J. Eur. Math. Soc. 17, 1293–1326

DOI 10.4171/JEMS/531

Daniela De Silva · Ovidiu Savin

Regularity of Lipschitz free boundaries for the thin one-phase problem

Received July 14, 2012 and in revised form March 19, 2014

Abstract. We study regularity properties of the free boundary for the thin one-phase problem which consists in minimizing the energy functional

$$E(u, \Omega) = \int_\Omega |\nabla u|^2 \, dX + \mathcal{H}^n(\{u > 0\} \cap \{x_{n+1} = 0\}), \qquad \Omega \subset \mathbb{R}^{n+1},$$

among all functions $u \geq 0$ which are fixed on $\partial\Omega$.

Keywords. Energy minimizers, one-phase free boundary problem, monotonicity formula

1. Introduction

In this paper we study minimizers u of the energy functional E associated to the *thin one-phase* problem,

$$E(u, \Omega) := \int_\Omega |\nabla u|^2 \, dX + \mathcal{H}^n(\{(x, 0) \in \Omega : u(x, 0) > 0\}), \tag{1.1}$$

where $\Omega \subset \mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$ and points in $\mathbb{R}^{n+1}$ are denoted by $X = (x, x_{n+1})$.

We are mainly concerned with the regularity of the free boundary of minimizers u, that is, the set

$$F(u) := \partial_{\mathbb{R}^n} \{u(x, 0) > 0\} \cap \Omega \subset \mathbb{R}^n.$$

We also consider viscosity solutions to the thin one-phase problem (see problem (1.2) below) and investigate the regularity of Lipschitz free boundaries.

Throughout this paper we consider only domains Ω and solutions u such that

$$\Omega \text{ is symmetric with respect to } \{x_{n+1} = 0\},$$
$$u \geq 0 \text{ is even with respect to } x_{n+1}.$$

D. De Silva: Department of Mathematics, Barnard College, Columbia University, New York, NY 10027, USA; e-mail: desilva@math.columbia.edu

O. Savin: Department of Mathematics, Columbia University, New York, NY 10027, USA; e-mail: savin@math.columbia.edu

Mathematics Subject Classification (2010): Primary 35B65; Secondary 35Q99

The thin one-phase problem is closely related to the classical Bernoulli free boundary problem (or one-phase problem) where the second term of the energy E is replaced by $\mathcal{H}^{n+1}(\{u > 0\})$. In our setting the set $\{u = 0\}$ occurs on the lower dimensional subspace $\mathbb{R}^n \times \{0\}$ and the free boundary is expected to be $n-1$-dimensional, whereas in the classical case the free boundary is n-dimensional (lying in $\mathbb{R}^{n+1}$). There is a wide literature on the regularity theory for the free boundary in the standard Bernoulli problem, which has similarities to the regularity theory of minimal surfaces; see for example [AC, ACF, C1, C2, C3, CJK, CS, DJ1, DJ2].

The thin one-phase problem was first introduced by Caffarelli, Roquejoffre and Sire [CRS] as a variational problem involving fractional H^s norms. Such problems are relevant in classical physical models in mediums where long range (non-local) interactions are present; see [CRS] for further motivation. For example, if u defined in $\mathbb{R}^{n+1}$ is a local minimizer of E, then its restriction to the n-dimensional space $\mathbb{R}^n \times \{0\}$ minimizes locally an energy of the type

$$c_n \|u\|^2_{H^{1/2}} + \mathcal{H}^n(\{u > 0\}).$$

In [CRS] the authors obtained the optimal regularity for minimizers u, the free boundary condition along $F(u)$ and proved that, in dimension $n = 2$, Lipschitz free boundaries are C^1. The question of the regularity of the free boundary in higher dimensions was left open. In [DR] De Silva and Roquejoffre studied viscosity solutions of the thin one-phase problem associated to the energy E and showed that flat free boundaries are $C^{1,\alpha}$. Motivated by the present paper, the current authors improved this result to $C^{2,\alpha}$ regularity. This estimate and some basic theorems for viscosity solutions were obtained in [DS] and they play a crucial role in the present paper (see Section 2).

The thin two-phase problem, that is, when u is allowed to change sign, was considered by Allen and Petrosyan [AP]. They showed that the positive and negative phases are always separated, so the problem reduces locally back to a one-phase problem. They also obtained a Weiss type monotonicity formula for minimizers and proved that, in dimension $n = 2$, the free boundary is C^1 in a neighborhood of a regular point.

The main difficulty in the thin-one phase problem occurs near the free boundary where all derivatives of u blow up and the problem becomes degenerate. The method developed by Caffarelli [C1, C2] for the $C^{1,\alpha}$ regularity of the free boundary in the standard one-phase problem does not seem to apply in this setting. The question of higher regularity is also delicate.

In this paper we obtain regularity results for Lipschitz free boundaries based on a Weiss type monotonicity formula and on $C^{2,\alpha}$ estimates for flat solutions. The monotonicity formula is used in a standard blow-up analysis near the free boundary and reduces the regularity question to the problem of classifying *global cones*, i.e. global solutions which are homogeneous of degree $1/2$. The $C^{2,\alpha}$ estimate for flat solutions allows us to show that all Lipschitz cones are trivial. This general strategy of obtaining regularity of Lipschitz solutions applies also to the classical one-phase problem and to the minimal surface equation, providing different proofs than the ones of Caffarelli [C1] for the one-phase, and of De Giorgi [DG] for the minimal surface equation.

Our first main result deals with the regularity of the free boundaries for minimizers. We show that $F(u)$ is a $C^{2,\alpha}$ surface except possibly on a small singular set.

Theorem 1.1. *Let u be a minimizer for E. The free boundary $F(u)$ is locally a $C^{2,\alpha}$ surface, except on a singular set $\Sigma_u \subset F(u)$ of Hausdorff dimension $n-3$, i.e.*

$$\mathcal{H}^s(\Sigma_u) = 0 \qquad \text{for } s > n-3.$$

Moreover, $F(u)$ has locally finite $\mathcal{H}^{n-1}$ measure.

As a corollary we obtain that in dimension $n = 2$, free boundaries of minimizers are always $C^{2,\alpha}$.

As mentioned above, we also study the regularity of Lipschitz free boundaries of viscosity solutions to the Euler–Lagrange equation associated to the minimization problem for E, that is, the following thin one-phase free boundary problem:

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \setminus \{(x,0) : u(x,0) = 0\}, \\ \dfrac{\partial u}{\partial U_0} = 1 & \text{on } F(u) := \partial_{\mathbb{R}^n}\{u(x,0) > 0\} \cap \Omega. \end{cases} \tag{1.2}$$

Here the free boundary condition reads

$$\frac{\partial u}{\partial U_0}(x_0) := \lim_{t\to 0^+} \frac{u(x_0 + t\nu(x_0), 0)}{\sqrt{t}}, \qquad x_0 \in F(u), \tag{1.3}$$

with $\nu(x_0)$ the normal to $F(u)$ at x_0 pointing toward $\{x : u(x,0) > 0\}$. We prove the following result (see Section 2 for the definition of viscosity solution).

Theorem 1.2. *Let u be a viscosity solution to (1.2) in B_1 with $0 \in F(u)$ and assume that $F(u)$ is a Lipschitz graph in the e_n direction with Lipschitz constant L. Then $F(u) \cap B_{1/2}$ is a $C^{2,\alpha}$ graph for any $\alpha < 1$ and its $C^{2,\alpha}$ norm is bounded by a constant that depends only on n, L and α.*

The paper is organized as follows. In Section 2 we introduce notation and recall definitions and some necessary results from [DS] about viscosity solutions to (1.2). Section 3 is devoted to minimizers of E. We prove general theorems which were obtained also in [CRS] and [AP], such as existence, optimal regularity, non-degeneracy, and compactness. We also show that minimizers are viscosity solutions to (1.2) (with 1 replaced by an appropriate constant). In Section 4 we prove a Weiss type monotonicity formula for minimizers of E and also for viscosity solutions to (1.2) which have Lipschitz free boundaries. Section 5 deals with minimal cones, that is, minimizers of E that are homogeneous of degree $1/2$. We establish that the only minimal cones in $\mathbb{R}^{2+1}$ are the trivial ones, and from that we deduce our main Theorem 1.1 by a dimension reduction argument. Finally, in the last section we use the flatness theorem and the monotonicity formula to prove Theorem 1.2.

2. Viscosity solutions

In this section we introduce notation and recall definitions and some necessary results from [DR, DS].

2.1. Notation

Throughout the paper, constants which depend only on the dimension n will be called *universal*. In general, small constants will be denoted by c and large constants by C, and they may change from line to line in the body of the proofs. The dependence on parameters other than n will be explicitly indicated.

A point $X \in \mathbb{R}^{n+1}$ will be denoted by $X = (x, x_{n+1}) \in \mathbb{R}^n \times \mathbb{R}$. A ball in $\mathbb{R}^{n+1}$ with radius r and center X is denoted by $B_r(X)$ and for simplicity $B_r = B_r(0)$. We use $B_r^+(X)$ to denote the upper ball

$$B_r^+(X) := B_r(X) \cap \{x_{n+1} > 0\}.$$

Also, we write

$$\mathcal{B}_r(X) = B_r(X) \cap \{x_{n+1} = 0\}.$$

Let $v \in C(\Omega)$ be a non-negative function on a bounded domain $\Omega \subset \mathbb{R}^{n+1}$. We associate to v the following sets:

$$\Omega^+(v) := \Omega \setminus \{(x, 0) : v(x, 0) = 0\} \subset \mathbb{R}^{n+1},$$
$$F(v) := \partial_{\mathbb{R}^n}\{v(x, 0) > 0\} \cap \Omega \subset \mathbb{R}^n.$$

Often subsets of $\mathbb{R}^n$ are embedded in $\mathbb{R}^{n+1}$, as will be clear from the context.

2.2. Definition and properties of viscosity solutions

We consider the *thin one-phase free boundary problem* ($u \geq 0$)

$$\begin{cases} \Delta u = 0 & \text{in } \Omega^+(u), \\ \dfrac{\partial u}{\partial U_0} = 1 & \text{on } F(u), \end{cases} \tag{2.1}$$

where

$$\frac{\partial u}{\partial U_0}(x_0) := \lim_{t \to 0^+} \frac{u(x_0 + t\nu(x_0), 0)}{\sqrt{t}}, \qquad X_0 = (x_0, 0) \in F(u). \tag{2.2}$$

Here $\nu(x_0)$ denotes the unit normal to $F(u)$, the *free boundary* of u, at x_0 pointing toward $\{u(x, 0) > 0\}$.

Our notation for the free boundary condition is justified by the following fact. If $F(u)$ is C^2 then any function $u \geq 0$ which is harmonic in $\Omega^+(u)$ has an asymptotic expansion at a point $X_0 = (x_0, 0) \in F(u)$,

$$u(X) = \alpha(x_0) U_0((x - x_0) \cdot \nu(x_0), x_{n+1}) + o(|X - X_0|^{1/2}),$$

where $U_0(t, s)$ is the real part of $\sqrt{z}$. Thus in the polar coordinates

$$t = r\cos\theta, \quad s = r\sin\theta, \quad r \geq 0, \quad -\pi \leq \theta \leq \pi,$$

U_0 is given by

$$U_0(t, s) = r^{1/2}\cos(\theta/2). \tag{2.3}$$

Then the limit in (2.2) represents the coefficient $\alpha(x_0)$ in the expansion above,

$$\frac{\partial u}{\partial U_0}(x_0) = \alpha(x_0),$$

and our free boundary condition requires that $\alpha \equiv 1$ on $F(u)$.

The precise result proved in [DS, Lemma 7.5] is stated below and will be often used in this paper.

Lemma 2.1 (Expansion at regular points from one side). *Let $w \in C^{1/2}(B_1)$ be 1/2-Hölder continuous, $w \geq 0$, with w harmonic in $B_1^+(w)$. If*

$$0 \in F(w), \quad \mathcal{B}_{1/2}\left(\tfrac{1}{2}e_n\right) \subset \{w(x,0) > 0\},$$

then

$$w = \alpha U_0 + o(|X|^{1/2}) \quad \textit{for some } \alpha > 0.$$

The same conclusion holds for some $\alpha \geq 0$ if

$$\mathcal{B}_{1/2}\left(-\tfrac{1}{2}e_n\right) \subset \{w = 0\}.$$

We now recall the notion of viscosity solutions to (2.1), introduced in [DR].

