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# A maximum principle for systems with variational structure and an application to standing waves

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**Abstract.** We establish via variational methods the existence of a standing wave together with an estimate on the convergence to its asymptotic states for a bistable system of partial differential equations on a periodic domain. The main tool is a replacement lemma which has as a corollary a maximum principle for minimizers.

**Keywords.** Vector Allen–Cahn equation, standing waves, periodic domains, maximum principle for (vector) minimizers

## 1. Introduction

We consider the elliptic system

$$\Delta u = W_u(u) \quad \text{for } u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m, \tag{1.1}$$

where $W : \mathbb{R}^m \to \mathbb{R}$ is a $C^2$ potential and $W_u(u) := (\partial W/\partial u_1, \dots, \partial W/\partial u_m)^\top$. Systems of type (1.1) have been studied in particular in [1, 10, 16, 5, 3, 13, 15], generally under symmetry hypotheses on the potential.

We assume:

**Hypothesis 1.** *There exist* $a_- \neq a_+ \in \mathbb{R}^m$ *such that*

$$0 = W(a_-) = W(a_+) < W(u) \quad \textit{for all } u \in \mathbb{R}^m \setminus \{a_-, a_+\}.$$

**Hypothesis 2.** *There is an* $r_0 > 0$ *such that, for* $\nu \in \mathbb{S}^{m-1}$*, where* $\mathbb{S}^{m-1} \subset \mathbb{R}^m$ *is the unit sphere, the map* $(0, r_0] \ni r \mapsto W(a + r\nu)$ *for* $a \in \{a_-, a_+\}$ *has a strictly positive first derivative.*

We are interested in globally bounded solutions of (1.1) and so growth conditions on $W$ at infinity are not relevant. Therefore, we have the following hypothesis.

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**Hypothesis 3.** *There exists $M > 0$ such that*

$$W(su) \geq W(u) \quad \text{for } s \geq 1 \text{ and } |u| = M.$$

We further assume that $\Omega \subset \mathbb{R}^n$ is a *periodic* domain (open connected) of class $C^{2,\alpha}$ with bounded cross section. We let $x = (s, y) \in \mathbb{R} \times \mathbb{R}^{n-1}$ be the typical element of $\mathbb{R}^n$.

**Hypothesis 4.** *There exist $L > 0$ and $R > 0$ such that*

$$(s, y) \in \Omega \quad \text{implies} \quad (s \pm L, y) \in \Omega \text{ and } |y| \leq R.$$

For fixed $s \in \mathbb{R}$ we denote by $\Omega^s := \Omega \cap (\{s\} \times \mathbb{R}^{n-1})$ the *cross section* of $\Omega$ with the plane $s =$ constant.

We also need the following technical hypothesis.

**Hypothesis 5.** *The set $\Omega^0$ is connected.*

This allows domains with a complicated topology with holes and other pathologies. We remark that there exist domains for which $\Omega^s = \Omega^0$ is the only connected cross section for $s \in (-L, L)$. Hypothesis 5 can be relaxed to

$$\Omega \cap \{(s, y) \mid s = \sigma(y) \text{ for } |y| \leq R\} \text{ is a connected set,} \tag{1.2}$$

where $\sigma : \{|y| \leq R\} \to \mathbb{R}$ is a smooth map.

For the boundary value problem

$$\begin{cases} \Delta u = W_u(u) & \text{in } \Omega, \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega, \\ \lim\limits_{s \to \pm\infty, (s,y) \in \Omega} u(s, y) = a_\pm, \end{cases} \tag{1.3}$$

where $n$ is the outward normal, we establish the following result.

**Theorem 1.** *Assume that $W$ and $\Omega$ satisfy Hypotheses 1–5. Then there exists a classical solution $u : \Omega \to \mathbb{R}^m$ to the boundary value problem (1.3). If $a_+$ is nondegenerate in the sense that the quadratic form $\langle D^2(W(a_+))z, z\rangle$ is positive definite, then we have exponential decay to $a_+$, that is, there exist $k_0, K_0 > 0$ such that*

$$|u(s, y) - a_+| \leq K_0 e^{-k_0 s} \quad \text{for } s > 0.$$

*A similar statement applies to $a_-$.*

We note here that for the equation $\Delta u = W_u(x, u)$, for potentials $W((s, y), u)$ periodic in $s$, a theorem analogous to Theorem 1 can also be established under a natural extension of the above hypotheses that take into account the $s$-dependence of the potential.

A basic feature of the problem implied by Hypothesis 4 is the $L$-translation invariance of the energy

$$J_\Omega(u) = \int_\Omega \left( \frac{1}{2}|\nabla u|^2 + W(u) \right) dx$$

in the $s$ direction. The class of domains $\Omega$ that satisfy Hypotheses 4 and 5 includes the case where $\Omega$ is a flat cylinder, which is naturally associated to the ODE version of (1.3), that is,

$$\begin{cases} u'' = W_u(u) & \text{for } s \in \mathbb{R}, \\ \lim_{s\to\pm\infty} u(s) = a_\pm. \end{cases} \tag{1.4}$$

Solutions to (1.4) are also known as *heteroclinic connections* (see [19] and [5]).

The present work bears a relation to the ODE system (1.4), similar to the relation that [9] bears to the traveling-wave problem for scalar parabolic equations. The difference is in the way higher dimensionality is introduced. In our case we assume periodicity of the domain but we keep the equation as before. In [9] the domain is a flat cylinder but the equation is modified by including spatial convection in the $s$ direction. In the scalar case $m = 1$, existence for the boundary value problem (1.3) was established in [8] and [18] for second-order and higher-order operators.

Our proof is variational and modeled after [4]. It proceeds by introducing an artificial constraint that restores compactness by eliminating the translation allowed by the periodicity of $\Omega$ and forces the appropriate behavior at infinity. The major effort is directed toward removing the constraint in the sense of showing that it is not saturated. The technique for doing so cannot invoke the usual maximum principle, which does not hold in the case at hand, but instead is purely variational. The main tool here is the Cut-Off Lemma, which is of independent interest and has as a corollary the following maximum principle.[1] We note that connectedness is crucial here.

**Theorem 2.** *Let $W : \mathbb{R}^m \to \mathbb{R}$ be $C^1$ and nonnegative. Assume that $W(a) = 0$ for some $a \in \mathbb{R}^m$ and that there is $r_0 > 0$ such that for $\nu \in \mathbb{S}^{m-1}$ the map*

$$(0, r_0] \ni r \mapsto W(a + r\nu)$$

*has a strictly positive derivative. Let $A \subset \mathbb{R}^n$ be an open, connected, bounded set, with $\partial A$ Lipschitz, and suppose that $\tilde{u} \in W^{1,2}(A; \mathbb{R}^m) \cap L^\infty(A; \mathbb{R}^m)$ minimizes*

$$J_A(u) = \int_A \left(\frac{1}{2}|\nabla u|^2 + W(u)\right) dx$$

*subject to the Dirichlet condition $u = \tilde{u}$ on $\partial A$. If*

$$|\tilde{u}(x) - a| \le r \quad \text{for } x \in \partial A,$$

*for some $r > 0$ with $2r \le r_0$, then also*

$$|\tilde{u}(x) - a| \le r \quad \text{for } x \in A.$$

**Remark.** The result above does not apply to general solutions of $\Delta u = W_u(u)$ which are not minimizers. For instance, given $r \in (0, 1)$, there is a periodic solution of $u_{xx} = u^3 - u$ that oscillates between $-1 + r$ and $1 - r$, for which obviously Theorem 2 fails.

[1] We would like to thank Haïm Brezis for pointing out to us this formulation of the Cut-Off Lemma in Section 2 and for his interest in this work.

The Cut-Off Lemma is a replacement result modeled after [4] and is presented in Section 2. In Section 3 we introduce the constrained variational problem. In Section 4 the constraint is removed in two stages. First, in Section 4.1, the constraint is removed at infinity by utilizing linear estimates and then, in Section 4.2, by invoking the Cut-Off Lemma, we conclude the proof that the constraint is not saturated and finish the proof of Theorem 1.

