \tag{4.17}$$

$$A_{\tau\tau} - A_{\xi\xi} = \frac{R_\tau A_\tau}{R} - \frac{R_\xi A_\xi}{R} + 4(A_\xi U_\xi - A_\tau U_\tau), \tag{4.18}$$

$$R_{\tau\tau} - R_{\xi\xi} = \frac{e^{2\nu}}{2R^3}K^2, \tag{4.19}$$

$$\nu_{\tau\tau} - \nu_{\xi\xi} = U_\xi^2 - U_\tau^2 + \frac{e^{4U}}{4R^2}(A_\tau^2 - A_\xi^2) - \frac{3e^{2\nu}}{4R^4}K^2. \tag{4.20}$$

3. *Two auxiliary equations:*

$$G_\tau + AH_\tau = 0, \qquad G_\tau = \frac{e^{2\nu}}{R^3}K. \tag{4.21}$$

4.4. Field equations in areal coordinates

Similarly, in the case of areal coordinates, we obtain the following equations.

[8] The variable P is now replaced by $U := P - \frac{1}{2}\ln R$, as this leads to some computational simplifications later on.

Proposition 4.4 (Weak version of the field equations in areal coordinates). *Let* $(\mathcal{M}, g)$ *be a weakly regular* T^2*-symmetric spacetime and let* (R, x, y, θ) *be areal admissible coordinates. Then the weak version* (2.30) *of the Einstein equations is equivalent to the following evolution and constraint equations:*

1. *Four evolution equations for the metric coefficients* U, A, η, a*:*

$$(Ra^{-1}U_R)_R - (RaU_\theta)_\theta = 2R\Omega^U, \tag{4.22}$$

$$(R^{-1}a^{-1}A_R)_R - (R^{-1}aA_\theta)_\theta = e^{-2U}\Omega^A, \tag{4.23}$$

$$(a^{-1}\eta_R)_R - (a\eta_\theta)_\theta = \Omega^\eta - R^{-3/2}(R^{3/2}(a^{-1})_R)_R, \tag{4.24}$$

$$(2\ln a)_R = -R^{-3}K^2e^{2\eta}, \tag{4.25}$$

where the right-hand sides are defined by

$$\Omega^U := (2R)^{-2}e^{4U}(a^{-1}A_R^2 - aA_\theta^2),$$
$$\Omega^A := 4R^{-1}e^{2U}(-a^{-1}U_RA_R + aU_\theta A_\theta),$$
$$\Omega^\eta := (-a^{-1}U_R^2 + aU_\theta^2) + (2R)^{-2}e^{4U}(a^{-1}A_R^2 - aA_\theta^2).$$

2. *Two constraint equations for the metric coefficient* η*:*

$$\eta_R + \frac{1}{4}R^{-3}e^{2\eta}K^2 = aRE, \qquad \eta_\theta = RF, \tag{4.26}$$

where

$$E := (a^{-1}U_R^2 + aU_\theta^2) + (2R)^{-2}e^{4U}(a^{-1}A_R^2 + aA_\theta^2),$$
$$F := 2U_RU_\theta + 2R^{-2}e^{2U}A_RA_\theta.$$

3. *Four auxiliary equations for the twists:*

$$\begin{aligned}(Re^{4U-2\eta}a(G_R + AH_R))_\theta = 0, &\quad (R^3e^{-2\eta}aH_R)_\theta = 0,\\ (Re^{4U-2\eta}a(G_R + AH_R))_R = 0, &\quad (R^3e^{-2\eta}aH_R)_R = 0.\end{aligned} \tag{4.27}$$

4. *Two equations for the metric coefficients* G, H*:*

$$G_R = -AKe^{2\eta}a^{-1}R^{-3}, \qquad H_R = Ke^{2\eta}a^{-1}R^{-3}. \tag{4.28}$$

5. First properties of weakly regular T^2-symmetric manifolds

5.1. Properties of the area function

In this section, we collect some properties of weakly regular T^2-symmetric manifolds which will be useful for the analysis of the initial value problem in Sections 6 and 7. First of all, we derive some properties of the area function which are immediate consequences of the field equations. The first one is an additional L^1 regularity for the second derivatives of R.

From the constraint equations (4.1)–(4.2) and the assumed regularity, we see that the second-order derivatives $R_{\theta\theta}$ and $R_{t\theta}$ may be written as a sum of functions that have L^1 regularity at least. Moreover, in view of the evolution equation (4.11), R_{tt} also has L^1 regularity. The additional regularity (5.1) will be crucial to prove local well-posedness of the system in Section 6. Furthermore, for sufficiently regular T^2-symmetric spacetimes, it is known that ∇R is timelike unless the spacetime is flat [8, 31]. That this is still true at our level regularity is the subject of the second statement below.

Proposition 5.1 (Properties of the area function). *Let $(\mathcal{M}, g)$ be a vacuum T^2-symmetric Lorentzian manifold and let (t, θ, x, y) be admisssible coordinates.*

1. *The area function $R = R(t, \theta)$ has the following* additional regularity properties:

$$R \in L^\infty_{\text{loc}}(W^{2,1}(S^1)), \qquad R_t \in L^\infty_{\text{loc}}(W^{1,1}(S^1)), \qquad R_{tt} \in L^\infty_{\text{loc}}(L^1(S^1)). \tag{5.1}$$

2. *If this manifold is nonflat, that is, g does not coincide with a smooth metric on $\mathcal{M}$ whose curvature tensor vanishes, then the* gradient ∇R is timelike, *i.e.*

$$g(\nabla R, \nabla R) < 0 \quad \text{in } \mathcal{M}. \tag{5.2}$$

This, in particular, establishes the *existence of an areal coordinate system* of class C^1 for any weakly regular T^2-symmetric spacetime. Note also that an alternative statement of the second item of Proposition 5.1 is as follows: for any weakly regular T^2-symmetric initial data set, one has either

$$\underset{0}{\overline{R}}^2 - \overline{R}_\xi^2 > 0,$$

or else the initial data is trivial, i.e. $\overline{R}, \overline{A}, \overline{U}$ are constants and $\underset{0}{\overline{R}}, \underset{0}{\overline{A}}, \underset{0}{\overline{U}}$ vanish identically.

Proof of Proposition 5.1. It remains to establish the second item. We follow here an argument due to Chrusciel [8] and Rendall [31] for sufficiently regular spacetimes. In our weak regularity class, it follows that the norm $g(\nabla R, \nabla R)$ is a measurable and bounded function defined almost everywhere, at least. However, it follows from the first item of this proposition that R is actually of class C^1 in both variables t, θ. Define $\lambda^\pm := \rho R_t \pm R_\theta$ and $H := \nu_\theta - P_\theta + \nu_t - P_t$. Taking the sum and the difference of the two constraint equations (4.1)–(4.2) leads to $Z(\lambda^\pm) = -\lambda^\pm H + N$, where N can be checked to belong to $L^\infty_{\text{loc}}(L^2(S^1))$ and be nonpositive almost everywhere. From the last two equations and the continuity, as well as the periodicity, of $\lambda^\pm$, it follows that either $\lambda^+ = 0$ or λ^+ never vanishes, as is clear from the integrated expression $\lambda(\theta) = e^{-\int_{\theta_0}^{\theta} H(\theta')\,d\theta'} \int_{\theta_0}^{\theta} e^{-\int_{\theta_0}^{\theta'} H(\theta'')\,d\theta''} N(\theta')\,d\theta'$. A similar conclusion holds for λ^-. Moreover, periodicity of R excludes the possibility that $\lambda^+ > 0$ and $\lambda^- < 0$, as well as the possibility that $\lambda^+ < 0$ and $\lambda^- > 0$. Thus, it follows that either $\lambda^+\lambda^- > 0$ or else $\lambda_\pm = 0$ and $N = 0$. In the latter case, U and A are constant functions and the spacetime is flat. ∎

5.2. *From conformal to areal coordinates*

To solve the initial value problem, we will need two different coordinate systems, one better suited for the local-in-time analysis (the conformal coordinate system) and the other better suited for the long-time control of the growth of the initial norms (the areal coordinate system). However, since the construction of these coordinates depends on the metric, the weak regularity of the metric imposes a restriction on the regularity of these coordinates as functions of the original coordinates. In this section, we prove that despite this difficulty, the weak regularity of the metric coefficients is invariant under such a transformation. We begin with the following technical result which establishes additional regularity in time.

Lemma 5.2 (Additional regularity in time). *Consider a* 2*-dimensional Lorentzian manifold* $(\mathcal{Q}, \widetilde{g})$ *with*

$$\mathcal{Q} := [t_0, t_1) \times S^1, \qquad \widetilde{g} := -\rho dt^2 + \rho^{-1} d\xi^2, \tag{5.3}$$

where $\rho = \rho(\tau, \xi)$ *is assumed to be of class* C^1*. Let*

$$f \in L^\infty_{\mathrm{loc}}([t_0, t_1), H^1(S^1)) \cap W^{1,\infty}_{\mathrm{loc}}([t_0, t_1), L^2(S^1))$$

be a weak solution to the wave equation

$$\Box_{\widetilde{g}} f = q,$$

where the right-hand side satisfies $q \in L^2_{\mathrm{loc}}([t_0, t_1), L^2(S^1))$*. Then* f *is actually more regular and belongs to* $C^0(H^1(S^1)) \cap C^1(L^2(S^1))$*.*

Proof. Let a time interval $[t_0, t_2] \subset [t_0, t_1)$ be fixed, let f_0^ϵ, $f_{t,0}^\epsilon$, and q^ϵ be smooth functions approximating $f(t_0, \cdot)$, $f_t(t_0, \cdot)$, and q in the topology of $H^1(S^1)$, $L^2(S^1)$, and $L^2([t_0, t_2] \times S^1)$, respectively. Let f^ϵ be the solution to the corresponding wave equation with source q^ϵ and initial data $(f_0^\epsilon, f_{t,0}^\epsilon)$. Observe that f^ϵ is of class C^1 at least, and set $\Delta f := f - f^\epsilon$, $\Delta q := q - q^\epsilon$, etc. Then a standard energy estimate implies that for all $t \in [t_0, t_2]$,

$$\|\Delta f_t(t)\|^2_{L^2} + \|\Delta f_\theta(t)\|^2_{L^2} \lesssim \|f_0 - f_0^\epsilon\|^2_{H^1(S^1)} + \|f_{t,0} - f^\epsilon_{t,0}\|^2_{L^2(S^1)} + \|\Delta f_t \Delta q\|_{L^1([t_0,t_2]\times S^1)},$$

where the implied constant depends on the Lipschitz constant of ρ and t_0, t_1. Applying Cauchy–Schwarz to the last term above, we arrive at a Lipschitz continuity estimate which implies convergence of f^ϵ toward f. □

Note that in conformal coordinates, $\rho = 1$ in (5.3) and hence is indeed C^1, while for areal coordinates $\rho = a^{-1}$ for which we prove $W^{2,1}$ (thus C^1) regularity in Section 7. Moreover, we will prove later in Section 6 that the source terms in the wave equations for R, U, A are indeed in L^2_{loc} so that the above lemma applies with $(\mathcal{Q}, \widetilde{g})$ chosen to be the quotient space $\mathcal{M}/T^2$ with its induced metric and differential structure given by either conformal or areal coordinates.

These observations lead us to the following important result which, in particular, shows that the regularity of the metric functions does not change under a change of coordinates from conformal to areal coordinates or vice versa.

