



Christian Böhning · Hans-Christian Graf von Bothmer ·
Ludmil Katzarkov · Pawel Sosna

Determinantal Barlow surfaces and phantom categories

Received May 16, 2013 and in revised form September 12, 2013

Abstract. We prove that the bounded derived category of the surface S constructed by Barlow admits a length 11 exceptional sequence consisting of (explicit) line bundles. Moreover, we show that in a small neighbourhood of S in the moduli space of determinantal Barlow surfaces, the generic surface has a semiorthogonal decomposition of its derived category into a length 11 exceptional sequence of line bundles and a category with trivial Grothendieck group and Hochschild homology, called a phantom category. This is done using a deformation argument and the fact that the derived endomorphism algebra of the sequence is constant. Applying Kuznetsov’s results on heights of exceptional sequences, we also show that the sequence on S itself is not full and its (left or right) orthogonal complement is also a phantom category.

Keywords. Derived categories, exceptional collections, semiorthogonal decompositions, Hochschild homology, Barlow surfaces

1. Introduction

A (geometric) phantom category is an admissible subcategory \mathcal{A} of the bounded derived category of coherent sheaves $D^b(X)$ on a smooth projective variety X with Hochschild homology $\mathrm{HH}_*(\mathcal{A}) = 0$ and Grothendieck group $K_0(\mathcal{A}) = 0$. Recently Katzarkov et al. [DKK, Conj. 4.1], [CKP, Conj. 29] conjectured that the derived category of the Barlow surface of [Bar1] should contain a phantom. Evidence for the possible existence of phantoms was given in the article [BBS12], where an admissible subcategory with vanishing Hochschild homology but with nonzero torsion Grothendieck group was produced in the derived category of the classical Godeaux surface. Later such “quasi-phantoms” were also found on Burniat surfaces in [A-O12], on Beauville surfaces in [GS] and on some surfaces isogenous to a product in [Lee].

Chr. Böhning: Fachbereich Mathematik der Universität Hamburg,
Bundesstraße 55, 20146 Hamburg, Germany; e-mail: christian.boehning@math.uni-hamburg.de

H.-Chr. Graf von Bothmer: Fachbereich Mathematik der Universität Hamburg,
Bundesstraße 55, 20146 Hamburg, Germany; e-mail: hans.christian.v.bothmer@uni-hamburg.de

L. Katzarkov: Department of Mathematics, University of Miami, Coral Gables, FL, 33146, USA,
and Fakultät für Mathematik, Universität Wien, 1090 Wien, Austria; e-mail: lkatzark@math.uci.edu

P. Sosna: Fachbereich Mathematik der Universität Hamburg,
Bundesstraße 55, 20146 Hamburg, Germany; e-mail: pawel.sosna@math.uni-hamburg.de

Mathematics Subject Classification (2010): Primary 14F05; Secondary 14J29, 18E30

In this article we prove the existence of a phantom on a generic determinantal Barlow surface S_t in a small neighbourhood of $S = S_0$ (the moduli space of determinantal Barlow surfaces is 2-dimensional, see [Cat81] or [Lee00]), as well as on the Barlow surface S itself. We think of t as a deformation parameter. More precisely, our main result is the following.

Theorem 1.1. *The derived category $D^b(S_t)$ of a generic determinantal Barlow surface S_t in a small neighbourhood of $S = S_0$ admits a semiorthogonal decomposition*

$$D^b(S_t) = \langle \mathcal{A}_t, \mathcal{L}_{1,t}, \dots, \mathcal{L}_{11,t} \rangle$$

where $(\mathcal{L}_{1,t}, \dots, \mathcal{L}_{11,t})$ is an exceptional sequence of line bundles and \mathcal{A}_t is a phantom category. Moreover, if S_{t_1} and S_{t_2} are two surfaces in a small neighbourhood of a generic point of this family, then the categories $\langle \mathcal{L}_{1,t_1}, \dots, \mathcal{L}_{11,t_1} \rangle$ and $\langle \mathcal{L}_{1,t_2}, \dots, \mathcal{L}_{11,t_2} \rangle$ are equivalent. Furthermore, $D^b(S)$ itself has a phantom.

After the discovery of the main results of this paper, we learned that Gorchinskiy and Orlov [GorOrl] very recently produced a phantom category in the bounded derived category of a product of two surfaces by an ingenious and totally different method.

Note that by [BM] two minimal surfaces of general type with equivalent derived categories are isomorphic (note that [BonOrl] is not applicable, since the canonical bundle of a determinantal Barlow surface is not ample), so $D^b(S_t)$ has to vary with the moduli of S_t . The way the moduli are encoded is analogous to what happens for Burniat surfaces in [A-O12] (which was very inspiring for our proof). We prove that the A_∞ -Yoneda algebra of the exceptional sequence does not vary in a neighbourhood of a generic determinantal Barlow, and deduce from this the existence of the phantoms. Note that $K_0(S_t) \simeq \mathbb{Z}^{11}$ is torsion free, so we cannot use torsion to prove that our exceptional sequence is not full. Likewise, we do not yet know how to exhibit explicit objects in \mathcal{A}_t as was done in [BBS12] (they all came from the fundamental group which is trivial here). It is an interesting topic for future investigations to try to “lay hands” on \mathcal{A}_t and produce explicit objects in it or even explicitly describe a strong generator.

Here is a short roadmap of the paper: In Section 2 we recall the features of Barlow’s construction of the surface S which we will need later. In Section 3 we describe the symmetry of the classes of line bundles in the exceptional sequence we are going to construct. Section 4 contains the construction of curves leading to an explicit integral basis in $\text{Pic}(S)$, and the description of the intersection theory pertaining to it. In Section 5 we explain how we obtain estimates for spaces of sections of line bundles on S and prove the existence of the length 11 exceptional sequence. In Section 6, we compute what we call cohomology data associated to this sequence, that is, the dimensions of extension groups (in the forward direction). Using a deformation argument, we prove existence of phantoms. In Section 7, we prove that S itself has a phantom using Kuznetsov’s recent results on heights for exceptional sequences.

We hope that the existence of phantom categories is not exclusively a pathology, but rather an interesting and potentially useful structure.

2. Notation and construction of the Barlow surface

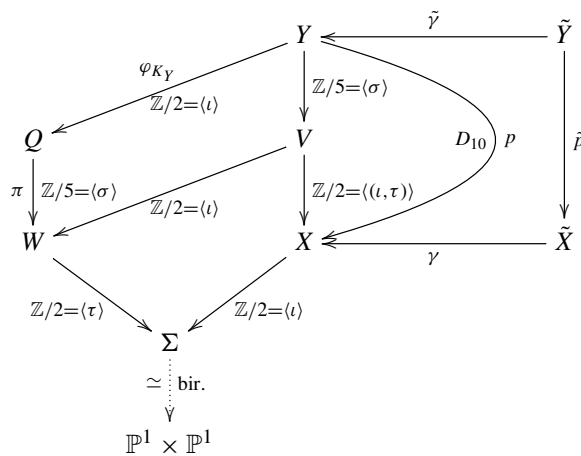
Let us recall the construction of determinantal Barlow surfaces in general. References are, for example, [Bar11], [Lee00] and [Lee01]. Let (x_1, \dots, x_4) be coordinates in \mathbb{P}^3 and consider an action of $D_{10} = \langle \sigma, \tau \rangle$ on \mathbb{P}^3 via

$$\begin{aligned} \sigma &: (x_1, x_2, x_3, x_4) \mapsto (\xi^1 x_1, \xi^2 x_2, \xi^3 x_3, \xi^4 x_4), \\ \tau &: (x_1, x_2, x_3, x_4) \mapsto (x_4, x_3, x_2, x_1), \end{aligned}$$

where ξ is a primitive fifth root of unity. Then D_{10} -invariant symmetric determinantal quintic surfaces Q in \mathbb{P}^3 can be given as the determinants of the following matrices (see e.g. [Lee00, p. 898]):

$$A = \begin{pmatrix} 0 & a_1 x_1 & a_2 x_2 & a_2 x_3 & a_1 x_4 \\ a_1 x_1 & a_3 x_2 & a_4 x_3 & a_5 x_4 & 0 \\ a_2 x_2 & a_4 x_3 & a_6 x_4 & 0 & a_5 x_1 \\ a_2 x_3 & a_5 x_4 & 0 & a_6 x_1 & a_4 x_2 \\ a_1 x_4 & 0 & a_5 x_1 & a_4 x_2 & a_3 x_3 \end{pmatrix}$$

where a_1, \dots, a_6 are parameters. The generic surface Q has an even set of 20 nodes, so that there is a double cover $\varphi_{K_Y} : Y \rightarrow Q$ with involution ι branched over the nodes. Here φ_{K_Y} is the canonical morphism. There is a twisted action of $D_{10} = \langle \sigma, (\tau, \iota) \rangle$ on Y which has a group of automorphisms $H = \langle \sigma, \tau \rangle \times \langle \iota \rangle = D_{10} \times \mathbb{Z}/2$. Then $X = Y/\langle \sigma, (\tau, \iota) \rangle$ is a surface with four nodes whose resolution \tilde{X} is a simply connected surface with $p_g = q = 0$ (a determinantal Barlow surface) and $W = Y/\langle \sigma, \iota \rangle$ is a determinantal Godeaux surface (with four nodes). This construction gives a 2-dimensional moduli space of determinantal Barlow surfaces. The geometry is summarized in the following diagram:



Here V is a numerical Campedelli surface or more precisely a Catanese surface, the double cover of the Godeaux surface W ramified in the even set of four nodes of W . Thus $p_g(V) = q(V) = 0, K_V^2 = 2, \pi_1(V) = \mathbb{Z}/5$.

