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Cloaking via anomalous localized resonance for doubly complementary media in the quasistatic regime

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Abstract. This paper is devoted to the study of cloaking via anomalous localized resonance (CALR) in the two- and three-dimensional quasistatic regimes. CALR associated with negative index materials was discovered by Milton and Nicorovici [21] for constant plasmonic structures in the two-dimensional quasistatic regime. Two key features of this phenomenon are the localized resonance, i.e., the fields blow up in some regions and remain bounded in some others, and the connection between the localized resonance and the blow up of the power of the fields as the loss of the material goes to 0. An important class of negative index materials for which the localized resonance might appear is the class of reflecting complementary media introduced in [24]. It was shown in [29] that the complementarity property is not enough to ensure a connection between the blow up of the power and the localized resonance. In this paper, we study CALR for a subclass of complementary media called doubly complementary media. This class is rich enough to allow us to cloak an arbitrary source concentrating on an arbitrary smooth bounded manifold of codimension 1 placed in an arbitrary medium via anomalous localized resonance; the cloak is independent of the source. The following three properties are established for doubly complementary media: P1. CALR appears if and only if the power blows up; P2. The power blows up if the source is located "near" the plasmonic structure; P3. The power remains bounded if the source is far away from the plasmonic structure. Property P2, the blow up of the power, is in fact established for reflecting complementary media. The proofs are based on several new observations and ideas. One of the difficulties is to handle the localized resonance. To this end, we extend the reflecting and removing localized singularity techniques introduced in [24–26], and implement the separation of variables for Cauchy problems for a general shell. The results in this paper are inspired by and imply recent ones of Ammari et al. [3] and Kohn et al. [16] in two dimensions and extend theirs to general non-radial core-shell structures in both two and three dimensions.

Keywords. Cloaking, anomalous localized resonance, negative index materials, complementary media

Contents

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1. Introduction

Negative index materials (NIMs) were first investigated theoretically by Veselago [36] and their theory was further developed by Nicorovici et al. [32] and Pendry [33]. The existence of such materials was confirmed by Shelby et al. [35]. The study of NIMs has attracted a lot of attention thanks to their many applications. One of the appealing ones is cloaking. There are at least three ways to do cloaking using NIMs. The first one is based on plasmonic structures introduced by Alu and Engheta [2]. The second one uses the concept of complementary media. This was suggested by Lai et al. [17] and confirmed theoretically in [25] for a slightly different scheme. The last one is based on the concept of anomalous localized resonance discovered by Milton and Nicorovici [21]. In this paper, we concentrate on the last method.

Cloaking via anomalous localized resonance (CALR) was discovered by Milton and Nicorovici [21]. Their work has roots in [32] (see also [23]) where the localized resonance was observed and established for constant symmetric plasmonic structures in the two-dimensional quasistatic regime. More precisely, in [21], the authors studied core-shell plasmonic structures in which a circular shell has permittivity $-1 + i\delta$ while the core and the matrix, the complement of the core and the shell, have permittivity 1. Here δ denotes the loss of the material in the shell. Let r_e and r_i be the outer and the inner radii of the shell. Milton and Nicorovici showed that there is a critical radius $r_* := (r_e^3 r_i^{-1})^{1/2}$ such that a dipole is not seen by an observer away from the core-shell structure, hence it is *cloaked*, if and only if the dipole is within distance r_* of the shell; moreover, the power $E_\delta(u_\delta)$ of the field u_δ, which is roughly speaking $\delta \|u_\delta\|_{H^1}^2$, blows up. They called this phenomenon *cloaking via anomalous localized resonance*. Two key features of this phenomenon are:

1. The *localized resonance*, i.e., the fields blow up in some regions and remain bounded in some others as the loss goes to 0.
2. The connection between the localized resonance and the blow up of the power as the loss goes to 0.

That work has led to a new method of cloaking and has been a source of inspiration for many investigations [3–8, 15, 16, 20, 22, 29, 31].

Let us discuss recent progress on CALR. In [6], Bouchitté and Schweizer proved that a small circular inclusion of radius $\gamma(\delta)$ (with $\gamma(\delta) \to 0$ fast enough) is cloaked by the core-shell plasmonic structure mentioned above in the two-dimensional quasistatic regime if the inclusion is located within distance r_* of the shell. Otherwise it is visible. Concerning the second feature of CALR, the blow up of the power was studied for a more general setting by Ammari et al. [3] and Kohn et al. [16]. More precisely, they considered non-radial core-shell structures in which the shell has permittivity $-1 + i\delta$ and the core and the matrix have permittivity 1. Ammari et al. [3] dealt with arbitrary shells in the two-dimensional quasistatic regime. They provided a characterization of sources for which the power blows up. Their characterization is based on the spectrum of a self-adjoint compact operator (Neumann–Poincaré type operator). Kohn et al. [16] considered core-shell structures in the two-dimensional quasistatic regime in which the matrix is radial symmetric but the core is not. Using a variational approach, they established the blow up of the power for a class of sources concentrated on circles within distance $r_* = (r_e^3 r_i^{-1})^{1/2}$ of the core-shell region B_{r_e} if the core is inside B_{r_i}. They also showed that the power remains bounded for a class of sources concentrated on circles outside B_{r_*} if the core is round, inside, and close to B_{r_i}. The localized resonance associated with CALR has been discussed so far only for simple geometries [3, 5, 8].

An important class of NIMs in which the localized resonance might appear is the class of reflecting complementary media [25, 26, 30]. The concept of reflecting complementary media for a general core-shell structure was introduced and studied in [24]. This class is inspired by the pivotal work of Nicorovici et al. [32] and by the important notion of complementary media suggested by Ramakrishna and Pendry [34]. Nevertheless, the complementarity property is not enough to ensure that CALR takes place, as discussed in [29]. Therefore, the study of the two features 1 and 2 together in CALR is of importance.

In this paper, we investigate CALR for a subclass of complementary media called the class of doubly complementary media for a core-shell structure, defined in Definition 1.2. This class is rich enough to allow us to cloak an **arbitrary source** concentrating on an **arbitrary smooth bounded manifold of codimension 1** placed in an **arbitrary medium** via anomalous localized resonance (see Section 5); the cloak is independent of the source. Roughly speaking, the shell is not only reflecting complementary to a part of the matrix but also to a part of the core. We establish the following three properties of CALR for doubly complementary media, which are what one would expect from a structure for which CALR takes place:

- P1. CALR appears if and only if the power blows up (Theorem 1.1).
- P2. The power blows up if the source is located "near" the shell (Theorem 1.2).
- P3. The power remains bounded if the source is far away from the shell (Theorem 1.3).

Property P2, the blow up of the power, is in fact established for **reflecting complementary media**. We also address qualitative estimates on the distance from the source to the shell for which CALR does or does not appear in various situations (Theorems 1.2 and 1.3).

We now describe the problem more precisely. Let $d = 2, 3$, and Ω be a smooth open bounded subset of $\mathbb{R}^d$, and let $0 < r_1 < r_2$ be such that $B_{r_2} \subset\subset \Omega$. Set, for $\delta > 0$,

$$s_\delta := \begin{cases} -1 + i\delta & \text{in } B_{r_2} \setminus B_{r_1}, \\ 1 & \text{otherwise.} \end{cases} \tag{1.1}$$

Let A be a symmetric uniformly elliptic matrix-valued function defined in Ω, i.e., A is symmetric and

$$\frac{1}{\Lambda}|\xi|^2 \le \langle A(x)\xi, \xi\rangle \le \Lambda|\xi|^2 \quad \forall \xi \in \mathbb{R}^d, \tag{1.2}$$

for a.e. $x \in \Omega$ and for some $1 \le \Lambda < \infty$. Let $f \in L^2(\Omega)$ with $\operatorname{supp} f \cap B_{r_2} = \emptyset$ and let $u_\delta \in H^1_0(\Omega)$ be the unique solution to

$$\operatorname{div}(s_\delta A \nabla u_\delta) = f \quad \text{in } \Omega. \tag{1.3}$$

The *power* $E_\delta(u_\delta)$ is defined by (see, e.g., [21])

$$E_\delta(u_\delta) = \delta \int_{B_{r_2}\setminus B_{r_1}} |\nabla u_\delta|^2.$$

Using the fact that $u_\delta = 0$ on $\partial\Omega$, one has[1]

$$\int_\Omega (|\nabla u_\delta|^2 + |u_\delta|^2) \le C\left(\int_{B_{r_2}\setminus B_{r_1}} |\nabla u_\delta|^2 + \|f\|^2_{L^2}\right) \tag{1.4}$$

for some positive constant C independent of f and $\delta \in (0, 1)$. Let $v_\delta \in H^1_0(\Omega)$ be the unique solution to

$$\operatorname{div}(s_\delta A \nabla v_\delta) = f_\delta \quad \text{in } \Omega. \tag{1.5}$$

Here $f_\delta = c_\delta f$, where c_δ is the normalization constant such that

$$\delta^{1/2} \int_{B_{r_2}\setminus B_{r_1}} |\nabla v_\delta|^2 = 1. \tag{1.6}$$

In this paper, we are interested in a class of matrices A, called doubly complementary media, for which CALR takes place. Before giving their definition for a general core-shell structure, let us recall the definition of reflecting complementary media introduced in [24, Definition 1].

Definition 1.1 (Reflecting complementary media). Let $r_1 < r_2 < r_3$. The media A in $B_{r_3} \setminus B_{r_2}$ and $-A$ in $B_{r_2} \setminus B_{r_1}$ are said to be *reflecting complementary* if there exists a diffeomorphism $F : B_{r_2} \setminus \overline{B_{r_1}} \to B_{r_3} \setminus \overline{B_{r_2}}$ such that

$$F_*A = A \quad \text{for } x \in B_{r_3} \setminus \overline{B_{r_2}}, \tag{1.7}$$
$$F(x) = x \quad \text{on } \partial B_{r_2}, \tag{1.8}$$

and the following two conditions hold:

[1] One way to obtain this inequality is to multiply (1.3) by $\bar{u}_\delta$ (the conjugate of u_δ), integrate on Ω, and consider the real part.

1. There exists a diffeomorphic extension of F, still denoted by F, from $B_{r_2} \setminus \{x_1\}$ to $\mathbb{R}^d \setminus \overline{B_{r_2}}$ for some $x_1 \in B_{r_1}$.
2. There exists a diffeomorphism $G : \mathbb{R}^d \setminus \overline{B_{r_3}} \to B_{r_3} \setminus \{x_2\}$ for some $x_2 \in B_{r_3}$ such that

$$G(x) = x \quad \text{on } \partial B_{r_3}, \tag{1.9}$$

$$G \circ F : B_{r_1} \to B_{r_3} \text{ is a diffeomorphism if one sets } G \circ F(x_1) = x_2. \tag{1.10}$$

Here and in what follows, if T is a diffeomorphism and a is a matrix-valued function, we denote

$$T_* a(y) = \frac{DT(x)a(x)DT(x)^T}{|\det DT(x)|} \quad \text{where } x = T^{-1}(y). \tag{1.11}$$

Remark 1.1. In (1.8) and (1.9), F and G denote some diffeomorphic extensions of F and G in a neighborhood of ∂B_{r_2} and of ∂B_{r_3}. As noted in [24], conditions (1.7) and (1.8) are the main assumptions in Definition 1.1. The term “reflecting” in Definition 1.1 comes from (1.8) and the fact that $B_{r_1} \subset B_{r_2} \subset B_{r_3}$. Conditions 1 and 2 are mild assumptions. Introducing G makes the analysis more accessible—see [24–26, 30] and the analysis presented in this paper.

Remark 1.2. The class of reflecting complementary media has played an important role in other applications of NIMs such as cloaking and superlensing using complementarity [25, 26, 30].

Remark 1.3. Taking $d = 2$, $A = I$ and $r_3 = r_2^2/r_1$, and letting F be the Kelvin transform with respect to ∂B_{r_2}, i.e., $F(x) = r_2^2 x/|x|^2$, one can verify that the core-shell structures considered by Milton et al. [21] and Kohn et al. [16] have the reflecting complementarity property.

We are ready to introduce the concept of doubly complementary media.

Definition 1.2. The medium $s_0 A$ is said to be *doubly complementary* if for some $r_3 > 0$ with $B_{r_3} \subset\subset \Omega$, the media A in $B_{r_3} \setminus B_{r_2}$ and $-A$ in $B_{r_2} \setminus B_{r_1}$ are reflecting complementary, and

$$F_* A = G_* F_* A = A \quad \text{in } B_{r_3} \setminus B_{r_2} \tag{1.12}$$

for some F and G coming from Definition 1.1 (see Figure 1).

Remark 1.4. The reason why media satisfying (1.12) are called doubly complementary media is that $-A$ in $B_{r_2} \setminus B_{r_1}$ is not only complementary to A in $B_{r_3} \setminus B_{r_2}$ but also to A in $(G \circ F)^{-1}(B_{r_3} \setminus \overline{B_{r_2}})$ (a subset of B_{r_1}) (see [27]).

Remark 1.5. Taking $d = 2$, $A = I$ and $r_3 = r_2^2/r_1$, and letting F and G be the Kelvin transform with respect to ∂B_{r_2} and ∂B_{r_3}, one can verify that the core-shell structures considered by Milton et al. [21] have the double complementarity property. The setting considered in [4] also has this property.

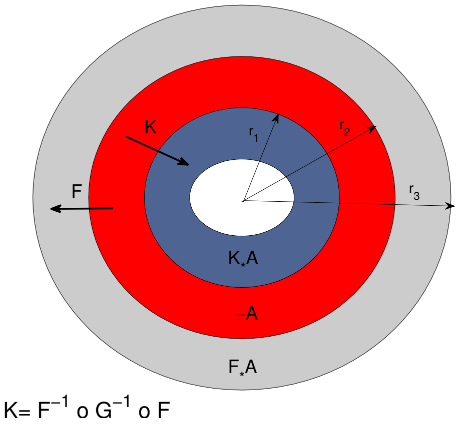

Fig. 1. $s_0 A$ is doubly complementary: $-A$ in $B_{r_2} \setminus B_{r_1}$ (red region) is complementary to $A = F_* A$ in $B_{r_3} \setminus B_{r_2}$ (grey region) and $A = K_* A$ with $K = F^{-1} \circ G^{-1} \circ F$ in $K(B_{r_2} \setminus B_{r_1})$ (blue grey region).

In what follows, we assume that

$$A \in [C^3(\overline{B_{r_3} \setminus B_{r_2}})]^{d \times d}. \tag{1.13}$$

This assumption, which can sometimes be weakened, is necessary for the use of a three spheres inequality, the unique continuation principle, and the separation of variables technique introduced later in this paper.

The following theorem is one of the main results of the paper. It gives the equivalence between the blow up of the power and CALR for doubly complementary media, which implies Property P1.

Theorem 1.1. *Let $d = 2, 3$, $f \in L^2(\Omega)$ with $\operatorname{supp} f \subset \Omega \setminus B_{r_2}$, $\delta_n \to 0$, and let $u_{\delta_n} \in H^1_0(\Omega)$ be the unique solution to*

$$\operatorname{div}(s_{\delta_n} A \nabla u_{\delta_n}) = f \quad \textit{in } \Omega.$$

Assume that $s_0 A$ is doubly complementary.

(i) *If $\lim_{n\to\infty} \delta_n \|\nabla u_{\delta_n}\|^2_{L^2(B_{r_2} \setminus B_{r_1})} = \infty$, then*

$$v_{\delta_n} \to 0 \quad \textit{weakly in } H^1(\Omega \setminus B_{r_3}), \tag{1.14}$$

where $v_\delta \in H^1_0(\Omega)$ is defined in (1.5).

(ii) *If $(\delta_n \|\nabla u_{\delta_n}\|^2_{L^2(B_{r_2} \setminus B_{r_1})})_{n \in \mathbb{N}}$ is bounded then*

$$u_{\delta_n} \to u \quad \textit{weakly in } H^1(\Omega \setminus B_{r_3}),$$

where $u \in H^1_0(\Omega)$ is the unique solution to

$$\operatorname{div}(\hat{A} \nabla u) = f \quad \textit{in } \Omega. \tag{1.15}$$

Here and in what follows, we denote

$$\hat{A} = \begin{cases} A & \text{in } \Omega \setminus B_{r_3}, \\ G_* F_* A & \text{in } B_{r_3}. \end{cases} \tag{1.16}$$

The proof of Theorem 1.1 is given in Section 4 where a stronger result (Proposition 4.1) is established.