Proposition 5.3 (From conformal to areal coordinates and vice versa). *Let $(\mathcal{M}_{\mathcal{C}}, g)$ be a weakly regular vacuum T^2-symmetric spacetime and assume that $\mathcal{C} = (\tau, \xi, x, y)$ are admissible conformal coordinates. It follows from Proposition 5.1 that there exists an areal coordinate system $\mathcal{A} = (R, \theta, x, y)$ (with ∇R timelike) that is $W^{2,1}$-compatible with $\mathcal{C} = (\tau, \xi, x, y)$. Let $\mathcal{M}_{\mathcal{A}}$ be the topological manifold $\mathcal{M}_{\mathcal{C}}$ endowed with the (unique) C^∞-differential structure compatible with (R, θ, x, y). Then (R, θ, x, y) are admissible coordinates for the manifold $(\mathcal{M}_{\mathcal{A}}, g)$ and, in particular, the Einstein field equations hold in areal coordinates.*

Similarly, let $(\mathcal{M}_{\mathcal{A}}, g)$ be a weakly regular vacuum T^2-symmetric spacetime and let $\mathcal{A} = (R, \theta, x, y)$ be admissible areal coordinates. It follows from Lemma 3.5 and the improved regularity of the coefficient a that there exists a conformal coordinate system $\mathcal{C} = (\tau, \xi, x, y)$ that is $W^{2,1}$-compatible with $\mathcal{A}$. Let $\mathcal{M}_{\mathcal{C}}$ be the topological manifold $\mathcal{M}_{\mathcal{A}}$ endowed with the (unique) C^∞-differential structure compatible with (τ, ξ, x, y). Then (τ, ξ, x, y) are admissible coordinates for the manifold $(\mathcal{M}_{\mathcal{C}}, g)$ and, in particular, the Einstein field equations hold in conformal coordinates.

Proof. We establish the result for the transformation from conformal to areal coordinates, the proof of the second statement being similar. Note first that since the change of coordinates is of class C^1, the measures of volume associated with (τ, ξ) and (R, θ) are equivalent, hence we may talk about L^p functions unambiguously. Lemma 6.9 below ensures that the assumptions of Lemma 5.2 are satisfied. (In this situation, the source terms q in Lemma 6.9 contain terms like $A_\tau^2 - A_\theta^2$ which are precisely $L^2_{t,\theta}$ thanks to Lemma 5.2.) Standard energy estimates and a density argument then show that U, A, as functions of (R, θ), are of class $C^0_R(H^1_\theta) \cap C^1_R(L^2_\theta)$. By density, the weak version of the Einstein equations must hold in areal coordinates. It then follows from the constraint equations that η and a are in $C^0_R(W^{1,1}_\theta) \cap C^1_R(L^1_\theta)$ and $C^1_{R,\theta}$, respectively. Recall here that $a \in W^{2,1}$ by Lemma 7.8. Note finally that, by construction, R is C^∞ in areal coordinates. □

5.3. Regularization of initial data sets with constant area of symmetry

We now establish that any given weakly regular T^2-symmetric initial data set with constant area $R = R_0$ can be uniformly approximated by smooth T^2-symmetric initial data set. In view of (4.26), the initial data for the functions G, H do not enter the constraint equations, hence we may suppress here any reference to these functions. Therefore, we set

$$\overline{X} := (\overline{U}_0, \overline{A}_0, \overline{U}_1, \overline{A}_1, \overline{a}, \overline{\eta}_0, \overline{\eta}_1),$$

which represents an initial data set for the reduced equations (4.22). We are interested in the existence of a suitable regularization of $\overline{X}$.

Lemma 5.4 (Regularization of initial data sets in areal coordinates). *Let $\overline{X}$ be an initial data set for the reduced Einstein equations, in particular satisfying the constraint equations (4.26) (with U_0 replaced by $\overline{U}_0$, etc.). Then there exists a sequence of smooth functions defined on S^1*

$$\overline{X}^n = (\overline{U}_0^n, \overline{A}_0^n, \overline{U}_1^n, \overline{A}_1^n, \overline{a}^n, \overline{\eta}_0^n, \overline{\eta}_1^n), \qquad n = 1, 2, \ldots,$$

referred to as a **regularized initial data set,** *such that* $\overline{X}^n$ *satisfies the reduced Einstein constraint equations* (4.26) *and converges almost everywhere* (*for the Lebesgue measure on* S^1) *with moreover* [9]

$$(\overline{U}_0^n, \overline{A}_0^n, \overline{U}_1^n, \overline{A}_1^n) \to (\overline{U}_0, \overline{A}_0, \overline{U}_1, \overline{A}_1) \quad \textit{in } L^2(S^1),$$
$$\overline{a}^n \to \overline{a} \quad \textit{weakly-star in } W^{1,\infty}(S^1),$$
$$(\overline{\eta}_0^n, \overline{\eta}_1^n) \to (\overline{\eta}_0, \overline{\eta}_1) \quad \textit{in } L^1(S^1).$$

Importantly, the method of proof of this lemma given now can also be applied to establish the existence of weakly regular T^2-symmetric initial data sets with constant R whose regularity is precisely the one introduced in Definitions 2.4 and 2.5, apart from the assumptions on R.

Proof of Lemma 5.4. By convolution of the data $\overline{X}$ and relying on the regularity assumed on the initial data set, one can define smooth functions $\overline{U}_0^n, \overline{A}_0^n, \overline{U}_1^n, \overline{A}_1^n, \overline{a}^n$ defined on S^1 such that, as $n \to \infty$, the functions $\overline{U}_0^n, \overline{A}_0^n, \overline{U}_1^n, \overline{A}_1^n$ converge in $L^2(S^1)$ toward $\overline{U}_0, \overline{A}_0, \overline{U}_1, \overline{A}_1$, respectively, while $\overline{a}^n$ converges to $\overline{a}$ in $W^{1,\infty}(S^1)$.

In order to obtain a complete set of regularized initial data, we also have to regularize the functions $\overline{\eta}_0$ and $\overline{\eta}_1$ in such a way that the constraint equations (4.26) hold for each integer n. To this end, to each regularized set $\overline{Y}^n := (\overline{U}_0^n, \overline{A}_0, \overline{U}_1^n, \overline{A}_1^n, \overline{a}^n)$, we associate the function and scalar

$$\omega[\overline{Y}^n] := 2R(\overline{U}_0^n \overline{U}_1^n + R^{-2} e^{2\overline{U}^n} \overline{A}_0^n \overline{A}_1^n), \qquad \Omega[\overline{Y}^n] := \int_{S^1} \omega[\overline{Y}^n]\, d\theta.$$

It follows that the function $\omega[\overline{Y}^n]$ converges in the space $L^1(S^1)$ toward $\overline{\eta}_1$, and that the sequence Ω^n (is uniformly bounded and) converges to 0.

Assuming first that we have been able to choose the regularization $\overline{Y}^n$ so that $\Omega[\overline{Y}^n] = 0$ for each integer n, and let us fix an arbitrary value $\theta_* \in S^1$. Then, by defining

$$\overline{\eta}^n(\theta) := \eta(\theta_*) + \int_{\theta_*}^{\theta} \omega[\overline{Y}^n]\, d\theta',$$

we see that the functions $\overline{\eta}^n$ converge in $W^{1,1}(S^1)$ toward the initial data $\overline{\eta}$. We can also define the function $\overline{\eta}_0^n$ by

$$\begin{aligned}
&(\overline{a}^n)^{-1}\overline{\eta}_0^n + (\overline{a}^n)^{-1}\frac{e^{2\overline{\eta}^n} K^2}{4R^3} = RE[\overline{Y}^n],\\
&E[\overline{Y}^n] := (\overline{a}^n)^{-1}(\overline{U}_0^n)^2 + \overline{a}^n(\overline{U}_1^n)^2 + (2R)^{-2} e^{4\overline{U}^n}\big((\overline{a}^n)^{-1}(\overline{A}_0^n)^2 + \overline{a}^n(\overline{A}_1^n)^2\big).
\end{aligned} \tag{5.4}$$

Here, the constant K is precisely the twist constant of the original initial data set. The right-hand side of (5.4) converges in $L^1(S^1)$ to the right-hand side of (4.26). We also claim

[9] In the application of this lemma, one could initially normalize the function a to be identically one.

that $(\overline{a}^n)^{-1} e^{2\overline{\eta}^n} K^2 R^{-3}$ converges to $\overline{a}^{-1} e^{2\overline{\eta}} K^2 R^{-3}$ in $L^1(S^1)$. Indeed, $\overline{a}^n$ converges to $\overline{a}$ in $W^{1,\infty}(S^1)$ and thus in $L^\infty(S^1)$, and moreover $e^{2\overline{\eta}^n}$ converges to $e^{2\overline{\eta}}$ in $L^1(S^1)$, as follows from the convergence of $\overline{\eta}^n$ in $W^{1,1}$, and thus in $L^\infty(S^1)$. Therefore, we see from (4.26) that $\overline{\eta}_0^n$ converges in $L^1(S^1)$ to $\overline{\eta}_0$, and passing to a subsequence if necessary, we may also assume almost everywhere convergence. Thus, $\overline{\eta}^n$ and $\overline{\eta}_0^n$ satisfy the requirement of the lemma.

It remains to determine a regularization such that $\Omega[\overline{Y}^n]$ vanishes. We start from an arbitrary regularized set $\overline{Y}^n$ that may not satisfy the constraints. Without loss of generality, we may assume that $\int_{S^1} (\overline{U}_1)^2$ or $\int_{S^1} (\overline{A}_1)^2 > 0$ (or both) are positive. For, if both of these terms vanish, $\overline{U}$ and $\overline{A}$ are almost everywhere constant, say $\overline{U} = U_*$, $\overline{A} = A_*$, and choosing for regularization $\mathcal{R} := (U_*, \overline{U}_0^n, A_*, \overline{A}_0^n, \overline{a}^n)$, we obtain $\Omega_{\mathcal{R}}^n = 0$.

Assume, for instance, that $\int_{S^1} (\overline{U}_1)^2 =: c$ is positive, the case of $\int_{S^1} (\overline{A}_\theta)^2$ positive being similar. For all sufficiently large n we have $\int_{S^1} (\overline{U}_1^n)^2 > c/2$ and, by assumption, $\Omega_{\mathcal{R}}^n$ goes to zero as $n \to \infty$. Setting

$$\delta^n := -\frac{\Omega[\overline{Y}^n]}{2R \int_{S^1} (\overline{U}_\theta^n)^2},$$

we now claim that

$$\overline{Y}' := (\overline{U}^n, \overline{U}_0^n + \delta^n \overline{U}_1^n, \overline{A}^n, \overline{A}_0^n, \overline{a}^n)$$

satisfies the constraints. Indeed, one can check that, by construction, $\Omega[\overline{Y}'] = 0$ and the conclusion follows from the estimate

$$|\delta^n| \leq \frac{|\Omega[\overline{Y}^n]|}{2c},$$

where the right-hand side converges to 0 as $n \to \infty$. □

5.4. Regularization of generic initial data sets

In passing, we now establish a stronger version of the previous regularization scheme which is of independent interest and applies to generic initial data sets. This result is not needed for our main result in this article, but is included for completeness.

Proposition 5.5 (Regularization of generic initial data sets). *Let (Σ, h, k) be a weakly regular T^2-symmetric Riemannian manifold satisfying the weak version of the vacuum constraint equations. Assume that either the area $\overline{R}$ of the symmetry orbits is constant on Σ, or the following condition holds (using the notation of Lemmas 3.1 and 3.2):*

$$\int_0^{2\pi} f(\xi) e^{-\int_{\xi'}^{2\pi} g \, d\xi''} \, d\xi' \neq 0, \tag{5.5}$$

where f and g are defined by

$$
\begin{aligned}
f &= \frac{\overline{R}_\xi}{\underset{0}{\overline{R}}^2 - \overline{R}_\xi^2}\,\frac{1}{2\overline{R}},\\
g &= 2(\underset{0}{\overline{R}}^2 - \overline{R}_\xi^2)^{-1}\left(-\overline{R}\,\overline{R}_\xi\left(\underset{0}{\overline{U}}^2 + \overline{U}_\xi^2 + \frac{e^{4\overline{U}}}{4\overline{R}^2}(\underset{0}{\overline{A}}^2 + \overline{A}_\xi^2) + \frac{\overline{R}_{\xi\xi}}{\overline{R}}\right)\right.\\
&\qquad\left. + \overline{R}\,\underset{0}{\overline{R}}\left(2\underset{0}{\overline{U}}\,\overline{U}_\xi + \frac{e^{4\overline{U}}}{2\overline{R}^2}\underset{0}{\overline{A}}\,\overline{A}_\xi + \frac{(\underset{0}{\overline{R}})_\xi}{\overline{R}}\right)\right),
\end{aligned}
$$

$\overline{U} = \overline{P} + 1/2$, and $\underset{0}{\overline{U}} = \underset{0}{\overline{P}} + \underset{0}{\overline{R}}/(2\overline{R})$. Then there exists a smooth family of T^2-symmetric metrics h^ϵ (parameterized by $\epsilon \in (0,1)$) invariant by the same T^2 action, together with a smooth family of T^2-symmetric, symmetric 2-tensors k^ϵ invariant by the same T^2 action such that the triple $(\Sigma, h^\epsilon, k^\epsilon)$ satisfies the constraints in the same conformal system of coordinates and (h^ϵ, k^ϵ) converges to (h,k) as ϵ goes to 0 in the following topology:

$$
\begin{aligned}
&\overline{U}^\epsilon, \overline{A}^\epsilon \to \overline{U}, \overline{A} \quad \text{in } H^1(S^1), &&\underset{0}{\overline{U}}^\epsilon, \underset{0}{\overline{A}}^\epsilon \to \underset{0}{\overline{U}}, \underset{0}{\overline{A}} \quad \text{in } L^2(S^1),\\
&\overline{\nu}^\epsilon \to \overline{\nu} \quad \text{in } W^{1,1}(S^1), &&\underset{0}{\overline{\nu}}^\epsilon \to \underset{0}{\overline{\nu}} \quad \text{in } L^1(S^1),\\
&\overline{R}^\epsilon \to \overline{R} \quad \text{in } W^{2,1}(S^1), &&\underset{0}{\overline{R}}^\epsilon \to \underset{0}{\overline{R}} \quad \text{in } W^{1,1}(S^1),
\end{aligned}
$$

and the twist coefficients associated with (h, k, X, Y) converge to the twists coefficients associated with $(h^\epsilon, k^\epsilon, X, Y)$.