Definition 2.2. Given g, v continuous, we say that v *touches g from below* (resp. *above*) at X_0 if $g(X_0) = v(X_0)$, and

$$g(X) \geq v(X) \quad (\text{resp. } g(X) \leq v(X)) \quad \text{in a neighborhood } O \text{ of } X_0.$$

If this inequality is strict in $O \setminus \{X_0\}$, we say that v *touches g strictly from below* (resp. *above*).

Definition 2.3. We say that $v \in C(\Omega)$ is a (*strict*) *comparison subsolution* to (2.1) if v is a non-negative function in Ω which is even with respect to x_{n+1} and

(i) v is C^2 and $\Delta v \geq 0$ in $\Omega^+(v)$;
(ii) $F(v)$ is C^2 and if $x_0 \in F(v)$ we have

$$v(x_0 + t\nu(x_0), 0) = \alpha(x_0)\sqrt{t} + o(\sqrt{t}) \quad \text{as } t \to 0^+,$$

with $\alpha(x_0) > 1$, where $\nu(x_0)$ is the unit normal at x_0 to $F(v)$ pointing toward $\{v(x,0) > 0\}$.

Similarly one can define a (strict) comparison supersolution.

Definition 2.4. We say that u is a *viscosity solution* to (2.1) if u is a continuous non-negative function in Ω which is even with respect to x_{n+1} and

(i) $\Delta u = 0$ in $\Omega^+(u)$;
(ii) no (strict) comparison subsolution (resp. supersolution) touches u from below (resp. from above) at a point $X_0 = (x_0, 0) \in F(u)$.

In [DS] we proved optimal regularity for viscosity solutions. Precisely, we have the following lemma.

Lemma 2.5 ($C^{1/2}$-Optimal regularity). *Assume u solves (2.1) in B_2 and $0 \in F(u)$. Then*

$$u(X) \leq C\,\mathrm{dist}(X, F(u))^{1/2} \qquad X \in \mathcal{B}_1.$$

Moreover,

$$\|u\|_{C^{1/2}(B_1)} \leq C(1 + u(e_{n+1})).$$

The main result in [DS] (see Theorem 1.1 there) is the following flatness theorem, which improves the previous $C^{1,\alpha}$ result obtained in [DR].

Theorem 2.6. *There exists $\bar{\epsilon} > 0$ small, depending only on n, such that if u is a viscosity solution to (2.1) in B_1 satisfying*

$$\{x \in \mathcal{B}_1 : x_n \leq -\bar{\epsilon}\} \subset \{x \in \mathcal{B}_1 : u(x, 0) = 0\} \subset \{x \in \mathcal{B}_1 : x_n \leq \bar{\epsilon}\},$$

then $F(u) \cap \mathcal{B}_{1/2}$ is a $C^{2,\alpha}$ graph for every $\alpha \in (0, 1)$ with $C^{2,\alpha}$ norm bounded by a constant depending on α and n.

We now recall the definition of a special family of functions $V_{\mathcal{S},a,b}$ introduced in [DS] which approximate solutions quadratically.

For any $a, b \in \mathbb{R}$ we define the following family of (two-dimensional) functions (given in polar coordinates (ρ, β)):

$$v_{a,b}(t, s) := \left(1 + \frac{a}{4}\rho + \frac{b}{2}t\right)\rho^{1/2} \cos\frac{\beta}{2}, \tag{2.4}$$

that is,

$$v_{a,b}(t, s) = \left(1 + \frac{a}{4}\rho + \frac{b}{2}t\right)U_0(t, s) = U_0(t, s) + o(\rho^{1/2}),$$

with U_0 defined in (2.3).

Given a surface $\mathcal{S} = \{x_n = h(x')\} \subset \mathbb{R}^n$, we denote by $\mathcal{P}_{\mathcal{S},X}$ the 2D plane passing through $X = (x, x_{n+1})$ and perpendicular to $\mathcal{S}$, that is, the plane containing X and generated by the x_{n+1} direction and the normal direction from $(x, 0)$ to $\mathcal{S}$.

We define the family of functions

$$V_{\mathcal{S},a,b}(X) := v_{a,b}(t, x_{n+1}), \qquad X = (x, x_{n+1}), \tag{2.5}$$

with $t = \rho\cos\beta$, $x_{n+1} = \rho\sin\beta$ respectively the first and second coordinate of X in the plane $\mathcal{P}_{\mathcal{S},X}$. In other words, t is the signed distance from x to $\mathcal{S}$ (positive above $\mathcal{S}$ in the x_n direction).

If

$$\mathcal{S} := \left\{x_n = \tfrac{1}{2}(x')^T M x'\right\}$$

for some $M \in S^{(n-1)\times(n-1)}$, we use the notation

$$V_{M,a,b}(X) := V_{\mathcal{S},a,b}(X).$$

We define the following class of functions:

$$\mathcal{V}^0_\Lambda := \{V_{M,a,b} : a + b - \operatorname{tr} M = 0,\ \|M\|, |a|, |b| \le \Lambda\}.$$

Notice that if we rescale $V = V_{M,a,b}$, that is, if we set

$$V_\lambda(X) = \lambda^{-1/2} V(\lambda X),$$

then it easily follows from our definition that

$$V_\lambda = V_{\lambda M,\lambda a,\lambda b}.$$

Moreover, it can be checked from the definition (see also [DS, Proposition 3.3]) that if $V \in \mathcal{V}^0_\Lambda$ then

$$|\Delta V(X)| \le C\Lambda^2 \quad \text{in } B_{1/2}(e_n). \tag{2.6}$$

In the course of the proof of the flatness theorem 2.6 we also showed that a solution u can be approximated in a $C^{2,\alpha}$ fashion near $0 \in F(u)$ by functions $V \in \mathcal{V}^0_\Lambda$. The precise statement can be formulated as follows [DS, Theorem 5.2]).

Theorem 2.7. *Assume $0 \in F(u)$ and $F(u)$ is a C^1 surface in a neighborhood of 0 with normal e_n pointing towards the positive side. Then, for any $\alpha \in (0, 1)$,*

$$V(X - \Lambda r^{2+\alpha} e_n) \le u(X) \le V(X + \Lambda r^{2+\alpha} e_n) \quad \text{in } B_r, \ \text{for all } r \text{ small,}$$

for some $V = V_{M,a,b} \in \mathcal{V}^0_\Lambda$, with Λ depending on u, n and α.

As a consequence of the theorem above we obtain the lemma below, which together with the monotonicity formula (Theorem 4.3) are the main ingredients to prove Theorem 1.2 (see Proposition 6.4). It is in this lemma that the $C^{2,\alpha}$ regularity of flat free boundaries is needed. For all the other arguments in this paper, $C^{1,\alpha}$ regularity is sufficient.

Lemma 2.8. *Assume $F(u)$ is C^1 in a neighborhood of $X_0 = (x_0, 0) \in F(u)$ and let $\nu \in \mathbb{R}^n \times \{0\}$ denote the unit normal vector at x_0 pointing towards $\{u > 0\}$. Then, for all $\alpha \in (0, 1)$, for all r small, and for some K depending on u, α, n,*

$$|\partial_\tau u(X_0 + r\nu)| \le K r^{1/2+\alpha}$$

where $\tau \in \mathbb{R}^n \times \{0\}$ is any unit tangent vector to $F(u)$ at X_0, that is, $\tau \cdot \nu = 0$.

Proof. Assume for simplicity that $X_0 = 0$, $\nu = e_n$. Then, by Theorem 2.7, we may assume that

$$V(X - \Lambda r^{2+\alpha} e_n) \le u(X) \le V(X + \Lambda r^{2+\alpha} e_n)$$

with $V = V_{M,a,b} \in \mathcal{V}^0_\Lambda$. The rescalings

$$u_r(X) = r^{-1/2} u(rX), \qquad V_r(X) = r^{-1/2} V(rX) = V_{rM,ra,rb}(X) \in \mathcal{V}^0_{\Lambda r}$$

satisfy

$$V_r(X - \Lambda r^{1+\alpha} e_n) \le u_r(X) \le V_r(X + \Lambda r^{1+\alpha} e_n).$$

In $B_{1/2}(e_n)$ we have

$$|u_r - V_r| \le \Lambda r^{1+\alpha}\partial_n(V_r) \le C(\Lambda)r^{1+\alpha}$$

and (see (2.6))

$$|\Delta(u_r - V_r)| \le |\Delta V_r| \le C(\Lambda)r^2.$$

Thus,

$$|\nabla u_r(e_n) - \nabla V_r(e_n)| \le C(\Lambda)r^{1+\alpha}.$$

Since $\nabla V_r(e_n) \in \mathrm{span}\{e_n, e_{n+1}\}$ and $\tau \cdot \nabla V_r(e_n) = 0$ if $\tau \in \mathbb{R}^n \times \{0\}$ and $\tau \perp e_n$, we infer from the previous inequality that

$$|\tau \cdot \nabla u_r(e_n)| \le K r^{1+\alpha}, \quad \text{that is,} \quad |\tau \cdot \nabla u(re_n)| \le K r^{1/2+\alpha}. \qquad \square$$

The next remark will be used in the proof of the monotonicity formula for viscosity solutions.

Remark 2.9. Using the $C^{1,\alpha}$ estimates of [DR], we can approximate u by U_0 (instead of V) in a $C^{1,\alpha}$ fashion and write in the proof above

$$U_0(X - \Lambda' r^{1+\alpha} e_n) \le u(X) \le U_0(X + \Lambda' r^{1+\alpha} e_n).$$

This leads to the conclusion

$$|\nabla u(X) - \nabla U_0(X)| \le K'|X|^{\alpha-1/2}$$

for all X in the two-dimensional plane generated by e_n and e_{n+1}.

We conclude this section by recalling the following compactness result [DS, Proposition 7.8].

Proposition 2.10 (Compactness). *Assume u_k solve (2.1) and converge uniformly to u_* in B_1, and $\{u_k = 0\}$ converges in the Hausdorff distance to $\{u_* = 0\}$. Then u_* solves (2.1) as well.*

3. Preliminaries on minimizers

In this section we prove general theorems about minimizers of the energy function E, defined by

$$E(u, \Omega) = \int_\Omega |\nabla u|^2 \, dX + \mathcal{H}^n(\{u > 0\} \cap \{x_{n+1} = 0\}). \tag{3.1}$$

Most of the results in this section are contained in [CRS] and [AP], such as existence, optimal regularity, non-degeneracy, and compactness. For completeness, we sketch their proofs. We also show that minimizers are viscosity solutions to problem (1.2) (with 1 replaced by an appropriate constant).

Definition 3.1. We say that u is a (local) *minimizer* for E in $\Omega \subset \mathbb{R}^{n+1}$ if $u \in H^1_{\mathrm{loc}}(\Omega)$ and for any domain $D \subset\subset \Omega$ and every function $v \in H^1_{\mathrm{loc}}(\Omega)$ which coincides with u in a neighborhood of $\Omega \setminus D$ we have

$$E(u, D) \le E(v, D).$$

Existence of minimizers with a given boundary data on $\partial\Omega$ follows easily from the lower semicontinuity of the energy E.

We remark that this minimization problem is invariant under the scaling

$$u_\lambda(X) = \lambda^{-1/2} u(\lambda X), \tag{3.2}$$

that is, u is a minimizer if and only if u_λ is a minimizer.

As already remarked in the introduction, throughout this paper we consider only domains Ω and minimizers u such that

$$\begin{aligned} &\Omega \text{ is symmetric with respect to } \{x_{n+1} = 0\},\\ &u \ge 0 \quad \text{ is even with respect to } x_{n+1}. \end{aligned}$$

We recall the notation

$$\mathcal{B}_r = B_r \cap \{x_{n+1} = 0\},$$

which will be often used in this section, and for any function $v \ge 0$ we denote

$$\mathcal{B}_r^+(v) := \{v > 0\} \cap \mathcal{B}_r.$$

Lemma 3.2. *If $u \ge 0$ is a minimizer for E in B_1 then u is subharmonic in B_1 and harmonic in B_1^+.*

Proof. Indeed, if $\varphi \ge 0$ is in $C_0^\infty(B_1)$ then

$$\mathcal{H}^n(\mathcal{B}_1^+(u)) \ge \mathcal{H}^n(\mathcal{B}_1^+(u - \epsilon\varphi)).$$

Thus the minimality of u,

$$E(u, B_1) \le E(u - \epsilon\varphi, B_1),$$

implies

$$\int |\nabla u|^2 \, dX \le \int |\nabla (u - \epsilon\varphi)|^2 \, dX \quad \text{and hence} \quad \int \nabla u \nabla \varphi \, dX \le 0,$$

that is, u is subharmonic in B_1. Similarly, taking $\varphi \in C_0^\infty(B_1^+)$ we show that u is harmonic in B_1^+. □

In view of Lemma 3.2 we can define u pointwise as

$$u(X) = \lim_{r \to 0} \fint_{B_r(X)} u \, dY.$$

Optimal regularity and non-degeneracy of a minimizer will follow from the next result.

