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Almost sure well-posedness for the periodic 3D quintic nonlinear Schrödinger equation below the energy space

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Abstract. We prove an almost sure local well-posedness result for the periodic 3D quintic nonlinear Schrödinger equation in the supercritical regime, that is, below the critical space $H^1(\mathbb{T}^3)$.

We also prove a long time existence result; more precisely, we show that for fixed $T > 0$ there exists a set Σ_T with $\mathbb{P}(\Sigma_T) > 0$ such that any data $\phi^\omega(x) \in H^\gamma(\mathbb{T}^3)$, $\gamma < 1$, $\omega \in \Sigma_T$, evolves up to time T into a solution $u(t)$ with $u(t) - e^{it\Delta}\phi^\omega \in C([0,T]; H^s(\mathbb{T}^3))$, $s = s(\gamma) > 1$. In particular we find a nontrivial set of data which gives rise to long time solutions below the critical space $H^1(\mathbb{T}^3)$, that is, in the supercritical scaling regime.

Keywords. Supercritical nonlinear Schrödinger equation, almost sure well-posedness, random data

1. Introduction

In this paper we continue the study of almost sure well-posedness for certain dispersive equations in a supercritical regime. In the last two decades there has been a burst of activity and significant progress in the field of nonlinear dispersive equations and systems. These range from the development of analytic tools in nonlinear Fourier and harmonic analysis combined with geometric ideas to address nonlinear estimates, to related deep functional-analytic methods and profile decompositions to study local and global well-posedness and singularity formation for these equations and systems. The thrust of this body of work has focused mostly on deterministic aspects of wave phenomena where sophisticated tools from nonlinear Fourier analysis, geometry and analytic number theory have played a crucial role in the methods employed. Building upon work by Bourgain [1, 2, 4] several works have appeared in which the well-posedness theory has been studied from a nondeterministic point of view relying on and adapting probabilistic ideas and tools as well (cf. [13, 14, 34, 28, 29, 35, 25, 27, 32, 17, 26, 12, 18, 19] and references therein).

It is by now well understood that randomness plays a fundamental role in a variety of fields. Situations when such a point of view is desirable include: when there

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still remains a gap between local and global well-posedness, when a certain type of ill-posedness is present, and in the very important supercritical regime when a deterministic well-posedness theory remains, in general, a major open problem in the field. A set of important and tractable problems is concerned with those (scaling) equations for which global well-posedness for *large data* is known at the *critical scaling level*. Of special interest is the case when the scale-invariant regularity s_c equals 1 (corresponding to the energy or the Hamiltonian functional). A natural question then is to understand the *supercritical* (relative to scaling) long time dynamics for the nonlinear Schrödinger equation in the defocusing case. Whether blow up occurs from classical data in the defocusing case remains a difficult open problem. However, what seems within reach at this time is to investigate these problems *from a nondeterministic viewpoint*, namely for *random data*.

In this paper we treat the energy-critical periodic quintic nonlinear Schrödinger equation (NLS), an especially important prototype in view of the results by Herr, Tzvetkov and Tataru [23] establishing small data global well-posedness in $H^1(\mathbb{T}^3)$, and of Ionescu and Pausader [24] proving *large data* global well-posedness in $H^1(\mathbb{T}^3)$ in the defocusing case, the first critical result for NLS on a compact manifold. Large data global well-posedness in $\mathbb{R}^3$ for the energy-critical quintic NLS had been previously established by Colliander, Keel, Staffilani, Takaoka and Tao [16].

Our main interest in this paper is to establish a local almost sure well-posedness for random data *below* $H^1(\mathbb{T}^3)$, that is, in the supercritical regime relative to scaling[1] and without any kind of symmetry restriction on the data. In a seminal paper, Bourgain [4] obtained almost sure global well-posedness for the 2D periodic defocusing (Wick ordered) cubic NLS with data *below* $L^2(\mathbb{T}^2)$, i.e. in a supercritical regime ($s_c = 0$).[2] Burq and Tzvetkov obtained similar results for the supercritical ($s_c = 1/2$) *radial* cubic NLW on compact Riemannian manifolds in 3D. Both global results rely on the existence and invariance of associated Gibbs measures. As it turns out, in many situations either the natural Gibbs or weighted Wiener construction does not produce an invariant measure, or (and this is particularly so in higher dimensions) a canonical construction is not expected. In the case of the ball or the sphere and if one restricts to the radial case, constructions of invariant measures are possible, as in [35, 14, 20, 21, 6, 7, 8]. Recently, a probabilistic method based on energy estimates has been used to obtain supercritical almost sure global results, thus circumventing the use of invariant measures and the restriction of radial symmetry. In this context Burq and Tzvetkov [15] and Burq, Thomann and Tzvetkov [11] considered the periodic cubic NLW, while Nahmod, Pavlović and Staffilani [26] treated the periodic Navier–Stokes equations. Colliander and Oh [17] also proved almost sure global well-posedness for the *subcritical* 1D periodic cubic NLS below L^2 in the absence of invariant measures by suitably adapting Bourgain's high-low method.

Extending the local solutions we obtain here to global ones is the next natural step; it is worth noting however that unlike the work of Bourgain [4] one would need to proceed in the absence of invariant measures; and unlike the work of Colliander and Oh [17] the

[1] That is, for Cauchy data in $H^s(\mathbb{T}^3)$, $s < s_c = 1$ for the quintic NLS in 3D.

[2] See Brydges and Slade [9] for a study of invariant measures associated to the 2D *focusing* cubic NLS.

smoother norm in our case, namely $H^1(\mathbb{T}^3)$, on which one would need to rest to extend the local theory to a global one is in fact *critical*. This forces the bounds on the Strichartz type norms to be of exponential type with respect to the energy, too large to close the argument. Nonetheless, as a byproduct of our local estimates we can show the existence of large data long time infinite energy solutions (see Section 10).

The problem we are considering here is the analogue of the supercritical local well-posedness result[3] proved by Bourgain in [4] for the periodic mass critical defocusing cubic NLS in 2D. Of course, Bourgain then constructed a 2D Gibbs measure and proved that for data in its statistical ensemble the local solutions obtained were in fact global, hence establishing almost sure global well-posedness in $H^{-\epsilon}(\mathbb{T}^2)$, $\epsilon > 0$.

There are several major complications in the work that we present below compared to the work of Bourgain: a quintic nonlinearity increases quite substantially the different cases that need to be treated when one analyzes the frequency interactions in the nonlinearity; the counting lemmata in a 3D lattice are much less favorable and the Wick ordering is not sufficient to remove certain resonant frequencies that need to be eliminated. The latter is not surprising, and in fact known within the context of quantum field renormalization (cf. Salmhofer's book [30]). In particular, to overcome this last obstacle, we introduce an appropriate gauge transformation, we work on the gauged problem and then transfer the result obtained back to the original problem; which as a consequence is studied through a contraction method applied in a certain metric space of functions. A similar approach was used by the second author in [31].

Finally, our estimates take place in the atomic function spaces where the only *deterministic global well-posedness* is known to hold at the H^1-critical level [23, 24]. Such choice of function spaces over the $X^{s,b}$ spaces[4] in [4] is natural given our result in this paper lays the foundation for an *almost sure global well-posedness* in the supercritical regime. In turn, such choice presupposes some care while working with the absolute value of the Fourier transform and various constraint equations of multilinear terms. This is because the norms of these atomic spaces are not defined through the absolute value of the Fourier transform, a property which is quite useful while working with the $X^{s,b}$ spaces; see Section 8.

In this work we consider the Cauchy initial value problem,

$$\begin{cases} iu_t + \Delta u = \lambda u|u|^4, \quad x \in \mathbb{T}^3, \\ u(0,x) = \phi(x), \end{cases} \tag{1.1}$$

where $\lambda = \pm 1$.

We randomize the data in the following sense:

Definition 1.1. Let $(g_n(\omega))_{n\in\mathbb{Z}^3}$ be a sequence of complex i.i.d. centered Gaussian random variables on a probability space $(\Omega, A, \mathbb{P})$. For $\phi \in H^s(\mathbb{T}^3)$, let (b_n) be its Fourier coefficients, that is,

$$\phi(x) = \sum_{n\in\mathbb{Z}^3} b_n e^{in\cdot x}, \qquad \sum_{n\in\mathbb{Z}^3} (1+|n|)^{2s}|b_n|^2 < \infty. \tag{1.2}$$

3 A.s. for data in $H^{-\beta}(\mathbb{T}^2)$, $\beta > 0$.

4 Our argument could also be carried out in the $X^{s,b}$ spaces.

The map from (Ω, A) to $H^s(\mathbb{T}^3)$ equipped with the Borel sigma algebra, defined by

$$\omega \mapsto \phi^\omega, \qquad \phi^\omega(x) := \sum_{n\in\mathbb{Z}^3} g_n(\omega) b_n e^{in\cdot x}, \tag{1.3}$$

is called a *map randomization*.

Remark 1.2. The map (1.3) is measurable and $\phi^\omega \in L^2(\Omega; H^s(\mathbb{T}^d))$ is an $H^s(\mathbb{T}^d)$-valued random variable. We recall that such a randomization does not introduce any H^s regularization (see [13, Lemma B.1] for a proof), indeed $\|\phi^\omega\|_{H^s} \sim \|\phi\|_{H^s}$. However, randomization gives improved L^p estimates almost surely.

Our setting to show almost sure local well-posedness is similar to that of Bourgain [4]. More precisely, we consider data $\phi \in H^{1-\alpha-\varepsilon}(\mathbb{T}^3)$ for any $\varepsilon > 0$ of the form

$$\phi(x) = \sum_{n\in\mathbb{Z}^3} \frac{1}{\langle n\rangle^{5/2-\alpha}} e^{in\cdot x} \tag{1.4}$$

whose randomization is

$$\phi^\omega(x) = \sum_{n\in\mathbb{Z}^3} \frac{g_n(\omega)}{\langle n\rangle^{5/2-\alpha}} e^{in\cdot x}. \tag{1.5}$$

Our main result can then be stated as follows:

Main Theorem 1.3 (Main Theorem). *Let* $0 < \alpha < 1/12$, $s \in (1+4\alpha, 3/2-2\alpha)$ *and* ϕ *as in* (1.4)*. Then there exist* $0 < \delta_0 \ll 1$ *and* $r = r(s,\alpha) > 0$ *such that for any* $\delta < \delta_0$ *there exists* $\Omega_\delta \in A$ *with*

$$\mathbb{P}(\Omega_\delta^c) < e^{-1/\delta^r}$$

such that for each $\omega \in \Omega_\delta$ *there exists a unique solution* u *of* (1.1) *in the space*

$$S(t)\phi^\omega + X^s([0,\delta))_d$$

where $S(t)\phi^\omega$ *is the linear evolution of the initial data* ϕ^ω *given by* (1.5).

Here we have denoted by $X^s([0,\delta))_d$ the metric space $(X^s([0,\delta)), d)$ where d is the metric defined by (2.21) in Section 2 and $X^s([0,\delta))$ is the space introduced in Definition 4.4 below.

The paper is organized as follows. In Section 2 we identify the problematic resonant terms and present an equivalent gauged Cauchy initial value problem where such terms are removed. Section 3 states the basic probabilistic results we rely upon. In Section 4 we first recall the atomic function spaces needed for the proof as well as their functional properties. Then we also prove two multilinear propositions which play a significant role in later sections. Section 5 contains statements on almost sure local well-posedness for the gauged Cauchy initial value problem, while in Section 6 we collect a few counting estimates and auxiliary lemmata. In Section 7 we prove all the trilinear and bilinear estimates needed for estimating certain nonlinear terms. Section 8 contains the main argument of the proof. In this section we prove the necessary quintilinear estimates for the top term in the nonlinearity and use the trilinear and bilinear estimates of Section 7 to control corresponding lower order nonlinear terms. Finally, in Section 9 we show how to extract a positive power of time from our estimates, which in turn allows us to close the argument via a contraction mapping principle and obtain our almost sure local well-posedness result.

2. Removing resonant frequencies: the gauged equation

The main idea in proving Theorem 1.3 goes back to Bourgain [4] and consists in proving that if u solves (1.1), then $w = u - S(t)\phi^{\omega}$ is smoother; see also [13, 17, 26]. In fact one reduces the problem to showing well-posedness for the initial value problem involving w, which is in fact treated as a deterministic function. The initial value problem that w solves does not become a subcritical one, but it is of a hybrid type involving also rougher but random terms, whose decay and moments play a fundamental role. For the NLS equation this argument can be carried out only after having removed certain resonant frequencies in the nonlinear part of the equation. In this section in fact we write the Fourier coefficients of the quintic expression $|u|^4u$ and we identify the resonant part that has to be removed if we want to take advantage of the moments coming from the randomized terms. We will go back to this concept in more detail in Remark 2.1 below.

Let us start by assuming that $\widehat{u}(n)(t) = a_n(t)$. We introduce the notation

$$\Gamma(n)_{[i_1,\ldots,i_r]} := \{(n_{i_1},\ldots,n_{i_r}) \in \mathbb{Z}^{3r} : n = n_{i_1} - n_{i_2} + \cdots + (-1)^{r+1} n_{i_r}\} \tag{2.1}$$

to indicate various hyperplanes, and $\Gamma(n)^c_{[i_1,\ldots,i_r]}$ is its complement.

