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A variational analysis of a gauged nonlinear Schrödinger equation

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Abstract. This paper is motivated by a gauged Schrödinger equation in dimension 2 including the so-called Chern–Simons term. The study of radial stationary states leads to the nonlocal problem

$$-\Delta u(x) + \left(\omega + \frac{h^2(|x|)}{|x|^2} + \int_{|x|}^{\infty} \frac{h(s)}{s} u^2(s)\, ds\right) u(x) = |u(x)|^{p-1} u(x),$$

where

$$h(r) = \frac{1}{2} \int_0^r s u^2(s)\, ds.$$

This problem is the Euler–Lagrange equation of a certain energy functional. We study the global behavior of that functional. We show that for $p \in (1, 3)$, the functional may be bounded from below or not, depending on ω. Quite surprisingly, the threshold value for ω is explicit. From this study we prove existence and non-existence of positive solutions.

Keywords. Gauged Schrödinger equations, Chern–Simons theory, variational methods, concentration compactness

1. Introduction

In this paper we are concerned with a planar gauged nonlinear Schrödinger equation

$$iD_0\phi + (D_1 D_1 + D_2 D_2)\phi + |\phi|^{p-1}\phi = 0. \tag{1}$$

Here $t \in \mathbb{R}$, $x = (x_1, x_2) \in \mathbb{R}^2$, $\phi : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{C}$ is a scalar field, $A_\mu : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ are the components of the gauge potential and $D_\mu = \partial_\mu + iA_\mu$ is the covariant derivative ($\mu = 0, 1, 2$).

The classical equation for the gauge potential A_μ is the Maxwell equation. However, the modified gauge field equation proposes to include the so-called Chern–Simons term into the equation (see for instance [23, Chapter 1]):

$$\partial_\mu F^{\mu\nu} + \tfrac{1}{2}\kappa\epsilon^{\nu\alpha\beta} F_{\alpha\beta} = j^\nu \quad \text{with} \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{2}$$

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In the above equation, κ is a parameter that measures the strength of the Chern–Simons term. As usual, $\epsilon^{\nu\alpha\beta}$ is the Levi-Civita tensor, and the superscripts are related to the Minkowski metric with signature $(1, -1, -1)$. Finally, j^μ is the conserved matter current,

$$j^0 = |\phi|^2, \qquad j^i = 2\,\mathrm{Im}(\bar{\phi} D_i \phi).$$

At low energies, the Maxwell term becomes negligible and can be dropped, giving rise to

$$\tfrac{1}{2}\kappa \epsilon^{\nu\alpha\beta} F_{\alpha\beta} = j^\nu. \tag{3}$$

See [7,8,12–14] for the discussion above.

For simplicity, fix $\kappa = 2$. Equations (1) and (3) lead to the problem

$$\begin{aligned} &i D_0\phi + (D_1 D_1 + D_2 D_2)\phi + |\phi|^{p-1}\phi = 0,\\ &\partial_0 A_1 - \partial_1 A_0 = \mathrm{Im}(\bar{\phi} D_2 \phi),\\ &\partial_0 A_2 - \partial_2 A_0 = -\,\mathrm{Im}(\bar{\phi} D_1 \phi),\\ &\partial_1 A_2 - \partial_2 A_1 = \tfrac{1}{2}|\phi|^2. \end{aligned} \tag{4}$$

As is usual in Chern–Simons theory, problem (4) is invariant under gauge transformation,

$$\phi \mapsto \phi e^{i\chi}, \qquad A_\mu \mapsto A_\mu - \partial_\mu \chi, \tag{5}$$

for any C^∞ function χ.

This model was first proposed and studied in [12–14], and is sometimes called the Chern–Simons–Schrödinger equation. The initial value problem, wellposedness, global existence and blow-up, scattering, etc. have been addressed in [2,9,11,18,19] for the case $p = 3$. See also [17] for a global existence result in the defocusing case.

The existence of stationary states for (4) and for general $p > 1$ has been studied recently in [4] (with respect to that paper, our notation interchanges the indices 1 and 2). By using the ansatz

$$\begin{aligned} \phi(t,x) &= u(|x|)e^{i\omega t}, & A_0(x) &= A_0(|x|),\\ A_1(t,x) &= -\frac{x_2}{|x|^2} h(|x|), & A_2(t,x) &= \frac{x_1}{|x|^2} h(|x|), \end{aligned}$$

in [4] it is found that u solves the equation

$$-\Delta u(x) + \left(\omega + \xi + \frac{h^2(|x|)}{|x|^2} + \int_{|x|}^{\infty} \frac{h(s)}{s} u^2(s)\,ds\right) u(x) = |u(x)|^{p-1}u(x), \qquad x \in \mathbb{R}^2, \tag{6}$$

where

$$h(r) = \frac{1}{2}\int_0^r s u^2(s)\,ds.$$

Here ξ in $\mathbb{R}$ is an integration constant of A_0, which takes the form

$$A_0(r) = \xi + \int_r^{\infty} \frac{h(s)}{s} u^2(s)\,ds.$$

Observe that (6) is a nonlocal equation. Moreover, in [4] it is shown that (6) is indeed the Euler–Lagrange equation of the energy functional

$$I_{\omega+\xi} : H^1_r(\mathbb{R}^2) \to \mathbb{R}$$

defined as

$$I_{\omega+\xi}(u) = \frac{1}{2}\int_{\mathbb{R}^2}\big(|\nabla u(x)|^2 + (\omega+\xi)u^2(x)\big)\,dx + \frac{1}{8}\int_{\mathbb{R}^2}\frac{u^2(x)}{|x|^2}\left(\int_0^{|x|} s u^2(s)\,ds\right)^2 dx - \frac{1}{p+1}\int_{\mathbb{R}^2}|u(x)|^{p+1}\,dx.$$

Here $H^1_r(\mathbb{R}^2)$ denotes the Sobolev space of radially symmetric functions. It is important to observe that the energy functional $I_{\omega+\xi}$ exhibits the competition between the nonlocal term and the local nonlinearity. The study of the behavior of the functional under this competition is one of the main motivations of this paper.

Given a stationary solution, and taking $\chi = ct$ in the gauge invariance (5), we obtain another stationary solution; the functions $u(x)$, $A_1(x)$, $A_2(x)$ are preserved, and

$$\omega \mapsto \omega + c, \qquad A_0(x) \mapsto A_0(x) - c.$$

Therefore, the constant $\omega + \xi$ is a gauge invariant of the stationary solutions of the problem. By the above discussion we can take $\xi = 0$ in what follows, that is,

$$\lim_{|x|\to\infty} A_0(x) = 0,$$

which was indeed assumed in [2, 14].

For $p > 3$, it is shown in [4] that I_ω is unbounded from below, so it exhibits a mountain-pass geometry. In a certain sense, in this case the local nonlinearity dominates the nonlocal term. However, the existence of a solution is not so direct, since for $p \in (3, 5)$ the (PS) property is not known to hold. This problem is bypassed in [4] by using a constrained minimization taking into account the Nehari and Pohozaev identities, in the spirit of [20]. Moreover, infinitely many solutions have been found in [10] for $p > 5$ (possibly sign-changing).

A special case in the above equation is $p = 3$: in this case, static solutions can be found by passing to a self-dual equation, which leads to a Liouville equation that can be solved explicitly. Those are the unique positive solutions, as proved in [4]. For more information on the self-dual equations, see [5, 14, 23].

In case $p \in (1, 3)$, solutions are found in [4] as minimizers on an L^2 sphere. Therefore, the value ω comes as a Lagrange multiplier, and it is not controlled. Moreover, the global behavior of the energy functional I_ω is not studied.

The main purpose of this paper is to study whether I_ω is bounded from below or not for $p \in (1, 3)$. In this case, the nonlocal term prevails over the local nonlinearity, in a certain sense. As we shall see, the situation is quite rich and unexpected a priori, and very different from the usual nonlinear Schrödinger equation. This situation also differs from the Schrödinger–Poisson problem (see [20]), which is another problem exhibiting the competition between local and nonlocal nonlinearities.

We shall prove the existence of a threshold value ω_0 such that I_ω is bounded from below if $\omega \geq \omega_0$, and it is not for $\omega \in (0, \omega_0)$. However, in our opinion, what is most surprising is that ω_0 has an explicit expression, namely

$$\omega_0 = \frac{3-p}{3+p}\, 3^{\frac{p-1}{2(3-p)}}\, 2^{\frac{2}{3-p}} \left(\frac{m^2(3+p)}{p-1}\right)^{-\frac{p-1}{2(3-p)}} \tag{7}$$

with

$$m = \int_{-\infty}^{\infty} \left(\frac{2}{p+1}\cosh^2\left(\frac{p-1}{2}r\right)\right)^{\frac{2}{1-p}} dr.$$

Let us give an idea of the proofs. It is not difficult to show that I_ω is coercive when the problem is posed on a bounded domain. So, there exists a minimizer u_n on the ball $B(0, n)$ with Dirichlet boundary conditions. To prove boundedness of u_n, the problem is the possible loss of mass at infinity as $n \to \infty$. The core of our proofs is a detailed study of the behavior of those masses. We are able to show that, if unbounded, the sequence u_n behaves as a soliton, if u_n is interpreted as a function of a single real variable. The proof uses a careful study of the level sets of u_n, which takes into account the effect of the nonlocal term. Then, the energy functional I_ω admits a natural approximation through a convenient limit functional. Finally, the solutions of that limit functional, and their energy, can be found explicitly, so we can find ω_0. See Section 2 for a heuristic explanation of the proof and a derivation of the limit functional.

