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On the motion of a curve by its binormal curvature

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Abstract. We propose a weak formulation for the binormal curvature flow of curves in $\mathbb{R}^3$. This formulation is sufficiently broad to consider integral currents as initial data, and sufficiently strong for the weak-strong uniqueness property to hold, as long as self-intersections do not occur. We also prove a global existence theorem in that framework.

Keywords. Binormal curvature flow, integral current, oriented varifold

I confess, I am skeptical about the stability of many of the motions which you appear to contemplate.

Stokes, letter to Kelvin, 1873

1. Introduction

The binormal curvature flow equation for a smooth family $(\gamma_t)_{t\in I}$ of curves in $\mathbb{R}^3$ is traditionally written in terms of an arc-length parametrization $\gamma : I \times \mathbb{R} \to \mathbb{R}^3$ by

$$\partial_t \gamma = \partial_s \gamma \times \partial_{ss} \gamma \tag{1}$$

where $t \in I$ is the time variable, $s \in \mathbb{R}$ is the arc-length parameter, and $\times$ denotes the vector product in $\mathbb{R}^3$. The arc-length parametrization condition

$$|\partial_s \gamma(t,s)|^2 = 1 \tag{2}$$

is indeed compatible with equation (1), since

$$\partial_t(|\partial_s \gamma|^2) = 2\partial_s \gamma \cdot \partial_{st} \gamma = 2\partial_s \gamma \cdot (\partial_s \gamma \times \partial_{sss} \gamma) = 0$$

whenever (1) is satisfied, at least for sufficiently smooth solutions. In particular, closed curves evolved by the binormal curvature flow equation (1) all have constant length. In more geometric terms, equation (1) takes its name from its equivalent form

$$\partial_t \gamma = \kappa b$$

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where κ and b are the curvature function and the binormal vector field along γ_t respectively.

It seems that equation (1) first appeared in the 1906 Ph.D. thesis of L. S. Da Rios [8], whose work was promoted in a series of lectures in 1931 in Paris by his advisor T. Levi-Civita [24]. The problem considered by Da Rios and Levi-Civita goes back to the celebrated 1858 paper of H. Helmholtz [14] on the motion of a three-dimensional incompressible fluid in rotation. Special attention was paid in the second part of [14] to configurations called "unendlich kleine Querschnitts", and translated in [15] by vortex filaments of indefinitely small cross-section: in such configurations, the vorticity field $\omega := \text{curl}(v)$ associated to the velocity field v of the fluid at a given time t is concentrated along a closed oriented curve γ_t, parallel to it and vanishing rapidly away from it, so that

$$\int_{\mathbb{R}^3} X(x) \cdot \omega(x,t)\, dx \simeq \int_{\gamma_t} X \cdot \tau_{\gamma_t}\, d\mathcal{H}^1$$

in some appropriate sense for any vector field $X \in \mathcal{D}(\mathbb{R}^3, \mathbb{R}^3)$. Helmholtz, like everybody since, failed to rigorously answer the question of the persistence in time of such vortex filaments under the Euler flow

$$\partial_t \omega + v \cdot \nabla\omega = \omega \cdot \nabla\omega.$$

Nevertheless, he obtained a number of important contributions in that direction, as well as suggestive evidences, which conducted him to study the question of the corresponding asymptotic motion law for the underlying curves γ_t in case of positive answer to the previous question. Because of mathematical obstacles related to the singularity of the Biot–Savart kernel involved in the reconstruction of v from ω when considering such vorticity measures, Helmholtz essentially restricted his mathematical study to the case of straight or circular vortex filaments, or combinations of those. Pursuing Helmholtz's work, Lord Kelvin announced in 1867 [19] and published in 1880 [20] the first result on linear stability of circular vortex filaments. The latter, also called vortex rings, correspond in the asymptotic of infinitely small cross-section to the traveling wave solutions of equation (1) given by

$$\gamma(t,s) = \gamma_{r,\vec{e}}(s) + \frac{t}{r}\vec{e},$$

where $\gamma_{r,\vec{e}}$ is an arc-length parametrization of a circle of radius r in a plane perpendicular to the unitary vector $\vec{e} \in \mathbb{R}^3$. Kelvin carefully described the neutral modes involved in small perturbations of such configurations, and which are today referred to as Kelvin waves. J. J. Thomson's 1883 treatise [28] and H. Poincaré's 1893 lecture notes [26] are also important sources regarding the state of the art for vortex filaments motion in incompressible fluids by the end of the nineteenth century. As already mentioned, it is only in 1906 with a careful use of potential theory that Da Rios formally obtained the speculated general motion law (1).

Letting aside the fact that it has never been rigorously derived from the Euler equations, and even though it is globally well-posed for initial data consisting of smooth closed curves, formulation (1) for binormal curvature flows has at least two limitations which we would like to address.

First, by essence this formulation is tailored for parametrized curves. In particular, and since it involves derivatives with respect to the parameters only, it is necessarily insensitive to self-intersections[1] in the curves γ_t. This property is surely unsatisfactory if one believes that such flows arise as limits from three-dimensional fluid dynamics. Instead, it would be desirable for a formulation to be able to detect such self-intersections, as well as possible collisions between elements of disconnected vortex filaments and changes of topology.

Second, there are presumably important configurations of curves which are too singular to be considered under formulation (1). Indeed, invoking distributional derivatives one can give a meaning to equation (1) in a variety of spaces, but those spaces just fail to include the case of curves which are barely Lipschitz. On the other hand, in numerical simulations of the Euler equation or the Gross–Pitaevskii equation for quantum fluids, it is observed (see e.g. [22] and [23]) that vortex filaments often tend to recombine by exchanging strands in cases of collisions or self-intersections. Those recombinations, when the intersections are transverse, inevitably create discontinuities of the tangent vector (see Figure 1 below).

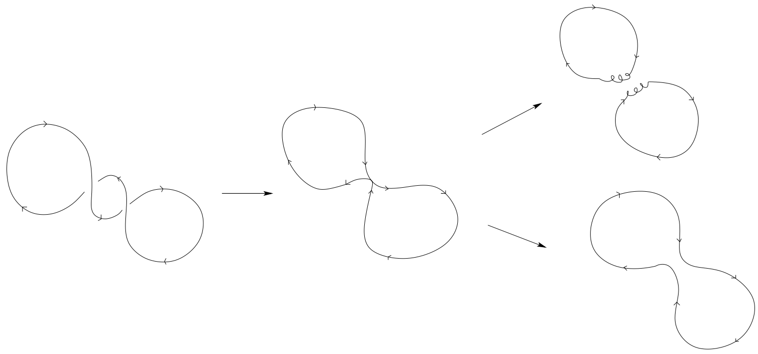


Fig. 1. Non-unique evolution through strands recombination and singularity formations.

Our starting point in trying to address these two important limitations is the following identity for smooth solutions of (1), which was remarked by the first author in [17] in a more general context.

Lemma 1 ([17]). *If γ is a smooth solution of (1) on $I \times \mathbb{T}^1$, where $I \subset \mathbb{R}$ is some open interval and $\mathbb{T}^1 = \mathbb{R}/\ell\mathbb{Z}$ for some $\ell > 0$, then for every vector field $X \in \mathcal{D}(\mathbb{R}^3, \mathbb{R}^3)$ and every $t \in I$,*

$$\frac{d}{dt} \int_{\gamma_t} X \cdot \tau_t \, d\mathcal{H}^1 = - \int_{\gamma_t} D(\mathrm{curl}(X)) : (\tau_t \otimes \tau_t) \, d\mathcal{H}^1, \tag{3}$$

where $\gamma_t \equiv \gamma(t, \cdot)$ and τ_t is the oriented tangent vector along γ_t.

[1] By self-intersection of γ_t we mean failure of injectivity of the map $\gamma(t, \cdot)$.

Notice that for fixed time, both sides of (3) involve, in terms of γ, only the tangent vectors τ_t, and therefore first order derivatives with respect to the arc-length. This suggests enlarging the definition of binormal curvature flows through an extension of formula (3) to one-dimensional objects that have well-defined tangent spaces, at least in a measure-theoretic sense. A tentative definition based entirely on integral currents of H. Federer and W. H. Fleming [12] was first proposed in [17]; an existence theory in that framework is still missing. The main difficulty in dealing with (3) in the framework of currents is that the right-hand side does not have good continuity properties for the usual topologies associated to currents, because of the presence of quadratic terms in the tangent vectors. Instead, such quantities seem more appropriate to be dealt with using the general framework of Young measures, and more specifically varifolds of F. J. Almgren [2] and W. K. Allard [1]. On the other hand, the left-hand side of (3) is more appropriate to currents than varifolds, in particular because the latter do not have an orientation. The strategy which we adopt here below tries in a sense to reconcile these two features, building both on integral currents and on a notion of oriented varifolds which can be viewed as the non-parametric version of what L. C. Young [29] and E. J. McShane [25] called generalized curves.

Integral currents. H. Federer and W. H. Fleming introduced integral currents of arbitrary dimension in [12]. One-dimensional currents have a simple characterization which we adopt as a definition (see [11, 4.2.25]).

A *simple closed oriented curve* in $\mathbb{R}^3$ is a vector valued distribution $T \in \mathcal{D}'(\mathbb{R}^3, \mathbb{R}^3)$ such that there exists a Lipschitz one-to-one function $\gamma : \mathbb{T}^1 \to \mathbb{R}^3$ satisfying

$$T(X) = \int_{\mathbb{T}^1} X(\gamma(s)) \cdot \gamma'(s)\, ds \qquad \forall X \in \mathcal{D}(\mathbb{R}^3, \mathbb{R}^3).$$

The *length* of a simple closed oriented curve T, denoted by $L(T)$, is given by $L(T) := \int_{\mathbb{T}^1} |\gamma'(s)|\, ds$, and we have the equality $L(T) = \sup\{T(X) : X \in \mathcal{D}(\mathbb{R}^3, \mathbb{R}^3),\ \|X\|_\infty \le 1\}$, so that in particular $L(T)$ is independent of the choice of parametrization γ.

The set $\mathcal{T}$ of *integral* 1*-currents in* $\mathbb{R}^3$ *without boundary* is the set of vector valued distributions $T \in \mathcal{D}'(\mathbb{R}^3, \mathbb{R}^3)$ such that $T = \sum_{j\in\mathbb{N}} T_j$ in $\mathcal{D}'(\mathbb{R}^3, \mathbb{R}^3)$ for a sequence $(T_j)_{j\in\mathbb{N}}$ of simple closed oriented curves in $\mathbb{R}^3$ such that $\sum_{j\in\mathbb{N}} L(T_j) < \infty$. The *mass* of an integral 1-current $T \in \mathcal{T}$ in $\mathbb{R}^3$ without boundary is defined as $\|T\| := \sup\{T(X) : X \in \mathcal{D}(\mathbb{R}^3, \mathbb{R}^3)\ ,\ \|X\|_\infty \le 1\}$, and in particular we have $\|T\| \le \sum_{j\in\mathbb{N}} L(T_j)$ whenever $T = \sum_{j\in\mathbb{N}} T_j$ for a sequence $(T_j)_{j\in\mathbb{N}}$ of simple closed oriented curves in $\mathbb{R}^3$.

Oriented integral varifolds. The set $\mathcal{V}$ of *oriented integral* 1*-varifolds* in $\mathbb{R}^3$ *without boundary* is defined[2] as the set of finite non-negative Radon measures $V \in \mathcal{M}(\mathbb{R}^3 \times S^2)$

[2] We emphasize that "integral" and "without boundary" actually refer not to the oriented varifold V_t but to its first moment T_{V_t}. This is arguably an abuse of language, but it is convenient here. As a result, although the terminologies look similar, our definition of integral oriented varifold allows for non-trivial measures with respect to ξ variables, whereas the definition of integral varifolds of Almgren and Allard does not.

whose first moment with respect to the S^2 variable

$$T_V : \mathcal{D}(\mathbb{R}^3, \mathbb{R}^3) \to \mathbb{R}, \qquad X \mapsto \int X(x) \cdot \xi \, dV(x, \xi),$$

is an integral 1-current in $\mathbb{R}^3$ without boundary. The *mass* of $V \in \mathcal{V}$ is defined as $\|V\| := \sup\{V(\psi) : \psi \in \mathcal{D}(\mathbb{R}^3 \times S^2, \mathbb{R}), \|\psi\|_\infty \leq 1\}$, and in particular we always have the inequality $\|T_V\| \leq \|V\|$.