The surface Y has an explicit description as follows [Cat81], [Reid81]: Let

$$R = \mathbb{C}[x_1, \dots, x_4, y_0, \dots, y_4]/I$$

where $\deg(x_i) = 1, \deg(y_j) = 2$ and the ideal I of relations is generated by

$$\sum_j A_{ij}y_j \quad (5 \text{ relations in degree } 3),$$

$$y_i y_j - B_{ij} \quad (15 \text{ relations in degree } 4),$$

where B_{ij} is the (i, j) -entry of the adjoint matrix of A (in particular, I contains $\det A$). Then Y is the subvariety in weighted projective space

$$Y = \text{Proj}(R) \subset \mathbb{P}(1^4, 2^5).$$

This is a smooth [Cat81, Prop. 2.11] surface of general type with $p_g = 4, q = 0, K^2 = 10$.

Remark 2.1. The special Barlow surface considered in [Bar1] corresponds to the choice of parameters

$$a_1 = a_2 = a_4 = a_5 = 1, \quad a_3 = a_6 = -4.$$

This can be seen by applying the base change

$$\begin{aligned} X_1 &= 5(x_1 + x_2 + x_3 + x_4), \\ X_2 &= 5(\xi x_1 + \xi^2 x_2 + \xi^3 x_3 + \xi^4 x_4), \\ X_3 &= 5(\xi^2 x_1 + (\xi^2)^2 x_2 + (\xi^3)^2 x_3 + (\xi^4)^2 x_4), \\ X_4 &= 5(\xi^3 x_1 + (\xi^2)^3 x_2 + (\xi^3)^3 x_3 + (\xi^4)^3 x_4), \\ X_5 &= 5(\xi^4 x_1 + (\xi^2)^4 x_2 + (\xi^3)^4 x_3 + (\xi^4)^4 x_4), \\ Y_0 &= \frac{1}{5}(y_0/6 + \xi^2 y_1 + \xi^4 y_2 + \xi y_3 + \xi^3 y_4), \\ Y_1 &= \frac{1}{5}(y_0/6 + \xi y_1 + \xi^2 y_2 + \xi^3 y_3 + \xi^4 y_4), \\ Y_2 &= \frac{1}{5}(y_0/6 + y_1 + y_2 + y_3 + y_4), \\ Y_3 &= \frac{1}{5}(y_0/6 + \xi^4 y_1 + \xi^3 y_2 + \xi^2 y_3 + \xi y_4), \\ Y_4 &= \frac{1}{5}(y_0/6 + \xi^3 y_1 + \xi y_2 + \xi^4 y_3 + \xi^2 y_4), \end{aligned}$$

to get the setup given in [Reid81]. We denote this special surface by S . It is distinguished by the fact that Q is even invariant under a larger group \mathfrak{S}_5 .

Remark 2.2. The numerical invariants of S are:

$$\begin{aligned} K_S^2 &= 1, \quad p_g = q = 0, \quad \pi_1(S) = \{1\}, \\ K_0(S) &\simeq \mathbb{Z}^{11}, \quad \text{Pic}(S) \simeq H^2(S, \mathbb{Z}) \simeq H_2(S, \mathbb{Z}) \simeq \mathbb{Z}^9. \end{aligned}$$

All integral cohomology classes on S are algebraic.

The least obvious statement that $K_0(S) \simeq \mathbb{Z}^{11}$ follows from the fact that $\text{Pic}(S) \simeq \mathbb{Z}^9$ and from the Bloch conjecture for S : $\text{CH}^2(S) \simeq \mathbb{Z}$ (this is known from [Bar12]). The argument is as follows: For surfaces we have (see [Ful, Ex. 15.3.6])

$$\begin{aligned} \text{rank} : F^0K(S)/F^1K(S) &\simeq \text{CH}^0(S) \simeq \mathbb{Z}, \\ \det : F^1K(S)/F^2K(S) &\simeq \text{Pic}(S), \quad c_2 : F^2K(S) \simeq \text{CH}^2(S), \end{aligned}$$

where $F^iK(S)$ is the filtration of $K_0(S)$ by codimension of support. Moreover, $\text{CH}^2(S)$ is generated by the structure sheaf \mathcal{O}_p of a point in S , and this is primitive in $K_0(S)$ (e.g. because $\chi(\mathcal{O}_p, \mathcal{O}_S) = 1$). Then, looking at the sequence of extensions given by the filtration steps, one sees that $K_0(S) \simeq \mathbb{Z}^{11}$.

Remark 2.3. The following are some basic facts in this set-up.

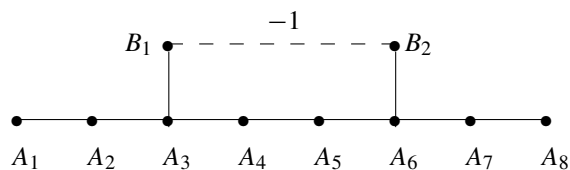
- (1) X and W have rational singularities, K_X and K_W are invertible, and if $\pi : Q \rightarrow W$ is the projection, $(\pi \circ \varphi_K)^*(K_W) = K_Y$. Moreover, $p^*K_X = K_Y$ and $\gamma^*(K_X) = K_{\tilde{X}}$.
- (2) Locally around the four fixed points of the group $\mathbb{Z}/2 = \langle (\iota, \tau) \rangle$, the quotient map $V \rightarrow X = V/(\mathbb{Z}/2)$ looks like $\mathbb{A}^2 \rightarrow \text{cone} \subset \mathbb{A}^3$ given by $(x, y) \mapsto (x^2, y^2, xy)$.
- (3) The bundle K_Y carries a canonical D_{10} -linearization corresponding to the D_{10} -action on $H^0(Y, K_Y) \simeq \langle x_1, \dots, x_4 \rangle$ given by the cycles σ and τ as above. In general, the action on $\bigoplus_{m \geq 0} H^0(Y, mK_Y)$ is the one described in Remark 4.1 on R : this is the canonical ring.

3. Lattice theory and semiorthonormal bases

We have $\text{Pic}(S) = \mathbf{1} \perp (-E_8)$ as a lattice. We recall some facts from [BBS12] which we will use.

Definition 3.1. A sequence l_1, \dots, l_N of classes in $K_0(S)$ is called *numerically exceptional* if $\chi(l_i, l_i) = 1$ for all i , and $\chi(l_i, l_j) = 0$ for $i > j$.

Let A_1, \dots, A_8 and B_1, B_2 be roots in $\text{Pic}(S)$ with the following intersection behaviour:



Here, if two nodes are joined by a solid line, the intersection is 1, otherwise it is zero. Moreover, B_1 and B_2 have intersection -1 .

Proposition 3.2. *The sequence*

$$\begin{aligned}
 & A_1, \\
 & A_1 + A_2, \\
 & k - B_1, \\
 & A_1 + A_2 + A_3, \\
 & A_1 + A_2 + A_3 + A_4, \\
 & A_1 + A_2 + A_3 + A_4 + A_5, \\
 & k - B_2, \\
 & A_1 + A_2 + A_3 + A_4 + A_5 + A_6, \\
 & A_1 + A_2 + A_3 + A_4 + A_5 + A_6 + A_7, \\
 & A_1 + A_2 + A_3 + A_4 + A_5 + A_6 + A_7 + A_8, \\
 & \mathcal{O}
 \end{aligned}$$

is numerically exceptional of length 11.

This is [BBS12, Prop. 5.6]. The exceptional sequence we will construct on S has this numerical behaviour. One advantage of this particular sequence is that the degrees of the differences of two classes in it are quite small, so there is a good chance to realize it as an actual exceptional sequence on S . The surface S is homeomorphic to \mathbb{P}^2 blown up in eight points, a del Pezzo surface of degree 1, and the numerics of full exceptional sequences on del Pezzo surfaces has been thoroughly investigated (see, for example, [KarNog]); also in this light, the sequence above seems to be most advantageous for our purposes.

4. Curves on the Barlow surface and an explicit basis of the Picard group

In this section we construct curves on the Barlow surface S . They will be used to make the intersection theory on the Barlow surface explicit. We will also use them to write down the exceptional sequence and to calculate sections of line bundles in Section 5. In a first step we construct D_{10} -invariant curves on Q , pull them back to Y and consider their images on X and strict transforms on \tilde{X} . These curves are of degree 1 and generate a $\mathbf{1} \oplus (-D_8)$ -sublattice of $\text{Pic}(\tilde{X})$. In a second step, using lattice theory, we find an effective divisor in the $\mathbf{1} \oplus (-D_8)$ -lattice which is divisible by 2 as an effective divisor. The resulting divisor is of degree 2. The degree 1 curves together with this degree 2 curve generate $\text{Pic}(\tilde{X})$ as a lattice. In a third step we use linkage and the automorphisms of Y to construct 32 curves of the same type as the degree 2 curve above. Finally, we calculate intersection numbers and write down our exceptional sequence and prove that the classes of the line bundles form a semiorthonormal basis of $K_0(\tilde{X})$.

The Macaulay2 scripts used to do the necessary calculations of this section and the following ones can be found at [BBKS12].

Remark 4.1. The D_{10} -action on Y (resp. the ambient $\mathbb{P}(1^4, 2^5)$) is given by

$$\begin{aligned} \sigma(x_i) &= \xi^i x_i & \text{and} & & \sigma(y_i) &= \xi^{-i} y_i, \\ \tau(x_i) &= x_{-i} & \text{and} & & \tau(y_i) &= y_{-i}, \\ \iota(x_i) &= x_i & \text{and} & & \iota(y_i) &= -y_i, \\ \alpha(x_i) &= x_{\alpha(i)} & \text{and} & & \alpha(y_i) &= y_{\alpha(i)}, \end{aligned}$$

with $\alpha = (1342)$. Moreover, we set $\beta = \iota \circ \tau$. Then $D_{10} = \langle \sigma, \beta \rangle$ operates on Y . The indices are interpreted as elements of $\mathbb{Z}/5$. The projection $Y \rightarrow Q$ is D_{10} -equivariant, where β acts as τ on Q . Moreover, α , τ and ι normalize the subgroup D_{10} . Hence they induce automorphisms on the quotient X and we have $\tau = \iota = \alpha^2$.