Regarding the existence of solutions, a priori, the global minimizer could correspond to the zero solution. And indeed this is the case for large ω. Instead, we show that $\inf I_\omega < 0$ if $\omega > \omega_0$ is close to the threshold value. Therefore, the global minimizer is not trivial, and corresponds to a positive solution. The mountain-pass theorem will provide the existence of a second positive solution.

If $\omega < \omega_0$, I_ω is unbounded from below, and hence the geometric assumptions of the mountain-pass theorem are satisfied. However, the boundedness of (PS) sequences seems to be a hard question in this case. Solutions are found for almost all values of $\omega \in (0, \omega_0)$ by using the well-known monotonicity trick of Struwe [22] (see also [15]).

Our main results are the following:

Theorem 1.1. *For ω_0 as given in (7):*

(i) *if $\omega \in (0, \omega_0)$, then I_ω is unbounded from below;*
(ii) *if $\omega = \omega_0$, then I_{ω_0} is bounded from below, not coercive and* $\inf I_{\omega_0} < 0$*;*
(iii) *if $\omega > \omega_0$, then I_ω is bounded from below and coercive.*

Regarding the existence of solutions, we obtain the following result:

Theorem 1.2. *Consider (6) with $\xi = 0$. There exist $\bar{\omega} > \tilde{\omega} > \omega_0$ such that:*

(i) *if $\omega > \bar{\omega}$, then (6) has no solutions different from zero;*
(ii) *if $\omega \in (\omega_0, \tilde{\omega})$, then (6) admits at least two positive solutions: one of them is a global minimizer for I_ω and the other is a mountain-pass solution;*
(iii) *for almost every $\omega \in (0, \omega_0)$, (6) admits a positive solution.*

The rest of the paper is organized as follows. Section 2 is devoted to some preliminary results. Moreover, we give a heuristic presentation of our proofs, which motivates the definition of the limit functional. This limit functional is studied in detail in Section 3. Finally, in Section 4 we prove Theorems 1.1 and 1.2.

2. Preliminaries

Let us first fix some notation. We denote by $H^1_r(\mathbb{R}^2)$ the Sobolev space of radially symmetric functions, and $\|\cdot\|$ its usual norm. Other norms, like Lebesgue norms, will be indicated with a subscript. In particular, $\|\cdot\|_{H^1(\mathbb{R})}$, $\|\cdot\|_{H^1(a,b)}$ are used to indicate the norms of the Sobolev spaces in dimension 1. If nothing is specified, strong and weak convergence of sequences of functions are considered in the space $H^1(\mathbb{R}^2)$.

In our estimates, we will frequently denote by $C, c > 0$ fixed constants, which may change from line to line, but are always independent of the variable under consideration. We also use $O(1)$, $o(1)$, $O(\varepsilon)$, $o(\varepsilon)$ to describe the asymptotic behavior of various quantities. Finally, the letters x, y indicate two-dimensional variables, and r, s denote one-dimensional variables.

Let us start with the following proposition, proved in [4]:

Proposition 2.1. *I_ω is a C^1 functional, and its critical points correspond to classical solutions of* (6).

The next result deals with the behavior of I_ω under weak limits in $H^1_r(\mathbb{R}^2)$. Even if it is not explicitly stated in this form, Proposition 2.2 follows easily from [4, Lemma 3.2] and the compactness of the embedding $H^1_r(\mathbb{R}^2) \hookrightarrow L^q(\mathbb{R}^2)$, $q \in (2,\infty)$ (see [21]).

Proposition 2.2. *If $u_n \rightharpoonup u$, then*

$$\int_{\mathbb{R}^2} \frac{u_n^2(x)}{|x|^2}\left(\int_0^{|x|} s u_n^2(s)\,ds\right)^2 dx \to \int_{\mathbb{R}^2} \frac{u^2(x)}{|x|^2}\left(\int_0^{|x|} s u^2(s)\,ds\right)^2 dx.$$

In particular, I_ω is weak lower semicontinuous. Moreover, if $u_n \rightharpoonup u$ then $I'_\omega(u_n)(\varphi) \to I'_\omega(u)(\varphi)$ for all $\varphi \in H^1_r(\mathbb{R}^2)$.

We now state an inequality which will prove to be fundamental in our analysis. It is proved in [4], where also maximizers are found.

Proposition 2.3. *For any $u \in H^1_r(\mathbb{R}^2)$,*

$$\int_{\mathbb{R}^2} |u(x)|^4\,dx \le 2\left(\int_{\mathbb{R}^2} |\nabla u(x)|^2\,dx\right)^{1/2}\left(\int_{\mathbb{R}^2} \frac{u^2}{|x|^2}\left(\int_0^{|x|} s u^2(s)\,ds\right)^2 dx\right)^{1/2}. \tag{8}$$

As mentioned in the introduction, this paper is concerned with boundedness of I_ω from below. Let us give a rough idea of our argument. First of all, consider a fixed function $u(r)$, and define $u_\rho(r) = u(r-\rho)$. Let us now estimate $I_\omega(u_\rho)$ as $\rho \to \infty$. We have

$$(2\pi)^{-1} I_\omega(u_\rho) = \frac{1}{2}\int_{-\rho}^{\infty} (|u'|^2 + \omega u^2)(r+\rho)\, dr \\ + \frac{1}{8}\int_{-\rho}^{\infty} \frac{u^2(r)}{r+\rho}\left(\int_{-\rho}^{r} (s+\rho)u^2(s)\, ds\right)^2 dr - \frac{1}{p+1}\int_{-\rho}^{\infty} |u|^{p+1}(r+\rho)\, dr.$$

We estimate the above expression by simply replacing $r+\rho$, $s+\rho$ with the constant ρ:

$$(2\pi)^{-1} I_\omega(u) \sim \rho\Bigg[\frac{1}{2}\int_{-\infty}^{\infty} (|u|'^2 + \omega u^2)\, dr + \frac{1}{8}\int_{-\infty}^{\infty} u^2(r)\left(\int_{-\infty}^{r} u^2(s)\, ds\right)^2 dr \\ - \frac{1}{p+1}\int_{-\infty}^{\infty} |u|^{p+1}\, dr\Bigg] \\ = \rho\left[\frac{1}{2}\int_{-\infty}^{\infty} (|u|'^2 + \omega u^2)\, dr + \frac{1}{24}\left(\int_{-\infty}^{\infty} u^2\, dr\right)^3 - \frac{1}{p+1}\int_{-\infty}^{\infty} |u|^{p+1}\, dr\right].$$

This estimate will be made rigorous in Lemma 4.1. Therefore, it is natural to define the limit functional $J_\omega : H^1(\mathbb{R}) \to \mathbb{R}$ by

$$J_\omega(u) = \frac{1}{2}\int_{-\infty}^{\infty} (|u'|^2 + \omega u^2)\, dr + \frac{1}{24}\left(\int_{-\infty}^{\infty} u^2\, dr\right)^3 - \frac{1}{p+1}\int_{-\infty}^{\infty} |u|^{p+1}\, dr. \quad (9)$$

As a consequence of the above argument, if J_ω attains negative values, then I_ω will be unbounded from below.

The converse is also true, but the proof is more delicate. We will show that if u_n is unbounded in $H_r^1(\mathbb{R}^2)$ and $I_\omega(u_n)$ is bounded from above, then somehow u_n contains a certain mass spreading to infinity, as u_ρ does. This will be made explicit in Proposition 4.2. But this will lead us to a contradiction if J_ω is positive on that mass. This argument is however far from trivial, and is the core of this paper.

Summing up, we are able to relate I_ω to the limit functional J_ω in the following way:

$$\inf I_\omega > -\infty \;\Leftrightarrow\; \inf J_\omega = 0.$$

Moreover, this characterization will give us the threshold value for ω, since the critical points of J_ω can be found explicitly, as will be shown in the next section.

3. The limit problem

In this section we deal with the limit functional $J_\omega : H^1(\mathbb{R}) \to \mathbb{R}$ of (9).

Clearly, the Euler–Lagrange equation of (9) is

$$-u'' + \omega u + \frac{1}{4}\left(\int_{-\infty}^{\infty} u^2(s)\, ds\right)^2 u = |u|^{p-1}u \quad \text{in } \mathbb{R} \quad (10)$$

Later, we will find explicit solutions of (10). But first let us study it from a variational point of view: this study will give us some further information on the solutions.

Before going on, we need a technical result; we think it must be well-known, but we have not been able to find an explicit reference.

Lemma 3.1. *Let $u_n \in H^1(\mathbb{R})$ be a sequence of even nonnegative functions which are decreasing in $r > 0$, and assume that $u_n \rightharpoonup u_0$ weakly in $H^1(\mathbb{R})$. Then u_0 is also even, nonnegative and decreasing in $r > 0$, and $u_n \to u_0$ in $L^q(\mathbb{R})$ for any $q \in (2, \infty)$.*

Proof. Observe that the set $A = \{u \in H^1(\mathbb{R}) : u$ is nonnegative, even and decreasing in $r > 0\}$ is a closed and convex subset of $H^1(\mathbb{R})$. As a consequence, $u_0 \in A$.