Next, for fixed time t, we take $\mathcal{F}$, the Fourier transform in space, and write

$$\begin{aligned}
\mathcal{F}(|u(t)|^4u(t))(n) &= \sum_{\Gamma(n)_{[1,\ldots,5]}} a_{n_1}(t)\overline{a}_{n_2}(t)a_{n_3}(t)\overline{a}_{n_4}(t)a_{n_5}(t)\\
&= \sum_{\Gamma(n)_{[1,\ldots,5]}\cap\Gamma(0)^c_{[1,2,3,4]}\cap\Gamma(0)^c_{[1,2,5,4]}\cap\Gamma(0)^c_{[3,2,5,4]}} a_{n_1}(t)\overline{a}_{n_2}(t)a_{n_3}(t)\overline{a}_{n_4}(t)a_{n_5}(t)\\
&\quad+ \sum_{\Gamma(n)_{[1,\ldots,5]}\cap\Gamma(0)_{[1,2,3,4]}} a_{n_1}(t)\overline{a}_{n_2}(t)a_{n_3}(t)\overline{a}_{n_4}(t)a_{n_5}(t)\\
&\quad+ \sum_{\Gamma(n)_{[1,\ldots,5]}\cap\Gamma(0)_{[1,2,5,4]}} a_{n_1}(t)\overline{a}_{n_2}(t)a_{n_3}(t)\overline{a}_{n_4}(t)a_{n_5}(t)\\
&\quad+ \sum_{\Gamma(n)_{[1,\ldots,5]}\cap\Gamma(0)_{[3,2,5,4]}} a_{n_1}(t)\overline{a}_{n_2}(t)a_{n_3}(t)\overline{a}_{n_4}(t)a_{n_5}(t)\\
&\quad- \sum_{\Gamma(n)_{[1,\ldots,5]}\cap\Gamma(0)_{[1,2,3,4]}\cap\Gamma(0)_{[1,2,5,4]}\cap\Gamma(0)^c_{[3,2,5,4]}} a_{n_1}(t)\overline{a}_{n_2}(t)a_{n_3}(t)\overline{a}_{n_4}(t)a_{n_5}(t)\\
&\quad- \sum_{\Gamma(n)_{[1,\ldots,5]}\cap\Gamma(0)_{[1,2,3,4]}\cap\Gamma(0)_{[3,2,5,4]}\cap\Gamma(0)^c_{[1,2,5,4]}} a_{n_1}(t)\overline{a}_{n_2}(t)a_{n_3}(t)\overline{a}_{n_4}(t)a_{n_5}(t)\\
&\quad- \sum_{\Gamma(n)_{[1,\ldots,5]}\cap\Gamma(0)_{[3,2,5,4]}\cap\Gamma(0)_{[1,2,5,4]}\cap\Gamma(0)^c_{[1,2,3,4]}} a_{n_1}(t)\overline{a}_{n_2}(t)a_{n_3}(t)\overline{a}_{n_4}(t)a_{n_5}(t)\\
&\quad- 2\sum_{\Gamma(n)_{[1,\ldots,5]}\cap\Gamma(0)_{[1,2,3,4]}\cap\Gamma(0)_{[3,2,5,4]}\cap\Gamma(0)_{[1,2,5,4]}} a_{n_1}(t)\overline{a}_{n_2}(t)a_{n_3}(t)\overline{a}_{n_4}(t)a_{n_5}(t)\\
&= \sum_{k=1}^{8} I_k.
\end{aligned}$$

We now rewrite each I_k using more explicitly the constraints in the hyperplanes. I_1 is the most complicated, and we start by rewriting it. To that end we set

$$\Lambda(n) := \Gamma(n)_{[1,\dots,5]} \cap \Gamma(0)^c_{[1,2,3,4]} \cap \Gamma(0)^c_{[1,2,5,4]} \cap \Gamma(0)^c_{[3,2,5,4]}, \tag{2.2}$$

$$\Sigma(n) := \{(n_1,n_2,n_3,n_4,n_5) \in \Lambda(n) : n_1,n_3,n_5 \neq n_2,n_4\}. \tag{2.3}$$

We have

$$\begin{aligned}
I_1 &= \sum_{\Lambda(n)} a_{n_1}(t)\overline{a}_{n_2}(t)a_{n_3}(t)\overline{a}_{n_4}(t)a_{n_5}(t)\\
&= \sum_{\Sigma(n)} a_{n_1}(t)\overline{a}_{n_2}(t)a_{n_3}(t)\overline{a}_{n_4}(t)a_{n_5}(t)\\
&\quad + 6\Big(\sum_{n_2} |a_{n_2}|^2\Big) \sum_{\Gamma(n)_{[3,4,5]},\, n_3,n_5\neq n_4} a_{n_3}(t)\overline{a}_{n_4}(t)a_{n_5}(t)\\
&\quad - 6|a_n|^2 \sum_{\Gamma(n)_{[3,4,5]},\, n_3,n_5\neq n_4} a_{n_3}(t)\overline{a}_{n_4}(t)a_{n_5}(t)\\
&\quad - 3 \sum_{\Gamma(n)_{[3,1,5]}, n_3,\, n_5\neq n_1} |a_{n_1}(t)|^2\overline{a}_{n_1}(t)a_{n_3}(t)a_{n_5}(t)\\
&\quad - 3|a_n|^4a_n(t) + 3|a_n|^2\overline{a}_n(t)\sum_{n_3+n_5=2n} a_{n_3}(t)a_{n_5}(t)\\
&\quad - 6 \sum_{\Gamma(n)_{[2,4,5]},\, n_2,n_5\neq n_4} |a_{n_2}(t)|^2 a_{n_2}(t)\overline{a}_{n_4}(t)a_{n_5}(t)\\
&\quad + 2 \sum_{n=2n_2-n_4,\, n_2\neq n_4} |a_{n_2}(t)|^2 a^2_{n_2}(t)\overline{a}_{n_4}(t).
\end{aligned} \tag{2.4}$$

Note here that we can write

$$\begin{aligned}
&|a_n|^2 \sum_{\Gamma(n)_{[3,4,5]},\, n_3,n_5\neq n_4} a_{n_3}(t)\overline{a}_{n_4}(t)a_{n_5}(t)\\
&\quad = -2|a_n|^2a_n\Big(\sum_{n_2}|a_{n_2}|^2\Big) + |a_n|^4a_n + |a_n|^2\sum_{\Gamma(n)_{[3,4,5]}} a_{n_3}(t)\overline{a}_{n_4}(t)a_{n_5}(t).
\end{aligned} \tag{2.5}$$

It is easier to see that for $i = 2, 3, 4$,

$$I_i = a_n(t)\sum_{\Gamma(0)_{[1,2,3,4]}} a_{n_1}(t)\overline{a}_{n_2}(t)a_{n_3}(t)\overline{a}_{n_4}(t) = \widehat{u}(n)(t)\int_{\mathbb{T}^3}|u|^4(x,t)\,dx, \tag{2.6}$$

while for $j = 5, 6, 7$,

$$I_j = -a^3_n(t)\sum_{n_2+n_4=2n}\overline{a}_{n_2}(t)\overline{a}_{n_4}(t) + a^2_n\sum_{n=n_2+n_4-n_1}\overline{a}_{n_2}(t)\overline{a}_{n_4}(t)a_{n_1}(t), \tag{2.7}$$

and finally

$$I_8 = -2a^3_n(t)\sum_{n_2+n_4=2n}\overline{a}_{n_2}(t)\overline{a}_{n_4}(t). \tag{2.8}$$

We summarize our findings from (2.4)–(2.8). In this part of the argument the time variable is not important, hence we will omit it for now. We write

$$\mathcal{F}\left(|u|^4 u - 3u\left(\int_{\mathbb{T}^3} |u|^4\, dx\right)\right)(n) = \sum_{k=1}^{7} J_k(n) \tag{2.9}$$

with

$$J_1(n) = \sum_{\Sigma(n)} a_{n_1}\overline{a}_{n_2}a_{n_3}\overline{a}_{n_4}a_{n_5}, \tag{2.10}$$

$$J_2(n) = 6m \sum_{\Gamma(n)_{[1,2,3]},\, n_3,n_1\neq n_2} a_{n_1}\overline{a}_{n_2}a_{n_3}, \tag{2.11}$$

$$\begin{aligned} J_3(n) = &-6 \sum_{\Gamma(n)_{[1,2,3]},\, n_1,n_3\neq n_2} |a_{n_1}|^2 a_{n_1}\overline{a}_{n_2}a_{n_3} \\ &- 3 \sum_{\Gamma(n)_{[1,2,3]},\, n_1,n_3,\neq n_2} a_{n_1}|a_{n_2}|^2\overline{a}_{n_2}a_{n_3}, \end{aligned} \tag{2.12}$$

$$J_4(n) = 2 \sum_{n=2n_1-n_2,} |a_{n_1}|^2 a_{n_1}^2\overline{a}_{n_2}, \tag{2.13}$$

$$J_5(n) = -6|a_n|^2 \sum_{\Gamma(n)_{[123]}} a_{n_1}\overline{a}_{n_2}a_{n_3} + 3a_n^2 \sum_{\Gamma(n)_{[214]}} \overline{a}_{n_2}a_{n_1}\overline{a}_{n_4}, \tag{2.14}$$

$$J_6(n) = -5a_n^3 \sum_{n=n_2+n_4} \overline{a}_{n_2}\overline{a}_{n_4} + 3|a_n|^2\overline{a}_n \sum_{n=n_1+n_3} a_{n_1}a_{n_3}, \tag{2.15}$$

$$J_7(n) = -11a_n|a_n|^4 + 12m|a_n|^2 a_n, \tag{2.16}$$

where $m = \int_{\mathbb{T}^3} |u(t,x)|^2\, dx$, the conserved mass.

Remark 2.1. In the calculations above we wrote the nonlinear terms in (1.1) in Fourier space, we isolated the term $u\int_{\mathbb{T}^3} |u|^4\, dx$ and we subtracted it from $|u|^4 u$ (see (2.9)). We show below that indeed in doing so we separated those terms that we claim are not suitable for smoother estimates from the ones that are. To understand this point let us replace $a_n = g_n(\omega)/\langle n\rangle^{5/2-\alpha}$, for α small, whose anti-Fourier transform barely misses to be in $H^1(\mathbb{T}^3)$. We want to claim that the randomness coming from $\{g_n(\omega)\}$ will increase the regularity of the nonlinearity in a certain sense, so that it can hold a bit more than one derivative. We realize immediately though that this claim cannot be true for the whole nonlinear term. For example the terms I_i, $i = 2, 3, 4$, have no chance to improve their regularity because they are simply linear with respect to a_n, hence they have to be removed. This same problem presented itself in the work of Bourgain [4] and Colliander–Oh [17] who considered the cubic NLS below L^2. In particular in their case the problematic term was of the type $a_n \int_{\mathbb{T}^d} |u|^2\, dx$ and the authors removed it by Wick ordering the Hamiltonian. An important ingredient in succeeding in this was that $\int_{\mathbb{T}^d} |u|^2\, dx$, that is, the mass, is independent of time. In our case, Wick ordering the Hamiltonian is not helpful since it does not remove the terms I_i, $i = 2, 3, 4$. As we mentioned before, the latter is not surprising, and in fact known within the context of quantum field renormalization (cf. Salmhofer's book [30]).

If we knew that $\int_{\mathbb{T}^3} |u|^4\,dx$ were constant in time, then we could simply relegate those terms to the linear part of the equation. Since this is obviously not the case, relegating these expressions to the main linear part of the equation would prevent us from using the simple form of the solution for a Schrödinger equation with constant coefficients. A similar situation to the one just described presented itself in [31] where a gauge transformation was used to remove the time dependent linear terms. We are able to use the same idea in the present context and this is the content of what follows in this section.

To prove Main Theorem 1.1 we proceed in two steps. First we consider the initial value problem

$$\begin{cases} iv_t + \Delta v = \mathcal{N}(v), & x \in \mathbb{T}^3, \\ v(0,x) = \phi(x), \end{cases} \tag{2.17}$$

where

$$\mathcal{N}(v) := \lambda\left(v|v|^4 - 3v\left(\int_{\mathbb{T}^3} |v|^4\,dx\right)\right) \tag{2.18}$$

with $\lambda = \pm 1$ and $\phi(x)$ the initial datum as in (1.1). To make the notation simpler set

$$\beta_v(t) := 3\int_{\mathbb{T}^3} |v|^4\,dx \tag{2.19}$$

and define

$$u(t,x) := e^{i\lambda \int_0^t \beta_v(s)\,ds} v(t,x). \tag{2.20}$$

We observe that u solves the initial value problem (1.1). Now suppose that one obtains well-posedness for the initial value problem (2.17) in a certain Banach space $(X, \|\cdot\|)$; then one can transfer those results to the initial value problem (1.1) by using a metric space $X_d := (X, d)$ where

$$d(u,v) := \|e^{-i\lambda \int_0^t \beta_u(s)\,ds} u(t,x) - e^{-i\lambda \int_0^t \beta_v(s)\,ds} v(t,x)\|. \tag{2.21}$$

The fact that this is indeed a metric follows from the properties of the norm $\|\cdot\|$ and the fact that if

$$e^{-i\lambda \int_0^t \beta_u(s)\,ds} u(t,x) = e^{-i\lambda \int_0^t \beta_v(s)\,ds} v(t,x)$$

then $\beta_v(t) = \beta_u(t)$ and hence $u = v$.

From this moment on, we work exclusively with the initial value problem (2.17). In particular, below we prove the following result:

Theorem 2.1. *Let* $0 < \alpha < 1/12$, $s \in (1+4\alpha, 3/2 - 2\alpha)$ *and* ϕ *as in* (1.4). *There exist* $0 < \delta_0 \ll 1$ *and* $r = r(s,\alpha) > 0$ *such that for any* $\delta < \delta_0$ *there exists* $\Omega_\delta \in A$ *with*

$$\mathbb{P}(\Omega_\delta^c) < e^{-1/\delta^r}$$

such that for each $\omega \in \Omega_\delta$ *there exists a unique solution* u *of* (2.17) *in the space*

$$S(t)\phi^\omega + X^s([0,\delta))$$

with initial condition ϕ^ω *given by* (1.5).

Here the space $X^s([0,\delta))$ is defined in Section 4.

Thanks to the transformation (2.20), Theorem 2.1 translates to Main Theorem 1.3.

3. Probabilistic set up

We first recall a classical result that goes back to Kolmogorov, Paley and Zygmund.

Lemma 3.1 ([13, Lemma 3.1]). *Let $\{g_n(\omega)\}$ be a sequence of complex i.i.d. zero mean Gaussian random variables on a probability space $(\Omega, A, \mathbb{P})$, and $(c_n) \in \ell^2$. Define*

$$F(\omega) := \sum_n c_n g_n(\omega). \tag{3.1}$$

Then there exists $C > 0$ such that for every $\lambda > 0$ we have

$$\mathbb{P}(\{\omega : |F(\omega)| > \lambda\}) \le \exp\left(\frac{-C\lambda^2}{\|F\|^2_{L^2(\Omega)}}\right). \tag{3.2}$$

As a consequence there exists $C > 0$ such that for every $q \ge 2$ and every $(c_n) \in \ell^2$,

$$\Big\|\sum_n c_n g_n\Big\|_{L^q(\Omega)} \le C\sqrt{q}\Big(\sum_n |c_n|^2\Big)^{1/2}.$$

We also recall the following basic probability results:

Lemma 3.2. *Let $1 \le m_1 < \cdots < m_k = m$ and let f_1 be a Borel measurable function of m_1 variables, f_2 one of $m_2 - m_1$ variables, $\dots$, f_k one of $m_k - m_{k-1}$ variables. If $\{X_1, \dots, X_m\}$ are real-valued independent random variables, then the k random variables $f_1(X_1, \dots, X_{m_1})$, $f_2(X_{m_1+1}, \dots, X_{m_2})$, $\dots$, $f_k(X_{m_{k-1}+1}, \dots, X_{m_k})$ are independent random variables.*

Lemma 3.3. *Let $k \ge 1$ and let $\{g_{n_j}\}_{1\le j\le k}, \{g_{n'_j}\}_{1\le j\le k} \in \mathcal{N}_{\mathbb{C}}(0,1)$ be complex $L^2(\Omega)$ normalized independent Gaussian random variables such that $n_i \ne n_j$ and $n'_i \ne n'_j$ for $i \ne j$. Then*

$$\left|\int_\Omega \prod_{j=1}^k g_{n_j}(\omega) \prod_{i=1}^k \overline{g}_{n'_i}(\omega)\, d\mathbb{P}(\omega)\right| \le \int_\Omega \prod_{\ell=1}^k |g_{n_\ell}(\omega)|^2\, d\mathbb{P}(\omega).$$

Proof. For every pair (n_ℓ, n'_i) such that $n_\ell = n'_i$ we write $K_{n_j}(\omega) := |g_{n_j}(\omega)|^2$ and note that thanks to the independence and normalization of $\{g_{n_j}\}$, for $n_j \ne n_i$, we have $\mathbb{E}(K_{n_j} g_{n_i}) = 0$. The latter together with Lemma 3.2 gives the desired conclusion. □

More generally, in the next sections we will repeatedly use a classical Fernique or large deviation-type result related to the product of $\{G_n\}_{1\le n\le d} \in \mathcal{N}_{\mathbb{C}}(0,1)$, complex L^2 normalized independent Gaussians. This result follows from the hypercontractivity property of the Ornstein–Uhlenbeck semigroup (cf. [35, 33] for a nice exposition) by writing $G_n = H_n + iL_n$ where $\{H_1, \dots, H_d, L_1, \dots, L_d\} \in \mathcal{N}_{\mathbb{R}}(0,1)$ are real centered independent Gaussian random variables with the same variance. Note that $\mathbb{E}(G_n^2) = \mathbb{E}(G_n) = 0$.