Proposition 4.2. *The determinantal quintic Q contains 15 lines:*

$$\begin{aligned} \overline{L}_i^0 &= \sigma^i(-t : -s : s : t), \\ \overline{L}_i^+ &= \sigma^i(s - \Phi t : -s + \Phi^{-1}t : -s : s + t), \\ \overline{L}_i^- &= \sigma^i(s + \Phi^{-1}t : -s - \Phi t : -s : s + t), \end{aligned}$$

where $(s : t) \in \mathbb{P}^1$ and $\Phi = (\sqrt{5} - 1)/2$ is the golden ratio.

Proof. \overline{L}_0^0 is the unique τ -invariant line on Q . A direct calculation also shows $\overline{L}_0^\pm \subset Q$ (see for example `determinantalGodeaux.m2` at [BBKS12]). The remaining lines lie on Q since Q is σ -invariant. A direct calculation on the Grassmannian shows that there are at most 15 lines on Q (see `linesQ.m2`). □

Lemma 4.3. *The lines of the σ -orbit $\langle \overline{L}_0^0 \rangle$ are disjoint. The lines in the σ -orbits $\langle \overline{L}_0^\pm \rangle$ form two pentagons.*

Proof. A direct calculation can be found in `determinantalGodeaux.m2` at [BBKS12]. □

Proposition 4.4. *The τ -invariant line $(s : t) \mapsto (t : s : s : t)$ intersects Q in five points. Over \mathbb{F}_{421} the coordinates of these points are*

$$\begin{aligned} \overline{P}_1 &= (-33 : 1 : 1 : -33), \\ \overline{P}_2 &= (1 : -33 : -33 : 1), \\ \overline{P}_3 &= (-50 : 1 : 1 : -50), \\ \overline{P}_4 &= (1 : -50 : -50 : 1), \\ \overline{P}_5 &= (1 : -1 : -1 : 1). \end{aligned}$$

In particular, we have $\overline{P}_5 \in \overline{L}_0^\pm$.

Proof. A direct calculation can be found in `determinantalGodeaux.m2` at [BBKS12]. □

We recall the classification result for $\mathbb{Z}/5$ -invariant elliptic quintics in \mathbb{P}^3 due to Reid.

Theorem 4.5 ([Reid91]). *Let $E \subset \mathbb{P}^3$ be a $\mathbb{Z}/5$ -invariant elliptic quintic curve not containing any coordinate points. Then*

- *the homogeneous ideal of E is generated by five cubics of the form*

$$\begin{aligned} R_0 &= ax_1^2x_3 - bx_1x_2^2 + cx_3^2x_4 - dx_2x_4^2, \\ R_1 &= asx_1x_2x_3 - atx_1^2x_4 - bsx_2^3 - ct x_3x_4^2, \\ R_2 &= asx_1x_3^2 - bsx_2^2x_3 - bt x_1x_2x_4 - dt x_4^3, \\ R_3 &= atx_1^3 + csx_2x_3^2 + ct x_1x_3x_4 - dsx_2^2x_4, \\ R_4 &= bt x_1^2x_2 + csx_3^3 - dsx_2x_3x_4 + dt x_1x_4^2, \end{aligned}$$

where a, b, c, d are nonzero constants and $(s : t) \in \mathbb{P}^1$. For E to be nonsingular, we must have $\frac{abc}{sad} \notin \{0, \infty, \frac{-11 \pm 5\sqrt{5}}{2} = (\frac{-1 \pm \sqrt{5}}{2})^5\}$. The set of all E is parametrized 1-to-1 by $(s : t) \in \mathbb{P}^1$ and the ratio $(a : b : c : d) \in \mathbb{P}^3$.

- *The vector space of $\mathbb{Z}/5$ -invariant quintic forms vanishing on E has a basis consisting of the seven elements*

$$x_1^2R_3, x_2^2R_1, x_3^2R_4, x_4^2R_2, x_1x_4R_0, x_2x_3R_0, x_3x_4R_3.$$

From this one gets

Proposition 4.6. *Q contains exactly eight D_{10} -invariant elliptic quintic curves. Their coordinates over \mathbb{F}_{421} in Reid’s parameter space are*

$$\begin{aligned} e_1 &= (-1 : 33 : -33 : 1, 1 : 202), & e_4^+ &= (-50 : 1 : -1 : 50, 133 : -1), \\ e_2 &= (-33 : 1 : -1 : 33, 202 : -1), & e_4^- &= (-50 : 1 : -1 : 50, -108 : -1), \\ e_3^+ &= (-1 : 50 : -50 : 1, 1 : 133), & e_5^+ &= (-1 : 1 : -1 : 1, 1 : 126), \\ e_3^- &= (-1 : 50 : -50 : 1, 1 : -108), & e_5^- &= (-1 : 1 : -1 : 1, 126 : -1). \end{aligned}$$

We denote by \overline{E}_i^\pm the elliptic quintic curve corresponding to e_i^\pm . We have $\overline{P}_i \in \overline{E}_j^\pm$ if and only if $i = j$. Furthermore $\overline{E}_5^\pm = \langle \overline{L}_0^\pm \rangle$ are two pentagons.

Proof. Using Reid’s setup we calculate the ideal of points on $\mathbb{P}^3 \times \mathbb{P}^1$ parametrizing $\mathbb{Z}/5\mathbb{Z}$ -invariant elliptic quintic curves in Q . It turns out that this ideal has degree 10 and two solution points appear with multiplicity 2. Over \mathbb{F}_{421} we obtain the same degrees and check that the above points are in the solution set by substitution. From the form of the solutions we see that the \overline{E}_i^\pm are also τ -invariant. Our script `determinantalGodeaux.m2` at [BBKS12] shows that \overline{E}_i^\pm are indeed elliptic curves on Q over \mathbb{F}_{421} . The fact that these are all such curves is checked in `allEllipticCurvesQ.m2`. □

Remark 4.7. The points e_1 and e_2 appear with multiplicity 2 on Reid’s parameter space (see `allEllipticCurvesQ.m2`).

Remark 4.8. The elliptic curves constructed in Proposition 4.6 are reductions of elliptic curves in characteristic 0 since a calculation over \mathbb{Q} shows that the number of such curves over \mathbb{C} is also 8 (see `allEllipticCurvesQ.m2`).

Proposition 4.9. *The preimages of \bar{P}_1 and \bar{P}_2 on Y are representatives of the branch locus of p . Their coordinates on Y are*

$$\begin{aligned} P_1^+ &= (-33 : 1 : 1 : -33 : 0 : -181 : 53 : -53 : 181), \\ P_1^- &= (-33 : 1 : 1 : -33 : 0 : 181 : -53 : 53 : -181), \\ P_2^+ &= (1 : -33 : -33 : 1 : 0 : 53 : 181 : -181 : -53), \\ P_2^- &= (1 : -33 : -33 : 1 : 0 : -53 : -181 : 181 : 53). \end{aligned}$$

Proof. A direct calculation can be found in `BarlowD8.m2` at [BBKS12]. □

Notation 4.10. We now pull back the curves constructed so far to Y and denote them by E_i^\pm and $\langle L \rangle$. Since they are D_{10} -invariant, they descend to X . We then denote their strict transforms in \tilde{X} by \tilde{L} and \tilde{E}_i^\pm . The nodes of X are at the images of $P_i^\pm, i = 1, 2$. We denote their preimages on \tilde{X} by \tilde{C}_i^\pm . The whole configuration of the elliptic curves and the (-2) -curves on \tilde{X} is visualized in Figure 1.

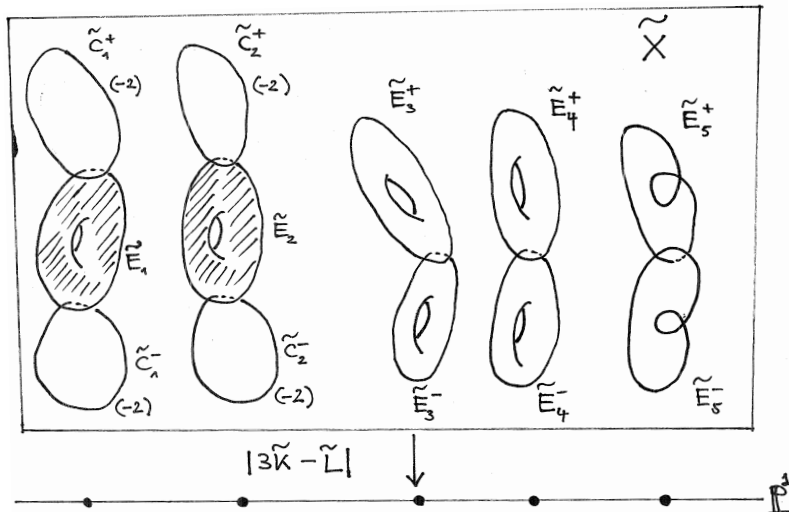


Fig. 1. The configuration of curves on the Barlow surface.

Lemma 4.11. *Let \tilde{D}_1, \tilde{D}_2 be two irreducible effective divisors on the Barlow surface \tilde{X} . Let I_j be the ideal of $D_j = \tilde{\gamma}(\tilde{p}^*(\tilde{D}_j))$ on Y . Set $I = I_1 + I_2$. We distinguish several cases:*

- (1) $V(I)$ is empty. Then $\tilde{D}_1 \cdot \tilde{D}_2 = 0$.
- (2) $V(I_i)$ are ramification points. Then $\tilde{D}_1 \cdot \tilde{D}_2 = (-2) \deg V(I)/5$.