Measurable and continuous families. In what follows, $I \subset \mathbb{R}$ denotes an interval such that $0 \in I$. A family $(T_t)_{t\in I}$ of integral 1-currents in $\mathbb{R}^3$ without boundary is called *continuous* if the map $t \mapsto T_t$ is continuous from I to $\mathcal{D}'(\mathbb{R}^3, \mathbb{R}^3)$. A family $(V_t)_{t\in I}$ of oriented integral 1-varifolds without boundary is called *measurable* if for every Borel subset $\mathcal{O} \subset \mathbb{R}^3 \times S^2$, the map $t \mapsto V_t(\mathcal{O})$ is measurable on I.

We are now in a position to state:

Definition 1. A measurable family $(V_t)_{t\in I}$ of oriented integral 1-varifolds in $\mathbb{R}^3$ without boundary is called a *generalized binormal curvature flow* on I if for any $X \in \mathcal{D}(\mathbb{R}^3, \mathbb{R}^3)$ the function $t \mapsto V_t(X \cdot \xi)$ is Lipschitz on I and satisfies

$$\frac{d}{dt} \int X \cdot \xi \, dV_t = - \int D(\mathrm{curl}(X)) : \xi \otimes \xi \, dV_t \tag{4}$$

for almost every $t \in I$.

Definition 2. A continuous family $(T_t)_{t\in I}$ of integral 1-currents in $\mathbb{R}^3$ without boundary is called a *weak binormal curvature flow on I with initial datum T_0* if there exists a generalized binormal curvature flow $(V_t)_{t\in I}$ on I such that

1. The first moment T_{V_t} of V_t coincides with T_t for every $t \in I$.
2. The mass $\|V_t\|$ satisfies $\|V_t\| \leq \|T_0\|$ for every $t \in I$.

For a generalized binormal curvature flow $(V_t)_{t\in I}$ on I, we call the family $(T_{V_t})_{t\in I}$ of first moments its family of *associated undercurrents*.

Remark 1. (i) Notice that Definition 1 is linear in V_t. In particular, the sum of two generalized binormal curvature flows is a generalized binormal curvature flow. Also, if $(T_t^1)_{t\in I}$ and $(T_t^2)_{t\in I}$ are two weak binormal curvature flows with initial data T_0^1 and T_0^2 respectively, and if moreover $\|T_0^1 + T_0^2\| = \|T_0^1\| + \|T_0^2\|$, then $(T_t^1 + T_t^2)_{t\in I}$ is a weak binormal curvature flow with initial datum $T_0^1 + T_0^2$.

(ii) Notice also that Definition 1 only involves, in terms of V_t, its first moment on the left-hand side of (4) and its second moment on the right-hand side of (4). As a result, a uniqueness or a Cauchy theory for generalized binormal curvature flows at the level of V_t is ruled out a priori. Further possible pathologies of generalized binormal curvature flows are illustrated by examples that we present in Remark 6, at the end of Section 5.2.

As we will see, the situation greatly improves for weak binormal curvature flows.

(iii) Finally observe that the equality (4) actually makes sense for a general measurable family of Radon measures $V_t \in \mathcal{M}(\mathbb{R}^3 \times S^2)$. Since we know only of artificial such examples of "diffuse" flows, we have preferred to stick with the actual Definition 1.

Note, however, that Theorems 2 and 3 below, which establish weak-strong uniqueness of weak binormal curvature flows together with a related stability result, do not require the full strength of the definition of weak binormal curvature flow. Indeed, the assumption that the undercurrents T_{V_t} be integral for every t is not used anywhere in these proofs.

In view of Lemma 1, we immediately deduce

Proposition 1 (Consistency). *Let $\ell > 0$ and $\gamma : I \times (\mathbb{R}/\ell\mathbb{Z}) \to \mathbb{R}^3$ denote a smooth classical solution of the binormal curvature flow equation (1). The family $(V_{\gamma,t})_{t\in I}$ defined by*

$$V_{\gamma,t}(\psi) := \int_0^\ell \psi(\gamma(t,s), \partial_s\gamma(t,s))\, ds \quad \forall \psi \in \mathcal{D}(\mathbb{R}^3 \times S^2, \mathbb{R})$$

is a generalized binormal curvature flow on I, and the family $(T_{\gamma,t})_{t\in I}$ defined by

$$T_{\gamma,t}(X) := \int_0^\ell X(\gamma(t,s)) \cdot \partial_s\gamma(t,s)\, ds \quad \forall X \in \mathcal{D}(\mathbb{R}^3, \mathbb{R}^3)$$

is a weak binormal curvature flow on I with initial datum $T_{\gamma,0}$ provided $\|T_{\gamma,0}\| = \ell$.

An advantage of Definitions 1 and 2 is that they lead rather directly to an existence theory globally in time.

Theorem 1 (Global existence). *For any integral 1-current T_0 in $\mathbb{R}^3$ without boundary, there exists a weak binormal curvature flow $(T_t)_{t\in\mathbb{R}}$ on $\mathbb{R}$ with initial datum T_0.*

Theorem 1 is proved using an approximation argument and compactness properties. We present some of these intermediate steps now, which, we believe, have their own independent interest.

Proposition 2. *Let $(V_t)_{t\in I}$ be a generalized binormal curvature flow on I and denote by $(T_{V_t})_{t\in I}$ its family of associated undercurrents. There exists a universal constant $C > 0$ such that for every $t_1, t_2 \in I$ we have the inequality*

$$d^*_{\mathcal{F}}(T_{V_{t_1}}, T_{V_{t_2}}) \le C\Big(\sup_{t\in I} \|V_t\|^{1/2}\Big)|t_1 - t_2|^{1/2},$$

where, for $T, \tilde{T} \in \mathcal{T}$,

$$d^*_{\mathcal{F}}(T, \tilde{T}) := \sup\{T(X) - \tilde{T}(X) : X \in \mathcal{D}(\mathbb{R}^3, \mathbb{R}^3),\ \|\mathrm{curl}(X)\|_\infty \le 1\}.$$

In particular, whenever $(T_t)_{t\in I}$ is a weak binormal curvature flow on I with initial datum T_0,

$$d^*_{\mathcal{F}}(T_{t_1}, T_{t_2}) \le C\|T_0\|^{1/2}\, |t_1 - t_2|^{1/2} \quad \forall t_1, t_2 \in I.$$

Remark 2. In geometric terms, the quantity $d^*_{\mathcal{F}}(T, \tilde{T})$ is exactly equal to the area of the two-dimensional minimal surface whose boundary is given by $T - \tilde{T}$ (see e.g. [11, 4.1.12]). The distance $d^*_{\mathcal{F}}$ is also much related to and actually slightly stronger than Whitney's flat metric. (In fact $d^*_{\mathcal{F}}$ can be thought of as a homogeneous flat metric.) It follows therefore from Proposition 2 that when $(T_t)_{t\in I}$ is a weak binormal curvature flow on I or the family of undercurrents associated to a generalized binormal curvature flow uniformly bounded in mass, the map $t \mapsto T_t$ is Hölder continuous with exponent $1/2$ from $I \subset \mathbb{R}$ to $\mathcal{T}$ equipped with Whitney's flat metric.

Proposition 3. *For each $n \in \mathbb{N}$, let $(V_t^n)_{t\in I}$ be a generalized binormal curvature flow on I. Assume that $\sup_{n\in\mathbb{N},\, t\in I} \|V_t^n\| < \infty$ and that*

$$V_t^n\, dt \rightharpoonup V \quad \text{in } \mathcal{M}(\mathbb{R}^3 \times S^2 \times I) \text{ as } n \to \infty.$$

Then $V = V_t\, dt$ in $\mathcal{M}(\mathbb{R}^2 \times S^2 \times I)$ where $(V_t)_{t\in I}$ is a generalized binormal curvature flow on I. Moreover, for every $t \in I$,

$$T_{V_t^n} \rightharpoonup T_{V_t} \quad \text{in } \mathcal{D}'(\mathbb{R}^3, \mathbb{R}^3) \text{ as } n \to \infty.$$

Proposition 3 implies in particular that every sequence of smooth binormal flows with uniform mass bounds and possibly highly oscillatory behavior converges, along subsequences, to a generalized flow. Examples of such limits which are not weak binormal curvature flows are provided in Section 5.2.

Corollary 1. *For each $n \in \mathbb{N}$, let $(T_t^n)_{t\in I}$ be a weak binormal curvature flow on I with initial datum T_0^n. Assume that for some $T_0 \in \mathcal{T}$ we have, as $n \to \infty$,*

$$T_0^n \rightharpoonup T_0 \quad \text{in } \mathcal{D}'(\mathbb{R}^3, \mathbb{R}^3) \quad \text{and} \quad \|T_0^n\| \to \|T_0\| \quad \text{in } \mathbb{R}.$$

Then there exist a subsequence $(n_k)_{k\in\mathbb{N}}$ and a weak binormal curvature flow $(T_t)_{t\in I}$ on I with initial datum T_0 such that for every $t \in I$,

$$T_t^{n_k} \rightharpoonup T_t \quad \text{in } \mathcal{D}'(\mathbb{R}^3, \mathbb{R}^3) \text{ as } k \to \infty,$$

The first part of Proposition 3 follows directly from Proposition 2 and the Arzelà–Ascoli theorem applied for a suitable localized version of the flat metric. The provided convergence is actually stronger than stated in Proposition 3 or Corollary 1 (see Section 3). Theorem 1 follows from Corollary 1 and the fact that integral 1-currents in $\mathbb{R}^3$ without boundary can be suitably approximated by finite sums of smooth closed curves, for which global existence of solutions to (1) can be used in conjunction with Proposition 1 and the linearity mentioned in Remark 1.

Uniqueness of weak binormal curvature flows for a given initial datum T_0 fails in general under Definition 2, and in particular it is necessary to consider a subsequence in the statement of Corollary 1. We believe however that Definition 2 is sufficiently strong to eliminate unrealistic sources of non-uniqueness, and that the remaining ones are probably intrinsic to any reasonable formulation of weak binormal curvature flows that requires self-intersections and collisions to possibly matter. A typical example of non-unique evolution is provided by an initial datum consisting of the sum of two circles of different radii (or else living in different planes) and that have exactly one intersection point. A first evolution is given by the sum of the independent evolutions of both circles, which are traveling wave solutions, and whose mutual distance will indefinitely increase since their speeds differ as vectors. A second evolution is obtained by approximating the initial datum by smooth simple closed curves T_0^n and applying Corollary 1 to their classical evolutions according to equation (1). In this second case, the solution at any time is supported in a Lipschitz image of $\mathbb{T}^1$, and therefore necessarily differs from the first evolution.

Still, we have

Theorem 2 (Weak-strong uniqueness). *Let $\ell > 0$ and $\gamma : I \times (\mathbb{R}/\ell\mathbb{Z}) \to \mathbb{R}^3$ denote a smooth classical solution of the binormal curvature flow equation (1), and assume that for any $t \in I$, the curve $\gamma_t := \gamma(t, \cdot)$ is without self-intersection. Then the weak binormal curvature flow $(T_{\gamma,t})_{t\in I}$ provided by Proposition 1 is the unique weak binormal curvature flow on I with initial datum $T_{\gamma,0}$.*

As a matter of fact, we deduce Theorem 2 from a stronger quantitative estimate. For that purpose, consider a compact subset $J \subset I$ containing 0 and set

$$r \equiv r(\gamma, J) := \tfrac{1}{2} \min_{t\in J} \min(\|\partial_{ss}\gamma(t,\cdot)\|_\infty^{-1}, r_s(t)) > 0,$$

where the security radius $r_s(t)$ is defined as the largest positive real number with the property that every point x satisfying $d(x, \gamma_t) < r_s(t)$ has a unique closest point $P_t(x)$ on γ_t. Define then the vector field $X_{\gamma,r}$ on $\mathbb{R}^3 \times J$ by[3]

$$X_{\gamma,r}(x,t) = f(d^2(x,\gamma_t))\tau_t(P_t(x)) \tag{5}$$

where τ_t is the oriented unit tangent vector along γ_t and[4]

$$f(d^2) = \begin{cases} (1-(d/r)^2)^3 & \text{for } 0 \le d^2 \le r^2, \\ 0 & \text{for } d^2 \ge r^2. \end{cases}$$

Theorem 3 (Control of instability). *Let $T_0 \in \mathcal{T}$ and let $(T_t)_{t\in J}$ be a weak binormal curvature flow on J with initial datum T_0. Define the non-negative functions F and G on J by*[5]

$$G(t) := \|T_0\| - \int X_{\gamma,r}(x,t)\cdot\xi \, dV_t(x,\xi) \ge F(t) := \int (1 - X_{\gamma,r}(x,t)\cdot\xi)\, dV_t(x,\xi) \ge 0.$$

Then G is Lipschitzian on J and

$$\left|\frac{d}{dt}G(t)\right| \le KF(t) \le KG(t)$$

almost everywhere on J, where $K \equiv K(r(\gamma, J), \|\partial_{sss}\gamma\|_{L^\infty(J\times\mathbb{T}^1)})$.