Proof. For simplicity of notation, we drop the bars and write R_τ for $\underset{0}{\overline{R}}$. Without loss of generality, we may also assume that the initial data are not trivial, in particular A, U are not constants and A_τ, U_τ do not vanish identically. Since R_τ, R_ξ are of class C^1 at least and $R_\tau^2 - R_\xi^2 > 0$ (cf. Proposition 5.1), by a continuity argument we obtain the lower bound $R_\tau^2 - R_\xi^2 \geq c > 0$ for some constant c. It follows that the constraint equations are equivalent to

$$
\begin{aligned}
0 &= R R_\xi \left(U_\tau^2 + U_\xi^2 + \frac{e^{4U}}{4R^2}(A_\tau^2 + A_\xi^2) + \frac{R_{\xi\xi}}{R} + \frac{e^{2\nu}K^2}{4R^2}\right) - \nu_\xi R_\xi^2 - \nu_\tau R_\tau R_\xi,\\
0 &= R R_\tau \left(2U_\tau U_\xi + \frac{e^{4U}}{2R^2}A_\tau A_\xi + \frac{R_{\xi\tau}}{R}\right) - \nu_\xi R_\tau^2 - \nu_\tau R_\tau R_\xi,
\end{aligned}
$$

where K denotes the twist constant associated with Y, the other twist constant being set to 0 (without loss of generality in view of Proposition 2.26). Taking the difference of the last two equations, we obtain

$$
\begin{aligned}
\nu_\xi + \frac{R R_\xi}{R_\tau^2 - R_\xi^2}\,\frac{K^2}{4R^2}e^{2\nu} &= (R_\tau^2 - R_\xi^2)^{-1}\left(-R R_\xi\left(U_\tau^2 + U_\xi^2 + \frac{e^{4U}}{4R^2}(A_\tau^2 + A_\xi^2) + \frac{R_{\xi\xi}}{R}\right)\right.\\
&\qquad\left. + R R_\tau\left(2U_\tau U_\xi + \frac{e^{4U}}{2R^2}A_\tau A_\xi + \frac{R_{\xi\tau}}{R}\right)\right).
\end{aligned}
$$

Let us rewrite this equation as $(e^{2\nu})_\xi e^{-2\nu} + K^2 f e^{2\nu} = g$ with

$$f = \frac{RR_\xi}{R_\tau^2 - R_\xi^2} \frac{1}{2R^2},$$
$$g = 2(R_\tau^2 - R_\xi^2)^{-1}\bigg(-RR_\xi\bigg(U_\tau^2 + U_\xi^2 + \frac{e^{4U}}{4R^2}(A_\tau^2 + A_\xi^2) + \frac{R_{\xi\xi}}{R}\bigg) + RR_\tau\bigg(2U_\tau U_\xi + \frac{e^{4U}}{2R^2}A_\tau A_\xi + \frac{R_{\xi\tau}}{R}\bigg)\bigg).$$

Setting $\phi = e^{-2\nu}$, we then have $\phi' + \phi g = K^2 f$, which may be solved as

$$e^{-2\nu}(\xi) = e^{-2\nu}(0)e^{-\int_0^\xi g(\xi')\,d\xi'} + K^2 \int_0^\xi f(\xi)e^{-\int_{\xi'}^\xi g\,d\xi''}\,d\xi'. \tag{5.6}$$

We now use the above formula to define the regularized coefficient ν, as follows. First, we regularize $R, R_\tau, U, U_\tau, A, A_\tau$ by a standard convolution.

If $R_\xi = 0$ uniformly, then $R = \text{const}$ initially and we may apply the regularization scheme developed in areal coordinates in the previous section. Thus, we can always assume that $R_\xi \neq 0$ so that the technical assumption of the lemma holds.

Next, let us define K^ϵ by

$$(K^\epsilon)^2 := e^{-2\nu(0)}(1 - e^{-\int_0^{2\pi} g^\epsilon(\xi')\,d\xi'})\bigg(\int_0^{2\pi} f^\epsilon(\xi)e^{-\int_{\xi'}^{2\pi} g^\epsilon\,d\xi''}\,d\xi'\bigg)^{-1}.$$

From the strong convergence of f^ϵ to f and g^ϵ to g it follows that $(K^\epsilon)^2$ is well-defined and converges to K^2 (as ϵ goes to 0). Define now ν^ϵ as

$$e^{-2\hat{\nu}^\epsilon}(\xi) = e^{-2\nu}(0)e^{-\int_0^\xi g^\epsilon(\xi')\,d\xi'} + (K^\epsilon)^2 \int_0^\xi f^\epsilon(\xi)e^{-\int_{\xi'}^\xi g^\epsilon\,d\xi''}\,d\xi'. \tag{5.7}$$

It follows from the definition of K^ϵ that ν^ϵ is periodic with period 2π (and so can be identified with a smooth function on S^1) and converges to ν in $W^{1,1}$ as ϵ goes to 0. Finally, we define ν_τ^ϵ so that the remaining constraint equation holds, i.e.

$$\nu_\tau^\epsilon = -\frac{1}{(R_\tau^\epsilon)^2 - (R_\xi^\epsilon)^2}\bigg(R^\epsilon R_\xi^\epsilon\bigg(2U_\tau^\epsilon U_\xi^\epsilon + \frac{e^{4U^\epsilon}}{2(R^\epsilon)^2}A_\tau^\epsilon A_\xi^\epsilon + R_{\xi\tau}^\epsilon\bigg) + R^\epsilon R_\tau^\epsilon\bigg((U_\tau^\epsilon)^2 + (U_\xi^\epsilon)^2 + \frac{e^{4U^\epsilon}}{4R^\epsilon}((A_\tau^\epsilon)^2 + (A_\xi^\epsilon)^2) + \frac{R_{\xi\xi}^\epsilon}{R^\epsilon} + \frac{e^{2\nu^\epsilon}(K^\epsilon)^2}{4(R^\epsilon)^2}\bigg)\bigg).$$

The convergence of the right-hand side and the given constraint equations then imply that ν_τ^ϵ converges in L^1 to ν_τ. □

6. Local geometry of weakly regular T^2-symmetric spacetimes

6.1. Strategy of proof

For the existence of weak solutions to the initial value problem associated with the Einstein equations under the assumed symmetry, we proceed as follows.

Step 1. *Local existence in conformal coordinates and blow-up criterion.* First, we prove a compactness property for solutions to the conformal equations. This yields, for any weakly regular initial data set, the existence of a local-in-time solution, defined on a sufficiently small interval of (conformal) time $[\tau_1, \tau_1 + \epsilon)$, where ϵ only depends on natural (energy-like) norms corresponding to the assumed (weak) regularity of the initial data. Together with this local existence result, we obtain a continuation criterion. This result is stated precisely in Theorem 6.1 below, and the rest of this section is devoted to its proof.

Step 2. *Local existence in areal coordinates.* We can always arrange that the condition $\tau = \tau_1$ coincides with $R = R_1$ (cf. the construction of conformal coordinates in Lemma 3.5), and since R is strictly increasing with τ and weak solutions to the conformal equations can be transformed to weak solutions to the areal equations (i.e. the equations derived in Proposition 5.3), we obtain a local solution to the equations in areal coordinates, defined on a small interval of areal time $[R_1, R_1 + \epsilon)$. Moreover, we also obtain a continuation criterion in areal coordinates which states the solution ceases to exist only if the natural energy-like norms are blowing up.

Step 3. *Global existence in areal coordinates.* Finally, performing a further analysis of the Einstein system in areal coordinates, we obtain a global-in-time control of the natural norms which will lead us to the desired global existence result. This step will be presented in Section 7.

The above strategy is motivated by the following observations. Due to the quasilinear structure of the equations in areal coordinates, one cannot directly estimate the difference of solutions. While we do obtain a priori estimates for solutions in Step 3, these estimates do not provide sufficiently strong compactness properties. A possible strategy (for general quasilinear systems) in order to cope with this difficulty would be to prove compactness in a weaker function space. However, under our weak regularity assumptions, the natural function spaces for U, A which one may think of would be L^2 (instead of H^1); however, one cannot control the behavior of the remaining metric coefficients a, ν by the L^2 norm of U, A. This is the reason why we propose here to rely on conformal coordinates (in which the equations become semilinear) in order to prove local well-posedness. However, in conformal coordinates, the natural energy associated with U, A fails to be a priori bounded, and this is why only local-in-time existence is obtained in conformal coordinates, one must introduce areal coordinates to get a global-in-time result. In the rest of this section, we discuss the issue of local existence in conformal coordinates.

6.2. Local existence

As explained above, the aim of this section is to prove the following result.

Theorem 6.1 (Local existence in conformal coordinates). *Let (Σ, h, K) be a weakly regular T^2-symmetric initial data set. Assume that (Σ, h, K) admits a regularization $(\Sigma^\epsilon, h^\epsilon, K^\epsilon)$ as described in Lemma 5.5, which, for instance, applies if the associated area function R is constant on Σ. Let (ξ, x, y) be admissible coordinates and $\underset{0}{\overline{R}}$ be defined in as in Lemma 3.2. Assume finally that*

$$M_0 := \inf_\Sigma |\underset{0}{\overline{R}} - \overline{R}_{\xi'}| \inf_\Sigma |\underset{0}{\overline{R}} + \overline{R}_{\xi'}|$$

is nonvanishing (which holds for nontrivial data in view of Proposition 5.1). Then there exists a weakly regular T^2-symmetric Lorentzian manifold $(\mathcal{M}, g)$ endowed with admissible conformal coordinates (τ, ξ, x, y) such that:

1. *$\mathcal{M} = [\tau_0, \tau_1) \times \Sigma$ for some $\tau_0 < \tau_1$, and the metric g takes the conformal form (3.4).*
2. *R is strictly increasing with τ.*
3. *$|\tau_1 - \tau_0| > 0$ depends only the initial norm N_0 of the initial data set, defined by*

$$\begin{aligned} N_0 := {} & \|\overline{U}, \overline{A}\|_{H^1(S^1)} + \|\underset{0}{\overline{U}}, \underset{0}{\overline{A}}\|_{L^2(S^1)} + \|\overline{\nu}\|_{W^{1,1}(S^1)} + \|\underset{0}{\overline{\nu}}\|_{L^1} \\ & + \|\overline{R}\|_{W^{2,1}(S^1)} + \|\underset{0}{\overline{R}}\|_{W^{1,1}(S^1)} + \|\overline{R}^{-1}\|_{L^\infty(S^1)} + \frac{1}{M_0}. \end{aligned} \tag{6.1}$$