## 2. The Cut-Off Lemma

### *2.1. The polar form*

Let $A$ be an open and bounded subset of $\mathbb{R}^n$. For $u \in W^{1,2}(A;\mathbb{R}^m) \cap L^\infty(A;\mathbb{R}^m)$ and $a \in \mathbb{R}^m$, if $\rho(x) := |u(x) - a| \neq 0$, we consider the *polar representation*

$$u(x) = a + |u(x) - a|\frac{u(x) - a}{|u(x) - a|} =: a + \rho(x)\nu(x), \tag{2.1}$$

where $|\cdot|$ is the Euclidean norm and $\nu(x) := (u(x) - a)/|u(x) - a|$. We call $\rho$ the *radial part* and $\nu$ the *angular part*.

The purpose of this subsection is to establish rigorously the appropriate version of the identity

$$\int_A |\nabla u|^2\, dx = \int_A |\nabla \rho|^2\, dx + \int_A \rho^2(x)|\nabla \nu|^2\, dx \tag{2.2}$$

for $u$ as above, and also show that modifying the radial part in (2.1) by setting

$$\tilde{u}(x) = a + f(\rho(x))\nu(x) \quad \text{ with } f(0) = 0$$

produces a $\tilde{u} \in W^{1,2}(A;\mathbb{R}^m) \cap L^\infty(A;\mathbb{R}^m)$ for locally Lipschitz $f : \mathbb{R} \to \mathbb{R}$ with a corresponding formula (2.2). The arguments that follow are well-known for Sobolev maps. We include them for completeness. For our purposes it suffices to take $f(s)/s$ locally Lipschitz. Without loss of generality we take $a = 0$, so that

$$\rho(x) = |u(x)| \quad \text{and} \quad \nu(x) = u(x)/\rho(x),$$

and we set

$$A_+ := \{x \in A \mid \rho > 0\} \quad \text{and} \quad A_0 := \{x \in A \mid \rho = 0\}.$$

**Proposition 1.** *Let $w_j : A \to \mathbb{R}^m$ be defined by*

$$w_j := \begin{cases} 0 & \text{on } A_0, \\ u_{,j} - \rho_{,j}\nu & \text{on } A_+, \end{cases}$$

*where $u_{,j} := \partial u/\partial x_j$, $\rho_{,j} := \partial\rho/\partial x_j$. Then*

(i) $w_j \in L^2(A;\mathbb{R}^m)$,
(ii) $\langle w_j, \nu\rangle = 0$ *on $A_+$, where $\langle\cdot,\cdot\rangle$ is the inner product in $\mathbb{R}^m$,*
(iii) *there exists a measurable map $\nu^{\cdot j} : A_+ \to \mathbb{R}^m$ such that $w_j = \rho\nu^{\cdot j}$ on $A_+$.*

Note that $\rho$ is in $W^{1,2}(A)$ (cf. for example [12, p. 130]) and that $\nu^{\cdot j}$ plays the role of $\nu_{,j}$.

*Proof.* Given $\varepsilon > 0$, the map $1/(\rho+\varepsilon)$, the composition of $\rho \in W^{1,2}(A) \cap L^\infty(A)$ and a Lipschitz map, belongs to $W^{1,2}(A) \cap L^\infty(A)$. From this, it follows that the map $v^\varepsilon := u/(\rho+\varepsilon)$ is in $W^{1,2}(A;\mathbb{R}^m)$, and moreover

$$v^\varepsilon_{,j} = \frac{u_{,j}}{\rho+\varepsilon} - \frac{u}{(\rho+\varepsilon)^2}\rho_{,j}. \tag{2.3}$$

Set $w^\varepsilon_j = \rho v^\varepsilon_{,j}$. After multiplication by $\rho$, equation (2.3) becomes

$$w^\varepsilon_j = \frac{\rho}{\rho+\varepsilon}u_{,j} - \frac{\rho}{(\rho+\varepsilon)^2}u\rho_{,j}. \tag{2.4}$$

Therefore,

$$\lim_{\varepsilon\to 0} w^\varepsilon_j = \rho \lim_{\varepsilon\to 0} v^\varepsilon_{,j} = \begin{cases} 0 & \text{on } A_0, \\ u_{,j} - \rho_{,j}v & \text{on } A_+. \end{cases}$$

On the other hand, from equation (2.4) we also have

$$|w^\varepsilon_j| \le \begin{cases} 0 & \text{on } A_0, \\ |u_{,j}| + |\rho_{,j}| & \text{on } A_+. \end{cases}$$

Thus, by Lebesgue's dominated convergence theorem,

$$\lim_{\varepsilon\to 0} w^\varepsilon_j = w_j \in L^2(A;\mathbb{R}^m).$$

This proves (i) and, since $w^\varepsilon_j = \rho v^\varepsilon_{,j}$, we have $\rho \lim_{\varepsilon\to 0} v^\varepsilon_{,j} = u_{,j} - \rho_{,j}v$ on $A_+$, and thus $v^{,j}$ is defined via $\lim_{\varepsilon\to 0} v^\varepsilon_{,j} =: v^{,j}$ and satisfies (iii). Finally, to show (ii) we observe that from (2.4) it follows that

$$\langle w^\varepsilon_j, u\rangle = \frac{\rho}{\rho+\varepsilon}\langle u_{,j}, u\rangle - \frac{\rho^2}{(\rho+\varepsilon)^2}\rho\rho_{,j} = \frac{\rho}{\rho+\varepsilon}\left(1 - \frac{\rho}{\rho+\varepsilon}\right)\langle u_{,j}, u\rangle. \tag{2.5}$$

Hence, passing to the limit in (2.5) gives $0 = \langle w_j, u\rangle = \langle w_j, \rho v\rangle$ on $A_+$, and (ii) follows. □

**Corollary 1.** *The following identity holds:*

$$\int_A |\nabla u|^2\,dx = \int_A |\nabla\rho|^2\,dx + \int_{A_+} \rho^2 \sum_j \langle v^{,j}, v^{,j}\rangle\,dx. \tag{2.6}$$

*Proof.* Since $u_{,j} = 0$ a.e. on $A_0$, we have

$$\begin{aligned}\int_A |\nabla u|^2\,dx &= \int_{A_+} |\nabla u|^2\,dx = \int_{A_+} \langle w_j + \rho_{,j}v, w_j + \rho_{,j}v\rangle\,dx \\ &= \int_{A_+} \Big(\sum_j |w_j|^2 + |\nabla\rho|^2\Big)\,dx = \int_{A_+} \Big(|\nabla\rho|^2 + \sum_j \rho^2\langle v^{,j}, v^{,j}\rangle\Big)\,dx. \quad \square\end{aligned}$$

We note that equation (2.6) gives a rigorous meaning to the representation formula (2.2).