Remark 3.1. Note that for $\{G_n(\omega)\}_n \in \mathcal{N}_{\mathbb{C}}(0,1)$, complex L^2 normalized independent Gaussians, if we write $|G_n(\omega)|^2 = (|G_n(\omega)|^2 - 1) + 1$, then thanks to the independence and normalization of G_n, $Y_n(\omega) := |G_n(\omega)|^2 - 1$ is a centered real Gaussian random variable such that $\mathbb{E}(Y_n G_{n'}) = 0 = \mathbb{E}(Y_n Y_{n'})$ for $n \ne n'$.

Proposition 3.1 ([33, Proposition 2.4] and [35, Lemma 4.5]). *Let $d \geq 1$ and $c(n_1, \dots, n_k) \in \mathbb{C}$. Let $\{G_n\}_{1 \leq n \leq d} \in \mathcal{N}_{\mathbb{C}}(0, 1)$ be complex centered L^2 normalized independent Gaussians. For $k \geq 1$ denote $A(k, d) := \{(n_1, \dots, n_k) \in \{1, \dots, d\}^k : n_1 \leq \dots \leq n_k\}$ and*

$$F_k(\omega) := \sum_{A(k,d)} c(n_1, \dots, n_k) G_{n_1}(\omega) \dots G_{n_k}(\omega). \tag{3.3}$$

Then for all $d \geq 1$ and $p \geq 2$,

$$\|F_k\|_{L^p(\Omega)} \lesssim \sqrt{k+1}\,(p-1)^{k/2} \|F_k\|_{L^2(\Omega)}.$$

As a consequence, from Chebyshev's inequality we have, for every $\lambda > 0$,

$$\mathbb{P}(\{\omega : |F_k(\omega)| > \lambda\}) \lesssim \exp\left(\frac{-C\lambda^{2/k}}{\|F_k\|_{L^2(\Omega)}^{2/k}}\right). \tag{3.4}$$

Remark 3.2. In Sections 7 and 8 we will rely repeatedly on Proposition 3.1, particularly (3.4), as well as on Lemma 3.1, and (3.2). Indeed, in proving our estimates we will encounter expressions of the following type. Let

$$\Sigma := \{(n_1, \dots, n_r, \ell_1, \dots, \ell_s) : |n_j| \sim N_j,\ |\ell_i| \sim L_i,\ n_j \neq \ell_i,\ 1 \leq j \leq r,\ 1 \leq i \leq s\}$$

and

$$F(\omega) := \sum_{(n_1, \dots, n_r, \ell_1, \dots, \ell_s) \in \Sigma} c_{n_1} \dots c_{n_r} b_{\ell_1} \dots b_{\ell_s} g_{n_1}(\omega) \dots g_{n_r}(\omega) \overline{g}_{\ell_1}(\omega) \dots \overline{g}_{\ell_s}(\omega)$$

where $\{g_{n_1}(\omega) \dots g_{n_r}(\omega), g_{\ell_1}(\omega) \dots g_{\ell_s}(\omega)\} \in \mathcal{N}_{\mathbb{C}}(0, 1)$ are complex centered L^2 normalized independent Gaussians. Then by Proposition 3.1, there exist $C, \gamma = \gamma(r, s) > 0$ such that for every $\lambda > 0$ we have

$$\mathbb{P}(\{\omega : |F(\omega)| > \lambda\}) \leq \exp\left(\frac{-C\lambda^{2/\gamma}}{\|F\|_{L^2(\Omega)}^{2/\gamma}}\right).$$

We will also apply Proposition 3.1 in the context of Remark 3.1.

Lemma 3.4. *Let $\{g_n(\omega)\}$ be a sequence of complex i.i.d. zero mean Gaussian random variables on a probability space $(\Omega, A, \mathbb{P})$. Then:*

(1) *For $1 \leq p < \infty$ there exists $c_p > 0$ (independent of n) such that $\|g_n\|_{L^p(\Omega)} \leq c_p$.*
(2) *Given $\varepsilon, \delta > 0$, for ω outside a set of measure $O(\delta)$,*

$$|g_n(\omega)| \lesssim \langle n \rangle^{\varepsilon}. \tag{3.5}$$

Proof. Part (1) follows from the fact that higher moments of $\{g_n(\omega)\}$ are uniformly bounded.

For part (2) first recall that if $\{X_j(\omega)\}_{j \geq 1}$ is a sequence of i.i.d. random variables such that $\mathbb{E}(|X_j|) = E < \infty$ then

$$\mathbb{P}(|X_j| \geq j) = \mathbb{P}(|X_1| \geq j) \tag{3.6}$$

and

$$\sum_j \mathbb{P}(|X_j| \geq j) = \sum_j \mathbb{P}(|X_1| \geq j) \leq \mathbb{E}(|X_1|) < \infty.$$

By Borel–Cantelli $\mathbb{P}(|X_j| \geq j$ for infinitely many $j) = 0$, whence one can show that $\lim_{j\to\infty} |X_j(\omega)|/j = 0$ almost surely in ω. Egoroff's Theorem then ensures that given $\delta > 0$,

$$\lim_{j\to\infty} \frac{|X_j(\omega)|}{j} = 0$$

uniformly outside a set of measure δ. Thus for j_0 sufficiently large,

$$\frac{|X_j(\omega)|}{j} \leq 1, \qquad j \geq j_0,$$

for ω outside an exceptional set of δ measure. If $\{g_n(\omega)\}$ is a sequence of i.i.d. complex Gaussian random variables given $\varepsilon > 0$, if we choose $r = 3/\varepsilon$ then $\mathbb{E}(|g_n|^r) < \infty$. For $n \mapsto j_n$ a one-to-one map $\mathbb{Z}^3 \to \mathbb{N}$ such that $j_n \sim |n|^3$, we let $X_{j_n}(\omega) := |g_n(\omega)|^r$ and reason as above. Note also that for $M \gg 1$ but fixed,

$$\mathbb{P}(|g_n(\omega)| \geq M^\varepsilon) = \mathbb{P}(|g_M(\omega)| \geq M^\varepsilon)$$

for all $|n| \leq M$; whence for $\mathcal{A} := \bigcup_{|n|\leq M-1}\{\omega : |g_M(\omega)| \geq M^\varepsilon\}$, we have $\mathbb{P}(\mathcal{A}) \leq C_M\delta$. We then have the desired conclusion (cf. [28, 17]). □

4. Function spaces

To establish our almost sure local well-posedness result, it suffices to work with X^s and Y^s, the atomic function spaces used by Herr, Tataru and Tzvetkov [23]. It is worth emphasizing that while working with these spaces, one should not rely on the notion of the norms depending on the absolute value of the Fourier transform, a feature that is quite useful when working within the context of $X^{s,b}$ spaces.

In this section we recall their definition and summarize the main properties by following the presentation in [23, Section 2]. In what follows, $\mathcal{H}$ is a separable Hilbert space over $\mathbb{C}$, and $\mathcal{Z}$ denotes the set of finite partitions $-\infty < t_0 < t_1 < \cdots < t_K \leq \infty$ of the real line, with the convention that if $t_k = \infty$ then $v(t_K) := 0$ for any function $v : \mathbb{R} \to \mathcal{H}$.

Definition 4.1 ([23, Definition 2.1]). Let $1 \leq p < \infty$. For $\{t_k\}_{k=0}^K \in \mathcal{Z}$ and $\{\phi_k\}_{k=0}^{K-1} \subset \mathcal{H}$ with $\sum_{k=0}^{K-1} \|\phi_k\|_{\mathcal{H}}^p = 1$, a *$U^p$-atom* is a piecewise defined function $a : \mathbb{R} \to \mathcal{H}$ of the form

$$a = \sum_{k=1}^K \chi_{[t_{k-1},t_k)}\phi_{k-1}.$$

The atomic Banach space $U^p(\mathbb{R}, \mathcal{H})$ is then defined to be the set of all functions $u : \mathbb{R} \to \mathcal{H}$ such that

$$u = \sum_{j=1}^\infty \lambda_j a_j \qquad \text{for } U^p\text{-atoms } a_j,\ \{\lambda_j\}_j \in \ell^1,$$

with the norm

$$\|u\|_{U^p} := \inf\Big\{\sum_{j=1}^{\infty} |\lambda_j| : u = \sum_{j=1}^{\infty} \lambda_j a_j,\ \lambda_j \in \mathbb{C},\ \text{and } a_j \text{ an } U^p\text{-atom}\Big\}.$$

Here χ_I denotes the indicator function of the set I. Note that for $1 \le p \le q < \infty$,

$$U^p(\mathbb{R}, \mathcal{H}) \hookrightarrow U^q(\mathbb{R}, \mathcal{H}) \hookrightarrow L^\infty(\mathbb{R}, \mathcal{H}), \tag{4.1}$$

and functions in $U^p(\mathbb{R}, \mathcal{H})$ are right continuous, $\lim_{t\to-\infty} u(t) = 0$.

Definition 4.2 ([23, Definition 2.2]). Let $1 \le p < \infty$. The Banach space $V^p(\mathbb{R}, \mathcal{H})$ is defined to be the set of all functions $v : \mathbb{R} \to \mathcal{H}$ such that

$$\|v\|_{V^p} := \sup_{\{t_k\}_{k=0}^K \in \mathcal{Z}} \Big(\sum_{k=1}^{K} \|v(t_k) - v(t_{k-1})\|_{\mathcal{H}}^p\Big)^{1/p} \quad \text{is finite.}$$

The Banach subspace of all right continuous functions $v : \mathbb{R} \to \mathcal{H}$ such that $\lim_{t\to-\infty} v(t) = 0$, endowed with the norm above, is denoted by $V^p_{\mathrm{rc}}(\mathbb{R}, \mathcal{H})$. Note that

$$U^p(\mathbb{R}, \mathcal{H}) \hookrightarrow V^p_{\mathrm{rc}}(\mathbb{R}, \mathcal{H}) \hookrightarrow L^\infty(\mathbb{R}, \mathcal{H}). \tag{4.2}$$

Definition 4.3 ([23, Definition 2.5]). For $s \in \mathbb{R}$ we let $U^p_\Delta H^s$, respectively $V^p_\Delta H^s$, be the space of all functions $u : \mathbb{R} \to H^s(\mathbb{T}^3)$ such that $t \mapsto e^{-it\Delta}u(t)$ is in $U^p(\mathbb{R}, H^s)$, respectively in $V^p_\Delta H^s$, with norm

$$\|u\|_{U^p_\Delta H^s} := \|e^{-it\Delta}u(t)\|_{U^p(\mathbb{R},H^s)}, \qquad \|u\|_{V^p_\Delta H^s} := \|e^{-it\Delta}u(t)\|_{V^p(\mathbb{R},H^s)}.$$

We will take $\mathcal{H}$ to be H^s. We refer the reader to [22], [23], and references therein for detailed definitions and properties of the U^p and V^p spaces.

Definition 4.4 ([23, Definition 2.6]). For $s \in \mathbb{R}$ we define X^s to be the space of all functions $u : \mathbb{R} \to H^s(\mathbb{T}^3)$ such that for every $n \in \mathbb{Z}^3$ the map $t \mapsto e^{it|n|^2}\widehat{u(t)}(n)$ is in $U^2(\mathbb{R}, \mathbb{C})$ and the norm

$$\|u\|_{X^s} := \Big(\sum_{n\in\mathbb{Z}^3} \langle n\rangle^{2s} \|e^{it|n|^2}\widehat{u(t)}(n)\|^2_{U^2_t}\Big)^{1/2} \quad \text{is finite.} \tag{4.3}$$

The X^s spaces are variations of the spaces $U^p_\Delta H^s$ and $V^p_\Delta H^s$ corresponding to the Schrödinger flow and defined as follows:

Definition 4.5 ([23, Definition 2.7]). For $s \in \mathbb{R}$ we define Y^s to be the space of all functions $u : \mathbb{R} \to H^s(\mathbb{T}^3)$ such that for every $n \in \mathbb{Z}^3$ the map $t \mapsto e^{it|n|^2}\widehat{u(t)}(n)$ is in $V^2_{\mathrm{rc}}(\mathbb{R}, \mathbb{C})$ and the norm

$$\|u\|_{Y^s} := \Big(\sum_{n\in\mathbb{Z}^3} \langle n\rangle^{2s} \|e^{it|n|^2}\widehat{u(t)}(n)\|^2_{V^2_t}\Big)^{1/2} \quad \text{is finite.} \tag{4.4}$$

Note that

$$U_\Delta^2 H^s \hookrightarrow X^s \hookrightarrow Y^s \hookrightarrow V_\Delta^2 H^s, \tag{4.5}$$

whence for any partition $\mathbb{Z}^3 := \bigcup_k C_k$,

$$\Big(\sum_k \|P_{C_k} u\|_{V_\Delta^2 H^s}^2\Big)^{1/2} \lesssim \|u\|_{Y^s}$$

(cf. [23, Section 2]).

Additionally, when $s = 0$ by orthogonality we have

$$\Big(\sum_k \|P_{C_k} u\|_{Y^0}^2\Big)^{1/2} = \|u\|_{Y^0}. \tag{4.6}$$

We also have the embedding

$$X^s \hookrightarrow Y^s \hookrightarrow L_t^\infty H_x^s \tag{4.7}$$

for $s \geq 0$ (cf. [24]).

Remark 4.6 ([23, Proposition 2.10]). From the atomic structure of the U^2 spaces one can immediately see that for $s \geq 0$, $T > 0$ and $\phi \in H^s(\mathbb{T}^3)$, the solution to the linear Schrödinger equation $u := e^{it\Delta}\phi$ belongs to $X^s([0,T))$ and $\|u\|_{X^s([0,T))} \leq \|\phi\|_{H^s}$.

Remark 4.7. Another important feature of the atomic structure of the U^2 spaces is the fact that just like the $X^{s,b}$ spaces they enjoy a 'transfer principle'. We recall in our context the precise statement below for completeness.