4. *The metric coefficients have the following regularity:*

$$\begin{aligned} & U, A \in C^0_\tau(H^1_\xi(S^1)) \cap C^1_\tau(L^2_\xi(S^1)), \qquad \nu \in C^0_\tau(W^{1,1}_\xi(S^1)) \cap C^1_\tau(L^1_\xi(S^1)), \\ & R \in C^0_\tau(W^{2,1}_\xi(S^1)) \cap C^1_\tau(W^{1,1}_\xi(S^1)). \end{aligned}$$

5. *Considering the embedding $\psi : \Sigma \to \mathcal{M}$, $(\xi, x, y) \mapsto (\tau_0, \xi, x, y)$, one has*

$$\begin{aligned} & (U, U_\tau, A, A_\tau, \nu, \nu_\tau, R, R_\tau, G, G_\tau, H, H_\tau)(\tau_0) \\ & \qquad = (\overline{U}, \underset{0}{\overline{U}}, \overline{A}, \underset{0}{\overline{A}}, \overline{\nu}, \underset{0}{\overline{\nu}}, \overline{R}, \underset{0}{\overline{R}}, \overline{G}, \underset{0}{\overline{G}}, \overline{H}, \underset{0}{\overline{H}}) \circ \psi. \end{aligned}$$

   Observe that this embedding respects the symmetry property.
6. *Let $(\mathcal{M}', g')$ be a weakly regular T^2-symmetric manifold with admissible conformal coordinates satisfying all of the conditions above but with another embedding $\psi' : \Sigma \to \mathcal{M}'$, $(\xi', x, y) \mapsto (\tau_0, \xi', x, y)$. Then there exists a neighborhood $\mathcal{U} \subset \mathcal{M}$ of $\psi(\Sigma)$, a neighborhood $\mathcal{U}' \subset \mathcal{M}'$ of $\psi'(\Sigma)$ and C^∞-diffeomorphism $\phi : \mathcal{U}' \to \mathcal{U}$ such that $g'|_{\mathcal{U}'} = \phi^* g|_{\mathcal{U}}$ and $\phi|_\Sigma \circ \psi' = \psi$.*

To establish this result, we are going first to derive a priori estimates for any given smooth solution, and next a priori estimates for the difference of two solutions. Compactness of the set of all solutions arising from a regularization of the initial data follows easily from these estimates. Interestingly, our estimate for the difference of two solutions requires a property of higher-order integrability on curved spacetimes with weakly regular geometry, inspired from Zhou [38] who treated a system of $(1+1)$-wave maps on the (flat) $(1+1)$-Minkowski background. The uniqueness statement in the above theorem also follows from our estimates on the difference of two solutions, once a system of conformal coordinates has been fixed, which is equivalent to fixing a system of admissible coordinates on Σ.

The derivation of a priori estimates for smooth solutions given below relies on a bootstrap argument. To establish energy estimates for the wave equations for U, A, we need an upper bound on the sup norm of the first derivatives of R as well as on the sup norm of ν. Thus, we first prove energy estimates depending on these bounds, and next use these energy estimates to improve the upper bounds, on sufficiently small time intervals at least.

6.3. A priori estimates for smooth solutions

We consider a smooth solution (U, A, R, ν) of the Einstein equations in conformal coordinates defined on some interval $[\tau_0, \tau_1)$ with $\tau_1 > \tau_0$. Moreover, we assume that the solution is nontrivial (i.e. does not lead to a flat spacetime) and that the time orientation has been chosen so that $R_\tau > 0$.

Lemma 6.2 (Monotonicity of the area function). *Both functions $R_\tau \pm R_\xi$ are strictly increasing along the integral curves of $\tau \mp \xi =$ const, as functions of $\tau \mp \xi$, respectively. Moreover, R is a strictly increasing function of τ and in particular, for all $\tau \geq \tau_0$,*

$$R(\tau, \xi) \geq \min_{\tau'=\tau_0} R.$$

Proof. Introducing the notation $\partial_u = \partial_\tau - \partial_\xi$, $\partial_v = \partial_\tau + \partial_\xi$, we observe that

$$R_{uv} \geq 0,$$

which, in view of our assumptions on the initial data, leads to the desired claims. □

From now on, we set

$$R_0 := \min_{\tau'=\tau_0} R, \tag{6.2}$$

and we work with the energy-like functional

$$\mathcal{E}_{\text{conf}}(\tau) := \int_{S^1} \left(R(U_\tau^2 + U_\xi^2) + \frac{e^{4U}}{4R}(A_\tau^2 + A_\xi^2) + \frac{e^{2\nu} K^2}{4R^3} \right).$$

Lemma 6.3 (Energy estimate). *For all $\tau \geq \tau_0$, one has*

$$\mathcal{E}_{\text{conf}}(\tau) \leq \mathcal{E}_{\text{conf}}(\tau_0) e^{C(R_0)(\|R\|_{C^1(\tau,\xi)}+1)(\tau-\tau_0)},$$

where $C(R_0) > 0$ depends only R_0.

Proof. From the constraint equations, it follows that

$$\mathcal{E}_{\text{conf}} = \int_{S^1} (-R_{\xi\xi} - \nu_\tau R_\tau - \nu_\xi R_\xi) = \int_{S^1} (-\nu_\tau R_\tau - \nu_\xi R_\xi)$$

and, after several integrations by parts,

$$\frac{d}{d\tau}\mathcal{E}_{\text{conf}} = \int_{S^1} \big(-\nu_\tau (R_{\tau\tau} - R_{\xi\xi}) - R_\tau (\nu_{\tau\tau} - \nu_{\xi\xi}) \big).$$

Using the wave equations for ν and R, we then obtain

$$\frac{d}{d\tau}\mathcal{E}_{\mathrm{conf}} \leq C(R_0)\|R\|_{C^1}\mathcal{E}_{\mathrm{conf}} - \int_{S^1} \nu_\tau e^{2\nu}\frac{K^2}{2R^3}.$$

The desired result then follows by integration in time, using Gronwall's lemma and integrating by parts to control the second term above. □

Lemma 6.4 (First-order estimates on the area function). *By defining*

$$M(R)(\tau) := \inf_{\xi\in S^1} R_u(\tau,\cdot)\inf_{\xi\in S^1} R_v(\tau,\cdot), \tag{6.3}$$

the area function satisfies

$$\begin{aligned}\|R\|_{C^1}(\tau) &\leq C\big((\tau-\tau_0)\|e^{2\nu}\|_{L^\infty[\tau_0,\tau]\times S^1} + \|R\|_{C^1}(\tau_0)\big),\\ (R_\tau^2 - R_\xi^2)(\tau) &\geq M(R)(\tau_0),\end{aligned}$$

where the constant $C = C(R_0, K) > 0$ only depends on R_0 and the twist constant K.

Proof. Both estimates are straightforward consequences of the wave equation satisfied by the function R. The second uses the fact that $R_\tau^2 - R_\xi^2 = R_u R_v$ and that R_u and R_v are increasing in respectively v and u. □

A direct consequence of the constraint equations is now stated.

Lemma 6.5 (First-order estimate on ν). *The metric coefficient ν satisfies*

$$\begin{aligned}&\|\nu_\xi\|_{L^1}(\tau) + \|\nu_\tau\|_{L^1}(\tau)\\ &\qquad\leq C\|R\|_{C^1([\tau_0,\tau]\times S^1)}\big(\mathcal{E}_{\mathrm{conf}}(\tau) + \|R_{\xi\xi}\|_{L^1}(\tau) + \|R_{\xi\tau}\|_{L^1}(\tau)\big),\end{aligned} \tag{6.4}$$

where $C = C(R_0, M(R)(\tau_0)) > 0$ is a constant.

Finally, we have the following additional estimate on R.

Lemma 6.6 (Higher-order estimates on the area function). *The area function satisfies the following second-order estimates:*

$$\begin{aligned}&\|R_{\xi\xi}\|_{L^1(S^1)}(\tau)\\ &\quad\leq C(R_0)\int_{\tau_0}^{\tau}(\|\nu_\xi\|_{L^1}(\tau') + \|R\|_{C^1([\tau_0,\tau']\times S^1)})\|e^{2\nu}\|_{L^\infty}\,d\tau' + \|R_{\xi\xi}\|_{L^1(S^1)}(\tau_0),\\ &\|R_{\xi\tau}\|_{L^1(S^1)}(\tau)\\ &\quad\leq C(R_0)\int_{\tau_0}^{\tau}(\|\nu_\tau\|_{L^1}(\tau') + \|R\|_{C^1([\tau_0,\tau']\times S^1)})\|e^{2\nu}\|_{L^\infty}\,d\tau' + \|R_{\xi\tau}\|_{L^1(S^1)}(\tau_0).\end{aligned}$$

Proof. This is a simple commutation argument for the wave equation of R. Recall that R satisfies an equation of the form $R_{uv} = \Omega_R$, hence we have

$$R_{\xi u}(\xi, v) = \int_v \partial_\xi \Omega_R + R_{\xi u}(\xi, v_0).$$

Similar expressions holds for $R_{\xi v}$, $R_{\tau u}$ and $R_{\tau v}$. Since $\Omega_R = e^{2\nu} K^2/(2R^3)$, the result follows. □

To close the argument and arrive at the desired uniform estimate, we consider the bootstrap assumptions

$$\begin{aligned}\|\nu\|_{L^\infty}(\tau) \\ &\leq 5C_1(\|\overline{R}\|_{C^1(S^1)} + \|\underset{0}{\overline{R}}\|_{C^0(S^1)})\big(\mathcal{E}_{\text{conf}}(\tau_0) + \|\overline{R}_{\xi\xi}\|_{L^1}(\tau_0) + \|(\underset{0}{\overline{R}})_\xi\|_{L^1}(\tau_0) + 1\big) \\ &\quad + \frac{1}{\pi}\|\overline{\nu}\|_{L^1},\end{aligned} \tag{6.5}$$

and

$$\|R\|_{C^1([\tau_0,\tau]\times S^1)} \leq 2(\|\overline{R}\|_{C^1(S^1)} + \|\underset{0}{\overline{R}}\|_{C^0(S^1)}), \tag{6.6}$$

where $C_1 = C_1(R_0, M_0) > 0$ is the constant arising in (6.4). Let $\delta > 0$ be fixed, and $\mathcal{B} \subset [\tau_0, \tau_0 + \delta]$ be the largest spacetime region which is included in $[\tau_0, \tau_0 + \delta]$ and in which (6.5)–(6.6) hold. Then $\mathcal{B}$ is clearly non-empty and open. We show that for all sufficiently small δ (in terms of the initial norm of the data (6.1) only) we can improve (6.5)–(6.6), namely the following holds.

Lemma 6.7. *If $\delta > 0$ is sufficiently small (depending only on the initial norm (6.1)) then $\mathcal{B}$ is closed.*

Proof. It follows from the previous estimates and the bootstrap assumptions that if δ is sufficiently small, depending only on the initial norm of the data, we have

$$\begin{aligned}&\|\nu_\xi\|_{L^1(S^1)}(\tau) + \|\nu_\tau\|_{L^1(S^1)}(\tau) \\ &\quad \leq 4C_1(\|\overline{R}\|_{C^1(S^1)} + \|\underset{0}{\overline{R}}\|_{C^0(S^1)})\big(\mathcal{E}_{\text{conf}}(\tau_0) + \|R_{\xi\xi}\|_{L^1}(\tau_0) + \|R_{\xi\tau}\|_{L^1}(\tau_0) + 1/2\big).\end{aligned}$$

Since

$$\|\nu\|_{L^\infty}(\tau) \leq \frac{1}{2\pi}\|\nu\|_{L^1}(\tau_0) + \|\nu_\xi\|_{L^1}(\tau) + \frac{1}{2\pi}(\tau - \tau_0)\|\nu_\tau\|_{L^1}(\tau),$$

we have improved (6.5), and then (6.6) is easily improved using the wave equation for R. □

Hence, we have established the following result.

Proposition 6.8 (A priori estimates in conformal coordinates). *There exists a real $\delta > 0$ depending only on the initial norm of the data (6.1) such that, on $[\tau_0, \tau + \delta]$,*

$$\begin{aligned}N(\tau) &:= \|U, A\|_{H^1(S^1)}(\tau) + \|U_\tau, A_\tau\|_{L^2(S^1)} + \|\nu\|_{W^{1,1}(S^1)} + \|\nu_\tau\|_{L^1} \\ &\quad + \|R\|_{W^{2,1}(S^1)} + \|R_\tau\|_{W^{1,1}(S^1)} + \|R^{-1}\|_{L^\infty(S^1)} + N(\nabla R)^{-1}(\tau) \leq C,\end{aligned} \tag{6.7}$$

where $C := C(N(\tau_0), M(R)(\tau_0))$ is a constant.