- (3) $V(I_1)$ are ramification points and $V(I_2)$ is a curve which is smooth at all ramification points, or vice versa. Then $\tilde{D}_1 \cdot \tilde{D}_2 = \deg V(I)/5$.
- (4) $V(I_1)$ and $V(I_2)$ are curves which are smooth at all ramification points and $V(I)$ is finite. Let I_r be the ideal of the ramification locus of p . Then

$$\tilde{D}_1 \cdot \tilde{D}_2 = \frac{\deg I - \deg(I + I_r)}{10}.$$

- (5) We have

$$\tilde{K} \cdot \tilde{D}_1 = \frac{\deg(I_1 + (x_1))}{10}.$$

- (6) $V(I_1) = V(I_2) = V(I) = D = D_1 = D_2$ is a curve. Then

$$\tilde{D}^2 = \frac{2p_a(D) - 2 - \deg V(I + I_r) - \deg V(I)}{10}.$$

Proof. Consider the diagram

$$\begin{array}{ccccc} Y & \xleftarrow{\tilde{\gamma}} & \tilde{Y} & \longleftarrow & \hat{C}_{i,j}^\pm \\ \downarrow p & & \downarrow \tilde{p} & & \\ X & \xleftarrow{\gamma} & \tilde{X} & \longleftarrow & \tilde{C}_i^\pm \end{array}$$

Here \tilde{C}_i^\pm are the four (-2) -curves on \tilde{X} and $\hat{C}_{i,j}^\pm, j = 1, \dots, 5$, are the twenty (-1) -curves lying over them. Assertions (1) and (2) are clear.

In case (3), \tilde{D}_1 is one of the (-2) -curves, and \tilde{D}_2 is a curve intersecting all \tilde{C}_i^\pm transversely. The number in (3) counts the intersection number of the (-2) -curve with \tilde{D}_2 .

For (4) we compute

$$\begin{aligned} \deg V(I) = D_1 \cdot D_2 &= \tilde{\gamma}^*(D_1) \cdot \tilde{\gamma}^*(D_2) \\ &= \left(\tilde{p}^*(\tilde{D}_1) + \sum \delta_{ij}^\pm \hat{C}_{ij}^\pm \right) \cdot \left(\tilde{p}^*(\tilde{D}_2) + \sum \epsilon_{ij}^\pm \hat{C}_{ij}^\pm \right) \\ &= \tilde{p}^*(\tilde{D}_1) \tilde{p}^*(\tilde{D}_2) + 2 \sum \delta_{ij}^\pm \cdot \epsilon_{ij}^\pm - \sum \delta_{ij}^\pm \cdot \epsilon_{ij}^\pm \\ &= \tilde{p}^*(\tilde{D}_1) \tilde{p}^*(\tilde{D}_2) + \sum \delta_{ij}^\pm \cdot \epsilon_{ij}^\pm = 10 \tilde{D}_1 \cdot \tilde{D}_2 + \sum \delta_{ij}^\pm \cdot \epsilon_{ij}^\pm \end{aligned}$$

where $\delta_{ij}^\pm = 1$ or 0 depending on whether D_1 passes through $\tilde{\gamma}(\hat{C}_{ij}^\pm)$ or not, and analogously for ϵ_{ij}^\pm . This proves (4).

For (5) note that the formula is correct for \tilde{D}_1 a (-2) -curve because $x_1 = 0$ contains none of the ramification points of p . If \tilde{D}_1 is not a (-2) -curve, then, since $\tilde{\gamma}^*(K_Y) = \tilde{p}^*(\tilde{K})$,

$$\tilde{K} \cdot \tilde{D}_1 = \frac{1}{10} \tilde{p}^*(\tilde{K}) \cdot \tilde{p}^*(\tilde{D}_1) = \frac{1}{10} \tilde{\gamma}^*(K_Y) \cdot \tilde{\gamma}^*(D_1) = \frac{1}{10} K_Y \cdot D_1 = \frac{1}{10} \deg V((x_1) + I_1).$$

The second equality holds because $\tilde{p}^*(\tilde{D}_1)$ is equal to $\tilde{\gamma}^*(D_1)$ up to exceptional divisors on which $\tilde{\gamma}^*(K_Y)$ is trivial.

In (6), $\tilde{D}_1 = \tilde{D}_2 =: \tilde{D}$. The genus formula for \tilde{D} yields

$$\tilde{D}^2 = 2p_a(\tilde{D}) - 2 - \tilde{K} \cdot \tilde{D}.$$

The Hurwitz formula gives

$$2p_a(D) - 2 = 2p_a(\tilde{p}^*(\tilde{D})) - 2 = 10(2p_a(\tilde{D}) - 2) + \deg V(I + I_r)$$

since D is smooth at the ramification points of p . It follows that

$$\tilde{D}^2 = \frac{2p_a(D) - 2 - \deg V(I + I_r) - \deg V(I)}{10} \quad \square$$

Proposition 4.12. *The intersection matrix of the curves*

$$\{\tilde{E}_1, \tilde{E}_2, \tilde{E}_3^+, \tilde{E}_3^-, \tilde{E}_4^+, \tilde{E}_4^-, \tilde{E}_5^+, \tilde{E}_5^-, \tilde{L}, \tilde{K}, \tilde{C}_1^+, \tilde{C}_1^-, \tilde{C}_2^+, \tilde{C}_2^-\},$$

where \tilde{K} is the canonical divisor on \tilde{X} , is

$$\begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 3 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 3 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 3 & 1 & 0 & 0 & 0 & 0 & 0 \\ 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & -3 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 \end{pmatrix}.$$

The rank of this matrix is 9. See also Figure 1.

Proof. We calculate the intersection numbers on Y . For this we first check that E_i^\pm , $\langle L \rangle$ are smooth at the P_i^\pm . By the D_{10} -invariance of the orbits this shows that they are smooth at all branch points of p . We also represent K by the curve $\{x_1 = 0\}$ and set $C_i^\pm = \langle P_i^\pm \rangle$. The assertion follows from a calculation in BarlowD8.m2 at [BKKS12] using Lemma 4.11. \square

Remark 4.13. The intersections are calculated over a finite field (namely, \mathbb{F}_{421}) and may potentially be different from those in characteristic zero. The way the argument works is however the following: we produce eventually an exceptional sequence

$$(\bar{\mathcal{L}}_1, \dots, \bar{\mathcal{L}}_{11})$$

of line bundles on the reduction of the Barlow surface to finite characteristic. However, the $\bar{\mathcal{L}}_i$ themselves are reductions of line bundles \mathcal{L}_i defined over an algebraic number field of characteristic 0. Hence by upper semicontinuity over $\text{Spec}(\mathfrak{D})$, where \mathfrak{D} is the ring of integers of this number field, the sequence

$$(\mathcal{L}_1, \dots, \mathcal{L}_{11})$$

will also be exceptional in characteristic 0.

Remark 4.14. We have 14 effective possibly reducible genus one curves of degree 1 on \tilde{X} . These are

$$\{\tilde{E}_i, \tilde{E}_i + \tilde{C}_i^+, \tilde{E}_i + \tilde{C}_i^-, \tilde{E}_i + \tilde{C}_i^+ + \tilde{C}_i^-\}$$

for $i = 1, 2$ and \tilde{E}_j^\pm for $j = 3, 4, 5$.

Proposition 4.15. On \tilde{X} we have the following roots (so far):

- $\tilde{F}_i - \tilde{F}_j$ with $\tilde{F}_i \cdot \tilde{F}_j = 0$ and \tilde{F}_i, \tilde{F}_j possibly reducible elliptic curves of degree one (84 of these).
- $\pm \tilde{K} \mp \tilde{F}_i$ with \tilde{F}_i as above (28 of these).

These 112 roots form a D_8 -root system. A D_8 -basis is given, for example, by the simple roots

$$\begin{aligned} \mathcal{D}_1 &= \tilde{K} - \tilde{E}_2 - \tilde{C}_2^-, & \mathcal{D}_5 &= \tilde{E}_3^- - \tilde{E}_4^-, \\ \mathcal{D}_2 &= \tilde{E}_2 + \tilde{C}_2^- - \tilde{E}_2, & \mathcal{D}_6 &= \tilde{E}_4^- - \tilde{E}_5^-, \\ \mathcal{D}_3 &= \tilde{E}_1 - \tilde{E}_2 - \tilde{C}_2^+ - \tilde{C}_2^-, & \mathcal{D}_7 &= \tilde{E}_1 + \tilde{C}_1^- - \tilde{E}_5^+, \\ \mathcal{D}_4 &= \tilde{E}_1 + \tilde{C}_1^+ + \tilde{C}_1^- - \tilde{E}_3^-, & \mathcal{D}_8 &= \tilde{E}_1 + \tilde{C}_1^+ - \tilde{E}_5^+, \end{aligned}$$

They have intersection matrix

$$\begin{pmatrix} -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -2 \end{pmatrix}.$$

Proof. Calculation using the above intersection matrix in Proposition 4.12. A Macaulay2 script checking this can be found in BarlowD8.m2 at [BBKS12]. \square

Remark 4.16. All effective curves constructed so far can be written in this D_8 -basis and \tilde{K} using integer coefficients.

Proposition 4.17. On \tilde{X} there exist 32 curves \tilde{B}_{ijk}^\pm , $i, j \in \mathbb{Z}/2\mathbb{Z}$ and $k \in \{0, 1, 2, 3\}$, of genus 3 and canonical degree 2 each intersecting two (-2) -curves, say \tilde{F}_{ijk}^\pm and \tilde{G}_{ijk}^\pm , such that

$$2\tilde{K} - \tilde{B}_{ijk}^\pm, \quad 2\tilde{K} - \tilde{B}_{ijk}^\pm - \tilde{F}_{ijk}^\pm, \quad 2\tilde{K} - \tilde{B}_{ijk}^\pm - \tilde{G}_{ijk}^\pm, \quad 2\tilde{K} - \tilde{B}_{ijk}^\pm - \tilde{F}_{ijk}^\pm - \tilde{G}_{ijk}^\pm$$

represent 128 additional roots. The total of $112 + 128 = 240$ roots forms an E_8 -lattice.