The equivalence between the blow up of the power and CALR can be obtained from Theorem 1.1 as follows. Suppose that the power blows up, i.e.,

$$\lim_{n\to\infty} \delta_n \|\nabla u_{\delta_n}\|^2_{L^2(B_{r_2}\setminus B_{r_1})} = \infty.$$

Then, by Theorem 1.1, $v_{\delta_n} \to 0$ in $\Omega \setminus B_{r_3}$, so the source $\alpha_{\delta_n} f$ is not seen by observers far away from the shell: the source is *cloaked*. We note that the localized resonance happens in this case since both (1.6) and (1.14) take place. If the power of u_{δ_n} remains bounded, then $u_{\delta_n} \to u$ weakly in $H^1(\Omega \setminus B_{r_3})$. Since $u \in H^1_0(\Omega)$ is the unique solution to (1.15), the source is not cloaked.

Theorem 1.1 is, to our knowledge, the first result providing the connection between the blow up of the power and the invisibility of a source in a general setting. The standard separation of variables is not available here.

We next show that CALR takes place if the source is located “near” the shell. This implies Property P2. In fact, we establish this property for **reflecting complementary media**. More precisely, we have the following result whose proof is given in Section 2.

Theorem 1.2. *Let $d = 2, 3$, $f \in L^2(\Omega)$ with $\operatorname{supp} f \subset \Omega \setminus B_{r_2}$, and let $u_\delta \in H^1_0(\Omega)$ be the unique solution to*

$$\operatorname{div}(s_\delta A \nabla u_\delta) = f \quad \text{in } \Omega.$$

Assume that A in $B_{\hat{r}_3} \setminus B_{r_2}$ and $-A$ in $B_{r_2} \setminus B_{\hat{r}_1}$ are reflecting complementary for some $r_1 \le \hat{r}_1 < r_2 < \hat{r}_3$, with $B_{\hat{r}_3} \subset\subset \Omega$. There exists a constant $r_* \in (r_2, \hat{r}_3)$, independent of δ and f, such that if there is **no** *$w \in H^1(B_{r_*} \setminus B_{r_2})$ with*

$$\operatorname{div}(A\nabla w) = f \quad \text{in } B_{r_*} \setminus B_{r_2}, \qquad w = 0 \text{ and } A\nabla w \cdot \eta = 0 \quad \text{on } \partial B_{r_2}, \tag{1.17}$$

then

$$\limsup_{\delta\to 0} \delta^{1/2} \|\nabla u_\delta\|_{L^2(B_{r_2}\setminus B_{r_1})} = \infty. \tag{1.18}$$

Assume in addition that $A = I$ in $B_{\hat{r}_3} \setminus B_{r_2}$. Then

$$r_* \text{ can be taken to be any number less than } \sqrt{\hat{r}_3 r_2}. \tag{1.19}$$

Here and in what follows, for D a smooth bounded open subset of $\mathbb{R}^d$, η denotes the outward unit normal vector on ∂D.

Concerning the boundedness of the power, we prove

Theorem 1.3. *Let $d = 2, 3$, $f \in L^2(\Omega)$, and let $u_\delta \in H^1_0(\Omega)$ be the unique solution to (1.3). Assume that $s_0 A$ is doubly complementary and* $\operatorname{supp} f \cap B_{r_3} = \emptyset$. *Then*

$$\limsup_{\delta\to 0} \|u_\delta\|_{H^1(\Omega)} < \infty. \tag{1.20}$$

Assume in addition that $A = I$ in $B_{r_3} \setminus B_{r_2}$. If there exists $w \in H^1(B_{r_0} \setminus B_{r_2})$ for some $r_0 > \sqrt{r_2 r_3}$ with

$$\operatorname{div}(A\nabla w) = f \quad \text{in } B_{r_0} \setminus B_{r_2}, \quad w = 0 \text{ and } A\nabla w \cdot \eta = 0 \quad \text{on } \partial B_{r_2},$$

then

$$\limsup_{\delta\to 0} \delta^{1/2} \|u_\delta\|_{H^1(\Omega)} < \infty. \tag{1.21}$$

It is clear that Theorem 1.3 implies Property P3. The proof of Theorem 1.3 is given in Section 3.

The analysis in this paper is based on several new observations and ideas. The proof of Theorem 1.1 (in Section 4) makes use of the reflecting and removing localized singularity techniques introduced in [24–26] to deal with the localized resonance. To develop these techniques for a general core-shell structure, we introduce and implement the separation of variables technique to solve Cauchy problems in a general shell (Proposition 4.2 in Section 4.2). The way to implement this technique is one of the cores of the analysis in this paper. The use of separation of variables to solve boundary value problems for the Laplace equation in an arbitrary domain was considered in the literature and was based on the integral method (see e.g. [14]). The analysis presented here is based on the idea of transformation optics and the reflecting technique. As a consequence, we obtain the existence of surface plasmons for general complementary media (Proposition 4.2). The proof of Theorem 1.2 (in Section 2) is based on a new observation for complementary media (Lemma 2.4) whose proof is based on a three spheres inequality. The idea of the proof of Theorem 1.3 (in Section 3) is as follows. The first part (1.20) is from [24]. The proof of the second part (1.21) is based on a kind of removing singularity technique and uses ideas of [24]. A key point is the construction of an auxiliary function W_δ in (3.9). Using Theorems 1.1 and 1.2, we can construct a cloaking device to cloak a general source concentrated on a manifold of codimension 1 in an arbitrary medium (see Section 5). The proof also makes use of the unique continuation principle.

By considering $A = I$ in Theorems 1.1, 1.2, and 1.3, one can recover the results of Milton and Nicorovici [21] and Kohn et al. [16], and the results of Ammari et al. [3] in the radial setting, in two dimensions. The results presented here extend theirs to general non-radial core-shell structures in both two and three dimensions.

The results of this paper were announced in [27]. The study of CALR in the finite frequency regime will be undertaken in [28].

2. A condition on the blow up of the power. Proof of Theorem 1.2

This section comprising two subsections is devoted to the proof of Theorem 1.2. In the first subsection, we present some useful lemmas. The proof of Theorem 1.2 is given in the second subsection.

2.1. Preliminaries

We first recall the following result, a change of variables formula, which follows immediately from [24, Lemma 2], and is used repeatedly in this paper.

Lemma 2.1. *Let $d = 2, 3$, $R > 0$, D_1 and D_2 be two smooth open subsets of $\mathbb{R}^d$ such that $D_1 \subset\subset B_R \subset\subset D_2$. Assume that T is a diffeomorphism from $B_R \setminus D_1$ onto $D_2 \setminus B_R$ and let $a \in [L^\infty(B_R \setminus D_1)]^{d\times d}$ be uniformly elliptic. Fix $u \in H^1(B_R \setminus D_1)$ and set $v = u \circ T^{-1}$. Then*

$$\operatorname{div}(a\nabla u) = 0 \text{ in } B_R \setminus D_1 \quad \text{if and only if} \quad \operatorname{div}(T_*a\nabla v) = 0 \text{ in } D_2 \setminus B_R.$$

Assume in addition that $T(x) = x$ on ∂B_R. Then

$$T_*a\nabla v \cdot \eta = -a\nabla u \cdot \eta \quad \text{on } \partial B_R. \tag{2.1}$$

We next recall the following three spheres inequality (see, e.g., [1, Theorem 2.3 and (2.10)]).

Lemma 2.2 (Three spheres inequality). *Let $d = 2, 3$, $0 < R_1 < R_2 < R_3$, and let M be a Lipschitz matrix-valued function defined in B_{R_3} such that M is symmetric and uniformly elliptic in B_{R_3}, and $M(0) = I$. Assume $v \in H^1(B_{R_3})$ is a solution to*

$$\operatorname{div}(M\nabla v) = 0 \quad \text{in } B_{R_3}.$$

There exist positive constants C and c, depending only on R_3 and the ellipticity and the Lipschitz constants of M, such that

$$\|v\|_{L^2(\partial B_{R_2})} \le C\|v\|^{\alpha}_{L^2(B_{\partial R_1})}\|v\|^{1-\alpha}_{L^2(\partial B_{R_3})},$$

where

$$\alpha = \frac{\ln(R_3/R_2)}{\ln(R_3/R_2) + c\ln(R_2/R_1)}, \tag{2.2}$$

In the case $M = I$ in B_{R_3}, one can take $c = 1$, i.e., $\alpha = \ln(R_3/R_2)/\ln(R_3/R_1)$.

Using Lemma 2.2, we can prove

Lemma 2.3. *Let $d = 2, 3$, $0 < R_1 < R_2 < R_3$, and let M be a Lipschitz matrix-valued function defined in B_{R_3} such that M is symmetric and uniformly elliptic in B_{R_3} and $M(0) = I$. Assume $v \in H^1(B_{R_3})$ is a solution to*

$$\operatorname{div}(M\nabla v) = 0 \quad \text{in } B_{R_3} \setminus B_{R_1}.$$

There exist positive constants C and c such that C depends only on R_1, R_3, the ellipticity and the Lipschitz constants of M, and c depends only on R_3, the ellipticity and the Lipschitz constants of M, and

$$\begin{aligned}\|v\|_{L^2(\partial B_{R_2})} \le C\Big(&(\|v\|_{H^{1/2}(\partial B_{R_1})} + \|M\nabla v\cdot\eta\|_{H^{-1/2}(\partial B_{R_1})})^{\alpha}\|v\|^{1-\alpha}_{L^2(\partial B_{R_3})}\\ &+ (\|v\|_{H^{1/2}(\partial B_{R_1})} + \|M\nabla v\cdot\eta\|_{H^{-1/2}(\partial B_{R_1})})\Big),\end{aligned} \tag{2.3}$$

where

$$\alpha = \frac{\ln(R_3/R_2)}{\ln(R_3/R_2) + c\ln(R_2/R_1)}. \tag{2.4}$$

In the case $M = I$ in B_{R_3}, one can take $c = 1$.

Proof. Let $w \in H^1(B_{R_3} \setminus \partial B_{R_1})$ be such that

$$\begin{aligned} &\operatorname{div}(M\nabla w) = 0 \quad \text{in } B_{R_3} \setminus \partial B_{R_1}, \qquad w = 0 \quad \text{on } \partial B_{R_3}, \\ &[w] = v \text{ and } [M\nabla w \cdot \eta] = M\nabla v \cdot \eta \quad \text{on } \partial B_{R_1}. \end{aligned}$$

Henceforth $[\cdot]$ denotes the jump across the boundary. It follows that

$$\|w\|_{H^1(B_{R_3}\setminus\partial B_{R_1})} \le C(\|v\|_{H^{1/2}(\partial B_{R_1})} + \|M\nabla v \cdot \eta\|_{H^{-1/2}(\partial B_{R_1})}). \tag{2.5}$$

Here and in what follows in this proof, C denotes a positive constant depending only on R_1, R_3, and the ellipticity and the Lipschitz constants of M. Define

$$V = \begin{cases} v - w & \text{in } B_{R_3} \setminus B_{R_1}, \\ -w & \text{in } B_{R_1}. \end{cases}$$

Then $V \in H^1(B_{R_3})$ and $\operatorname{div}(M\nabla V) = 0$ in B_{R_3}. Applying Lemma 2.2, we obtain

$$\|V\|_{L^2(\partial B_{R_2})} \le C\|V\|^{\alpha}_{L^2(\partial B_{R_1})}\|V\|^{1-\alpha}_{L^2(\partial B_{R_3})}.$$

The conclusion follows from (2.5) and the definition of V. □

The following result provides the key ingredient for the proof of Theorem 1.2.

Lemma 2.4. *Let $d = 2, 3$, $0 < R_1 < R_2 < \infty$, let M be a symmetric uniformly elliptic matrix-valued function defined in $B_{R_2} \setminus B_{R_1}$, and let $g, h \in L^2(B_{R_2} \setminus B_{R_1})$. Assume that M is Lipschitz and $U_\delta, V_\delta \in H^1(B_{R_2} \setminus B_{R_1})$ satisfy*

$$\begin{aligned} &\operatorname{div}(M\nabla U_\delta) = g \text{ and } \operatorname{div}(M\nabla V_\delta) = h \quad \text{in } B_{R_2} \setminus B_{R_1}, \\ &U_\delta = V_\delta \text{ and } M\nabla U_\delta \cdot \eta = (1 - i\delta)M\nabla V_\delta \cdot \eta \quad \text{on } \partial B_{R_1}. \end{aligned}$$

There exists a constant $R_* \in (R_1, R_2)$, depending only on R_1, R_2, and the ellipticity and the Lipschitz constants of M, but independent of δ, g, and h, such that if there is **no** *$W \in H^1(B_{R_*} \setminus B_{R_1})$ with*

$$\operatorname{div}(M\nabla W) = g - h \quad \text{in } B_{R_*} \setminus B_{R_1}, \quad W = 0 \text{ and } M\nabla W \cdot \eta = 0 \quad \text{on } \partial B_{R_1}, \tag{2.6}$$

then

$$\limsup_{\delta \to 0} \delta^{1/2}(\|U_\delta\|_{H^1(B_{R_2}\setminus B_{R_1})} + \|V_\delta\|_{H^1(B_{R_2}\setminus B_{R_1})}) = \infty. \tag{2.7}$$

Assume in addition that $M = I$ in $B_{R_2} \setminus B_{R_1}$. Then

$$R_* \text{ can be taken to be any number less than } \sqrt{R_1 R_2}. \tag{2.8}$$

Proof. For notational ease, we denote $U_{2^{-n}}$ and $V_{2^{-n}}$ by U_n and V_n. We have

$$\begin{aligned} &\operatorname{div}(M\nabla U_n) = g \text{ and } \operatorname{div}(M\nabla V_n) = h \quad \text{in } B_{R_2}\setminus B_{R_1},\\ &U_n = V_n \text{ and } M\nabla U_n\cdot\eta = (1-i2^{-n})M\nabla V_n\cdot\eta \quad \text{on } \partial B_{R_1}. \end{aligned}$$

Let $\hat{M}$ be an extension on M in B_{R_2} such that $\hat{M}$ is Lipschitz and uniformly elliptic in B_{R_2}, and $\hat{M}(0) = I$.[2] Let c be the constant in Lemma 2.3 corresponding to $\hat{M}$ and the shell $B_{R_2}\setminus B_{R_1}$. Define

$$\alpha(r) = \frac{\ln(R_2/r)}{\ln(R_2/r) + c\ln(r/R_1)} \quad \forall r\in(R_1,R_2).$$

Fix R_* such that $\alpha(R_*) > 1/2$ (this holds if R_* is chosen close to R_1). There exists $\gamma\in(0,1)$ (close to 1) such that

$$\alpha(r) > (\alpha(R_*) + 1/2)/2 \quad \text{for } r\in(\gamma R_*, (2-\gamma)R_*). \tag{2.9}$$

We prove that

$$\limsup_{n\to\infty} 2^{-n/2}(\|U_n\|_{H^1(B_{R_2}\setminus B_{R_1})} + \|V_n\|_{H^1(B_{R_2}\setminus B_{R_1})}) = \infty. \tag{2.10}$$

Assume for contradiction that

$$m := \sup_n 2^{-n/2}(\|U_n\|_{H^1(B_{R_2}\setminus B_{R_1})} + \|V_n\|_{H^1(B_{R_2}\setminus B_{R_1})}) < \infty. \tag{2.11}$$

Define

$$W_n = U_n - V_n \quad \text{in } B_{R_2}\setminus B_{R_1}, \qquad \Phi_n = -i2^{-n}M\nabla V_n\cdot\eta \quad \text{on } \partial B_{R_1}.$$

Then

$$\operatorname{div}(M\nabla W_n) = g - h \quad \text{in } B_{R_2}\setminus B_{R_1}, \qquad W_n = 0 \text{ and } M\nabla W_n\cdot\eta = \Phi_n \quad \text{on } \partial B_{R_1}.$$

We claim that (W_n) is a Cauchy sequence in $H^1(B_{R_*}\setminus B_{R_1})$.