**Corollary 2.** *Let $u$ be as in (2.1) above and $r \geq 0$. Let also*

$$\tilde{u}(x) = \begin{cases} a + \min\{\rho(x), r\}\nu(x) & \text{for } x \in A_+, \\ a & \text{for } x \in A_0. \end{cases}$$

*Then $\tilde{u} \in W^{1,2}(A; \mathbb{R}^m) \cap L^\infty(A; \mathbb{R}^m)$, and we have the following explicit representation of the energy:*

$$\int_A |\nabla \tilde{u}|^2 \, dx = \int_A |\nabla \tilde{\rho}|^2 \, dx + \int_{A_+} \tilde{\rho}^2 \sum_j \langle \nu^{,j}, \nu^{,j} \rangle \, dx, \tag{2.7}$$

*where $\tilde{\rho}(x) = |\tilde{u}(x) - a| = \min\{\rho(x), r\}$ on $A$.*

*Proof.* On $A_+$ we have

$$\tilde{u}(x) = a + \frac{\min\{\rho(x), r\}}{\rho(x)}(u(x) - a).$$

Thus, if we define

$$g(s) := \begin{cases} 1 & \text{for } s \leq 0, \\ \dfrac{s + r - |s - r|}{2s} & \text{for } s > 0, \end{cases} \tag{2.8}$$

then since $\min\{a, b\} = \frac{1}{2}(a + b - |a - b|)$ for $a, b \in \mathbb{R}$, we have

$$\tilde{u}(x) = a + g(\rho(x))(u(x) - a) \quad \text{for } x \in A.$$

Since $g$ is Lipschitz (and $\rho$ bounded) it follows that $g(\rho(\cdot))$ is in $W^{1,2}(A) \cap L^\infty(A)$, and therefore $\tilde{u} \in W^{1,2}(A; \mathbb{R}^m) \cap L^\infty(A; \mathbb{R}^m)$, and (2.7) follows by Corollary 1. □

**Remark.** We will also need the cut-off function

$$\alpha(\tau) := \begin{cases} 1 & \text{for } \tau \leq r \quad (r > 0), \\ (2r - \tau)/r & \text{for } r \leq \tau \leq 2r, \\ 0 & \text{for } \tau \geq 2r. \end{cases}$$

Let

$$\tilde{u}(x) = \begin{cases} a + \min\{\rho(x), r\}\alpha(\rho(x))\nu(x) & \text{for } x \in A_+ \cap \{\rho < 2r\}, \\ a & \text{for } x \in A_0 \cup \{\rho \geq 2r\}. \end{cases}$$

Set $\tilde{\rho}(x) := |\tilde{u}(x) - a|$ and $\tilde{A}_0 = A_0 \cup \{\rho \geq 2r\}$, $\tilde{A}_+ = A_+ \cap \{\rho < 2r\}$. Then

$$\tilde{u} \in W^{1,2}(A; \mathbb{R}^m) \cap L^\infty(A; \mathbb{R}^m), \tag{2.9}$$

and the analogue to (2.7) holds. Indeed,

$$\begin{aligned} \tilde{\rho}(x) &= \min\{\rho(x), r\}\alpha(\rho(x)) \quad \text{on } \tilde{A}_+, \\ \tilde{u}(x) &= a + g(\rho(x))\alpha(\rho(x))(u(x) - a), \end{aligned}$$

with $g$ as in (2.8). Since $\alpha$ is Lipschitz, the same argument applies and yields (2.9).

*2.2. The lemma*

We are now ready to state the main technical tool of the paper.

**Cut-Off Lemma.** *Let $W : \mathbb{R}^m \to \mathbb{R}$ be a $C^2$ potential. Assume that the map*

$$r \mapsto W(a + r\nu) \text{ is strictly increasing} \tag{H}$$

*for $r \in (0, r_0]$, $r_0 > 0$, with $W(a) = 0$ and $W \geq 0$ otherwise. Set*

$$J_\Omega(u) := \int_\Omega \left(\frac{1}{2}|\nabla u|^2 + W(u)\right) dx,$$

*where $\Omega$ is an open and bounded subset of $\mathbb{R}^n$, and $A \subset \Omega$ is an open, bounded, connected, and Lipschitz set with $\partial A \cap \Omega \neq \emptyset$. Suppose that*

(i) $u \in W^{1,2}(\Omega; \mathbb{R}^m) \cap L^\infty(\Omega; \mathbb{R}^m)$,
(ii) $|u(x) - a| \leq r$ *on* $\partial A \cap \Omega$ (*in the sense of the trace*) *for some* $r$ *with* $2r \in (0, r_0]$,
(iii) $\big|\{x \in A \mid |u(x) - a| > r\}\big| > 0$ (*when applied to sets,* $|\cdot|$ *stands for Lebesgue measure*).

*Then there exists $\tilde{u} \in W^{1,2}(\Omega; \mathbb{R}^m) \cap L^\infty(\Omega; \mathbb{R}^m)$ such that*

$$\begin{cases} \tilde{u}(x) = u(x) & \text{on } \Omega \setminus A, \\ |\tilde{u}(x) - a| \leq r & \text{on } A, \\ J_\Omega(\tilde{u}) < J_\Omega(u). \end{cases}$$

**Remarks.** The hypothesis (H) is a very mild nondegeneracy hypothesis for the minimum $a$. Notice that it allows $C^\infty$ contact. This can be useful in applying the lemma to certain situations where degeneracy is natural (see [7]).

The proof utilizes variations (replacements) of the map $u$ which are obtained by deforming the radial part but keeping the angular part fixed. The previous subsection guarantees that these variations are in $W^{1,2}(\Omega; \mathbb{R}^m)$ and provides a convenient formula for calculating their energy. Note that the replacements are not necessarily local since $|u(x) - a|$ is not *a priori* restricted.

The way the lemma is implemented is as follows: If $u$ is a minimizer, then $|u(x) - a| \leq r$ on $A$. The idea is that $u(x)$ cannot make an excursion far away from $a$ and benefit by entering a low energy region of $W$ because the energy required to get outside a small neighborhood of $a$ exceeds the energy needed to bring $u$ down to $a$ and keep it there.

We remark explicitly that Lemma 2.2 applies to *global* minimizers $\tilde{u}$ and not to *local* minimizers where local means in the linearized sense. To illustrate this point we recall that (cf. [17]) for $\Omega$ a dumbbell domain, say the union $B_- \cup C \cup B_+$ of two balls $B_\pm$ and a sufficiently narrow neck $C$, there exists a stable solution $\tilde{u}$ of the equation $\Delta u = u^3 - u$ which away from the neck satisfies

$$|\tilde{u} \pm 1| \leq r \quad \text{on } B_\mp. \tag{2.10}$$

Lemma 2.2 confirms that this solution is only a local minimizer. Indeed, otherwise from (2.10) that implies $|\tilde{u}+1| \leq r$ on $B_-$ away from the neck, and Lemma 2.2, we could derive

$$|\tilde{u}+1| \leq r \quad \text{on } B_- \cup C \cup B_+. \tag{2.11}$$

Hypothesis (ii) is the most difficult to verify. In Section 4.2, in the proof of Theorem 1, we give an explicit construction of a set $A$. We observe that an $L^\infty$ bound on $|\nabla \tilde{u}|$ implies that, for a minimizer $\tilde{u}$, from the existence of $(\bar{s}, \bar{y}) \in \Omega$ such that

$$\min_{a \in \{a_- a_+\}} |\tilde{u}(\bar{s}, \bar{y}) - a| \geq r$$

it follows that

$$J_{\Omega_\delta^{\bar{s}}}(\tilde{u}) \geq w_0, \qquad \Omega_\delta^{\bar{s}} = \bigcup_{s \in (\bar{s}-\delta, \bar{s}+\delta)} \Omega^s,$$

for some $w_0 > 0$ and $\delta \in (0, L/2)$. This and the *a priori* bound $J_\Omega(\tilde{u}) \leq J_\Omega(\bar{u})$ on $J_\Omega(\tilde{u})$ imply that, in each interval of values of $h$ of size $2J_\Omega(\bar{u})/w_0$, there is an $\bar{h}$ with

$$\min_{a \in \{a_- a_+\}} |\tilde{u}(\bar{h}L, y) - a| < r \quad \text{for all } (\bar{h}L, y) \in \Omega^{\bar{h}L},$$

and since, by assumption, $\Omega^{\bar{h}L}$ is connected we finally obtain

$$|\tilde{u}(\bar{h}, y) - a| < r \quad \text{for all } (\bar{h}, y) \in \Omega^{\bar{h}L}, \text{ for some } a \in \{a_-, a_+\}, \tag{2.12}$$

and we see that hypothesis (ii) is satisfied for $A = \bigcup_{s \in (\bar{h}L, \hat{h}L)} \Omega^s$ whenever $\hat{h} \neq \bar{h}$ is such that

$$|\tilde{u}(\hat{h}L, y) - a| < r \quad \text{for all } (\hat{h}L, y) \in \Omega^{\hat{h}L}.$$

*Proof of the Cut-Off Lemma.* We utilize the polar representation in the first subsection above.