Proposition 4.1 ([22, Proposition 2.19]). *Let $T_0 : L^2 \times \cdots \times L^2 \to L^1_{\text{loc}}$ be an m-linear operator. Assume that for some $1 \leq p, q \leq \infty$,*

$$\|T_0(e^{it\Delta}\phi_1, \ldots, e^{it\Delta}\phi_m)\|_{L^p(\mathbb{R}, L_x^q(\mathbb{T}^3))} \lesssim \prod_{i=1}^m \|\phi_i\|_{L^2(\mathbb{T}^3)}. \tag{4.8}$$

Then there exists an extension $T : U_\Delta^p \times \cdots \times U_\Delta^p \to L^p(\mathbb{R}, L^q(\mathbb{T}^3))$ satisfying

$$\|T(u_1, \ldots, u_m)\|_{L^p(\mathbb{R}, L_x^q(\mathbb{T}^3))} \lesssim \prod_{i=1}^m \|u_i\|_{U_\Delta^p} \tag{4.9}$$

and such that $T(u_1, \ldots, u_m)(t, \cdot) = T_0(u_1(t), \ldots, u_m(t))(\cdot)$ a.e. In other words, one can reduce estimates for multilinear operators on functions in U_Δ^p to similar estimates on solutions to the linear Schrödinger equation.

We will use the following interpolation result at the end of Section 8 to obtain bounds in terms of the X^s spaces from those in $U_\Delta^2 H^s$ and $U_\Delta^p H^s$, just as in [23]. Its proof relies solely on linear interpolation [22, 23].

Proposition 4.2 ([22, Proposition 2.20] and [23, Lemma 2.4]). *Let $q_1, \dots, q_m > 2$ where $m = 1, 2,$ or 3, E be a Banach space, and $T : U^{q_1} \times \cdots \times U^{q_m} \to E$ be a bounded m-linear operator with*

$$\|T(u_1, \dots, u_m)\|_E \le C \prod_{i=1}^{m} \|u_i\|_{U_\Delta^{q_i}}. \tag{4.10}$$

In addition assume there exists $0 < C_2 \le C$ such that

$$\|T(u_1, \dots, u_m)\|_E \le C_2 \prod_{i=1}^{m} \|u_i\|_{U_\Delta^{2}}. \tag{4.11}$$

Then

$$\|T(u_1, \dots, u_m)\|_E \lesssim C_2 \left(\ln \frac{C}{C_2} + 1\right)^m \prod_{i=1}^{m} \|u_i\|_{V^2}, \qquad u_i \in V^2_{\mathrm{rc}},\ i = 1, \dots, m, \tag{4.12}$$

where V^2_{rc} denotes the closed subspace of V^2 of all right continuous functions of t such that $\lim_{t \to -\infty} v(t) = 0$.

Finally, we state two results from [23] we rely on in the next sections. In what follows, $\mathcal{I}$ denotes the Duhamel operator,

$$\mathcal{I}(f)(t) := \int_0^t e^{i(t-t')\Delta} f(t')\,dt', \qquad t \ge 0, \tag{4.13}$$

defined for $f \in L^1_{\mathrm{loc}}([0, \infty), L^2(\mathbb{T}^3))$.

Proposition 4.3 ([23, Proposition 2.11]). *Let $s \ge 0$, $T > 0$. For $f \in L^1([0, T), H^s(\mathbb{T}^3))$ we have $\mathcal{I}(f) \in X^s([0, T))$ and*

$$\|\mathcal{I}(f)\|_{X^s([0,T))} \le \sup_{v \in Y^{-s}([0,T)) :\, \|v\|_{Y^{-s}} = 1} \left| \int_0^T \int_{\mathbb{T}^3} f(t, x) \overline{v(t, x)}\,dx\,dt \right|.$$

As a consequence,

$$\|\mathcal{I}(f)\|_{X^s([0,T))} \lesssim \|f\|_{L^1([0,T), H^s(\mathbb{T}^3))}. \tag{4.14}$$

Proposition 4.4 ([23, Proposition 4.1]). *Fix $s \ge 1$. Then for all $T \in (0, 2\pi]$ and $u_k \in X^s([0, T))$, $k = 1, \dots, 5$,*

$$\left\| \mathcal{I}\Big(\prod_{k=1}^{5} \tilde{u}_k\Big) \right\|_{X^s([0,T))} \lesssim \sum_{j=1}^{5} \|u_j\|_{X^s([0,T))} \prod_{k=1,\, k \ne j}^{5} \|u_k\|_{X^1([0,T))}, \tag{4.15}$$

where $\tilde{u}_k$ denotes either $\overline{u}_k$ or u_k. In particular, (4.15) follows from the estimate for the multilinear form:

$$\left| \int_{[0,T) \times \mathbb{T}^3} \prod_{k=0}^{5} \tilde{u}_k\,dx\,dt \right| \lesssim \|u_0\|_{Y^{-s}([0,T))} \sum_{j=1}^{5} \Big(\|u_j\|_{X^s([0,T))} \prod_{k=1,\, k \ne j}^{5} \|u_k\|_{X^1([0,T))} \Big)$$

where $u_0 := P_{\le N} v$.

Next, we recall the $L^p(\mathbb{T} \times \mathbb{T}^3)$ Strichartz-type estimates of Bourgain's [5] in this context. First recall the usual Littlewood–Paley decomposition of periodic functions. For $N \ge 1$

a dyadic number, we denote by $P_{\le N}$ the rectangular Fourier projection operator

$$P_{\le N} f := \sum_{n=(n_1,n_2,n_3)\in\mathbb{Z}^3:\, |n_i|\le N} \widehat{f}(n) e^{in\cdot x}.$$

Then $P_N = P_{\le N} - P_{\le N-1}$ so that $P_{\le N} = \sum_{M=1}^{N} P_M$ and $P_N^{\perp} := I - P_N$. We then have

$$\|f\|_{H^s(\mathbb{T}^3)} := \|D^s f\|_{L^2(\mathbb{T}^3)} = \Big(\sum_{n\in\mathbb{Z}^3} \langle n\rangle^{2s} |\widehat{f}(n)|^2\Big)^{1/2} \equiv \Big(\sum_{N\ge 1} N^{2s} \|P_N(f)\|^2_{L^2(\mathbb{T}^3)}\Big)^{1/2},$$

where $(D^s f)(n) = \langle n\rangle^s \widehat{f}(n)$.

Definition 4.8. For $N \ge 1$, we denote by $\mathcal{C}_N$ the collection of cubes C in $\mathbb{Z}^3$ with sides parallel to the axes of sidelength N.

Proposition 4.5 ([23, Proposition 3.1, Corollary 3.2], cf. [5]). *Let $p > 4$. For all $N \ge 1$,*

$$\|P_N e^{it\Delta}\phi\|_{L^p(\mathbb{T}\times\mathbb{T}^3)} \lesssim N^{3/2-5/p} \|P_N \phi\|_{L^2(\mathbb{T}^3)}, \tag{4.16}$$

$$\|P_C e^{it\Delta}\phi\|_{L^p(\mathbb{T}\times\mathbb{T}^3)} \lesssim N^{3/2-5/p} \|P_C \phi\|_{L^2(\mathbb{T}^3)}, \tag{4.17}$$

$$\|P_C u\|_{L^p(\mathbb{T}\times\mathbb{T}^3)} \lesssim N^{3/2-5/p} \|P_C u\|_{U^p_\Delta L^2}, \tag{4.18}$$

where P_C is the Fourier projection operator onto $C \in \mathcal{C}_N$ defined by the multiplier χ_C, the characteristic function of C.

Finally, we prove two propositions which will play an important role in Sections 7 and 8.

Proposition 4.6. *Let u, v and w be functions of x and t such that*

$$\widehat{u}(n,t) = a_n^1(t) a_n^2(t) a_n^3(t),$$
$$\widehat{v}(n,t) = a_n^1(t) a_n^2(t) a_n^3(t) a_n^4(t) a_n^5(t),$$
$$\widehat{w}(n,t) = a_n^1(t) a_n^2(t) a_n^3(t) \sum_m a_m^4 a_{n-m}^5,$$

and $|n| \sim N$. Assume that $J \subseteq \{1,2,3,4,5\}$ and if $i \in J$ then

$$a_n^i(t) = \frac{g_n(\omega)}{|n|^{3/2+\varepsilon}} e^{it|n|^2},$$

while if $i \notin J$ then there is a deterministic function f_i such that $\widehat{f_i}(n,t) = a_n^i(t)$. Then

$$\|P_N u\|_{L^p(\mathbb{T}\times\mathbb{T}^3)} \lesssim \prod_{i\notin J\cap\{1,2,3\}} \|P_N f_i\|_{Y^0}, \qquad p > 4, \tag{4.19}$$

$$\|P_N u\|_{L^2(\mathbb{T}\times\mathbb{T}^3)} \lesssim \prod_{i\notin J\cap\{1,2,3\}} \|P_N f_i\|_{Y^0}, \tag{4.20}$$

$$\|P_N v\|_{L^2(\mathbb{T}\times\mathbb{T}^3)} \lesssim \prod_{i\notin J} \|P_N f_i\|_{Y^0}, \tag{4.21}$$

$$\|P_N w\|_{L^2(\mathbb{T}\times\mathbb{T}^3)} \lesssim \prod_{i\notin J,\, i\neq 4,5} \|P_N f_i\|_{Y^0} \prod_{j\notin J,\, j=4,5} \|f_j\|_{Y^0}. \tag{4.22}$$

Proof. To prove (4.19) we write $u = k_1 * k_2 * k_3$, where the convolution is only with respect to the space variable. Then by Young's inequality in the space variable followed by Hölder's inequality and the embedding (4.7) we have the desired inequality.

To prove (4.20) we use Plancherel

$$\|P_N u\|_{L^2(\mathbb{T}\times\mathbb{T}^3)} \lesssim \left\| \|\chi_{|n|\sim N} a_n^1 a_n^2 a_n^3\|_{\ell^2} \right\|_{L^\infty(\mathbb{T})} \lesssim \left\| \prod_{i=1}^3 \|\chi_{|n|\sim N} a_n^i\|_{\ell^2} \right\|_{L^\infty(\mathbb{T})}$$
$$\lesssim \left\| \prod_{i=1}^3 \|P_N f_i\|_{L^2_x} \right\|_{L^\infty(\mathbb{T})} \lesssim \prod_{i\notin J\cap\{1,2,3\}} \|P_N f_i\|_{L^\infty(\mathbb{T},L^2(\mathbb{T}^3))},$$

and the conclusion follows from the embedding (4.7).

To prove (4.21) we proceed in a similar manner.

To prove (4.22) we first write

$$\|P_N w\|_{L^2(\mathbb{T}\times\mathbb{T}^3)} \sim \|P_N(k_1 * k_2 * k_3 * (k_4 k_5))\|_{L^2(\mathbb{T}\times\mathbb{T}^3)},$$

and by the Young, Hölder and Cauchy–Schwarz inequalities we continue with

$$\lesssim \left\| \prod_{i=1}^3 \|P_N k_i\|_{L^2} \|P_N(k_4 k_5)\|_{L^1} \right\|_{L^2(\mathbb{T})} \lesssim \left\| \prod_{i=1}^3 \|P_N k_i\|_{L^2} \|k_4\|_{L^2} \|k_5\|_{L^2} \right\|_{L^2(\mathbb{T})}$$
$$\lesssim \prod_{i\notin J,\, i\neq 4,5} \|P_N f_i\|_{L^\infty(\mathbb{T},L^2(\mathbb{T}^3))} \prod_{j\notin J,\, j=4,5} \|f_j\|_{L^\infty(\mathbb{T},L^2(\mathbb{T}^3))}. \qquad \square$$

We now state a trilinear L^2 estimate that is similar to Proposition 3.5 in [23] but in which some of the functions may be linear evolution of random data.

Proposition 4.7. *Assume that $N_1 \geq N_2 \geq N_3$ and $C \in \mathcal{C}_{N_2}$, a cube of sidelength N_2. Assume also that $J \subseteq \{1,2,3\}$ and if $j \in J$ then $\widehat{u}_j(n) = e^{i|n|^2 t} g_n(\omega)/|n|^{3/2+\varepsilon}$ for $\varepsilon > 0$ small. Then*

$$\|P_C P_{N_1}\tilde{u}_1 P_{N_2}\tilde{u}_2 P_{N_3}\tilde{u}_3\|_{L^2(\mathbb{T}\times\mathbb{T}^3)} \lesssim N_2 N_3 \prod_{j\notin J} \|P_{N_j} u_j\|_{U^4_\Delta L^2}, \tag{4.23}$$

$$\|P_C P_{N_1}\tilde{u}_1 P_{N_2}\tilde{u}_2\|_{L^2(\mathbb{T}\times\mathbb{T}^3)} \lesssim N_2^{1/2+\varepsilon} \prod_{j\notin J} \|P_{N_j} u_j\|_{U^4_\Delta L^2}, \tag{4.24}$$

where $\tilde{u}_k$ denotes either $\overline{u}_k$ or u_k.

Moreover (4.23) and (4.24) also hold with the Y^0 norms on the right hand side.

Proof. To prove (4.23) we follow [23, proof of (24)]. We write

$$\|P_C P_{N_1}\tilde{u}_1 P_{N_2}\tilde{u}_2 P_{N_3}\tilde{u}_3\|_{L^2(\mathbb{T}\times\mathbb{T}^3)} \lesssim \|P_C P_{N_1} u_1\|_{L^p} \|P_{N_2} u_2\|_{L^p} \|P_{N_3} u_3\|_{L^q}$$

where $2/p + 1/q = 1/2$ and $4 < p < 5$. Then we use (4.17) for the random linear functions and (4.18) for the deterministic functions to obtain

$$\|P_C P_{N_1}\tilde{u}_1 P_{N_2}\tilde{u}_2 P_{N_3}\tilde{u}_3\|_{L^2(\mathbb{T}\times\mathbb{T}^3)} \lesssim N_2 N_3 \left(\frac{N_3}{N_2}\right)^{-2+10/p} \prod_{j\notin J} \|P_{N_j} u_j\|_{U^4_\Delta L^2},$$

where we have used the embedding (4.1).

To prove (4.24) we use Hölder's inequality to write

$$\|P_C P_{N_1}\tilde{u}_1 P_{N_2}\tilde{u}_2\|_{L^2(\mathbb{T}\times\mathbb{T}^3)} \lesssim \|P_C P_{N_1}u_1\|_{L^{4+\varepsilon}}\|P_{N_2}u_2\|_{L^{4+\varepsilon}}; \tag{4.25}$$

we then use (4.17), (4.18) and the embedding (4.1) to continue with

$$\lesssim N_2^{1/2+\varepsilon}\prod_{j\notin J}\|P_{N_j}u_j\|_{U^4_\Delta L^2}.$$

To obtain the Y^0 on the right hand side we use the interpolation Proposition 4.2 and the embedding (4.1). □

5. Almost sure local well-posedness for the initial value problem (2.17)

We define

$$v_0^\omega(t,x) := S(t)\phi^\omega(x) \tag{5.1}$$

where $\phi^\omega(x)$ is as in (1.5), and instead of solving the initial value problem (2.17) we solve the one for $w := v - v_0^\omega$:

$$\begin{cases} iw_t + \Delta w = \mathcal{N}(w + v_0^\omega), & x \in \mathbb{T}^3, \\ w(0,x) = 0, \end{cases} \tag{5.2}$$

where $\mathcal{N}(\cdot)$ was defined in (2.18). To understand the nonlinear term of (5.2) we express it in terms of its spatial Fourier transform. Let $a_n := \widehat{v}(n)$, $\theta_n^\omega := \mathcal{F}(S(t)\phi^\omega)(n)$; then $b_n := \widehat{w}(n) = a_n - \theta_n^\omega$. Now recall (2.9) and replace in it a_n with $b_n + \theta_n^\omega$. Then

$$\mathcal{F}(\mathcal{N}(w + v_0^\omega))(n) = \sum_{k=1}^{7} J_k(b_n + \theta_n^\omega) \tag{5.3}$$

where $J_k(b_n + \theta_n^\omega)$ means that the terms J_k defined in (2.10)–(2.16) are evaluated for the sequence $b_n + \theta_n^\omega$ instead of a_n.