Proof. Let d_1, \dots, d_8 be a system of simple roots in a D_8 -lattice. If this lattice is a sublattice of an E_8 -lattice, the Borel–Siebenthal algorithm (see e.g. [MT, 13.2, p. 109]) gives the highest root in the E_8 -lattice as

$$e = \frac{1}{2}(d_1 + 2d_2 + 3d_3 + 4d_4 + 5d_5 + 6d_6 + 3d_7 + 4d_8),$$

and $\{e, d_8, \dots, d_2\}$ is a system of simple roots in the E_8 -lattice. Observe that

$$\chi(e + 2\tilde{K}) = \frac{(e + 2\tilde{K}) \cdot (e + \tilde{K}) + 2}{2} = \frac{-2 + 2 + 2}{2} = 1.$$

Since $H^2(e + 2\tilde{K}) = H^0(-\tilde{K} - e) = 0$, this implies that $H^0(e + 2\tilde{K}) \neq 0$. We will construct a curve $B \in |e + 2\tilde{K}|$. By construction B is not in the subgroup of $\text{Pic } \tilde{X}$ considered so far, but $2B$ is, i.e. there exists a nonreduced curve in the linear system $|2B|$ whose support is B . To represent $|2B|$ in a computer, we write

$$2B \equiv 8\tilde{K} - \tilde{L} - \tilde{E}_3^+ - \tilde{E}_4^+ - \tilde{E}_5^+ + \tilde{C}_1^+ - \tilde{C}_2^+,$$

and therefore consider D_{10} -invariant polynomials of degree 8 that lie in the ideal I of $\langle L \rangle \cup E_3^+ \cup E_4^+ \cup E_5^+ \cup \langle P_1^+ \rangle$ in Y . A computation shows that there is a \mathbb{P}^3 of such polynomials. By restricting to lines we find the unique such polynomial F that is nonreduced on a curve outside of $L \cup E_3^+ \cup E_4^+ \cup E_5^+$. The ideal of the curve B is then obtained as $\text{rad}(((F) + I(Y)) : I)$. It is D_{10} -invariant and hence descends to X . We denote its strict transform by \tilde{B}_{000}^+ .

The intersection of \tilde{B}_{000}^+ with the effective curves of Proposition 4.12 can be calculated to be

$$\{2, 2, 3, 2, 3, 2, 3, 2, 1, 2, 0, 1, 1, 0\}.$$

Now let \tilde{E} be an elliptic curve and \tilde{B} be a genus 3 curve of degree 2 with $\tilde{E} \cdot \tilde{B} = 3$ on \tilde{X} . Then

$$\chi(5\tilde{K} - \tilde{E} - \tilde{B}) = 1.$$

We thus have an effective curve $\tilde{B}' \in |5\tilde{K} - \tilde{E} - \tilde{B}|$, and since $\tilde{B}^2 = 2$, we have $g(\tilde{B}') = 3$ and $\tilde{B}' \cdot \tilde{K} = 2$. We say that \tilde{B}' is *linked via $5\tilde{K}$ to $\tilde{B} + \tilde{E}$* . Performing this construction with \tilde{B}_0^+ and $\tilde{E}_3^+, \tilde{E}_4^+$ and \tilde{E}_5^+ , we obtain further genus 2 curves $\tilde{B}_{001}^+, \tilde{B}_{002}^+, \tilde{B}_{003}^+$. Observe that, furthermore,

$$\chi(4\tilde{K} - \tilde{B}) = 1,$$

and for $\tilde{B}' \in |4\tilde{K} - \tilde{B}|$ we also have $g(\tilde{B}') = 3$ and $\tilde{B}' \cdot \tilde{K} = 2$. Therefore we can link \tilde{B}_{00k}^+ via $4\tilde{K}$ to \tilde{B}_{00k}^- . Now we set

$$\tilde{B}_{ijk}^\pm = t^i(\alpha^j(\tilde{B}_{00k}^\pm))$$

and obtain a total of 32 curves. With a computer we can check that all of the 32 constructed curves are distinct and each of them intersects exactly two (-2) -curves.

For the calculation of the roots we choose an effective \mathbb{Z} -basis of $\text{Pic}(\tilde{X})$, for example

$$\{\tilde{E}_1, \tilde{E}_2, \tilde{E}_3^+, \tilde{E}_3^-, \tilde{E}_4^+, \tilde{E}_5^+, \tilde{K}, \tilde{C}_1^+, \tilde{B}_{000}^+\}.$$

With respect to this basis we can calculate the numerical class of all effective curves constructed so far, by using Lemma 4.11. The remaining assertions of the proposition are then simple calculations with numerical classes.

All computations needed in this proof can be found in `BarLowE8.m2` at [BBKS12].

□

Proposition 4.18. *We have the following intersections on \tilde{X} :*

$$(\tilde{B}_{ijk}^\pm - 2\tilde{K}) \cdot \tilde{L} = \mp 1.$$

Proof. Calculation using numerical classes (see BarlowE8.m2). □

Proposition 4.19. *The exceptional curves \tilde{C}_i^\pm have the following intersections with $\tilde{B}_{ijk}^\pm - 2\tilde{K}$ on \tilde{X} :*

	$i = 0$		$i = 1$	
	$j = 0$	$j = 1$	$j = 0$	$j = 1$
\tilde{C}_1^+	0	1	0	1
\tilde{C}_1^-	1	0	1	0
\tilde{C}_2^+	1	0	0	1
\tilde{C}_2^-	0	1	1	0

Proof. Calculation using numerical classes (see BarlowE8.m2). □

Proposition 4.20. *We have the following intersections with the elliptic curves \tilde{E}_1 and \tilde{E}_2 :*

$$(\tilde{B}_{ijk}^+ - 2\tilde{K}) \cdot \tilde{E}_i = 0, \quad (\tilde{B}_{ijk}^- - 2\tilde{K}) \cdot \tilde{E}_i = -1.$$

For the elliptic curves \tilde{E}_3^\pm and \tilde{E}_4^\pm we have:

		$k = 0$	$k = 1$	$k = 2$	$k = 3$
$\tilde{B}_{0jk}^\pm - 2\tilde{K}$	\tilde{E}_3^+	± 1	± 1	0	0
	\tilde{E}_3^-	0	0	± 1	± 1
	\tilde{E}_4^+	± 1	0	± 1	0
	\tilde{E}_4^-	0	± 1	0	± 1
$\tilde{B}_{1jk}^\pm - 2\tilde{K}$	\tilde{E}_3^-	± 1	0	± 1	0
	\tilde{E}_3^+	0	± 1	0	± 1
	\tilde{E}_4^-	± 1	± 1	0	0
	\tilde{E}_4^+	0	0	± 1	± 1

For the elliptic curves \tilde{E}_5^\pm we have:

	$k = 0$	$k = 1$	$k = 2$	$k = 3$
$(\tilde{B}_{0jk}^\pm - 2\tilde{K}) \cdot \tilde{E}_5^-$	0	± 1	± 1	0
$(\tilde{B}_{1jk}^\pm - 2\tilde{K}) \cdot \tilde{E}_5^+$	0	± 1	± 1	0
$(\tilde{B}_{1jk}^\pm - 2\tilde{K}) \cdot \tilde{E}_5^-$	± 1	0	0	± 1
$(\tilde{B}_{0jk}^\pm - 2\tilde{K}) \cdot \tilde{E}_5^+$	± 1	0	0	± 1

Proof. Calculation using numerical classes (see BarlowE8.m2). □

Proposition 4.21. *Let $i, j, i', j' \in \{0, 1\}$ and $q, q' \in \{+1, -1\}$. If $i = i'$, then*

$$(\tilde{B}_{ijk}^q \cdot \tilde{B}_{i'j'k'}^{q'})_{k=0,\dots,3, k'=0,\dots,3} = \begin{pmatrix} b & a & a & a \\ a & b & a & a \\ a & a & b & a \\ a & a & a & b \end{pmatrix}$$

with $a = 3 + ((j + j') \bmod 2)$ and $b = a - qq'$. If $i \neq i'$ then

$$(\tilde{B}_{ijk}^q \cdot \tilde{B}_{i'j'k'}^{q'})_{k=0,\dots,3, k'=0,\dots,3} = \begin{pmatrix} a & a & a & b \\ a & b & a & a \\ a & a & b & a \\ b & a & a & a \end{pmatrix}$$

with $a = 4 - (qq' + 1)/2$ and $b = 3 + (qq' + 1)/2$.

Proof. Calculation using numerical classes (see BarlowE8.m2). □

Using these roots, we can apply the method of Section 3 to obtain an explicit numerically semiorthogonal sequence of line bundles:

Proposition 4.22. *The following sequence of line bundles is numerically semiorthogonal:*

$$\begin{aligned} \mathcal{L}_1 &= \tilde{E}_1 - \tilde{E}_2, & \mathcal{L}_5 &= 2\tilde{K} - \tilde{B}_{012}^- - \tilde{C}_1^+, & \mathcal{L}_9 &= 2\tilde{K} - \tilde{B}_{101}^- - \tilde{C}_1^-, \\ \mathcal{L}_2 &= \tilde{E}_3^+ - \tilde{E}_2, & \mathcal{L}_6 &= 2\tilde{K} - \tilde{B}_{002}^- - \tilde{C}_1^-, & \mathcal{L}_{10} &= \mathcal{O}, \\ \mathcal{L}_3 &= 2\tilde{K} - \tilde{E}_4^+, & \mathcal{L}_7 &= 2\tilde{K} - \tilde{E}_5^+, & \mathcal{L}_{11} &= \tilde{K} - \tilde{E}_2. \\ \mathcal{L}_4 &= \tilde{E}_4^- - \tilde{E}_2, & \mathcal{L}_8 &= 2\tilde{K} - \tilde{B}_{111}^- - \tilde{C}_1^+, \end{aligned}$$

Proof. The following matrix contains $\chi(\mathcal{L}_i - \mathcal{L}_j)$ at the (i, j) -th entry:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & -1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

This can be checked using numerical classes and Riemann–Roch (see BarlowE8.m2 once again). □

5. Sections of line bundles and the exceptional sequence

Here we explain how we calculate sections of line bundles on S , or rather, obtain upper bounds for the dimensions of the spaces of sections of those line bundles.