**Step 1.** We begin by establishing the lemma under the additional hypothesis

$$\rho(x) \leq 2r \leq r_0 \quad \text{a.e. in } A. \tag{2.13}$$

Set

$$\tilde{u}(x) = \begin{cases} a & \text{for } x \in A_0 = \{x \in A \mid \rho = 0\}, \\ a + \min\{\rho(x), r\}\nu(x) & \text{for } x \in A_+ = \{x \in A \mid \rho > 0\} \\ u(x) & \text{for } x \in \Omega \setminus A. \end{cases} \tag{2.14}$$

By Corollary 2, and since $\tilde{u} = u$ on $\partial A \cap \Omega$, we have $\tilde{u} \in W^{1,2}(\Omega; \mathbb{R}^m) \cap L^\infty(\Omega; \mathbb{R}^m)$. Thus,

$$\int_\Omega |\nabla \tilde{u}|^2 \, dx = \int_A |\nabla \tilde{u}|^2 \, dx + \int_{\Omega \setminus A} |\nabla u|^2 \, dx. \tag{2.15}$$

On the other hand, via (2.7),

$$\int_A |\nabla \tilde{u}|^2\,dx = \int_A |\nabla \tilde{\rho}|^2\,dx + \int_{A_+} \tilde{\rho}^2 \sum_j \langle \nu^{,j}, \nu^{,j}\rangle\,dx$$
$$\leq \int_A |\nabla \rho|^2\,dx + \int_{A_+} \rho^2 \sum_j \langle \nu^{,j}, \nu^{,j}\rangle\,dx = \int_A |\nabla u|^2\,dx. \tag{2.16}$$

Thus, (2.15) and (2.16) give

$$\int_\Omega |\nabla \tilde{u}|^2\,dx \leq \int_\Omega |\nabla u|^2\,dx. \tag{2.17}$$

Next we treat the potential term in $J$. We write

$$\int_\Omega W(\tilde{u}(x))\,dx = \int_A W(\tilde{u}(x))\,dx + \int_{\Omega\setminus A} W(u(x))\,dx,$$

and we have

$$\int_A W(\tilde{u}(x))\,dx = \int_A W(a + \tilde{\rho}(x)\nu(x))\,dx$$
$$= \int_{A\cap\{\rho\leq r\}} W(a + \tilde{\rho}(x)\nu(x))\,dx + \int_{A\cap\{\rho>r\}} W(a + \tilde{\rho}(x)\nu(x))\,dx$$
$$= \int_{A\cap\{\rho\leq r\}} W(u(x))\,dx + \int_{A\cap\{\rho>r\}} W(a + \tilde{\rho}(x)\nu(x))\,dx. \tag{2.18}$$

By (H), (2.13) and (iii) above we have

$$\int_{A\cap\{\rho>r\}} W(a + \tilde{\rho}(x)\nu(x))\,dx < \int_{A\cap\{\rho>r\}} W(a + \rho(x)\nu(x))\,dx$$
$$= \int_{A\cap\{\rho>r\}} W(u(x))\,dx, \tag{2.19}$$

and so by (2.18), (2.19),

$$\int_A W(\tilde{u}(x))\,dx < \int_A W(u(x))\,dx.$$

Thus, the lemma is established under (2.13).

**Step 2.** Now we can assume that (2.13) does not hold, hence

$$|\{x \in A \mid \rho(x) > 2r\}| > 0.$$

Set (cf. the Remark following the proof of Corollary 2)

$$\tilde{u}(x) = \begin{cases} a & \text{for } x \in \tilde{A}_0, \\ a + \min\{\rho(x), r\}\alpha(\rho(x))\nu(x) & \text{for } x \in \tilde{A}_+, \\ u(x) & \text{for } x \in \Omega \setminus A, \end{cases}$$

where $\alpha$ is defined in the aforementioned remark and $\tilde{\rho}(x) = |\tilde{u}(x) - a|$, $\tilde{A}_0 = \{x \in A \mid \tilde{\rho}(x) = 0\}$, and $\tilde{A}_+ = \{x \in A \mid \tilde{\rho}(x) > 0\}$. Just as for the function (2.14), we conclude that $\tilde{u} \in W^{1,2}(A; \mathbb{R}^m) \cap L^\infty(A; \mathbb{R}^m)$.

Moreover,

$$\int_\Omega |\nabla \tilde{u}|^2 \, dx = \int_A |\nabla \tilde{u}|^2 \, dx + \int_{\Omega \setminus A} |\nabla u|^2 \, dx,$$
$$\int_A |\nabla \tilde{u}|^2 \, dx = \int_A |\nabla \tilde{\rho}|^2 \, dx + \int_{\tilde{A}_+} \tilde{\rho}^2 \sum_j \langle \nu_{,j}, \nu_{,j} \rangle \, dx.$$

Note that $\tilde{\rho}(x) = \min\{\rho(x), r\}\alpha(\rho(x))$ on $\tilde{A}_+$ and $\tilde{\rho}(x) = r\alpha(\rho(x))$ on $\tilde{A}_+ \cap \{r \leq \rho \leq 2r\}$. We have

$$\int_A |\nabla \tilde{\rho}|^2 \, dx = \int_{\tilde{A}_+} |\nabla \tilde{\rho}|^2 \, dx = \int_{\tilde{A}_+ \cap \{r \leq \rho \leq 2r\}} |\nabla \tilde{\rho}|^2 \, dx + \int_{\tilde{A}_+ \cap \{\rho < r\}} |\nabla \rho|^2 \, dx. \quad (2.20)$$

On $\tilde{A}_+ \cap \{r \leq \rho \leq 2r\}$,

$$|\nabla \tilde{\rho}(x)|^2 = |r\alpha'(\rho)\nabla\rho(x)|^2 \leq |\nabla \rho(x)|^2,$$

hence

$$\int_{\tilde{A}_+ \cap \{r \leq \rho \leq 2r\}} |\nabla \tilde{\rho}|^2 \, dx + \int_{\tilde{A}_+ \cap \{\rho < r\}} |\nabla \rho|^2 \, dx$$
$$\leq \int_{\tilde{A}_+ \cap \{r \leq \rho \leq 2r\}} |\nabla \rho|^2 \, dx + \int_{\tilde{A}_+ \cap \{\rho < r\}} |\nabla \rho|^2 \, dx \leq \int_{\tilde{A}_+} |\nabla \rho|^2 \, dx, \quad (2.21)$$

and therefore by (2.20), (2.21) we have

$$\int_A |\nabla \tilde{\rho}|^2 \, dx \leq \int_A |\nabla \rho|^2 \, dx,$$

and since $\tilde{\rho} \leq \rho$,

$$\int_A |\nabla \tilde{u}|^2 \, dx \leq \int_A |\nabla u|^2 \, dx.$$

Next, we consider the potential on $A \cap \{r \leq \rho \leq 2r\}$. We have

$$W(\tilde{u}(x)) = W\big(a + r\alpha(\rho(x))\nu(x)\big) \leq W(a + r\nu(x))$$
$$\leq W(a + \rho(x)\nu(x)) = W(u(x)), \quad (2.22)$$

where (H) was utilized in the last two inequalities. Note that $W(\tilde{u}(x)) = W(u(x))$ on $A \cap \{\rho < r\}$.