[3] The function $f(d^2(\cdot,\gamma_t))$ vanishes where P_t is undefined, so that $X_{\gamma,r}$ is globally well-defined.

[4] The analytic form of f does not really matter, but it is important that $f(d^2)$ has a non-degenerate maximum at $d = 0$. See in particular (6) and (7) below.

[5] Notice that the definitions of F and G only depend on the first moments T_{V_t} of V_t; therefore F and G are uniquely determined by $T_t = T_{V_t}$ and well-defined.

As noted earlier, this result, and hence Theorem 2 as well, remains true if we drop the assumption (contained in the definition of a generalized binormal curvature flow) that T_{V_t} be an *integral* 1-current.

The function F which appears in the statement of Theorem 3 may be understood as a measure of the discrepancy between γ_t and T_t. To get some insight into its geometric meaning, we express the integral 1-current T_t as $T_t = (\Gamma_t, \theta_t, \xi_t)$, where Γ_t, θ_t and ξ_t are respectively the geometrical support, the multiplicity and the orientation of T_t, and then define $\Gamma_t^{\text{in}} = \{x \in \Gamma_t : d(x, \gamma_t) < r\}$ and $\Gamma_t^{\text{out}} = \Gamma_t \setminus \Gamma_t^{\text{in}}$. For $x \in \Gamma_t^{\text{in}}$ such that $\tau_t(P_t(x)) \cdot \xi_t(x) \geq 0$, we have $1 - X_{\gamma,r}(x,t) \cdot \xi_t(x) \geq 1 - \tau_t(P(x)) \cdot \xi_t(x) = \frac{1}{2}|\tau_t(P_t(x)) - \xi_t(x)|^2$, while for $x \in \Gamma^{\text{in}}$ such that $\tau_t(P_t(x)) \cdot \xi_t(x) < 0$, we have $1 - X_{\gamma,r}(x,t) \cdot \xi_t(x) \geq 1 \geq \frac{1}{2}|\tau_t(P_t(x)) - \xi_t(x)|^2$. It follows in particular that

$$F(t) \geq \int_{\Gamma_t^{\text{in}}} \frac{1}{2}|\tau_t \circ P_t - \xi_t|^2\, \theta_t\, d\mathcal{H}^1 + \int_{\Gamma_t^{\text{out}}} \theta_t\, d\mathcal{H}^1. \tag{6}$$

In a different direction, for $x \in \Gamma_t$ we also have $1 - X_{\gamma,r}(x,t) \cdot \xi_t(x) \geq 1 - f(d^2(x, \gamma_t)) \geq \min(d^2(x, \gamma_t), r^2)$, from which it follows that

$$F(t) \geq \int_{\Gamma_t} \min(d^2(\cdot, \gamma_t), r^2)\theta_t\, d\mathcal{H}^1. \tag{7}$$

Upper bounds on $F(t)$ therefore provide upper bounds on the right-hand sides of (6) and (7), which together correspond to an H^1 or tilt excess type measure of the discrepancy between γ_t and T_t. Notice however that T_t may have multiple components, some of which, of small total length, could be located arbitrarily far from γ_t even if $F(t)$ is small. We refer to [18] for the additional information that can be derived from F when T_t is itself a classical mean curvature flow for a parametrized curve.

Going back to Theorem 1, we mention that, whereas it is not difficult to produce weak binormal curvature flows for which $\|T_t\| < \|T_0\|$ for t in some interval of positive length, e.g. by collision and annihilation of circles of opposite speeds, we do not know of any such example for a flow constructed as a limit of smooth flows of single curves (see Section 5.1 and the notion of almost parametric flows). On the other hand, we have not been able to prove the contrary either, nor the fact that the ξ part of the measures V_t are always reduced to single Dirac masses. We believe that it would be of interest to obtain further insight into these questions.

We would also like to stress that we have here only considered weak binormal curvature flows for finite mass currents. In view of the fact that the quantity $1 - X_{\gamma,r} \cdot \xi$ involved in the definition of F in Theorem 3 is pointwise non-negative, it is not unreasonable to expect that part of the analysis could be carried out as well for integral 1-currents of locally finite mass, at least under suitable assumptions on their behavior at infinity. Such an extension would be of particular interest when considering the special solutions that have been recently studied in a series of interesting works by V. Banica and L. Vega [3, 4], using quite different methods, and which correspond to perturbations of an infinitely extended broken line.

To conclude this introduction, we mention that integral formulas of a nature somewhat similar to (3) have been known and used in the past in related, yet very different, contexts including the mean curvature flow and the incompressible Euler equations. Notably, the works of Brakke [5] and Ilmanen [16] have established existence and in some cases weak-strong uniqueness for mean curvature flows in the frameworks of integral varifolds and integral currents. As we deal with a Hamiltonian flow rather than a gradient flow, it turns out that the existence part is simpler here in some aspects. We have voluntarily stressed some analogies between the two situations in the way we stated Definitions 1 and 2, in particular regarding Brakke's definition of varifold mean curvature flow [5] and Ilmanen's definition of enhanced motion [16]. Regarding the Euler equations, a related integral formula has been used by DiPerna and Majda [10] to define and study a class of measure-valued solutions, and a weak-strong uniqueness theorem in this framework has recently been established by Brenier, de Lellis, and Székelyhidi [6]. Although there are some analogies between our work and that of [10, 6], probably reflecting the fluid dynamical roots of the binormal curvature flow, it seems difficult in practice to directly relate the two approaches; as already noted, this has been an open problem since the work of Helmholtz in the 1850s.

We present the proofs of Lemma 1 and Proposition 1 in Section 2, of Proposition 2, Proposition 3, Corollary 1 and Theorem 1 in Section 3, and of Theorems 2 and 3 in Section 4. In Section 5, we gather some additional results as well as some examples and open questions.

2. Proofs of Lemma 1 and Proposition 1

Proof of Lemma 1. We expand both sides of (3) in coordinates and use the convention of summation over repeated indices. Concerning the left-hand side of (3), we first have

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{T}^1}(X\circ\gamma)\cdot\partial_s\gamma\,ds &= \int_{\mathbb{T}^1}((\partial_i X^j)\circ\gamma)\partial_t\gamma^i\partial_s\gamma^j\,ds + \int_{\mathbb{T}^1}(X^j\circ\gamma)\partial_{st}\gamma^j\,ds\\ &= \int_{\mathbb{T}^1}((\partial_i X^j)\circ\gamma)(\partial_t\gamma^i\partial_s\gamma^j - \partial_s\gamma^i\partial_t\gamma^j)\,ds.\end{aligned}$$

By definition of the vector product,

$$(\partial_t\gamma^i\partial_s\gamma^j - \partial_t\gamma^j\partial_s\gamma^i) = \varepsilon_{ijk}(\partial_t\gamma\times\partial_s\gamma)^k,$$

where ε_{ijk} is the permutation symbol, so that

$$\frac{d}{dt}\int_{\mathbb{T}^1}(X\circ\gamma)\cdot\partial_s\gamma\,ds = \varepsilon_{ijk}\int_{\mathbb{T}^1}((\partial_i X^j)\circ\gamma)(\partial_t\gamma\times\partial_s\gamma)^k\,ds. \tag{8}$$

Concerning the right-hand side of (3), we write in coordinates

$$\begin{aligned}&\int_{\mathbb{T}^1}D(\mathrm{curl}(X))(\gamma(t,s)) : (\partial_s\gamma(t,s)\otimes\partial_s\gamma(t,s))\,ds\\ &\qquad\qquad = \int_{\mathbb{T}^1}((\partial_l(\mathrm{curl}(X))^k)\circ\gamma)\partial_s\gamma^l\partial_s\gamma^k\,ds.\end{aligned}$$

By definition of the rotation and the chain rule,

$$((\partial_l(\mathrm{curl}(X))^k)\circ\gamma)\partial_s\gamma^l\partial_s\gamma^k = \varepsilon_{ijk}((\partial_{il}X^j)\circ\gamma)\partial_s\gamma^l\partial_s\gamma^k = \varepsilon_{ijk}\partial_s((\partial_i X^j)\circ\gamma))\partial_s\gamma^k.$$

Integration by parts therefore yields

$$\int_{\mathbb{T}^1} D(\mathrm{curl}(X))(\gamma(t,s)) : (\partial_s\gamma(t,s)\otimes\partial_s\gamma(t,s))\,ds$$
$$= -\varepsilon_{ijk}\int_{\mathbb{T}^1}(\partial_i X^j)\circ\gamma)\partial_{ss}\gamma^k\,ds. \qquad (9)$$

Finally, since (1) holds we have $\partial_t\gamma \times \partial_s\gamma = \partial_{ss}\gamma$, and the conclusion then follows by combining (8) and (9). □

Proof of Proposition 1. It is nothing more than a rephrasing of Lemma 1 in the frameworks of Definitions 1 and 2. □

3. Proofs of Propositions 2 and 3, Corollary 1 and Theorem 1

The point of the next proof is to interpolate between uniform bounds on $\|T_{V_t}\|$ and the Lipschitz continuity of $t \mapsto T_{V_t}$ with respect to a weak norm (roughly speaking, the norm dual to $\|D(\mathrm{curl}(X))\|_\infty$), implicit in the definition of a generalized binormal curvature flow.

Proof of Proposition 2. Let $X \in \mathcal{D}(\mathbb{R}^3,\mathbb{R}^3)$ and $t_1 \neq t_2 \in I$. Let $\varepsilon > 0$ whose actual value will be determined at the end of the proof, and set $\rho_\varepsilon(x) := \varepsilon^{-3}\rho(x/\varepsilon)$, where $\rho(x) = \zeta(|x|)$ is a fixed non-negative radially symmetric function in $\mathcal{D}(\mathbb{R}^3,\mathbb{R})$, compactly supported in $B(0,1)$, and such that $\int\rho = 1$. Define $X_\varepsilon := \rho_\varepsilon * X$. We have

$$T_{V_{t_1}}(X) - T_{V_{t_2}}(X) = T_{V_{t_1}}(X - X_\varepsilon) - T_{V_{t_2}}(X - X_\varepsilon) + T_{V_{t_1}}(X_\varepsilon) - T_{V_{t_2}}(X_\varepsilon). \qquad (10)$$

We first estimate, in view of Definition 1,

$$\begin{aligned} T_{V_{t_1}}(X_\varepsilon) - T_{V_{t_2}}(X_\varepsilon) &= -\int_{t_1}^{t_2}\int_{\mathbb{R}^3\times S^2} D(\mathrm{curl}(X_\varepsilon))(x) : \xi\otimes\xi\,dV_t(x,\xi)\,dt \\ &\leq 3|t_2 - t_1|\,\|D(\mathrm{curl}(X_\varepsilon))\|_\infty\Big(\sup_{t\in I}\|V_t\|\Big) \\ &\leq 3|t_2 - t_1|\frac{C_1}{\varepsilon}\|\mathrm{curl}(X)\|_\infty\Big(\sup_{t\in I}\|V_t\|\Big), \end{aligned} \qquad (11)$$

where $C_1 := \int|\nabla\rho| < \infty$ is a fixed constant. Next, for any $x \in \mathbb{R}^3$ and $j \in 1, 2, 3$, we write

$$X^j(x) - X^j_\varepsilon(x) = \int_0^\varepsilon \varepsilon^{-3}\zeta\Big(\frac{r}{\varepsilon}\Big)\Big(\int_{\partial B(x,r)}[X^j(y) - X^j(x)]\,d\mathcal{H}^2\Big)\,dr.$$