We are now ready to state the almost sure well-posedness result for the initial value problem (5.2).

Theorem 5.1. *Let $0 < \alpha < 1/12$, $s \in (1+4\alpha, 3/2-2\alpha)$. There exist $0 < \delta_0 \ll 1$ and $r = r(s,\alpha) > 0$ such that for any $\delta < \delta_0$, there exists $\Omega_\delta \in A$ with*

$$\mathbb{P}(\Omega_\delta^c) < e^{-1/\delta^r}$$

such that for each $\omega \in \Omega_\delta$ there exists a unique solution w of (5.2) in the space $X^s([0,\delta)) \cap C([0,\delta), H^s(\mathbb{T}^3))$.

This theorem follows from the following two propositions via a contraction mapping argument.

Proposition 5.1. *Let $0 < \alpha < 1/12$, $s \in (1+4\alpha, 3/2-2\alpha)$, $\delta \ll 1$ and $r > 0$. Assume N_i, $i = 0, \dots, 5$, are dyadic numbers and $N_1 \geq \dots \geq N_5$. Then there exist $\rho = \rho(s, \alpha)$, $\mu > 0$, and $\Omega_\delta \in A$ with $\mathbb{P}(\Omega_\delta^c) < e^{-1/\delta^r}$ such that for $\omega \in \Omega_\delta$ we have:*

- *If $N_1 \gg N_0$ or $P_{N_1} w = P_{N_1} v_0^\omega$ then*

$$\left| \int_0^{2\pi} \int_{\mathbb{T}^3} D^s(\mathcal{N}(P_{N_i}(w + v_0^\omega)))\overline{P_{N_0} h}\, dx\, dt \right| \lesssim \delta^{-\mu r} N_1^{-\rho} \| P_{N_0} h\|_{Y^{-s}} \Big(1 + \prod_{i \notin J} \|P_{N_i} w\|_{X^s}\Big). \tag{5.4}$$

- *If $N_1 \sim N_0$ and $P_{N_1} w \neq P_{N_1} v_0^\omega$ then*

$$\left| \int_0^{2\pi} \int_{\mathbb{T}^3} D^s(\mathcal{N}(P_{N_i}(w + v_0^\omega)))\overline{P_{N_0} h}\, dx\, dt \right| \lesssim \delta^{-\mu r} N_2^{-\rho} \| P_{N_0} h\|_{Y^{-s}} \|P_{N_1} w\|_{X^s} \Big(1 + \prod_{i \notin J,\, i \neq 1} \|\psi_\delta P_{N_i} w\|_{X^s}\Big). \tag{5.5}$$

Here v_0^ω is as in (5.1), $w \in X^s([0, 2\pi])$, and $J \subseteq \{1, 2, 3, 4, 5\}$ denote those indices corresponding to random functions.

Proposition 5.2. *Let $0 < \alpha < 1/12$, $s \in (1+4\alpha, 3/2-2\alpha)$ and $\delta \ll 1$. Let v_0^ω be defined as in (5.1) and assume $w \in X^s([0, 2\pi])$. Then there exist $\theta = \theta(s, \alpha) > 0$, $r = r(s, \alpha)$ and $\Omega_\delta \in A$ with $\mathbb{P}(\Omega_\delta^c) < e^{-1/\delta^r}$ such that for $\omega \in \Omega_\delta$,*

$$\|\mathcal{I}(\psi_\delta \mathcal{N}(w + v_0^\omega))\|_{X^s([0,2\pi])} \lesssim \delta^\theta (1 + \|\psi_\delta w\|^5_{X^s([0,2\pi])}) \tag{5.6}$$

where $\mathcal{N}(\cdot)$ was defined in (2.18) and ψ_δ is a smooth time cut-off of the interval $[0, \delta]$.

The proof of Proposition 5.1 is the content of Sections 7 and 8, while Proposition 5.2 is proved in Section 9.

6. Auxiliary lemmata and further notation

We begin by recalling some counting estimates for integer lattice sets (cf. Bourgain [5]).

Lemma 6.1. *Let S_R be a sphere of radius R, B_r be a ball of radius r, and $\mathcal{P}$ be a plane in $\mathbb{R}^3$. Then*

$$|\mathbb{Z}^3 \cap S_R| \lesssim R, \tag{6.1}$$

$$|\mathbb{Z}^3 \cap B_r \cap S_R| \lesssim \min(R, r^2), \tag{6.2}$$

$$|\mathbb{Z}^3 \cap B_r \cap \mathcal{P}| \lesssim r^2, \tag{6.3}$$

where $|\cdot|$ denotes cardinality.

Next, we state a result we will invoke when the higher frequencies correspond to deterministic terms and one can afford to ignore the moments given by the lower frequency random terms as well as rely on Strichartz estimates.

Lemma 6.2. *Assume $N_i, i = 0, \dots, 5$, are dyadic numbers and $N_1 \sim N_0$ and $N_1 \geq \dots \geq N_5$. Let $\{C\}$ be a partition of $\mathbb{Z}^3$ into cubes $C \in \mathcal{C}_{N_2}$, and let $\{Q\}$ be a partition of $\mathbb{Z}^3$ into cubes $Q \in \mathcal{C}_{N_3}$. Then*

$$\sum_{N_i,\, i=0,\dots,5} \left| \int_0^1 \int_{\mathbb{T}^3} P_{N_1} f_1 P_{N_2} f_2 P_{N_3} f_3 P_{N_4} f_4 P_{N_5} f_5 \overline{P_{N_0} h}\, dx\, dt \right|$$
$$\lesssim \sum_{N_i,\, i=0,\dots,5} \Big(\sup_{\tilde{C}} \| P_{\tilde{C}} P_{N_1} f_1 P_{N_2} f_2 P_{N_\ell} f_\ell \|^2_{L^2_{xt}} \sum_{C,Q} \| P_Q P_C \overline{P_{N_0} h} P_{N_3} f_3 P_{N_r} f_r \|^2_{L^2_{xt}} \Big)^{1/2} \tag{6.4}$$

where $\ell \neq r \in \{4, 5\}$ and $\tilde{C}$ are cubes whose sidelength is $10N_2$.

Proof. This follows from orthogonality arguments. □

Just as Bourgain [4], in the course of the proof we will use the following classical result about matrices, which we state as a lemma for convenience.

Lemma 6.3. *Let $\mathcal{A} = (A_{ik})_{\substack{1\leq i\leq N\\ 1\leq k\leq M}}$ be an $N \times M$ matrix with adjoint $\mathcal{A}^* = (\overline{A}_{kj})_{\substack{1\leq k\leq M\\ 1\leq j\leq N}}$. Then*

$$\|\mathcal{A}\mathcal{A}^*\| \leq \max_{1\leq j\leq N} \sum_{k=1}^{M} |A_{jk}|^2 + \Big(\sum_{i\neq j} \Big| \sum_{k=1}^{M} A_{ik} \overline{A}_{jk} \Big|^2 \Big)^{1/2} \tag{6.5}$$

where $\|\cdot\|$ means the 2-norm.

Proof. Decompose $\mathcal{A}\mathcal{A}^*$ into the sum of a diagonal matrix D plus an off-diagonal one F. Then note the 2-norm of D is bounded by the square root of the largest eigenvalue of DD^*, which, since D is diagonal, is the maximum of the absolute values of the diagonal entries of D. This gives the first term in (6.5). Bounding the 2-norm of F by the Frobenius norm of F gives the second term in (6.5). □

Notation. Given k-tuples $(n_1, \dots, n_k) \in \mathbb{Z}^{3k}$, a set $\mathscr{C}$ of constraints on them, and a subset $\{i_1, \dots, i_h\} \subseteq \{1, \dots, k\}$, we denote by $S_{(n_{i_1},\dots,n_{i_h})}$ the set of $(k-h)$-tuples $(n_{j_1}, \dots, n_{j_{k-h}})$ with $\{j_1, \dots j_{k-h}\} = \{1, \dots, k\} \setminus \{i_1, \dots, i_h\}$ which satisfy the constraints $\mathscr{C}$ for fixed $(n_{i_1}, \dots, n_{i_h})$. We also denote by $|S_{(n_{i_1},\dots,n_{i_h})}|$ its cardinality.

7. The trilinear and bilinear estimates

In this section, we denote by $D_j := e^{it\Delta} P_{N_j}\phi$ solutions to the linear equation for data ϕ in L^2 localized at frequency N_j, and by R_k the function defined by

$$\widehat{R_k}(n) = \chi_{\{|n|\sim N_k\}}(n) \frac{g_n(\omega)}{\langle n\rangle^{3/2}} e^{it|n|^2}, \tag{7.1}$$

and representing the linear evolution of a random function of type (1.5), localized at frequency N_k and *almost* L^2 normalized.

7.1. Trilinear estimates

We prove certain trilinear estimates which serve as building blocks for the proof in Section 8. Their proofs are of the same flavor as those presented by Bourgain [4]. For N_j, $j = 1, 2, 3$, dyadic numbers, let $\alpha_j = 0$ or 1 for $j = 1, 2, 3$ and define

$$\Upsilon(n,m) := \left\{ (n_1, m_1; n_2, m_2; n_3, m_3) : \begin{array}{l} n = (-1)^{\alpha_1} n_1 + (-1)^{\alpha_2} n_2 + (-1)^{\alpha_3} n_3, \\ n_k \neq n_\ell \text{ whenever } \alpha_k \neq \alpha_\ell, \\ |n_j| \sim N_j,\ j = 1,2,3, \\ m = (-1)^{\alpha_1} m_1 + (-1)^{\alpha_2} m_2 + (-1)^{\alpha_3} m_3 \end{array} \right\}. \tag{7.2}$$

Then define T_Υ to be the multilinear operator with multiplier χ_Υ.

Proposition 7.1. *Fix $N_1 \geq N_2 \geq N_3$, $r, \delta > 0$ and $C \in \mathcal{C}_{N_2}$. Then there exist $\mu, \varepsilon > 0$ and a set $\Omega_\delta \in A$ with $\mathbb{P}(\Omega_\delta^c) \leq e^{-1/\delta^r}$ such that for any $\omega \in \Omega_\delta$ we have the following space-time estimates:*

$$\|T_\Upsilon(P_C \bar{R}_1, \tilde{D}_2, R_3)\|_{L^2} \lesssim \delta^{-\mu r} N_2^{5/4} N_1^{-1/2} \|P_{N_2}\phi\|_{L^2_x}, \tag{7.3}$$

$$\|T_\Upsilon(P_C \bar{R}_1, \tilde{D}_2, \bar{R}_3)\|_{L^2} \lesssim \delta^{-\mu r} N_2^{5/4} N_1^{-1/2} \|P_{N_2}\phi\|_{L^2_x}, \tag{7.4}$$

$$\|T_\Upsilon(P_C \tilde{D}_1, \bar{R}_2, R_3)\|_{L^2} \lesssim \delta^{-\mu r} N_2^{3/4} \|P_C P_{N_1}\phi\|_{L^2_x}, \tag{7.5}$$

$$\|T_\Upsilon(P_C \tilde{D}_1, R_2, R_3)\|_{L^2} \lesssim \delta^{-\mu r} N_2^{3/4} \|P_C P_{N_1}\phi\|_{L^2_x}, \tag{7.6}$$

$$\|T_\Upsilon(P_C \bar{R}_1, R_2, \tilde{D}_3)\|_{L^2} \lesssim \delta^{-\mu r} [N_1^{-3/4} N_2^{1/2} N_3^{5/4} + N_1^{-1/2} N_2^{1/2} N_3^{3/4}] \|P_{N_3}\phi\|_{L^2_x}, \tag{7.7}$$

$$\|T_\Upsilon(P_C \bar{R}_1, \bar{R}_2, \tilde{D}_3)\|_{L^2} \lesssim \delta^{-\mu r} [N_1^{-3/4} N_2^{1/2} N_3^{5/4} + N_1^{-1/2} N_2^{1/2} N_3^{3/4}] \|P_{N_3}\phi\|_{L^2_x}, \tag{7.8}$$

$$\begin{aligned} \|T_\Upsilon(P_C R_1, \tilde{D}_2, \tilde{D}_3)\|_{L^2} \lesssim\ & \delta^{-\mu r} N_2^{1/2+3\theta/4} N_1^{-1/2+\varepsilon} \min(N_1, N_2^2)^{1-\theta/2} N_3^{3/2} \\ & \times \|P_{N_2}\phi\|_{L^2_x} \|P_{N_3}\phi\|_{L^2_x}, \quad 0 \leq \theta \leq 1, \end{aligned} \tag{7.9}$$

$$\|T_\Upsilon(P_C \tilde{D}_1, R_2, \tilde{D}_3)\|_{L^2} \lesssim \delta^{-\mu r} N_2^{1/2+\varepsilon} N_3^{3/2} \|P_{N_1}\phi\|_{L^2_x} \|P_{N_3}\phi\|_{L^2_x}, \tag{7.10}$$

$$\|T_\Upsilon(P_C \bar{R}_1, \bar{R}_2, R_3)\|_{L^2} \lesssim \delta^{-\mu r} N_1^{-1/2} N_2^{1/2}, \tag{7.11}$$

$$\|T_\Upsilon(P_C \bar{R}_1, R_2, \bar{R}_3)\|_{L^2} \lesssim \delta^{-\mu r} N_1^{-1/2} N_2^{1/2}, \tag{7.12}$$

$$\|T_\Upsilon(P_C \bar{R}_1, R_2, R_3)\|_{L^2} \lesssim \delta^{-\mu r} N_1^{-1/2} N_2^{1/2}, \tag{7.13}$$

where $L^2 = L^2(\mathbb{T} \times \mathbb{T}^3)$. Note that here the bar $-$ indicates complex conjugate while the tilde $\sim$ indicates both complex conjugate or not. Also, without writing it explicitly, we always assume that if $\widehat{\bar{R}}(n_1)$ and $\widehat{R}(n_2)$ appear in the trilinear expressions on the left hand side, then $n_1 \neq n_2$.

Remark 7.1. In using the trilinear estimates above, sometimes it is convenient to interpret a random term as deterministic and choose the minimum estimate possible. For example, in considering $\|P_C\bar{R}_1\bar{R}_2R_3\|_{L^2}$ we have a choice between (7.11) and (7.8) by thinking of R_3 as an 'almost' L^2 normalized $\tilde{D}_3$ function.