By examining the inequalities above we observe that $J_\Omega(\tilde{u}) < J_\Omega(u)$ will follow once we prove the following strict inequality:

$$|A \cap \{r < \rho \leq 2r\}| > 0 \quad \text{(by (2.22), (H))}.$$

Suppose for the sake of contradiction that the inequality above is violated, that is,

$$|A \cap \{r < \rho \leq 2r\}| = 0.$$

From the hypothesis of Step 2, we have

$$|A \cap \{\rho > 2r\}| > 0.$$

Let us partition $A$ into the three sets

$$E_1 := A \cap \{\rho \leq r\}, \quad E_2 := A \cap \{r < \rho \leq 2r\}, \quad E_3 := A \cap \{\rho > 2r\}.$$

According to what precedes, we have $|E_2| = 0$ and $|E_3| > 0$. We now consider[2] the following Sobolev functions defined on the open and connected set $A$:

$$\sigma(x) = \min\{\rho(x), 2r\} = \begin{cases} \rho(x) & \text{for } x \in E_1, \\ 2r & \text{for } x \in E_3, \end{cases}$$

$$\tau(x) = \max\{\sigma, r\} - r = \begin{cases} 0 & \text{for } x \in E_1, \\ r & \text{for } x \in E_3. \end{cases}$$

Since $\tau$ is equal almost everywhere to a multiple of the characteristic function of $E_3$, and since it is a Sobolev function, it follows that $\nabla\tau = 0$ a.e. in $A$. By connectedness, $\tau = r$ a.e. in $A$ (cf. [11, p. 307]), $|E_1| = 0$, and $\rho > 2r$ a.e. in $A$. This is a contradiction since we have assumed that $\rho \leq r$ on $\partial A \cap \Omega$ in the sense of the trace. The proof of the lemma is complete. □

## 3. The constrained variational problem

For fixed $N \geq 1$ consider the set $X_N$ of maps defined by

$$X_N := \left\{u \in W^{1,2}_{\text{loc}}(\Omega; \mathbb{R}^m) \;\middle|\; |u((s, y)) - a_\pm| \leq r_0/2 \text{ for } \pm s \geq NL\right\},$$

where $r_0$ is the constant in Hypothesis 2. We will minimize the energy $J$ in the class $X_N$. Notice that the constant maps $u \equiv a_-$, $u \equiv a_+$ are not allowed in $X_N$. Existence of minimizers of $J_\Omega$ in $X_N$ is rather standard but we will give the details for the convenience of the reader. We remark that due to the presence of the constraint we cannot claim *a priori* that minimizers satisfy the Euler–Lagrange equation.

**Proposition 2.** *Assume that* $W : \mathbb{R}^m \to \mathbb{R}$ *is of class* $C^2$ *and satisfies Hypothesis 3. Then there exists* $u^N \in X_N$ *such that*

$$J_\Omega(u^N) = \min_{X_N} J_\Omega(u).$$

[2] We thank Panayotis Smyrnelis for simplifying significantly our previous lengthy argument.

*Proof.* Define the affine map

$$\bar{u}(s,y) = \begin{cases} a_- & \text{for } s \le -L, \\ \dfrac{1-s/L}{2}a_- + \dfrac{1+s/L}{2}a_+ & \text{for } s \in (-L, L), \\ a_+ & \text{for } s \ge L. \end{cases}$$

Clearly $\bar{u} \in X_N$ and $J_\Omega(\bar{u}) < \infty$. Thus

$$0 \le \inf_{X_N} J_\Omega(u) \le J_\Omega(\bar{u}) < \infty. \tag{3.1}$$

Given $u \in X_N$ that satisfies $J_\Omega(u) \le J_\Omega(\bar{u})$ set $u_M = 0$ if $u = 0$ and

$$u_M = \min\{|u|, M\}\frac{u}{|u|}$$

otherwise. Then Hypothesis 3 implies

$$\begin{aligned} J_\Omega(u_M) &= \int_{\{|u|<M\}} \left(\frac{1}{2}|\nabla u|^2 + W(u)\right) dx + \int_{\{|u|\ge M\}} \left(\frac{1}{2}|\nabla u_M|^2 + W\left(\frac{M}{|u|}u\right)\right) dx \\ &\le J_\Omega(u) \end{aligned}$$

where we have used Corollary 1 that implies

$$\int_{\{|u|\ge M\}} |\nabla u_M|^2\, dx \le \int_{\{|u|\ge M\}} |\nabla u|^2\, dx.$$

It follows that we can restrict to $X_N \cap \{\|u\|_{L^\infty(\Omega;\mathbb{R}^m)} \le M\}$ and therefore we may assume that $W(u) \ge c^2|u|^2$ for $|u| \ge M+1$, for some $c > 0$. Let $\{u_j\}_{j=1}^\infty \subset X_N \cap \{\|u\|_{L^\infty(\Omega;\mathbb{R}^m)} \le M\}$ be a minimizing sequence. From (3.1) we have

$$\int_\Omega \frac{1}{2}|\nabla u_j|^2\, dx \le J_\Omega(u_j) \le J_\Omega(\bar{u}).$$

Hence, using also the fact that $\|u_j\|_{L^\infty(\Omega;\mathbb{R}^m)} \le M$, we find that, possibly by passing to a subsequence,

$$u_j \rightharpoonup u^N \quad \text{in } W^{1,2}_{\text{loc}}(\Omega;\mathbb{R}^m),$$

by weak compactness. By compactness of the embedding we can assume that $u_j \to u^N$ strongly in $L^2_{\text{loc}}(\Omega;\mathbb{R}^m)$ and therefore, along a further subsequence,

$$\lim_{j\to\infty} u_j(s,y) = u^N(s,y) \quad \text{a.e. in } \Omega.$$

Weak lower semicontinuity of the $L^2$ norm gives

$$\liminf_{j\to\infty} \int_\Omega \frac{1}{2}|\nabla u_j|^2\, dx \ge \int_\Omega \frac{1}{2}|\nabla u^N|^2\, dx,$$

and by Fatou's lemma,

$$\liminf_{j\to\infty} \int_\Omega W(u_j)\, dx \ge \int_\Omega W(u^N)\, dx.$$

The proof is complete. □

## 4. Removing the constraint

In this section we conclude the proof of Theorem 1 by showing that if $N$ is taken sufficiently large, then a minimizer $u^N \in X_N$ does not saturate the constraints and satisfies

$$\lim_{s\to\pm\infty} u^N(s,y) = a_\pm. \tag{4.1}$$

In Subsection 4.1 we prove (4.1) and that the only points where a minimizer $u^N \in X_N$ can touch the constraints are at $s = \pm NL$. In Subsection 4.2 we show that also this possibility can be excluded by taking $N$ large. To sketch the idea of the proof of (4.1) and of

$$\rho(s,y) := |u^N(s,y) - a_+| < r_0/2 \quad \text{for } s > NL, \tag{4.2}$$

assume that $a_+$ is nondegenerate and let $\varphi = \varphi((s,y),t)$ be the solution of the problem

$$\begin{cases} \Delta\varphi = c^2\varphi & \text{in } \omega = \bigcup_{s\in(-L,L)} \Omega^s, \\ \varphi = t & \text{on } \partial^b\omega, \\ \dfrac{\partial\varphi}{\partial n} = 0 & \text{on } \partial^l\omega, \end{cases} \tag{4.3}$$

where $t > 0$ is a parameter and $\partial^b\omega := \partial\omega \cap (\{-L, L\} \times \mathbb{R}^{n-1})$, $\partial^l\omega = \partial\omega \setminus \partial^b\omega$ is the lateral boundary of $\omega$, and $n$ is the outward normal.

From the linearity of (4.3) and the maximum principle it follows that

$$\begin{aligned} &\varphi((s,y),t) < t \quad \text{for } (s,y) \in \omega, \\ &\max_{(0,y)\in\overline{\Omega^0}} \varphi((0,y),t) \le \theta t \quad \text{for some } \theta \in (0,1). \end{aligned} \tag{4.4}$$

We set $t_0 = r_0^2/4$ and via a comparison argument (cr. Lemma 3 below) we show that

$$\rho^2(s,y) \le \varphi((s-(N+k)L, y), t_0), \quad \text{for } s \in ((N+k-1)L, (N+k+1)L),\ k = 1,2,\ldots, \tag{4.5}$$

and therefore from (4.4) we obtain (4.2). To prove (4.1) we observe that from (4.5) and (4.4) it follows that

$$\rho^2((N+k)L, y) \le \theta t_0 \quad \text{for } k = 1,2,\ldots, \tag{4.6}$$

From this and the same argument leading to (4.5) we obtain

$$\rho^2(s,y) \le \varphi((s-(N+k)L, y), \theta t_0), \quad \text{for } s \in ((N+k-1)L, (N+k+1)L),\ k = 2,3,\ldots, \tag{4.7}$$

which implies

$$\rho^2(s,y) \le \theta t_0 \quad \text{for } s \ge (N+1)L$$

and by iterating the procedure

$$\rho^2(s,y) \le \theta^j t_0 \quad \text{for } s \ge (N+j)L,\ j = 1,2,\ldots,$$

which implies (4.1) with exponential decay.