Indeed, set

$$w_n = W_{n+1} - W_n \quad \text{in } B_{R_2}\setminus B_{R_1}, \qquad \phi_n = \Phi_{n+1} - \Phi_n \quad \text{on } \partial B_{R_1}.$$

We have

$$\operatorname{div}(M\nabla w_n) = 0 \quad \text{in } B_{R_2}\setminus B_{R_1}, \qquad w_n = 0 \text{ and } \nabla w_n\cdot\eta = \phi_n \quad \text{on } \partial B_{R_1}.$$

[2] One can choose $\hat{M}$ as follows: $\hat{M}(x) = (2r/R_1 - 1)M(R_1\sigma) + (2-2r/R_1)I$ if $x\in B_{R_1}\setminus B_{R_1/2}$ and $\hat{M}(x) = I$ if $x\in B_{R_1/2}$, where $r = |x|$ and $\sigma = x/|x|$. In the case $M = I$ in $B_{R_2}\setminus B_{R_1}$, we choose $\hat{M} = I$ in B_{R_1}.

From (2.11), we derive that

$$\|w_n\|_{H^1(B_{R_2}\setminus B_{R_1})} \le Cm2^{n/2}, \quad \|\phi_n\|_{H^{1/2}(\partial B_{R_1})} \le Cm2^{-n/2}.$$

In this proof, C denotes a constant independent of n. Applying Lemma 2.3, we obtain

$$\|w_n\|_{L^2(\partial B_r)} \le C(\|\phi_n\|_{H^{-1/2}(\partial B_{R_1})}^{\alpha(r)}\|w_n\|_{L^2(\partial B_{R_2})}^{1-\alpha(r)} + \|\phi_n\|_{H^{-1/2}(\partial B_{R_1})}) \le Cm2^{-n\beta(r)},$$

where

$$\beta(r) = (2\alpha(r) - 1)/2.$$

From (2.9),

$$\beta(r) > (\alpha(R_*) - 1/2)/2 > 0 \quad \text{for } r \in (\gamma R_*, (2-\gamma)R_*).$$

Since $\operatorname{div}(M\nabla w_n) = 0$ in $B_{R_2} \setminus B_{R_1}$, by the regularity theory of elliptic equations,

$$\|w_n\|_{H^{1/2}(\partial B_{R_*})} \le Cm2^{-n(\alpha(R_*)-1/2)/2}.$$

Since $\operatorname{div}(M\nabla w_n) = 0$ in $B_{R_*} \setminus B_{R_1}$ and $w_n = 0$ on ∂B_{R_1}, it follows that

$$\|w_n\|_{H^1(B_{R_*}\setminus B_{R_1})} \le Cm2^{-n(\alpha(R_*)-1/2)/2}.$$

Hence (W_n) is a Cauchy sequence in $H^1(B_{R_*} \setminus B_{R_1})$. Let $W \in H^1(B_{R_*} \setminus B_{R_1})$ be its limit. Then

$$\operatorname{div}(M\nabla W) = g - h \quad \text{in } B_{R_*} \setminus B_{R_1}, \qquad W = 0 \text{ and } M\nabla W \cdot \eta = 0 \quad \text{on } \partial B_{R_1}.$$

This contradicts the non-existence of such a W. Hence (2.10) holds. □

2.2. *Proof of Theorem 1.2*

Set

$$u_{1,\delta} = u_\delta \circ F^{-1} \quad \text{in } B_{\hat{r}_3} \setminus B_{r_2}.$$

Since $F_*A = A$ in $B_{\hat{r}_3} \setminus B_{r_2}$ and $F(x) = x$ on ∂B_{r_2}, it follows from Lemma 2.1 that

$$\operatorname{div}(A\nabla u_{1,\delta}) = 0 \quad \text{in } B_{\hat{r}_3} \setminus B_{r_2},$$
$$u_\delta = u_{1,\delta} \text{ and } A\nabla u_\delta \cdot \eta = (1 - i\delta)A\nabla u_{1,\delta} \cdot \eta \quad \text{on } \partial B_{r_2}.$$

Recall that $\operatorname{div}(A\nabla u_\delta) = f$ in $B_{\hat{r}_3} \setminus B_{r_2}$. Applying Lemma 2.4 with $U_\delta = u_\delta$, $V_\delta = u_{1,\delta}$, $R_1 = r_2$, and $R_2 = \hat{r}_3$, we find that there exists a constant $r_* \in (r_2, r_3)$, independent of δ and f, such that if there is no solution $w \in H^1(B_{r_*} \setminus B_{r_2})$ to (1.17), then

$$\limsup_{\delta\to 0} \delta^{1/2}(\|u_\delta\|_{H^1(B_{\hat{r}_3}\setminus B_{r_2})} + \|u_{1,\delta}\|_{H^1(B_{\hat{r}_3}\setminus B_{r_2})}) = \infty.$$

This implies, by (1.4),

$$\limsup_{\delta\to 0} \delta^{1/2}\|\nabla u_\delta\|_{L^2(B_{r_2}\setminus B_{r_1})} = \infty.$$

In the case $A = I$ in $B_{\hat{r}_3} \setminus B_{r_2}$, by Lemma 2.4, r_* can be taken to be any number less than $\sqrt{\hat{r}_3 r_2}$. □

3. A condition on the boundedness of the power. Proof of Theorem 1.3

This section comprises two subsections. In the first subsection, we present two lemmas used in the proof of Theorem 1.3. The proof of Theorem 1.3 is given in the second subsection.

3.1. *Two useful lemmas*

The first lemma was established in [24, Lemma 1].

Lemma 3.1. *Let* $d = 2, 3$, $\delta \in (0, 1)$, *and* $f \in H^{-1}(\Omega)$ *and let* $u_\delta \in H^1_0(\Omega)$ *be the unique solution to*

$$\operatorname{div}(s_\delta A \nabla u_\delta) = f \quad \text{in } \Omega.$$

Then

$$\|u_\delta\|_{H^1(\Omega)} \le \frac{C}{\delta} \|f\|_{H^{-1}(\Omega)}$$

for some positive constant C *independent of* f *and* δ.

Here is the second lemma whose proof has roots in [24].

Lemma 3.2. *Let* $d = 2, 3$, $\delta \in (0, 1)$, *and let* $f \in L^2(\Omega)$, $g \in H^{1/2}(\partial B_{r_3})$, *and* $h \in H^{-1/2}(\partial B_{r_3})$. *Assume that* $s_0 A$ *is doubly complementary and* $\operatorname{supp} f \subset \Omega \setminus B_{r_3}$, *and let* $V_\delta \in H^1(\Omega \setminus \partial B_{r_3})$ *be the unique solution to*

$$\begin{cases} \operatorname{div}(s_\delta A \nabla V_\delta) = f & \text{in } \Omega \setminus \partial B_{r_3}, \\ [V_\delta] = g \text{ and } [A \nabla V_\delta \cdot \eta] = h & \text{on } \partial B_{r_3}, \\ V_\delta = 0 & \text{on } \partial \Omega. \end{cases}$$

Then

$$\|V_\delta\|_{H^1(\Omega \setminus \partial B_{r_3})} \le C(\|f\|_{L^2(\Omega)} + \|g\|_{H^{1/2}(\partial B_{r_3})} + \|h\|_{H^{-1/2}(\partial B_{r_3})})$$

for some positive constant C *independent of* δ, f, g, *and* h.

Remark 3.1. The case $g = h = 0$ was considered in [24, Theorem 1 and Corollary 1].

Proof of Lemma 3.2. Let $U \in H^1(\Omega \setminus \partial B_{r_3})$ be the unique solution to

$$\begin{cases} \operatorname{div}(\hat{A} \nabla U) = f & \text{in } \Omega \setminus \partial B_{r_3}, \\ [U] = g \text{ and } [\hat{A} \nabla U \cdot \eta] = h & \text{on } \partial B_{r_3}, \\ U = 0 & \text{on } \partial \Omega, \end{cases}$$

where $\hat{A}$ is defined in (1.16). Then

$$\|U\|_{H^1(\Omega \setminus \partial B_{r_3})} \le C(\|f\|_{L^2(\Omega)} + \|g\|_{H^{1/2}(\partial B_{r_3})} + \|h\|_{H^{-1/2}(\partial B_{r_3})}). \tag{3.1}$$

Define $V_0 \in H^1(\Omega \setminus \partial B_{r_3})$ as follows:

$$V_0 = \begin{cases} U & \text{in } \Omega \setminus B_{r_2}, \\ U \circ F & \text{in } B_{r_2} \setminus B_{r_1}, \\ U \circ G \circ F & \text{in } B_{r_1}. \end{cases} \tag{3.2}$$

Using (1.12) and applying Lemma 2.1, as in [24, Step 2 in Section 3.2.2] one can verify that $V_0 \in H^1(\Omega \setminus \partial B_{r_3})$ is a solution to

$$\begin{cases} \operatorname{div}(s_0 A \nabla V_0) = f & \text{in } \Omega \setminus \partial B_{r_3}, \\ [V_0] = g \text{ and } [A \nabla V_0 \cdot \eta] = h & \text{on } \partial B_{r_3}, \\ V_0 = 0 & \text{on } \partial \Omega. \end{cases}$$

Set

$$W_\delta = V_\delta - V_0 \quad \text{in } \Omega. \tag{3.3}$$

Then $W_\delta \in H_0^1(\Omega)$ is the unique solution to

$$\operatorname{div}(s_\delta A \nabla W_\delta) = -\operatorname{div}(i\delta A \nabla V_0 1_{B_{r_2} \setminus B_{r_1}}) \quad \text{in } \Omega.$$

Here and in what follows, for a subset D of $\mathbb{R}^d$, 1_D denotes the characteristic function of D. Applying Lemma 3.1, we have

$$\|W_\delta\|_{H^1(\Omega)} \le C \|V_0\|_{H^1(B_{r_2} \setminus B_{r_1})}. \tag{3.4}$$

The conclusion follows from (3.1)–(3.4). □

3.2. Proof of Theorem 1.3

Proof of (1.20). This is a consequence of Lemma 3.2 with $g = h = 0$.

Proof of (1.21). Without loss of generality, one might assume that $r_2 = 1$. As in [24], define

$$u_{1,\delta} = u_\delta \circ F^{-1} \quad \text{in } \mathbb{R}^d \setminus B_{r_3}, \qquad u_{2,\delta} = u_{1,\delta} \circ G^{-1} \quad \text{in } B_{r_3}.$$

Let $\phi \in H_0^1(B_{r_3} \setminus B_{r_2})$ be the unique solution to

$$\Delta \phi = f \quad \text{in } B_{r_3} \setminus B_{r_2}, \tag{3.5}$$

and set

$$W = w - \phi \quad \text{in } B_{r_0} \setminus B_{r_2}.$$

Then $W \in H^1(B_{r_0} \setminus B_{r_2})$ satisfies

$$\Delta W = 0 \quad \text{in } B_{r_0} \setminus B_{r_2}, \qquad W = 0 \text{ and } \partial_r W = -\partial_r \phi \quad \text{on } \partial B_{r_2}. \tag{3.6}$$

We now consider the cases $d = 2$ and $d = 3$ separately.

Case 1: $d = 2$. Since $r_2 = 1$ and $W = 0$ on ∂B_{r_2}, it follows that

$$W = g_0 \ln r + \sum_{\ell=1}^{\infty} \sum_{\pm} g_{\ell,\pm}(r^{\ell} - r^{-\ell}) e^{\pm i \ell \theta} \quad \text{in } B_{r_0} \setminus B_{r_2}, \tag{3.7}$$

for some $g_0, g_{\ell,\pm} \in \mathbb{C}$ $(\ell \geq 1)$. It is clear that, since $r_2 = 1 < r_0$,

$$\|W\|^2_{H^1(B_{r_0} \setminus B_{r_2})} \sim |g_0|^2 + \sum_{\ell=1}^{\infty} \sum_{\pm} \ell |g^2_{\ell,\pm}| r_0^{2\ell} < \infty. \tag{3.8}$$

One of the key points in the proof is the construction of $W_\delta \in H^1(B_{r_3} \setminus B_{r_2})$ which is defined as follows:

$$W_\delta = g_0 \ln r + \sum_{\ell=1}^{\infty} \sum_{\pm} \frac{g_{\ell,\pm}}{1 + \xi_\ell}(r^{\ell} - r^{-\ell}) e^{\pm i \ell \theta} \quad \text{in } B_{r_3} \setminus B_{r_2}, \tag{3.9}$$

where

$$\xi_\ell = \delta^{1/2} (r_3 / r_0)^{\ell} \quad \text{for } \ell \geq 1. \tag{3.10}$$

Roughly speaking, W_δ is the main part of the singularity of u_δ. From the definition of W_δ,

$$\Delta W_\delta = 0 \quad \text{in } B_{r_3} \setminus \overline{B_{r_2}}, \qquad W_\delta = 0 \quad \text{on } \partial B_{r_2}, \tag{3.11}$$

and

$$\|W_\delta\|^2_{H^1(B_{r_3} \setminus B_{r_2})} \sim |g_0|^2 + \sum_{\ell=1}^{\infty} \sum_{\pm} \frac{\ell |g_{\ell,\pm}|^2}{1 + \xi_\ell^2} r_3^{2\ell}. \tag{3.12}$$

By (3.10), if $\xi_\ell \leq 1$ then

$$\frac{\ell |g_{\ell,\pm}|^2}{1 + \xi_\ell^2} r_3^{2\ell} \leq \ell |g_{\ell,\pm}|^2 r_3^{2\ell} \leq \delta^{-1} \ell |g_{\ell,\pm}|^2 r_0^{2\ell}, \tag{3.13}$$

and if $\xi_\ell \geq 1$ then

$$\frac{\ell |g_{\ell,\pm}|^2}{1 + \xi_\ell^2} r_3^{2\ell} \leq \ell |g_{\ell,\pm}|^2 r_3^{2\ell} \xi_\ell^{-2} = \delta^{-1} \ell |g_{\ell,\pm}|^2 r_0^{2\ell}. \tag{3.14}$$

A combination of (3.8), (3.12), (3.13), and (3.14) yields

$$\|W_\delta\|_{H^1(B_{r_3} \setminus B_{r_2})} \leq C \delta^{-1/2}. \tag{3.15}$$

Let $W_{1,\delta} \in H^1(\Omega)$ be the unique solution to

$$\begin{cases} \operatorname{div}(s_\delta A \nabla W_{1,\delta}) = 0 & \text{in } \Omega \setminus \partial B_{r_2}, \\ [s_\delta A \nabla W_{1,\delta} \cdot \eta] = (-1 + i\delta) h_\delta & \text{on } \partial B_{r_2}, \\ W_{1,\delta} = 0 & \text{on } \partial \Omega, \end{cases}$$

where

$$h_\delta = -\partial_r(\phi + W_\delta) \quad \text{on } \partial B_{r_2},$$

and let $W_{2,\delta} \in H^1(\Omega \setminus \partial B_{r_3})$ be the unique solution to

$$\begin{cases} \operatorname{div}(s_\delta A \nabla W_{2,\delta}) = f 1_{\Omega \setminus B_{r_3}} & \text{in } \Omega \setminus \partial B_{r_3}, \\ [W_{2,\delta}] = \phi + W_\delta \text{ and } [A\nabla W_{2,\delta} \cdot \eta] = \partial_r \phi + \partial_r W_\delta & \text{on } \partial B_{r_3}, \\ W_{2,\delta} = 0 & \text{on } \partial\Omega. \end{cases}$$

Recall that, for a subset D of $\mathbb{R}^d$, 1_D denotes the characteristic function of D. From (3.5), (3.11), and the fact $A = I$ in $B_{r_3} \setminus B_{r_2}$, we have

$$u_\delta - (\phi + W_\delta) 1_{B_{r_3} \setminus B_{r_2}} = W_{1,\delta} + W_{2,\delta} \quad \text{in } \Omega. \tag{3.16}$$