Lemma 3.3. *Assume that u minimizes E in B_2. If $u(0) \geq C > 0$ with C universal then $B_1 \subset \{u > 0\}$, and u is harmonic in B_1.*

Before the proof we recall the following Sobolev trace inequality. If $\phi \in H^1(\mathbb{R}^{n+1})$ then

$$\int_{\mathbb{R}^{n+1}} |\nabla \phi|^2 \, dX \geq c(n) \left(\int_{\mathbb{R}^n \times \{0\}} \phi^{2(1+\delta)} \, dx \right)^{1/(1+\delta)}, \qquad \delta = \frac{1}{n-1}. \tag{3.3}$$

Proof of Lemma 3.3. Denote

$$a(r) = \mathcal{H}^n(\{u = 0\} \cap \mathcal{B}_r), \qquad 1 \leq r \leq 2.$$

Let v be the harmonic replacement of u in B_r. By minimality,

$$\int_{B_r} |\nabla u|^2 \, dX \leq \int_{B_r} |\nabla v|^2 \, dX + a(r). \tag{3.4}$$

We have

$$\int_{B_r} |\nabla u|^2 \, dX = \int_{B_r} \left(|\nabla v|^2 + 2\nabla v \cdot \nabla(u - v) + |\nabla(u - v)|^2 \right) dX,$$

and hence since v is harmonic and equals u on ∂B_r,

$$\int_{B_r} |\nabla u|^2 \, dX = \int_{B_r} (|\nabla v|^2 + |\nabla(u - v)|^2) \, dX.$$

Thus, by the Sobolev inequality (3.3) and (3.4), the inequality above gives

$$\begin{aligned} a(r) \geq \int_{B_r} |\nabla(u - v)|^2 \, dX &\geq c \left(\int_{\mathbb{R}^n \times \{0\}} (v - u)^{2(1+\delta)} \, dx \right)^{1/(1+\delta)} \\ &\geq c \left(\int_{\{u=0\} \cap \mathcal{B}_r} v^{2(1+\delta)} \, dx \right)^{1/(1+\delta)}. \end{aligned} \tag{3.5}$$

Since $v \geq 0$ is harmonic in B_r we have

$$v(X) \geq c v(0) r^{-1} \operatorname{dist}(X, \partial B_r).$$

Thus, since $v(0) \geq u(0)$ and $1 \leq r \leq 2$, in the set $\{u = 0\} \cap \mathcal{B}_{r-2^{-k}}$ we have

$$v \geq c\, 2^{-k} u(0).$$

Hence from (3.5) we get

$$a(r) \geq c\, 2^{-2k} u(0)^2 a(r - 2^{-k})^{1/(1+\delta)}.$$

We denote

$$a_k := a(1 + 2^{-k+1}),$$

thus

$$a_{k+1} \le C\, 2^{4k} u(0)^{-2(1+\delta)} a_k^{1+\delta}, \quad a_1 \le C. \tag{3.6}$$

By De Giorgi iteration, if $u(0) \ge C$ is sufficiently large then $a_k \to 0$ as $k \to \infty$. Thus $a(1) = 0$ and in view of (3.4) we conclude that u is harmonic in B_1. □

By the scaling (3.2), Lemma 3.3 shows that if u is a minimizer in $B_{2r}(X_0)$ with $X_0 \in \{x_{n+1} = 0\}$ and $u(X_0) \ge Cr^{1/2}$ then $B_r(X_0) \subset \{u > 0\}$. Thus we immediately obtain the following corollary.

Corollary 3.4. *Assume u is a minimizer in B_2. Then u is continuous in B_2 and thus harmonic in $B_2^+(u)$. Moreover, if $F(u) \cap \mathcal{B}_1 \neq \emptyset$, then*

$$u(x,0) \le C \operatorname{dist}(x, F(u))^{1/2}, \quad \forall x \in \mathcal{B}_1, \tag{3.7}$$

with C universal.

We now easily obtain $C^{1/2}$-optimal regularity of minimizers.

Corollary 3.5 (Optimal regularity). *Let u be a minimizer in B_2. Then*

$$\|u\|_{C^{1/2}(B_1)} \le C(1 + u(e_{n+1})), \tag{3.8}$$

with C universal.

Proof. Assume that $F(u) \cap \mathcal{B}_1 \neq \emptyset$; otherwise the statement is trivial. We write $u = v + w$ with v, w harmonic in $B_{3/2}^+$ and

$$v = 0 \quad \text{on } \{x_{n+1} = 0\}, \qquad v = u \quad \text{on } \partial B_{3/2}^+ \cap \{x_{n+1} > 0\},$$
$$w = u \quad \text{on } \{x_{n+1} = 0\}, \qquad w = 0 \quad \text{on } \partial B_{3/2}^+ \cap \{x_{n+1} > 0\}.$$

Then

$$\|v\|_{C^{1/2}(B_1^+)} \le C v(e_{n+1}) \le C u(e_{n+1}),$$

and by Corollary 3.4,

$$\|w\|_{C^{1/2}(B_1^+)} \le \|u\|_{C^{1/2}(\mathcal{B}_{3/2})} \le C.$$ □

Remark 3.6. The optimal regularity gives $u(X) \le K \operatorname{dist}(X, \{u = 0\})^{1/2}$ for some $K > 0$. From this and the fact that u is harmonic in $\{u > 0\}$, we deduce that $|\nabla u(X)| \le C(K) \operatorname{dist}(X, \{u = 0\})^{-1/2}$. This implies that u^2 is a Lipschitz function.

We now prove non-degeneracy of a minimizer.

Lemma 3.7 (Non-degeneracy). *Assume u is a minimizer and $B_1 \subset \{u > 0\}$. Then*

$$u(0) \ge c > 0$$

with c universal.

Proof. Let $\varphi \in C_0^\infty(B_{1/2})$ with $\varphi \equiv 1$ in $B_{1/4}$. Since u is harmonic in B_1,

$$\|u\|_{L^\infty(B_{1/2})}, \|\nabla u\|_{L^\infty(B_{1/2})} \leq Cu(0),$$

and we obtain

$$\int_{B_1} |\nabla u|^2 \, dX \geq \int_{B_1} |\nabla(u(1-\varphi))|^2 \, dX - Cu(0)^2.$$

Also,

$$\mathcal{H}^n(\mathcal{B}_1^+(u)) \geq \mathcal{H}^n(\mathcal{B}_1^+(u(1-\varphi))) + c_0.$$

In conclusion, by the minimality of u we have $0 \geq -Cu(0)^2 + c_0$, that is, $u(0) \geq c$. □

Again by the scaling (3.2), the lemma above implies that if u is a minimizer in B_2 then

$$u(X_0) \geq C \operatorname{dist}(X_0, \{u=0\})^{1/2}, \quad \forall X_0 \in \mathcal{B}_1.$$

In the next lemma, we prove that minimizers satisfy a slightly different type of non-degeneracy which will be used to prove density estimates for the zero phase.

Lemma 3.8. *Assume $v \geq 0$ is defined in B_1, harmonic in $B_1^+(v)$. Assume that there is a small constant $\eta > 0$ such that*

$$\|v\|_{C^{1/2}(B_1)} \leq \eta^{-1}, \tag{3.9}$$

and v satisfies the non-degeneracy condition on $\mathcal{B}_1$,

$$v(X) \geq \eta d(X)^{1/2}, \quad X \in \mathcal{B}_1, \ d(X) = \operatorname{dist}(X, \{v=0\}).$$

Then whenever $0 \in F(v)$, we have

$$\max_{\mathcal{B}_r} v \geq c(\eta) r^{1/2}, \quad \forall r \leq 1.$$

Proof. The proof follows the lines of one in [C3, Lemma 7] (see also [CRS]). Given a point $X_0 \in \mathcal{B}_1^+(v)$ (to be chosen close to 0) we construct a sequence of points $X_k \in \mathcal{B}_1$ such that

$$v(X_{k+1}) = (1+\delta)v(X_k), \quad |X_{k+1} - X_k| \leq C(\eta) d(X_k),$$

with δ small depending on η.

Then using the fact that $d(X_k) \sim v(X_k)^2$ and that $v(X_k)$ grows geometrically we find

$$\begin{aligned} |X_{k+1} - X_0| &\leq \sum_{i=0}^{k} |X_{i+1} - X_i| \leq C \sum_{i=0}^{k} d(X_i) \\ &\leq C \sum_{i=0}^{k} v(X_i)^2 \leq C v(X_{k+1})^2 \sim d(X_{k+1}). \end{aligned}$$

Hence for a sequence of r_k's of size $v(X_k)^2$ we find that

$$\sup_{\mathcal{B}_{r_k}(X_0)} v \geq c r_k^{1/2},$$

from which we obtain

$$\sup_{\mathcal{B}_r(X_0)} v \geq c r^{1/2} \quad \text{for all } r \geq |X_0|.$$

The conclusion follows by letting X_0 go to 0.

We now show that the sequence of X_k's exists. Assume we have constructed X_k. After scaling we may suppose that

$$v(X_k) = 1.$$

We let Y_k be the point where the distance from X_k to $\{v = 0\}$ is achieved. By the assumptions on v ($C^{1/2}$ bound and non-degeneracy),

$$c(\eta) \leq d(X_k) = |X_k - Y_k| \leq C(\eta).$$

Assume for contradiction that we cannot find X_{k+1} in $\mathcal{B}_M(X_k)$ with M large to be specified later, with

$$v(X_{k+1}) \geq 1 + \delta.$$

Then

$$v \leq 1 + \delta + w,$$

with w harmonic in $B_M^+(X_k)$,

$$w = 0 \quad \text{on } \{x_{n+1} = 0\}, \qquad w = v \quad \text{on } \partial B_M(X_k) \cap \{x_{n+1} > 0\}.$$

We have

$$w \leq C(n) \frac{x_n}{M} \sup_{B_M^+(X_k)} v \leq C \eta^{-1} x_n M^{-1/2} \leq \delta \quad \text{in } B := B_{d(X_k)}(X_k),$$

if M is chosen large depending on δ. Thus,

$$v \leq 1 + 2\delta \quad \text{in } B. \tag{3.10}$$

On the other hand, $v(Y_k) = 0$ and $Y_k \in \partial B$. Thus from the Hölder continuity of v we find

$$v \leq 1/2 \quad \text{in } B_{c(\eta)}(Y_k). \tag{3.11}$$

If δ is sufficiently small, (3.10)–(3.11) contradict the equality $1 = v(X_k) = ⨍_B v$. □

Next we prove a density estimate for the zero phase of minimizers.

Corollary 3.9. *If u is a minimizer in B_2 and $0 \in F(u)$ then*

$$\sup_{\mathcal{B}_r} u \geq \mu r^{1/2} \tag{3.12}$$

and

$$1 - \mu \geq \frac{\mathcal{H}^n(\{u = 0\} \cap \mathcal{B}_r)}{\mathcal{H}^n(\mathcal{B}_r)} \geq \mu,$$

where μ depends on n and $u(e_{n+1})$.

Proof. By scaling it suffices to prove the corollary only for $r = 1$. The first statement is contained in Lemma 3.8, in view of the optimal regularity and non-degeneracy of minimizers. This easily implies the left inequality in the density estimate. We now prove the other inequality.

From (3.12), for some $X_0 \in \mathcal{B}_{1/8}$ we have $u(X_0) \geq \mu/2$. Then from the proof of Lemma 3.2 with $u(X_0)$ replacing $u(0)$ we see that if

$$\mathcal{H}^n(\{u = 0\} \cap \mathcal{B}_{1/2}(X_0)) \leq \mathcal{H}^n(\{u = 0\} \cap \mathcal{B}_1) \leq \delta$$

for δ sufficiently small depending on μ, then by the De Giorgi iteration argument (see (3.6))

$$B_{1/4}(X_0) \subset \{u > 0\}.$$

This contradicts $0 \in F(u) \cap \mathcal{B}_{1/4}(X_0)$. □

From the density estimate we immediately obtain the following corollary.