Look at the natural commutative diagram of G -varieties (recall $G = D_{10}$ here)

$$\begin{array}{ccc} Y & \xleftarrow{\tilde{\gamma}} & \tilde{Y} \\ p \downarrow & & \downarrow \tilde{p} \\ X & \xleftarrow{\gamma} & S \end{array}$$

We write

$$D \equiv nK_S - P$$

where P is an effective divisor and $n \in \mathbb{N}$. Note that numerical equivalence coincides with linear equivalence on S . We find P using integer programming [BBKS12].

Write $P = P' + \sum a_i \tilde{C}_i$ where \tilde{C}_i are the (-2) -curves on S (these are the \tilde{C}_k^\pm of Section 4; here i runs from 1 to 4), and P' does not contain (-2) -curves as components.

Let P'' be the strict transform of $\tilde{p}^*(P')$ on Y .

Proposition 5.1. *We have*

$$\dim(H^0(Y, \mathcal{O}_Y(nK_Y - P''))^G) \geq \dim H^0(S, \mathcal{O}_S(D)).$$

If \mathfrak{p} is some prime number and we denote reduction by subscripts \mathfrak{p} (all data are defined over \mathbb{Z}), an analogous inequality holds:

$$\dim(H^0(Y_{\mathfrak{p}}, \mathcal{O}_{Y_{\mathfrak{p}}}(nK_{Y_{\mathfrak{p}}} - P''_{\mathfrak{p}}))^G) \geq \dim H^0(S, \mathcal{O}_S(D)).$$

Proof. First note that K_X, K_Y and K_S are all line bundles (Cartier), and $\gamma^*(K_X) = K_S, p^*(K_X) = K_Y$. We want to calculate

$$\dim H^0(S, \mathcal{O}_S(D)) = \dim H^0\left(S, \mathcal{O}_S\left(nK_S - P' - \sum_i a_i \tilde{C}_i\right)\right),$$

and this is bounded by $\dim H^0(S, \mathcal{O}_S(nK_S - P'))$. Now $H^0(S, \mathcal{O}_S(nK_S - P'))$ is the subspace of sections in $H^0(S, \mathcal{O}_S(nK_S))$ which vanish along the components of P' (with prescribed multiplicities). Since X is normal, we have $\gamma_*\mathcal{O}_S = \mathcal{O}_X$, hence by the projection formula $H^0(S, nK_S) = H^0(S, \gamma^*(nK_X)) = H^0(X, nK_X)$, and sections in nK_S vanishing along P' map to sections of nK_X vanishing along the strict transform P'_X of P' on X (by which we mean the Weil divisor on X whose irreducible components are the images of the components of P' on S , and each component of P'_X has the multiplicity of the component of P' of which it is the image). Now sections in nK_X inject into G -invariant sections in $p^*(nK_X) = nK_Y$ because p is a quotient map (in fact, the two spaces of sections are equal because the line bundles K_Y and \mathcal{O}_Y , together with the G -linearizations we use here, are pulled back from the base). Under this correspondence, sections in nK_X vanishing along P'_X map to G -invariant sections in nK_Y vanishing along P'' , where P'' is as defined above, or equivalently, the unique divisor on Y which restricts to $(p|_V)^*((P'_X)|_{p(V)})$ on the complement V of the ramification points (P'_X is Cartier off the nodes of X and the divisorial pull-back from the complement U of the nodes in X to the complement $V = p^{-1}(U)$ of the ramification points upstairs makes sense). This proves the first inequality, and the second follows using upper semicontinuity over $\text{Spec}(\mathbb{Z})$. □

We will use the second inequality in Proposition 5.1 to obtain bounds on dimensions of spaces of sections: we will set $p = 421$ and compute (with Macaulay2) the dimension of the space of degree n polynomials in the coordinates $x_1, \dots, x_4, y_0, \dots, y_4$ on the weighted projective space $\mathbb{P}(1^4, 2^5)$ vanishing on $p^*(P'_p)$ modulo the space of degree n polynomials in the homogeneous ideal of Y in $\mathbb{P}(1^4, 2^5)$. Note that $\mathbb{C}[x_1, \dots, x_4, y_1, \dots, y_5]/I$ is indeed exactly the canonical ring of Y , thus its elements give the pluricanonical sections on Y .

We obtain the following vanishing theorem:

Proposition 5.2. *Let $\tilde{R} \in \text{Pic } \tilde{X}$ be a root, i.e. $\tilde{R} \cdot \tilde{K} = 0$ and $\tilde{R}^2 = -2$. Then:*

- $h^0(\tilde{R}) \neq 0$ if and only if $\tilde{R} \simeq C_i^\pm$. In this case $h^1(R) = 1$ and $h^2(R) = 0$.
- $h^2(\tilde{R}) \neq 0$ if and only if $\tilde{K} - \tilde{R}$ is numerically equivalent to one of the 14 elliptic curves of Remark 4.14. In this case $h^1(\tilde{R}) = 1$ and $h^0(\tilde{R}) = 0$.

Proof. The “if” part is obvious. The reverse can be checked for all 240 roots by calculating directly over \mathbb{F}_{421} using Proposition 5.1. This is done in BarlowSections.m2. \square

We obtain furthermore

Theorem 5.3. *The sequence of line bundles given in Proposition 4.22 is exceptional on S : $\text{RHom}^\bullet(\mathcal{L}_j, \mathcal{L}_i) = 0$ for $j > i$.*

Proof. Most of the \mathcal{L}_i and the differences $\mathcal{L}_i - \mathcal{L}_j$ are roots by construction. For these the vanishing of all cohomology follows from Proposition 5.2 by comparing numerical classes. For the remaining differences we calculate the cohomology over \mathbb{F}_{421} using Proposition 5.1. This is done in BarlowSections.m2. \square

6. The deformation argument and existence of phantoms

In this section we will prove the existence of phantom categories in $D^b(S_t)$ where S_t is generic in the moduli space of determinantal Barlow surfaces in a small neighbourhood of the distinguished Barlow surface $S = S_0$ of Section 2.

Lemma 6.1. *Let S_t be a generic determinantal Barlow surface in a small neighbourhood of S . Then there is an exceptional sequence $(\mathcal{L}_{1,t}, \dots, \mathcal{L}_{11,t})$ in $D^b(S_t)$ consisting of line bundles $\mathcal{L}_{i,t}$ which are deformations of the \mathcal{L}_i .*

Proof. Consider a small nontrivial deformation of S (one deformation parameter t for simplicity) among determinantal Barlow surfaces:

$$\begin{array}{ccc}
 S & \xlongequal{\quad} & S_0 \hookrightarrow S \\
 & & \downarrow \quad \downarrow \\
 & & 0 \hookrightarrow B
 \end{array}$$

The line bundles \mathcal{L}_i deform to line bundles $\mathcal{L}_{i,t}$ and by upper semicontinuity, the $\mathcal{L}_{i,t}$ are also an exceptional sequence. \square

Consider now $\mathbb{L}_t = \bigoplus_{i=1}^{11} \mathcal{L}_{i,t}$ and the differential graded algebra $\mathfrak{A}_t = \mathrm{RHom}^\bullet(\mathbb{L}_t, \mathbb{L}_t)$ of derived endomorphisms of the exceptional sequence $(\mathcal{L}_{1,t}, \dots, \mathcal{L}_{11,t})$ above. It has a minimal model in the sense of [Keller01, Sect. 3.3], that is, we consider the Yoneda algebra $H^*(\mathfrak{A}_t)$ together with its A_∞ -structure such that $m_1 = 0, m_2 =$ Yoneda multiplication and there is a quasi-isomorphism of A_∞ -algebras $\mathfrak{A}_t \simeq H^*(\mathfrak{A}_t)$ lifting the identity of $H^*(\mathfrak{A}_t)$. We will show

Proposition 6.2. *In the neighbourhood of a generic value of the deformation parameter t , the algebra $H^*(\mathfrak{A}_t)$ is constant. Hence, the subcategories $\langle \mathcal{L}_{1,t}, \dots, \mathcal{L}_{11,t} \rangle$ are all equivalent in a neighbourhood of a generic value of t .*

In fact, we are dealing here with an A_∞ -category on the eleven objects $\mathcal{L}_{i,t}$. Let us recall now some facts about A_∞ -categories which we need to prove Proposition 6.2. A possible reference is the first chapter in Seidel’s book [Seid]. In particular, in an A_∞ -category we are given a set of objects X_i with a graded vector space $\mathrm{hom}(X_0, X_1)$ for any pair of objects, and composition maps of every order $d \geq 1$

$$\mathrm{hom}(X_0, X_1) \otimes \mathrm{hom}(X_1, X_2) \otimes \dots \otimes \mathrm{hom}(X_{d-1}, X_d) \rightarrow \mathrm{hom}(X_0, X_d)[2 - d]$$

satisfying the A_∞ -associativity equations, whose precise form we actually need not know here. The important point is that m_d is homogeneous of degree $2 - d$. Another important point is (cf. [Seid, Lem. 2.1]) that any homotopy unital A_∞ -category is quasi-isomorphic to a strictly unital one, i.e. we may assume

$$m_d(a_0 \otimes \dots \otimes a_{i-1} \otimes \mathrm{id} \otimes a_{i+1} \otimes \dots \otimes a_d) = 0, \quad d \geq 3,$$

which means $m_d, d \geq 3$, is zero as soon as one of its arguments is a homothetic automorphism of an object.