Using (3.6), (3.7), and (3.9), we obtain

$$h_\delta = -\partial_r(\phi + W_\delta) = \partial_r(W - W_\delta) = \partial_r \left(\sum_{\ell=1}^{\infty} \sum_{\pm} \frac{\xi_\ell g_{\ell,\pm}}{1+\xi_\ell} (r^\ell - r^{-\ell}) e^{\pm i \ell \theta} \right) \quad \text{on } \partial B_{r_2}.$$

Since $r_2 = 1$, it follows that

$$\|h_\delta\|^2_{H^{-1/2}(\partial B_{r_2})} \sim \sum_{\ell=1}^{\infty} \sum_{\pm} \frac{\ell |\xi_\ell|^2 |g_{\ell,\pm}|^2}{1 + |\xi_\ell|^2}. \tag{3.17}$$

By (3.10), if $\xi_\ell \le 1$ then

$$\frac{\ell|\xi_\ell|^2}{1+|\xi_\ell|^2} |g_{\ell,\pm}|^2 \le \delta \ell |g_{\ell,\pm}|^2 (r_3/r_0)^{2\ell} = \delta \ell |g_{\ell,\pm}|^2 r_0^{2\ell} (r_3/r_0^2)^{2\ell} \le \delta \ell |g_{\ell,\pm}|^2 r_0^{2\ell}, \tag{3.18}$$

since $r_0 > \sqrt{r_2 r_3} = \sqrt{r_3}$, and if $\xi_\ell \ge 1$ then

$$\frac{\ell|\xi_\ell|^2}{1+|\xi_\ell|^2} |g_{\ell,\pm}|^2 \le \ell |g_{\ell,\pm}|^2 = \ell |g_{\ell,\pm}|^2 r_0^{2\ell} r_0^{-2\ell} \le \delta \ell |g_{\ell,\pm}|^2 r_0^{2\ell}, \tag{3.19}$$

since $\delta^{1/2} r_0^\ell > \delta^{1/2} (r_3/r_0)^\ell \ge 1$. A combination of (3.17)–(3.19) yields

$$\|h_\delta\|_{H^{-1/2}(\partial B_{r_2})} \le C \delta^{1/2} \|W\|_{H^{1/2}(\partial B_{r_0})} \le C \delta^{1/2}.$$

Applying Lemma 3.1, we have

$$\|W_{1,\delta}\|_{H^1(\Omega)} \le (C/\delta) \delta^{1/2} = C \delta^{-1/2}. \tag{3.20}$$

On the other hand, from (3.15) and Lemma 3.2, we obtain

$$\|W_{2,\delta}\|_{H^1(\Omega \setminus \partial B_{r_3})} \le C \delta^{-1/2}. \tag{3.21}$$

The conclusion in the case $d = 2$ now follows from (3.15), (3.16), (3.20), and (3.21).

Case 2: $d = 3$. Since $r_2 = 1$ and $W = 0$ on ∂B_{r_2}, it follows that

$$W = g_0 + \frac{\hat{g}_0}{r} + \sum_{\ell=1}^{\infty} \sum_{k=-\ell}^{\ell} g_{\ell,k}(r^{\ell} - r^{-\ell-1}) Y_{\ell}^{k}(x/|x|) \quad \text{in } B_{r_0} \setminus B_{r_2},$$

for some $g_0, \hat{g}_0, g_{\ell,k} \in \mathbb{C}$. Here Y_ℓ^k is the spherical harmonic function of degree ℓ and of order k. Define $W_\delta \in H^1(B_{r_3} \setminus B_{r_2})$ as follows:

$$W_\delta = g_0 + \frac{\hat{g}_0}{r} + \sum_{\ell=1}^{\infty} \sum_{k=-\ell}^{\ell} \frac{g_{\ell,k}}{1+\xi_\ell}(r^{\ell} - r^{-\ell-1}) Y_{\ell}^{k}(x/|x|) \quad \text{in } B_{r_3} \setminus B_{r_2},$$

where $\xi_\ell = \delta^{1/2}(r_3/r_0)^\ell$ for $\ell \geq 1$. The proof now follows as in the two-dimensional case. The details are left to the reader. □

4. A connection between the blow up of the power and CALR. Proof of Theorem 1.1

We establish a stronger result than Theorem 1.1:

Proposition 4.1. *Let $d = 2, 3$, let $\delta_n \to 0$, $(g_n) \subset L^2(\Omega)$ with* supp $g_n \subset \Omega \setminus B_{r_2}$, *and let $v_n \in H_0^1(\Omega)$ be the unique solution to*

$$\operatorname{div}(s_{\delta_n} A \nabla v_n) = g_n \quad \text{in } \Omega.$$

Assume that $s_0 A$ is doubly complementary. Suppose that $g_n \to g$ weakly in $L^2(\Omega)$ for some $g \in L^2(\Omega)$, and

$$\lim_{n\to\infty} \delta_n \|\nabla v_n\|_{L^2(B_{r_2}\setminus B_{r_1})} = 0. \tag{4.1}$$

Then $v_n \to v$ weakly in $H^1(\Omega \setminus B_{r_3})$, where $v \in H_0^1(\Omega)$ is the unique solution to

$$\operatorname{div}(\hat{A} \nabla v) = g \quad \text{in } \Omega.$$

Granting Proposition 4.1, we first give

Proof of Theorem 1.1. (i) Since $\delta_n \|\nabla v_{\delta_n}\|^2_{L^2(B_{r_2}\setminus B_{r_1})} = 1$, it follows from (1.4) that

$$\lim_{n\to\infty} \delta_n \|\nabla v_{\delta_n}\|_{L^2(B_{r_2}\setminus B_{r_1})} = 0.$$

On the other hand, since $\lim_{n\to\infty} \delta_n \|\nabla u_{\delta_n}\|^2_{L^2(B_{r_2}\setminus B_{r_1})} = \infty$, we have

$$\lim_{n\to\infty} \|f_{\delta_n}\|_{L^2(\Omega)} = 0.$$

The conclusion now follows from Proposition 4.1.

(ii) Since $(\delta_n \|\nabla u_{\delta_n}\|^2_{L^2(B_{r_2}\setminus B_{r_1})})$ is bounded, it follows from (1.4) that

$$\lim_{n\to\infty} \delta_n \|\nabla u_{\delta_n}\|_{L^2(B_{r_2}\setminus B_{r_1})} = 0.$$

The conclusion follows again from Proposition 4.1. □

The rest of this section comprising three subsections is devoted to the proof of Proposition 4.1. In the first subsection, we present the proof in the case $A = I$ in $B_{r_3} \setminus B_{r_2}$. This situation is already non-trivial since A can be arbitrarily uniformly elliptic outside B_{r_3}; the standard separation of variables cannot be applied. Taking this simple but representative setting, we present the ideas of the proof of Proposition 4.1. The proof essentially uses the reflecting and removing localized singularity techniques introduced in [24–26]. The way to remove localized singularities in this context will lead us to develop a separation of variables technique for solving Cauchy problems in a general shell in Section 4.2. In Section 4.3, we give the proof of Proposition 4.1 in the form stated. To this end, we follow the strategy presented in Section 4.1 and make essential use of the results of Section 4.2. Due to the lack of the orthogonality of plasmon modes, the analysis is more delicate.

4.1. Proof of Proposition 4.1 in the case $A = I$ in $B_{r_3} \setminus B_{r_2}$

Without loss of generality, we may assume that $r_3 = 1$. Using (1.4), we derive from (4.1) that

$$\lim_{n\to\infty} \delta_n \|v_n\|_{H^1(\Omega)} = 0. \tag{4.2}$$

We now consider the cases $d = 2$ and $d = 3$ separately.

Case 1: $d = 2$. Define

$$v_{1,n} = v_n \circ F^{-1} \quad \text{in } \mathbb{R}^d \setminus B_{r_2}, \qquad v_{2,n} = v_{1,n} \circ G^{-1} \quad \text{in } B_{r_3}.$$

It follows from (1.12) and Lemma 2.1 that

$$\operatorname{div}(A\nabla v_{1,n}) = \operatorname{div}(A\nabla v_{2,n}) = 0 \quad \text{in } B_{r_3} \setminus B_{r_2}.$$

Since $A = I$ in $B_{r_3} \setminus B_{r_2}$, one can represent $v_{1,n}$ and $v_{2,n}$ in $B_{r_3} \setminus B_{r_2}$ as follows:

$$v_{1,n} = c_0 + d_0 \ln r + \sum_{\ell=1}^{\infty} \sum_{\pm} (c_{\ell,\pm} r^\ell + d_{\ell,\pm} r^{-\ell}) e^{\pm i\ell\theta}, \tag{4.3}$$

$$v_{2,n} = e_0 + f_0 \ln r + \sum_{\ell=1}^{\infty} \sum_{\pm} (e_{\ell,\pm} r^\ell + f_{\ell,\pm} r^{-\ell}) e^{\pm i\ell\theta}, \tag{4.4}$$

for some $c_0, d_0, e_0, f_0, c_{\ell,\pm}, d_{\ell,\pm}, e_{\ell,\pm}, f_{\ell,\pm} \in \mathbb{C}$ $(\ell \geq 1)$. By Lemma 2.1, we have

$$v_{1,n} = v_{2,n} \text{ and } \partial_r v_{1,n} = \frac{1}{1 - i\delta_n} \partial_r v_{2,n} \quad \text{on } \partial B_{r_3}.$$

Since $r_3 = 1$, it follows that

$$c_{\ell,\pm} + d_{\ell,\pm} = e_{\ell,\pm} + f_{\ell,\pm}, \qquad c_{\ell,\pm} - d_{\ell,\pm} = \frac{1}{1 - i\delta_n}(e_{\ell,\pm} - f_{\ell,\pm}) \quad \text{for } \ell \geq 1,$$

$$c_0 = e_0, \qquad d_0 = \frac{1}{1 - i\delta_n} f_0.$$

This implies, for $\ell \geq 1$,

$$c_{\ell,\pm} = \frac{2 - i\delta_n}{2(1 - i\delta_n)} e_{\ell,\pm} - \frac{i\delta_n}{2(1 - i\delta_n)} f_{\ell,\pm}, \qquad d_{\ell,\pm} = \frac{2 - i\delta_n}{2(1 - i\delta_n)} f_{\ell,\pm} - \frac{i\delta_n}{2(1 - i\delta_n)} e_{\ell,\pm}.$$

We derive from (4.3) and (4.4) that

$$\begin{aligned} v_{1,n} - v_{2,n} = & \frac{i\delta_n}{1 - i\delta_n} f_0 \ln r \\ & + \frac{i\delta_n}{2(1 - i\delta_n)} \sum_{\ell=1}^{\infty} \sum_{\pm} (e_{\ell,\pm} - f_{\ell,\pm})(r^{\ell} - r^{-\ell}) e^{\pm i\ell\theta} \quad \text{in } B_{r_3} \setminus B_{r_2}. \end{aligned} \tag{4.5}$$

It follows from (4.2) that

$$\lim_{n\to\infty} \delta_n^2 (\|v_{2,n}\|^2_{H^{1/2}(\partial B_{r_3})} + \|\partial_r v_{2,n}\|^2_{H^{-1/2}(\partial B_{r_3})}) = 0,$$
$$\lim_{n\to\infty} \delta_n^2 (\|v_{2,n}\|^2_{H^{1/2}(\partial B_{r_2})} + \|\partial_r v_{2,n}\|^2_{H^{-1/2}(\partial B_{r_2})}) = 0.$$

Using (4.4), we obtain

$$\lim_{n\to\infty} \delta_n^2 \Big(|e_0|^2 + \sum_{\ell=1}^{\infty} \sum_{\pm} \ell |e_{\ell,\pm}|^2 r_3^{2\ell} + |f_0|^2 + \sum_{\ell=1}^{\infty} \sum_{\pm} \ell |f_{\ell,\pm}|^2 r_3^{-2\ell} \Big) = 0, \tag{4.6}$$

$$\lim_{n\to\infty} \delta_n^2 \Big(|e_0|^2 + \sum_{\ell=1}^{\infty} \sum_{\pm} \ell |e_{\ell,\pm}|^2 r_2^{2\ell} + |f_0|^2 + \sum_{\ell=1}^{\infty} \sum_{\pm} \ell |f_{\ell,\pm}|^2 r_2^{-2\ell} \Big) = 0. \tag{4.7}$$

We now use the removing localized singularity technique. Set

$$\hat{v}_n = -\frac{i\delta_n}{1 - i\delta_n} f_0 \ln r - \frac{i\delta_n}{2(1 - i\delta_n)} \sum_{\ell=1}^{\infty} \sum_{\pm} (e_{\ell,\pm} - f_{\ell,\pm}) r^{-\ell} e^{\pm i\ell\theta} \quad \text{in } B_{r_3} \setminus B_{r_2}, \tag{4.8}$$

and define V_n in Ω as follows:

$$V_n = \begin{cases} v_n & \text{in } \Omega \setminus B_{r_3}, \\ v_n - \hat{v}_n & \text{in } B_{r_3} \setminus B_{r_2}, \\ v_{2,n} & \text{in } B_{r_2}. \end{cases} \tag{4.9}$$

Since $A = F_*A = G_*F_*A = I$ in $B_{r_3} \setminus B_{r_2}$, we have, by Lemma 2.1,

$$\operatorname{div}(\hat{A} \nabla V_n) = g_n \quad \text{in } \Omega \setminus (\partial B_{r_2} \cup \partial B_{r_3}), \tag{4.10}$$

where $\hat{A}$ is defined in (1.16).

We claim that

$$\|[V_n]\|_{H^{1/2}(\partial B_{r_3})} + \|[\hat{A} \nabla V_n \cdot \eta]\|_{H^{-1/2}(\partial B_{r_3})} = o(1), \tag{4.11}$$

$$\|[V_n]\|_{H^{1/2}(\partial B_{r_2})} + \|[\hat{A} \nabla V_n \cdot \eta]\|_{H^{-1/2}(\partial B_{r_2})} = o(1). \tag{4.12}$$

Here and in what follows, $o(1)$ denotes a quantity converging to 0 as $n \to \infty$.

Granting the claim, we continue the proof. Combining (4.10)–(4.12) and using the fact that $V_n = 0$ on $\partial\Omega$ and $g_n \to g$ weakly in $L^2(\Omega)$, we obtain

$$V_n \to v \quad \text{weakly in } H^1(\Omega \setminus (\partial B_{r_3} \cup \partial B_{r_2})),$$

by the definition of v. The conclusion follows since $v_n = V_n$ in $\Omega \setminus B_{r_3}$.

It remains to prove the claim.

Proof of (4.11). Since $r_3 = 1$, we have, on ∂B_{r_3},

$$[V_n] = \hat{v}_n = -\sum_{\ell=1}^{\infty}\sum_{\pm} \frac{i\delta_n}{2(1 - i\delta_n)}(e_{\ell,\pm} - f_{\ell,\pm}) r_3^{-\ell} e^{\pm i\ell\theta}.$$

Since $r_3 = 1$, it follows from (4.6) and (4.7) that

$$\|[V_n]\|_{H^{1/2}(\partial B_{r_3})} = o(1). \tag{4.13}$$

Similarly,

$$\|[\hat{A}\nabla V_n \cdot \eta]\|_{H^{-1/2}(\partial B_{r_3})} = o(1). \tag{4.14}$$

Claim (4.11) is now a consequence of (4.13) and (4.14).