Corollary 3.10. *Let u be a minimizer. Then $\mathcal{H}^n(F(u)) = 0$.*

Remark 3.11. We remark that if $u \in C^{1/2}(B_1) \cap H^1(B_1)$, u is harmonic in $B_1^+(u)$ and $\mathcal{H}^n(F(u)) = 0$ then u satisfies the following integration by parts identity:

$$\int_{B_1} |\nabla u|^2 \, dX = \int_{\partial B_1} u u_\nu \, d\sigma.$$

To justify this equality we notice that since u is harmonic in B_1^+,

$$\int_{B_1} |\nabla u|^2 \, dX = \lim_{\epsilon \to 0} \int_{B_1 \setminus \{|x_{n+1}| \leq \epsilon\}} |\nabla u|^2 \, dX = \int_{\partial B_1} u u_\nu \, d\sigma + \lim_{\epsilon \to 0} \int_{|x_{n+1}| = \epsilon} u u_\nu.$$

However,

$$\lim_{\epsilon \to 0} \int_{|x_{n+1}| = \epsilon} u u_\nu = 0,$$

since $u|\nabla u| \leq K$ (see Remark 3.6), $\mathcal{H}^n(F(u)) = 0$ and

$$\lim_{\epsilon \to 0} u u_\nu(x, \epsilon) = 0 \quad \text{if } x \notin F(u).$$

We now prove a compactness result for minimizers.

Theorem 3.12. *Assume u_k are minimizers of E in Ω and $u_k \to u$ uniformly locally. Then u is a minimizer of E, $\chi_{\{u_k>0\}} \to \chi_{\{u>0\}}$ locally in L^1 and $F(u_k) \to F(u)$ locally in the Hausdorff distance.*

Proof. Assume for simplicity $\Omega = B_2$. Since the $u_k(e_{n+1})$ are uniformly bounded, the u_k are uniformly non-degenerate and $C^{1/2}$ in B_1 in view of Corollary 3.5.

First we show that $F(u_k) \to F(u)$ locally in the Hausdorff distance. If $X_0 \in \mathcal{B}_1$ and $\mathcal{B}_\epsilon(X_0) \subset \{u > 0\}$ then by the uniform convergence of the u_k, $\mathcal{B}_{\epsilon/2}(X_0) \subset \{u_k > 0\}$ for all large k.

If $\mathcal{B}_\epsilon(X_0) \subset \{u = 0\}$ then $\mathcal{B}_{\epsilon/2}(X_0) \subset \{u_k = 0\}$. Otherwise, by Lemma 3.7,

$$F(u_k) \cap \mathcal{B}_{\epsilon/2}(X_0) \neq \emptyset.$$

Select $Y_k \in F(u_k) \cap \mathcal{B}_{\epsilon/2}(X_0)$. Then by the non-degeneracy of the u_k,

$$\sup_{\mathcal{B}_\epsilon(X_0)} u_k \geq \sup_{\mathcal{B}_{\epsilon/2}(Y_k)} u_k \geq \mu\epsilon^{1/2},$$

which contradicts the uniform convergence of u_k to u.

In particular

$$\chi_{\{u_k>0\}}(x) \to \chi_{\{u>0\}}(x) \quad \text{for all } x \notin F(u). \tag{3.13}$$

On the other hand, it follows from non-degeneracy that if B_ϵ does not intersect $F(u_k)$ for a subsequence, then B_ϵ lies outside of $F(u)$. Thus $F(u_k) \to F(u)$ locally in the Hausdorff distance.

Next, we show that $\mathcal{H}^n(F(u)) = 0$, hence the convergence in (3.13) holds $\mathcal{H}^n$-a.e.

Indeed, assume $X_0 \in F(u) \cap \mathcal{B}_1$. Then we can find $Y_k \in F(u_k)$ such that $Y_k \to X_0$. From Corollary 3.9 applied to the u_k on balls centered at the Y_k and the uniform convergence of the u_k we deduce that the limit u satisfies the same estimates in the conclusion of Corollary 3.9.

We now prove that u is a minimizer for E. First we notice that $u_k \to u$ in $H^1(B_1)$. Indeed, since $u_k \to u$ uniformly, we have $\nabla u_k \rightharpoonup \nabla u$ weakly in $H^1(B_1)$ and by Remark 3.11 and Lebesgue's dominated convergence theorem,

$$\int_{B_1} |\nabla u_k|^2 \to \int_{B_1} |\nabla u|^2.$$

Let $v \in H^1(B_1)$ with $v = u$ outside $B_{1-\delta}$, and let φ be a cut-off function with $\varphi = 1$ in $B_{1-\delta}$ and $\varphi = 0$ outside $B_{1-\delta/2}$. Define

$$v_k = \varphi v + (1 - \varphi)u_k;$$

then, by the minimality of the u_k,

$$E(v_k, B_1) \geq E(u_k, B_1).$$

We let $k \to \infty$ in this inequality and use that

$$v_k \to v \quad \text{in } H^1, \qquad \chi_{\{v_k>0\}} \to \chi_{\{v>0\}} \quad \mathcal{H}^n\text{-a.e.}$$

to obtain the desired inequality $E(v, B_1) \geq E(u, B_1)$. □

Next, we want to prove that minimizers are viscosity solutions. For this purpose we need the following proposition, which we will also use later in our dimension reduction argument in Section 5.

Proposition 3.13. *Assume u is constant in the e_1 direction, i.e.*

$$u(x_1, x_2, \dots, x_{n+1}) = v(x_2, \dots x_{n+1}).$$

Then u is a minimizer in $\mathbb{R}^{n+1}$ if and only if v is a minimizer in $\mathbb{R}^n$.

Proof. Assume u is a minimizer in $\mathbb{R}^{n+1}$ and let $w(x_2, \dots, x_{n+1})$ be a function which coincides with v outside $B_K \subset \mathbb{R}^n$. Then define

$$\tilde{u} := \varphi(x_1) w(x_2, \dots, x_{n+1}) + (1 - \varphi(x_1)) v(x_2, \dots, x_{n+1}),$$

with

$$\varphi(x_1) = \begin{cases} 1 & \text{if } |x_1| \leq R - 1, \\ 0 & \text{if } |x_1| \geq R. \end{cases}$$

Then $\tilde{u}$ coincides with u outside of $\Omega := [-R, R] \times B_K$. Hence, $E(u, \Omega) \leq E(\tilde{u}, \Omega)$, which implies

$$2RE(v, B_K) \leq 2(R - 1)E(w, B_K) + M$$

with M depending on w and v but not on R. We let $R \to \infty$ to obtain

$$E(v, B_K) \leq E(w, B_K).$$

Vice versa, assume that v is a minimizer in $\mathbb{R}^n$. Then if $w = u$ outside of Ω with Ω as above,

$$E(w, \Omega) \geq \int_{-R}^{R} E(w(x_1, \cdot), B_K)\, dx_1.$$

Since v is a minimizer, we have

$$E(w, \Omega) \geq \int_{-R}^{R} E(v(x_2, \dots, x_{n+1}), B_K)\, dx_1 = E(u, \Omega). \qquad \square$$

Proposition 3.14. *If u is a minimizer for E then u is a viscosity solution to*

$$\begin{cases} \Delta u = 0 & \text{in } \{u > 0\}, \\ \dfrac{\partial u}{\partial U_0} = \sqrt{2/\pi} & \text{on } F(u). \end{cases}$$

Proof. The fact that u is harmonic in the set where it is positive is already proved in Corollary 3.4. We need to verify the free boundary condition. Assume that we touch $F(u)$ at 0 with $B_\delta(\delta e_n)$ from the positive side (or the zero side). Then by Lemma 2.1, u has an expansion

$$u(X) = \alpha U_0(x_n, x_{n+1}) + o(|X|^{1/2}),$$

with $\alpha > 0$ in view of the non-degeneracy (3.12) (see (2.3) for the definition of U_0). It suffices to prove that

$$\alpha = \sqrt{2/\pi}.$$

The rescaled solutions $\lambda^{-1/2}u(\lambda X)$ converge uniformly to αU_0, so by Theorem 3.12 and Proposition 3.13, αU_0 is a minimizer in $\mathbb{R}^2$. The following computations are two-dimensional. We perturb U_0 as

$$V(X) = U_0(X - \epsilon\varphi(X)e_1), \quad \varphi \in C_0^\infty(B_2), \quad \varphi \equiv 1 \quad \text{in } B_{3/2}.$$

Then,

$$\begin{aligned}\int_{B_1} |\nabla V|^2 - \int_{B_1} |\nabla U_0|^2 &= \int_{B_1(-\epsilon e_1)} |\nabla U_0|^2 - \int_{B_1} |\nabla U_0|^2 \\ &= -\epsilon \int_{\partial B_1} |\nabla U_0|^2 \nu \cdot e_1 + O(\epsilon^2) = O(\epsilon^2)\end{aligned}$$

because $|\nabla U_0|$ is constant on ∂B_1. Since $V = U_0 - \epsilon\varphi(U_0)_1 + O(\epsilon^2)$, where $(U_0)_\tau$ denotes the derivative of U_0 in the τ direction, we have

$$\begin{aligned}\int_{B_2\setminus B_1} |\nabla V|^2 - \int_{B_2\setminus B_1} |\nabla U_0|^2 &= \int_{B_2\setminus B_1} \left(2\nabla U_0 \cdot \nabla(V - U_0) + 2|\nabla(V - U_0)|^2\right) dX \\ = 2\epsilon \int_{\partial B_1} (U_0)_\nu (U_0)_1 + O(\epsilon^2) &= \frac{\epsilon}{2} \int_{\partial B_1} \left(\cos\frac{\theta}{2}\right)^2 + O(\epsilon^2) = \epsilon\frac{\pi}{2} + O(\epsilon^2).\end{aligned}$$

In the equality above we have used the equality (see formula (2.3))

$$(U_0)_1 = (U_0)_\nu = \tfrac{1}{2} r^{-1/2} \cos(\theta/2).$$

Finally, since

$$\mathcal{H}^1(\{V > 0\} \cap B_2) - \mathcal{H}^1(\{U_0 > 0\} \cap B_2) = -\epsilon$$

we obtain

$$E(\alpha V, B_2) - E(\alpha U_0, B_2) = \epsilon(\alpha^2\pi/2 - 1) + O(\epsilon^2),$$

from which we conclude that $\alpha^2\pi/2 - 1 = 0$, that is, $\alpha = \sqrt{2/\pi}$, as desired. □

4. Monotonicity formula

In this section we prove a Weiss type monotonicity formula (see [W]) for minimizers of the energy functional E and also for viscosity solutions to the thin one-phase problem (2.1) which have Lipschitz free boundaries. In the case of minimizers this result is also contained in [AP].

Theorem 4.1 (Monotonicity formula for minimizers). *If u is a minimizer of E in B_R, then the function*

$$\Phi_u(r) := r^{-n} E(u, B_r) - \frac{1}{2} r^{-n-1} \int_{\partial B_r} u^2 \, d\sigma, \qquad 0 < r \le R,$$

is increasing in r. Moreover Φ_u is constant if and only if u is homogeneous of degree $1/2$.

Before the proof, we remark that the rescaling $u_\lambda(X) := \lambda^{-1/2} u(\lambda X)$ satisfies

$$\Phi_{u_\lambda}(r) = \Phi_u(\lambda r). \tag{4.1}$$

Proof. For a.e. r we have

$$\frac{d}{dr}\left(\int_{B_r} |\nabla u|^2 \, dX\right) = \int_{\partial B_r} |\nabla u|^2 \, d\sigma, \tag{4.2}$$

$$\frac{d}{dr}(\mathcal{H}^n(\{u > 0\} \cap \mathcal{B}_r)) = \mathcal{H}^{n-1}(\{u > 0\} \cap \partial \mathcal{B}_r), \tag{4.3}$$

$$\frac{d}{dr}\left(r^{-n-1} \int_{\partial B_r} u^2 \, d\sigma\right) = r^{-n-2} \int_{\partial B_r} (2r u u_\nu - u^2) \, d\sigma, \tag{4.4}$$

where in (4.4) we have used that u^2 is a Lipschitz function (see Remark 3.6). Recall that B_r denotes the $(n+1)$-dimensional ball.