6.4. Higher integrability in spacetime

In order to prove compactness of sequences of solutions, we need a better control over the source terms arising in the equations satisfied by the metric coefficients U, A. To this end, we now establish a higher integrability property in spacetime for these source terms, which is motivated by Zhou [38] who treated $(1+1)$-wave maps. The following result is actually stated in a more general form than needed for the proof of local well-posedness in conformal coordinates in the present section, but the full statement will be relevant in our analysis in areal coordinates (cf. Section 7). This lemma is essential for the existence theory for the Einstein equations with weak regularity, since it will allow us to compare two arbitrary solutions and eventually establish a compactness property.

Lemma 6.9 (Spacetime higher integrability estimate). *Let $w_-, w_+ : [R_0, R^\star] \times \mathbb{R} \to \mathbb{R}$ be weak solutions in $L^\infty_t L^2_\theta$ to the equations $\partial_R w_\pm \pm \partial_\theta(a w_\pm) = h_\pm$, respectively, where the coefficient $a : [R_0, R^\star] \times \mathbb{R} \to \mathbb{R}$ belongs to L^∞ and satisfies $0 < a_0 \le a \le a_1$ and $h_\pm : [R_0, R^\star] \times \mathbb{R} \to \mathbb{R}$ in $L^\infty_t L^1_\theta$ are given functions. Then for each $L > a_1 R$ one has*

$$\frac{d}{dR} N^I + 2a_0 N^{II} \le N^{III},$$

with

$$\begin{aligned}
N^I(R) &:= \int_{-L+a_1R}^{L-a_1R} \int_{\theta_+}^{L-a_1R} |w_+(R,\theta_+)|\,|w_-(R,\theta_-)|\,d\theta_+\,d\theta_-,\\
N^{II}(R) &:= \int_{-L+a_1R}^{L-a_1R} |w_+(R,\cdot)|\,|w_-|(R,\cdot)\,d\theta,\\
N^{III}(R) &:= \sum_\pm \int_{-L+a_1R}^{L-a_1R} |h_\pm(R,\cdot)|\,d\theta \int_{-L+a_1R}^{L-a_1R} |w_\mp(R,\cdot)|\,d\theta.
\end{aligned}$$

Proof. It is not difficult to check that

$$\partial_R|w_\pm| \pm \partial_\theta(a|w_\pm|) \le |h_\pm|.$$

On the other hand, from the definitions, we obtain

$$\begin{aligned}
\frac{d}{dR} N^I(R) \le &\int_{-L+a_1R}^{L-a_1R} \int_{\theta_+}^{L-a_1R} \big(-\partial_\theta(a|w_+|) + |h_+|\big)(R,\theta_+)|w_-(R,\theta_-)|\,d\theta_-\,d\theta_+\\
&+ \int_{-L+a_1R}^{L-a_1R} \int_{\theta_+}^{L-a_1R} |w_+(R,\theta_+)|\big(\partial_\theta(a|w_-|) + |h_-|\big)(R,\theta_-)\,d\theta_-\,d\theta_+\\
&- a_1 \int_{-L+a_1R}^{L-a_1R} |w_+(R,-L+a_1R)|\,|w_-(R,\theta)|\,d\theta\\
&- a_1 \int_{-L+a_1R}^{L-a_1R} |w_+(R,\theta)|\,|w_-(R,L-a_1R)|\,d\theta
\end{aligned}$$

and therefore

$$\begin{aligned}\frac{d}{dR}N^I(R) &\le -2\int_{-L+a_1R}^{L-a_1R} a(R,\theta)|w_+(R,\theta)|\,|w_-|(R,\theta)\,d\theta\\&+\int_{-L+a_1R}^{L-a_1R}\int_{\theta_+}^{L-a_1R}\big(|h_+|(R,\theta_+)|w_-(R,\theta_-)|+|w_+|(R,\theta_+)|h_-(R,\theta_-)|\big)\,d\theta_-\,d\theta_+\\&-\int_{-L+a_1R}^{L-a_1R}(a_1-a(R,\theta))|w_+(R,-L+a_1R)|\,|w_-(R,\theta)|\,d\theta\\&-\int_{-L+a_1R}^{L-a_1R}(a_1-a(R,\theta))|w_+(R,\theta)|\,|w_-(R,L-a_1R)|\,d\theta.\end{aligned}$$

Using the lower and upper bounds of the function a, we obtain the desired estimate. □

6.5. *Well-posedness theory for weak solutions*

We are now in a position to complete the proof of Theorem 6.1 concerning local existence of solutions for the system (4.14)–(4.20) by establishing estimates for the difference of two solutions. Let $(U^{\epsilon_1}, A^{\epsilon_1}, \nu^{\epsilon_1}, R^{\epsilon_1}, K^{\epsilon_1})$ and $(U^{\epsilon_2}, A^{\epsilon_2}, \nu^{\epsilon_2}, R^{\epsilon_2}, K^{\epsilon_2})$ be two C^∞ solutions to the system (4.14)–(4.20), with respective twist constant K^{ϵ_1} and K^{ϵ_2}, defined on a cylinder $[\tau_0,\tau_1]\times S^1$, where $\tau_1=\tau_0+\delta$, with δ small enough so that the uniform estimates of the previous section hold for both solutions. Denote by $N^i(\tau)$ $(i=1,2)$ the norms of the solutions at time τ, i.e.

$$\begin{aligned}N^i(\tau) &= \|U^{\epsilon_i}, A^{\epsilon_i}\|_{H^1(S^1)}(\tau)+\|U^{\epsilon_i}_\tau, A^{\epsilon_i}_\tau\|_{L^2(S^1)}+\|\nu^{\epsilon_i}\|_{W^{1,1}(S^1)}+\|\nu^{\epsilon_i}_\tau\|_{L^1}\\&\quad+\|R^{\epsilon_i}\|_{W^{2,1}(S^1)}+\|R^{\epsilon_i}_\tau\|_{W^{1,1}(S^1)}+\|(R^{\epsilon_i})^{-1}\|_{L^\infty(S^1)}+\frac{1}{M(R^{\epsilon_i})(\tau)}.\end{aligned}$$

From the uniform estimates established above, it follows that for $i=1,2$, there exists a positive constant C^i, depending only on $N^i(\tau_0)$, such that

$$N^i(\tau)\le C^i.$$

We define $\Delta U := U^{\epsilon_2}-U^{\epsilon_1}$, $\Delta A := A^{\epsilon_2}-A^{\epsilon_1}, \ldots$ and we set

$$\begin{aligned}N^\Delta(\tau) &:= \|\Delta U,\Delta A\|_{H^1(S^1)}(\tau)+\|\Delta\nu,\Delta R_\xi,\Delta R_\tau\|_{W^{1,1}}(\tau)+\|\Delta U_\tau,\Delta A_\tau\|_{L^2(S^1)}(\tau)\\&\quad+\|\nu_\tau\|_{L^1}(\tau)+\|\Delta R,\Delta(R^{-1})\|_{C^1(S^1)}(\tau)+\|\Delta R_\tau\|_{C^0(S^1)}(\tau).\end{aligned}$$

Then ΔU, ΔA, etc. satisfy the equations

$$\begin{aligned}\Delta U_{\tau\tau}-\Delta U_{\xi\xi}&=\Omega^{\Delta U}, &\quad \Delta A_{\tau\tau}-\Delta A_{\xi\xi}&=\Omega^{\Delta A},\\ \Delta\nu_{\tau\tau}-\Delta\nu_{\xi\xi}&=\Omega^{\Delta\nu}, &\quad \Delta R_{\tau\tau}-\Delta R_{\xi\xi}&=\Omega^{\Delta R},\end{aligned}$$

with error terms given by

$$\Omega^{\Delta U} = -\frac{R_\tau^{\epsilon_2}}{R^{\epsilon_2}}U_\tau^{\epsilon_2} + \frac{R_\tau^{\epsilon_1}}{R^{\epsilon_1}}U_\tau^{\epsilon_1} + \frac{R_\xi^{\epsilon_2}}{R^{\epsilon_2}}U_\xi^{\epsilon_2} - \frac{R_\xi^{\epsilon_1}}{R^{\epsilon_1}}U_\xi^{\epsilon_1} + \frac{e^{4U^{\epsilon_2}}}{2(R^{\epsilon_2})^2}((A_\tau^{\epsilon_2})^2 - (A_\xi^{\epsilon_2})^2) - \frac{e^{4U^{\epsilon_1}}}{2(R^{\epsilon_1})^2}((A_\tau^{\epsilon_1})^2 - (A_\xi^{\epsilon_1})^2), \tag{6.8}$$

$$\Omega^{\Delta A} = \frac{R_\tau^{\epsilon_2}}{R^{\epsilon_2}}A_\tau^{\epsilon_2} - \frac{R_\tau^{\epsilon_1}}{R^{\epsilon_1}}A_\tau^{\epsilon_1} - \frac{R_\xi^{\epsilon_2}}{R^{\epsilon_2}}A_\xi^{\epsilon_2} + \frac{R_\xi^{\epsilon_1}}{R^{\epsilon_1}}A_\xi^{\epsilon_1} + 4(A_\xi^{\epsilon_2}U_\xi^{\epsilon_2} - A_\tau^{\epsilon_2}U_\tau^{\epsilon_2}) - 4(A_\xi^{\epsilon_1}U_\xi^{\epsilon_1} - A_\tau^{\epsilon_1}U_\tau^{\epsilon_1}), \tag{6.9}$$

and

$$\Omega^{\Delta\nu} = (U_\xi^{\epsilon_2})^2 - (U_\tau^{\epsilon_2})^2 - (U_\xi^{\epsilon_1})^2 + (U_\tau^{\epsilon_1})^2 + \frac{e^{4U^{\epsilon_2}}}{4(R^{\epsilon_2})^2}((A_\tau^{\epsilon_2})^2 - (A_\xi^{\epsilon_2})^2) - \frac{e^{4U^{\epsilon_1}}}{4(R^{\epsilon_1})^2}((A_\tau^{\epsilon_1})^2 - (A_\xi^{\epsilon_1})^2) - \frac{3(K^{\epsilon_2})^2}{4(R^{\epsilon_2})^4}e^{2\nu^{\epsilon_2}} + \frac{3(K^{\epsilon_1})^2}{4(R^{\epsilon_1})^4}e^{2\nu^{\epsilon_1}},$$

$$\Omega^{\Delta R} = -\frac{(K^{\epsilon_2})^2}{2(R^{\epsilon_2})^3}e^{2\nu^{\epsilon_2}} + \frac{(K^{\epsilon_2})^2}{2(R^{\epsilon_1})^2}e^{2\nu^{\epsilon_1}}.$$