Proof of (4.12). We have

$$[V_n] = v_n - \hat{v}_n - v_{2,n} \quad \text{on } \partial B_{r_2}.$$

This implies, since $v_n = v_{1,n}$ on ∂B_{r_2},

$$[V_n] = v_{1,n} - v_{2,n} - \hat{v}_n \quad \text{on } \partial B_{r_2}.$$

It follows from (4.5) and (4.8) that

$$\|[V_n]\|_{H^{1/2}(\partial B_{r_2})} \le \left\| \frac{i\delta_n}{2(1 - i\delta_n)} \sum_{\ell=1}^{\infty}\sum_{\pm}(e_{\ell,\pm} - f_{\ell,\pm}) r^{\ell} e^{\pm i\ell\theta} \right\|_{H^{1/2}(\partial B_{r_2})}.$$

Since $r_3 = 1$, we derive from (4.6) and (4.7) that

$$\|[V_n]\|_{H^{1/2}(\partial B_{r_2})} = o(1). \tag{4.15}$$

Similarly, from the fact that $\partial_r v_n = (1 - i\delta_n)\partial_r v_{1,n}$ and $\lim_{n\to\infty} \delta_n \|v_n\|_{H^1(\Omega)} = 0$,

$$\|[\hat{A}\nabla V_n \cdot \eta]\|_{H^{-1/2}(\partial B_{r_2})} = o(1). \tag{4.16}$$

A combination of (4.15) and (4.16) yields (4.12).

Case 2: $d = 3$. The proof is similar to the one in the two-dimensional case. We just note that, in three dimensions, $v_{1,n}$ and $v_{2,n}$ can be represented in $B_{r_3} \setminus B_{r_2}$ as follows:

$$v_{1,n} = c_{0,0} + \frac{d_{0,0}}{r} + \sum_{\ell=1}^{\infty}\sum_{k=-\ell}^{\ell}(c_{\ell,k} r^{\ell} + d_{\ell,k} r^{-\ell-1}) Y_{\ell}^{k}(x/|x|),$$

$$v_{2,n} = e_{0,0} + \frac{f_{0,0}}{r} + \sum_{\ell=1}^{\infty}\sum_{k=-\ell}^{\ell}(e_{\ell,k} r^{\ell} + f_{\ell,k} r^{-\ell-1}) Y_{\ell}^{k}(x/|x|),$$

for some $c_{\ell,k}, d_{\ell,k}, e_{\ell,k}, f_{\ell,k} \in \mathbb{C}$. □

4.2. Separation of variables for Cauchy problems in a general shell

In this section, we state variants of (4.3) and (4.4) for a general core-shell structure, i.e., A is not required to be I in $B_{r_3} \setminus B_{r_2}$. Using these variants, we will extend the method used in Section 4.1 to a general core-shell structure in Section 4.3. We have

Proposition 4.2. *Let $d = 2, 3$, $0 < R_1 < R_2$, and let $a \in [C^3(\overline{B_{R_2} \setminus B_{R_1}})]^{d\times d}$ be symmetric and uniformly elliptic. Set $R_3 = R_2^2/R_1$ and let $K : B_{R_2} \setminus B_{R_1} \to B_{R_3} \setminus B_{R_2}$ be the Kelvin transform with respect to ∂B_{R_2}, i.e., $K(x) = xR_2^2/|x|^2$. Define*

$$a_1 = \begin{cases} K_* a & \text{in } B_{R_3} \setminus B_{R_2}, \\ a & \text{in } B_{R_2} \setminus B_{R_1}, \\ I & \text{in } B_{R_1}. \end{cases} \tag{4.17}$$

Let $v_\ell \in H^1(B_{R_3})$ $(\ell \geq 1)$ be a solution to

$$\operatorname{div}(a_1 \nabla v_\ell) = 0 \quad \text{in } B_{R_3},$$

and set $v_0 = 1$ in B_{R_3}. Let $w_\ell \in H^1(B_{R_2} \setminus B_{R_1})$ $(\ell \geq 1)$ be the reflection of v_ℓ through ∂B_{R_2} by K^{-1}, i.e.,

$$w_\ell = v_\ell \circ K \quad \text{in } B_{R_2} \setminus B_{R_1},$$

and denote by $w_0 \in H^1(B_{R_3} \setminus B_{R_2})$ the unique solution to

$$\operatorname{div}(a \nabla w_0) = 0 \quad \text{in } B_{R_2} \setminus B_{R_1}, \qquad w_0 = 1 \quad \text{on } \partial B_{R_2}, \qquad w_0 = 0 \quad \text{on } \partial B_{R_1}.$$

Then, for $\ell \geq 1$,

$$\operatorname{div}(a \nabla w_\ell) = \operatorname{div}(a \nabla v_\ell) = 0 \quad \text{in } B_{R_2} \setminus B_{R_1}, \tag{4.18}$$

$$w_\ell = v_\ell \text{ and } a \nabla w_\ell \cdot \frac{x}{|x|} = -a \nabla v_\ell \cdot \frac{x}{|x|} \quad \text{on } \partial B_{R_2}. \tag{4.19}$$

Assume that $\{v_\ell\}_{\ell=0}^\infty$ is dense in $H^{1/2}(\partial B_{R_3})$. Then, in the $H^1(B_{R_2} \setminus B_{R_1})$-norm:

1. *$\{v_\ell - w_\ell;\ \ell \geq 0\}$ is dense in $\{v \in H^1(B_{R_2} \setminus B_{R_1});\ \operatorname{div}(a\nabla v) = 0$ and $v = 0$ on $\partial B_{R_2}\}$.*
2. *$\{1\} \cup \{v_\ell + w_\ell;\ \ell \geq 1\}$ is dense in $\{v \in H^1(B_{R_2} \setminus B_{R_1});\ \operatorname{div}(a\nabla v) = 0$ and $a\nabla v \cdot \eta = 0$ on $\partial B_{R_2}\}$.*
3. *$\{v_\ell, w_\ell;\ \ell \geq 0\}$ is dense in $\{v \in H^1(B_{R_2} \setminus B_{R_1});\ \operatorname{div}(a\nabla v) = 0\}$.*

The proof of Proposition 4.2 is given in the appendix.

The existence of v_ℓ and w_ℓ, their density properties, and (4.18) and (4.19) can be considered as the existence of surface plasmons for complementary media, a fact which can be used elsewhere; see e.g. [11, 12, 19] for discussions on surface plasmons and their applications. The choice of a_1 is to ensure such properties.

4.3. Proof of Proposition 4.1

Using (1.4), we derive from (4.1) that

$$\lim_{n\to\infty} \delta_n \|v_n\|_{H^1(\Omega)} = 0. \tag{4.20}$$

Define

$$v_{1,n} = v_n \circ F^{-1} \quad \text{in } B_{r_4} \setminus B_{r_3}, \qquad v_{2,n} = v_{1,n} \circ G^{-1} \quad \text{in } B_{r_3}.$$

Using (1.12) and applying Lemma 2.1, we obtain

$$\operatorname{div}(A\nabla v_{1,n}) = \operatorname{div}(A\nabla v_{2,n}) = 0 \quad \text{in } B_{r_3} \setminus B_{r_2},$$
$$v_{1,n} = v_{2,n} \text{ and } A\nabla v_{1,n} \cdot \eta = \frac{1}{1 - i\delta_n} A\nabla v_{2,n} \cdot \eta \quad \text{on } \partial B_{r_3}.$$

Set $\hat{r} = r_3^2/r_2$ and let $K : B_{r_3} \setminus B_{r_2} \to B_{\hat{r}} \setminus B_{r_3}$ be the Kelvin transform with respect to ∂B_{r_3}. Define

$$A_1 = \begin{cases} K_*A & \text{in } B_{\hat{r}} \setminus B_{r_3}, \\ A & \text{in } B_{r_3} \setminus B_{r_2}, \\ I & \text{in } B_{r_2}. \end{cases} \tag{4.21}$$

Let $v_\ell \in H^1(B_{\hat{r}})$ $(\ell \ge 1)$ be a solution to $\operatorname{div}(A_1\nabla v_\ell) = 0$ in $B_{\hat{r}}$, and set $v_0 = 1$ in $B_{\hat{r}}$. Define $w_\ell \in H^1(B_{r_3} \setminus B_{r_2})$ $(\ell \ge 1)$ the reflection of v_ℓ through ∂B_{r_3} by K^{-1}, i.e.,

$$w_\ell = v_\ell \circ K \quad \text{in } B_{r_3} \setminus B_{r_2}, \tag{4.22}$$

and denote $w_0 \in H^1(B_{r_3} \setminus B_{r_2})$ the unique solution to

$$\operatorname{div}(A\nabla w_0) = 0 \quad \text{in } B_{r_3} \setminus B_{r_2}, \qquad w_0 = 1 \quad \text{on } \partial B_{r_3}, \qquad w_0 = 0 \quad \text{on } \partial B_{r_2}.$$

We assume in addition that $\{v_\ell\}_{\ell=0}^\infty$ is an orthogonal basis of $H^{1/2}(\partial B_{\hat{r}})$. In particular,

$$\int_{\partial B_{\hat{r}}} v_\ell = 0 \quad \text{for } \ell \ge 1.\text{[3]} \tag{4.23}$$

For $m \ge 0$, let P_m be the projection from $H^1(B_{r_3} \setminus B_{r_2})$ to $\operatorname{span}\{v_\ell, w_\ell;\ 0 \le \ell \le m\}$ with respect to the $H^1(B_{r_3} \setminus B_{r_2})$-norm. By Proposition 4.2, there exists m such that

$$\|v_{1,n} - P_m v_{1,n}\|_{H^1(B_{r_3}\setminus B_{r_2})} + \|v_{2,n} - P_m v_{2,n}\|_{H^1(B_{r_3}\setminus B_{r_2})} \le \delta_n^2. \tag{4.24}$$

We have, in $B_{r_3} \setminus B_{r_2}$,

$$P_m v_{1,n} = \sum_{\ell=0}^m (c_\ell v_\ell + d_\ell w_\ell), \tag{4.25}$$

$$P_m v_{2,n} = \sum_{\ell=0}^m (e_\ell v_\ell + f_\ell w_\ell), \tag{4.26}$$

[3] In the case $d = 2$ and $r_3 = 1$, v_ℓ and w_ℓ can be seen as a replacement of $r^\ell e^{\pm i\ell\theta}$ and $r^{-\ell} e^{\pm i\ell\theta}$ respectively.

for some $c_\ell, d_\ell, e_\ell, f_\ell \in \mathbb{C}$ $(0 \le \ell \le m)$. Define $(D_\ell)_0^m, (N_\ell)_0^m \subset \mathbb{C}$ as follows:

$$c_\ell + d_\ell = e_\ell + f_\ell + D_\ell, \quad c_\ell - d_\ell = \frac{1}{1 - i\delta_n}(e_\ell - f_\ell) + N_\ell \quad \text{for } 1 \le \ell \le m, \tag{4.27}$$

$$c_0 + d_0 = e_0 + f_0 + D_0, \quad d_0 = \frac{1}{1 - i\delta_n} f_0 + N_0. \tag{4.28}$$

It follows from (4.19) that on ∂B_{r_3},

$$P_m v_{1,n} - P_m v_{2,n} = \sum_{\ell=0}^{m} D_\ell v_\ell, \tag{4.29}$$

$$a\nabla P_m v_{1,n} \cdot \eta - \frac{1}{1 - i\delta_n} a\nabla P_m v_{2,n} \cdot \eta = N_0\, a\nabla w_0 \cdot \eta + \sum_{\ell=1}^{m} N_\ell\, a\nabla v_\ell \cdot \eta. \tag{4.30}$$

From (4.27) and (4.28), we have, for $1 \le \ell \le m$,

$$\begin{aligned} c_\ell &= \frac{2 - i\delta_n}{2(1 - i\delta_n)} e_\ell - \frac{i\delta_n}{2(1 - i\delta_n)} f_\ell + \frac{D_\ell + N_\ell}{2}, \\ d_\ell &= \frac{2 - i\delta_n}{2(1 - i\delta_n)} f_\ell - \frac{i\delta_n}{2(1 - i\delta_n)} e_\ell + \frac{D_\ell - N_\ell}{2}, \end{aligned}$$

and

$$c_0 = e_0 - \frac{i\delta_n}{1 - i\delta_n} f_0 + D_0 - N_0, \quad d_0 = \frac{1}{1 - i\delta_n} f_0 + N_0.$$

We derive from (4.25) and (4.26) that

$$\begin{aligned} P_m v_{1,n} - P_m v_{2,n} = {} & \frac{i\delta_n}{2(1 - i\delta_n)} \sum_{\ell=1}^{m} (e_\ell - f_\ell)(v_\ell - w_\ell) + \sum_{\ell=1}^{m} \left(\frac{D_\ell + N_\ell}{2} v_\ell + \frac{D_\ell - N_\ell}{2} w_\ell \right) \\ & + \left(-\frac{i\delta_n}{1 - i\delta_n} f_0 + D_0 - N_0 \right) + \left(\frac{i\delta_n}{1 - i\delta_n} f_0 + N_0 \right) w_0. \end{aligned} \tag{4.31}$$

From (4.20) and (4.24), we have

$$\| P_m v_{2,n} \|_{H^{1/2}(\partial B_{r_3})} = \delta_n^{-1} o(1), \quad \| P_m v_{2,n} \|_{H^{1/2}(\partial B_{r_2})} = \delta_n^{-1} o(1). \tag{4.32}$$

Since $v_\ell = w_\ell$ on ∂B_{r_3} for $\ell \ge 1$, it follows from (4.26) and (4.32) that

$$\left\| \sum_{\ell=0}^{m} (e_\ell + f_\ell) v_\ell \right\|_{H^{1/2}(\partial B_{r_3})} = \delta_n^{-1} o(1), \tag{4.33}$$

$$\begin{aligned} \left\| \sum_{\ell=0}^{m} (e_\ell v_\ell + f_\ell w_\ell) \right\|_{H^{1/2}(\partial B_{r_2})} &= \left\| \sum_{\ell=0}^{m} (e_\ell + f_\ell) v_\ell + \sum_{\ell=0}^{m} f_\ell (w_\ell - v_\ell) \right\|_{H^{1/2}(\partial B_{r_2})} \\ &= \delta_n^{-1} o(1). \end{aligned} \tag{4.34}$$

Since, for $\ell \ge 0$, we have $\operatorname{div}(A_1 \nabla v_\ell) = 0$ in B_{r_3}, it follows that

$$\Big\| \sum_{\ell=0}^{m} (e_\ell + f_\ell) v_\ell \Big\|_{H^{1/2}(\partial B_{r_2})} \le C \Big\| \sum_{\ell=0}^{m} (e_\ell + f_\ell) v_\ell \Big\|_{H^{1/2}(\partial B_{r_3})}. \tag{4.35}$$

Here and in what follows in this proof, C denotes a positive constant independent of δ_n, u_n, g_n, and ℓ. A combination of (4.33)–(4.35) yields

$$\Big\| \sum_{\ell=0}^{m} f_\ell (w_\ell - v_\ell) \Big\|_{H^{1/2}(\partial B_{r_2})} = \delta_n^{-1} o(1). \tag{4.36}$$

Using (4.23) and applying Lemma 4.1 below with $v = -\sum_{\ell \ge 1}^{m} f_\ell v_\ell$, $c = f_0$, $R_1 = r_2$, and $R_2 = r_3$, we deduce from (4.36) that

$$|f_0| + \Big\| \sum_{\ell=1}^{m} f_\ell v_\ell \Big\|_{H^{1/2}(\partial B_{r_3})} + \Big\| \sum_{\ell=0}^{m} f_\ell w_\ell \Big\|_{H^{1/2}(\partial B_{r_2})} = \delta_n^{-1} o(1). \tag{4.37}$$

We also use here the fact that $w_0 = 0$ on ∂B_{r_2}. This implies, by (4.33),

$$\Big\| \sum_{\ell=0}^{m} e_\ell v_\ell \Big\|_{H^{1/2}(\partial B_{r_3})} = \delta_n^{-1} o(1). \tag{4.38}$$

From (4.37) and (4.38), we obtain

$$\Big\| \sum_{\ell=0}^{m} e_\ell v_\ell \Big\|_{H^{1/2}(\partial B_{r_3})} + |f_0| + \Big\| \sum_{\ell=0}^{m} f_\ell w_\ell \Big\|_{H^{1/2}(\partial B_{r_2})} = \delta_n^{-1} o(1). \tag{4.39}$$