For each $r > 0$, we expand

$$\begin{aligned}\int_{\partial B(x,r)} [X^j(y) - X^j(x)]\, d\mathcal{H}^2 &= \int_{\partial B(x,r)} \int_0^1 \nabla X^j(sy + (1-s)x) \cdot (y-x)\, ds\, d\mathcal{H}^2 \\ &= \int_0^1 \int_{\partial B(x,sr)} \nabla X^j(z) \cdot \frac{z-x}{rs}\, d\mathcal{H}^2 \frac{r}{s^2}\, ds = \int_0^1 \int_{B(x,sr)} \Delta X^j(z)\, dz \frac{r}{s^2}\, ds \\ &= \int_{\mathbb{R}^3} \Delta X^j(z) k_r(|z-x|)\, dz,\end{aligned}$$

where $k_r(\tau) = \int_{\tau/r}^{\max(\tau/r,1)} (r/s^2)\, ds = r(r/\tau - 1)^+$. It follows that

$$X(x) - X_\varepsilon(x) = K_\varepsilon * \Delta X, \quad \text{where} \quad K_\varepsilon(y) := \varepsilon^{-3} \int_0^\varepsilon \zeta(r/\varepsilon) k_r(y)\, dr.$$

Hence, for $i = 1, 2$ and summing over repeated indices, we obtain

$$\begin{aligned}T_{V_{t_i}}(X - X_\varepsilon) &= T_{V_{t_i}}(K_\varepsilon * \Delta X) = T_{V_{t_i}}\big(K_\varepsilon * (\nabla \mathrm{div}(X) + \mathrm{curl}\,\mathrm{curl}(X))\big) \\ &= T_{V_{t_i}}\big(\nabla(K_\varepsilon * \mathrm{div}(X))\big) + \varepsilon_{jk\ell} T_{V_{t_i}}\big(\partial_j K_\varepsilon * (\mathrm{curl}(X) \cdot e_k) e_\ell\big) \\ &\leq 0 + 6\Big(\sup_{t \in I} \|V_t\|\Big) \|DK_\varepsilon\|_1 \|\mathrm{curl}(X)\|_\infty, \end{aligned} \tag{12}$$

where we have used the fact that $T_{V_{t_i}}$ is boundary free. Inspection of K_ε yields the estimate $\|DK_\varepsilon\|_1 \leq C_2/\varepsilon$ where $C_2 > 0$ depends only on ρ, and therefore from (10)–(12) we deduce

$$T_{V_{t_1}}(X) - T_{V_{t_2}}(X) \leq \left(3\frac{C_1}{\varepsilon}|t_2 - t_1| + 6C_2\varepsilon\right) \|\mathrm{curl}(X)\|_\infty \Big(\sup_{t \in I} \|V_t\|\Big).$$

The conclusion follows by choosing $\varepsilon := |t_2 - t_1|^{1/2}$ and $C := 3C_1 + 6C_2$. □

Proof of Proposition 3. First, it follows from the convergence $V_t^n dt \rightharpoonup V$ that

$$V(\mathbb{R}^3 \times S^2 \times (a,b)) \leq \Big(\sup_{n \in \mathbb{N},\, t \in I} \|V_t^n\|\Big)|b - a| \quad \forall a, b \in I,$$

and therefore we may disintegrate V as $V = V_t\, dt$ where the measurable family $(V_t)_{t \in I}$ of non-negative Radon measures on $\mathbb{R}^3 \times S^2$, uniquely defined for almost every $t \in I$, satisfies

$$\sup_{t \in I} \|V_t\| \leq \liminf_{n \to \infty} \sup_{t \in I} \|V_t^n\|. \tag{13}$$

Next, for $m \geq 1$ and $T, \tilde{T} \in \mathcal{T}$, set

$$d_{\mathcal{F},m}(T, \tilde{T}) := \sup\{T(X) - \tilde{T}(X) : \|X\|_\infty \leq 1,\ \|\mathrm{curl}(X)\|_\infty \leq 1,\ \mathrm{supp}(X) \subset B(0,m)\},$$

where $X \in \mathcal{D}(\mathbb{R}^3, \mathbb{R}^3)$, and define

$$d_{\mathcal{F},\mathrm{loc}}(T, \tilde{T}) := \sum_{m=1}^\infty 2^{-m} \frac{d_{\mathcal{F},m}(T, \tilde{T})}{d_{\mathcal{F},m}(T, \tilde{T}) + 1}.$$

By the Federer and Fleming compactness theorem (see e.g. [11, 4.2.17]), for every $R > 0$ the set $Y := \{T \in \mathcal{T} : \|T\| \leq R\}$ equipped with the metric $d_{\mathcal{F},\mathrm{loc}}$ is compact. In what follows, we fix $R := \sup_{n\in\mathbb{N},\, t\in I} \|V_t^n\|$. In view of the inequality $\|T_{V_t^n}\| \leq \|V_t^n\|$, the definition of R and Proposition 2, it follows that the sequence of maps $t \mapsto T_{V_t^n}$, $n \in \mathbb{N}$, is equibounded and equicontinous in $\mathcal{C}(I, Y)$. By the Arzelà–Ascoli theorem, we infer that there exists a subsequence $(n_k)_{k\in\mathbb{N}}$ and a family $(T_t)_{t\in I}$ in $\mathcal{C}(I, Y)$ such that $t \mapsto T_t^{n_k}$ converge to $t \mapsto T_t$ in $\mathcal{C}(J, Y)$ as $k \to \infty$ for any compact subset $J \subset I$.

Let $h \in \mathcal{D}(I, \mathbb{R})$ and $X \in \mathcal{D}(\mathbb{R}^3, \mathbb{R}^3)$ be given. On the one hand we have

$$\lim_{k\to\infty} \int \int h(t)X(x) \cdot \xi \, dV_t^{n_k} \, dt = \lim_{k\to\infty} \int h(t)T_{V_t^{n_k}}(X) \, dt = \int h(t)T_t(X) \, dt,$$

and on the other hand we also have

$$\lim_{k\to\infty} \int \int h(t)X(x) \cdot \xi \, dV_t^{n_k} \, dt = \int \int h(t)X(x) \cdot \xi \, dV_t \, dt = \int h(t)T_{V_t}(X) \, dt.$$

It follows from those last two equalities and the du Bois-Reymond lemma that $T_{V_t}(X) = T_t(X)$ for almost every $t \in I$. Considering a countable family of vector fields X in $\mathcal{D}(\mathbb{R}^3, \mathbb{R}^3)$, dense in $\mathcal{D}(\mathbb{R}^3, \mathbb{R}^3)$ for the topology of uniform convergence, it follows next that $T_{V_t} = T_t$ in $\mathcal{D}'(\mathbb{R}^3, \mathbb{R}^3)$ for almost every $t \in I$. In turn, this implies that the only cluster point of the family $t \mapsto T_t^n$ in $\mathcal{C}(J, Y)$ is given by $t \mapsto T_t$, and therefore that the convergence of $t \mapsto T_t^n$ to $t \mapsto T_t$ in $\mathcal{C}(J, Y)$ holds without need to take a subsequence. Finally, we redefine $(V_t)_{t\in I}$ for a negligible set of t in such a way that $T_{V_t} = T_t$ in $\mathcal{D}'(\mathbb{R}^3, \mathbb{R}^3)$ now holds for all $t \in I$, and that (13) is still valid.

It remains to verify that $(V_t)_{t\in I}$ is a generalized binormal curvature flow. Let thus $X \in \mathcal{D}(\mathbb{R}^3, \mathbb{R}^3)$. For each $m \in \mathbb{N}$, by assumption the function $t \mapsto T_t^m(X)$ is Lipschitz on I and

$$\frac{d}{dt}T_t^m(X) = -\int D(\mathrm{curl}(X)) : \xi \otimes \xi \, dV_t^m$$

for almost every $t \in I$. In particular, $\left\| \frac{d}{dt} T_t^m(X) \right\|_\infty \leq C(X) \sup_{n\in\mathbb{N},\, t\in I} \|V_t^n\|$ depends possibly on X but not on m. Since the function $t \mapsto T_t(X)$ is the pointwise limit of the functions $t \mapsto T_t^m(X)$ as $m \to \infty$, the previous estimate implies that $t \mapsto T_t(X)$ is Lipschitz on I. For any $h \in \mathcal{D}(I, \mathbb{R})$, passing to the limit in the equality

$$\int T_t^m(X)h'(t) \, dt = \int \int D(\mathrm{curl}(X)) : \xi \otimes \xi \, dV_t^m \, h(t) \, dt,$$

we obtain

$$\int \int X \cdot \xi \, dV_t \, h'(t) \, dt = \int T_t(X)h'(t) \, dt = \int \int D(\mathrm{curl}(X)) : \xi \otimes \xi \, dV_t \, h(t) \, dt,$$

and since $t \mapsto T_t(X)$ is Lipschitz this finally implies that

$$\frac{d}{dt} \int X \cdot \xi \, dV_t = -\int D(\mathrm{curl}(X)) : \xi \otimes \xi \, dV_t$$

for almost every $t \in I$. □

Proof of Corollary 1. For each $n \in \mathbb{N}$, let $(V_t^n)_{t\in I}$ be a generalized binormal curvature flow whose family of undercurrents is given by $(T_t^n)_{t\in\mathbb{N}}$ and such that $\sup_{t\in I}\|V_n^t\| \le \|T_0^n\|$. In view of the assumption $\|T_0^n\| \to \|T_0\|$, we infer that $\sup_{n\in\mathbb{N},\, t\in I}\|V_t^n\| < \infty$. By the de la Vallée Poussin theorem, there exist a subsequence $(n_k)_{k\in\mathbb{N}}$ and a non-negative Radon measure $V \in \mathcal{M}(\mathbb{R}^3 \times S^2 \times I)$ such that $V_t^{n_k}dt \rightharpoonup V$ in $\mathcal{M}(\mathbb{R}^3 \times S^2 \times I)$ as $k \to \infty$. The conclusion then follows from Proposition 3 and the inequality

$$\sup_{t\in I}\|V_t\| \le \liminf_{k\to\infty}\sup_{t\in I}\|V_t^{n_k}\| \le \liminf_{k\to\infty}\|T_0^{n_k}\| = \|T_0\|. \qquad \square$$

Proof of Theorem 1. We proceed by approximation. Let $T_0 \in \mathcal{T}$. By Federer's approximation theorem [11, 4.2.20], there exists a sequence $(T_0^n)_{n\in\mathbb{N}}$ in $\mathcal{T}$ such that $T_0^n \rightharpoonup T_0$ in $\mathcal{D}'(\mathbb{R}^3, \mathbb{R}^3)$ and $\|T_0^n\| \to \|T_0\|$ in $\mathbb{R}$ as $n \to \infty$, and such that for each $n \in \mathbb{N}$, T_0^n has the following structure: there exists a finite collection $(\gamma_{j,0}^n)_{j\in J(n)}$ of smooth closed oriented curves in $\mathbb{R}^3$ such that

$$T_0^n = \sum_{j\in J(n)} T_{\gamma_{j,0}^n} \quad \text{and} \quad \|T_0^n\| = \sum_{j\in J(n)} \|T_{\gamma_{j,0}^n}\|. \tag{14}$$

For each $n \in \mathbb{N}$ and $j \in J(n)$, let γ_j^n denote the global classical solutions of equation[6] (1) with initial data $\gamma_{j,0}^n$ and set $\gamma_{j,t}^n := \gamma_j^n(t, \cdot)$. By Proposition 1, Remark 1, and (14), we infer that for each $n \in \mathbb{N}$ the map $t \mapsto T_t^n := \sum_{j\in J(n)} T_{\gamma_{j,t}^n}$ defines a weak binormal curvature flow with initial datum T_0^n. The conclusion then follows from Corollary 1. $\square$

4. Proofs of Theorems 3 and 2

We first prove the following key estimate:[7]

Proposition 4. *Assume the hypotheses of Theorem 2. For any $\xi_0 \in S^2 \subset \mathbb{R}^3$, the estimate*

$$|\partial_t X_{\gamma,r} \cdot \xi_0 - D(\mathrm{curl}(X)_{\gamma,r}) : (\xi_0 \otimes \xi_0)| \le K(1 - X_{\gamma,r} \cdot \xi_0) \tag{15}$$

holds on $\mathbb{R}^3 \times J$, where the vector field $X_{\gamma,r}$ was defined in (5) and

$$K := \frac{54}{r^2} + 14\|\partial_{sss}\gamma\|_{L^\infty(J\times\mathbb{T}^1)}.$$

Proof. First notice that since f vanishes otherwise, we may restrict our attention to points $(x_0, t_0) \in \mathbb{R}^3 \times J$ such that $d(x_0, \gamma_{t_0}) \le r$. Let $s_0 \in \mathbb{T}^1$ be uniquely defined by $P_{t_0}(x_0) = \gamma(t_0, s_0)$. In particular, we have

$$|x_0 - \gamma(t_0, s_0)|\,|\partial_{ss}\gamma(t_0, s_0)| \le 1/2 \tag{16}$$

[6] Existence of classical solutions of (1) for smooth data is well-known; one of the earlier proofs is given in [27], in a slightly different setting.