In Subsection 4.2, to complete the proof that the constraints are not saturated for a minimizer $u^N \in X_N$ for large $N$, we detail the argument sketched in the remarks following the statement of the Cut-Off Lemma. As described there, if $N$ is sufficiently large, there exist $-N < \bar{h} < N$ and $a \in \{a_-, a_+\}$ such that

$$|u^N(\bar{h}L, y) - a| < r < r_0/2 \quad \text{for } (\bar{h}L, y) \in \Omega^{\bar{h}L}.$$

On the other hand, (4.1) implies

$$|u^N(\hat{h}L, y) - a| < r < r_0/2 \quad \text{for } (\hat{h}L, y) \in \Omega^{\hat{h}L}$$

for some $\hat{h} \in (-\infty, -N) \cup (N, \infty)$. Then the Cut-Off Lemma applied to

$$A = \bigcup_{s \in (\bar{h}L, \hat{h}L)} \Omega^s$$

yields

$$|u^N(s, y) - a| < r < r_0/2 \quad \text{for } (s, y) \in A,$$

and the constraint is removed at $s = \pm NL$ if $a = a_\pm$. To remove the constraint also at the other side we translate the map $u^N$ by one period in the direction of $a$. By (4.1) the translated map touches the constraints at neither end and is still a minimizer since has the same energy as $u^N$.

### *4.1. Removing the constraint for $s \in (-\infty, -NL) \cup (NL, \infty)$*

Removing the constraint in the interior of the cylinders

$$\{|u - a_\pm| \le r_0/2 \text{ for } \pm s \ge NL\}$$

is easier since linearization about the minima $a_\pm$ is available. So, in this subsection the Cut-Off Lemma is *not* utilized.

Let $g : [0, r_0] \to \mathbb{R}$ be defined by

$$g(r) := \min_{r \le r' \le r_0} \min_{\substack{\nu \in \mathbb{S}^{m-1} \\ a \in \{a_-, a_+\}}} \langle W_u(a + r'\nu), \nu \rangle \quad \text{for } r \in [0, r_0].$$

From Hypotheses 1 and 2 we find that $g(0) = 0$ and $g$ is strictly increasing. Let $f : [0, r_0] \to [0, \infty]$ be a strictly increasing function that satisfies $f(0) = 0$ and

$$0 \le f(r^2) \le 2rg(r) \quad \text{for } r \in [0, r_0]. \tag{4.8}$$

Observe that if $a_\pm$ is nondegenerate, then $g$ is bounded below by a linear map. Therefore, in that case we can assume that

$$f(t) = c^2 t \quad \text{for } t \in [0, r_0^2],$$

for some constant $c > 0$.

The reason for introducing the function $f$ as in (4.8) will become apparent in Lemma 2.

**Lemma 1.** *Let $f$ be as in (4.8) and let $\omega = \bigcup_{s\in(-L,L)} \Omega^s$. For $t \in (0, r_0^2]$ let $\varphi : \omega \times (0, r_0^2] \to \mathbb{R}$ be the solution of the problem*

$$\begin{cases} \Delta\varphi = f(\varphi) & \text{in } \omega, \\ \varphi = t & \text{on } \partial^b\omega, \\ \dfrac{\partial\varphi}{\partial n} = 0 & \text{on } \partial^l\omega, \end{cases} \tag{4.9}$$

*where $\partial^b\omega := \partial\omega \cap (\{-L, L\} \times \mathbb{R}^{n-1})$, $\partial^l\omega = \partial\omega \setminus \partial^b\omega$ is the lateral boundary of $\omega$, and $n$ is the outward normal. Then:*

(i) *$\varphi((s, y), t) < t$ for $(s, y) \in \omega$, and $\hat{t} := \max_{(0,y)\in\overline{\Omega^0}} \varphi((0, y), t) < t$.*
(ii) *$\lim_{j\to\infty} t_j = 0$, where $\{t_j\}$ is defined by $t_0 = t$ and $t_j = \hat{t}_{j-1}$ for $j = 1, 2, \ldots$.*
(iii) *If $f$ is linear, that is, $f(t) = c^2 t$ for some $c > 0$, then there is a $\theta \in (0, 1)$ such that*

$$t_j = \theta^j t \quad \text{for } j = 1, 2, \ldots.$$

*Proof.* With the change of variables $\varphi = t + \psi$, problem (4.9) becomes

$$\begin{cases} \Delta\psi = \tilde{f}(\psi) := f(t + \psi) & \text{in } \omega, \\ \psi = 0 & \text{on } \partial^b\omega, \\ \dfrac{\partial\psi}{\partial n} = 0 & \text{on } \partial^l\omega. \end{cases} \tag{4.10}$$

Let $W^{1,2}_\sharp(\omega)$ be the closure of $\{\psi \in C^\infty(\bar{\omega}) \mid \psi^+ = 0 \text{ on } \partial^b\omega\}$ in $W^{1,2}(\omega)$, where $\psi^+ = \max\{0, \psi\}$. We can assume that $f$ is extended to a nondecreasing nonnegative function $f : \mathbb{R} \to [0, \infty]$. Since $\tilde{f}(-t) = f(0) = 0$, the function $\underline{\psi} \equiv -t$ is a weak subsolution of (4.10), that is,

$$\int_\omega \big(\nabla\underline{\psi}\nabla z + \tilde{f}(\underline{\psi})z\big)\, dx \le 0 \quad \text{for } z \in W^{1,2}_\sharp(\omega) \text{ with } z \ge 0.$$

Similarly, the fact that $f$ is nonnegative implies that $\overline{\psi} \equiv 0$ is a weak supersolution of (4.10). Moreover,

$$\underline{\psi}|_{\partial^b\omega} < 0, \qquad \overline{\psi}_{\partial^b\omega} = 0, \qquad \text{and} \quad \underline{\psi} < \overline{\psi} \quad \text{a.e. in } \omega. \tag{4.11}$$

The existence of weak sub- and supersolutions $\underline{\psi}$ and $\overline{\psi}$ that satisfy (4.11) implies the existence of a weak solution $\psi \in W^{1,2}_\sharp(\omega)$ of (4.10) such that

$$\underline{\psi} \le \psi \le \overline{\psi} \qquad \text{a.e. on } \omega.$$

This can be proved as in [11, p. 543]. From elliptic regularity, $\psi$ is a $C^2$ map away from $\partial\Omega \cap (\{\pm L\} \times \mathbb{R}^{n-1})$. Therefore, the Hopf boundary lemma and the strong maximum principle imply $\psi < 0$ in $\omega \cup \partial^l\omega$, and therefore (i) is established.