Since $\operatorname{div}(A_1 \nabla v_\ell) = 0$ in $B_{\hat r}$ for $\ell \ge 1$, $v_0 = 1$, and $A_1 = A$ in $B_{r_3} \setminus B_{r_2}$,

$$\Big\| \sum_{\ell=1}^{m} e_\ell A \nabla v_\ell \cdot \eta \Big\|_{H^{-1/2}(\partial B_{r_3})} \le C \Big\| \sum_{\ell=0}^{m} e_\ell v_\ell \Big\|_{H^{1/2}(\partial B_{r_3})}. \tag{4.40}$$

From (4.21) and (4.22), we have

$$\begin{aligned} \Big\| \sum_{\ell=1}^{m} f_\ell A \nabla w_\ell \cdot \eta \Big\|_{H^{-1/2}(\partial B_{r_2})} &\le C \Big\| \sum_{\ell=1}^{m} f_\ell A_1 \nabla v_\ell \cdot \eta \Big\|_{H^{-1/2}(\partial B_{\hat r})} \\ &\le C \Big\| \sum_{\ell=1}^{m} f_\ell v_\ell \Big\|_{H^{1/2}(\partial B_{\hat r})} \le C \Big\| \sum_{\ell=1}^{m} f_\ell w_\ell \Big\|_{H^{1/2}(\partial B_{r_2})}. \end{aligned} \tag{4.41}$$

Recall that $w_0 = 0$ on ∂B_{r_2}. A combination of (4.39)–(4.41) yields

$$\Big\| \sum_{\ell=1}^{m} e_\ell A \nabla v_\ell \cdot \eta \Big\|_{H^{-1/2}(\partial B_{r_3})} + \Big\| \sum_{\ell=1}^{m} f_\ell A \nabla w_\ell \cdot \eta \Big\|_{H^{-1/2}(\partial B_{r_2})} = \delta_n^{-1} o(1). \tag{4.42}$$

We are ready to remove localized singularities. Set, in $B_{r_3} \setminus B_{r_2}$,

$$\hat{v}_n = -\sum_{\ell=1}^{m} \frac{i\delta_n}{2(1-i\delta_n)}(e_\ell - f_\ell)w_\ell + \sum_{\ell=1}^{m}\left(\frac{D_\ell + N_\ell}{2}v_\ell + \frac{D_\ell - N_\ell}{2}w_\ell\right)$$
$$+\left(-\frac{i\delta_n}{1-i\delta_n}f_0 + D_0 - N_0\right) + \left(\frac{i\delta_n}{1-i\delta_n}f_0 + N_0\right)w_0 - \frac{i\delta_n}{2(1-i\delta_n)}(e_0 - f_0)v_0.$$

It follows from (4.31) that

$$P_m v_{1,n} - P_m v_{2,n} = \frac{i\delta_n}{2(1-i\delta_n)}\sum_{\ell=0}^{m}(e_\ell - f_\ell)v_\ell + \hat{v}_n \quad \text{in } B_{r_3} \setminus B_{r_2}. \tag{4.43}$$

Define

$$V_n = \begin{cases} v_n & \text{in } \Omega \setminus B_{r_3}, \\ v_n - \hat{v}_n & \text{in } B_{r_3} \setminus B_{r_2}, \\ v_{2,n} & \text{in } B_{r_2}. \end{cases} \tag{4.44}$$

We have

$$\operatorname{div}(\hat{A}\nabla V_n) = g_n \quad \text{in } \Omega \setminus (\partial B_{r_2} \cup \partial B_{r_3}). \tag{4.45}$$

We claim that

$$\|[V_n]\|_{H^{1/2}(\partial B_{r_3})} + \|[\hat{A}\nabla V_n \cdot \eta]\|_{H^{1/2}(\partial B_{r_3})} = o(1), \tag{4.46}$$
$$\|[V_n]\|_{H^{1/2}(\partial B_{r_2})} + \|[\hat{A}\nabla V_n \cdot \eta]\|_{H^{1/2}(\partial B_{r_2})} = o(1). \tag{4.47}$$

Granting (4.46) and (4.47), we derive that $V_n \to v$ weakly in $H^1(\Omega \setminus (\partial B_{r_2} \cup \partial B_{r_3}))$ as in Section 4.1. The conclusion now follows from (4.44).

It remains to prove (4.46) and (4.47).

Proof of (4.46). We have, on ∂B_{r_3},

$$[V_n] = \hat{v}_n = -\sum_{\ell=0}^{m} \frac{i\delta_n}{2(1-\delta_n)}(e_\ell - f_\ell)v_\ell + \sum_{\ell=0}^{m} D_\ell v_\ell.$$

Here we use the fact that $w_\ell = v_\ell$ ($\ell \geq 0$) on ∂B_{r_3}. We derive from (4.24), (4.29), (4.37), and (4.38) that

$$\|[V_n]\|_{H^{1/2}(\partial B_{r_3})} = o(1). \tag{4.48}$$

Similarly, using the fact that $A\nabla v_\ell \cdot \eta = -A\nabla w_\ell \cdot \eta$ on ∂B_{r_3} for $\ell \geq 1$, and $f_0 = \delta_n^{-1}o(1)$, we derive from (4.24), (4.30), and (4.42) that

$$\|[A\nabla V_n \cdot \eta]\|_{H^{1/2}(\partial B_{r_3})} = o(1). \tag{4.49}$$

A combination of (4.48) and (4.49) yields (4.46).

Proof of (4.47). On ∂B_{r_2}, we have $[V_n] = v_n - \hat{v}_n - v_{2,n}$. It follows that, on ∂B_{r_2},

$$[V_n] = v_n - v_{1,n} + v_{1,n} - P_m v_{1,n} + P_m v_{1,n} - P_m v_{2,n} + P_m v_{2,n} - v_{2,n} - \hat{v}_n.$$

Since $v_n = v_{1,n}$ on ∂B_{r_2}, we derive from (4.24) and (4.43) that

$$\|[V_n]\|_{H^{1/2}(\partial B_{r_2})} \le \delta_n^2 + \Big\| \frac{i\delta_n}{2(1-i\delta_n)} \sum_{\ell=0}^{m} (e_\ell - f_\ell) v_\ell \Big\|_{H^{1/2}(\partial B_{r_2})}.$$

From (4.37) and (4.39), we obtain

$$\|[V_n]\|_{H^{1/2}(\partial B_{r_2})} = o(1). \tag{4.50}$$

Similarly,

$$\|[\hat{A}\nabla V_n \cdot \eta]\|_{H^{-1/2}(\partial B_{r_2})} = o(1). \tag{4.51}$$

A combination of (4.50) and (4.51) yields (4.47). □

In the proof of Proposition 4.1, we used the following lemma.

Lemma 4.1. *Let $d = 2, 3$, $0 < R_1 < R_2$, and let a be a uniformly elliptic matrix-valued function defined in $B_{R_2} \setminus B_{R_1}$. Set $R_3 = R_2^2/R_1$ and let $K : B_{R_2} \setminus B_{R_1} \to B_{R_3} \setminus B_{R_2}$ be the Kelvin transform with respect to ∂B_{R_2}. Define*

$$a_1 = \begin{cases} K_* a & \text{in } B_{R_3} \setminus B_{R_2}, \\ a & \text{in } B_{R_2} \setminus B_{R_1}, \\ I & \text{in } B_{R_1}. \end{cases}$$

Let $v \in H^1(B_{R_3})$ be such that $\int_{\partial B_{R_3}} v = 0$ and $\operatorname{div}(a_1 \nabla v) = 0$ in B_{R_3}, and let $w \in H^1(B_{R_2} \setminus B_{R_1})$ be the reflection of w by K^{-1} through ∂B_{R_2}, i.e., $w = v \circ K$ in $B_{R_2} \setminus B_{R_1}$. Then, for all $c \in \mathbb{C}$,

$$\|v\|_{H^{1/2}(\partial B_{R_2})} + |c| \le C \|v - w + c\|_{H^{1/2}(\partial B_{R_1})},$$

where C is a positive constant independent of v and c.

Proof. Assume that the conclusion is not true. Then there are sequences $(v_n) \subset H^1(B_{R_3})$ and $(c_n) \subset \mathbb{C}$ such that

$$\operatorname{div}(a_1 \nabla v_n) = 0 \quad \text{in } B_{R_3}, \tag{4.52}$$

$$\int_{\partial B_{R_3}} v_n = 0, \quad \|v_n\|_{H^{1/2}(\partial B_{R_2})} + |c_n| = 1, \quad \lim_{n\to\infty} \|v_n - w_n + c_n\|_{H^{1/2}(\partial B_{R_1})} = 0. \tag{4.53}$$

Here w_n is the reflection of v_n with respect to ∂B_{R_2} by K^{-1}. From (4.53), we have

$$\|v_n + c_n\|_{H^{1/2}(\partial B_{R_1})} \le C.$$

In this proof, C denotes a positive constant independent of n. It follows from (4.53) that $\|w_n\|_{H^{1/2}(\partial B_{R_1})} \le C$, which implies, by the definition of w_n, that $\|v_n\|_{H^{1/2}(\partial B_{R_3})} \le C$.

Without loss of generality, one might assume that $v_n \to v$ weakly in $H^1(B_{R_3})$, $v_n \to v$ in $H^1_{loc}(B_{R_3})$, and $c_n \to c \in \mathbb{C}$. Moreover, from (4.52) and (4.53), we have

$$\operatorname{div}(a_1 \nabla v) = 0 \quad \text{in } B_{R_3}, \tag{4.54}$$

$$\int_{\partial B_{R_3}} v = 0, \qquad \|v\|_{H^{1/2}(\partial B_{R_2})} = 1. \tag{4.55}$$

Let w be the reflection of v with respect to ∂B_{R_2} by K^{-1}. Since $v_n \to v$ in $H^1(B_{R_2})$, it follows from (4.53) that

$$\lim_{n\to\infty} \|w_n - w\|_{H^{1/2}(\partial B_{R_1})} = 0,$$

which implies

$$\lim_{n\to\infty} \|v_n - v\|_{H^{1/2}(\partial B_{R_3})} = 0.$$

From (4.53), we have $v - w + c = 0$ on ∂B_{R_1}. It follows from Lemma A1 in the appendix that $v = 0$ and $c = 0$. Here we use the fact that $\int_{\partial B_{R_3}} v = 0$. This contradicts (4.55). □

5. Cloaking a source via anomalous localized resonance

In this section, we describe how to use the CALR theory discussed previously to cloak a source f concentrating on an arbitrary bounded smooth manifold of codimension 1 in an arbitrary medium. Without loss of generality, one may assume that the medium is contained in $B_{r_3} \setminus B_{r_2}$ and characterized by a matrix a which is assumed to be smooth and uniformly elliptic in $\overline{B_{r_3} \setminus B_{r_2}}$ for some $0 < r_2 < r_3$. Assume that f concentrates on ∂D for some bounded smooth open subset $D \subset\subset B_{r_3} \setminus B_{r_2}$. One might assume as well that $D \subset\subset B_{r_*}$ where r_* is the constant coming from Theorem 1.2, since one can choose r_3 large enough (see [25, Lemma 1]). Define $r_1 = r_2^2/r_3$. Let $F : B_{r_2} \setminus \{0\} \to \mathbb{R}^d \setminus B_{r_2}$ and $G : \mathbb{R}^d \setminus B_{r_3} \to B_{r_3} \setminus \{0\}$ be the Kelvin transforms with respect to ∂B_{r_2} and ∂B_{r_3} respectively. Note that $G \circ F(x) = (r_2^2/r_1^2)x$. Define

$$A = \begin{cases} a & \text{in } B_{r_3} \setminus B_{r_2}, \\ F_*^{-1} a & \text{in } B_{r_2} \setminus B_{r_1}, \\ F_*^{-1} G_*^{-1} a & \text{in } B_{r_1} \setminus B_{r_1^2/r_2}, \\ I & \text{otherwise.} \end{cases} \tag{5.1}$$

It is clear that $s_0 A$ is doubly complementary. Applying Theorems 1.1 and 1.2, we have

Proposition 5.1. *Let $d = 2, 3$, $\delta > 0$, and $D \subset\subset B_{r_*} \setminus B_{r_2}$, and let $f \in L^2(\partial D)$. Assume that u_δ and v_δ are defined by (1.3) and (1.5) where A is given in (5.1). There exists a sequence $\delta_n \to 0$ such that*

$$\lim_{n\to\infty} E_{\delta_n}(u_{\delta_n}) = \infty.$$

Moreover, $v_{\delta_n} \to 0$ weakly in $H^1(\Omega \setminus B_{r_3})$.

Proof. By Theorems 1.1 and 1.2, it suffices to prove that there is no $W \in H^1(B_{r_*} \setminus B_{r_2})$ such that

$$\operatorname{div}(A\nabla W) = f \quad \text{in } B_{r_*} \setminus B_{r_2}, \qquad W = A\nabla W \cdot \eta = 0 \quad \text{on } \partial B_{r_2}.$$

In fact, Theorems 1.1 and 1.2 only deal with the case $f \in L^2(\Omega)$, but the same results hold for f as here, and the proofs are unchanged. Suppose that such a W exists. Since $\operatorname{div}(A\nabla W) = 0$ in $(B_{r_*} \setminus B_{r_2}) \setminus \bar{D}$ and $W = A\nabla W \cdot \eta = 0$ on ∂B_{r_2}, it follows from the unique continuation principle that $W = 0$ in $(B_{r_*} \setminus B_{r_2}) \setminus \bar{D}$. Hence $W = 0$ in D since $W \in H^1(B_{r_*} \setminus B_{r_2})$, $W = 0$ on ∂D, and $\operatorname{div}(A\nabla W) = 0$ in D. We deduce that $W = 0$ in $B_{r_*} \setminus B_{r_2}$. Hence $W = 0$ in $B_{r_*} \setminus B_{r_2}$. This contradicts the fact that $\operatorname{div}(A\nabla W) = f \neq 0$ in $B_{R_*} \setminus B_{r_2}$. □

Appendix: Proof of Proposition 4.2

This appendix comprising two subsections is devoted to the proof of Proposition 4.2. Some useful lemmas are established in the first section and the proof of Propositions 4.2 is given in the second subsection.

A.1. Preliminaries

In this section, we assume that

- $a \in [C^3(\overline{B_{R_2} \setminus B_{R_1}})]^{d\times d}$ is uniformly elliptic symmetric,
- $K : B_{R_2} \setminus B_{R_1} \to B_{R_3} \setminus B_{R_2}$ is defined by $K(x) = xR_2^2/|x|^2$,
- a_1 is given by (4.17):

$$a_1 = \begin{cases} K_*a & \text{in } B_{R_3} \setminus B_{R_2}, \\ a & \text{in } B_{R_2} \setminus B_{R_1}, \\ I & \text{in } B_{R_1}. \end{cases}$$

Lemma A1. *Let $d = 2, 3$, $v \in H^1(B_{R_3})$ be a solution to* $\operatorname{div}(a_1\nabla v) = 0$ *in B_{R_3}, and w be the reflection of v through ∂B_{R_2} by K^{-1}, i.e., $w = v \circ K$ in $B_{R_2} \setminus B_{R_1}$. Assume that*

$$v - w + c = 0 \quad \text{on } \partial B_{R_1}, \tag{A1}$$

for some $c \in \mathbb{C}$. Then

$$v \text{ is constant and } c = 0. \tag{A2}$$

Proof. By considering the real part and the imaginary part separately, one may assume that v, w, and c are real. We first prove that $c = 0$. Assume that $c \neq 0$. From the definition of w and (A1), we have

$$v(R_1\sigma) = v(R_3\sigma) - c \quad \forall \sigma \in \partial B_1. \tag{A3}$$

By the standard theory of elliptic equations, $\sup_{\sigma\in\partial B_1} |v(R_1\sigma)| < \infty$, which implies, by (A3),

$$\sup_{\sigma\in\partial B_1} |v(R_3\sigma)| < \infty. \tag{A4}$$

Set, for $t \in \mathbb{R}$,

$$b(t) = \sup_{\sigma \in \partial B_1} |v(R_3\sigma) + t|.$$

Applying the maximum principle, we derive from (A3) that

$$\sup_{\sigma \in \partial B_1} |v(R_3\sigma) + t| = \sup_{\sigma \in \partial B_1} |v(R_1\sigma) + (t + c)| \le \sup_{\sigma \in \partial B_1} |v(R_3\sigma) + (t + c)|;$$

this reads $b(t) \le b(t + c)$, so $b(-mc) \le b(0)$ for all $m \ge 1$; this a contradiction by (A4). Hence $c = 0$. From (A3) and the maximum principle, we derive that v is constant. □

Lemma A2. *Let $d = 2, 3$, $v \in H^1(B_{R_3})$ be a solution to $\operatorname{div}(a_1 \nabla v) = 0$ in B_{R_3}, and w be the reflection of v through ∂B_{R_2} by K^{-1}, i.e., $w = v \circ K$ in $B_{R_2} \setminus B_{R_1}$. Set*

$$V = v + w.$$

Assume that

$$a\nabla V \cdot \eta = c \quad \text{on } \partial B_{R_1},$$

for some $c \in \mathbb{C}$. Then

$$v \text{ is constant and } c = 0. \tag{A5}$$

Proof. From the definition of a_1, by Lemma 2.1, we have

$$\operatorname{div}(a\nabla V) = 0 \quad \text{in } B_{R_2} \setminus B_{R_1}, \tag{A6}$$

$$V = 2v \text{ and } a\nabla V \cdot \eta = 0 \quad \text{on } \partial B_{R_2}. \tag{A7}$$

Integrating (A6) in $B_{R_2} \setminus B_{R_1}$ and using (A7), we obtain

$$\int_{\partial B_{R_1}} a\nabla V \cdot \eta = 0,$$

which implies $c = 0$. Hence, $a\nabla V \cdot \eta = 0$ on $\partial B_{R_1} \cup \partial B_{R_2}$. It follows from (A6) that V is constant in $B_{R_2} \setminus B_{R_1}$. We derive from (A7) that v is constant on ∂B_{R_2}; hence v is constant in B_{R_3} by the unique continuation principle. □

The following lemma is one of the main ingredients in the proof of statement (1) of Proposition 4.2 in two dimensions.