[7] A very similar estimate, with a nearly identical proof, is also presented in our companion paper [18], which is more suitable to binormal curvature flows in parametric form only.

and

$$(x_0 - \gamma(t_0, s_0)) \cdot \partial_s \gamma(t_0, s_0) = 0. \tag{17}$$

The mapping $\Psi : \mathbb{R}^3 \times J \times \mathbb{T}^1 \to \mathbb{R}$ given by

$$(x, t, s) \mapsto (x - \gamma(t, s)) \cdot \partial_s \gamma(t, s)$$

satisfies $\Psi(x_0, t_0, s_0) = 0$ and

$$\partial_s \Psi(x_0, t_0, s_0) = -|\partial_s \gamma(t_0, s_0)|^2 + (x_0 - \gamma(t_0, s_0)) \cdot \partial_{ss} \gamma(t_0, s_0) \le -1/2, \tag{18}$$

where we have used (2) and (16) for the last inequality. From the implicit function theorem, we infer that there exist an open neighborhood $\mathcal{U}$ of (x_0, t_0) in $\mathbb{R}^4$ and a smooth function $\zeta : \mathcal{U} \to \mathbb{R}$ such that

$$\Psi(x, t, \zeta(x, t)) = 0 \quad \forall (x, t) \in \mathcal{U}. \tag{19}$$

We may assume that $\mathcal{U} \subset \{(x, t) : d(x, \gamma_t) < \frac{3}{2} r\}$, so that $P_t(x)$ is defined for $(x, t) \in \mathcal{U}$. By uniqueness of the nearest-point projection, we therefore infer that

$$P_t(x) = \gamma(t, \zeta(x, t)) \quad \forall (x, t) \in \mathcal{U},$$

and also that

$$X_{\gamma,r}(x, t) = f\big(|x - \gamma(t, \zeta(x, t))|^2\big) \partial_s \gamma(t, \zeta(x, t)) \quad \forall (x, t) \in \mathcal{U}, \tag{20}$$

and finally that

$$\rho(x, t) := 1 - \big(x - \gamma(t, \zeta(x, t))\big) \cdot \partial_{ss} \gamma(t, \zeta(x, t)) > 0 \quad \text{in } \mathcal{U}. \tag{21}$$

We fix some notation to keep subsequent expressions of reasonable size. For a function Y with values in $\mathbb{R}^3$, and $i \in \{1, 2, 3\}$, we write Y^i to denote the i-th component of Y. We write d^2 to denote the function $(x, t) \mapsto |x - \gamma(t, \zeta(x, t))|^2$, $\gamma(\zeta)$ to denote the function $(x, t) \mapsto \gamma(t, \zeta(x, t))$, and similarly for $\partial_t \gamma(\zeta)$, $\partial_{ts} \gamma(\zeta)$, $\partial_s \gamma(\zeta)$, $\partial_{ss} \gamma(\zeta)$ and $\partial_{sss} \gamma(\zeta)$. When it does not lead to possible confusion, we also denote by x the function $(x, t) \mapsto x$. Each of these functions is defined on $\mathcal{U}$.

Step 1: First computation of $D(\mathrm{curl}(X)) : (\xi_0 \otimes \xi_0)$. Differentiating (20) we obtain, pointwise on $\mathcal{U}$ and for $i, j \in \{1, 2, 3\}$,

$$\partial_j X^i = \partial_j (f(d^2)) \partial_s \gamma(\zeta)^i + f(d^2) \partial_{ss} \gamma(\zeta)^i \partial_j \zeta \tag{22}$$

for the space derivatives, and

$$\partial_t X^i = \partial_t (f(d^2)) \partial_s \gamma(\zeta)^i + f(d^2) [\partial_{ss} \gamma(\zeta)^i \partial_t \zeta + \partial_{ts} \gamma(\zeta)] \tag{23}$$

for the time derivative. Also, for $i, j, \ell \in \{1, 2, 3\}$,

$$\begin{aligned} \partial_{\ell j} X^i &= \partial_{\ell j} (f(d^2)) \partial_s \gamma(\zeta)^i + \partial_{ss} \gamma(\zeta)^i [\partial_\ell (f(d^2)) \partial_j \zeta + \partial_j (f(d^2)) \partial_\ell \zeta] \\ &\quad + f(d^2) \partial_{sss} \gamma(\zeta)^i \partial_\ell \zeta \partial_j \zeta + f(d^2) \partial_{ss} \gamma(\zeta)^i \partial_{\ell j} \zeta. \end{aligned}$$

In particular, we may write

$$D(\mathrm{curl}(X)) : (\xi_0 \otimes \xi_0) =: A = A_1 + A_2 + A_3 + A_4, \tag{24}$$

where

$$\begin{aligned} A_1 &:= \epsilon_{ijk}\partial_{\ell i}(f(d^2))\partial_s\gamma(\zeta)^j\,\xi_0^k\xi_0^\ell, \\ A_2 &:= \epsilon_{ijk}\partial_{ss}\gamma(\zeta)^j[\partial_\ell(f(d^2))\partial_i\zeta + \partial_i(f(d^2))\partial_\ell\zeta]\xi_0^k\xi_0^\ell, \\ A_3 &:= \epsilon_{ijk}f(d^2)\partial_{sss}\gamma(\zeta)^j\partial_\ell\zeta\partial_i\zeta\xi_0^k\xi_0^\ell, \\ A_4 &:= \epsilon_{ijk}f(d^2)\partial_{ss}\gamma(\zeta)^j\partial_{\ell i}\zeta\xi_0^k\xi_0^\ell, \end{aligned} \tag{25}$$

in which ϵ_{ijk} is the Levi-Civita symbol and we sum over repeated indices.

Step 2: Expressing derivatives of ζ in terms of γ. Recall that by definition of ζ, we have

$$(x - \gamma(t, \zeta(x,t))) \cdot \partial_s\gamma(t, \zeta(x,t)) = 0 \tag{26}$$

for every $(x,t) \in \mathcal{U}$. For $j \in \{1,2,3\}$, differentiating (26) with respect to x_j and using (2) we find

$$\partial_s\gamma^j(\zeta) - \partial_j\zeta + (x - \gamma(\zeta)) \cdot \partial_{ss}\gamma(\zeta)\partial_j\zeta = 0. \tag{27}$$

In view of (21), we may rewrite (27) as

$$\partial_j\zeta = \frac{1}{\rho}\partial_s\gamma^j(\zeta). \tag{28}$$

For $\ell \in \{1,2,3\}$, differentiating (27) with respect to x_ℓ and using (18), we obtain

$$\begin{aligned} \partial_{\ell j}\zeta = \frac{1}{\rho}\Big(\partial_{ss}\gamma(\zeta)^j\frac{\partial_s\gamma(\zeta)^\ell}{\rho} + \partial_{ss}\gamma(\zeta)^\ell\frac{\partial_s\gamma(\zeta)^j}{\rho} \\ + (x - \gamma(\zeta)) \cdot \partial_{sss}\gamma(\zeta)\frac{\partial_s\gamma(\zeta)^j\partial_s\gamma(\zeta)^\ell}{\rho^2}\Big). \end{aligned} \tag{29}$$

Finally, differentiating (26) with respect to t we obtain

$$\partial_t\zeta = \frac{1}{\rho}\big(-\partial_t\gamma(\zeta) \cdot \partial_s\gamma(\zeta) + (x - \gamma(\zeta)) \cdot \partial_{ts}\gamma(\zeta)\big). \tag{30}$$

In particular, taking into account (1) it follows from (30) that, at the point (x_0, t_0),

$$\partial_t\zeta = \frac{1}{\rho}(x - \gamma(\zeta)) \cdot (\partial_s\gamma(\zeta) \times \partial_{sss}\gamma(\zeta)). \tag{31}$$

Step 3: Expressing derivatives of d^2 in terms of γ. In view of the definition of d^2, we have, for $j \in \{1,2,3\}$,

$$\partial_j d^2 = 2(x - \gamma(\zeta))^j - 2(x - \gamma(\zeta)) \cdot \partial_s\gamma(\zeta)\partial_j\zeta = 2(x - \gamma(\zeta))^j, \tag{32}$$

where the last equality follows from (26). For $\ell \in \{1, 2, 3\}$, differentiating (32) with respect to x_ℓ and using (27), we obtain

$$\partial_{\ell j} d^2 = -2(\delta_{j\ell} - \partial_s \gamma(\zeta)^j \partial_\ell \zeta) = -2\left(\delta_{j\ell} - \frac{\partial_s \gamma(\zeta)^j \partial_s \gamma(\zeta)^\ell}{\rho}\right), \tag{33}$$

where $\delta_{j\ell}$ is the Kronecker symbol. Also from the definition of d^2, we have

$$\partial_t d^2 = -2(x - \gamma(\zeta)) \cdot (\partial_t \gamma(\zeta) + \partial_s \gamma(\zeta) \partial_t \zeta). \tag{34}$$

In particular, taking into account (26) and (1) it follows from (34) that, at the point (x_0, t_0),

$$\partial_t d^2 = -2(x - \gamma(\zeta)) \cdot (\partial_s \gamma(\zeta) \times \partial_{ss} \gamma(\zeta)). \tag{35}$$

Step 4: A reduced expression for $D(\mathrm{curl}(X)) : (\xi_0 \otimes \xi_0)$. We substitute, in the terms A_1, A_2, A_3 and A_4 defined in Step 1, the expressions for the derivatives of d^2 and ζ which we obtained in Steps 2 and 3. Some cancellations occur.

Examining A_1, we first expand

$$\begin{aligned}\partial_{\ell i}(f(d^2)) &= f''(d^2) \partial_\ell d^2 \, \partial_i d^2 + f'(d^2) \partial_{\ell i} d^2 \\ &= 4 f''(d^2)(x - \gamma(\zeta))^\ell (x - \gamma(\zeta))^i + \frac{2}{\rho} f'(d^2) \partial_s \gamma(\zeta)^\ell \partial_s \gamma(\zeta)^i - 2 f'(d^2) \delta_{\ell i},\end{aligned}$$

where we have used (32) and (33) for the second equality. Next, we write

$$\begin{aligned}\epsilon_{ijk}(x - \gamma(\zeta))^\ell (x - \gamma(\zeta))^i \partial_s \gamma(\zeta)^j \xi_0^k \xi_0^\ell &= \big(\epsilon_{ijk}(x - \gamma(\zeta))^i \partial_s \gamma(\zeta)^j \xi_0^k\big)((x - \gamma(\zeta))^\ell \xi_0^\ell) \\ &= \big((x - \gamma(\zeta)) \cdot (\partial_s \gamma(\zeta) \times \xi_0)\big)((x - \gamma(\zeta)) \cdot \xi_0).\end{aligned}$$

Similarly,

$$\epsilon_{ijk} \partial_s \gamma(\zeta)^\ell \, \partial_s \gamma(\zeta)^i \partial_s \gamma(\zeta)^j \xi_0^k \xi_0^\ell = \big(\partial_s \gamma(\zeta) \cdot (\partial_s \gamma(\zeta) \times \xi_0)\big)(\partial_s \gamma(\zeta) \cdot \xi_0) = 0,$$

and

$$\epsilon_{ijk} \delta_{\ell i} \partial_s \gamma(\zeta)^j \xi_0^k \xi_0^\ell = \epsilon_{ijk} \xi_0^i \partial_s \gamma(\zeta)^j \, \xi_0^k = \xi_0 \cdot (\partial_s \gamma(\zeta) \times \xi_0) = 0.$$