Then, for any $r \in \mathbb{R}$, $r \neq 0$,

$$C \geq \left| \int_0^r u_n^2(s)\, ds \right| \geq u_n^2(r)|r|, \quad \text{so} \quad u_n(r) \leq \frac{C}{\sqrt{|r|}},$$

and the same estimate works for u_0. With this inequality, we can estimate

$$\begin{aligned} \int_{-\infty}^{\infty} |u_n - u_0|^q\, dr &\leq \int_{-R}^{R} |u_n - u_0|^q\, dr + 2C \int_{|r|>R} r^{-q/2}\, dr \\ &= \int_{-R}^{R} |u_n - u_0|^q\, dr + 4C \frac{2}{2-q} R^{(2-q)/2}. \end{aligned}$$

Taking into account that, by the Rellich–Kondrashov Theorem, $u_n \to u_0$ in $L^q(-R, R)$ for any $R > 0$ fixed, the above inequality implies that $u_n \to u_0$ in $L^q(\mathbb{R})$. □

Some properties of the functional J_ω are discussed below:

Proposition 3.2. *Consider the functional J_ω with $p \in (1, 3)$ and $\omega > 0$. Then:*

(a) *J_ω is coercive and attains its infimum.*
(b) *0 is a local minimum of J_ω. Indeed, there exists $r_0 > 0$ with the following property: for any $r \in (0, r_0)$, there exists $\alpha > 0$ such that $J_\omega(u) > \alpha$ for any $u \in H^1(\mathbb{R})$ with $\|u\|_{H^1(\mathbb{R})} = r$.*
(c) *There exists $\omega_0 > 0$ such that* $\min J_\omega < 0$ *if and only if $\omega \in [0, \omega_0)$.*

Proof. (a) To prove coercivity, we use the Gagliardo–Nirenberg inequality:

$$\|u\|_{L^4} \leq C \|u'\|_{L^2}^{1/4} \|u\|_{L^2}^{3/4}.$$

Hence

$$\int_{-\infty}^{\infty} u^4\, dr \leq \frac{C}{2} \left[\int_{-\infty}^{\infty} |u'|^2\, dr + \left(\int_{-\infty}^{\infty} u^2\, dr \right)^3 \right].$$

Then

$$J_\omega(u) \geq \frac{1}{4} \int_{-\infty}^{\infty} |u'|^2\, dr + \frac{1}{48} \left(\int_{-\infty}^{\infty} u^2\, dr \right)^3 + c \int_{-\infty}^{\infty} u^4\, dr - \frac{1}{p+1} \int_{-\infty}^{\infty} |u|^{p+1}\, dr. \tag{11}$$

Observe that for any $C > 0$ we can choose $D > 0$ so that $t^3 \geq Ct - D$ for every $t \geq 0$. Applying this with $t = \int_{-\infty}^{\infty} u^2\, dr$ to (11), and renaming C, we obtain

$$J_\omega(u) \geq \frac{1}{4}\int_{-\infty}^{\infty} |u'|^2\, dr + \int_{-\infty}^{\infty}\left(Cu^2 + cu^4 - \frac{1}{p+1}|u|^{p+1}\right) dr - D.$$

Now, it suffices to take C so that $Cu^2 + cu^4 - \frac{1}{p+1}|u|^{p+1} \geq 0$ for any $u \in \mathbb{R}$.

Take now u_n such that $J_\omega(u_n) \to \inf J_\omega$. From coercivity, it follows that u_n is bounded. Consider now the sequence $v_n = |u_n|^*$ of nonnegative symmetrized functions. Clearly, v_n is also bounded, and it is easy to observe that $\inf J_\omega \leq J_\omega(v_n) \leq J_\omega(u_n) \to \inf J_\omega$.

Assume, passing to a subsequence, that $v_n \rightharpoonup v$ weakly in $H^1(\mathbb{R})$. By Lemma 3.1, $v_n \to v$ in $L^{p+1}(\mathbb{R})$. The weak lower semicontinuity of the norm allows us to conclude that u is a minimizer of J_ω.

(b) The proof is quite standard: by the Sobolev inequality,

$$J_\omega(u) \geq \tfrac{1}{2}\min\{1, \omega\}\|u\|^2_{H^1(\mathbb{R})} - C\|u\|^{p+1}_{H^1(\mathbb{R})}.$$

(c) Define $\phi : [0, \infty) \to \mathbb{R}$ by $\phi(\omega) = \min J_\omega$. It is easy to check that ϕ is increasing and continuous. Moreover, $\phi(\omega) \leq 0$ for all ω (observe that $J_\omega(0) = 0$).

We claim that $\phi(\omega) = 0$ for large ω. Indeed, by the same arguments of the proof of (a),

$$J_\omega(u) \geq \int_{-\infty}^{\infty}\left(\frac{\omega}{2}u^2 + cu^4 - \frac{1}{p+1}|u|^{p+1}\right) dr.$$

For ω sufficiently large, $\frac{\omega}{2}u^2 + cu^4 - \frac{1}{p+1}|u|^{p+1} \geq 0$ for any $u \in \mathbb{R}$. Hence $J_\omega(u) \geq 0$ for any $u \in H^1(\mathbb{R})$, proving the claim.

We now show that $\phi(0) < 0$. To this end, fix $u \in H^1(\mathbb{R})$ and define $u_\lambda(r) = \lambda^{2/(p-1)}u(\lambda r)$. Then

$$J_0(u_\lambda) = \frac{1}{2}\lambda^{\frac{p+3}{p-1}}\int_{-\infty}^{\infty}|u'|^2\, dr + \frac{1}{24}\lambda^{\frac{3(5-p)}{p-1}}\left(\int_{-\infty}^{\infty} u^2\, dr\right)^3 - \frac{1}{p+1}\lambda^{\frac{p+3}{p-1}}\int_{-\infty}^{\infty}|u|^{p+1}\, dr.$$

Therefore, for λ sufficiently small, $J_0(u_\lambda)$ has the sign of the term

$$\frac{1}{2}\int_{-\infty}^{\infty}|u'|^2\, dr - \frac{1}{p+1}\int_{-\infty}^{\infty}|u|^{p+1}\, dr.$$

It suffices to take u such that this quantity is negative to conclude.

So, we can define $\omega_0 = \min\{\omega \geq 0 : \phi(\omega) = 0\} > 0$. □

As a consequence of the previous result, for $\omega \in [0, \omega_0)$ there exists a nontrivial solution for (10), which corresponds to a global minimum of J_ω. As announced in the introduction, the expression for ω_0 will be found later on.

We now turn to finding explicit solutions of problem (10). For any $k > 0$ we denote by $w_k \in H^1(\mathbb{R})$ the unique positive radial solution of

$$-w_k'' + k w_k = w_k^p \quad \text{in } \mathbb{R}. \tag{12}$$

Let us state some well-known properties of this equation. First, the Hamiltonian of w_k is equal to 0, that is,

$$-\frac{1}{2}|w_k'(r)|^2 + \frac{k}{2} w_k^2(r) - \frac{1}{p+1} w_k^{p+1}(r) = 0 \quad \text{for all } r \in \mathbb{R}. \tag{13}$$

It is also known that any solution of (12) is of the form $u(x) = \pm w_k(x - y)$ for some $y \in \mathbb{R}$. Moreover,

$$w_k(r) = k^{1/(p-1)} w_1(\sqrt{k}\, r), \quad \text{where} \quad w_1(r) = \left(\frac{2}{p+1}\cosh^2\left(\frac{p-1}{2} r\right)\right)^{\frac{1}{1-p}}. \tag{14}$$

In what follows we define

$$m = \int_{-\infty}^{\infty} w_1^2 \, dr.$$

The following relations are also well known, and can be deduced from (13):

$$\int_{-\infty}^{\infty} |w_1'|^2 \, dr = \frac{p-1}{p+3} m, \qquad \int_{-\infty}^{\infty} w_1^{p+1} \, dr = \frac{2(p+1)}{p+3} m. \tag{15}$$

Proposition 3.3. *Consider the equation*

$$k = \omega + \tfrac{1}{4} m^2 k^{\frac{5-p}{p-1}}, \qquad k > 0. \tag{16}$$

Then u is a nontrivial solution of (10) if and only if $u(r) = w_k(r - \xi)$ for some $\xi \in \mathbb{R}$ and k a root of (16).

Define

$$\omega_1 = \left(\frac{(5-p)m^2}{4(p-1)}\right)^{-\frac{p-1}{2(3-p)}} - \frac{m^2}{4}\left(\frac{(5-p)m^2}{4(p-1)}\right)^{-\frac{(5-p)}{2(3-p)}}. \tag{17}$$

Then:

- *if $\omega > \omega_1$, equation (16) has no solution and there is no nontrivial solution of (10);*
- *if $\omega = \omega_1$, equation (16) has a unique solution k_0, and $w_{k_0}(r)$ is the only nontrivial solution of (10) (up to translations);*
- *if $\omega \in (0, \omega_1)$, equation (16) has two solutions $k_1(\omega) < k_2(\omega)$ and $w_{k_1}(r)$, $w_{k_2}(r)$ are the only two nontrivial solutions of (10) (up to translations).*

Proof. Let u be a nontrivial solution of (10), and define $k = \omega + \frac{1}{4}(\int_{-\infty}^{\infty} u^2\,dr)^2$. Then u is a solution of $-u'' + ku = u^p$, so $u(r) = w_k(r - \xi)$ for some $\xi \in \mathbb{R}$. By using (14), we obtain

$$k = \omega + \frac{1}{4}\left(\int_{-\infty}^{\infty} w_k^2(r)\,dr\right)^2 = \omega + \frac{1}{4}k^{4/(p-1)}\left(\int_{-\infty}^{\infty} w_1^2(\sqrt{k}\,r)\,dr\right)^2.$$

A change of variables leads us to equation (16).

Moreover,

$$1 < p < 3, \quad \text{so} \quad \frac{5-p}{p-1} > 1.$$

Therefore, the function $(0,\infty) \ni k \mapsto k^{(5-p)/(p-1)}$ is convex. Hence there exists $\omega_1 > 0$ with the properties indicated.

In order to get the exact value of ω_1, observe that the function $k \mapsto \omega_1 + \frac{1}{4}m^2 k^{(5-p)/(p-1)} - k$ has a degenerate zero. Then ω_1 solves the system

$$\begin{cases} \omega + \frac{1}{4}m^2 k^{\frac{5-p}{p-1}} = k, \\ \dfrac{5-p}{4(p-1)} m^2 k^{\frac{5-p}{p-1}-1} = 1. \end{cases}$$

From this one obtains formula (17). □

In our next result, we deduce information from Proposition 3.3.