Lemma A3. *Let $d = 2$, let $v_{\ell,\pm} \subset H^1(B_{R_3})$ $(\ell \ge 1)$ be the unique solution to*

$$\operatorname{div}(a_1 \nabla v_{\ell,\pm}) = 0 \quad \text{in } B_{R_3}, \qquad v_{\ell,\pm} = e^{\pm i\ell\theta} \quad \text{on } \partial B_{R_3}, \tag{A8}$$

and set $v_0 = 1$ in B_{R_3}. Define $w_{\ell,\pm} \in H^1(B_{R_2} \setminus B_{R_1})$ $(\ell \ge 1)$ to be the reflection of $v_{\ell,\pm}$ through ∂B_{R_2} by K^{-1}, i.e.,

$$w_{\ell,\pm} = v_{\ell,\pm} \circ K \quad \text{in } B_{R_2} \setminus B_{R_1}, \tag{A9}$$

and denote by $w_0 \in H^1(B_{R_2} \setminus B_{R_1})$ the unique solution to

$$\operatorname{div}(a_1 \nabla w_0) = 0 \quad \text{in } B_{R_2} \setminus B_{R_1}, \qquad w_0 = 1 \quad \text{on } \partial B_{R_2}, \qquad w_0 = 0 \quad \text{on } \partial B_{R_1}. \tag{A10}$$

Then

$$\{v_0 - w_0\} \cup \{v_{\ell,\pm} - w_{\ell,\pm};\ \ell \ge 1\} \text{ is a dense subset of } H^{1/2}(\partial B_{R_1}). \tag{A11}$$

Proof. Let $G(x, y)$ be the fundamental solution to $\operatorname{div}(a_1 \nabla u) = 0$ in B_{R_3} with the zero Dirichlet boundary condition, i.e.,

$$\operatorname{div}_y(a_1(y)\nabla_y G(x, y)) = \delta_x \quad \text{in } B_{R_3}, \qquad G(x, y) = 0 \quad \text{on } \partial B_{R_3}.$$

We have, by the Green formula,

$$v_{\ell,\pm}(x) = \int_{\partial B_{R_3}} a_1(y)\nabla_y G(x, y) \cdot \eta_y \, v_{\ell,\pm}(y)\, dy, \tag{A12}$$

and (see e.g. [9])[4]

$$|G(x, y)| \le C \quad \text{ for } x \in B_{R_2}, y \in B_{R_3} \setminus B_{(R_2+R_3)/2}. \tag{A13}$$

Here and in what follows in this proof, C denotes a positive constant independent of x, y, and ℓ. It follows from (A13) that, for $|\alpha| \le 2$ (see, e.g., [10, Theorems 6.2 and 6.6]),

$$|D^\alpha G(x, y)| \le C \quad \text{ for } x \in B_{R_2},\ y \in B_{R_3} \setminus B_{(R_2+R_3)/2}, \tag{A14}$$

since $a_1 \in [C^3(\overline{B_{R_3} \setminus B_{(R_2+R_3)/2}})]^{2\times 2}$. A combination of (A12) and (A14) yields

$$|\nabla v_{\ell,\pm}(x)| \le C/\ell \quad \text{ for } x \in B_{R_3},\ \ell \ge 1. \tag{A15}$$

We claim that, for $\ell_0 \in \mathbb{N}$ large enough,

$$\{e^{\pm i\ell\theta};\ 0 \le \ell \le \ell_0 - 1\} \cup \{v_{\ell,\pm} - w_{\ell,\pm};\ \ell \ge \ell_0\} \text{ is dense in } H^{1/2}(\partial B_{R_1}). \tag{A16}$$

Consider the linear transformations $\mathcal{J}, \mathcal{P} : H^{1/2}(\partial B_{R_1}) \to H^{1/2}(\partial B_{R_1})$ defined by

$$\mathcal{J}(e^{\pm i\ell\theta}) = \begin{cases} -e^{\pm i\ell\theta} & \text{if } 0 \le \ell < \ell_0, \\ v_{\ell,\pm} - w_{\ell,\pm} & \text{if } \ell \ge \ell_0, \end{cases} \qquad \mathcal{P}(e^{\pm i\ell\theta}) = \begin{cases} 0 & \text{if } 0 \le \ell < \ell_0, \\ v_{\ell,\pm} & \text{if } \ell \ge \ell_0. \end{cases}$$

Since $w_{\ell,\pm} = e^{\pm i\ell\theta}$ on ∂B_{R_1}, it follows that $\mathcal{J} = -\mathcal{I} + \mathcal{P}$, where $\mathcal{I}$ denotes the identity transformation.

Any $f \in H^{1/2}(\partial B_{R_1})$ can be represented as

$$f = \alpha_0 + \sum_{\ell=1}^{\infty} \sum_{\pm} \alpha_{\ell,\pm} e^{\pm i\ell\theta} \quad \text{ on } \partial B_{R_1},$$

for some $\alpha_0, \alpha_{\ell,\pm} \in \mathbb{C}$ $(\ell \ge 1)$. We have

$$|\alpha_0|^2 + \sum_{\ell \ge 1} \sum_{\pm} \ell |\alpha_{\ell,\pm}|^2 \le C \|f\|^2_{H^{1/2}(\partial B_{R_1})}.$$

From the definition of $\mathcal{P}$,

$$\mathcal{P}(f) = \sum_{\ell \ge \ell_0} \sum_{\pm} \alpha_{\ell,\pm} v_{\ell,\pm} \quad \text{ on } \partial B_{R_1}.$$

[4] The corresponding result in three dimensions can be found in [13].

We derive from (A15) that

$$\|\mathcal{P}(f)\|_{H^{1/2}(\partial B_{R_1})} \le C\sum_{\ell \ge \ell_0}\sum_{\pm} |\alpha_{\ell,\pm}|/\ell \le C\Big(\sum_{\ell\ge\ell_0}\sum_{\pm}\ell|\alpha_{\ell,\pm}|^2\Big)^{1/2}\Big(\sum_{\ell\ge\ell_0}\sum_{\pm}1/\ell^3\Big)^{1/2}$$
$$\le C\ell_0^{-1}\|f\|_{H^{1/2}}.$$

Thus, for ℓ_0 large enough, $\|\mathcal{P}\| \le 1/2$. Hence $\mathcal{J}$ is invertible and (A16) follows.

Fix ℓ_0 such that (A16) holds. Using (A16), we derive that the dimension of the orthogonal complement of $\{v_{\ell,\pm} - w_{\ell,\pm};\ \ell \ge \ell_0\}$ in $H^{1/2}(\partial B_{R_1})$ is less than or equal to $2\ell_0 - 1$. Hence, to obtain the conclusion, it suffices to prove that

$$\{U_0\} \cup \{U_{\ell,\pm}\}_{1\le\ell<\ell_0} \text{ is linearly independent in } H^{1/2}(\partial B_{R_1}), \tag{A17}$$

where U_0 and $U_{\ell,\pm}$ $(1 \le \ell < \ell_0)$ are respectively the projection of $v_0 - w_0$ and $v_{\ell,\pm} - w_{\ell,\pm}$ into $(\text{span}\{v_{\ell,\pm} - w_{\ell,\pm};\ \ell \ge \ell_0\})^{\perp}$ with respect to the $H^{1/2}(\partial B_{R_1})$ scalar product. Indeed, let $\alpha_0, \alpha_{\ell,\pm} \in \mathbb{C}$ $(1 \le \ell < \ell_0)$ be such that

$$\alpha_0 U_0 + \sum_{\ell=1}^{\ell_0-1}\sum_{\pm}\alpha_{\ell,\pm}U_{\ell,\pm} = 0 \quad \text{on } \partial B_{R_1}. \tag{A18}$$

We have to prove that $\alpha_0 = \alpha_{\ell,\pm} = 0$ for $1 \le \ell \le \ell_0 - 1$. From (A18), we have

$$\alpha_0(v_0 - w_0) + \sum_{\ell=1}^{\ell_0-1}\sum_{\pm}\alpha_{\ell,\pm}(v_{\ell,\pm} - w_{\ell,\pm}) = v - w \quad \text{on } \partial B_{R_1},$$

for some $v \in \text{closure}\{\text{span}\{v_{\ell,\pm};\ \ell \ge \ell_0\}\}$ with respect to the $H^1(B_{R_3})$-norm. Here w is the reflection of v through ∂B_{R_2} by K^{-1}. Set

$$V = \sum_{\ell=1}^{\ell_0-1}\sum_{\pm}\alpha_{\ell,\pm}v_{\ell,\pm} - v \quad \text{in } B_{R_3}, \tag{A19}$$

and denote by W the reflection of V through ∂B_{R_2} by K^{-1}. It follows that

$$\alpha_0(v_0 - w_0) + V - W = 0 \quad \text{on } \partial B_{R_1}.$$

Applying Lemma A1, we find that $\alpha_0 = 0$ and V is constant. We derive from the definition of V in (A19) that $\alpha_{\ell,\pm} = 0$ for $1 \le \ell \le \ell_0 - 1$. The proof of (A17) is complete. □

For D an open subset of $\mathbb{R}^d$, we denote

$$H^1_\sharp(D) = \Big\{v \in H^1(D);\ \int_D v = 0\Big\}.$$

The following result, which is a variant of Lemma A3 when the Neumann data on ∂B_{R_1} is considered, plays an important role in the proof of statement (2) of Proposition 4.2.

Lemma A4. *Let $d = 2$ and let $v_{\ell,\pm} \subset H^1_\sharp(B_{R_3})$ ($\ell \geq 1$) be the unique solution to*

$$\operatorname{div}(a_1 \nabla v_{\ell,\pm}) = 0 \quad \text{in } B_{R_3}, \qquad a_1 \nabla v_{\ell,\pm} \cdot \eta = e^{\pm i\ell\theta} \quad \text{on } \partial B_{R_3}, \tag{A20}$$

Define $w_{\ell,\pm} \in H^1_\sharp(B_{R_2} \setminus B_{R_1})$ to be the reflection of $v_{\ell,\pm}$ through ∂B_{R_2} by K^{-1}, i.e.,

$$w_{\ell,\pm} = v_{\ell,\pm} \circ K \quad \text{in } B_{R_2} \setminus B_{R_1}. \tag{A21}$$

Then

$$\{1\} \cup \{a\nabla(v_{\ell,\pm} + w_{\ell,\pm}) \cdot \eta;\ \ell \geq 1\} \text{ is a dense subset of } H^{-1/2}(\partial B_{R_1}). \tag{A22}$$

Remark A.1. Since $\int_{\partial B_{R_3}} e^{\pm i\ell\theta} = 0$ for $\ell \geq 1$, it follows that $v_{\ell,\pm}$ is well-defined.

Proof of Lemma A4. The proof is in the same spirit as that of Lemma A3. As in the previous proof, we also show that

$$\{1\} \cup \{e^{\pm i\ell\theta};\ 1 \leq \ell < \ell_0\} \cup \{a\nabla(v_\ell + w_\ell) \cdot \eta;\ \ell \geq \ell_0\} \text{ is dense in } H^{-1/2}(\partial B_{R_1}), \tag{A23}$$

for some $\ell_0 > 1$ (large). It follows that the dimension of the orthogonal complement of closure$\{\text{span}\{a\nabla(v_\ell + w_\ell) \cdot \eta;\ \ell \geq \ell_0\}\}$ in $H^{-1/2}(\partial B_{R_1})$ is less than or equal to $2\ell_0 - 1$. Hence, to obtain the conclusion, it suffices to prove that

$$\{U_0\} \cup \{U_{\ell,\pm}\}_{1 \leq \ell < \ell_0} \text{ is independent in } H^{-1/2}(\partial B_{R_1}), \tag{A24}$$

where $U_0 = 1$ and $U_{\ell,\pm}$ ($1 \leq \ell < \ell_0$) is the projection of $a\nabla(v_{\ell,\pm} + w_{\ell,\pm}) \cdot \eta$ into $(\text{closure}\{\text{span}\{a\nabla(v_{\ell,\pm} + w_{\ell,\pm}) \cdot \eta;\ \ell \geq \ell_0\}\})^\perp$ with respect to the $H^{-1/2}(\partial B_{R_1})$ scalar product.

Let $\alpha_0, \alpha_{\ell,\pm} \in \mathbb{C}$ ($1 \leq \ell \leq \ell_0 - 1$) be such that

$$\alpha_0 + \sum_{\ell=1}^{\ell_0-1} \sum_{\pm} \alpha_{\ell,\pm} U_{\ell,\pm} = 0 \quad \text{on } \partial B_{R_1}. \tag{A25}$$

We will prove that $\alpha_0 = \alpha_{\ell,\pm} = 0$ for $1 \leq \ell \leq \ell_0 - 1$. From (A25), we have

$$\alpha_0 + \sum_{\ell=1}^{\ell_0-1} \sum_{\pm} \alpha_{\ell,\pm} a\nabla(v_{\ell,\pm} + w_{\ell,\pm}) \cdot \eta = a\nabla(v + w) \cdot \eta \quad \text{on } \partial B_{R_1}, \tag{A26}$$

for some $v \in \text{closure}\{\text{span}\{v_{\ell,\pm};\ \ell \geq \ell_0\}\}$ in $H^1_\sharp(B_{R_3})$. Here w is the reflection of v through ∂B_{R_2} by K^{-1}. Set

$$V = \sum_{\ell=1}^{\ell_0-1} \sum_{\pm} \alpha_{\ell,\pm} v_{\ell,\pm} - v \quad \text{in } B_{R_3}, \tag{A27}$$

and denote by W the reflection of V through ∂B_{R_2} by K^{-1}. It follows from (A26) that

$$\alpha_0 + a\nabla(V + W) \cdot \eta = 0 \quad \text{on } \partial B_{R_1}.$$

Applying Lemma A2, we have $\alpha_0 = 0$ and V is constant. Hence $V = 0$ since $V \in H^1_\sharp(B_{R_3})$. We derive from the definition of V in (A27) and of $v_{\ell,\pm}$ that $\alpha_{\ell,\pm} = 0$ for $1 \le \ell \le \ell_0 - 1$. The proof of (A24) is complete. □

Here are variants of Lemmas A3 and A4 in three dimensions. The first one is the variant of Lemma A3.