Moreover, from the constraint equations we also have

$$\Delta\nu_\tau = \Omega^{\Delta\nu_\tau}, \qquad \Delta\nu_\xi = \Omega^{\Delta\nu_\xi},$$

where $\Omega^{\Delta\nu_\tau}$ and $\Omega^{\Delta\nu_\xi}$ are obtained from the equations

$$\nu_\tau^{\epsilon_i} = -\frac{1}{(R_\tau^{\epsilon_i})^2 - (R_\xi^{\epsilon_i})^2}\Bigg(R^{\epsilon_i}R_\xi^{\epsilon_i}\left(2U_\tau^{\epsilon_i}U_\xi^{\epsilon_i} + \frac{e^{4U^{\epsilon_i}}}{2(R^{\epsilon_i})^2}A_\tau^{\epsilon_i}A_\xi^{\epsilon_i} + R_{\xi\tau}^{\epsilon_i}\right) + R^{\epsilon_i}R_\tau^{\epsilon_i}\left((U_\tau^{\epsilon_i})^2 + (U_\xi^{\epsilon_i})^2 + \frac{e^{4U^{\epsilon_i}}}{4R^{\epsilon_i}}((A_\tau^{\epsilon_i})^2 + (A_\xi^{\epsilon_i})^2) + \frac{R_{\xi\xi}^{\epsilon_i}}{R^{\epsilon_i}} + \frac{e^{2\nu^{\epsilon_i}}(K^{\epsilon_i})^2}{4(R^{\epsilon_i})^2}\right)\Bigg)$$

and

$$\nu_\xi^{\epsilon_i} = -\frac{R^{\epsilon_i}R_\xi^{\epsilon_i}}{(R^{\epsilon_i})_\tau^2 - (R^{\epsilon_i})_\xi^2}\,\frac{(K^{\epsilon_i})^2}{4(R^{\epsilon_i})^2}e^{2\nu^{\epsilon_i}} - ((R^{\epsilon_i})_\tau^2 - (R^{\epsilon_i})_\xi^2)^{-1}R^{\epsilon_i}R_\xi^{\epsilon_i}\left((U^{\epsilon_i})_\tau^2 + (U^{\epsilon_i})_\xi^2 + \frac{e^{4U^{\epsilon_i}}}{4(R^{\epsilon_i})^2}((A^{\epsilon_i})_\tau^2 + (A^{\epsilon_i})_\xi^2) + \frac{(R^{\epsilon_i})_{\xi\xi}}{(R^{\epsilon_i})}\right) + ((R^{\epsilon_i})_\tau^2 - (R^{\epsilon_i})_\xi^2)^{-1}R^{\epsilon_i}R_\tau^{\epsilon_i}\left(2U_\tau^{\epsilon_i}U_\xi^{\epsilon_i} + \frac{e^{4U^{\epsilon_i}}}{2(R^{\epsilon_i})^2}A_\tau^{\epsilon_i}A_\xi^{\epsilon_i} + \frac{R_{\xi\tau}^{\epsilon_i}}{R^{\epsilon_i}}\right).$$

We now arrive at one of our key estimates, i.e. a Lipschitz continuity property for solutions to the Einstein equations in terms of their initial data. Note that the small-time restriction below is made for convenience of application of Lemma 6.9. Using the following proposition, we obtain the existence of a solution when $\epsilon \to 0$, thus completing our proof of Theorem 6.1.

Proposition 6.10 (Continuous dependence on initial data). *Provided that* $\tau_1 - \tau_0 \le \pi$, *one has*

$$N^\Delta(\tau_1) \le C N^\Delta(\tau_0),$$

where $C > 0$ *only depends on the constants* C^i.

Proof. We apply Lemma 6.9, first with $w_+ = (\Delta A_+)^2 = (\Delta A_\tau + \Delta A_\xi)^2$ and $w_- = A_-^2 = (A_\tau - A_\xi)^2$, where A stands for any of the components A^{ϵ_i}. Then w_+ and w_- satisfy

$$\partial_\tau w_+ - \partial_\xi w_+ = 2\Delta A_+ \Omega^{\Delta A}, \qquad \partial_\tau w_- + \partial_\xi w_- = 2A_- \Omega^A,$$

with $\Omega^{\Delta A}$ given by (6.9) and Ω^A given by

$$\Omega^A = \frac{R_\tau A_\tau}{R} - \frac{R_\xi A_\xi}{R} + 4(A_\xi U_\xi - A_\tau U_\tau),$$

using (4.18). This leads to

$$\|(\Delta A_+)A_-\|^2_{L^2([\tau_0,\tau]\times S^1)} \le 4 \sum_\pm \int_{\tau_0}^\tau \int_{S^1} |h_\pm|\, d\xi \int_{S^1} |w_\mp|\, d\xi$$

for any $\tau \in [\tau_0, \tau_1]$, with

$$h_+ = 2(\Delta A_+)\Omega^{\Delta A}, \qquad |w_+| \le 2(\Delta A_\tau)^2 + 2(\Delta A_\xi)^2,$$
$$h_- = 2A_-\Omega^A, \qquad |w_-| \le 2A_\tau^2 + 2A_\xi^2.$$

Thus, we have

$$\|(\Delta A_+)A_-\|^2_{L^2([\tau_0,\tau]\times S^1)} \le C N^2 \int_{\tau_0}^\tau \int_{S^1} \Delta A_+ |\Omega^{\Delta A}|\, d\xi\, d\tau' + \int_{\tau_0}^\tau (N^\Delta)^2(\tau') \int_{S^1} A_- |\Omega^A|\, d\xi\, d\tau', \qquad (6.10)$$

where N is the maximum of $N^1(\tau_0)$ and $N^2(\tau_0)$ and where $C > 0$ is a constant.

For the second term on the right-hand side, recall also the estimate

$$\int_{\tau_0}^\tau \int_{S^1} A_-\Omega^A \le CN\big(\|A_-R_+\|_{L^2_{\tau,\xi}} \|A_-\|_{L^2_{\tau,\xi}} + \|R_-A_+\|_{L^2_{\tau,\xi}} \|A_-\|_{L^2_{\tau,\xi}} + \|A_-U_+\|_{L^2_{\tau,\xi}} \|A_-\|_{L^2_{\tau,\xi}} + \|U_-A_+\|_{L^2_{\tau,\xi}} \|A_-\|_{L^2_{\tau,\xi}}\big),$$

where A stands for any of the A^{ϵ_i} and where $\|\cdot\|_{L^2_{\tau,\xi}}$ stands for $\|\cdot\|_{L^2([\tau_0,\tau_1]\times S^2)}$. Together with the a priori estimate in Lemma 6.9, we then obtain

$$\int_{\tau_0}^{\tau_1} \int_{S^1} A_-\Omega^A \le CN^3.$$

For the first term on the right-hand side of (6.10), we note that

$$|\Omega^{\Delta A}| \le \sum_{\pm}\big(\tfrac{1}{2}R_0^{-1}|\Delta R_\pm|\,|A_\mp^{\epsilon_2}| + \tfrac{1}{2}R_0^{-1}|\Delta A_\pm|\,|R_\mp^{\epsilon_1}| + \tfrac{1}{2}|\Delta R^{-1}|\,|A_\pm R_\mp| + 2|\Delta A_\pm|\,|U_\mp^{\epsilon_2}| + 2|A_\pm^{\epsilon_1}|\,|\Delta U_\mp|\big),$$

where $R_0 > 0$ is the minimum of $(R_i^\epsilon)^{-1}$ for $i = 1, 2$ on the initial data. Then we have

$$|\Omega^{\Delta A}| \le CN\sum_{\pm}\big(|\Delta R_\pm|\,|A_\mp^{\epsilon_2}| + |\Delta A_\pm|\,|R_\mp^{\epsilon_1}| + |\Delta R^{-1}|\,|A_\pm R_\mp| + 2|\Delta A_\pm|\,|U_\mp^{\epsilon_2}| + 2|A_\pm^{\epsilon_1}|\,|\Delta U_\mp|\big)$$

for some constant $C > 0$. Thus, using the Cauchy–Schwarz inequality, we find

$$\begin{aligned}
\int_{\tau_0}^{\tau}\int_{S^1} \Delta A_+|\Omega^{\Delta A}|\,d\xi\,d\tau' &\le CN\big(\|\Delta A_+A_-^{\epsilon_2}\|_{L^2_{\tau,\xi}}\|\Delta R_+\|_{L^2_{\tau,\xi}} \\
&+\|\Delta A_+A_-^{\epsilon_2}\|_{L^2_{\tau,\xi}}\|\Delta R^{-1}R_+\|_{L^2_{\tau,\xi}} + \|\Delta A_+U_-^{\epsilon_2}\|_{L^2_{\tau,\xi}}\|\Delta A_+\|_{L^2_{\tau,\xi}} \\
&+\|\Delta A_-U_+^{\epsilon_2}\|_{L^2_{\tau,\xi}}\|\Delta A_+\|_{L^2_{\tau,\xi}} + \|\Delta A_+A_-^{\epsilon_2}\|_{L^2_{\tau,\xi}}\|\Delta U_+\|_{L^2_{\tau,\xi}} + \|\Delta U_-A_+^{\epsilon_2}\|_{L^2_{\tau,\xi}}\|\Delta A_+\|_{L^2_{\tau,\xi}}\big) \\
&+\int_{\tau_0}^{\tau}\big(\|A_+^{\epsilon_2}\Delta R_-\|_{L^2_\xi}\|\Delta A_+\|_{L^2_\xi} + \|\Delta A_+R_-^{\epsilon_2}\|_{L^2_\xi}\|\Delta A_+\|_{L^2_\xi} + \|\Delta A_-R_+^{\epsilon_2}\|_{L^2_\xi}\|\Delta A_+\|_{L^2_\xi} \\
&+ \|\Delta R^{-1}\|_{L^\infty_\xi}\|\Delta A_+R_-^{\epsilon_2}\|_{L^2_\xi}\|A_+\|_{L^2_\xi}\big)\,d\tau'. \qquad (6.11)
\end{aligned}$$

On the right-hand side of the previous inequality, we have two set of terms, those which contain spacetime L^2 norms of null products of ΔA, A, ΔU or U, and those which contains a null product involving always a factor of $\Delta R_\pm$ or $R_\pm$ and which have been estimating using Cauchy–Schwarz in the spatial variable only. For these last terms, we consider each of the products

$$\|A_\pm\Delta R_\mp\|_{L^2_\xi}\|\Delta A_\pm\|_{L^2_\xi}, \qquad \|\Delta A_\mp R_\pm\|_{L^2_\xi}\|\Delta A_\pm\|_{L^2_\xi},$$
$$\|\Delta R^{-1}\|_{L^\infty_\xi}\|\Delta A_+R_-^{\epsilon_2}\|_{L^2_\xi}\|A_+\|_{L^2_\xi},$$

and estimate the $R_\pm$ and $\Delta R_\pm$ terms in the uniform norm

$$\begin{aligned}
\|A_\pm^{\epsilon_2}\Delta R_\mp\|_{L^2(\xi)}(\tau)\|\Delta A_\pm\|_{L^2(\xi)}(\tau) &\le C\|\Delta R_\pm\|_{C^0(\xi)}(\tau)N\|\Delta A_\pm\|_{L^2(\xi)}(\tau) \\
&\le CN(N^\Delta)^2(\tau),
\end{aligned}$$

and similarly

$$\begin{aligned}
\|\Delta A_\mp R_\pm\|_{L^2(\xi)}(\tau)\|\Delta A_\pm\|_{L^2(\xi)}(\tau) &\le CN(N^\Delta)^2(\tau), \\
\|\Delta R^{-1}\|_{C^0(\xi)}(\tau)\|\Delta A_+R_-^{\epsilon_2}\|_{L^2(\xi)}(\tau)\|A_+\|_{L^2(\xi)}(\tau) &\le CN^2(N^\Delta)^2(\tau).
\end{aligned} \qquad (6.12)$$

Similar estimates hold with $+$ replaced by $-$, A by U, and ΔA by ΔU, and so we have

$$\sum_{i,j}\sum_{\pm}\|u_\pm^i\Delta u_\mp^j\|^2_{L^2_{\tau,\xi}} \le C(N^3+N^2)\int_{\tau_0}^{\tau}(N^\Delta)^2(\tau')\,d\tau'. \qquad (6.13)$$

where $(u_1, u_2) = (U, A)$ stands for either $(U^{\epsilon_1}, A^{\epsilon_1})$ or $(U^{\epsilon_2}, A^{\epsilon_2})$ and $\Delta u^j = (\Delta U, \Delta A)$. In view of

$$\frac{d}{d\tau}\int_{S^1}(\Delta A_\tau^2 + \Delta A_\xi^2) = \int_{S^1} 2\Delta A_\tau \Omega^{\Delta A},$$

we have proved that

$$\begin{aligned}&\left(\int_{S^1}(\Delta A_\tau^2 + \Delta U_\tau^2 + \Delta \nu_\tau^2 + \Delta A_\xi + \Delta U_\xi^2 + \Delta \nu_\xi^2)\, d\xi\right)(\tau)\\ &\leq \left(\int_{S^1}(\Delta A_\tau^2 + \Delta U_\tau^2 + \Delta \nu_\tau^2 + \Delta A_\xi^2 + \Delta U_\xi^2 + \Delta \nu_\xi^2)\, d\xi\right)(\tau_0) + C_N \int_{\tau_0}^{\tau}(N^\Delta)^2(\tau')\, d\tau',\end{aligned}$$

where $C_N > 0$ only depends on N.