Assume that the equalities above are satisfied at $r = 1$. Define

$$v_\epsilon(X) = \begin{cases} (1-\epsilon)^{1/2} u\left(\frac{X}{1-\epsilon}\right) & \text{if } |X| \le 1-\epsilon, \\ |X|^{1/2} u\left(\frac{X}{|X|}\right) & \text{if } 1-\epsilon < |X| \le 1. \end{cases}$$

We have

$$\begin{aligned} E(v_\epsilon, B_1) = & \int_{B_{1-\epsilon}} (1-\epsilon)^{-1} |\nabla u((1-\epsilon)^{-1} X)|^2 \, dX + (1-\epsilon)^n (\mathcal{H}^n(\{u > 0\} \cap \mathcal{B}_1)) \\ & + \epsilon \int_{\partial B_1} \left(\frac{1}{4} u^2 + u_\tau^2\right) d\sigma + \epsilon \mathcal{H}^{n-1}(\{u > 0\} \cap \partial \mathcal{B}_1) + o(\epsilon), \end{aligned}$$

with the sum of the first two terms on the right hand side equaling $(1-\epsilon)^n E(u, B_1)$. In the equality above, u_τ denotes the tangential gradient of u on ∂B_1. Also,

$$\begin{aligned} E(u, B_1) = & \int_{B_{1-\epsilon}} |\nabla u|^2 \, dX + \mathcal{H}^n(\{u > 0\} \cap \mathcal{B}_{1-\epsilon}) \\ & + \epsilon \left(\int_{\partial B_1} |\nabla u|^2 \, d\sigma + \mathcal{H}^{n-1}(\{u > 0\} \cap \partial \mathcal{B}_1)\right) + o(\epsilon), \end{aligned}$$

with $|\nabla u|^2 = u_\nu^2 + u_\tau^2$. The inequality $E(u, B_1) \le E(v_\epsilon, B_1)$ then implies

$$o(\epsilon) + \epsilon \int_{\partial B_1} \left(u_\nu^2 - \frac{1}{4} u^2\right) d\sigma + E(u, B_{1-\epsilon}) \le (1-\epsilon)^n E(u, B_1).$$

Hence, dividing by $(1-\epsilon)^n$ and letting $\epsilon \to 0$ we obtain

$$\frac{d}{dr}(r^{-n}E(u, B_r))\Big|_{r=1} \geq \int_{\partial B_1}\left(u_\nu^2 - \frac{1}{4}u^2\right)d\sigma.$$

Using (4.4), this shows that

$$\frac{d}{dr}\Phi_u(r)\Big|_{r=1} \geq \int_{\partial B_1}\left(u_\nu - \frac{1}{2}u\right)^2 d\sigma \geq 0.$$

Thus,

$$\frac{d}{dr}\Phi_u(r) \geq 0 \quad \text{for a.e. } r,$$

and the conclusion follows since Φ_u is absolutely continuous in r.

From the above we see that Φ_u is constant if and only if

$$u_\nu = \frac{1}{2|X|}u \quad \text{a.e.},$$

which implies that u is homogeneous of degree $1/2$. □

Remark 4.2. We have used the minimality only up to first order in ϵ, which shows that the formula remains valid for critical points of E. Indeed, we only need to require that u is critical for E under domain variations (see [AP, W]).

Next we show that the monotonicity formula is also valid for viscosity solutions with Lipschitz free boundary. The proof is technical since we need to justify a certain integration by parts.

Theorem 4.3 (Monotonicity formula for viscosity solutions). *Let u be a viscosity solution to*

$$\begin{cases} \Delta u = 0 & \text{in } B_R^+(u), \\ \dfrac{\partial u}{\partial U_0} = \sqrt{2/\pi} & \text{on } F(u), \end{cases}$$

with $F(u)$ a Lipschitz graph. Then the function

$$\Phi_u(r) := r^{-n}E(u, B_r) - \frac{1}{2}r^{-n-1}\int_{\partial B_r} u^2\, d\sigma, \qquad 0 < r \leq R,$$

is increasing in r. Moreover Φ_u is constant if and only if u is homogeneous of degree $1/2$.

Proof. First we remark that since $\{u = 0\}$ is a Caccioppoli set in $\mathbb{R}^n$,

$$\mathcal{H}^{n-1}(F(u) \cap \partial\mathcal{B}_r) = 0 \quad \text{for a.e. } r.$$

We assume that $r = 1$ is a regular value for Φ_u in the sense of (4.2)–(4.4) and also that the equality above holds for it, i.e. $\mathcal{H}^{n-1}(F(u) \cap \partial\mathcal{B}_1) = 0$. We compute

$$\Phi_u'(1) = \int_{\partial B_1} |\nabla u|^2 \, d\sigma + \mathcal{H}^{n-1}(\{u > 0\} \cap \partial\mathcal{B}_1) - n \int_{B_1} |\nabla u|^2 \, dX \\ - n\mathcal{H}^n(\{u > 0\} \cap \mathcal{B}_1) + \int_{\partial B_1} \left(-u u_\nu + \frac{1}{2}u^2\right) d\sigma.$$

Next we want to prove that

$$(n-1) \int_{B_1} |\nabla u|^2 \, dX = \int_{\partial B_1} (|\nabla u|^2 - 2u_\nu^2) \, d\sigma \\ - n\mathcal{H}^n(\{u > 0\} \cap \mathcal{B}_1) + \mathcal{H}^{n-1}(\{u > 0\} \cap \partial\mathcal{B}_1). \tag{4.5}$$

Using this identity together with the identity (see Remark 3.11)

$$\int_{B_1} |\nabla u|^2 \, dX = \int_{\partial B_1} u u_\nu \, d\sigma, \tag{4.6}$$

in the formula above for $\Phi_u'(1)$, we obtain

$$\Phi_u'(1) = 2 \int_{\partial B_1} \left(u_\nu - \frac{1}{2}u\right)^2 d\sigma \geq 0.$$

Analogously for a.e. r we get

$$\Phi_u'(r) = 2 \int_{\partial B_r} \left(u_\nu - \frac{1}{2}u\right)^2 d\sigma \geq 0,$$

from which our conclusion follows.

Let $\Gamma := F(u)$. To prove (4.5), we need to show that

$$(n-1) \int_{B_1} |\nabla u|^2 \, dX = \int_{\partial B_1} (|\nabla u|^2 - 2u_\nu^2) \, d\sigma + \int_{\Gamma \cap B_1} y \cdot \nu_\Gamma \, d\mathcal{H}^{n-1}, \tag{4.7}$$

with ν_Γ the normal to Γ in $\mathbb{R}^n$ pointing toward the positive phase. Then, by the divergence theorem,

$$\int_{\Gamma \cap B_1} y \cdot \nu_\Gamma \, d\mathcal{H}^{n-1} = -n\mathcal{H}^n(\{u > 0\} \cap \mathcal{B}_1) + \mathcal{H}^{n-1}(\{u > 0\} \cap \partial\mathcal{B}_1).$$

This combined with (4.7) gives us (4.5).

To prove (4.7), let us denote

$$T_\epsilon := \{X \in \mathbb{R}^{n+1} : \operatorname{dist}(X, \Gamma) \leq \epsilon\}, \qquad \Omega_\epsilon := B_1^+(u) \setminus T_\epsilon.$$

Notice that Ω_ϵ is a Caccioppoli set and u is a smooth function outside $T_\epsilon \cup \{u = 0\}$. Thus we can use integration by parts. Precisely,

$$\int_{\Omega_\epsilon} \nabla u \cdot \nabla(\nabla u \cdot X)\, dX = \int_{\partial^*\Omega_\epsilon} u_\nu \nabla u \cdot X\, d\sigma, \tag{4.8}$$

where $\partial^*\Omega_\epsilon$ denotes the reduced boundary of Ω_ϵ and ν denotes the exterior normal to $\partial^*\Omega_\epsilon$.

On the other hand, again using integration by parts we get

$$\begin{aligned}\int_{\Omega_\epsilon} \nabla u \cdot \nabla(\nabla u \cdot X)\, dX &= \int_{\Omega_\epsilon} (u_i u_{ij} x_j + u_i^2)\, dX \\ &= \int_{\Omega_\epsilon} \left(-\frac{n+1}{2}|\nabla u|^2 + |\nabla u|^2\right) dX + \int_{\partial^*\Omega_\epsilon} \frac{1}{2}|\nabla u|^2 X \cdot \nu\, d\sigma. \end{aligned}\tag{4.9}$$

From (4.8)–(4.9) we find that

$$(n-1)\int_{\Omega_\epsilon} |\nabla u|^2\, dX = \int_{\partial^*\Omega_\epsilon} (|\nabla u|^2 X \cdot \nu - 2u_\nu \nabla u \cdot X)\, d\sigma. \tag{4.10}$$

We need to show that (4.7) follows from the equality above by letting $\epsilon \to 0$. We remark that since $u(X) \le C\, \mathrm{dist}(X, F(u))^{1/2}$ (see Lemma 2.5) we have

$$|\nabla u|^2 \le C\epsilon^{-1} \qquad \text{on } \partial T_\epsilon,$$

and since Γ is Lipschitz,

$$\mathcal{H}^n(\partial T_\epsilon \cap B_r(X_0)) \le C r^{n-1}\epsilon, \qquad X_0 \in \Gamma.$$

Combining these two inequalities we obtain

$$\left|\int_{\partial T_\epsilon \cap B_r} (|\nabla u|^2 X \cdot \nu - 2u_\nu \nabla u \cdot X)\, d\sigma\right| \le C r^{n-1}. \tag{4.11}$$

Next we claim that if Γ is a $C^{2,\alpha}$ surface in a neighborhood of $X_0 \in \Gamma$ then for r small (depending on the $C^{2,\alpha}$ norm) we have

$$\lim_{\epsilon\to 0} \int_{\partial T_\epsilon \cap B_r(X_0)} (|\nabla u|^2 X \cdot \nu - 2u_\nu \nabla u \cdot X)\, d\sigma = \int_{\Gamma \cap B_r(X_0)} y \cdot \nu_\Gamma\, d\mathcal{H}^{n-1} \tag{4.12}$$

with ν the interior normal to ∂T_ϵ and ν_Γ the normal to Γ in $\mathbb{R}^n$ pointing toward the positive phase. To obtain (4.12) we parametrize T_ϵ by the map

$$(y, \theta) \mapsto X = y + \epsilon(\nu_\Gamma \cos\theta + e_{n+1} \sin\theta), \qquad (y, \theta) \in \Gamma \times [-\pi, \pi].$$

Then, on ∂T_ϵ,

$$\begin{aligned} d\sigma &= (1 + O(\epsilon))\epsilon\, dy\, d\theta, \\ X &= y + O(\epsilon), \\ \nabla u(X) &= \sqrt{2/\pi}\, (\nu_\Gamma (U_0)_1 + e_{n+1}(U_0)_2) + o(\epsilon^{-1/2}), \end{aligned}$$

where in the last equality (which follows from Remark 2.9) the derivatives of U_0 are evaluated at $\epsilon\omega$ with $\omega := (\cos\theta, \sin\theta)$.

Using these identities, for a fixed $y \in \Gamma$ we compute

$$\epsilon \int_{-\pi}^{\pi} (|\nabla u|^2 X \cdot \nu - 2u_\nu \nabla u \cdot X)\, d\theta = \epsilon \int_{-\pi}^{\pi} (|\nabla u|^2 y \cdot \nu - 2u_\nu \nabla u \cdot y)\, d\theta + O(\epsilon)$$
$$= \epsilon \frac{2}{\pi} \int_{-\pi}^{\pi} \left(|\nabla U_0|^2 \cos\theta\, y \cdot \nu_\Gamma + 2(U_0)_\omega (U_0)_1 y \cdot \nu_\Gamma\right) d\theta + O(\epsilon) = y \cdot \nu_\Gamma + O(\epsilon)$$

where again the derivatives of U_0 are evaluated at $\epsilon\omega$, and in the last equality we have used the equality (see the proof of Proposition 3.14)

$$\int_{-\pi}^{\pi} \left(|\nabla U_0|^2 \cos\theta + 2(U_0)_\omega (U_0)_1\right) d\theta = \epsilon^{-1}\pi/2.$$

In conclusion,

$$\epsilon \int_{-\pi}^{\pi} (|\nabla u|^2 X \cdot \nu - 2u_\nu \nabla u \cdot X)\, d\theta = y \cdot \nu_\Gamma + O(\epsilon),$$

and integrating this identity over Γ we obtain (4.12).