Proposition 3.4. *Let $\omega_0,\ \omega_1$ be the values defined in Propositions 3.2 and 3.3 . Then:*

- *$\omega_0 < \omega_1$, and ω_0 has the expression*

$$\omega_0 = \frac{3-p}{3+p}\, 3^{\frac{p-1}{2(3-p)}}\, 2^{\frac{2}{3-p}} \left(\frac{m^2(3+p)}{p-1}\right)^{-\frac{p-1}{2(3-p)}}, \tag{18}$$

  where m is as in (3).
- *For any $\omega \in (0,\omega_1)$, $J_\omega(w_{k_1}) > J_\omega(w_{k_2})$. In particular, for any $\omega \in (0,\omega_0)$, w_{k_2} is a global minimizer of J_ω.*

Proof. We consider the energy functional J_ω evaluated on the curve $k \mapsto w_k$. In the computations that follow we use (14) and a change of variables. We have

$$\begin{aligned} \psi(k) := J_\omega(w_k) = {} & \frac{k^{\frac{3+p}{2(p-1)}}}{2}\int_{-\infty}^{\infty} |w_1'(r)|^2\,dr + \omega\frac{k^{\frac{5-p}{2(p-1)}}}{2}\int_{-\infty}^{\infty} w_1^2(r)\,dr \\ & + \frac{k^{\frac{3(5-p)}{2(p-1)}}}{24}\left(\int_{-\infty}^{\infty} w_1^2(r)\,dr\right)^3 - \frac{k^{\frac{3+p}{2(p-1)}}}{p+1}\int_{-\infty}^{\infty} |w_1(r)|^{p+1}\,dr. \end{aligned}$$

Plugging (15) into that expression, we get

$$\psi(k) = m\left[\frac{p-5}{2(3+p)}k^{\frac{3+p}{2(p-1)}} + \frac{\omega}{2}k^{\frac{5-p}{2(p-1)}} + \frac{m^2}{24}k^{\frac{3(5-p)}{2(p-1)}}\right].$$

Then
$$\frac{d}{dk}\psi(k) = mk^{\frac{7-3p}{2(p-1)}}\frac{5-p}{4(p-1)}\left[-k+\omega+\frac{1}{4}m^2k^{\frac{5-p}{p-1}}\right].$$
In particular, the roots of (16) are exactly the critical points of ψ. Observe that
$$\frac{5-p}{2(p-1)} < \frac{3+p}{2(p-1)} < \frac{3(5-p)}{2(p-1)}.$$
Hence ψ is increasing near 0 (for $\omega > 0$) and near infinity. Therefore, for $\omega \in (0, \omega_1)$, its first root corresponds to a local maximum of ψ and the second one to a local minimum, so $J(w_{k_1}) > J(w_{k_2})$. Take now $\omega \in (0, \omega_0)$. Since in this case the minimizer is nontrivial, it must correspond to w_{k_2}. Moreover, $\omega_0 < \omega_1$.

In order to get the value of ω_0, observe that $J_{\omega_0}(w_{k_2}) = 0$. Therefore, $\omega_0 > 0$ solves
$$\begin{cases} \omega + \frac{1}{4}m^2k^{\frac{5-p}{p-1}} = k, \\ \frac{p-5}{2(3+p)}k^{\frac{3+p}{2(p-1)}} + \frac{\omega}{2}k^{\frac{5-p}{2(p-1)}} + \frac{m^2}{24}k^{\frac{3(5-p)}{2(p-1)}} = 0. \end{cases}$$
From this, expression (18) follows. □

Remark 3.5. Observe that the map ψ defined in the proof of Proposition 3.4 gives us a quite clear interpretation of the functional J_ω. Indeed, k is a critical point of ψ if and only if w_k is a critical point of J_ω. Moreover, the following hold:

- If $\omega > \omega_1$, then ψ is positive and increasing without critical points.
- If $\omega = \omega_1$, then ψ is still positive and increasing, but it has an inflection point at $k = k_0$.
- If $\omega \in (0, \omega_1)$, then ψ has a local maximum and minimum attained at k_1 and k_2, respectively.
- If $\omega = \omega_0$, then $\psi(k_2) = 0$. In this case, the minimum of J_{ω_0} is 0, and is attained at 0 and w_{k_2}.
- If $\omega \in [0, \omega_0)$, then $\psi(k_2) < 0$ and w_{k_2} is the unique global minimizer, with $J_\omega(w_{k_2}) < 0$.

Remark 3.6. In general, we cannot obtain a more explicit expression of m depending on p, but it can be easily approximated by using some software. In Figure 1 the maps $\omega_0(p)$ and $\omega_1(p)$ have been plotted.

For some specific values of p, m can be explicitly computed, and hence ω_0 and ω_1. For instance, if $p = 2$, $m = 6$, then $\omega_1 = \frac{2}{9\sqrt{3}}$ and $\omega_0 = \frac{2}{5\sqrt{15}}$.

We finish this section with a technical result that will be of use later in the proof of Theorem 1.1.

Proposition 3.7. *Assume $\omega \geq \omega_0$, and $u_n \in H^1(\mathbb{R})$ are such that $J_\omega(u_n) \to 0$. Then:*

- *if $\omega > \omega_0$, then $u_n \to 0$ in $H^1(\mathbb{R})$;*
- *if $\omega = \omega_0$, then, up to a subsequence, either $u_n \to 0$ or $u_n(\cdot - x_n) \to w_{k_2}$ in $H^1(\mathbb{R})$, for some sequence $x_n \in \mathbb{R}$.*

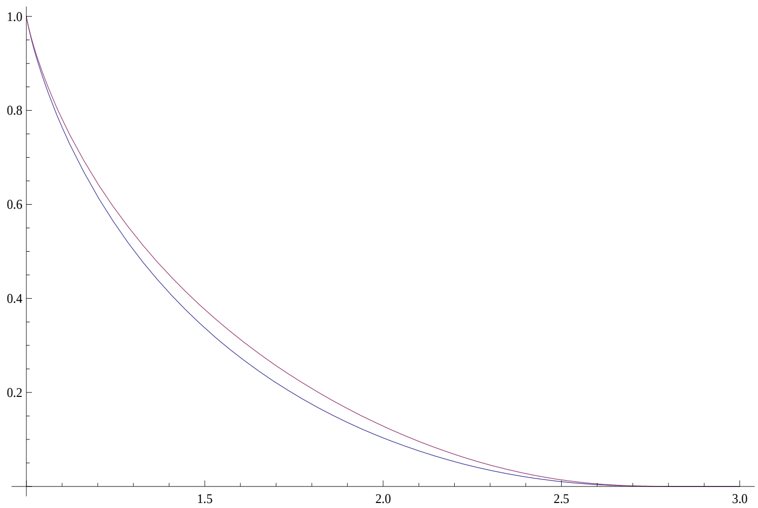

Fig. 1. The values $\omega_0(p) < \omega_1(p)$ for $p \in (1, 3)$.

Proof. Since J_ω is coercive, we know that u_n is bounded. If $u_n \to 0$ in $H^1(\mathbb{R})$, we are done. Otherwise, we have

$$o_n(1) = J_\omega(u_n) \geq \frac{1}{2}\int_{-\infty}^{\infty}\big(|u_n'(r)|^2 + \omega u_n^2(r)\big)dr - \frac{1}{p+1}\int_{-\infty}^{\infty}|u_n(r)|^{p+1}\,dr.$$

Thus, $u_n \not\to 0$ in $L^{p+1}(\mathbb{R})$. The concentration-compactness lemma (see [16, Lemma I.1]) shows that there exists $\xi_n \in \mathbb{R}$ such that $\int_{\xi_n-1}^{\xi_n+1} u_n^2 \geq \varepsilon > 0$. Therefore, $\tilde{u}_n(r) = u_n(r-\xi_n) \rightharpoonup u \neq 0$ weakly in $H^1(\mathbb{R})$. Define $v_n = \tilde{u}_n - u$, which clearly converges weakly to 0 in $H^1(\mathbb{R})$.

Step 1: $v_n \to 0$ **in** $L^2(\mathbb{R})$**.** We just compute

$$\begin{aligned} o_n(1) &= J_\omega(u_n) = J_\omega(\tilde{u}_n) = J_\omega(v_n + u) \\ &= \frac{1}{2}\int_{-\infty}^{\infty}(|v_n'|^2 + |u'|^2 + 2v_n'u')\,dr + \frac{\omega}{2}\int_{-\infty}^{\infty}(v_n^2 + u^2 + 2v_n u)\,dr \\ &\quad + \frac{1}{8}\Bigg[\left(\int_{-\infty}^{\infty} v_n^2\,dr\right)^3 + \left(\int_{-\infty}^{\infty} u^2\,dr\right)^3 + 3\left(\int_{-\infty}^{\infty} v_n^2\,dr\right)^2\left(\int_{-\infty}^{\infty} u^2\,dr\right) \\ &\quad + 3\left(\int_{-\infty}^{\infty} v_n^2\,dr\right)\left(\int_{-\infty}^{\infty} u^2\,dr\right)^2\Bigg] - \frac{1}{p+1}\int_{-\infty}^{\infty}|v_n + u|^{p+1}\,dr + o_n(1). \end{aligned}$$

Here the mixed products converge to zero, since $v_n \rightharpoonup 0$. Passing to a subsequence, we can assume that $v_n \to 0$ almost everywhere. Then the well-known Brezis–Lieb lemma [3] implies that

$$\int_{-\infty}^{\infty}|v_n + u|^{p+1}\,dr - \int_{-\infty}^{\infty}(|v_n|^{p+1} + |u|^{p+1})\,dr \to 0.$$

Then

$$o_n(1) = J_\omega(u_n) = J_\omega(v_n) + J_\omega(u) + \frac{3}{8}\Big[\Big(\int_{-\infty}^{\infty} v_n^2\,dr\Big)^2\Big(\int_{-\infty}^{\infty} u^2\,dr\Big) + \Big(\int_{-\infty}^{\infty} v_n^2\,dr\Big)\Big(\int_{-\infty}^{\infty} u^2\,dr\Big)^2\Big] + o_n(1).$$

It is here that the assumption $\omega \geq \omega_0$ is crucial. Indeed, it implies that $J_\omega(v_n) \geq 0$ and $J_\omega(u) \geq 0$. Recall that $u \neq 0$ to conclude the proof of Step 1.