For R, we proceed as before, by integration along null lines, to check that

$$\begin{aligned}\|\Delta R_\pm\|_{C^0} \leq& C_N \int_{\tau_0}^{\tau}(\|\Delta R^{-1}\|_{L^\infty_\xi} + \|\Delta \nu\|_{L^\infty_\xi})\, d\tau'\\ \leq& C_N \int_{\tau_0}^{\tau}(\|\Delta R^{-1}\|_{L^\infty_\xi} + \|\Delta \nu\|_{W^{1,1}} + \|\Delta \nu_\tau\|_{L^1})\, d\tau'.\end{aligned}$$

Similar estimates for higher derivatives hold in L^1 after following the same strategy as in the previous section. Using

$$\left|\frac{1}{x_1^2 - y_1^2} - \frac{1}{x_2^2 - y_2^2}\right| \leq \frac{|x_1^2 - x_2^2| + |y_1^2 - y_2^2|}{|x_1^2 - y_1^2|\,|x_2^2 - y_2^2|}$$

to estimate the differences for the terms containing $1/((R_\tau^{\epsilon_i})^2 - (R_\xi^{\epsilon_i})^2)$, we also easily obtain the necessary estimates for $\Omega^{\Delta \nu_\xi}$ and $\Omega^{\Delta \nu_\tau}$.

Finally, we trivially have the following estimates for ΔA, ΔU in L^2 (and not derivatives thereof):

$$\frac{d}{d\tau}\|U, A\|_{L^2}^2 \leq (N^\Delta)^2(\tau);$$

similarly, we have estimates on $\Delta \nu$ and ΔR simply from the definition of N^Δ. Thus, putting everything together, we have the following estimate from which the result follows:

$$(N^\Delta)^2(\tau) \leq (N^\Delta)^2(\tau_0) + C_N \int_{\tau_0}^{\tau}(N^\Delta)^2(\tau')\, d\tau'. \qquad \square$$

7. Global geometry of weakly regular T^2-symmetric spacetimes

7.1. Continuation criterion

We are now in a position to complete the proof of Theorem 1.2. Combining Theorem 6.1 and Proposition 5.3, we deduce that, for any weakly regular T^2-symmetric initial data with constant $R = R_0$, there exists a weakly regular T^2-symmetric Lorenztian manifold arising from this data with admissible areal coordinates. Consider one such development and let R_1 denote the final time of existence of this solution. Note that, in conformal coordinates, we have the following lower bound:

$$R_\tau \geq \frac{1}{2}\Big(\inf_{\tau=\tau_0} R_u + \inf_{\tau=\tau_0} R_v\Big), \tag{7.1}$$

where we have used the notation of the previous section. Since the conformal time of existence given by Theorem 6.1 only depends on the initial norm (6.1), it follows that the areal time of existence of the solution is bounded below by a constant depending only on (6.1). Hence, we have the following continuation criterion.

Lemma 7.1 (Continuation criterion). *Let (U, A, η, a) be a solution to the equations (4.22)–(4.26) with the regularity $U, A \in C^0_R(H^1_\theta(S^1)) \cap C^1_R(L^2_\theta(S^1))$, $\eta \in C^0_R(W^{1,1}_\theta(S^1)) \cap C^1_R(L^1_\theta(S^1))$, $a, a^{-1} \in C^0_R(W^{2,1}_\theta(S^1)) \cap C^1_R(W^{1,1}_\theta(S^1))$, and defined on an interval of time $R \in [R_0, R_1)$. Assume that $R_1 < \infty$ and that the norm*

$$\begin{aligned} N := {} & \|U, A\|_{H^1}(R) + \|U_R, A_R\|_{L^2}(R) + \|\eta, a_R, a_\theta\|_{W^{1,1}}(R) \\ & + \|\eta_R, a_{RR}, a_{R\theta}, a_{\theta\theta}\|_{L^1}(R) + \|a, a^{-1}\|_{L^\infty}(R) \end{aligned}$$

is uniformly bounded on the interval $[R_0, R_1)$. Then the solution can be extended beyond R_1 with the same regularity.

As a consequence, we can prove the existence of global solutions in areal coordinates provided we derive uniform estimates on the above norm, as we do in the rest of this section. Moreover, since one can approximate (locally in time, at least) weakly regular solutions by smooth solutions, we consider, in the rest of this section, a smooth solution (U, A, η, a) to (4.22)–(4.26) defined on $[R_0, R^\star)$ for some $R^\star > R_0$. We search for bounds that are uniform on $[R_0, R^\star)$. Constants that depend on the (natural norms of the) initial data only are denoted by C, while those that also depend on $R_\star$ are denoted by $C^\star$.

7.2. Uniform energy estimates in areal coordinates

Both energy-like functionals

$$\mathcal{E}(R) := \int_{S^1} E(R, \theta)\, d\theta, \qquad E := a^{-1}(U_R)^2 + a(U_\theta)^2 + \frac{e^{4U}}{4R^2}(a^{-1}(A_R)^2 + a(A_\theta)^2)$$

and

$$\mathcal{E}_K(R) := \int_{S^1} E_K(R, \theta)\, d\theta, \qquad E_K := E + \frac{K^2}{4R^4} e^{2\eta} a^{-1}$$

are nonincreasing in time, since

$$\begin{aligned} \frac{d}{dR}\mathcal{E}(R) &= -\frac{K^2}{2R^3}\int_{S^1} E e^{2\eta}\, d\theta - \frac{2}{R}\int_{S^1}\Big(a^{-1}(U_R)^2 + \frac{1}{4R^2}e^{4U}a(A_\theta)^2\Big)\, d\theta, \\ \frac{d}{dR}\mathcal{E}_K(R) &= -\frac{K^2}{R^5}\int_{S^1} a^{-1}e^{2\eta}\, d\theta - \frac{2}{R}\int_{S^1}\Big(a^{-1}(U_R)^2 + \frac{e^{4U}}{4R^2}a(A_\theta)^2\Big)\, d\theta. \end{aligned}$$

These functionals yields a uniform control for all times $R \geq R_0$.

Lemma 7.2 (Energy estimates). *We have the energy bounds*

$$\sup_{R\in[R_0,R^\star)} \mathcal{E}(R) \leq \mathcal{E}(R_0), \qquad \sup_{R\in[R_0,R^\star)} \mathcal{E}_K(R) \leq \mathcal{E}_K(R_0),$$

as well as the spacetime bounds

$$\int_{R_0}^{\infty}\int_{S^1} \big(c_1(U_R)^2 a^{-1} + c_2(U_\theta)^2 a + c_3(A_R)^2 a^{-1} + c_4(A_\theta)^2 a\big)\, d\theta\, dR \leq \mathcal{E}(R_0)$$

with

$$c_1 := \frac{2}{R} + \frac{K^2}{2R^3}e^{2\eta}, \qquad c_2 := \frac{K^2}{2R^3}e^{2\eta},$$
$$c_3 := \frac{K^2}{8R^5}e^{4U+2\eta}, \qquad c_4 := \frac{1}{2R^3}e^{4U} + \frac{K^2}{8R^5}e^{4U+2\eta},$$

and

$$\int_{R_0}^{\infty}\int_{S^1} \frac{K^2}{R^5}e^{2\eta}a^{-1}\, d\theta\, dR \leq \mathcal{E}_K(R_0).$$

Moreover, since the function a is bounded above and below on the initial slice $R = R_0$, the initial energy $\mathcal{E}(R_0)$ is comparable with the H^1 norm of the data $\overline{U}, \overline{A}$, that is,

$$C_1\mathcal{E}(R_0) \leq \|(\overline{U}, \underset{0}{U}, \overline{A}, \underset{0}{A})\|_{L^2(S^1)} \leq C_2\mathcal{E}(R_0)$$

for constants $C_1, C_2 > 0$ depending on the *sup norm* of the data at time $R = R_0$ only. To have similar inequalities at arbitrary times R requires a sup-norm bound on the other metric coefficients, which we derive below.

We now derive direct consequences of the energy estimate in Lemma 7.2.

Lemma 7.3 (Upper bound for the function a). *The function a satisfies the upper bound*

$$\sup_{[R_0,R^\star)\times S^1} a \leq \sup_{S^1} \overline{a},$$

as well as

$$\frac{1}{2R}\int_{S^1} |(1/a)_R|\, d\theta \leq \mathcal{E}_K(R_0).$$

Proof. From (4.25) we see that a decreases when R increases, which implies the desired sup-norm bound for a. The other estimate follows immediately from the equations (4.25) and (4.26), since

$$0 \leq -2a_R a^{-1} \leq \frac{K^2}{R^3}e^{2\eta}a^{-1} = 4R(E_K - E) \leq 4RE_K. \qquad \square$$

Lemma 7.4 (Estimates for the function η). *The function η satisfies the integral estimates*

$$\frac{1}{R}\int_{S^1} |\eta_R| a^{-1}\, d\theta \le \mathcal{E}_K(R_0), \qquad \frac{1}{R}\int_{S^1} |\eta_\theta|\, d\theta \le \mathcal{E}(R) \le \mathcal{E}(R_0)$$

and the pointwise estimate

$$|\eta(R,\theta)| \le R\mathcal{E}(R_0) + \left|\int_{S^1} \overline{\eta}\, d\theta'\right| + \left(\sup_{S^1} \overline{a}\right)\frac{R^2 - R_0^2}{2}\mathcal{E}_K(R_0).$$

Proof. We have

$$|\eta_\theta| \le RE, \qquad |\eta_R| a^{-1} \le RE + \frac{a^{-1}}{4R^3} e^{2\eta} K^2 = RE_K.$$

On the other hand, in view of Lemma 7.3, for any $\theta, \theta' \in S^1$ we have

$$|\eta(R,\theta) - \eta(R,\theta')| \le R\mathcal{E}(R).$$

Thus, by integrating in θ', we find

$$\int_{S^1} \eta(R,\theta')\, d\theta' - 2\pi R\mathcal{E}(R) \le 2\pi\eta(R,\theta) \le 2\pi R\mathcal{E}(R) + \int_{S^1} \eta(R,\theta')\, d\theta'.$$

On the other hand, we have

$$\left|\int_{S^1} \eta(R,\theta')\, d\theta'\right| \le \left|\int_{S^1}\int_{R_0}^{R} \eta_R(R,\theta')\, d\theta'\right| + \left|\int_{S^1} \overline{\eta}\, d\theta'\right|,$$

and we can evaluate the second term on the right-hand side above from Lemma 7.3, as follows:

$$\left|\int_{S^1}\int_{R_0}^{R} \eta_R(R,\theta')\, d\theta'\right| \le \left(\sup_{S^1} a(R,\cdot)\right)\frac{R^2 - R_0^2}{2}\mathcal{E}_K(R).$$

The desired conclusion then follows from the energy estimates in Lemma 7.2 and the upper bound on a in Lemma 7.3. □

7.3. Conclusion of the proof of Theorem 1.2

We already know that a is nonincreasing, and so bounded above, but the lower bound is less obvious and is now discussed.

Lemma 7.5 (Lower bound for the function a). *The function a satisfies*

$$a^{-1} \le C^\star.$$

Proof. Using Lemma 7.4, we find $(a^{-2})_R \le CR^{-3}e^{CR^2}$ and by integration,

$$a(R,\theta)^{-2} - \overline{a}(\theta)^{-2} \le \int_{R_0}^{R} C \frac{e^{CR'^2}}{R'^3}\, dR' \le \int_{R_0}^{R} C \frac{2R' e^{CR'^2}}{2R'^4}\, dR' \le C_1(e^{CR^2} - e^C).$$

By estimating $\overline{a}(\theta)^{-2}$, this concludes the proof. □

Lemma 7.6 (Estimates of the functions U, A). *The functions U, A satisfy the integral estimate*

$$\int_{S^1} (U_t^2 + A_t^2 + U_\theta^2 + A_\theta^2)\, d\theta \le C^\star,$$

and the pointwise estimate

$$\sup_{[R_0, R^\star]\times S^1} (|U| + |A|) \le C^\star.$$

Proof. It follows immediately from the energy estimates and the estimates for a and a^{-1} that

$$\int_{S^1} (U_\theta^2 + e^{4U} A_\theta^2)\, d\theta \le C^\star, \qquad \int_{S^1} (U_t^2 + e^{4U} A_t^2)\, d\theta \le C.$$ □

Lemma 7.7 (Additional estimate for the function a). *The mixed derivative of the metric coefficient a is controlled by the energy density*

$$|(\ln a)_{R\theta}| \le \frac{K^2}{2R^2} e^{2\eta} E,$$

and therefore its θ-derivative satisfies the pointwise estimate

$$|a_\theta| \le C^\star.$$

Proof. Taking the θ-derivative of $(\ln a)_R = -e^{2\eta}K^2/(2R^3)$, we obtain

$$|(\ln a)_{R\theta}| = \left| \frac{K^2}{4R^3} e^{2\eta} 2\eta_\theta \right| \le \frac{K^2}{2R^3} e^{2\eta} RE,$$

since $|\eta_\theta| \le RE$. From the identity

$$(a^{-1}e^{2\eta})_R = 2Re^{2\eta}E, \tag{7.2}$$

we obtain

$$|(\ln a)_{R\theta}| \le \frac{K^2}{4R^3} (a^{-1}e^{2\eta})_R.$$

The second statement follows immediately by integration and using Lemmas 7.5 and 7.4. □

Finally, we obtain further control on the metric coefficient a.