From our flatness theorem 2.6 we know that Γ is $C^{2,\alpha}$ except on a closed set Σ of $\mathcal{H}^{n-1}$ measure zero and also recall that $\mathcal{H}^{n-1}(\Gamma \cap \partial\mathcal{B}_1) = 0$. We use a standard covering argument for $\Sigma \cup (\Gamma \cap \partial\mathcal{B}_1)$ with balls of small radius on which we apply the inequality (4.11). On the remaining part of Γ we use (4.12) and obtain the desired conclusion

$$(n-1) \int_{B_1} |\nabla u|^2\, dX = \int_{\partial B_1} (|\nabla u|^2 - 2u_\nu^2)\, d\sigma + \int_{\Gamma \cap B_1} y \cdot \nu_\Gamma\, d\mathcal{H}^{n-1}$$

by passing to the limit as $\epsilon \to 0$ in (4.10). □

Remark 4.4. If u_k are minimizers which converge uniformly to u on compact sets, then it follows from the proof of the compactness theorem 3.12 that $\Phi_{u_k}(r) \to \Phi_u(r)$. This is also true if the u_k are viscosity solutions with Lipschitz free boundaries with uniform Lipschitz bound.

Remark 4.5. If u satisfies the assumptions of either Theorem 4.1 or Theorem 4.3 then $\Phi_u(r)$ is bounded below as $r \to 0$. Indeed, by scaling we only need to check that $\Phi_u(1)$ is bounded, which follows from the formula (see Remark 3.11)

$$\Phi_u(1) = \int_{\partial B_1} \left(u u_\nu - \frac{1}{2}u^2\right) d\sigma + \mathcal{H}^n(\{u > 0\} \cap \mathcal{B}_1).$$

This means that

$$\Phi_u(0^+) = \lim_{r \to 0^+} \Phi_u(r) = \lim_{r \to 0^+} r^{-n} \mathcal{H}^n(\{u > 0\} \cap \mathcal{B}_r) \quad \text{exists},$$

and any blow-up sequence u_λ converges uniformly on compact sets (up to a subsequence) to a solution U homogeneous of degree $1/2$ (see (4.1)).

Definition 4.6. A minimizer U of E which is homogeneous of degree 1/2 is called a *minimal cone*. Analogously a viscosity solution to (2.1) which is homogeneous of degree 1/2 and has Lipschitz free boundary is called a *Lipschitz viscosity cone.*

Let U be a (minimal or viscosity) cone. We denote by Φ_U its energy (which is a constant for all r)

$$\Phi_U = \mathcal{H}^n(\{U > 0\} \cap \mathcal{B}_1) \in (0, \omega_n), \tag{4.13}$$

where ω_n denotes the volume of the n-dimensional unit ball.

We say that a cone U is *trivial* if it coincides (up to a rotation) with the cone $U_0(X) = U_0(x_n, x_{n+1})$ (defined in (2.3)), and therefore its free boundary is a hyperplane. The energy of the trivial cone is $\omega_n/2$.

5. Minimal cones

This section is devoted to the study of minimal cones. First we prove an "energy gap" result in the spirit of the analogue for minimal surfaces. We then show that in dimension $n = 2$ the only minimal cone is the trivial cone U_0 (see (2.3)). Finally, by a standard dimension reduction argument we prove our main Theorem 1.1.

Lemma 5.1. *Minimal cones are uniformly* $C^{1/2}$.

Proof. Let U be a minimal cone. From the proof of the $C^{1/2}$ bound (see Corollary 3.5) we obtain

$$\frac{|U(X) - U(Y)|}{|X - Y|^{1/2}} \leq C(1 + U(e_{n+1})|X - Y|^{1/2}), \qquad X, Y \in B_1,$$

with C universal. Writing this estimate for the rescaling

$$U_R(\tilde{X}) = R^{-1/2} U(R\tilde{X}), \qquad X = R\tilde{X}, \tilde{X} \in B_2,$$

we obtain

$$\frac{|U(X) - U(Y)|}{|X - Y|^{1/2}} \leq C\left(1 + \frac{1}{R} U(Re_{n+1})|X - Y|^{1/2}\right).$$

Since U is homogeneous of degree $1/2$,

$$\frac{1}{R} U(Re_{n+1}) \to 0 \quad \text{as } R \to \infty,$$

and we obtain the desired bound. □

Definition 5.2. Given a minimizer u for E in $\Omega \subset \mathbb{R}^{n+1}$, we say that a point $X \in F(u)$ is a *regular point* if there exists a blow-up sequence of u centered at X which converges to the trivial cone. The points of $F(u)$ which are not regular will be called *singular*, and the set of all singular points of $F(u)$ is denoted by Σ_u.

We notice that in view of our flatness theorem 2.6, $F(u)$ is a $C^{2,\alpha}$ surface in a neighborhood of any regular point, and moreover Σ_u is a closed set in Ω.

Proposition 5.3 (Energy gap). *Let U be a non-trivial minimal cone. Then there exists a universal $\delta > 0$ such that*

$$\Phi_U \geq \omega_n/2 + \delta.$$

Proof. First we show that $\Phi_U > \omega_n/2$. Assume for contradiction that this does not hold and let $X_0 \in F(U)$ be a point where we can touch $F(U)$ with a ball completely contained in $\{U > 0\}$. Set

$$\Phi_U(r, X_0) = \Phi_{\bar{U}}(r), \qquad \bar{U}(X) = U(X - X_0).$$

Then by (4.1) and the fact that U is a cone we obtain

$$\Phi_U(r, X_0) = \Phi_{U_r}(1, X_0/r) = \Phi_U(1, X_0/r).$$

Thus,

$$\lim_{r\to\infty} \Phi_U(r, X_0) = \Phi_U \leq \omega_n/2.$$

On the other hand, from the expansion of U near X_0 (see Theorem 2.1) the blow-up energy is

$$\lim_{r\to 0} \Phi_U(r, X_0) = \omega_n/2.$$

By the monotonicity of $\Phi_U(r, X_0)$ we obtain

$$\Phi_U(r, X_0) \equiv \omega_n/2,$$

and hence U is a cone with respect to X_0, thus U is the trivial cone, a contradiction.

Now we prove the existence of δ by compactness. If no such δ exists then we can find a sequence of cones U_k with $\Phi_{U_k} \to \omega_n/2$. By Lemma 5.1 we may assume that $U_k \to U_*$ uniformly on compact sets. Thus $\Phi_{U_*} = \omega_n/2$ and hence U_* is the trivial cone in view of the preceding argument. By the flatness theorem 2.6 and the compactness theorem 3.12, $F(U_k)$ are smooth in B_1 for all large k, a contradiction. □

Lemma 5.4. *Assume U is a minimal cone in $\mathbb{R}^{n+1}$ and $X_0 = e_1 \in F(U)$. Then any blow-up sequence*

$$V_\lambda(X) = \lambda^{-1/2} U(X_0 + \lambda X)$$

has a subsequence V_{λ_k} with $\lambda_k \to 0$ which converges uniformly on compact sets to $v(x_2, \ldots, x_{n+1})$ with V a minimal cone in $\mathbb{R}^{(n-1)+1}$. Moreover if X_0 is a singular point for $F(U)$, then V is a non-trivial cone.

Proof. In view of Remark 4.5 and Proposition 3.13, we only need to show that V is constant in the e_1 direction.

From the fact that U is homogeneous of degree 1/2 and from the formula for V_λ we get

$$\begin{aligned} V_\lambda(X) &= \lambda^{-1/2}(1 + t\lambda)^{-1/2} U((1 + t\lambda)(X_0 + \lambda X)) \\ &= (1 + t\lambda)^{-1/2} V_\lambda(tX_0 + (1 + t\lambda)X). \end{aligned}$$

Letting $\lambda = \lambda_k \to 0$ we obtain

$$V(X) = V(tX_0 + X) \qquad \text{for all } t.$$

Thus, V is constant in the $X_0 = e_1$ direction.

The final statement follows from the flatness theorem 2.6. □

Assume that U is a non-trivial minimal cone in $\mathbb{R}^{n+1}$ for some dimension n. Then by Lemma 5.4, if $F(U)$ has a singular point different from the origin, then there exists a non-trivial minimal cone in $\mathbb{R}^{(n-1)+1}$. By repeating this dimension reduction argument, we can assume that there is a dimension $k \le n$ and a non-trivial cone in $\mathbb{R}^{k+1}$ which is regular at all points except 0.

Clearly, all minimal cones in dimension $n = 1$ are trivial. In the next theorem we show that there are no non-trivial minimal cones in $\mathbb{R}^{2+1}$.

Theorem 5.5. *If $n = 2$, all minimal cones are trivial.*

Proof. We follow the strategy in [SV], where the authors proved that non-local minimal cones (defined in [CRSa]) are trivial in $\mathbb{R}^2$.

Let U be a minimal cone. By the discussion above, $\Sigma_U = 0$. Define

$$\psi_R(t) := \begin{cases} 1, & 0 \le t \le R, \\ 2 - \dfrac{\log t}{\log R}, & R < t \le R^2, \\ 0, & t \ge R^2. \end{cases}$$

Then ψ_R is a Lipschitz continuous function with compact support in $\mathbb{R}$. Notice that

$$\psi_R'(t) = \begin{cases} 0, & t \in (0, R) \cup (R^2, \infty), \\ \dfrac{-1}{t \log R}, & t \in (R, R^2). \end{cases}$$

We define a bi-Lipschitz change of coordinates:

$$Y := X + \psi_R(|X|)e_1$$

and let

$$U_R^+(Y) = U(X).$$

Next we estimate $E(U_R^+, B_{R^2})$ in terms of $E(U, B_{R^2})$. We have

$$D_X Y = I + A$$

with

$$A(X) = \psi_R'(|X|) \begin{pmatrix} \frac{x_1}{|X|} & \frac{x_2}{|X|} & \cdots & \frac{x_{n+1}}{|X|} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \quad \text{and} \quad \|A\| \le |\psi_R'(X)| \ll 1.$$

Notice that

$$D_Y X = (I + A)^{-1} = I - \frac{1}{1 + \operatorname{tr} A} A.$$

We have

$$\nabla_Y U_R^+ = \nabla_X U \; D_Y X, \qquad dY = (1 + \operatorname{tr} A)\, dX,$$

thus

$$|\nabla U_R^+|^2 \, dY = \nabla U \left(I(1 + \operatorname{tr} A) - (A + A^T) + \frac{1}{1 + \operatorname{tr} A} A A^T \right) (\nabla U)^T \, dX,$$

and

$$\mathcal{H}^n(\{U_R^+ > 0\} \cap \mathcal{B}_{R^2}) = \int_{\{U>0\} \cap \mathcal{B}_{R^2}} (1 + \operatorname{tr} A)\, dx.$$

Writing the same equalities for U_R^- which is defined just like U_R^+ but with ψ_R replaced $-\psi_R$, and thus A by $-A$, we obtain

$$E(U_R^+, B_{R^2}) + E(U_R^-, B_{R^2}) \leq 2E(U, B_{R^2}) + C \int_{B_{R^2}} |\nabla U|^2 \|A\|^2 \, dX$$

with

$$\int_{B_{R^2}} |\nabla U|^2 \|A\|^2 \, dX = \int_R^{R^2} \left(\int_{\partial B_r} |\nabla U|^2 \|A\|^2 \, d\sigma \right) dr$$
$$\leq \int_R^{R^2} C r^2 r^{-1} \left(\frac{r^{-1}}{\log R} \right)^2 dr \leq \frac{C}{\log R} \to 0 \quad \text{as } R \to \infty.$$

The inequality above is the crucial step where we have used the assumption $n = 2$. In conclusion, since $E(U_R^\pm, B_{R^2}) \geq E(U, B_{R^2})$ we get

$$E(U_R^+, B_{R^2}) \leq E(U, B_{R^2}) + \delta(R)$$

with $\delta(R) \to 0$ as $R \to \infty$. Now the proof continues as in [SV]. We sketch it for completeness. Since

$$E(\underline{w}, B_{R^2}) + E(\overline{w}, B_{R^2}) = E(U, B_{R^2}) + E(U_R^+, B_{R^2}),$$

with

$$\underline{w} := \min\{U, U_R^+\}, \qquad \overline{w} = \max\{U, U_R^+\},$$

the inequality above shows that

$$E(\underline{w}, B_{R^2}) \leq E(U, B_{R^2}) + \delta(R). \tag{5.1}$$

We remark that $\{U = 0\}$ consists of a finite number of closed sectors, since $\Sigma_U = 0$.