The sequence $\{t_j\}$ is decreasing and bounded below, so the limit in (ii) exists. Assume that $\lim_{j\to\infty} t_j = t_\infty > 0$ and let $\varphi_j$ be the solution of (4.9) corresponding to $t_j$. Since $f$ is increasing, the difference $w := \varphi_j - \varphi_{j+1}$ satisfies the linear equation

$$\begin{cases} \Delta w - c^2 w = 0 & \text{in } \omega, \\ w = t_j - t_{j+1} > 0 & \text{on } \partial^b \omega, \\ \dfrac{\partial w}{\partial n} = 0 & \text{on } \partial^l \omega, \end{cases}$$

where

$$c^2 = \begin{cases} \dfrac{f(\varphi_j) - f(\varphi_{j+1})}{\varphi_j - \varphi_{j+1}} & \text{if } \varphi_j - \varphi_{j+1} \neq 0, \\ f'(\varphi_{j+1}) & \text{if } \varphi_j - \varphi_{j+1} = 0, \end{cases}$$

therefore the comparison principle implies that $w \geq 0$ in $\omega$ and we have $\varphi_{j+1} \leq \varphi_j$. Since the sequence $\{\varphi_j\}$ of continuous functions in $\omega$ is bounded, we conclude that, as $j \to \infty$, $\varphi_j$ converges uniformly to a map $\varphi_\infty$. Actually, $\varphi_\infty \in W^{1,2}_\sharp(\omega)$. To see this, we note that from the fact that $f$ is bounded and the sequence $\{\varphi_j\}$ is bounded, it follows that also $\|f(\varphi_j)\|_{L^2}$ is uniformly bounded. This and the fact that $\varphi_j$ is a weak solution of (4.9) imply a uniform bound for $\|\varphi_j\|_{W^{1,2}(\omega)}$. It follows that $\varphi_\infty$ is in $W^{1,2}_\sharp(\omega)$ as a weak limit of the sequence $\{\varphi_j\}$ in $W^{1,2}_\sharp(\omega)$, and $\varphi_\infty$ is a weak solution of (4.9). By elliptic regularity, $\varphi_\infty$ is $C^2$ away from $\partial\Omega \cap (\{\pm L\} \times \mathbb{R}^{n-1})$. Therefore, uniform convergence of $\varphi_j$ to $\varphi_\infty$ implies that

$$t_\infty = \lim_{j\to\infty} t_j = \lim_{j\to\infty} \max_{\overline{\Omega^0}} \varphi_j = \max_{\overline{\Omega^0}} \varphi_\infty.$$

Hence, the strong maximum principle yields $\varphi_\infty \equiv t_\infty$, but this and $f(t_\infty) > 0$ contradict (4.9). This contradiction establishes (ii).

To prove (iii) we note that if $f$ is linear, system (4.9) is also linear and therefore $\varphi(\cdot, t) = t\varphi(\cdot, 1)$. This implies $\hat{t} = \hat{t}_1$ and therefore we can take $\theta = \hat{t}_1 < 1$ and $t_j = \theta^j$. $\square$

**Lemma 2.** *Let $u^N$ be a minimizer as in Proposition 2. Set*

$$\rho = |u^N - a_+|$$

*and let $\omega_k = \bigcup_{s\in(-L,L)} \Omega^{s+(N+k)L}$ for $k = 1, 2, \ldots$. Then*

$$\int_{\omega_k} (\langle \nabla(\rho^2), \nabla p\rangle + f(\rho^2)p) \leq 0$$

*for all $p \geq 0$ in $W^{1,2}(\omega_k) \cap L^\infty(\omega_k)$ such that $p = 0$ on $\partial^b \omega_k$.*

*Proof.* For $p \in W^{1,2}(\omega_k) \cap L^\infty(\omega_k)$ as above and for $\varepsilon > 0$ small, let $u_\varepsilon$ be the variation of $u^N$ defined by

$$u_\varepsilon = \begin{cases} u^N - \varepsilon p \rho \nu = a_+ + (1-\varepsilon p)\rho\nu = a_+ + (1-\varepsilon p)(u^N - a_+) & \text{on } \omega_k, \\ u^N & \text{on } \Omega \setminus \overline{\omega_k}, \end{cases}$$

where $\nu = (u^N - a_+)/|u^N - a_+|$. Note that for $\varepsilon > 0$ sufficiently small we have $0 \leq 1 - \varepsilon p \leq 1$ and therefore $u_\varepsilon$ satisfies the constraint $|u_\varepsilon - a_+| = (1-\varepsilon p)\rho \leq 2r$ on $\omega_k$. This and the minimality of $u^N$ imply

$$\begin{aligned} 0 &\leq \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} J_{\omega_k}(u_\varepsilon) \\ &= \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \int_{\omega_k} \bigg(\frac{1}{2}\Big(|\nabla((1-\varepsilon p)\rho)|^2 + ((1-\varepsilon p)\rho)^2 \sum_j \langle \nu^{,j}, \nu^{,j}\rangle\Big) \\ &\qquad\qquad + W(a_+ + (1-\varepsilon p)\rho\nu)\bigg), \end{aligned}$$

where we have also used the polar form (2.2) of the energy. It follows that

$$\begin{aligned} &-\int_{\omega_k} \Big(\langle \nabla\rho, \nabla(p\rho)\rangle + p\rho^2 \sum_j \langle \nu^{,j}, \nu^{,j}\rangle + p\rho\langle W_u(a_+ + \rho\nu), \nu\rangle\Big) \\ &= -\int_{\omega_k} \bigg(\frac{1}{2}\langle \nabla(\rho^2), \nabla p\rangle + p\Big(|\nabla\rho|^2 + \rho^2 \sum_j \langle \nu^{,j}, \nu^{,j}\rangle\Big) + p\rho\langle W_u(a_+ + \rho\nu), \nu\rangle\bigg) \geq 0, \end{aligned}$$

and since $p(|\nabla\rho|^2 + \rho^2 \sum_j \langle \nu^{,j}, \nu^{,j}\rangle) \geq 0$ and, by the definition of $f$, we have $2\rho\langle W_u(a_+ + \rho\nu), \nu\rangle \geq f(\rho^2)$, we conclude that

$$-\int_{\omega_k} (\langle \nabla(\rho^2), \nabla p\rangle + f(\rho^2)p) \geq 0. \qquad \square$$

**Remark.** At first sight, the more natural variation would be $u_\varepsilon = u^N + \varepsilon p \nu$, which formally leads to $\Delta\rho \geq g(\rho)$. The problem is that $u_\varepsilon$ does not, in general, vanish when $\rho$ vanishes, and therefore $u_\varepsilon$ may not be a $W^{1,2}$ map.

**Lemma 3.** *Let $k = 1, 2, \dots$ be given and assume that $\rho^2 \leq t$ on $\partial^b\omega_k$. Then*

$$\rho^2(s, y) \leq \varphi((s - (N+k)L, y), t) < t \quad \textit{for } (s, y) \in \omega_k,$$

*and*

$$\rho^2 \leq \hat{t} \quad \textit{on } \Omega^{(N+k)L} \textit{ for } k = 1, 2, \dots,$$

*with $\rho$ as in Lemma 2.*

*Proof.* Set $\varphi_k((s,y),t) := \varphi((s-(N+k)L,y),t)$ for $(s,y) \in \omega_k$; then $\varphi_k$ satisfies all the statements in Lemma 1 with $\omega$ replaced by $\omega_k$. Therefore, from (4.9) and integration by parts we get

$$-\int_{\omega_k} (\langle \nabla \varphi_k, \nabla p \rangle + f(\varphi_k)p) = 0 \tag{4.12}$$

for all $p \in W^{1,2}(\omega_k) \cap L^\infty(\omega_k)$ such that $p = 0$ on $\partial^b \Omega_k$. From (4.12) and Lemma 2 it follows that

$$\int_{\omega_k} \big(\langle \nabla(\rho^2 - \varphi_k), \nabla p \rangle + (f(\rho^2) - f(\varphi_k))p\big) \le 0 \tag{4.13}$$

for all $p \ge 0$ in $W^{1,2}(\omega_k) \cap L^\infty(\omega_k)$ such that $p = 0$ on $\partial^b \omega_k$. In particular, for $p = (\rho^2 - \varphi_k)^+$, (4.13) yields

$$\int_{\omega_k \cap \{\rho^2 > \varphi_k\}} \big(|\nabla(\rho^2 - \varphi_k)^+|^2 + (f(\rho^2) - f(\varphi_k))(\rho^2 - \varphi_k)^+\big) \le 0. \tag{4.14}$$