Hence,

$$A_1 = 4 f''(d^2)\big((x - \gamma(\zeta)) \cdot (\partial_s \gamma(\zeta) \times \xi_0)\big)((x - \gamma(\zeta)) \cdot \xi_0). \tag{36}$$

In the same way, for A_2, (28) and (32) yield

$$\begin{aligned}A_2 &= \frac{2}{\rho} f'(d^2) \epsilon_{ijk} \xi_0^k \xi_0^\ell \partial_{ss} \gamma(\zeta)^j \big((x - \gamma(\zeta))^\ell \partial_s \gamma(\zeta)^i + (x - \gamma(\zeta))^i \partial_s \gamma(\zeta)^\ell\big) \\ &= \frac{2}{\rho} f'(d^2) \partial_{ss} \gamma(\zeta) \cdot (\partial_s \gamma(\zeta) \times \xi_0)(\xi_0 \cdot (x - \gamma(\zeta))) \\ &\quad + \frac{2}{\rho} f'(d^2)(x - \gamma(\zeta)) \cdot (\partial_{ss} \gamma(\zeta) \times \xi_0)(\xi_0 \cdot \partial_s \gamma(\zeta)) \\ &=: A_{2,1} + A_{2,2}.\end{aligned} \tag{37}$$

For A_3, we invoke (28) to substitute $\partial_\ell \zeta$ and $\partial_i \zeta$ and obtain

$$A_3 = \frac{1}{\rho^2} f(d^2)(\partial_s\gamma(\zeta)\cdot\xi_0)\partial_s\gamma(\zeta)\cdot(\partial_{sss}\gamma(\zeta)\times\xi_0). \tag{38}$$

For A_4 finally, we invoke (29) to substitute $\partial_{\ell i}\zeta$ and obtain

$$\begin{aligned} A_4 &= \frac{1}{\rho^2} f(d^2)\partial_{ss}\gamma(\zeta)\cdot(\partial_{ss}\gamma(\zeta)\times\xi_0)(\partial_s\gamma(\zeta)\cdot\xi_0)\\ &\quad + \frac{1}{\rho^2} f(d^2)\partial_s\gamma(\zeta)\cdot(\partial_{ss}\gamma(\zeta)\times\xi_0)(\partial_{ss}\gamma(\zeta)\cdot\xi_0)\\ &\quad + \frac{1}{\rho^3} f(d^2)\big((x-\gamma(\zeta))\cdot\partial_{sss}\gamma(\zeta)\big)\partial_s\gamma(\zeta)\cdot(\partial_{ss}\gamma(\zeta)\times\xi_0)(\partial_s\gamma(\zeta)\cdot\xi_0)\\ &=: 0 + A_{4,1} + A_{4,2}. \end{aligned} \tag{39}$$

Step 5: Computation of $\partial_t X\cdot\xi_0$. We expand (23) as

$$\partial_t X^i = f'(d^2)\partial_t d^2 \partial_s\gamma(\zeta)^i + f(d^2)[\partial_{ss}\gamma(\zeta)^i\partial_t\zeta + \partial_{ts}\gamma(\zeta)]. \tag{40}$$

Therefore, *at the point* (x_0, t_0), we obtain from (1), (30) and (35)

$$\partial_t X\cdot\xi_0 =: B = B_1 + B_2 + B_3, \tag{41}$$

where

$$\begin{aligned} B_1 &:= -2f'(d^2)\big((x-\gamma(\zeta))\cdot(\partial_s\gamma(\zeta)\times\partial_{ss}\gamma(\zeta))\big)(\partial_s\gamma(\zeta)\cdot\xi_0),\\ B_2 &:= \frac{1}{\rho} f(d^2)\big((x-\gamma(\zeta))\cdot(\partial_s\gamma(\zeta)\times\partial_{sss}\gamma(\zeta))\big)(\partial_{ss}\gamma(\zeta)\cdot\xi_0),\\ B_3 &:= f(d^2)(\partial_s\gamma(\zeta)\times\partial_{sss}\gamma(\zeta))\cdot\xi_0. \end{aligned} \tag{42}$$

Step 6: Proof of Proposition 4 completed. We write, *at the point* (x_0, t_0),

$$\begin{aligned} |B-A| &= |\partial_t X\cdot\xi_0 - D(\mathrm{curl}(X)) : (\xi_0\otimes\xi_0)|\\ &\le |A_1| + |A_{2,1}| + |A_{2,2}-B_1| + |A_3-B_3| + |A_{4,1}| + |A_{4,2}| + |B_2|, \end{aligned} \tag{43}$$

and we will estimate each of the terms in the last line separately. We first observe the following elementary facts that hold at the point (x_0, t_0) (when they involve functions):

$\lvert\xi_0^\perp\rvert, \lvert\xi_0\rvert, \lvert\partial_s\gamma(\zeta)\rvert \le 1$	(indeed, $\xi_0\in S^2$ and (2) holds),
$\lvert f'(d^2)\rvert \le 3/r^2$, $\lvert f''(d^2)\rvert \le 6/r^4$	(from the definition of f),
$\rho\ge 1/2$, $\lvert 1-1/\rho\rvert \le d/r$, $\lvert 1-1/\rho^2\rvert \le 3d/r$	(from (16) and the definition (21) of ρ).

For convenience, set $\Sigma = \|\partial_{sss}\gamma\|_{L^\infty(J\times\mathbb{T}^1)}$. Taking into account (17), direct inspection yields

$$|A_1| \le 24 d^2 r^{-4}, \qquad |A_{2,1}| \le 6 d r^{-3}|\xi_0^\perp|, \qquad |A_{4,1}| \le \tfrac{1}{2} r^{-2}|\xi_0^\perp|^2, \tag{44}$$

as well as

$$|A_{4,2}| \le 4dr^{-1}\Sigma|\xi_0^\perp|, \qquad |B_2| \le dr^{-1}\Sigma|\xi_0^\perp|. \tag{45}$$

Next, we write

$$\begin{aligned}|B_1 - A_{2,2}| &= \left|\frac{2}{\rho} f'(d^2)(\partial_s\gamma(\zeta)\cdot\xi_0)\big((x-\gamma(\zeta))\times\partial_{ss}\gamma(\zeta)\big)\cdot\left(\partial_s\gamma(\zeta)-\frac{\xi_0}{\rho}\right)\right| \\ &\le 6dr^{-3}(|\partial_s\gamma(\zeta)-\xi_0| + dr^{-1}),\end{aligned} \tag{46}$$

and

$$\begin{aligned}|B_3 - A_3| &= \left|f(d^2)[(\partial_s\gamma(\zeta)\times\partial_{sss}\gamma(\zeta))\cdot\xi_0]\left(1-\frac{1}{\rho^2}\xi_0\cdot\partial_s\gamma(\zeta)\right)\right| \\ &\le \Sigma|\xi_0^\perp|\big((1-\partial_s\gamma(\zeta)\cdot\xi_0)+dr^{-1}\big) \\ &\le \Sigma\big((1-\partial_s\gamma(\zeta)\cdot\xi_0)+dr^{-1}|\xi_0^\perp|\big).\end{aligned} \tag{47}$$

It remains to bound d, $|\xi_0^\perp|$, $|\partial_s\gamma(\zeta)-\xi_0|$ and $|1-\partial_s\gamma(\zeta)\cdot\xi_0|$ in terms of $1-X_{\gamma,r}\cdot\xi_0$. For that purpose, first recall from the definition of f, from the fact that $|\xi_0| = |\partial_s\gamma(\zeta)| = 1$, and from the assumption $d \le r$, that

$$1 - X_{\gamma,r}\cdot\xi_0 \ge 1 - f(d^2) \ge d^2 r^{-2}. \tag{48}$$

Also, if $X_{\gamma,r}\cdot\xi_0 \ge 0$ then $1 - X_{\gamma,r}\cdot\xi_0 \ge 1-\partial_s\gamma(\zeta)\cdot\xi_0$, and if $X_{\gamma,r}\cdot\xi_0 < 0$ then $1 - X_{\gamma,r}\cdot\xi_0 \ge 1 \ge (1-\partial_s\gamma(\zeta)\cdot\xi_0)/2$. In any case, we have

$$1 - X_{\gamma,r}\cdot\xi_0 \ge \tfrac{1}{2}(1-\partial_s\gamma(\zeta)\cdot\xi_0). \tag{49}$$

Finally, by Hilbert's projection theorem

$$|\xi_0^\perp|^2 \le |\xi_0 - \partial_s\gamma(\zeta)|^2 = 2(1-\partial_s\gamma(\zeta)\cdot\xi_0) \le 4(1-X_{\gamma,r}\cdot\xi_0). \tag{50}$$

Inserting (48), (49), or (50) in (44)–(47), and writing $x \preceq y$ for $x \le y(1-X_{\gamma,r}\cdot\xi_0)$, we obtain

$$\begin{gathered}|A_1| \preceq 24/r^2, \quad |A_{2,1}| \preceq 12/r^2, \quad |A_{4,1}| \preceq 2/r^2, \quad |A_{4,2}| \preceq 8\Sigma, \\ |B_2| \preceq 2\Sigma, \quad |B_1 - A_{2,2}| \preceq 16/r^2, \quad |B_3 - A_3| \preceq 4\Sigma,\end{gathered}$$

and summation according to (43) yields the claim. □

Proof of Theorem 3. Since the map $t \mapsto \|V_t\|$ is bounded on J, since f is of class $\mathcal{C}^2$ and since $D(\mathrm{curl}(X_{\gamma,r}))$ is a continuous function, we infer from Definition 1 that the function G is Lipschitz on J and that

$$\frac{d}{dt}G(t) = -\frac{d}{dt}\int X_{\gamma,r}\cdot\xi\, dV_t = -\int \partial_t X_{\gamma,r}\cdot\xi - D(\mathrm{curl}(X_{\gamma,r})) : (\xi\otimes\xi)\, dV_t.$$

The conclusion follows directly from Proposition 4. □

Proof of Theorem 2. Let $t \mapsto T_t$ be a weak binormal curvature flow on J with initial datum $T_{\gamma,0}$. By Theorem 3 and the Gronwall inequality, we infer that G and F vanish identically on any compact subinterval of J containing 0, and therefore vanish on J. Fix $t \in J$. Since $X_{\gamma,r}(x,t) \cdot \xi = 0$ if and only if $x = \gamma(t,s)$ for some $s \in \mathbb{T}^1$ and $\xi = \partial_s \gamma(t,s)$, we deduce from the identity $F(t) = 0$ that for $\mathcal{H}^1$-a.e. x in the geometrical support of T_t we have $x \in \gamma_t$. It follows from the Federer–Fleming constancy theorem [11, 4.1.31] that $T_t = aT_{\gamma,t}$ for some $a \in \mathbb{Z}$, and then from the identity $F(t) = 0$ that $a = 1$. □

5. Additional results, examples and open questions

5.1. Control of average speed and conserved quantities

In general, the convergence stated in Proposition 3 or Corollary 1, and involved in the construction of a solution in Theorem 1, does not imply that there is no mass loss at infinity, and it could be that $\|V_t\|$ is not constant in time. In the following, we present a sufficient condition to rule out this possibility, and we deduce conservation of momentum and angular momentum in that case.

Definition 3. A weak binormal curvature flow $(T_t)_{t\in I}$ is called *almost parametric* if there exists a sequence $(T_{\gamma^n,t})_{t\in I}$, $n \in \mathbb{N}$, of binormal curvature flows associated to smooth solutions $(\gamma^n)_{n\in\mathbb{N}}$ of (1) according to Proposition 1, such that

$$\|T_0\| = \lim_{n\to\infty} \|T_{\gamma^n,0}\|$$

and for all $t \in I$,

$$T_{\gamma^n,t} \rightharpoonup T_t \quad \text{in } \mathcal{D}'(\mathbb{R}^3, \mathbb{R}^3).$$

Remark 3. (i) It follows from Proposition 3 that given any current T_0 associated to a Lipschitz function $\gamma_0 : \mathbb{R}/\ell\mathbb{Z} \to \mathbb{R}^3$ by the formula

$$T_0(X) := \int_0^\ell X(\gamma_0(s)) \cdot \partial_s \gamma_0(s)\, ds \qquad \forall X \in \mathcal{D}(\mathbb{R}^3, \mathbb{R}^3),$$

there exists an almost parametric binormal curvature flow with initial datum T_0.