Step 2: Conclusion. By interpolation,

$$\|v_n\|_{L^{p+1}} \leq \|v_n\|_{L^2}^{\alpha}\|v_n\|_{L^{p+2}}^{1-\alpha}$$

for some $\alpha \in (0, 1)$. Since v_n is bounded in $H^1(\mathbb{R})$, all norms above are bounded. Hence, by Step 1, $\|v_n\|_{L^{p+1}} \to 0$. In other words, $\tilde{u}_n \to u$ in $L^{p+1}(\mathbb{R})$.

From this it is easy to conclude the proof. Indeed,

$$o_n(1) = J_\omega(\tilde{u}_n) = \frac{1}{2}\int_{-\infty}^{\infty}(|\tilde{u}_n'|^2+\omega\tilde{u}_n^2)\,dr + \frac{1}{8}\Big(\int_{-\infty}^{\infty}\tilde{u}_n^2\,dr\Big)^3 - \frac{1}{p+1}\int_{-\infty}^{\infty}|\tilde{u}_n|^{p+1}\,dr,$$

$$0 \leq J_\omega(u) = \frac{1}{2}\int_{-\infty}^{\infty}(|u'|^2+\omega u^2)\,dr + \frac{1}{8}\Big(\int_{-\infty}^{\infty}u^2\,dr\Big)^3 - \frac{1}{p+1}\int_{-\infty}^{\infty}|u|^{p+1}\,dr.$$

Thus, $\|\tilde{u}_n\|_{H^1(\mathbb{R})} \to \|u\|_{H^1(\mathbb{R})}$. This implies $\tilde{u}_n \to u$ in $H^1(\mathbb{R})$, finishing the proof. □

4. Proof of Theorems 1.1 and 1.2

Lemma 4.1. *Let $U \in H^1(\mathbb{R})$ be an even function which decays to zero exponentially at infinity, and define $U_\rho(r) = U(r-\rho)$. Then there exists $C > 0$ such that*

$$I_\omega(U_\rho) = 2\pi\rho J_\omega(U) - C + o_\rho(1).$$

Proof. We have

$$(2\pi)^{-1}I_\omega(U_\rho) = \frac{1}{2}\int_0^{\infty}(|U_\rho'|^2 + \omega U_\rho^2)r\,dr + \frac{1}{8}\int_0^{\infty}\frac{U_\rho^2(r)}{r}\Big(\int_0^r sU_\rho^2(s)\,ds\Big)^2 dr - \frac{1}{p+1}\int_0^{\infty}|U_\rho|^{p+1}r\,dr. \tag{19}$$

Let us first evaluate the local terms. By the evenness and the exponential decay of U, we get

$$\int_0^{\infty}|U_\rho'|^2 r\,dr = \int_{-\infty}^{\infty}|U'(r-\rho)|^2(r-\rho)\,dr + \rho\int_{-\infty}^{\infty}|U'(r-\rho)|^2\,dr + o_\rho(1) = \rho\int_{-\infty}^{\infty}|U'|^2\,dr + o_\rho(1). \tag{20}$$

Analogously,

$$\int_0^\infty U_\rho^2 r\,dr = \rho\int_{-\infty}^\infty U^2\,dr + o_\rho(1), \tag{21}$$

$$\int_0^\infty |U_\rho|^{p+1} r\,dr = \rho\int_{-\infty}^\infty |U|^{p+1}\,dr + o_\rho(1). \tag{22}$$

For the nonlocal term, we have

$$\begin{aligned}
&\int_0^\infty \frac{U_\rho^2(r)}{r}\Big(\int_0^r sU_\rho^2(s)\,ds\Big)^2 dr - \rho\int_0^\infty U_\rho^2(r)\Big(\int_0^r U_\rho^2(s)\,ds\Big)^2 dr\\
&\qquad = \underbrace{\int_0^\infty U_\rho^2(r)\Big(\frac{1}{r}-\frac{1}{\rho}\Big)\Big(\int_0^r sU_\rho^2(s)\,ds\Big)^2 dr}_{(I)}\\
&\qquad\quad + \underbrace{\frac{1}{\rho}\int_0^\infty U_\rho^2(r)\Big[\Big(\int_0^r sU_\rho^2(s)\,ds\Big)^2 - \Big(\int_0^r \rho U_\rho^2(s)\,ds\Big)^2\Big] dr}_{(II)}.
\end{aligned}$$

Let us study the term (I):

$$\begin{aligned}
(I) &= \int_{-\infty}^\infty U_\rho^2(r)\frac{\rho-r}{r\rho}\Big(\int_{-\infty}^r sU_\rho^2(s)\,ds\Big)^2 dr + o_\rho(1)\\
&= -\int_{-\infty}^\infty U^2(r)\frac{r}{(\rho+r)\rho}\Big(\int_{-\infty}^r (s+\rho)U^2(s)\,ds\Big)^2 dr + o_\rho(1)\\
&= \int_0^\infty U^2(r)\frac{r}{(\rho-r)\rho}\Big(\int_{-\infty}^{-r} (s+\rho)U^2(s)\,ds\Big)^2 dr\\
&\quad - \int_0^\infty U^2(r)\frac{r}{(\rho+r)\rho}\Big(\int_{-\infty}^r (s+\rho)U^2(s)\,ds\Big)^2 dr + o_\rho(1)\\
&= \int_0^\infty U^2(r)\Big(\frac{r}{(\rho-r)\rho} - \frac{r}{(\rho+r)\rho}\Big)\Big(\int_{-\infty}^{-r} (s+\rho)U^2(s)\,ds\Big)^2 dr\\
&\quad + \int_0^\infty U^2(r)\frac{r}{(\rho+r)\rho}\Big[\Big(\int_{-\infty}^{-r} (s+\rho)U^2(s)\,ds\Big)^2 - \Big(\int_{-\infty}^r (s+\rho)U^2(s)\,ds\Big)^2\Big] dr\\
&\quad + o_\rho(1)\\
&= \frac{1}{\rho}\int_0^\infty U^2(r)\Big(\frac{2r^2\rho^2}{(\rho-r)(\rho+r)}\Big)\Big(\int_{-\infty}^{-r} \frac{s+\rho}{\rho}\,U^2(s)\,ds\Big)^2 dr\\
&\quad + \int_0^\infty U^2(r)\frac{r\rho}{(\rho+r)}\Big[\Big(\int_{-\infty}^{-r} \frac{s+\rho}{\rho}\,U^2(s)\,ds\Big)^2 - \Big(\int_{-\infty}^r \frac{s+\rho}{\rho}\,U^2(s)\,ds\Big)^2\Big] dr\\
&\quad + o_\rho(1).
\end{aligned}$$

Passing to the limit by the Lebesgue Theorem, we obtain

$$(I) = \int_0^\infty U^2(r) r \left[\left(\int_{-\infty}^{-r} U^2(s)\,ds \right)^2 - \left(\int_{-\infty}^{r} U^2(s)\,ds \right)^2 \right] dr + o_\rho(1)$$
$$= -C_I + o_\rho(1).$$

Let us study the term (II):

$$(II) = \frac{1}{\rho} \int_0^\infty U_\rho^2(r) \left(\int_0^r (s+\rho) U_\rho^2(s)\,ds \right) \left(\int_0^r (s-\rho) U_\rho^2(s)\,ds \right) dr$$
$$= \frac{1}{\rho} \int_{-\infty}^\infty U_\rho^2(r) \left(\int_{-\infty}^r (s+\rho) U_\rho^2(s)\,ds \right) \left(\int_{-\infty}^r (s-\rho) U_\rho^2(s)\,ds \right) dr + o_\rho(1)$$
$$= \int_{-\infty}^\infty U^2(r) \left(\int_{-\infty}^r \frac{s+2\rho}{\rho} U^2(s)\,ds \right) \left(\int_{-\infty}^r s U^2(s)\,ds \right) dr + o_\rho(1).$$

Again by the Lebesgue Theorem,

$$(II) = 2 \int_{-\infty}^\infty U^2(r) \left(\int_{-\infty}^r U^2(s)\,ds \right) \left(\int_{-\infty}^r s U^2(s)\,ds \right) dr + o_\rho(1) = -C_{II} + o_\rho(1).$$

Observe that the above expression is negative since the function $r \mapsto \int_{-\infty}^r s U^2(s)\,ds$ is negative. Therefore, denoting $C = C_I + C_{II} > 0$, we have

$$\int_0^\infty \frac{U_\rho^2(r)}{r} \left(\int_0^r s U_\rho^2(s)\,ds \right)^2 dr = \rho \int_0^\infty U_\rho^2(r) \left(\int_0^r U_\rho^2(s)\,ds \right)^2 dr - C + o_\rho(1). \tag{23}$$

Hence the conclusion follows from (19)–(23). □

In our next result we study the behavior of unbounded sequences with energy bounded from above. This will be essential for the proof of Theorems 1.1 and 1.2.