We use

Lemma 6.3. *The following matrix describes the $\mathrm{Ext}^n(\mathcal{L}_{j,t}, \mathcal{L}_{i,t})$ arising from the exceptional sequence. More precisely, the (i, j) -entry of the matrix is*

$$[\dim \mathrm{Hom}(\mathcal{L}_{j,t}, \mathcal{L}_{i,t}), \dim \mathrm{Ext}^1(\mathcal{L}_{j,t}, \mathcal{L}_{i,t}), \dim \mathrm{Ext}^2(\mathcal{L}_{j,t}, \mathcal{L}_{i,t})].$$

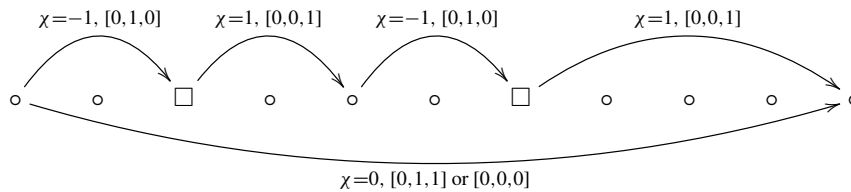
We call this triple a cohomology datum for short. We just write 0 for the trivial cohomology datum $[0, 0, 0]$.

$$\begin{pmatrix} [1, 0, 0] & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & [1, 0, 0] & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ [0, 1, 0] & [0, 1, 0] & [1, 0, 0] & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & [0, 0, 1] & [1, 0, 0] & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & [0, 0, 1] & 0 & [1, 0, 0] & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & [0, 0, 1] & 0 & 0 & [1, 0, 0] & 0 & 0 & 0 & 0 & 0 \\ [0, 1, 0] & [0, 1, 0] & 0 & [0, 1, 0] & [0, 1, 0] & [0, 1, 0] & [1, 0, 0] & 0 & 0 & 0 & 0 \\ 0 & 0 & [0, 0, 1] & 0 & 0 & 0 & [0, 0, 1] & [1, 0, 0] & 0 & 0 & 0 \\ 0 & 0 & [0, 0, 1] & 0 & 0 & 0 & [0, 0, 1] & 0 & [1, 0, 0] & 0 & 0 \\ 0 & 0 & [0, 0, 1] & 0 & 0 & 0 & [0, 0, 1] & 0 & 0 & [1, 0, 0] & 0 \\ * & * & [0, 0, 1] & * & * & * & [0, 0, 1] & * & * & * & [1, 0, 0] \end{pmatrix}.$$

Here $*$ means either $[0, 0, 0]$ or $[0, 1, 1]$.

Proof. We check this for the sequence $(\mathcal{L}_1, \dots, \mathcal{L}_{11})$ by explicit calculation over \mathbb{F}_{421} in BarlowSections.m2. The general statement follows by upper semicontinuity. \square

Remark 6.4. It is helpful to think of the degree 0 line bundles $\mathcal{L}_{i,t}$, $i \neq 3, 7$, as “circles” and of the degree 1 line bundles $\mathcal{L}_{3,t}$, $\mathcal{L}_{7,t}$ as “squares”. Then the assertion of Lemma 6.3 can be paraphrased as saying that there are no derived homomorphisms between circles except from the first to the last (eleventh) circle where one can have the cohomology datum $[0, 1, 1]$. The squares are completely orthogonal, and derived homomorphisms of a circle into a square (in the forward direction) have $\chi = -1$ and cohomology datum $[0, 1, 0]$, whereas derived homomorphisms of a square into a circle (in the forward direction) have $\chi = 1$ and cohomology datum $[0, 0, 1]$. In the proof of Proposition 6.2 we will first show that $H^*(\mathfrak{A}_t)$ has no higher multiplication, and then that the algebra structure is also fixed in the neighbourhood of a generic point. It is instructive to think of pictures like the following illustrating a potential composition for m_4 :



Proof of Proposition 6.2. We choose t in a neighbourhood of a generic point in the moduli space of determinantal Barlow surfaces; hence we can assume that the matrix of cohomology data in Lemma 6.3 is constant (i.e. there are no changes of the entries $*$ in the matrix).

We think of the $\mathcal{L}_{i,t}$ as the objects of our A_∞ -category. It is clear that

$$m_2 : \text{hom}(X_0, X_1) \otimes \text{hom}(X_1, X_2) \rightarrow \text{hom}(X_0, X_2)$$

is always the zero map in our case if X_0, X_1, X_2 are pairwise different; in fact, the only way to get a potentially nonzero composition would be to compose a morphism from a circle to a square with a morphism from that square to the last circle, which is impossible because this is a degree 3 morphism; or to compose a morphism from a square to a circle with a morphism from that circle to the last circle, but this is also at least of degree 3.

Hence it suffices to prove that there is no higher multiplication, i.e. $m_i = 0$ for $i \geq 3$. Then the endomorphism algebra of our category is just a usual graded algebra, and the algebra structure is completely determined and does not deform.

Clearly, $m_d = 0$ for $d \geq 6$: in fact, if $i < j < k < l < m < n < o$, one of the spaces

$$\begin{aligned} & \text{RHom}^\bullet(\mathcal{L}_{i,t}, \mathcal{L}_{j,t}), & \text{RHom}^\bullet(\mathcal{L}_{j,t}, \mathcal{L}_{k,t}), & \text{RHom}^\bullet(\mathcal{L}_{k,t}, \mathcal{L}_{l,t}), \\ & \text{RHom}^\bullet(\mathcal{L}_{l,t}, \mathcal{L}_{m,t}), & \text{RHom}^\bullet(\mathcal{L}_{m,t}, \mathcal{L}_{n,t}), & \text{RHom}^\bullet(\mathcal{L}_{n,t}, \mathcal{L}_{o,t}) \end{aligned}$$

is the zero space.

Now look at m_5 : the smallest degree of a nonzero element in a space

$$\text{hom}(\mathcal{L}_{i,t}, \mathcal{L}_{j,t}) \otimes \text{hom}(\mathcal{L}_{j,t}, \mathcal{L}_{k,t}) \otimes \text{hom}(\mathcal{L}_{k,t}, \mathcal{L}_{l,t}) \otimes \text{hom}(\mathcal{L}_{l,t}, \mathcal{L}_{m,t}) \otimes \text{hom}(\mathcal{L}_{m,t}, \mathcal{L}_{n,t})$$

for $i < j < k < l < m < n$ is equal to 7. But m_5 lowers the degree by 3, and there are no Ext^4 's.

For m_4 resp. m_3 we argue similarly: the lowest degrees of nonzero elements in the spaces of four resp. three composable morphisms are 6 resp. 4; but m_4 lowers the degree by 2 and m_3 lowers the degree by 1. \square

We will need the following special case of a result by Voisin [Voi].

Theorem 6.5. *The generic determinantal Barlow surface \tilde{X} of Section 2 satisfies the Bloch conjecture $\text{CH}^2(\tilde{X}) = \mathbb{Z}$.*

Corollary 6.6. *The exceptional sequence $(\mathcal{L}_{1,t}, \dots, \mathcal{L}_{11,t})$ is not full, in other words, there exists a phantom category \mathcal{A}_t in $\text{D}^b(S_t)$ as in Theorem 1.1 for a surface S_t which is generic in a small neighbourhood of the Barlow surface $S = S_0$ in the moduli space of determinantal Barlow surfaces.*

Proof. By Proposition 6.2, the subcategories $\langle \mathcal{L}_{1,t}, \dots, \mathcal{L}_{11,t} \rangle$ are all equivalent. However, by [BM], two minimal surfaces of general type whose derived categories are equivalent are isomorphic. It follows that the sequence $(\mathcal{L}_{1,t}, \dots, \mathcal{L}_{11,t})$ cannot be full, i.e. there is a nontrivial complement \mathcal{A}_t (generically). Since $\text{K}_0(S_t)$ is isomorphic to \mathbb{Z}^{11} , it follows that \mathcal{A}_t is a phantom. Here we use Theorem 6.5 for the generic determinantal Barlow surface to have $\text{K}_0(S_t) \simeq \mathbb{Z}^{11}$. \square

7. Heights of exceptional collections and a phantom on the Barlow surface

This section contains a proof that the Barlow surface S itself contains a phantom (whereas Corollary 6.6 gives this for a general determinantal Barlow surface somewhere in the moduli space). For this, we will use results of Kuznetsov [Kuz12] concerning (pseudo)heights of exceptional collections. First we need to recall some notions.

Given objects F, F' in a triangulated category \mathcal{T} , we define their relative height to be

$$e(F, F') := \min\{p \in \mathbb{Z} \mid \text{Hom}(F, F'[p]) \neq 0\}.$$

Consider an exceptional collection (E_1, \dots, E_n) in the bounded derived category of coherent sheaves on some smooth projective variety Z . The *anticanonical pseudoheight* of this exceptional collection is defined as follows. For a sequence $\underline{a} = (a_0, \dots, a_p)$ of integers with $1 \leq a_0 < a_1 < \dots < a_p \leq n$ consider the number

$$e(\underline{a}) = e(E_{a_0}, E_{a_1}) + \dots + e(E_{a_{p-1}}, E_{a_p}) + e(E_{a_p}, E_{a_0} \otimes \omega_Z^{-1}) - p.$$

Now the *anticanonical pseudoheight* is given by

$$\text{ph}_{\text{ac}}(E_1, \dots, E_n) = \min_{\underline{a}} e(\underline{a}).$$

The *pseudoheight* ph of the exceptional collection is given by $\text{ph}_{\text{ac}} = \text{ph} - \dim Z$.

It is proved in [Kuz12, Cor. 6.2] that if $\text{ph}_{\text{ac}}(E_1, \dots, E_n) > -2$, then the collection is not full. We can apply this to the Barlow surface.