Proposition 7.2. *Let D_j and R_k be as above and fix $N_1 \geq N_2 \geq N_3$, $r, \delta > 0$ and $C \in \mathcal{C}_{N_2}$. Then there exists $\mu > 0$ and a set $\Omega_\delta \in A$ with $\mathbb{P}(\Omega_\delta^c) \leq e^{-1/\delta^r}$ such that for any $\omega \in \Omega_\delta$ we have (7.3) and (7.4).*

Proof. As in [23] we will first assume that the deterministic functions D_i are localized linear solutions, that is, $D_i = P_{N_i}S(t)\psi$ and $\widehat{\psi}(n) = a_n$. Once an estimate is proved with $\|\chi_{N_i}(n)a_n\|_{\ell^2}$ on the right hand side, we invoke the transfer principle of Proposition 4.1 to complete the proof.

We start by estimating (7.3). Without any loss of generality we assume that $\tilde{D}_2 = D_2$. By using Fourier transform to write the left hand side we note that it is enough to estimate

$$\mathcal{T} := \sum_{m\in\mathbb{Z},\, n\in\mathbb{Z}^3}\Bigg|\sum_{\substack{n=-n_1+n_2+n_3\\ n_1\neq n_2,n_3\\ m=-|n_1|^2+|n_2|^2+|n_3|^2}} \chi_C(n_1)\frac{\overline{g}_{n_1}(\omega)}{|n_1|^{3/2}}a_{n_2}\frac{g_{n_3}(\omega)}{|n_3|^{3/2}}\Bigg|^2 \tag{7.14}$$

where we recall that C is a cube of sidelength N_2. We are going to use duality and a change of variable since, as will be apparent below, the counting with respect to the time frequency will be more favorable.

Using duality we find that

$$\mathcal{T} = \Bigg[\sup_{\|\gamma\otimes k\|_{\ell^2}\leq 1}\Bigg|\sum_{m,n}k(n)\gamma(m)\sum_{\substack{n=-n_1+n_2+n_3\\ n_1\neq n_2,n_3\\ m=-|n_1|^2+|n_2|^2+|n_3|^2}}\chi_C(n_1)\frac{\overline{g}_{n_1}(\omega)}{|n_1|^{3/2}}a_{n_2}\frac{g_{n_3}(\omega)}{|n_3|^{3/2}}\Bigg|\Bigg]^2.$$

Letting $\zeta := m - |n_2|^2 = -|n_1|^2 + |n_3|^2$, we continue with

$$\mathcal{T} = \Bigg[\sup_{\|\gamma\otimes k\|_{\ell^2}\leq 1}\Bigg|\sum_{n_2}a_{n_2}\sum_{\zeta}\gamma(\zeta+|n_2|^2)\sum_{\substack{n=-n_1+n_2+n_3\\ n_1\neq n_2,n_3\\ \zeta=-|n_1|^2+|n_3|^2}}\chi_C(n_1)\frac{\overline{g}_{n_1}(\omega)}{|n_1|^{3/2}}\frac{g_{n_3}(\omega)}{|n_3|^{3/2}}k_n\Bigg|\Bigg]^2$$

$$\lesssim \sup_{\|\gamma\otimes k\|_{\ell^2}\leq 1}\|a_{n_2}\|^2_{\ell^2_{n_2}}\|\gamma\|^2_{\ell^2_\zeta}\sum_{n_2,\zeta}\Bigg|\sum_{\substack{n=-n_1+n_2+n_3\\ n_1\neq n_2,n_3\\ \zeta=-|n_1|^2+|n_3|^2}}\chi_C(n_1)\frac{\overline{g}_{n_1}(\omega)}{|n_1|^{3/2}}\frac{g_{n_3}(\omega)}{|n_3|^{3/2}}k_n\Bigg|^2.$$

All in all, we then have to estimate, uniformly for $\|\gamma\otimes k\|_{\ell^2}\leq 1$,

$$\|a_{n_2}\|^2_{\ell^2}\|\gamma\|^2_{\ell^2}\sum_{n_2}\sum_{|\zeta|\leq N_1N_2}\Bigg|\sum_n\sigma_{n_2,n}k_n\Bigg|^2 \tag{7.15}$$

where

$$\sigma_{n_2,n} := \sum_{\substack{n_2=n_1+n-n_3,\, n_1\neq n_2,n_3\\ \zeta=-|n_1|^2+|n_3|^2}}\chi_C(n_1)\frac{\overline{g}_{n_1}(\omega)}{|n_1|^{3/2}}\frac{g_{n_3}(\omega)}{|n_3|^{3/2}}.$$

Note that σ_{n,n_2} also depends on ζ but we estimate it independently of ζ. If we denote by $\mathcal{G}$ the matrix of entries $\sigma_{n_2,n}$, and we recall that the variation in ζ is at most N_1N_2, we are reduced to estimating

$$\|a_{n_2}\|^2_{\ell^2} N_1 N_2 \|\mathcal{G}\mathcal{G}^*\|.$$

We note that by Lemma 6.3,

$$\|\mathcal{G}\mathcal{G}^*\| \lesssim \max_{n_2} \sum_n |\sigma_{n_2,n}|^2 + \Big(\sum_{n_2\neq n_2'} \Big|\sum_{n\in\tilde{C}} \sigma_{n_2,n}\overline{\sigma}_{n_2',n}\Big|^2\Big)^{1/2} =: M_1 + M_2,$$

where $\tilde{C}$ is a cube of sidelength approximately N_2.

To estimate M_1 we first define

$$S_{(\zeta,n_2)} := \{(n_1,n,n_3) : n_2 = n_1 + n - n_3,\ n_1 \neq n_2, n_3,\ \zeta = -|n_1|^2 + |n_3|^2\},$$

with $|S_{(\zeta,n_2)}| \lesssim N_3^3 N_1$, where we use (6.1) for fixed n_3. Then we have

$$M_1 \lesssim \sup_{(n_2,\zeta)} \Bigg| \sum_{\substack{n_2=n_1+n-n_3,\, n_1\neq n_2,n_3\\ \zeta=-|n_1|^2+|n_3|^2}} \chi_C(n_1) \frac{\overline{g}_{n_1}(\omega)}{|n_1|^{3/2}} \frac{g_{n_3}(\omega)}{|n_3|^{3/2}}\Bigg|^2.$$

Now we use (3.4) with $\lambda = \delta^{-r}\|F_2\|_{L^2}$ and Lemma 3.3 to obtain, for ω outside a set of measure e^{-1/δ^r}, the bound

$$\begin{aligned} M_1 &\lesssim \sup_{(n_2,\zeta)} \delta^{-2r} \sum_{S_{(\zeta,n_2)}} \sum_{S_{(\zeta,n_2)}} \frac{1}{|n_1|^{3/2}} \frac{1}{|n_3|^{3/2}} \frac{1}{|\xi_1|^{3/2}} \frac{1}{|\xi_3|^{3/2}} \\ &\qquad\qquad \times \Big| \int_\Omega \overline{g}_{n_1}(\omega) g_{n_3}(\omega) g_{\xi_1}(\omega) \overline{g}_{\xi_3}(\omega)\, d\mathbb{P}(\omega)\Big| \\ &\lesssim \sup_{(n_2,\zeta)} \delta^{-2r} \sum_{S_{(\zeta,n_2)}} \frac{1}{|n_1|^3} \frac{1}{|n_3|^3} \lesssim \delta^{-2r} N_1^{-3} N_3^{-3} N_3^3 N_1 \sim \delta^{-2r} N_1^{-2}. \end{aligned} \tag{7.16}$$

To estimate M_2 we first write

$$M_2^2 = \sum_{n_2\neq n_2'} \Big|\sum_{n\in\tilde{C}} \sigma_{n_2,n}\overline{\sigma}_{n_2',n}\Big|^2 \sim \sum_{n_2\neq n_2'} \Bigg|\sum_{S_{(n_2,n_2')}} \frac{\overline{g}_{n_1}(\omega)}{|n_1|^{3/2}} \frac{g_{n_3}(\omega)}{|n_3|^{3/2}} \frac{g_{n_1'}(\omega)}{|n_1'|^{3/2}} \frac{\overline{g}_{n_3'}(\omega)}{|n_3'|^{3/2}}\Bigg|^2$$

where

$$S_{(n_2,n_2',\zeta)} := \left\{ (n,n_1,n_3,n_1',n_3') : \begin{array}{c} n_2 = n_1 + n - n_3,\ n_2' = n_1' + n - n_3', \\ n_1 \neq n_2, n_3,\ n_1' \neq n_2', n_3',\ n \in \tilde{C}, \\ \zeta = -|n_1|^2 + |n_3|^2,\ \zeta = -|n_1'|^2 + |n_3'|^2 \end{array} \right\}.$$

We need to organize the estimates according to whether some frequencies are the same or not; in all we have six cases:

- **Case** β_1**:** n_1, n_1', n_3, n_3' are all different.
- **Case** β_2**:** $n_1 = n_1'$; $n_3 \neq n_3'$.
- **Case** β_3**:** $n_1 \neq n_1'$; $n_3 = n_3'$.
- **Case** β_4**:** $n_1 \neq n_3'$; $n_3 = n_1'$.
- **Case** β_5**:** $n_1 = n_3'$; $n_3 \neq n_1'$.
- **Case** β_6**:** $n_1 = n_3'$; $n_3 = n_1'$.

Case β_1**.** We define the set

$$S_{(\zeta)} := \left\{ (n_2, n_2', n, n_1, n_3, n_1', n_3') : \begin{array}{l} n_2 = n_1 + n - n_3,\ n_2' = n_1' + n - n_3', \\ n_1 \neq n_2, n_3,\ n_1' \neq n_2', n_3',\ n_1, n_1' \in C, \\ \zeta = -|n_1|^2 + |n_3|^2,\ \zeta = -|n_1'|^2 + |n_3'|^2 \end{array} \right\}$$

and we note that $|S_{(\zeta)}| \lesssim N_1^2 N_3^6 N_2^3$ since $n \in \tilde{C}$ and for fixed n_3 and n_3' we use (6.1) to count n_1 and n_1'. Using (3.4) with $\lambda = \delta^{-2r} \|F_4\|_{L^2}$ and again Lemma 3.3 we can write, for ω as above,

$$\begin{aligned} M_2^2 &\lesssim \delta^{-4r} \sum_{n_2 \neq n_2'} \sum_{S_{(n_2, n_2', \zeta)}} \frac{1}{|n_1|^3} \frac{1}{|n_3|^3} \frac{1}{|n_1'|^3} \frac{1}{|n_3'|^3} \\ &\lesssim \delta^{-4r} N_1^{-6} N_3^{-6} N_1^2 N_3^6 N_2^3 \sim \delta^{-4r} N_1^{-4} N_2^3. \end{aligned}$$

Case β_2**.** First define

$$S_{(n_2, n_2', n_3, n_3', \zeta)} := \left\{ (n, n_1) : \begin{array}{l} n_2 = n_1 + n - n_3,\ n_2' = n_1 + n - n_3', \\ n_1 \neq n_2, n_2', n_3, n_3',\ n \in \tilde{C}, \\ \zeta = -|n_1|^2 + |n_3|^2,\ \zeta = -|n_1|^2 + |n_3'|^2 \end{array} \right\}.$$

To compute $|S_{(n_2, n_2', n_3, n_3', \zeta)}|$ we count n_1; then n is determined. Since n_1 sits on a sphere, by (6.1) we have $|S_{(n_2, n_2', n_3, n_3', \zeta)}| \lesssim N_1$. Then we set

$$S_{(\zeta)} := \left\{ (n_2, n_2', n, n_1, n_3, n_3') : \begin{array}{l} n_2 = n_1 + n - n_3,\ n_2' = n_1 + n - n_3', \\ n_1 \neq n_2, n_2', n_3, n_3',\ n \in \tilde{C}, \\ \zeta = -|n_1|^2 + |n_3|^2,\ \zeta = -|n_1|^2 + |n_3'|^2 \end{array} \right\}$$

with $|S_{(\zeta)}| \lesssim N_1 N_3^6 N_2^3$, where we have again used that $n \in \tilde{C}$ and (6.1). Now, we find that

$$M_2^2 \sim \sum_{n_2 \neq n_2'} \left| \sum_{S_{(n_2, n_2', \zeta)}} \frac{|g_{n_1}(\omega)|^2}{|n_1|^3} \frac{g_{n_3}(\omega)}{|n_3|^{3/2}} \frac{\overline{g}_{n_3'}(\omega)}{|n_3'|^{3/2}} \right|^2 \lesssim Q_1 + Q_2 \tag{7.17}$$

where

$$Q_1 := \sum_{n_2 \neq n_2'} \left| \sum_{S_{(n_2, n_2', \zeta)}} \frac{|g_{n_1}(\omega)|^2 - 1}{|n_1|^3} \frac{g_{n_3}(\omega)}{|n_3|^{3/2}} \frac{\overline{g}_{n_3'}(\omega)}{|n_3'|^{3/2}} \right|^2, \tag{7.18}$$

$$\mathcal{Q}_2 := \sum_{n_2 \neq n_2'} \left| \sum_{S_{(n_2,n_2',\zeta)}} \frac{1}{|n_1|^3} \frac{g_{n_3}(\omega)}{|n_3|^{3/2}} \frac{\overline{g}_{n_3'}(\omega)}{|n_3'|^{3/2}} \right|^2. \tag{7.19}$$

We first estimate $\mathcal{Q}_2$. We rewrite

$$\mathcal{Q}_2 \sim \sum_{n_2 \neq n_2'} \left| \sum_{n_3,n_3'} \left[\sum_{S_{(n_2,n_2',n_3,n_3',\zeta)}} \frac{1}{|n_1|^3} \frac{1}{|n_3|^{3/2}} \frac{1}{|n_3'|^{3/2}} \right] g_{n_3}(\omega) \overline{g}_{n_3'}(\omega) \right|^2. \tag{7.20}$$

We now proceed as in (7.16) above. We use (3.4) with $\lambda = \delta^{-r} \|F_2\|_{L^2}$, Lemma 3.3 and (3.5) to deduce that for ω outside a set of measure e^{-1/δ^r} one has

$$\begin{aligned}
(7.20) &\lesssim \delta^{-2r} \sum_{n_2 \neq n_2'} \sum_{n_3,n_3'} \left[\sum_{S_{(n_2,n_2',n_3,n_3',\zeta)}} \frac{1}{|n_1|^3} \frac{1}{|n_3|^{3/2}} \frac{1}{|n_3'|^{3/2}} \right]^2 \\
&\lesssim \delta^{-2r} N_1^{-6} N_3^{-6} \sum_{n_2 \neq n_2'} \sum_{n_3,n_3'} |S_{(n_2,n_2',n_3,n_3',\zeta)}|^2 \\
&\lesssim \delta^{-2r} N_1^{-6} N_3^{-6} N_1 \sum_{n_2 \neq n_2'} \sum_{n_3,n_3'} |S_{(n_2,n_2',n_3,n_3',\zeta)}| \\
&\lesssim \delta^{-2r} N_1^{-6} N_3^{-6} N_1 |S_{(\zeta)}| \sim \delta^{-2r} N_1^{-4} N_2^3.
\end{aligned} \tag{7.21}$$