Proposition 4.2. *Assume $\omega > 0$ and $u_n \in H_r^1(\mathbb{R}^2)$ are such that $\|u_n\|$ is unbounded but $I_\omega(u_n)$ is bounded from above. Then there exists a subsequence (still denoted by u_n) such that:*

(i) *for all $\varepsilon > 0$, $\int_{\varepsilon \|u_n\|^2}^\infty (|u_n'|^2 + u_n^2)\,dr \le C$;*

(ii) *there exists $\delta \in (0,1)$ such that $\int_{\delta \|u_n\|^2}^{\delta^{-1}\|u_n\|^2} (|u_n'|^2 + u_n^2)\,dr \ge c > 0$;*

(iii) $\|u_n\|_{L^2(\mathbb{R}^2)} \to \infty$.

Proof. The beginning of the proof follows the ideas of [20, Theorem 4.3]. The main difference is that here we cannot conclude directly that I_ω is bounded from below, and indeed this fact depends on ω. The proof of Theorem 1.1 will require much more work.

We start by using inequality (8) and the Cauchy–Schwarz inequality to estimate I_ω:

$$I_\omega(u) \ge \frac{\pi}{2} \int_0^\infty (|u'|^2 + \omega u^2) r\,dr + \frac{\pi}{8} \int_0^\infty \frac{u^2(r)}{r} \left(\int_0^r s u^2(s)\,ds \right)^2 dr$$
$$+ 2\pi \int_0^\infty \left(\frac{\omega}{4} u^2 + \frac{1}{8} u^4 - \frac{1}{p+1} |u|^{p+1} \right) r\,dr. \tag{24}$$

Define

$$f : \mathbb{R}_+ \to \mathbb{R}, \qquad f(t) = \frac{\omega}{4}t^2 + \frac{1}{8}t^4 - \frac{1}{p+1}t^{p+1}.$$

Then the set $\{t > 0 : f(t) < 0\}$ is of the form (α, β), where α, β are positive constants depending only on p, ω. Moreover, we denote $-c_0 = \min f < 0$.

For each function u_n, we define

$$A_n = \{x \in \mathbb{R}^2 : u_n(x) \in (\alpha, \beta)\}, \qquad \rho_n = \sup\{|x| : x \in A_n\}.$$

With these definitions, we can rewrite (24) in the form

$$I_\omega(u_n) \geq \frac{\pi}{2}\int_0^\infty (|u_n'|^2 + \omega u_n^2) r\, dr + \frac{\pi}{8}\int_0^\infty \frac{u_n^2(r)}{r}\left(\int_0^r s u_n^2(s)\, ds\right)^2 dr - c_0|A_n|. \tag{25}$$

In particular this implies that $|A_n|$ must diverge, and hence so does ρ_n. This already proves (iii).

By Strauss's Lemma [21], we have

$$\alpha \leq u_n(\rho_n) \leq \frac{\|u_n\|}{\sqrt{\rho_n}}, \quad \text{so} \quad \|u_n\|^2 \geq \alpha^2 \rho_n. \tag{26}$$

We now estimate the nonlocal term. For that, define

$$B_n = A_n \cap B(0, \gamma_n) \quad \text{for } \gamma_n \in (0, \rho_n) \text{ such that } |B_n| = \tfrac{1}{2}|A_n|. \tag{27}$$

Then

$$\begin{aligned}\int_0^\infty \frac{u_n^2(r)}{r}\left(\int_0^r s u_n^2(s)\, ds\right)^2 dr &\geq \frac{1}{4\pi^2}\int_{\gamma_n}^\infty \frac{u_n^2(r)}{r}\left(\int_{B_n} u_n^2(x)\, dx\right)^2 dr \\ &\geq c|A_n|^2 \int_{\gamma_n}^\infty \frac{u_n^2(r)}{r}\, dr \geq c|A_n|^2 \int_{A_n\setminus B_n} \frac{u_n^2(x)}{|x|^2}\, dx \\ &\geq c\frac{|A_n|^2}{\rho_n^2}\int_{A_n\setminus B_n} u_n^2(x)\, dx \geq c\frac{|A_n|^3}{\rho_n^2}. \end{aligned} \tag{28}$$

Hence, by (24), (26) and (28), we get

$$I_\omega(u_n) \geq c\rho_n + c\frac{|A_n|^3}{\rho_n^2} - c_0|A_n| = \rho_n\left(c + c\frac{|A_n|^3}{\rho_n^3} - c_0\frac{|A_n|}{\rho_n}\right).$$

Observe that $t \mapsto c + ct^3 - c_0 t$ is strictly positive near zero and goes to ∞ as $t \to \infty$. Then we can assume, passing to a subsequence, that $|A_n| \sim \rho_n$. In other words, there exists $m > 0$ such that $\rho_n |A_n|^{-1} \to m$ as $n \to \infty$.

Taking into account (25) we conclude that up to a subsequence, $\|u_n\|^2 \sim \rho_n$. Moreover, for any fixed $\varepsilon > 0$, we have

$$C\rho_n \geq \|u_n\|_{L^2}^2 \geq \int_{\varepsilon\rho_n}^\infty u_n^2 r\, dr \geq \varepsilon\rho_n \int_{\varepsilon\rho_n}^\infty u_n^2\, dr.$$

An analogous estimate works also for $\int_{\varepsilon\rho_n}^\infty |u_n'|^2\, dr$. This proves (i).

We now show that for some $\delta > 0$, $\|u_n\|_{H^1(\delta\rho_n,\rho_n)} \nrightarrow 0$, which implies (ii). First, recall the definition of B_n and γ_n in (27). Then

$$\int_{\gamma_n}^{\rho_n} u_n^2(r)\,dr \geq \rho_n^{-1}\int_{\gamma_n}^{\rho_n} u_n^2(r)r\,dr \geq \rho_n^{-1}\int_{A_n\setminus B_n} u_n^2(x)dx \geq \rho_n^{-1}|A_n\setminus B_n|\alpha^2 > c > 0.$$

To conclude it suffices to show that $\gamma_n \sim \rho_n$. Indeed, define

$$C_n = B_n \cap B(0,\tau_n) \quad \text{for } \tau_n \in (0,\gamma_n) \text{ such that } |C_n| = \tfrac{1}{2}|B_n|. \tag{29}$$

We can repeat the estimate (28) with A_n, B_n replaced with B_n, C_n respectively to obtain

$$\int_0^\infty \frac{u_n^2(r)}{r}\left(\int_0^r s u_n^2(s)\,ds\right)^2 dr \geq c\frac{|B_n|^3}{\gamma_n^2}.$$

Hence,

$$I_\omega(u_n) \geq c\rho_n + c\frac{|A_n|^3}{\gamma_n^2} - c_0|A_n| = \gamma_n\left(c\frac{\rho_n}{\gamma_n} + c\frac{|A_n|^3}{\gamma_n^3} - c_0\frac{|A_n|}{\gamma_n}\right).$$

And we are done since $I_\omega(u_n)$ is bounded from above. □

Proof of Theorem 1.1. If $\omega \in (0,\omega_0)$, then $J_\omega(w_{k_2}) < 0$ (see Proposition 3.2): applying Lemma 4.1 to $U = w_{k_2}$ we deduce assertion (i).

We now prove (ii) and (iii). Let us denote by $H^1_{0,r}(B(0,R))$ the Sobolev space of radial functions with zero boundary value. Given any $n \in \mathbb{N}$, Proposition 4.2 implies that $I_\omega|_{H^1_{0,r}(B(0,n))}$ is coercive (indeed, this is an immediate consequence of (24)). So, there exists a minimizer u_n for $I_\omega|_{H^1_{0,r}(B(0,n))}$. Moreover,

$$I_\omega(u_n) \to \inf I_\omega \quad \text{as } n \to \infty.$$

If u_n is bounded, then $I_\omega(u_n)$ is also bounded and therefore $\inf I_\omega$ is finite. In what follows we assume that u_n is an unbounded sequence. Then it satisfies the hypotheses of Proposition 4.2. Let $\delta > 0$ be given by that proposition.

The proof will be divided into several steps.

Step 1: $\int_{(\delta/2)\|u_n\|^2}^{(2/\delta)\|u_n\|^2} |u_n|^{p+1}\,dr \nrightarrow 0$. By Proposition 4.2(i), we have

$$\sum_{k=1}^{[(\delta/2)\|u_n\|^2]} \int_{(\delta/2)\|u_n\|^2+k-1}^{(\delta/2)\|u_n\|^2+k} (|u_n'|^2 + u_n^2)\,dr \leq \int_{(\delta/2)\|u_n\|^2}^{\delta\|u_n\|^2} (|u_n'|^2 + u_n^2)\,dr \leq C.$$

Taking the smaller summand on the left hand side we find x_n such that

$$\frac{\delta}{2}\|u_n\|^2 \leq x_n \leq \delta\|u_n\|^2 - 1, \qquad \|u_n\|^2_{H^1(x_n,x_n+1)} \leq \frac{C}{\|u_n\|^2}.$$

Reasoning in an analogous way, we can choose y_n such that

$$\delta^{-1}\|u_n\|^2+1\le y_n\le 2\delta^{-1}\|u_n\|^2, \qquad \|u_n\|^2_{H^1(y_n,y_n+1)}\le \frac{C}{\|u_n\|^2}.$$

Observe that if $\delta^{-1}\|u_n\|^2\ge n$, the choice of y_n can be arbitrary, but this is not necessary. Let $\phi_n:[0,\infty]\to[0,1]$ be a C^∞ function such that

$$\phi_n(r)=\begin{cases}0 & \text{if } r\le x_n,\\ 1 & \text{if } x_n+1\le r\le y_n,\\ 0 & \text{if } r\ge y_n+1.\end{cases} \qquad |\phi_n'(r)|\le 2.$$

We have

$$\begin{aligned}0=I'_\omega(u_n)[\phi_n u_n]&\ge 2\pi\int_{x_n}^{y_n}(|u_n'|^2+\omega u_n^2)r\,dr-2\pi\int_{x_n}^{y_n}|u_n|^{p+1}r\,dr+O(1)\\ &\ge \|u_n\|^2\left(\frac{\delta}{2}\int_{x_n}^{y_n}(|u_n'|^2+\omega u_n^2)\,dr-\frac{2}{\delta}\int_{x_n}^{y_n}|u_n|^{p+1}\,dr\right)+O(1).\end{aligned}$$

Since $\|u_n\|_{H^1(x_n,y_n)}\nrightarrow 0$, this concludes the proof of Step 1.