Since $f$ is strictly increasing, we have $f(\rho^2) - f(\varphi_k) > 0$ for $\rho^2 > \varphi_k$, and therefore (4.14) implies $\rho \le \varphi_k$ a.e. on $\omega_k$. This and Lemma 1 conclude the proof. □

**Proposition 3.** *Let $t_0 = r_0^2/4$ and $t_j = \hat{t}_{j-1}$ for $j = 1, 2, \dots$. Then*

$$\begin{cases} \rho^2 \le t_j & \text{on } \Omega^s \text{ for } s > (N+j)L, \ j = 1, 2, \dots, \\ \rho^2 < t_{j-1} & \text{on } \Omega^s \text{ for } s > (N+j-1)L, \ j = 1, 2, \dots, \end{cases}$$

*with $\rho$ as in Lemma 2.*

*Proof.* Since $u^N$ satisfies the constraint, we have $\rho^2 \le t_0$ on $\Omega^s$ for $s \ge NL$. Therefore, Lemma 3 with $t = t_0$ and $k = 1, 2, \dots$ yields

$$\rho^2 \le t_1 \quad \text{on } \Omega^{(N+k)L} \text{ for } k = 1, 2, \dots.$$

This and Lemma 3 with $t = t_0$ and $k = 2, 3, \dots$ imply

$$\rho^2 \le t_1 \quad \text{on } \Omega^s \text{ for } s \ge (N+1)L,$$
$$\rho^2 \le t_2 \quad \text{on } \Omega^{(N+k)L} \text{ for } k = 2, \dots.$$

Induction on $j$ concludes the proof of the first inequality. The second inequality follows from the first and from Lemma 3, which imply

$$\rho^2 < t_{j-1} \quad \text{on } \omega_j, \ j = 1, 2, \dots.$$ □

Obviously Proposition 3 implies

$$\lim_{\substack{s\to\infty \\ (s,y)\in\Omega}} \rho(s,y) = a_+ \tag{4.15}$$

and, by statement (iii) of Lemma 1, if $a_+$ is nondegenerate then

$$|u^N(s,y) - a_+| \le K_0 e^{-k_0 s} \quad \text{for } s > 0 \text{ with } (s,y) \in \Omega.$$

The analogous statements concerning $a_-$ are proved in a similar way.

Thus, in this subsection we have established that $u^N$, for $N \geq 1$, does not realize the constraint in $(-\infty, NL) \cup (NL, \infty)$ and hence satisfies in this set the equation, the Neumann condition, and also the asymptotic condition in (1.3), which takes the form (1.4) for nondegenerate $a_\pm$.

### *4.2. Removing the constraint at $s = \pm NL$*

In this part of the proof we use the Cut-Off Lemma developed in Section 2. The minimizer $u^N$ is a classical solution of (1.1) in $\Omega_N := \bigcup_{s\in(-NL,NL)} \Omega^s$. Moreover, from Hypothesis 3 it follows that

$$\|u^N\|_{L^\infty(\Omega_N;\mathbb{R}^m)} \leq M,$$

with $M$ independent of $N$. Therefore, linear elliptic theory implies that

$$|\nabla u^N| \leq M' \quad \text{on } \overline{\Omega_{N-1}}, \tag{4.16}$$

for some $M' > 0$ independent of $N \geq 2$.

**Lemma 4.** *Let $r \in (0, r_0/2)$ be fixed. Then there exist $w_0 > 0$ and $\delta \in (0, L/2)$ such that*

$$(\bar{s}, \bar{y}) \in \Omega_{N-2} \quad \text{and} \quad \min_{a\in\{a_-,a_+\}} |u^N(\bar{s}, \bar{y}) - a| \geq r$$

*imply*

$$J_{\Omega_\delta^{\bar{s}}}(u^N) \geq w_0, \quad \text{where} \quad \Omega_\delta^{\bar{s}} := \bigcup_{s\in(\bar{s}-\delta,\bar{s}+\delta)} \Omega^s.$$

*Proof.* Since $\Omega$ is periodic and of class $C^2$, it satisfies the interior sphere condition with a ball of fixed radius. It follows that there is $\delta > 0$ such that each $x \in \Omega$ belongs to $B_{x',\delta/2} \subset \Omega$ for some $x' \in \Omega$.

Assume that there exists $(\bar{s}, \bar{y}) \in \Omega_{N-2}$ such that

$$\min_{a\in\{a_-,a_+\}} |u^N(\bar{s}, \bar{y}) - a| \geq r.$$

Observe that if we take $\delta < L/2$, we have

$$B_{(\bar{s},\bar{y})',\delta/2} \subset \Omega_{N-1},$$

where $(\bar{s}, \bar{y})'$ is the point $x' = (\bar{s}, \bar{y})'$ corresponding to $x = (\bar{s}, \bar{y})$ and we can apply (4.16). Therefore, if we restrict the choice of $\delta > 0$ to $\delta < \min\{r/(2M'), L/2\}$, the bound (4.16) implies that

$$\min_{a\in\{a_-,a_+\}} |u^N(s, y) - a| > r/2 \quad \text{for } (s, y) \in B_{(\bar{s},\bar{y})',\delta/2}.$$

From this and from the properties of $W$ it follows that

$$\int_{B_{(\bar{s},\bar{y})',\delta/2}} W(u^N) \geq w_0$$

for some $w_0 > 0$. Therefore,

$$J_{\Omega_\delta^{\bar{s}}}(u^N) \geq \int_{\Omega_\delta^{\bar{s}}} W(u^N) \geq \int_{B_{(\bar{s},\bar{y})',\delta/2}} W(u^N) \geq w_0,$$

which concludes the proof of the lemma. □

*Conclusion of the proof of Theorem 1.* Assume $\delta \in (0, L/4)$ in Lemma 4 and observe that we then have

$$\Omega_\delta^{hL} \cap \Omega_\delta^{kL} = \emptyset \quad \text{for } h \neq k, \text{ with } h, k \in [-(N-2), N-2]. \tag{4.17}$$

Let $Z \leq 2N-3$ be the number of integers $h \in [-(N-2), N-2]$ such that

$$\min_{a \in \{a_-, a_+\}} |u^N(hL, y) - a| \geq r \quad \text{for some } (hL, y) \in \Omega^{hL}.$$

From Lemma 4, the *a priori* estimate (3.1), and (4.17) we have

$$Z \leq J_\Omega(\bar{u})/w_0,$$

therefore $2N-3 > J_\Omega(\bar{u})/w_0$ is a sufficient condition for the existence of $\bar{h} \in [-(N-2), N-2]$ such that

$$\min_{a \in \{a_-, a_+\}} |u^N(\bar{h}L, y) - a| < r \quad \text{for all } (\bar{h}L, y) \in \Omega^{\bar{h}L}.$$

Since by Hypothesis 5 the cross section $\Omega^{\bar{h}L}$ is connected and $u^N$ is smooth in $\Omega_{N-1}$, there exists $a \in \{a_-, a_+\}$ such that

$$|u^N(\bar{h}L, y) - a| < r \quad \text{for all } (\bar{h}L, y) \in \Omega^{\bar{h}L}.$$

Assume for definiteness that $a = a_+$ (if $a = a_-$ the argument is completely analogous) and use (4.15) to fix $\hat{h} > N+2$ such that

$$|u^N(\hat{h}L, y) - a_+| < r \quad \text{for all } (\hat{h}L, y) \in \Omega^{\hat{h}L}.$$

Then, the minimality of $u^N$ and the Cut-Off Lemma imply $|u^N(s, y) - a_+| \leq r < r_0/2$ for all $s \in [\bar{h}L, \hat{h}L]$, $(s, y) \in \Omega$, that is, the constraint is not saturated at $s = NL$. To remove the constraint also at $s = -NL$ we use the analogue of (4.15) for $a_-$ and Proposition 3 which together with $\bar{h} < N-1$ implies that the translation $u^N(\cdot + L, \cdot)$ of $u^N$ by one period to the right realizes the constraint neither at $s = NL$ nor at $s = -NL$. The proof of Theorem 1 is complete. □

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