$(L_j, L_7 - K)$, $8 \leq j < 11$. Thus, we only need to consider the spaces $\text{Hom}(L_{11}, L_3 - K_S)$ and $\text{Hom}(L_{11}, L_7 - K_S)$ or, to put it differently, we have to check whether the divisors $L_3 - K_S - L_{11}$ and $L_7 - K_S - L_{11}$ are sums of irreducible effective (-2) -curves. Since the first divisor is $\tilde{E}_2 - \tilde{E}_4^+$ and the second is $\tilde{E}_2 - \tilde{E}_5^+$, their intersection with \tilde{C}_1^\pm is 0 and the intersection with \tilde{C}_2^\pm is 1. Hence, they cannot be sums of exceptional curves. Furthermore, one can directly check that, in fact, $\text{Hom}(L_{11}, (L_7 - K_S)[1]) = 0$, so the above argument readily gives the desired statement. \square

Remark 7.2. Notice that our result implies the existence of rational fourfolds whose derived category contains a phantom. Namely, we can embed the Barlow surface S into \mathbb{P}^5 and find a generic projection to \mathbb{P}^4 such that the image $\bar{S} \subset \mathbb{P}^4$ has improper double points as only singularities (which look like two planes meeting transversally in one point locally). Blowing up the double points, we have an embedding $S \subset \tilde{\mathbb{P}}^4$. Now consider the fourfold $Z = \text{Bl}_S(\tilde{\mathbb{P}}^4)$.

Acknowledgments. We would like to thank Denis Auroux, Fedor Bogomolov, Fabrizio Catanese, Igor Dolgachev, Sergey Galkin, Sergey Gorchinskiy, Sergei Gukov, Vladimir Guletskii, Fabian Haiden, Dmitry Kaledin, Alexander Kuznetsov, Maxim Kontsevich, Stefan Müller-Stach, Dmitri Orlov, Miles Reid, Eric Sharpe and Cumrun Vafa for useful discussions, helpful suggestions and remarks. Special thanks to Claire Voisin for making the results of [Voi] available to us, thereby allowing us to render the treatment in Section 6 complete.

Christian Böhning was supported by Heisenberg-Stipendium BO 3699/1-1 of the DFG (German Research Foundation); Ludmil Katzarkov was supported by grants NSF DMS0600800, NSF FRG DMS-0652633, NSF FRG DMS-0854977, NSF DMS-0854977, NSF DMS-0901330, grants FWF P 24572-N25 and FWF P20778 and an ERC grant — GEMIS; Pawel Sosna was supported by the RTG 1670 of the DFG (German Research Foundation).

References

- [A-O12] Alexeev, V., Orlov, O.: Derived categories of Burniat surfaces and exceptional collections. *Math. Ann.* **357**, 743–759 (2013) [Zbl 1282.14030](#) [MR 3096524](#)
- [Barl0] Barlow, R.: Some new surfaces with $p_g = 0$. *Duke Math. J.* **51**, 889–904 (1984) [Zbl 0576.14038](#) [MR 0771386](#)
- [Barl1] Barlow, R.: A simply connected surface of general type with $p_g = 0$. *Invent. Math.* **79**, 293–301 (1985) [Zbl 0561.14015](#) [MR 0778128](#)
- [Barl2] Barlow, R.: Rational equivalence of zero cycles for some more surfaces with $p_g = 0$. *Invent. Math.* **79**, 303–308 (1985) [Zbl 0584.14002](#) [MR 0778129](#)
- [BPV] Barth, W., Hulek, K., Peters, C., van de Ven, A.: *Compact Complex Surfaces*. 2nd ed., *Ergeb. Math. Grenzgeb.* 4, Springer, Berlin (2004) [Zbl 1036.14016](#) [MR 2030225](#)
- [BBS12] Böhning, Chr., Graf von Bothmer, H.-Chr., Sosna, P.: On the derived category of the classical Godeaux surface. *Adv. Math.* **243**, 203–231 (2013) [Zbl 1299.14015](#) [MR 3062745](#)
- [BBKS12] Böhning, Chr., Graf von Bothmer, H.-Chr., Katzarkov, L., Sosna, P.: Macaulay2 scripts for determinantal Barlow surfaces and phantom categories. <http://www.math.uni-hamburg.de/home/boehning/research/BarlowM2/M2scripts>
- [BonOrl] Bondal, A., Orlov, D.: Reconstruction of a variety from the derived category and groups of autoequivalences. *Compos. Math.* **125**, 327–344 (2001) [Zbl 0994.18007](#) [MR 1818984](#)

- [Bour] Bourbaki, N.: Elements of Mathematics. Lie Groups and Lie Algebras. Chapters 4–6. Springer, Berlin (2002) [Zbl 0983.17001](#) [MR 1890629](#)
- [BM] Bridgeland, T., Maciocia, A.: Complex surfaces with equivalent derived categories. *Math. Z.* **236**, 677–697 (2001) [Zbl 1081.14023](#) [MR 1827500](#)
- [Cat81] Catanese, F.: Babbage’s conjecture, contact of surfaces, symmetric determinantal varieties and applications. *Invent. Math.* **63**, 433–465 (1981) [Zbl 0472.14024](#) [MR 0620679](#)
- [CKP] Cheltsov, I., Katzarkov, L., Przyjalkowski, V.: Birational geometry via moduli spaces. In: *Birational Geometry, Rational Curves, and Arithmetic*, Springer, New York, 93–132 (2013) [Zbl 1302.14035](#) [MR 3114925](#)
- [DKK] Diemer, C., Katzarkov, L., Kerr, G.: Compactifications of spaces of Landau–Ginzburg models. *Izv. Ross. Akad. Nauk Ser. Mat.* **77**, no. 3, 55–76 (2013) (in Russian); English transl.: *Izv. Math.* **77**, 487–508 (2013) [Zbl 1277.14015](#) [MR 3098787](#)
- [Ful] Fulton, W.: *Intersection Theory*. 2nd ed., *Ergeb. Math. Grenzgeb.* 2, Springer, Berlin (1998) [Zbl 0885.14002](#) [MR 1644323](#)
- [GS] Galkin, S., Shinder, E.: Exceptional collections of line bundles on the Beauville surface. *Adv. Math.* **244**, 1033–1050 (2013) [Zbl 06264359](#) [MR 3077896](#)
- [GorOrl] Gorchinskiy, S., Orlov, D.: Geometric phantom categories. *Publ. Math. Inst. Hautes Études Sci.* **117**, 329–349 (2013) [Zbl 1285.14018](#) [MR 3090263](#)
- [Humph] Humphreys, J. E.: *Introduction to Lie Algebras and Representation Theory*. *Grad. Texts in Math.* 9, Springer, New York (1972) [Zbl 0254.17004](#) [MR 0323842](#)
- [KarNog] Karpov, B. V., Nogin, D. Yu.: Three-block exceptional sets on del Pezzo surfaces. *Izv. Ross. Akad. Nauk Ser. Mat.* **62**, no. 3, 3–38 (1998) (in Russian); English transl.: *Izv. Math.* **62**, 429–463 (1998) [Zbl 0949.14026](#) [MR 1642152](#)
- [Keller01] Keller, B.: Introduction to A-infinity algebras and modules. *Homology Homotopy Appl.* **3**, 1–35 (2001) [Zbl 0989.18009](#) [MR 1854636](#)
- [Kuz09] Kuznetsov, A.: Hochschild homology and semiorthogonal decompositions. [arXiv:0904.4330v1](#) [math.AG] (2009)
- [Kuz] Kuznetsov, A.: Derived categories and rationality of cubic fourfolds. <http://www.mi.ras.ru/~akuznet/5ecm.pdf>
- [Kuz12] Kuznetsov, A.: Height of exceptional collections and Hochschild cohomology of quasiphantom categories. *J. Reine Angew. Math.*, to appear; [arXiv:1211.4693v2](#) [math.AG]
- [Lee] Lee, K.-S.: Derived categories of surfaces isogenous to a higher product. [arXiv:1303.0541](#) [math.AG] (2013)
- [Lee00] Lee, Y.: Bicanonical pencil of a determinantal Barlow surface. *Trans. Amer. Math. Soc.* **353**, 893–905 (2001) [Zbl 0967.14027](#) [MR 1707700](#)
- [Lee01] Lee, Y.: Interpretation of the deformation space of a determinantal Barlow surface via smoothings. *Proc. Amer. Math. Soc.* **130**, 963–969 (2002) [Zbl 0996.14016](#) [MR 1873768](#)
- [MT] Malle, G., Testerman, D.: *Linear Algebraic Groups and Finite Groups of Lie Type*. *Cambridge Stud. Adv. Math.* 133, Cambridge Univ. Press, Cambridge (2011) [Zbl 1256.20045](#) [MR 2850737](#)
- [Ok] Okawa, S.: Semi-orthogonal decomposability of the derived category of a curve. *Adv. Math.* **228**, 2869–2873 (2011) [Zbl 1230.14020](#) [MR 2838062](#)
- [Orl93] Orlov, D.: Projective bundles, monoidal transformations and derived categories of coherent sheaves. *Izv. Akad. Nauk SSSR Ser. Mat.* **56**, 852–862 (1992) (in Russian); English transl.: *Math. USSR Izv.* **38**, 133–141 (1993) [Zbl 0798.14007](#) [MR 1208153](#)

- [Reid81] Reid, M.: A simply connected surface of general type with $p_g = 0$, $K^2 = 1$, due to Rebecca Barlow. Warwick preprint, <http://homepages.warwick.ac.uk/~masda/surf/more/Barlow.pdf> (1981)
- [Reid91] Reid, M.: Campedelli versus Godeaux. In: Problems in the Theory of Surfaces and their Classification (Cortona, 1988), Sympos. Math. 32, Academic Press, London, 309–365 (1991) [Zbl 0878.14029](#) [MR 1273384](#)
- [Seid] Seidel, P.: Fukaya categories and Picard–Lefschetz theory. Zurich Lecture Notes in Adv. Math., Eur. Math. Soc. (2008) [Zbl 1159.53001](#) [MR 2441780](#)
- [Voi] Voisin, C.: Bloch’s conjecture for Catanese and Barlow surfaces. J. Differential Geom. **97**, 149–175 (2014) [Zbl 06322514](#) [MR 3229054](#)