Lemma A5. *Let $d = 3$ and let $v_\ell^k \subset H^1(B_{R_3})$ ($\ell \ge 1$, $-\ell \le k \le \ell$) be the unique solution to*

$$\operatorname{div}(a_1 \nabla v_\ell^k) = 0 \quad \text{in } B_{R_3}, \qquad v_\ell^k = Y_\ell^k \quad \text{on } \partial B_{R_3}, \tag{A28}$$

and set $v_0^0 = 1$. Here Y_ℓ^k is the spherical harmonic function of degree ℓ and of order k. Define $w_\ell^k \in H^1(B_{R_2} \setminus B_{R_1})$ to be the reflection of v_ℓ^k through ∂B_{R_2} by K^{-1}, i.e.,

$$w_\ell^k = v_\ell^k \circ K \quad \text{in } B_{R_2} \setminus B_{R_1}, \tag{A29}$$

and denote by $w_0^0 \in H^1(B_{R_2} \setminus B_{R_1})$ the unique solution to

$$\operatorname{div}(a_1 \nabla w_0^0) = 0 \quad \text{in } B_{R_2} \setminus B_{R_1}, \qquad w_0^0 = 1 \quad \text{on } \partial B_{R_2}, \qquad w_0^0 = 0 \quad \text{on } \partial B_{R_1}. \tag{A30}$$

Then

$$\{v_\ell^k - w_\ell^k;\ \ell \ge 0,\ -\ell \le k \le \ell\} \text{ is a dense subset of } H^{1/2}(\partial B_{R_1}). \tag{A31}$$

Proof. The proof is similar to the one of Lemma A3. The details are left to the reader. □

The second lemma is the variant of Lemma A4.

Lemma A6. *Let $d = 3$ and let $v_{k,\ell} \subset H^1_\sharp(B_{R_3})$ ($\ell \ge 1$, $-\ell \le k \le \ell$) be the unique solution to*

$$\operatorname{div}(a_1 \nabla v_\ell^k) = 0 \quad \text{in } B_{R_3}, \qquad a_1 \nabla v_\ell^k \cdot \eta = Y_\ell^k \quad \text{on } \partial B_{R_3}. \tag{A32}$$

Define $w_\ell^k \in H^1_\sharp(B_{R_2} \setminus B_{R_1})$ ($\ell \ge 1$) to be the reflection of v_ℓ^k through ∂B_{R_2} by K^{-1}, i.e.,

$$w_\ell^k = v_\ell^k \circ K \quad \text{in } B_{R_2} \setminus B_{R_1}. \tag{A33}$$

Then

$$\{1\} \cup \{a_1 \nabla (v_\ell^k + w_\ell^k) \cdot \eta;\ \ell \ge 1,\ -\ell \le k \le \ell\} \text{ is a dense subset of } H^{-1/2}(\partial B_{R_1}). \tag{A34}$$

Proof. Since $\int_{\partial B_{R_3}} Y_\ell^m = 0$ for $\ell \ge 1$ and $-\ell \le k \le \ell$, it follows that v_ℓ^k is well-defined. The proof is similar to the one of Lemma A4. The details are left to the reader. □

A.2. Proof of Proposition 4.2

Statements (4.18) and (4.19) are consequences of Lemma 2.1. It remains to prove statements (1)–(3). The proof is divided into two steps.

Step 1: We prove that if one of (1)–(3) holds for a (particular) dense set $\{v_\ell\}_{\ell\ge 0}$, then it also holds for all dense sets $\{v_\ell\}_{\ell\ge 0}$.

We will only show this for statement (1), the other cases being similar. Assume that (1) holds for a specific sequence $\{v_\ell\}_{\ell\ge 0}$ which satisfies the assumptions of Proposition 4.2. We will prove that (1) holds for any sequence $\{\hat v_\ell\}_{\ell\ge 0}$ satisfying those assumptions. Let $v \in H^1(B_{R_2} \setminus B_{R_1})$ be such that $\operatorname{div}(a\nabla v) = 0$ in $B_{R_2} \setminus B_{R_1}$ and $v = 0$ on ∂B_{R_2}. For $\varepsilon > 0$, there exist $\ell_\varepsilon > 0$ and $(\alpha_\ell)_{\ell=0}^{\ell_\varepsilon} \subset \mathbb{C}$ such that

$$\Big\| v - \sum_{\ell=0}^{\ell_\varepsilon} \alpha_\ell (v_\ell - w_\ell) \Big\|_{H^1(B_{R_2}\setminus B_{R_1})} \le \varepsilon, \tag{A35}$$

since (1) holds for (v_ℓ). On the other hand, there exist $\hat\ell_\varepsilon$ and $(\hat\alpha_\ell)_{\ell=0}^{\hat\ell_\varepsilon} \subset \mathbb{C}$ such that

$$\Big\| \sum_{\ell=0}^{\ell_\varepsilon} \alpha_\ell v_\ell - \sum_{\ell=0}^{\hat\ell_\varepsilon} \hat\alpha_\ell \hat v_\ell \Big\|_{H^{1/2}(\partial B_{R_3})} \le \varepsilon,$$

by the density of $\{\hat v_\ell\}_{\ell=0}^{\infty}$. This implies

$$\Big\| \sum_{\ell=0}^{\ell_\varepsilon} \alpha_\ell v_\ell - \sum_{\ell 0}^{\hat\ell_\varepsilon} \hat\alpha_\ell \hat v_\ell \Big\|_{H^1(B_{R_3})} \le \varepsilon. \tag{A36}$$

Let $\hat w_\ell$ be the reflection of $\hat v_\ell$ through ∂B_{R_2} by K^{-1} for $\ell \ge 1$. Note that if w is the reflection of v through ∂B_{R_2} by K^{-1}, then

$$\|w\|_{H^1(B_{R_2}\setminus B_{R_1})} \le C\|v\|_{H^1(B_{R_3})}. \tag{A37}$$

Here and in what follows, C denotes a positive constant depending only on a, R_1, and R_2. A combination of (A36) and (A37) yields

$$\Big\| \sum_{\ell=1}^{\ell_\varepsilon} \alpha_\ell w_\ell - \sum_{\ell=1}^{\hat\ell_\varepsilon} \hat\alpha_\ell \hat w_\ell + (\alpha_0 - \hat\alpha_0) \Big\|_{H^1(B_{R_2}\setminus B_{R_1})} \le C\varepsilon. \tag{A38}$$

We derive from (A36) and (A38) that

$$\Big\| \sum_{\ell=1}^{\ell_\varepsilon} \alpha_\ell (v_\ell - w_\ell) - \sum_{\ell=1}^{\hat\ell_\varepsilon} \hat\alpha_\ell (\hat v_\ell - \hat w_\ell) \Big\|_{H^1(B_{R_2}\setminus B_{R_1})} \le C\varepsilon. \tag{A39}$$

From (A35) and (A39), we obtain

$$\Big\| v - \sum_{\ell=1}^{\hat\ell_\varepsilon} \hat\alpha_\ell (\hat v_\ell - \hat w_\ell) - \alpha_0 (v_0 - w_0) \Big\|_{H^1(B_{R_2}\setminus B_{R_1})} \le C\varepsilon.$$

Hence statement (1) holds for $(\hat v_\ell)$.

Step 2: Proof of statements (1)–(3). We only establish these statements in two dimensions. The three-dimensional case follows similarly, with Lemmas A5 and A6 applied instead of Lemmas A3 and A4.

Assume $d = 2$. Let $v_{\ell,\pm} \subset H^1(B_{R_3})$ ($\ell \geq 1$) be the unique solution to

$$\operatorname{div}(a_1 \nabla v_{\ell,\pm}) = 0 \quad \text{in } B_{R_3}, \qquad v_{\ell,\pm} = e^{\pm i\ell\theta} \quad \text{on } \partial B_{R_3}, \tag{A40}$$

and set

$$v_0 = 1 \quad \text{in } B_{R_3}. \tag{A41}$$

Let $w_{\ell,\pm} \in H^1(B_{R_2} \setminus B_{R_1})$ ($\ell \geq 1$) be the reflection of $v_{\ell,\pm}$ through ∂B_{R_2} by K^{-1}, i.e.,

$$w_{\ell,\pm} = v_{\ell,\pm} \circ K \quad \text{in } B_{R_2} \setminus B_{R_1}, \tag{A42}$$

and denote by $w_0 \in H^1(B_{R_3} \setminus B_{R_2})$ the unique solution to

$$\operatorname{div}(a \nabla w_0) = 0 \quad \text{in } B_{R_2} \setminus B_{R_1}, \qquad w_0 = 1 \quad \text{on } \partial B_{R_2}, \qquad w_0 = 0 \quad \text{on } \partial B_{R_1}.$$

By Step 1, it suffices to prove (1)–(3) for $\{v_0, w_0\} \cup \{v_{\ell,\pm}, w_{\ell,\pm}\}_{\ell \geq 1}$.

Proof of statement (1). This statement is a consequence of the fact that $v = 0$ if $v \in H^1(B_{R_2} \setminus B_{R_1})$ satisfies

$$\operatorname{div}(a \nabla v) = 0 \quad \text{in } B_{R_2} \setminus B_{R_1}, \qquad v = 0 \quad \text{on } \partial B_{R_2}, \tag{A43}$$

$$\int_{B_{R_2} \setminus B_{R_1}} a \nabla v \nabla (\bar{v}_{\ell,\pm} - \bar{w}_{\ell,\pm}) = 0 \quad \forall \ell \geq 1, \tag{A44}$$

$$\int_{B_{R_2} \setminus B_{R_1}} a \nabla v \nabla (\bar{v}_0 - \bar{w}_0) = 0. \tag{A45}$$

Indeed, using (A43), we derive from (A44) and (A45) that

$$\int_{\partial B_{R_1}} a \nabla v \cdot \eta \, (\bar{v}_{\ell,\pm} - \bar{w}_{\ell,\pm}) = 0 \quad \forall \ell \geq 1, \tag{A46}$$

$$\int_{\partial B_{R_1}} a \nabla v \cdot \eta \, (\bar{v}_0 - \bar{w}_0) = 0. \tag{A47}$$

Since, by Lemma A3, $\{v_0 - w_0\} \cup \{v_{\ell,\pm} - w_{\ell,\pm};\ \ell \geq 1\}$ is dense in $H^{1/2}(\partial B_{R_1})$ it follows from (A46) and (A47) that $a \nabla v \cdot \eta = 0$ on ∂B_{R_1}. We then derive from (A43) that $v = 0$ in $B_{R_2} \setminus B_{R_1}$, and statement (1) is proved.

Proof of statement (2). This statement is a consequence of the fact that v is constant if $v \in H^1(B_{R_2} \setminus B_{R_1})$ satisfies

$$\operatorname{div}(a \nabla v) = 0 \quad \text{in } B_{R_2} \setminus B_{R_1}, \qquad a \nabla v \cdot \eta = 0 \quad \text{on } \partial B_{R_2}, \tag{A48}$$

$$\int_{B_{R_2} \setminus B_{R_1}} a \nabla v \nabla (\bar{v}_{\ell,\pm} + \bar{w}_{\ell,\pm}) = 0 \quad \forall \ell \geq 1. \tag{A49}$$

Indeed, since $a\nabla v_{\ell,\pm} \cdot \eta = -a\nabla w_{\ell,\pm} \cdot \eta$ on ∂B_{R_2} for $\ell \geq 1$ by (4.19), it follows from (A49) that

$$\int_{\partial B_{R_1}} a\nabla(\bar{v}_{\ell,\pm} + \bar{w}_{\ell,\pm}) \cdot \eta v = 0 \quad \forall \ell \geq 1. \tag{A50}$$

By Lemma A4 and Step 1,

$$\{1\} \cup \{a\nabla(v_{\ell,\pm} + w_{\ell,\pm}) \cdot \eta;\ \ell \geq 1\} \text{ is a dense subset of } H^{-1/2}(\partial B_{R_1}). \tag{A51}$$

We derive from (A50) that v is constant on ∂B_{R_1}. This implies, by (A48), that v is constant in $B_{R_2} \setminus B_{R_1}$, and statement (2) is proved.

Proof of statement (3). This statement is a consequence of the fact that v is constant if $v \in H^1(B_{R_2} \setminus B_{R_1})$ satisfies

$$\operatorname{div}(a\nabla v) = 0 \quad \text{in } B_{R_2} \setminus B_{R_1}, \tag{A52}$$

$$\int_{B_{R_2}\setminus B_{R_1}} a\nabla v \nabla \bar{v}_{\ell,\pm} = \int_{B_{R_2}\setminus B_{R_1}} a\nabla v \nabla \bar{w}_{\ell,\pm} = 0 \quad \forall \ell \geq 1, \tag{A53}$$

$$\int_{B_{R_2}\setminus B_{R_1}} a\nabla v \nabla \bar{v}_0 = \int_{B_{R_2}\setminus B_{R_1}} a\nabla v \nabla \bar{w}_0 = 0. \tag{A54}$$

In fact, a combination of (A52)–(A54) yields

$$\int_{\partial B_{R_2}\cup\partial B_{R_1}} a\nabla v \cdot \eta\ \bar{v}_{\ell,\pm} = \int_{\partial B_{R_2}\cup\partial B_{R_1}} a\nabla v \cdot \eta\ \bar{w}_{\ell,\pm} = 0 \quad \forall \ell \geq 1, \tag{A55}$$

$$\int_{\partial B_{R_2}\cup\partial B_{R_1}} a\nabla v \cdot \eta\ \bar{v}_0 = \int_{\partial B_{R_2}\cup\partial B_{R_1}} a\nabla v \cdot \eta\ \bar{w}_0 = 0. \tag{A56}$$

Since $v_0 = w_0 = 1$ and $v_{\ell,\pm} = w_{\ell,\pm}$ on ∂B_{R_2} for $\ell \geq 1$, it follows from (A55) that

$$\int_{\partial B_{R_1}} a\nabla v \cdot \eta\ (\bar{v}_{\ell,\pm} - \bar{w}_{\ell,\pm}) = 0 \quad \forall \ell \geq 1, \tag{A57}$$

and, since $w_0 = 0$ on ∂B_{R_1},

$$\int_{\partial B_{R_1}} a\nabla v \cdot \eta = 0. \tag{A58}$$

From (4.18), (A53), and the symmetry of a, we also have

$$\int_{\partial B_{R_2}\cup\partial B_{R_1}} a\nabla \bar{v}_{\ell,\pm} \cdot \eta\ \bar{v} = \int_{\partial B_{R_2}\cup\partial B_{R_1}} a\nabla \bar{w}_{\ell,\pm} \cdot \eta\ v = 0 \quad \forall \ell \geq 1,$$

which yields, since $a\nabla v_{\ell,\pm} \cdot \eta = -a\nabla w_{\ell,\pm} \cdot \eta$ for $\ell \geq 1$,

$$\int_{\partial B_{R_1}} a\nabla(\bar{v}_{\ell,\pm} + \bar{w}_{\ell,\pm}) \cdot \eta\ v = 0 \quad \forall \ell \geq 1. \tag{A59}$$

Using Lemma A3 and (A51), we derive from (A57)–(A59) that

$$a\nabla v \cdot \eta = 0, \quad v - \fint_{\partial B_{r_1}} v = 0 \quad \text{on } \partial B_{R_1}. \tag{A60}$$

A combination of (A52) and (A60) shows that v is constant in $B_{R_2} \setminus B_{R_1}$ by the unique continuation principle. Statement (3) is proved. □

Remark A.2. In Proposition 4.2, if one assumes in addition that $\{v_\ell\}_{\ell=0}^{\infty}$ is a basis of $H^{1/2}(\partial B_{R_3})$, then

- $\{v_\ell, w_\ell;\ \ell \geq 0\}$ is linearly independent in $H^1(B_{R_2} \setminus B_{R_1})$,
- $\{v_\ell;\ \ell \geq 0\}$ is linearly independent in $H^{1/2}(\partial B_{R_2})$,
- $\{1\} \cup \{a\nabla w_\ell \cdot \eta;\ \ell \geq 1\}$ is linearly independent in $H^{-1/2}(\partial B_{R_2})$.

These facts can be derived from Lemma A1.

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