(ii) From the convergence $T_{\gamma^n,t} \rightharpoonup T_t$ it follows that T_t is compactly supported for every $t \in I$.

For a smooth solution $\gamma : I \times \mathbb{R}/\ell\mathbb{Z} \to \mathbb{R}^3$ of (1), the momentum $P(\gamma(t,\cdot))$ and the angular momentum $Q(\gamma(t,\cdot))$ defined respectively by

$$P(\gamma(t,\cdot)) := \int_0^\ell \gamma(t,s) \times \partial_s \gamma(t,s)\, ds,$$
$$Q(\gamma(t,\cdot)) := \int_0^\ell \gamma(t,s) \times \big(\gamma(t,s) \times \partial_s \gamma(t,s)\big)\, ds,$$

are independent of time.

Notice that

$$P(\gamma(t,\cdot)) = (T_{\gamma,t}(X_1), T_{\gamma,t}(X_2), T_{\gamma,t}(X_3)),$$
$$Q(\gamma(t,\cdot)) = (T_{\gamma,t}(Y_1), T_{\gamma,t}(Y_2), T_{\gamma,t}(Y_3)),$$

where the vector fields X_1, X_2, X_3 and Y_1, Y_2, Y_3 on $\mathbb{R}^3$ are given by $X_1 := (0, -x_3, x_2)$, $X_2 := (x_3, 0, -x_1)$, $X_3 := (-x_2, x_1, 0)$, $Y_1 := (-x_2^2 - x_3^2, x_1x_2, x_1x_3)$, $Y_2 := (x_1x_2, -x_1^2 - x_3^2, x_2x_3)$, and $Y_3 := (x_1x_3, x_2x_3, -x_1^2 - x_2^2)$.

Definition 4. Let T be a compactly supported integral 1-current without boundary in $\mathbb{R}^3$. The *momentum* of T, denoted by $P(T)$, and the *angular moment* of T, denoted by $Q(T)$, are the vectors in $\mathbb{R}^3$ defined by $P(T) := (T(X_1), T(X_2), T(X_3))$ and $Q(T) := (T(Y_1), T(Y_2), T(Y_3))$.

The sufficient condition which we rely on amounts to non-vanishing of the momentum.

Proposition 5. *Let $(T_t)_{t\in I}$ be an almost parametric binormal curvature flow on I with initial datum T_0, and assume that $P(T_0) \neq 0$. There exists a universal constant $C > 0$ such that for every $t \in I$, either* $\mathrm{supp}(T_t)$ *remains at a distance at most* $2\|T_0\|$ *of* $\mathrm{supp}(T_0)$, *or*

$$\mathrm{supp}(T_t) \subseteq \mathrm{supp}(T_0) + B(V_0|t|, 0), \quad \textit{where} \quad V_0 := C\frac{\|T_0\|^3}{P(T_0)^2}.$$

Corollary 2. *Let $(T_t)_{t\in I}$ be an almost parametric binormal curvature flow on I with initial datum T_0, and assume $P(T_0) \neq 0$. Then the momentum $P(T_t)$ and the angular momentum $Q(T_t)$ are independent of time.*

Remark 4. Up to gradient vector fields, the family $\{X_1, X_2, X_3, Y_1, Y_2, Y_3\}$ is maximal for smooth globally defined and linearly independent vector fields such that $D(\mathrm{curl}(X))$ is pointwise an anti-symmetric matrix. In particular, there are no other "first order" invariants of this form. In contrast, smooth binormal curvature flows are known to possess infinitely many higher order invariants (see Hasimoto [13]).

Concerning Proposition 5, notice that a circle of radius $\varepsilon > 0$ gives rise to a traveling wave solution of (1) with speed $1/\varepsilon$. On the other hand, the current associated to such a solution (given an orientation) has a mass equal to $2\pi\varepsilon$ and a momentum equal to $\pi\varepsilon^2$. This shows that the upper bound on the speed given by Proposition 5, except for the value of C, is in some sense optimal. Actually, even a curve of length of order one but small momentum may travel at a very large speed, as shown by the "bullet" $\gamma_0(s) := \left(\frac{1}{n}\cos(ns), \frac{1}{n}\sin(ns), 0\right)$ for $s \in \mathbb{R}/2\pi\mathbb{Z}$. In that case, the associated current T_0 has mass $\|T_0\| = 2\pi$, its momentum satisfies $|P(T_0)| = 2\pi/n$, and its speed is equal to n. This suggests raising the following:

Question 1. Given a smooth solution $\gamma : \mathbb{R} \times \mathbb{R}/\ell\mathbb{Z} \to \mathbb{R}^3$ of (1) such that the image of $\gamma(0,\cdot)$ is not entirely contained in any ball of radius $r > 0$, is it possible to bound its average speed (i.e. similar to the statement of Proposition 5) by a function V_0 depending only on r?

Proof of Proposition 5. In view of Definition 3 and Remark 3(ii), it suffices to consider the case of a binormal curvature flow associated to a single smooth solution of (1). Assume that $\mathrm{supp}(T_t)$ extends to a distance bigger than $2\|T_0\|$ from $\mathrm{supp}(T_0)$, fix arbitrary $a \in \mathrm{supp}(T_0)$ and $b \in \mathrm{supp}(T_t)$, and set

$$X(x) := \chi(\|x-a\|)X_i(x-a) - \chi(\|x-b\|)X_i(x-b),$$

where $i \in \{1,2,3\}$ is chosen such that $|T_0(X_i)| \geq \frac{1}{\sqrt{3}}|P(T_0)|$, and $\chi : [0,\infty) \to [0,1]$ is a smooth cut-off function such that $\chi \equiv 1$ on $[0, \|T_0\|/2]$, $\chi \equiv 0$ outside $[0, \|T_0\|]$ and $\|\chi'\|_\infty \leq 3/\|T_0\|$. By assumption and by construction, $X = X_i$ on $\mathrm{supp}(T_0)$ and $X = -X_i$ on $\mathrm{supp}(T_t)$, so that

$$T_0(X) = P_i(T_0) \quad \text{and} \quad T_t(X) = -P_i(T_t) = -P_i(T_0),$$

where the last equality is a consequence of the conservation of momentum for smooth binormal curvature flows. On the other hand, by Proposition 2, we have

$$|T_0(X) - T_t(X)| \leq C|t|^{1/2}\|T_0\| \, \|\mathrm{curl}(X)\|_\infty \leq 4C|t|^{1/2}\|T_0\|.$$

Hence,

$$|t| \geq \frac{P(T_0)^2}{12C^2\|T_0\|^2}.$$

The conclusion follows by splitting the whole time interval into subintervals on which $\mathrm{supp}(T_t)$ moves by a distance $2\|T_0\|$. □

Proof of Corollary 2. It suffices to use the conservation of P and Q at the level of the approximating smooth flows γ^n, to consider cut-offs of X_1, X_2, X_3 and Y_1, Y_2, Y_3 sufficiently far at infinity so that the cut-off does not occur on the supports of T_t and $T_{\gamma^n,t}$, and to invoke pointwise in time convergence in $\mathcal{D}'(\mathbb{R}^3, \mathbb{R}^3)$. □

5.2. Oscillations and generalized binormal curvature flows

The undercurrents associated to generalized binormal curvature flows, even when they can be identified with smooth parametrized curves, need not be solutions of the classical binormal curvature flow equation (1). We present here a family of typical such examples, for which the speed is modified by a constant multiplicative factor, and we question about its occurrence as an almost parametrized flow according to Definition 3.

Proposition 6. *Let $\gamma : \mathbb{R} \times (\mathbb{R}/\ell\mathbb{Z}) \to \mathbb{R}^3$ be a smooth solution of (1), for some $\ell > 0$, and let $(V_{\gamma,t})_{t\in\mathbb{R}}$ and $(T_{\gamma,t})_{t\in\mathbb{R}}$ denote the associated generalized and weak binormal curvature flows, respectively, as described in Proposition 1. Then for any $m > 1$ and any $a \in [a_m, m]$, where $a_m := \frac{1}{2}(3/m - m)$, there exists a generalized binormal curvature flow $(V_t^{m,a})_{t\in\mathbb{R}}$ such that the associated undercurrents are given by*

$$T_t^{m,a} := T_{V_t^{m,a}} = T_{\gamma,at}, \tag{51}$$

and

$$\text{for every Borel } O \subset \mathbb{R}^3, \qquad V_t^{m,a}(O \times S^2) = m V_{\gamma,at}(O \times S^2). \tag{52}$$

Condition (52) can be thought of as asserting that these generalized solutions have "mass $m > 1$ per unit arc-length". Heuristically, one may think of the extra mass $m - 1$ as corresponding to microscopic oscillations. Note in particular that there exist generalized solutions with $a < 0$ as soon as $m > \sqrt{3}$.

Proof of Proposition 6. We first show that for $m > 1$ and $a \in [a_m, m]$, and for any $\xi_0 \in S^2$, there exists a measure $W^{m,a}[\xi_0]$ on S^2 such that

$$\int_{S^2} \xi \, dW^{m,a}[\xi_0] = \xi_0, \qquad \int_{S^2} \xi \otimes \xi \, dW^{m,a}[\xi_0] = a\xi_0 \otimes \xi_0 + \frac{m-a}{3} \operatorname{Id}, \tag{53}$$

where Id denotes the identity matrix. Note that the second identity above implies that

$$W^{m,a}[\xi_0](S^2) = \int_{S^2} |\xi|^2 \, dW^{m,a}[\xi_0] = \operatorname{Tr}\Big(\int_{S^2} \xi \otimes \xi \, dW^{m,a}[\xi_0]\Big) = m.$$

In general, measures $W^{m,a}[\xi_0]$ are of course not uniquely determined by these moment conditions; the explicit examples we write down are chosen just for convenience.

For $\xi_0 \in S^2$ and $\alpha \in (0, 1]$, we define the sets

$$S(\xi_0, \alpha) := \{\xi \in S^2 : \xi \cdot \xi_0 = \alpha\},$$

and the positive Radon measures $\mu[\xi_0, \alpha] \in \mathcal{M}(S^2)$ where

$$\int_{S^2} f(\xi) \, d\mu[\xi_0, \alpha](\xi) := \frac{1}{\alpha} ⨍_{S(\xi_0,\alpha)} f(\xi) \, d\mathcal{H}^1(\xi) \qquad \forall f \in \mathcal{C}(S^2, \mathbb{R})$$

if $\alpha < 1$, and $\mu[\xi_0, 1] = \delta_{\xi_0}$ for $\alpha = 1$. For $\beta \geq 0$, further define

$$\mu[\xi_0, \alpha, \beta] := (1 + \beta)\mu[\xi_0, \alpha] + \beta\delta_{-\xi_0}.$$

One checks that for all $\alpha \in (0, 1]$ and $\beta \geq 0$,

$$\int_{S^2} \xi \, d\mu[\xi_0, \alpha, \beta] = \xi_0$$

and

$$\int_{S^2} \xi \otimes \xi \, d\mu[\xi_0, \alpha, \beta] = \Big[(1 + \beta)\frac{3\alpha^2 - 1}{2\alpha} + \beta\Big]\xi_0 \otimes \xi_0 + \frac{(1 + \beta)(1 - \alpha^2)}{2\alpha} \operatorname{Id}.$$

Then a computation shows that $\mu[\xi_0, \alpha, \beta]$ satisfies the second identity in (53) if

$$\alpha = \frac{2a + 3 + m}{3(1 + m)}, \qquad \beta = \frac{m\alpha - 1}{1 + \alpha}.$$

Note that since $\beta \geq 0$, we must have $\alpha \geq 1/m$, and clearly $\alpha \leq 1$. The requirement $\alpha \in [1/m, 1]$ gives rise to the restriction $a \in [a_m, m]$.