Step 2: Exponential decay. At this point we can apply the concentration-compactness principle (see [16, Lemma 1.1]): there exists $\sigma>0$ such that

$$\sup_{\xi\in[x_n,\ y_n]}\int_{\xi-1}^{\xi+1}u_n^2\,dr\ge 2\sigma>0.$$

Define

$$D_n=\left\{\xi>0:\int_{\xi-1}^{\xi+1}(|u_n'|^2+u_n^2)\,dr\ge\sigma\right\}\ne\emptyset, \qquad \xi_n=\max D_n\in[x_n,n+1). \tag{30}$$

Observe that $\xi_n\sim\|u_n\|^2$; indeed, $\xi_n\ge x_n\ge c\|u_n\|^2$, and moreover

$$\|u_n\|^2\ge c\int_{\xi_n-1}^{\xi_n+1}(|u_n'|^2+u_n^2)r\,dr\ge c(\xi_n-1)\int_{\xi_n-1}^{\xi_n+1}(|u_n'|^2+u_n^2)\,dr\ge c(\xi_n-1).$$

By definition, $\int_{\zeta-1}^{\zeta+1}(|u_n'|^2+u_n^2)\,dr<\sigma$ for all $\zeta>\xi_n$. By embedding of $H^1(\zeta-1,\zeta+1)$ in L^∞, we have $0<u_n(\zeta)<C\sigma$ for any $\zeta>\xi_n$. From this we will get exponential decay of u_n. Indeed, u_n is a solution of

$$-u_n''(r)-\frac{u'(r)}{r}+\omega u_n(r)+f_n(r)u_n(r)=|u_n(r)|^{p-1}u_n(r)$$

with

$$f_n(r)=\frac{h_n^2(r)}{r^2}+\int_r^n\frac{h_n(s)}{s}u_n^2(s)\,ds, \qquad h_n(r)=\frac{1}{2}\int_0^r u_n^2(s)s\,ds.$$

It is important to observe that $0 \leq f_n(r) \leq C$ for all $r > \delta\|u_n\|^2$. Then, by taking a smaller σ if necessary, we can conclude that there exists $C > 0$ such that

$$|u_n(r)| < C\exp(-\sqrt{\omega}(r-\xi_n)) \qquad \text{for all } r > \xi_n.$$

The local C^1 regularity theory for the Laplace operator (see [6, Section 3.4]) implies a similar estimate for $u_n'(r)$. In other words,

$$|u_n(r)| + |u_n'(r)| < C\exp(-\sqrt{\omega}(r-\xi_n)) \qquad \text{for all } r > \xi_n. \tag{31}$$

Step 3: Splitting of $I_\omega(u_n)$. Reasoning as at the beginning of Step 1, we can take z_n such that

$$\xi_n - 3\|u_n\| \leq z_n \leq \xi_n - 2\|u_n\|, \qquad \|u_n\|^2_{H^1(z_n, z_n+1)} \leq \frac{C}{\|u_n\|}.$$

Define $\psi_n : [0,\infty] \to [0,1]$ to be a smooth function such that

$$\psi_n(r) = \begin{cases} 0 & \text{if } r \leq z_n, \\ 1 & \text{if } r \geq z_n + 1, \end{cases} \qquad |\psi_n'(r)| \leq 2. \tag{32}$$

In what follows we want to estimate $I_\omega(u_n)$ with $I_\omega(\psi_n u_n)$ and $I_\omega((1-\psi_n)u_n)$. Let us start by evaluating the local terms:

$$\int_0^n |u_n'|^2 r\,dr = \int_0^n |(u_n\psi_n)'|^2 r\,dr + \int_0^n |(u_n(1-\psi_n))'|^2 r\,dr + O(\|u_n\|),$$
$$\int_0^n u_n^2 r\,dr = \int_0^n |u_n\psi_n|^2 r\,dr + \int_0^n |u_n(1-\psi_n)|^2 r\,dr + O(\|u_n\|),$$
$$\int_0^n |u_n|^{p+1} r\,dr = \int_0^n |u_n\psi_n|^{p+1} r\,dr + \int_0^n |u_n(1-\psi_n)|^{p+1} r\,dr + O(\|u_n\|).$$

Let us now study the nonlocal term:

$$\begin{aligned}
\int_0^n \frac{u_n^2(r)}{r}\left(\int_0^r s u_n^2(s)\,ds\right)^2 dr &= \int_0^n \frac{u_n^2(r)\psi_n^2(r)}{r}\left(\int_0^r s u_n^2(s)\psi_n^2(s)\,ds\right)^2 dr \\
&\quad + \int_0^n \frac{u_n^2(r)(1-\psi_n(r))^2}{r}\left(\int_0^r s u_n^2(s)(1-\psi_n(s))^2\,ds\right)^2 dr \\
&\quad + \underbrace{\int_0^n \frac{u_n^2(r)\psi_n^2(r)}{r}\left(\int_0^r s u_n^2(s)(1-\psi_n(s))^2\,ds\right)^2 dr}_{(I)} \\
&\quad + \underbrace{2\int_0^n \frac{u_n^2(r)\psi_n^2(r)}{r}\left(\int_0^r s u_n^2(s)\psi_n^2(s)\,ds\right)\left(\int_0^r s u_n^2(s)(1-\psi_n(s))^2\,ds\right) dr}_{(II)} \\
&\quad + O(\|u_n\|).
\end{aligned}$$

We now estimate:

$$
\begin{aligned}
(I) &\geq 0, \\
(II) &= \int_{z_n}^{n} \frac{u_n^2(r)\psi_n^2(r)}{r}\left(\int_{z_n}^{r} s u_n^2(s)\psi_n^2(s)\,ds\right)\left(\int_0^{z_n+1} s u_n^2(s)(1-\psi_n(s))^2\,ds\right) dr \\
&\quad + O(\|u_n\|) \\
&\geq c_n \|u_n(1-\psi_n)\|^2_{L^2(\mathbb{R}^2)} + O(\|u_n\|),
\end{aligned}
$$

where

$$
c_n = \int_{z_n}^{n} \frac{u_n^2(r)\psi_n^2(r)}{r}\left(\int_{z_n}^{r} s u_n^2(s)\psi_n^2(s)\,ds\right) dr \geq c > 0.
$$

Therefore, we get

$$
I_\omega(u_n) \geq I_\omega(u_n\psi_n) + I_\omega(u_n(1-\psi_n)) + c\|u_n(1-\psi_n)\|^2_{L^2(\mathbb{R}^2)} + O(\|u_n\|). \tag{33}
$$

Step 4: The following estimate holds:

$$
I_\omega(u_n\psi_n) = 2\pi\xi_n J_\omega(u_n\psi_n) + O(\|u_n\|). \tag{34}
$$

Indeed, by taking into account Proposition 4.2, (31) and the definition (32) of ψ_n, we have

$$
\begin{aligned}
&\left|\int_0^n (u_n\psi_n)^2 r\,dr - \xi_n \int_0^n (u_n\psi_n)^2\,dr\right| \leq \int_0^n (u_n\psi_n)^2 |r-\xi_n|\,dr \\
&\quad \leq \int_{\xi_n-3\|u_n\|}^{\xi_n+\|u_n\|} u_n^2|r-\xi_n|\,dr + o(1) \leq O(\|u_n\|)\int_{\xi_n-3\|u_n\|}^{\xi_n+\|u_n\|} u_n^2\,dr + o(1) = O(\|u_n\|).
\end{aligned}
$$

The estimates for the other local terms of I_ω are similar. For the nonlocal term, we get

$$
\begin{aligned}
&\int_0^n \frac{(u_n\psi_n)^2(r)}{r}\left(\int_0^r s(u_n\psi_n)^2(s)\,ds\right)^2 dr - \xi_n \int_0^n (u_n\psi_n)^2(r)\left(\int_0^r (u_n\psi_n)^2(s)\,ds\right)^2 dr \\
&\quad = \underbrace{\int_0^n (u_n\psi_n)^2(r)\left(\frac{1}{r}-\frac{1}{\xi_n}\right)\left(\int_0^r s(u_n\psi_n)^2(s)\,ds\right)^2 dr}_{(I)} \\
&\qquad + \underbrace{\frac{1}{\xi_n}\int_0^n (u_n\psi_n)^2(r)\left[\left(\int_0^r s(u_n\psi_n)^2(s)\,ds\right)^2 - \left(\int_0^r \xi_n(u_n\psi_n)^2(s)\,ds\right)^2\right] dr}_{(II)},
\end{aligned}
$$

where

$$
(I) \leq \int_{\xi_n-3\|u_n\|}^{\xi_n+\|u_n\|} u_n^2(r)\frac{|\xi_n-r|}{r\xi_n}\left(\int_{\xi_n-3\|u_n\|}^{\xi_n+\|u_n\|} s u_n^2(s)\,ds\right)^2 dr + o(1) = O(\|u_n\|) \tag{35}
$$

and

$$(II) \leq \frac{1}{\xi_n} \int_{\xi_n - 3\|u_n\|}^{\xi_n + \|u_n\|} u_n^2(r) \left| \int_{\xi_n - 3\|u_n\|}^{\xi_n + \|u_n\|} (s + \xi_n) u_n^2(s)\, ds \right| \left| \int_{\xi_n - 3\|u_n\|}^{\xi_n + \|u_n\|} (s - \xi_n) u_n^2(s)\, ds \right| dr$$
$$+ o(1)$$
$$= O(\|u_n\|). \tag{36}$$

Step 5: Conclusion for $\omega > \omega_0$. By (33) and (34), we have

$$I_\omega(u_n) \geq 2\pi \xi_n J_\omega(u_n \psi_n) + I_\omega(u_n(1 - \psi_n)) + c\|u_n(1 - \psi_n)\|^2_{L^2(\mathbb{R}^2)} + O(\|u_n\|). \tag{37}$$

Recall that $\|u_n \psi_n\|^2_{H^1(\mathbb{R})} \geq \sigma > 0$. By Proposition 3.7, we have $J_\omega(u_n \psi_n) \to c > 0$ up to a subsequence. Since $\xi_n \sim \|u_n\|^2$, it turns out from (37) that $I_\omega(u_n) > I_\omega(u_n(1-\psi_n))$. But this contradicts the definition of u_n, proving that $\inf I_\omega > -\infty$.