Now, assume for contradiction that U is non-trivial. Then we can find a direction (say e_1) and either a point $P \in \{U = 0\}^o$ such that $P \pm e_1 \in \{U > 0\}$ or a point

$P \in \{U > 0\}$ such that $P \pm e_1 \in \{U = 0\}^o$. Assume for simplicity that we are in the first case. This implies that

$$\underline{w} = U < U_R^+ \quad \text{in a neighborhood of } P,$$
$$\underline{w} = U_R^+ < U \quad \text{in a neighborhood of } P - e_1.$$

In conclusion, $\underline{w}$ is not harmonic in $B^+_{|P|+2}$ and therefore we can modify $\underline{w}$ inside this ball without changing its values on $\{x_{n+1} = 0\}$ so that the resulting function v satisfies

$$E(v, B_{|P|+2}) \leq E(\underline{w}, B_{|P|+2}) - \eta$$

with η small independent of R.

In conclusion, using (5.1) we obtain

$$E(v, B_{R^2}) \leq E(U, B_{R^2}) + \delta(R) - \eta,$$

which contradicts the minimality of U for R large enough. □

By our flatness theorem 2.6, Remark 4.5 and the compactness theorem 3.12, we immediately obtain the following corollary.

Corollary 5.6. *Minimizers of E in $\mathbb{R}^{2+1}$ have $C^{2,\alpha}$ free boundaries.*

In the next two lemmas, we follow the dimension reduction argument due to Federer for minimal surfaces (see also [CRSa]), and prove the first claim in Theorem 1.1, that is,

$$\mathcal{H}^s(\Sigma_u) = 0, \qquad s > n - 3,$$

for all minimizers u of E in $\Omega \subset \mathbb{R}^{n+1}$.

Lemma 5.7. *Assume that for some $s > 0$, $\mathcal{H}^s(\Sigma_U) = 0$ for all minimal cones U in $\mathbb{R}^{n+1}$. Then $\mathcal{H}^s(\Sigma_u) = 0$ for all minimizers u of E defined on $\Omega \subset \mathbb{R}^{n+1}$.*

Proof. First we show the following *property* (P): for every $Y \in \Sigma_u$ there exists $d_Y > 0$ such that for any $\delta \leq d_Y$, any subset D of $\Sigma_u \cap \mathcal{B}_\delta(Y)$ can be covered by a finite number of balls $B_{r_i}(Y_i)$ with $Y_i \in D$ such that

$$\sum_i r_i^s \leq \delta^s/2.$$

Property (P) follows by compactness. Indeed, given $Y \in \Sigma_u$, assume that the conclusion does not hold for a sequence $\delta_k \to 0$. By possibly passing to a subsequence, we may assume that the sequence u_{δ_k} converges uniformly to a minimal cone U where

$$u_\lambda(X) = \lambda^{-1/2} u(Y + \lambda X).$$

By our hypothesis, we can cover $\Sigma_U \cap \mathcal{B}_1$ by a finite number of balls $\mathcal{B}_{r_i/4}(X_i)$ with radius $r_i/4$ so that

$$\sum_i r_i^s \leq 1/2.$$

On the other hand, by the flatness theorem 2.6,

$$\Sigma_{u_{\delta_k}} \cap \mathcal{B}_1 \subset \bigcup_i \mathcal{B}_{r_i/2}(X_i)$$

for all large k. Thus, after scaling, u satisfies the conclusion in $\mathcal{B}_{\delta_k}$ for all large k and we reach a contradiction.

Next, denote by D_k the set of $Y \in \Sigma_u$ with $d_Y \geq 1/k$. Fix $Y_0 \in D_k$. By property (P), we can cover $D_k \cap \mathcal{B}_{r_0}(Y_0)$ where $r_0 = 1/k$ with a finite number of balls $\mathcal{B}_{r_i}(Y_i)$ with $Y_i \in D_k$ and

$$\sum_i r_i^s \leq r_0^s/2.$$

Now, we repeat the same argument for each ball $\mathcal{B}_{r_i}(Y_i)$ and cover it with balls $\mathcal{B}_{r_{ij}}(Y_{ij})$ with $Y_{ij} \in D_k$ and

$$\sum_j r_{ij}^s \leq r_i^s/2.$$

By repeating this argument m times we obtain $\mathcal{H}^s(D_k \cap \mathcal{B}_{r_0}(Y_0)) = 0$, hence $\mathcal{H}^s(D_k) = 0$ and the conclusion follows by letting $k \to \infty$. □

Lemma 5.8. *Assume that for some $s > 0$, $\mathcal{H}^s(\Sigma_U) = 0$ for all minimal cones U in $\mathbb{R}^{n+1}$. Then $\mathcal{H}^{s+1}(\Sigma_V) = 0$ for all minimal cones V defined in $\mathbb{R}^{n+2}$.*

Proof. It suffices to show that $\mathcal{H}^s(\Sigma_V \cap \partial\mathcal{B}_1) = 0$. Using our assumption we can deduce by the same compactness argument in the previous lemma that when restricted to $\partial\mathcal{B}_1$, $\Sigma_V \cap \partial\mathcal{B}_1$ satisfies the same property (P) as above. The conclusion now follows again with the same argument as in Lemma 5.7. □

In dimension $n = 3$, in view of Theorem 5.5, $\mathcal{H}^s(\Sigma_U) = 0$ for all $s > 0$, for all minimal cones U. This fact, combined with the previous two lemmas, gives the desired claim that

$$\mathcal{H}^s(\Sigma_u) = 0, \qquad s > n - 3,$$

for all minimizers u in $\mathbb{R}^{n+1}$.

Next we show the second claim in Theorem 1.1, that is, $F(u)$ has locally finite $\mathcal{H}^{n-1}$ measure for all minimizers u in $\mathbb{R}^{n+1}$.

Lemma 5.9. *Assume u is a minimizer in B_2 with $\|u\|_{C^{1/2}} \leq M$. Then there exists $C(M)$ large depending on M such that*

$$\mathcal{H}^{n-1}\Big((F(u) \cap \mathcal{B}_1) \setminus \bigcup_{i=1}^m \mathcal{B}_{\delta_i}(X_i)\Big) \leq C(M)$$

for some finite collection of balls $\mathcal{B}_{\delta_i}(X_i)$ with

$$\sum_{i=1}^m \delta_i^{n-1} \leq 1/2.$$

Proof. Assume for contradiction that we can find u_k such that $\|u_k\|_{C^{1/2}} \leq M$ and

$$\mathcal{H}^{n-1}\Big((F(u_k) \cap \mathcal{B}_1) \setminus \bigcup_{i=1}^{m} \mathcal{B}_{\delta_i}(X_i)\Big) \geq k \tag{5.2}$$

for any collection of balls with

$$\sum_{i=1}^{m} \delta_i^{n-1} \leq 1/2.$$

We may assume that u_k converges uniformly on compact subsets of B_2 to a minimizer u. Since $\mathcal{H}^{n-1}(\Sigma_u) = 0$ and Σ_u is closed,

$$\Sigma_u \cap \mathcal{B}_1 \subset \bigcup_{i=1}^{m} \mathcal{B}_{\delta_i/2}(X_i), \qquad \sum_{i=1}^{m} \delta_i^{n-1} \leq 1/2,$$

for some collection of balls.

Since $F(u) \setminus \Sigma_u$ is locally a $C^{2,\alpha}$ surface, we conclude from the flatness theorem that $(F(u_k) \cap \mathcal{B}_1) \setminus \bigcup_{i=1}^{m} \mathcal{B}_{\delta_i}(X_i)$ is a $C^{2,\alpha}$ surface which converges in the C^2 norm to $(F(u) \cap \mathcal{B}_1) \setminus \bigcup_{i=1}^{m} \mathcal{B}_{\delta_i}(X_i)$, contradicting (5.2). □

Lemma 5.10. *Assume u is a minimizer in B_2 with $\|u\|_{C^{1/2}} \leq M$. Then*

$$\mathcal{H}^{n-1}(F(u) \cap \mathcal{B}_1) \leq 2C(M).$$

Proof. By Lemma 5.9,

$$F(u) \cap \mathcal{B}_1 \subset \Gamma \cup \bigcup_{i=1}^{m} \mathcal{B}_{\delta_i}(X_i)$$

with $\mathcal{H}^{n-1}(\Gamma) \leq C(M)$, and

$$\sum_{i=1}^{m} \delta_i^{n-1} \leq 1/2.$$

For each ball $\mathcal{B}_{\delta_i}(X_i)$ we again apply Lemma 5.9 rescaled to obtain

$$F(u) \cap \mathcal{B}_{\delta_i}(X_i) \subset \Gamma_i \cup \bigcup_{j=1}^{m_i} \mathcal{B}_{\delta_{ij}}(X_{ij})$$

with $\mathcal{H}^{n-1}(\Gamma_i) \leq C(M)\delta_i^{n-1}$, and

$$\sum_{j=1}^{m_i} \delta_{ij}^{n-1} \leq \frac{1}{2}\delta_i^{n-1}.$$

Now for each ball $\mathcal{B}_{\delta_{ij}}(X_{ij})$ we apply the same argument and after l such steps we find that

$$F(u) \cap \mathcal{B}_1 \subset \tilde{\Gamma} \cup \bigcup_{q=1}^{r} \mathcal{B}_{\delta_q}(X_q)$$

with

$$\mathcal{H}^{n-1}(\tilde{\Gamma}) \le C(M)\left(1 + \frac{1}{2} + \cdots + \frac{1}{2^{l-1}}\right) \quad \text{and} \quad \sum_q \delta_q^{n-1} \le 2^{-l},$$

which implies the conclusion. □

Remark 5.11. The same argument can be used to show that the non-local minimal surfaces defined in [CRSa] have locally finite $\mathcal{H}^{n-1}$ measure.

Proof of Theorem 1.1. The conclusion follows from Theorem 5.5 and Lemmas 5.7–5.10. □

6. Viscosity solutions with Lipschitz free boundaries

In this section we prove our main Theorem 1.2, that is, that Lipschitz thin free boundaries are $C^{2,\alpha}$. First we prove non-degeneracy of viscosity solutions with Lipschitz free boundaries.

Lemma 6.1. *Assume u is a viscosity solution in B_2 with $F(u)$ a Lipschitz graph in the e_n direction with Lipschitz constant L and $0 \in F(u)$. Then*

$$\|u\|_{C^{1/2}(B_1)} \le C(L) \quad \textit{and} \quad \max_{B_r} u \ge c(L) r^{1/2} \quad \textit{for all } r \le 1.$$

Proof. Since

$$u(e_n) \le C \operatorname{dist}(e_n, F(u))^{1/2} \le C$$

we can apply the Harnack inequality to obtain

$$u(e_{n+1}) \le C(L),$$

which gives the first inequality of the claim (in view of Lemma 2.5).

By scaling, it suffices to prove the second inequality for $r = 1$.

Let μ be small depending on L and $X_0 \in \{u = 0\} \cap B_{1/2}$ be such that

$$\mathcal{B}_\mu(X_0) \subset \{u = 0\}$$

and it is tangent to $F(u)$ at Y_0. Let w be the harmonic function in $B_{2\mu}(X_0) \setminus \mathcal{B}_\mu(X_0)$ which is zero on $\mathcal{B}_\mu(X_0)$ and equals 1 on $\partial B_{2\mu}(X_0)$. Then, by the maximum principle,

$$w \max_{B_1} u \ge u \quad \text{on } B_{2\mu}(X_0).$$

Hence, since Y_0 is a regular point for $F(u)$, from the free boundary condition at Y_0 we obtain

$$\max_{B_1} u \, \frac{\partial w}{\partial U_0}(Y_0) \ge 1, \quad \text{so} \quad \max_{B_1} u \ge c(\mu).$$ □

In view of Proposition 2.10 and the previous lemma we obtain the following compactness result for viscosity solutions with Lipschitz free boundaries.

Corollary 6.2. *Let u_k be a sequence of viscosity solutions in B_2 with $F(u_k)$ uniformly Lipschitz, and $0 \in F(u_k)$. Then there exists a subsequence u_{k_l} such that*

$$u_{k_l} \to u_*, \qquad F(u_{k_l}) \to F(u_*) \qquad \textit{uniformly in } B_1$$

with u_* a viscosity solution in B_1.