To estimate $\mathcal{Q}_1$ we let

$$S_{(n_2,n_2',n_1,n_3,n_3',\zeta)} := \left\{ n : \begin{array}{c} n_2 = n_1 + n - n_3,\ n_2' = n_1 + n - n_3', \\ n_1 \neq n_2, n_2', n_3, n_3',\ n \in \tilde{C}, \\ \zeta = -|n_1|^2 + |n_3|^2,\ \zeta = -|n_1|^2 + |n_3'|^2 \end{array} \right\}, \tag{7.22}$$

and note that its cardinality is 1 since n is determined for fixed $(n_2, n_2', n_1, n_3, n_3')$. We have

$$\begin{aligned}
\mathcal{Q}_1 \sim \sum_{n_2 \neq n_2'} \Bigg| \sum_{n_1 \neq n_2, n_2', n_3, n_3 \neq n_3'} &\left[\sum_{S^2_{(n_2,n_2',n_1,n_3,n_3')}} \frac{1}{|n_1|^3} \frac{1}{|n_3|^{3/2}} \frac{1}{|n_3'|^{3/2}} \right] \\
&\times (|g_{n_1}(\omega)|^2 - 1) g_{n_3}(\omega) \overline{g}_{n_3'}(\omega) \Bigg|^2.
\end{aligned}$$

Proceeding as above, we find that for ω outside a set of measure e^{-1/δ^r},

$$\mathcal{Q}_1 \lesssim \delta^{-2r} N_1^{-6} N_3^{-6} |S_{(\zeta)}| \sim \delta^{-2r} N_1^{-5} N_2^3,$$

which is a better estimate. Hence all in all we conclude that

$$M_2^2 \lesssim \delta^{-2r} N_1^{-4} N_2^3. \tag{7.23}$$

Case β_3. In this case we first define

$$S_{(n_2,n_2',n_1,n_1',\zeta)} := \left\{ (n, n_3) : \begin{array}{c} n_2 = -n_1 + n - n_3,\ n_2' = n_1' + n - n_3, \\ n_3, n_2, n_2' \neq n_1, n_1',\ n \in \tilde{C}, \\ \zeta = -|n_1|^2 + |n_3|^2,\ \zeta = -|n_1'|^2 + |n_3|^2 \end{array} \right\}$$

with $|S_{(n_2,n_2',n_1,n_1',\zeta)}| \lesssim N_3^2$ by (6.2) since n is determined by n_3 and the latter lies on a sphere of radius at most N_1 intersected with a ball of radius N_3. If we now define

$$S_{(\zeta)} := \left\{ (n_2, n_2', n, n_1, n_1', n_3) : \begin{array}{c} n_2 = -n_1 + n - n_3,\ n_2' = n_1' + n - n_3, \\ n_3, n_2, n_2' \neq n_1, n_1',\ n \in \tilde{C}, \\ \zeta = -|n_1|^2 + |n_3|^2,\ \zeta = -|n_1'|^2 + |n_3|^2 \end{array} \right\},$$

then $|S_{(\zeta)}| \lesssim N_1^2 N_3^3 N_2^3$, since again n ranges over a cube of size N_2 and we use (6.1) to count n_1 and n_1'. We follow the argument used above in (7.17)–(7.23) to bound M_2^2 but now with the couple (n_1, n_1') and corresponding sums $\mathcal{Q}_1$ and $\mathcal{Q}_2$. Just as in Case β_2 above, the bound for $\mathcal{Q}_2$ is larger. We then obtain, for ω outside a set of measure e^{-1/δ^r},

$$\begin{aligned} M_2^2 &\lesssim \delta^{-2r} N_1^{-6} N_3^{-6} \sum_{n_2 \neq n_2'} \sum_{n_1, n_1'} |S_{(n_2,n_2',n_1,n_1',\zeta)}|^2 \\ &\lesssim \delta^{-2r} N_1^{-6} N_3^{-6} N_3^2 \sum_{n_2 \neq n_2'} \sum_{n_1, n_1'} |S_{(n_2,n_2',n_1,n_1',\zeta)}| \\ &\lesssim \delta^{-2r} N_1^{-6} N_3^{-6} N_3^2 |S_{(\zeta)}| \sim \delta^{-2r} N_1^{-4} N_3^{-1} N_2^3. \end{aligned}$$

Case β_4. In this case note that $N_1 \sim N_3 \sim N_2$. We define two sets. First,

$$S_{(n_2,n_2',n_1,n_3',\zeta)} := \left\{ (n, n_3) : \begin{array}{c} n_2 = n_1 + n - n_3,\ n_2' = n_3 + n - n_3', \\ n_2, n_2', n_3, n_3' \neq n_1,\ n \in \tilde{C}, \\ \zeta = -|n_1|^2 + |n_3|^2,\ \zeta = |n_3|^2 + |n_3'|^2 \end{array} \right\},$$

and since n_3 lives on a sphere of radius at most N_1, from (6.1) we have $|S_{(n_2,n_2',n_1,n_3',\zeta)}| \lesssim N_1$. Next, the set

$$S_{(\zeta)} := \left\{ (n_2, n_2', n, n_1, n_3', n_3) : \begin{array}{c} n_2 = n_1 + n - n_3,\ n_2' = n_3 + n - n_3', \\ n_2, n_2', n_3, n_3' \neq n_1,\ n \in \tilde{C}, \\ \zeta = -|n_1|^2 + |n_3|^2,\ \zeta = -|n_3|^2 + |n_3'|^2 \end{array} \right\}$$

has $|S_{(\zeta)}| \lesssim N_1 N_2^3 N_3^6$. Just as in Case β_3 and following the argument in (7.17)–(7.23) but with the couple (n_1, n_3') we obtain, for ω outside a set of measure e^{-1/δ^r},

$$\begin{aligned} M_2^2 &\lesssim \delta^{-2r} N_1^{-6} N_3^{-6} \sum_{n_2 \neq n_2'} \sum_{n_1, n_3'} |S_{(n_2,n_2',n_1,n_3',\zeta)}|^2 \\ &\lesssim \delta^{-2r} N_1^{-6} N_3^{-6} N_1 \sum_{n_2 \neq n_2'} \sum_{n_1, n_3'} |S_{(n_2,n_2',n_1,n_3',\zeta)}| \\ &\lesssim \delta^{-2r} N_1^{-6} N_3^{-6} N_1 |S_{(\zeta)}| \sim \delta^{-2r} N_1^{-4} N_2^{-3}. \end{aligned}$$

Case β_5. By symmetry this case is exactly the same as Case β_4.

We now put all the estimates above together and bound $\mathcal{T}$ in Cases β_1–β_5:

$$\begin{aligned} \mathcal{T} &\lesssim \|a_{n_2}\|_{\ell^2}^2 N_1 N_2 \|\mathcal{G}\mathcal{G}^*\| \lesssim \|a_{n_2}\|_{\ell^2}^2 N_1 N_2 (M_1 + M_2) \\ &\lesssim \|a_{n_2}\|_{\ell^2}^2 \delta^{-2r} N_1 N_2 N_1^{-2} N_2^{3/2} \lesssim \delta^{-2r} N_2^{5/2} N_1^{-1} \|a_{n_2}\|_{\ell^2}^2. \end{aligned}$$

Case β_6**.** In this case we set

$$S_{(n_2,n_2',\zeta)} := \left\{ (n,n_1,n_3) : \begin{array}{l} n_2 = n_1 + n - n_3,\ n_2' = n_3 + n - n_1, \\ n_1 \neq n_2, n_2', n_3,\ |n_1|^2 = |n_3|^2,\ n \in \tilde{C} \end{array} \right\}.$$

Notice that the summation over ζ is eliminated and in this case $N_1 \sim N_2 \sim N_3$, so $|S_{(n_2,n_2',\zeta)}| \sim N_3^4$. Using (3.5) we have, for ω outside a set of measure e^{-1/δ^r},

$$\begin{aligned} M_2^2 &= \sum_{n_2 \neq n_2'} \Big| \sum_{n \in \tilde{C}} \sigma_{n_2,n} \overline{\sigma}_{n_2',n} \Big|^2 \sim \sum_{n_2 \neq n_2'} \Big| \sum_{S_{(n_2,n_2',\zeta)}} \frac{|g_{n_1}(\omega)|^2}{|n_1|^3} \frac{|g_{n_3}(\omega)|^2}{|n_3|^3} \Big|^2 \\ &\lesssim \sum_{n_2 \neq n_2'} N_1^{-6+\varepsilon} N_3^{-6} |S_{(n_2,n_2',\zeta)}|^2 \lesssim N_1^{-6+\varepsilon} N_3^{-6} N_3^4 |S_{(\zeta)}| \end{aligned} \tag{7.24}$$

where

$$S_{(\zeta)} := \left\{ (n_2,n_2',n,n_1,n_3) : \begin{array}{l} n_2 = n_1 + n - n_3,\ n_2' = n_3 + n - n_1, \\ n_1 \neq n_3, n_2, n_2',\ |n_1|^2 = |n_3|^2,\ n \in \tilde{C} \end{array} \right\}$$

and $|S_{(\zeta)}| \lesssim N_2^3 N_3^4$. Hence $M_2 \lesssim N_1^{-3+\varepsilon} N_2^{5/2}$ and as a consequence

$$\mathcal{T} \lesssim \|a_{n_2}\|_{\ell^2}^2 N_1^{-3+\varepsilon} N_2^{5/2}.$$

We now notice that to prove (7.4) we first have to consider the case when $n_1 = n_3$, which is not excluded here, and then we can use exactly the same argument as above since a plus or minus sign in front of n_3 does not change any of the counting.

Consider now (7.4) with $n_1 = n_3$. Note that $N_1 \sim N_2 \sim N_3$. We now set

$$\mathcal{T} := \sum_{m \in \mathbb{Z},\, n \in \mathbb{Z}^3} \Big| \sum_{\substack{n = -2n_1 + n_2 \\ m = -2|n_1|^2 + |n_2|^2}} \frac{(\overline{g}_{n_1}(\omega))^2}{|n_1|^3} a_{n_2} \Big|^2. \tag{7.25}$$

Let $S_{(m,n)} := \{(n_1,n_2) : n = -2n_1 + n_2, m = -2|n_1|^2 + |n_2|^2\}$, and note that $|S_{(m,n)}| \lesssim N_1$. Then

$$\mathcal{T} \lesssim N_1 \sum_{m,n} \sum_{S_{(m,n)}} \frac{|\overline{g}_{n_1}(\omega)|^4}{|n_1|^6} |a_{n_2}|^2 \sim N_1 \sum_{n,n_1 \in \mathbb{Z}^3} \frac{|\overline{g}_{n_1}(\omega)|^4}{|n_1|^6} |a_{n+2n_1}|^2 \lesssim N_1^{-2+\varepsilon} \|a_{n_2}\|_{l^2}^2,$$

where we use (3.5) for ω outside a set of measure e^{-1/δ^r}. □

Proposition 7.3. *Let D_j and R_k be as above and fix $N_1 \geq N_2 \geq N_3$, $r, \delta > 0$ and $C \in \mathcal{C}_{N_2}$. Then there exists $\mu > 0$ and a set $\Omega_\delta \in A$ with $\mathbb{P}(\Omega_\delta^c) \leq e^{-1/\delta^r}$ such that for any $\omega \in \Omega_\delta$ we have (7.5) and (7.6).*

Proof. We start by estimating (7.5) where without any loss of generality we assume that $\tilde{D}_1 = D_1$. We now set

$$\mathcal{T} := \sum_{m\in\mathbb{Z},\, n\in\mathbb{Z}^3} \Bigg| \sum_{\substack{n=n_1-n_2+n_3 \\ n_2\neq n_3 \\ m=|n_1|^2-|n_2|^2+|n_3|^2}} \chi_C(n_1) a_{n_1} \frac{\overline{g}_{n_2}(\omega)}{|n_2|^{3/2}} \frac{g_{n_3}(\omega)}{|n_3|^{3/2}} \Bigg|^2. \tag{7.26}$$

We are going to use duality and change of variables with $\zeta := m - |n_1|^2 = -|n_2|^2 + |n_3|^2$ again. Note though that if n_1 is in a cube C of size N_2 then also n will be in a cube $\tilde{C}$ of approximately the same size. Hence just as in (7.15) we need to estimate

$$\|\chi_C a_{n_1}\|_{\ell^2}^2 \|\gamma\|_{\ell^2}^2 \sum_{n_1} \sum_{|\zeta|\leq N_2^2} \Big| \sum_n \sigma_{n_1,n} \chi_{\tilde{C}}(n) k_n \Big|^2$$

where

$$\sigma_{n_1,n} = \sum_{\substack{n_1=n_2+n-n_3,\, n_2\neq n_1,n_3 \\ \zeta=-|n_2|^2+|n_3|^2}} \frac{\overline{g}_{n_2}(\omega)}{|n_2|^{3/2}} \frac{g_{n_3}(\omega)}{|n_3|^{3/2}}.$$

If we denote by $\mathcal{G}$ the matrix of entries $\sigma_{n_1,n}$, and we recall that the variation in ζ is at most N_2^2, we are reduced to estimating

$$\|\chi_C a_{n_1}\|_{\ell^2}^2 N_2^2 \|\mathcal{G}\mathcal{G}^*\|.$$

We note that by Lemma 6.3,

$$\|\mathcal{G}\mathcal{G}^*\| \lesssim \max_{n_1} \sum_n |\sigma_{n_1,n}|^2 + \Bigg(\sum_{n_1\neq n_1'} \Big| \sum_{n\in\tilde{C}} \sigma_{n_1,n} \overline{\sigma}_{n_1',n} \Big|^2 \Bigg)^{1/2} =: M_1 + M_2$$

where $\tilde{C}$ is a cube of sidelength approximately N_2.

From this point on, the proof is similar to the one already provided for (7.3) where n_2 is replaced by n_1. We still go through the argument though, since the sizes of n_1 and n_2 are different.

To estimate M_1 we first define

$$S_{(\zeta,n_1)} := \{(n_2, n, n_3) : n_2 \neq n_1, n_3,\ n_2 = n_1 - n + n_3,\ \zeta = -|n_2|^2 + |n_3|^2\}.$$

Applying (6.1) for each fixed n_3, we find that $|S_{(\zeta,n_1)}| \lesssim N_3^3 N_2$ since n_2 sits on a sphere of radius approximately N_2 . Then we proceed as in (7.16) to obtain, for ω outside a set of measure e^{-1/δ^r}, the bound

$$M_1 \lesssim \delta^{-r} N_2^{-3} N_3^{-3} N_3^3 N_2 \sim \delta^{-2r} N_2^{-2}.$$

To estimate M_2 we first write

$$M_2^2 = \sum_{n_1\neq n_1'} \Big| \sum_{n\in\tilde{C}} \sigma_{n_1,n} \overline{\sigma}_{n_1',n} \Big|^2 \sim \sum_{n_1\neq n_1'} \Bigg| \sum_{S_{(n_1,n_1',\zeta)}} \frac{\overline{g}_{n_2}(\omega)}{|n_2|^{3/2}} \frac{g_{n_3}(\omega)}{|n_3|^{3/2}} \frac{g_{n_2'}(\omega)}{|n_2'|^{3/2}} \frac{\overline{g}_{n_3'}(\omega)}{|n_3'|^{3/2}} \Bigg|^2$$

where

$$S_{(n_1,n_1',\zeta)} = \left\{ (n, n_2, n_3, n_2', n_3') : \begin{array}{l} n_2 = n_1 - n + n_3,\ n_2' = n_1' - n + n_3', \\ n_2 \neq n_1, n_3,\ n_2' \neq n_1', n_3',\ n \in \tilde{C}, \\ \zeta = -|n_2|^2 + |n_3|^2,\ \zeta = -|n_2'|^2 + |n_3'|^2 \end{array} \right\}.$$

We organize once again the estimates according to whether some frequencies are the same or not. As before, we have six cases:

- **Case** β_1**:** n_2, n_2', n_3, n_3' are all different.
- **Case** β_2**:** $n_2 = n_2'$; $n_3 \neq n_3'$.
- **Case** β_3**:** $n_2 \neq n_2'$; $n_3 = n_3'$.
- **Case** β_4**:** $n_2 \neq n_3'$; $n_3 = n_2'$.
- **Case** β_5**:** $n_2 = n_3'$; $n_3 \neq n_2'$.
- **Case** β_6**:** $n_2 = n_3'$; $n_3 = n_2'$.