Let us now show that I_ω is coercive. Indeed, take an unbounded sequence $u_n \in H^1(\mathbb{R}^2)$, and assume that $I_\omega(u_n)$ is bounded from above. By Proposition 4.2(iii), we obtain $I_{\hat{\omega}}(u_n) \to -\infty$ for any $\omega_0 < \hat{\omega} < \omega$, a contradiction.

Step 6: Conclusion for $\omega = \omega_0$. As above, (37) gives a contradiction unless $J_\omega(u_n \psi_n) \to 0$. Proposition 3.7 now implies that $\psi_n u_n(\cdot - t_n) \to w_{k_2}$ up to a subsequence, for some $t_n \in (0, \infty)$. Since $\xi_n \in D_n$ (see definition in (30)), we see that $|t_n - \xi_n|$ is bounded. With this extra information, we have a better estimate of the decay of the solutions: indeed,

$$|u_n(r)| + |u_n'(r)| < C \exp(-\sqrt{\omega}\, |r - \xi_n|) \quad \text{for all } r > \xi_n - 2\|u_n\|. \tag{38}$$

This allows us to do the cut-off in a much more accurate way. Indeed, take

$$\tilde{z}_n = \xi_n - \|u_n\|.$$

Then (38) implies that

$$\|u_n\|^2_{H^1(\tilde{z}_n, \tilde{z}_n + 1)} \leq C \exp(-\sqrt{\omega}\, \|u_n\|). \tag{39}$$

Define $\tilde{\psi}_n : [0, \infty] \to [0, 1]$ accordingly:

$$\tilde{\psi}_n(r) = \begin{cases} 0 & \text{if } r \leq \tilde{z}_n, \\ 1 & \text{if } r \geq \tilde{z}_n + 1, \end{cases} \qquad |\tilde{\psi}_n'(r)| \leq 2.$$

The advantage is that, in the estimate of $I_\omega(u_n)$, the errors are now exponentially small. Indeed, by repeating the estimates of Step 3 with the new information (39), we obtain

$$I_\omega(u_n) \geq I_\omega(u_n \tilde{\psi}_n) + I_\omega(u_n(1 - \tilde{\psi}_n)) + c\|u_n(1 - \tilde{\psi}_n)\|^2_{L^2(\mathbb{R}^2)} + o(1).$$

Let us show that in this case (34) becomes

$$I_\omega(u_n \tilde{\psi}_n) = 2\pi \xi_n J_\omega(u_n \tilde{\psi}_n) + O(1).$$

Indeed, by (38) and (39), we have

$$\left| \int_0^n (u_n \tilde{\psi}_n)^2 r \, dr - \xi_n \int_0^n (u_n \tilde{\psi}_n)^2 \, dr \right| \leq \int_{-\infty}^{\infty} (u_n \tilde{\psi}_n)^2 |r - \xi_n| \, dr \leq C;$$

the other local terms can be estimated similarly. For the nonlocal term, we repeat the arguments of the previous case using in (35) and (36) the information contained in (38) and (39). Thus,

$$\begin{aligned} I_\omega(u_n) &\geq I_\omega(u_n \tilde{\psi}_n) + I_\omega(u_n(1 - \tilde{\psi}_n)) + c\|u_n(1 - \tilde{\psi}_n)\|^2_{L^2(\mathbb{R}^2)} + O(1) \\ &= 2\pi \xi_n J_\omega(u_n \tilde{\psi}_n) + I_\omega(u_n(1 - \tilde{\psi}_n)) + c\|u_n(1 - \tilde{\psi}_n)\|^2_{L^2(\mathbb{R}^2)} + O(1) \\ &\geq I_{\omega+2c}(u_n(1 - \tilde{\psi}_n)) + O(1). \end{aligned}$$

But, by Step 1, we already know that $I_{\omega+2c}$ is bounded from below, and hence $\inf I_{\omega_0} > -\infty$.

Finally, applying Lemma 4.1 to $U = w_{k_2}$, we readily see that I_{ω_0} is not coercive. □

Proof of Theorem 1.2. We shall prove each statement separately.

(i) Let u be a solution of (6). We multiply (6) by u and integrate: taking into account (8), we get

$$\begin{aligned} 0 &= \int_{\mathbb{R}^2} (|\nabla u|^2 + \omega u^2) \, dx + \frac{3}{4} \int_{\mathbb{R}^2} \frac{u^2(x)}{|x|^2} \left(\int_0^{|x|} s u^2(s) \, ds \right)^2 dx - \int_{\mathbb{R}^2} |u|^{p+1} \, dx \\ &\geq \frac{1}{4} \int_{\mathbb{R}^2} |\nabla u|^2 \, dx + \int_{\mathbb{R}^2} \left(\omega u^2 + \frac{3}{4} u^4 - |u|^{p+1} \right) dx. \end{aligned}$$

Observe that there exists $\bar{\omega} > 0$ such that, for $\omega > \bar{\omega}$, the function $t \mapsto \omega t^2 + \frac{3}{4} t^4 - |t|^{p+1}$ is nonnegative. Therefore u must be identically zero.

(ii) First, we observe that since $\inf I_{\omega_0} < 0$, there exists $\tilde{\omega} > \omega_0$ such that $\inf I_\omega < 0$ if and only if $\omega \in (\omega_0, \tilde{\omega})$. Since, by Theorem 1.1 and Proposition 2.2, I_ω is coercive and weakly lower semicontinuous, we infer that the infimum is attained.

Clearly, 0 is a local minimum for I_ω. Next, if $\omega \in (\omega_0, \tilde{\omega})$, the functional satisfies the geometrical assumptions of the mountain-pass theorem [1]. Since I_ω is coercive, (PS) sequences are bounded. By the compact embedding of $H_r^1(\mathbb{R}^2)$ into $L^{p+1}(\mathbb{R}^2)$ and Proposition 2.2, standard arguments show that I_ω satisfies the Palais–Smale condition and so we can find a second solution which is at a positive energy level.

(iii) Let now consider $\omega \in (0, \omega_0)$. Performing the rescaling $u \mapsto u_\omega = \sqrt{\omega}\, u(\sqrt{\omega}\, \cdot)$, we get

$$\begin{aligned} I_\omega(u_\omega) = \omega \Bigg[\frac{1}{2} \int_{\mathbb{R}^2} (|\nabla u|^2 + u^2) \, dx + \frac{1}{8} \int_{\mathbb{R}^2} \frac{u^2(x)}{|x|^2} \left(\int_0^{|x|} s u^2(s) \, ds \right)^2 dx \\ - \frac{\omega^{(p-3)/2}}{p+1} \int_{\mathbb{R}^2} |u|^{p+1} \, dx \Bigg]. \end{aligned}$$

Define $\lambda = \omega^{(p-3)/2}$ and $\mathcal{I}_\lambda : H^1_r(\mathbb{R}^2) \to \mathbb{R}$ as

$$\mathcal{I}_\lambda(u) = \frac{1}{2}\int_{\mathbb{R}^2}(|\nabla u|^2 + u^2)\,dx + \frac{1}{8}\int_{\mathbb{R}^2}\frac{u^2(x)}{|x|^2}\left(\int_0^{|x|} s u^2(s)\,ds\right)^2 dx - \frac{\lambda}{p+1}\int_{\mathbb{R}^2}|u|^{p+1}\,dx.$$

Since $\mathcal{I}_\lambda$ satisfies the geometrical assmptions of the mountain-pass theorem, from [15, Theorem 1.1] we infer that, for almost every λ, the functional $\mathcal{I}_\lambda$ has a bounded Palais–Smale sequence u_n. Assume $u_n \rightharpoonup u$; Proposition 2.2 and standard arguments imply that u is a critical point of $\mathcal{I}_\lambda$. Making the change of variables back we obtain a solution of (6) for almost every $\omega \in (0, \omega_0)$.

Finally, in order to find positive solutions of (6), we simply observe that the whole argument applies to the functional $I^+_\omega : H^1_r(\mathbb{R}^2) \to \mathbb{R}$ given by

$$I^+_\omega(u) = \frac{1}{2}\int_{\mathbb{R}^2}(|\nabla u|^2 + \omega u^2)\,dx + \frac{1}{8}\int_{\mathbb{R}^2}\frac{u^2(x)}{|x|^2}\left(\int_0^{|x|} s u^2(s)\,ds\right)^2 dx - \frac{1}{p+1}\int_{\mathbb{R}^2}(u^+)^{p+1}\,dx.$$

Due to the maximum principle, the critical points of I^+_ω are positive solutions of (6). □

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