Next we show that positive harmonic functions v (not necessarily viscosity solutions) are monotone in the e_n direction in a neighborhood of $F(v)$ if $F(v)$ is a Lipschitz graph.

Proposition 6.3 (Monotonicity around $F(v)$). *Assume that $v \geq 0$ solves $\Delta v = 0$ in $B_1^+(v)$, and that $F(v)$ is a Lipschitz graph in the e_n direction in $\mathcal{B}_1$ with Lipschitz constant L and $0 \in F(v)$. Then v is monotone in the e_n direction in B_δ, with δ depending on L and n.*

Proof. Assume by scaling that v is defined in B_{8L}. Let w be the harmonic function in

$$\Omega := \{|(x', 0, x_{n+1})| \leq 1,\ |x_n| \leq 2L\} \setminus \{v = 0\}$$

such that

$$w = 0 \quad \text{on } \partial\Omega \setminus \{x_n = 2L\}, \qquad w = 1 \quad \text{on } \{x_n = 2L\} \cap \partial\Omega\ .$$

Then w is strictly increasing in the e_n direction in Ω (by the maximum principle $w(X) \leq w(X + \epsilon e_n)$). By the boundary Harnack inequality ([CFMS])

$$v/w \in C^\alpha(B_{1/2}).$$

After multiplying v by an appropriate constant we may assume that $\frac{v}{w}(0) = 1$, and obtain

$$\left|\frac{v}{w} - 1\right| \leq \epsilon \quad \text{ in } B_{2\delta},$$

for some ϵ small to be made precise later and δ depending on ϵ, L and n. For each $r \leq \delta$, let

$$\tilde{v}(X) = \frac{v(rX)}{w(re_n)}, \qquad \tilde{w}(X) = \frac{w(rX)}{w(re_n)}.$$

Hence

$$\left|\frac{\tilde{v}}{\tilde{w}} - 1\right| \leq \epsilon \quad \text{in } B_2, \qquad \tilde{w}(e_n) = 1.$$

In the region

$$\mathcal{C}_{\mu_0} := \{|x'| < \mu_0,\ 1 - \mu_0 < |(x_n, x_{n+1})| < 1 + \mu_0\} \setminus \{(x, 0) : x_n < 0\}$$

with μ_0 small depending on L, we have (by the Harnack inequality for $\tilde{w}$)

$$|\tilde{v} - \tilde{w}| \leq \epsilon\tilde{w} \leq C(L)\epsilon.$$

Since $\tilde{v} - \tilde{w}$ is harmonic we obtain

$$|\tilde{v}_n - \tilde{w}_n| \le C(L)\epsilon \quad \text{in } \mathcal{C}_{\frac{3}{4}\mu_0}.$$

Since $\tilde{v}_n - \tilde{w}_n$ and $\tilde{w}_n$ are harmonic functions which vanish on

$$\partial\mathcal{C}_{\mu_0} \cap \{x_n \le 0,\ x_{n+1} = 0\}$$

and $\tilde{w}_n \ge 0$ and $\tilde{w}_n(e_n) \ge c(L) > 0$, we obtain

$$|\tilde{v}_n - \tilde{w}_n| \le C(L)\epsilon\tilde{w}_n \quad \text{in } \mathcal{C}_{\mu_0/2}. \tag{6.1}$$

The bound $\tilde{w}_n(e_n) \ge c(L) > 0$ follows from the Harnack inequality for $\tilde{w}_n$. Indeed, $\tilde{w}(e_n) = 1$ and $\tilde{w}(-e_n) = 0$, so we can find a point $\bar{X}$ on the line segment

$$[-e_n + \eta e_{n+1}, e_n + \eta e_{n+1}], \qquad \eta \text{ small},$$

where $\tilde{w}_n(\bar{X}) \ge c > 0$ for some c, η depending on L.

From (6.1) we get

$$\tilde{v}_n \ge \tilde{w}_n(1 - C(L)\epsilon) > 0 \quad \text{in } \mathcal{C}_{\mu_0/2},$$

provided that ϵ is chosen small depending on L. This inequality applied for all $r \le \delta$ easily implies the conclusion. □

The key step in the proof of Theorem 1.2 is to show that there are no non-trivial Lipschitz viscosity cones. By the dimension reduction argument in the previous section, it suffices to prove that there are no non-trivial cones with $C^{2,\alpha}$ free boundary outside of the origin. Indeed, we remark that Proposition 3.13 also holds for viscosity solutions, which can be easily checked directly from Definition 2.4. Therefore, Lemma 5.4 also holds for Lipschitz viscosity cones (see Remark 4.5).

Proposition 6.4. *All Lipschitz viscosity cones are trivial.*

Proof. Let U be a viscosity cone with Lipschitz free boundary and denote by L the Lipschitz norm of $F(U)$, as a graph in the e_n direction. We want to show that U is trivial. By the discussion above we can assume that $F(U)$ is $C^{2,\alpha}$ outside of the origin.

Now we prove the proposition by induction on n. The case $n = 1$ is obvious. Assume the statement holds for $n - 1$.

By Proposition 6.3, U is monotone in the cone of directions $(\xi, 0) \in \mathcal{C} \times \{0\}$ with

$$\mathcal{C} := \{\xi = (\xi', \xi_n) \in \mathbb{R}^n : \xi_n \ge L|\xi'|\},$$

since $F(U)$ is a Lipschitz graph with respect to any direction $\xi \in \mathcal{C}^o$. Moreover there is a direction $\tau \in \partial\mathcal{C}$ with $|\tau| = 1$ such that τ is tangent to $F(U)$ at some point X_0 in $F(U) \setminus \{0\}$. Then

$$U_\tau \ge 0 \quad \text{in } \{U > 0\}.$$

If $U_\tau = 0$ at some point in $\{U > 0\}$ then $U_\tau \equiv 0$, thus U is constant in the τ direction, and by dimension reduction we can reduce the problem to $n - 1$ dimensions, so by the induction assumption U is trivial. Otherwise $U_\tau > 0$ in $\{U > 0\}$ and by the boundary Harnack inequality,

$$U_\tau \geq \delta U \quad \text{in a neighborhood of } X_0, \text{ for some } \delta > 0.$$

This contradicts Lemma 2.8 since for all r small,

$$\frac{\delta}{2} r^{1/2} \leq \delta U(X_0 + \nu r) \leq U_\tau(X_0 + \nu r) \leq K r^{1/2+\alpha}. \qquad \square$$

Remark 6.5. As mentioned in the introduction, the argument above also works for the classical one-phase problem and the minimal surface equation. In the classical one-phase problem we need to use the Hopf lemma, and in the minimal surface equation we use the strong maximum principle.

We are now finally ready to exhibit the proof of our main Theorem 1.2.

Proof of Theorem 1.2. First, we show that given a viscosity solution u with Lipschitz free boundary in B_1 with $0 \in F(u)$, we can find $\sigma > 0$ small depending on u such that $F(u)$ is a $C^{2,\alpha}$ graph in B_σ. Indeed, there exists a blow-up sequence u_{λ_k} which converges to a Lipschitz viscosity cone (see Remark 4.5), which in view of the previous lemma is trivial. The conclusion now follows from our flatness theorem 2.6 and Corollary 6.2.

Next we use compactness to show that σ depends only on the Lipschitz constant L of $F(u)$. For this we need to show that $F(u)$ is $\bar{\epsilon}$-flat in B_r for some $r \geq \sigma$ depending on L. Indeed, if no such σ exists then we can find a sequence of solutions u_k and a sequence $\sigma_k \to 0$ such that u_k is not $\bar{\epsilon}$-flat in any B_r with $r \geq \sigma_k$. Then the u_k converge uniformly (up to a subsequence) to a solution u_*, and we reach a contradiction since $F(u_*)$ is $C^{2,\alpha}$ in a neighborhood of 0 by the first part of the proof. $\square$

Acknowledgments. D. D. is supported by NSF grant DMS-1301535. O. S. is supported by NSF grant DMS-1200701.

References

[AP] Allen, M., Petrosyan, A.: A two phase problem with a lower dimensional free boundary. Interfaces Free Bound. **14**, 307–342 (2012) Zbl 1260.35256 MR 2995409

[AC] Alt, H. W., Caffarelli, L. A.: Existence and regularity for a minimum problem with free boundary. J. Reine Angew. Math **325**, 105–144 (1981) Zbl 0449.35105 MR 0618549

[ACF] Alt, H. W., Caffarelli, L. A., Friedman, A.: Variational problems with two phases and their free boundaries. Trans. Amer. Math. Soc. **282**, 431–461 (1984) Zbl 0844.35137 MR 0732100

[C1] Caffarelli, L. A.: A Harnack inequality approach to the regularity of free boundaries. Part I: Lipschitz free boundaries are $C^{1,\alpha}$. Rev. Mat. Iberoamer. **3**, 139–162 (1987) Zbl 0676.35085 MR 0990856

[C2] Caffarelli, L. A.: A Harnack inequality approach to the regularity of free boundaries. Part II: Flat free boundaries are Lipschitz. Comm. Pure Appl. Math. **42**, 55–78 (1989) Zbl 0676.35086 MR 0973745

[C3] Caffarelli, L. A.: A Harnack inequality approach to the regularity of free boundaries. Part III: Existence theory, compactness, and dependence on X. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **15**, 583–602 (1989) (1988) Zbl 0702.35249 MR 1029856

[CFMS] Caffarelli, L. A., Fabes, E., Mortola, S., Salsa, S.: Boundary behavior of nonnegative solutions of elliptic operators in divergence form. Indiana Univ. Math. J. **30**, 621–640 (1981) Zbl 0512.35038 MR 0620271

[CJK] Caffarelli, L. A., Jerison, D., Kenig, C. E.: Global energy minimizers for free boundary problems and full regularity in three dimensions. In: Noncompact Problems at the Intersection of Geometry, Analysis, and Topology, Contemp. Math. 350, Amer. Math. Soc., Providence, RI, 8397 (2004) Zbl 02166797 MR 2082392

[CRSa] Caffarelli, L., Roquejoffre, J.-M., Savin, O.: Nonlocal minimal surfaces. Comm. Pure Appl. Math. **63**, 1111–1144 (2010) Zbl 1248.53009 MR 2675483

[CRS] Caffarelli, L. A., Roquejoffre, J.-M., Sire, Y.: Variational problems with free boundaries for the fractional Laplacian. J. Eur. Math. Soc. **12**, 1151–1179 (2010) Zbl 1221.35453 MR 2677613

[CS] Caffarelli, L. A., Salsa, S.: A Geometric Approach to Free Boundary Problems. Grad. Stud. Math. 68, Amer. Math. Soc., Providence, RI (2005) Zbl 1083.35001 MR 2145284

[DG] De Giorgi, E.: Sulla differenziabilità e l'analiticità delle estremali degli integrali multipli regolari. Mem. Accad. Sci. Torino Cl. Sci. Fis. Mat. Nat. (3) **3**, 25–43 (1957) Zbl 0084.31901 MR 0093649

[DJ1] De Silva, D., Jerison, D.: A singular energy minimizing free boundary. J. Reine Angew. Math. **635**, 1–21 (2009) Zbl 1185.35050 MR 2572253

[DJ2] De Silva, D., Jerison, D.: A gradient bound for free boundary graphs. Comm. Pure Appl. Math. **64**, 538–555 (2011) Zbl 1216.35179 MR 2796515

[DR] De Silva, D., Roquejoffre, J.-M.: Regularity in a one-phase free boundary problem for the fractional Laplacian. Ann. Inst. H. Poincaré Anal. Non Linéaire **29**, 335–367 (2012) Zbl 1251.35178 MR 2926238

[DS] De Silva, D., Savin, O.: $C^{2,\alpha}$ regularity of flat free boundaries for the thin one-phase problem. J. Differential Equations **253**, 2420–2459 (2012) Zbl 1248.35238 MR 2950457

[SV] Savin, O., Valdinoci, E.: Regularity of nonlocal minimal cones in dimension 2. Calc. Var. Partial Differential Equations **48**, 33–39 (2013) Zbl 1275.35065 MR 3090533

[W] Weiss, E. S.: Partial regularity for weak solutions of an elliptic free boundary problem. Comm. Partial Differential Equations **23**, 439–455 (1998) Zbl 0897.35017 MR 1620644