Now define

$$\int \psi(x, \xi) \, dV_t^{m,a} := \int_{\mathbb{R}/\ell\mathbb{Z}} \Big(\int_{S^2} \psi(\gamma(s), \xi) \, dW^{m,a}[\partial_s \gamma(at, s)]\Big) \, ds.$$

It then follows directly from (53) that (52) is satisfied and that for every compactly supported vector field X,

$$\int X \cdot \xi \, dV_t^{m,a} = \int X \cdot \xi \, dV_{\gamma,at},$$

which just says that (51) holds. In addition, since $D(\mathrm{curl}(X)) : \mathrm{Id} \equiv 0$ for every X, we deduce from (53) and the definitions that

$$\int D(\mathrm{curl}(X)) : \xi \otimes \xi \, dV_t^{m,a} = a \int D(\mathrm{curl}(X)) : \xi \otimes \xi \, dV_{\gamma,at}.$$

It follows from these last two identities and Proposition 1 that $(V_t^{m,a})_{t\in\mathbb{R}}$ is a generalized binormal curvature flow. □

Remark 5. We remark that if $W^{m,a}[\xi_0]$ is any measure on S^2 satisfying (53), then

$$\begin{aligned} 1 &= \int \xi_0 \cdot \xi \, dW^{m,a}[\xi_0] \leq \left(\int (\xi_0 \cdot \xi)^2 \, dW^{m,a}[\xi_0] \int 1 \, dW^{m,a}[\xi_0] \right)^{1/2} \\ &= \sqrt{\left(a + \frac{m-a}{3}\right) m}, \end{aligned}$$

and it follows that $a \geq a_m$. Clearly $a \leq m$, so the restriction on the range of a in (53) is optimal. In addition, if $a = a_m$, then the above calculation implies that $\xi_0 \cdot \xi$ is $W^{m,a}[\xi_0]$ a.e. constant, and from this one can check that $W^{m,a}[\xi_0]$ is supported on $S(\xi_0, \alpha)$. Thus the extremal case $a = a_m$ corresponds, heuristically, to microscopic oscillations whose tangents form a constant angle with the tangents of macroscopic smooth curves.

Varifolds with non-trivial (i.e. not reduced to a single Dirac mass) dependence on ξ are typically associated to limits of wild oscillations. Indeed, the generalized binormal curvature flows described in the previous proposition may be obtained as limits of smooth solutions of (1) (of course without the mass convergence of the currents), at least in the case of the traveling circles with $a = a_m$.

Proposition 7. *Let $\ell > 0$ and $\gamma : \mathbb{R} \times (\mathbb{R}/\ell\mathbb{Z}) \to \mathbb{R}^3$ be a smooth solution of* (1) *corresponding to a traveling circle at speed $2\pi/\ell$. For every $m > 1$, there exists a sequence $(\gamma_n)_{n\in\mathbb{N}}$, $\gamma_n : \mathbb{R} \times (\mathbb{R}/\ell_n\mathbb{Z}) \to \mathbb{R}^3$, of smooth solutions of* (1) *such that $\ell_n \to m\ell$ and for all $t \in \mathbb{R}$,*

$$T_{\gamma_n,t} \rightharpoonup T_{\gamma,a_m t} \quad \text{in } \mathcal{D}'(\mathbb{R}^3, \mathbb{R}^3) \text{ as } n \to \infty.$$

Proof. It turns out that one may actually even require the approximating solutions γ_n to be exact traveling wave solutions of (1). The latter have been extensively studied by Kida [21] and the particular asymptotic required for the present proof (the γ_n correspond to a curve with small helices wrapped around a circle) have been carefully detailed in [18, Section 8]. In fact the proof shows that the generalized binormal curvature flows associated to γ_n converge to $(V_t^{m,a_m})_{t\in\mathbb{R}}$ constructed in the proof of Proposition 6. □

Question 2. Given a smooth binormal curvature flow $\gamma : \mathbb{R} \times (\mathbb{R}/\ell\mathbb{Z}) \to \mathbb{R}^3$ and numbers $m > 1$ and $a \in [a_m, m]$, does there exist a sequence $\gamma_n : \mathbb{R} \times (\mathbb{R}/\ell_n\mathbb{Z}) \to \mathbb{R}^3$ of smooth solutions of (1) such that $\ell_n \to m\ell$ and $T_{\gamma_n,t} \rightharpoonup T_{\gamma,at}$ in the sense of distributions?

Even though one could expect strong instability for highly oscillatory data, the numerics in fact tend to suggest that the answer could be positive, at least in the case $a = a_m$, and that corresponding choices of initial data for γ_n would be obtained by wrapping helices around the initial smooth curve $\gamma(0, \cdot)$, as is the case for the construction in Proposition 7.

Remark 6. One can use the generalized solutions of Proposition 6 to create rather pathological examples.

For example, fix $m > 1$, and let $a : \mathbb{R} \to [a_m, m]$ be a measurable function that does not change sign and is a.e. bounded away from 0. Define $t(\tau) = a(\tau)^{-1} \int_0^\tau a(s)\, ds$, and let $V_\tau := V_{t(\tau)}^{m,a(t)}$ for $V_t^{m,a}$ as constructed above. Then it is straightforward to verify that $(V_\tau)_{\tau\in\mathbb{R}}$ is a generalized binormal curvature flow with associated undercurrents $(T_{\gamma,t(\tau)})_{\tau\in\mathbb{R}}$. This illustrates quite dramatically the ill-posedness of the initial value problem for generalized binormal curvature flows, even if we impose the condition that $t \mapsto V_t(\mathbb{R}^3 \times S^2)$ is constant.

In a different direction, fix $m > 1$, let $\rho : [a_m, m] \to [0, \infty)$ be a smooth function such that $\int_{a_m}^m \rho(a)\, da = 1$, and define

$$V_t = \int_{a_m}^m V_t^{m,a} \rho(a)\, da.$$

Then $T_{V_0} = \int_{a_m}^m T_{V_0^{m,a}} \rho(a)\, da = T_{\gamma,0}$, and it is easy to see that $(V_t)_{t\in\mathbb{R}}$ satisfies (4) and has no boundary in the sense that $\int \nabla\psi \cdot \xi\, dV_t = 0$ for all $\psi \in C_c^\infty(\mathbb{R}^3)$. But V_t is not integral for times $t > 0$, in the sense that the associated undercurrent is not integral. Thus the balance law (4) is not by itself enough to preserve integrality.

5.3. Numerical curiosities

Our existence theory in Theorem 1 allows considering initial curves that have corners, and in particular polygons. There are a number of open questions about the behavior of weak binormal curvature flows with polygonal initial data, many of which (uniqueness, loss of mass, etc.) are special cases of more general open questions about almost parametrized weak binormal curvature flows. In order to possibly obtain some insight into these questions, we have performed numerical simulations according to an algorithm of Buttke [7], and we have observed some phenomena which we did not expect, which we believe are worth mentioning, and for which we have no explanation[8] beyond obscure appeals to integrability (discovered for long by Hasimoto [13] for (1), but which is not well adapted to a non-smooth setting).

[8] After all one cannot rule out a priori that the numerics are completely misleading, even if we do not believe it is the case here.

If γ is a solution to (1), the corresponding tangent vector $u := \partial_s \gamma : I \times (\mathbb{R}/\ell\mathbb{Z}) \to S^2$ satisfies the Schrödinger map equation

$$\partial_t u = u \times \partial_{ss} u. \tag{54}$$

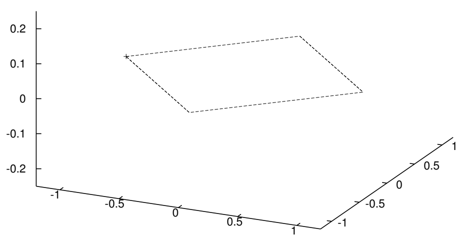
Buttke's algorithm simulates the binormal curvature flow equation (1) by the Crank–Nicolson type discretization

$$\frac{u_n^{j+1} - u_n^j}{\Delta t} = \left(\frac{u_n^j + u_n^{j+1}}{2} \right) \times \left(\frac{u_{n-1}^j + u_{n+1}^j}{2(\Delta x)^2} + \frac{u_{n-1}^{j+1} + u_{n+1}^{j+1}}{2(\Delta x)^2} \right)$$

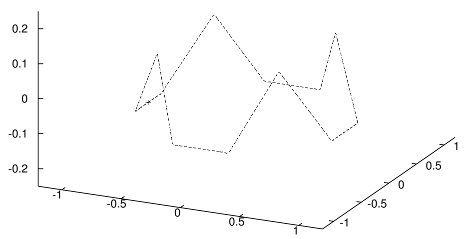
of (54), and numerical integration to recover γ from u. The implicit scheme for u can be resolved by a fixed point method if $\Delta t < \sigma(\Delta x)^2$ for some explicit $\sigma > 0$; it has the advantage that the constraint $|u_n^j| = 1$, the mean $\sum_n u_n^j$, and the discrete squared $\dot{H}^1$ norm $\sum_n |u_n^j - u_{n+1}^j|^2$ are conserved quantities of the scheme.

In the following pictures, we present the shape of the simulated solution at different (well chosen) times for a 5000 points discretization of a unit square parallel to the xy-plane as initial datum.

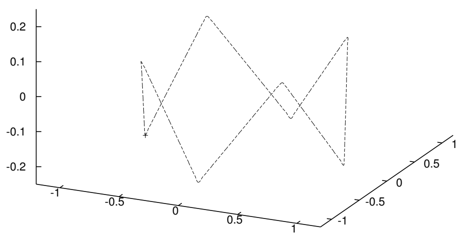
Binormal flow, $T = 0$



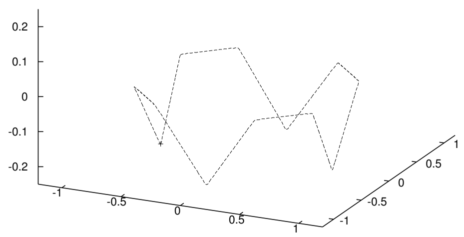
Binormal flow, $T = 0.05296$



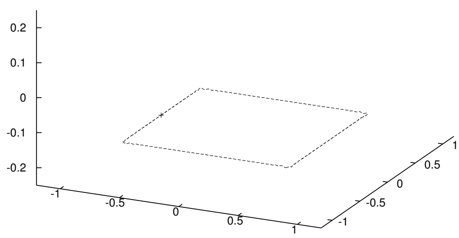
Binormal flow, $T = 0.07948$



Binormal flow, $T = 0.10591$



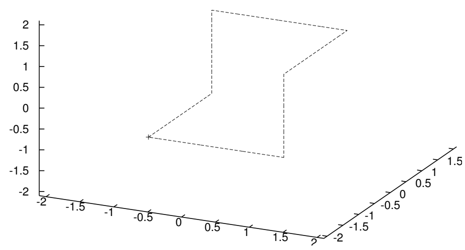
Binormal flow, $T = 0.15878$



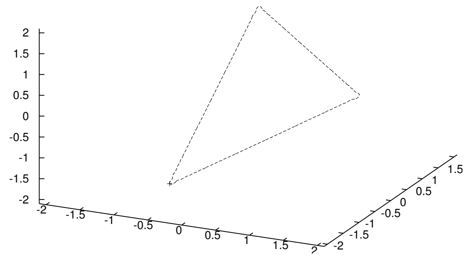
As it may suggest, at some times with rational ratios, the (or “a”) solution could become again polygonal. Notice that the symmetries of the square are preserved (intermediate shapes have eight or twelve sides), and that the square in the last picture is rotated by $\pi/4$ with respect to the initial one. At times intermediate between those special moments the simulated solution looks quite jerky and has not been represented. Also, running the simulation further in time suggests that this sequence is reproduced in a (quasi)periodic manner. This is reminiscent of known phenomena for the *linear* Schrödinger equation with step functions as initial data. Being *nonlinear* but integrable, it is perhaps tempting to believe that solitons could play a role here; on the other hand polygons are the worst possible examples for the Hasimoto transform (the solution is not smooth and the curvature vanishes almost everywhere!).

This kind of phenomena seems rather robust to some changes in the initial polygon, in particular for rectangles or non-planar initial data like the following “half-cube”:

Binormal flow, $T = 0$



Binormal flow, $T = 0.62595$



(other additional times between 0 and 0.62595 seem to correspond to different non-planar polygons, all with the symmetries of the equilateral triangle; we have not included them in the picture because they are less distinctive on small size graphics).

Question 3. Does there exist an almost parametric binormal curvature flow $(T_t)_{t\in\mathbb{R}}$ for which T_t is the integral 1-current associated to an oriented polygon for at least two (and possibly an infinite sequence) of different times $t \in \mathbb{R}$? In case of positive answer, how to give an interpretation of those solutions in terms of the Hasimoto transform and the cubic Schrödinger equation with Dirac masses?

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