Lemma 7.8 (Higher-order estimates on a). *The following uniform estimates hold:*

$$\|a_{R\theta}, a_{RR}, a_{\theta\theta}\|_{L^1}(R) \le C^\star.$$

Proof. For the mixed derivative $a_{R\theta}$, this follows from the pointwise estimate derived in the previous lemma and the energy bounds. For the derivative a_{RR}, this follows from the L^1 uniform estimate on η_R by commuting the evolution equation for a. For $a_{\theta\theta}$, we proceed as follows. Note first that

$$(a^{-2}(e^{2\eta})_R)_R - (e^{2\eta})_{\theta\theta} = 4e^{2\eta}(a^{-2}\eta_R^2 - \eta_\theta^2) + 2e^{2\eta}((a^{-2}\eta_R)_R - \eta_{\theta\theta}).$$

The second term on the right-hand side is known to be uniformly bounded in L^1, using the wave equation for η. For the first term, we note that it involves the product $(a^{-1}\eta_R + \eta_\theta)(a^{-1}\eta_R - \eta_\theta)$. This is a null product which rewritten in terms of U and A and up to uniformly bounded factors is the sum of uniformly bounded functions and the null products $(a^{-1}U_R + aU_\theta)^2(a^{-1}U_R - aU_\theta)^2$, $(a^{-1}A_R + aA_\theta)^2(a^{-1}A_R - aA_\theta)^2$. However, these are bounded in spacetime L^1 as an application of Lemma 6.9. On the other hand,

$$(\ln a)_{R\theta\theta} = \frac{K^2}{2R^3}(-e^{2\eta})_{\theta\theta} = \frac{K^2}{2R^3}\big((a^{-2}(e^{2\eta})_R)_R + F\big),$$

where F is a function uniformly bounded in $L^1([R_0, R^*] \times S^1)$. The result then follows by integration of the previous equation, using an integration by parts and the L^1 estimate on η_R to control the term arising from $(a^{-2}(e^{2\eta})_R)_R$. □

This completes the derivation of global-in-time uniform estimates, and hence the proof of Theorem 1.2. We can now reformulate our existence result in coordinates.

Theorem 7.9 (Global existence in areal coordinates). *For any weakly regular initial data set with constant area $R = R_0 > 0$, the system of partial differential equations describing T^2-symmetric spacetimes in areal coordinates admits a weak solution U, A, ν, a, G, H, satisfying the regularity conditions (3.8), defined on the whole interval $[R_0, \infty)$ and which is unique among the set of functions satisfying (3.8). The solution constructed has the following regularity:*

$$U, A \in C^0_R(H^1_\theta(S^1)) \cap C^1_R(L^2_\theta(S^1)), \qquad \eta \in C^0_R(W^{1,1}_\theta(S^1)) \cap C^1_R(L^1_\theta(S^1)),$$
$$a, a^{-1} \in C^0_R(W^{2,1}_\theta(S^1)) \cap C^1_R(W^{1,1}_\theta(S^1)),$$
$$G, H \in C^0_R(L^\infty(S^1)), \qquad G_R, H_R \in C^0_R(W^{1,1}_\theta(S^1)) \cap C^1_R(L^1_\theta(S^1)).$$

We emphasize that *additional regularity* of the metric is established here, which was not required to express Einstein's field equations in the weak sense, but was deduced from the structure of the Einstein equations under the assumed symmetry.

8. Geometric uniqueness and maximal development

In this section, we discuss the issue of geometric uniqueness and the notion of maximal development associated with a given initial data set. First of all, we introduce the following concept of T^2-symmetric development.

Definition 8.1. Given a weakly regular T^2-symmetric initial data set (Σ, h, K), a **weakly regular T^2-symmetric development** of (Σ, h, K) is a weakly regular T^2-symmetric Lorentzian manifold $(\mathcal{M}, g)$ together with a smooth embedding ϕ of Σ onto one of the hypersurfaces Σ_t (where t is as in Definition 2.8) and such that (h, K) coincides with $(h(t), K(t))$. The development is called **one-sided** if $\phi(\Sigma)$ coincides with the boundary of $\mathcal{M}$.

Recall that in the classical case, if $(\mathcal{M}, g)$ is diffeomorphic to $I_t \times \Sigma$, with Σ a compact manifold without boundary and $t \in I_t$ a time function, each $t = \text{const}$ hypersurface is a Cauchy hypersurface for $(\mathcal{M}, g)$. Hence, the notion of development as introduced above contains a natural replacement for global hyperbolicity. On the other hand, in the above definition, the symmetry is imposed on the development, while in the classical case, it can be propagated from the data. Naturally, if enough regularity is imposed, a one-sided development is a past or a future development. Next, we also introduce the following partial order relation.

Definition 8.2. Given two developments $(\mathcal{M}, g)$ and $(\mathcal{M}', g')$, one says that $(\mathcal{M}, g)$ **is an extension of** $(\mathcal{M}', g')$ if there exists a C^1 isometric embedding of $(\mathcal{M}', g')$ into a proper subset of $(\mathcal{M}, g)$. A **maximal development** is a development admitting no proper extension.

Based on this definiton, we have the following result.

Theorem 8.3 (Uniqueness theory for weakly regular T^2-symmetric developments). *For any weakly regular T^2-symmetric initial data set (Σ, h, K) with constant area of symmetry R_0, there exists a unique (up to C^1-diffeomorphisms) and maximal (for the order relation induced by the notion of extension in Definition 8.2) one-sided weakly regular T^2-symmetric development with $R \geq R_0$, which coincides with the solution constructed in areal coordinates in Theorem 7.9.*

The requirement that R be constant is made for convenience only. The inequality for R is a replacement for the time orientation. Interestingly, we do not need to use the geodesics construction as in [6], since the solution in Theorem 7.9 is necessarily maximal. This follows since T^2-symmetric spacetimes always admit an areal foliation and since the solution of Theorem 7.9 exhausts all values of R.

Proof of Theorem 8.3. Let (Σ, h, K) be a weakly regular initial data set with constant area of symmetry R_0. Let $(\mathcal{M}, g)$ be a one-sided development of (Σ, h, K) with $R \geq R_0$. From Proposition 5.1, the function R enjoys additional regularity and, in particular, is C^1. Hence, we may introduce a new coordinate system, with R as the time coordinate, which is C^1-compatible with the original differential structure of $(\mathcal{M}, g)$, and such that the regularity of all metric functions is preserved, as in Proposition 5.3. Hence, areal coordinates may be introduced on $(\mathcal{M}, g)$ and the reduced Einstein equations (4.22)–(4.28) hold. It then follows from the uniqueness of the solution of the reduced system that $(\mathcal{M}, g)$ can be identified with a subset of the solution obtained from Theorem 7.9. □

Appendix. Compatible connections of weakly regular T^2-symmetric manifolds

We establish here that the standard properties of the Levi-Civita connection of a Riemannian manifold may be extended to our weak regularity framework.

Definition A.1. Let (Σ, h) be a weakly regular T^2-symmetric Riemannian manifold and (X, Y, Z) be an adapted frame. Define the space $C^\infty_{X,Y,Z}$ of (X, Y, Z)**-smooth vector fields** to be the set of all vector fields on Σ whose components in the basis (X, Y, Z) are smooth functions defined on Σ.

Since X, Y are smooth and Z is L^∞, vector fields in $C^\infty_{X,Y,Z}$ have L^∞ components in any smooth frame (such as (X, Y, Θ)). Note also that we can extend the definition of the commutator $[\cdot, \cdot]$ to vector fields in $C^\infty_{X,Y,Z}$ by computing its components in the basis (X, Y, Z). We denote this commutator by $[\cdot, \cdot]_{X,Y,Z}$.

Lemma A.2. *$C^\infty_{X,Y,Z}$ and $[\cdot, \cdot]_{X,Y,Z}$ are independent of the choice of the adapted frame. Moreover, if V, W are in $C^\infty_{X,Y,Z}$, then $h(V, W)$ is a $W^{1,1}(\Sigma)$ function at least.*

Proof. The Killing fields X, Y are uniquely determined up to a linear combination. Moreover, the orthogonal supplement of X, Y is one-dimensional, and so Z is uniquely determined up to multiplication by smooth functions. Consequently, $C^\infty_{X,Y,Z}$ is independent of the choice of the adapted frame. The second claim follows immediately. For the last claim, we compute $h(V, W)$ in the basis (X, Y, Z) and easily check that the term with the weakest regularity is $h_{ZZ} V^Z W^Z$, which is in $W^{1,1}$. □

With some abuse of notation, from now on, we simply write $[\cdot, \cdot]$ instead of $[\cdot, \cdot]_{X,Y,Z}$. We can now introduce the desired concept of connection.

Definition A.3. Let (Σ, h) be a weakly regular T^2-symmetric Riemannian manifold. An L^1 **connection compatible with the** T^2**-symmetry** is a bilinar map ∇, mapping a pair of vector fields $(V, W) \in C^\infty_{X,Y,Z} \times C^\infty_{X,Y,Z}$ to an L^1 vector field $\nabla_V W$ and satisfying, for any smooth function f, $\nabla_{fV} W = f \nabla_V W$ and $\nabla_V fW = f\nabla_V W + V(f)W$. The connection is said to be **torsion free** if moreover $\nabla_V W - \nabla_W V = [V, W]$.

Definition A.4. Let (Σ, h) be a weakly regular T^2-symmetric Riemannian manifold and $\mathcal{A} = (X, Y, Z)$ be an adapted frame. Let Γ^i_{jk} be the $L^1(\Sigma)$ or $L^2(\Sigma)$ functions introduced earlier in Proposition 2.12. Then one may define an L^1 connection compatible with the T^2-symmetry, denoted ${}^{\mathcal{A}}\nabla$, via

$$({}^{\mathcal{A}}\nabla_V W)^i = V^j (W^i_{,j} + \Gamma^i_{jk} W^k),$$

which, precisely, defines the components of ${}^{\mathcal{A}}\nabla_V W$ in the basis (X, Y, Z).

We then have the following existence result.

Proposition A.5. *The operator* ${}^{\mathcal{A}}\nabla$ *is the unique, torsion free,* L^1 *connection compatible with the* T^2*-symmetry satisfying*

$$S(h(V, W)) = h(\nabla_S V, W) + h(V, \nabla_S W) \tag{A.1}$$

for any $(S, V, W) \in (C^{\infty}_{X,Y,Z})^3$*. In particular,* ${}^{\mathcal{A}}\nabla$ *is independent of the choice of the adapted frame.*

Proof. Since $h(V, W)$ belongs to $W^{1,1}$ and S has L^∞ components with respect to any smooth frame, the expression $S(h(V, W)) = S^i(h(V, W))_{,i}$ is well-defined as a sum of L^∞ times L^1 products. Hence, the uniqueness may be established as in the regular case. What remains to be proven is that ${}^{\mathcal{A}}\nabla$ indeed satisfies (A.1). One easily finds that this is equivalent to saying $h_{jk,i} = h_{lj}\Gamma^l_{ik} + h_{lk}\Gamma^l_{ij}$, which can be checked from Proposition 2.12. □

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