Case β_1**.** We define

$$S_{(\zeta)} := \left\{ (n_1, n_1', n, n_2, n_3, n_2', n_3') : \begin{array}{l} n_2 = n_1 - n + n_3,\ n_2' = n_1' - n + n_3' \\ n_2 \neq n_1, n_3,\ n_2' \neq n_1', n_3',\ n_1, n_1' \in C, \\ \zeta = -|n_2|^2 + |n_3|^2,\ \zeta = -|n_2'|^2 + |n_3'|^2 \end{array} \right\},$$

and note that $|S_{(\zeta)}| \lesssim N_2^2 N_3^6 N_2^3$ by Lemma 6.1 since for n_3 fixed, n_2 and n_2' sit on a sphere of radius $\sim N_2$, and $n \in \tilde{C}$, a cube of sidelength approximately N_2. Hence, for ω outside a set of measure e^{-1/δ^r}, we obtain

$$M_2^2 \lesssim \delta^{-4r} N_2^{-6} N_3^{-6} N_2^2 N_3^6 N_2^3 \sim \delta^{-4r} N_2^{-1}.$$

Case β_2**.** In this case we define two sets. We start with

$$S_{(n_1,n_1',n_3,n_3',\zeta)} := \left\{ (n, n_2) : \begin{array}{l} n_2 = n_1 - n + n_3,\ n_2 = n_1' - n + n_3', \\ n_2 \neq n_1, n_1', n_3, n_3',\ n \in \tilde{C}, \\ \zeta = -|n_2|^2 + |n_3|^2,\ \zeta = -|n_2|^2 + |n_3'|^2 \end{array} \right\}.$$

To compute $|S_{(n_1,n_1',n_3,n_3',\zeta)}|$, it is enough to count n_2; then n is determined. Since n_2 sits on a sphere of radius $\sim N_2$, by (6.1) we have $|S_{(n_1,n_1',n_3,n_3',\zeta)}| \lesssim N_2$. Then we set

$$S_{(\zeta)} := \left\{ (n_1, n_1', n, n_2, n_3, n_3') : \begin{array}{l} n_2 = n_1 - n + n_3,\ n_2 = n_1' - n + n_3', \\ n_2 \neq n_1, n_1', n_3, n_3',\ n \in \tilde{C}, \\ \zeta = -|n_2|^2 + |n_3|^2,\ \zeta = -|n_2|^2 + |n_3'|^2 \end{array} \right\},$$

for which $|S_{(\zeta)}| \lesssim N_2 N_3^6 N_2^3$, where we have used again that $n \in \tilde{C}$. Arguing as in (7.17)–(7.23), we then find that for ω outside a set of measure e^{-1/δ^r},

$$
\begin{aligned}
M_2^2 &\lesssim \delta^{-2r} N_2^{-6} N_3^{-6} \sum_{n \neq n'} \sum_{n_3, n_3'} |S_{(n_1, n_1', n_3, n_3', \zeta)}|^2 \\
&\lesssim \delta^{-2r} N_2^{-6} N_3^{-6} N_2 \sum_{n_1 \neq n_1'} \sum_{n_3, n_3'} |S_{(n_1, n_1', n_3, n_3', \zeta)}| \\
&\lesssim \delta^{-2r} N_2^{-6} N_3^{-6} N_2 |S_{(\zeta)}| \sim \delta^{-2r} N_2^{-1}.
\end{aligned}
$$

Case β_3. In this case we define first

$$
S_{(n_2, n_2', n_1, n_1', \zeta)} := \left\{ (n, n_3) : \begin{array}{l} n_2 = n_1 - n + n_3,\ n_2' = n_1' - n + n_3, \\ n_2, n_2' \neq n_3, n_1, n_1',\ n \in \tilde{C}, \\ \zeta = -|n_2|^2 + |n_3|^2,\ \zeta = -|n_2'|^2 + |n_3|^2 \end{array} \right\}
$$

for which $|S_{(n_2, n_2', n_1, n_1', \zeta)}| \lesssim N_3^2$, since n is determined by n_3 and the latter lies on a sphere of radius at most N_1 intersected with a ball of radius N_3 (see Lemma 6.1). Then we define

$$
S_{(\zeta)} := \left\{ (n_2, n_2', n, n_1, n_1', n_3) : \begin{array}{l} n_2 = n_1 - n + n_3,\ n_2' = n_1' - n + n_3, \\ n_2, n_2' \neq n_3, n_1, n_1',\ n \in \tilde{C}, \\ \zeta = -|n_2|^2 + |n_3|^2,\ \zeta = -|n_2'|^2 + |n_3|^2 \end{array} \right\}
$$

for which $|S_{(\zeta)}| \lesssim N_2^2 N_3^3 N_2^3$, since again n ranges over a cube of size N_2. We then find, as usual using (3.4) and (3.5) as above, that for ω outside a set of measure e^{-1/δ^r},

$$
\begin{aligned}
M_2^2 &\lesssim \delta^{-2r} N_2^{-6} N_3^{-6} \sum_{n_1 \neq n_1'} \sum_{n_2, n_2'} |S_{(n_2, n_2', n_1, n_1', \zeta)}|^2 \\
&\lesssim \delta^{-2r} N_2^{-6} N_3^{-6} N_3^2 \sum_{n_1 \neq n_1'} \sum_{n_2, n_2'} |S_{(n_2, n_2', n_1, n_1', \zeta)}| \\
&\lesssim \delta^{-2r} N_2^{-6+\varepsilon} N_3^{-6} N_3^2 |S_{(\zeta)}| \sim \delta^{-2r} N_2^{-1+\varepsilon} N_3^{-1}.
\end{aligned}
$$

Case β_4. In this case note that $N_3 \sim N_2$. We define two sets:

$$
S_{(n_1, n_1', n_2, n_3', \zeta)} := \left\{ (n, n_3) : \begin{array}{l} n_2 = n_1 - n + n_3,\ n_3 = n_1' - n + n_3', \\ n_2 \neq n_1, n_3,\ n_3 \neq n_3', n_1',\ n \in \tilde{C}, \\ \zeta = -|n_2|^2 + |n_3|^2,\ \zeta = -|n_3|^2 + |n_3'|^2 \end{array} \right\}
$$

with $|S_{(n_1, n_1', n_2, n_3', \zeta)}| \lesssim N_2$ since n_3 lives on a sphere of radius at most N_2; and

$$
S_{(\zeta)} := \left\{ (n_1, n_1', n, n_2, n_3', n_3) : \begin{array}{l} n_2 = n_1 - n + n_3,\ n_3 = n_1' - n + n_3', \\ n_2 \neq n_1, n_3,\ n_3 \neq n_3', n_1',\ n \in \tilde{C}, \\ \zeta = -|n_2|^2 + |n_3|^2,\ \zeta = -|n_3|^2 + |n_3'|^2 \end{array} \right\}
$$

with $|S_{(\zeta)}| \lesssim N_2 N_2^3 N_3^6$ since for fixed n_3, n_3', the frequencies n_2 sit on a sphere of radius at most N_2 and $n \in \tilde{C}$ (see Lemma 6.1). Then as above, for ω outside a set of measure e^{-1/δ^r},

$$M_2^2 \lesssim \delta^{-2r} N_2^{-6} N_3^{-6} \sum_{n\neq n'} \sum_{n_2, n_3'} |S_{(n_1,n_1',n_2,n_3',\zeta)}|^2$$
$$\lesssim \delta^{-2r} N_2^{-6} N_3^{-6} N_2 \sum_{n\neq n'} \sum_{n_2, n_3'} |S_{(n_1,n_1',n_2,n_3',\zeta)}|$$
$$\lesssim \delta^{-2r} N_2^{-6} N_3^{-6} N_2 |S_{(\zeta)}| \sim \delta^{-2r} N_2^{-1}.$$

Case β_5. By symmetry this case is exactly the same as Case β_4.

We now put all the estimates together and bound $\mathcal{T}$ in Cases β_1–β_5:

$$\mathcal{T} \lesssim \|\chi_C a_{n_1}\|_{\ell^2}^2 N_2^2 \|\mathcal{G}\mathcal{G}^*\| \lesssim \|a_{n_1}\|_{\ell^2}^2 N_2^2 (M_1 + M_2)$$
$$\lesssim \|\chi_C a_{n_1}\|_{\ell^2}^2 \delta^{-2r} N_2^2 N_2^{-1/2} \sim \|\chi_C a_{n_1}\|_{\ell^2}^2 \delta^{-2r} N_2^{3/2}.$$

Case β_6. In this case

$$S_{(n_1,n_1',\zeta)} := \left\{ (n, n_2, n_3) : \begin{array}{l} n_2 = n_1 - n + n_3,\ n_3 = n_1' - n + n_2, \\ n_2 \neq n_3, n_1,\ |n_2|^2 = |n_3|^2,\ n \in \tilde{C} \end{array} \right\}.$$

Notice that $\Delta\zeta = 1$ and in this case $N_2 \sim N_3$, so $|S_{(n_1,n_1',\zeta)}| \sim N_3^4$. Then, as in (7.24),

$$M_2^2 \lesssim N_2^{-6+\epsilon} N_3^{-6} N_3^4 |S_{(\zeta)}|$$

where

$$S_{(\zeta)} := \left\{ (n_1, n_1', n, n_2, n_3) : \begin{array}{l} n_2 = n_1 - n + n_3,\ n_3 = n_1' - n + n_2, \\ n_2 \neq n_3, n_1,\ |n_2|^2 = |n_3|^2,\ n \in \tilde{C} \end{array} \right\}$$

and $|S_{(\zeta)}| \lesssim N_2^3 N_3^4$. Hence, all in all, for ω outside a set of measure e^{-1/δ^r}, we have $M_2 \lesssim N_2^{-1/2+\epsilon}$, and as a consequence

$$\mathcal{T} \lesssim \|\chi_C a_{n_1}\|_{\ell^2}^2 N_2^{-1/2+\epsilon},$$

which is a better bound.

To prove (7.6) we write

$$\mathcal{T} := \sum_{m\in\mathbb{Z},\, n\in\mathbb{Z}^3} \Bigg| \sum_{\substack{n=n_1+n_2+n_3 \\ m=|n_1|^2+|n_2|^2+|n_3|^2}} \chi_C(n_1) a_{n_1} \frac{g_{n_2}(\omega)}{|n_2|^{3/2}} \frac{g_{n_3}(\omega)}{|n_3|^{3/2}} \Bigg|^2. \tag{7.27}$$

We can repeat the argument above after checking the case $n_2 = n_3$. In this case (7.27) becomes

$$\mathcal{T} = \sum_{m\in\mathbb{Z},\, n\in\mathbb{Z}^3} \Bigg| \sum_{\substack{n=n_1+2n_2 \\ m=|n_1|^2+2|n_2|^2}} \chi_C(n_1) a_{n_1} \frac{(g_{n_2}(\omega))^2}{|n_2|^3} \Bigg|^2.$$

Let $S_{(m,n)} := \{(n_1, n_2) : n = n_1 + 2n_2,\ m = |n_1|^2 + 2|n_2|^2\}$, and note that by Lemma 6.1, $|S_{(m,n)}| \lesssim \min(N_1, N_2^2)$. Then

$$\mathcal{T} \lesssim \min(N_1, N_2^2) \sum_{m,n} \sum_{S_{(m,n)}} \frac{|g_{n_2}(\omega)|^4}{|n_2|^6} |\chi_C a_{n_1}|^2$$
$$\sim \min(N_1, N_2^2) \sum_{n,n_1} \frac{|g_{(n-n_1)/2}(\omega)|^4}{|(n-n_1)/2|^6} |\chi_C a_{n_1}|^2 \lesssim \min(N_1, N_2^2) N_2^{-3+\varepsilon} \|\chi_C a_{n_1}\|_{l^2}^2,$$

where we have used (3.5) for ω outside a set of measure e^{-1/δ^r}. □

Proposition 7.4. *Let D_j and R_k be as above and fix $N_1 \geq N_2 \geq N_3$, $r, \delta > 0$ and $C \in \mathcal{C}_{N_2}$. Then there exists $\mu > 0$ and a set $\Omega_\delta \in A$ with $\mathbb{P}(\Omega_\delta^c) \leq e^{-1/\delta^r}$ such that for any $\omega \in \Omega_\delta$ we have (7.7) and (7.8).*

Proof. Without loss of generality we assume that $\tilde{D}_3 = D_3$. We write

$$\mathcal{T} := \sum_{m \in \mathbb{Z},\, n \in \mathbb{Z}^3} \Bigg| \sum_{\substack{n = -n_1 + n_2 + n_3 \\ n_1 \neq n_2, n_3 \\ m = -|n_1|^2 + |n_2|^2 + |n_3|^2}} \chi_C(n_1) \frac{\overline{g}_{n_1}(\omega)}{|n_1|^{3/2}} \frac{g_{n_2}(\omega)}{|n_2|^{3/2}} a_{n_3} \Bigg|^2 \tag{7.28}$$

where $C \in \mathcal{C}_{N_2}$. Let us now define

$$\sigma_{n,n_3} := \sum_{\substack{n = -n_1 + n_2 + n_3,\, n_1 \neq n_2, n_3 \\ m = -|n_1|^2 + |n_2|^2 + |n_3|^2}} \chi_C(n_1) \frac{\overline{g}_{n_1}(\omega)}{|n_1|^{3/2}} \frac{g_{n_2}(\omega)}{|n_2|^{3/2}}.$$

If we denote by $\mathcal{G}$ the matrix with entries σ_{n,n_3}, since the variation in m is at most $N_1 N_2$ we can continue the estimate of $\mathcal{T}$ in (7.28) by

$$\mathcal{T} \lesssim \|a_{n_3}\|_{\ell^2}^2 N_1 N_2 \|\mathcal{G}\mathcal{G}^*\|.$$

Once again by Lemma 6.3,

$$\|\mathcal{G}\mathcal{G}^*\| \lesssim \max_n \sum_{n_3} |\sigma_{n,n_3}|^2 + \Bigg(\sum_{n \neq n'} \Bigg| \sum_{n_3} \sigma_{n,n_3} \overline{\sigma}_{n',n_3} \Bigg|^2 \Bigg)^{1/2} =: M_1 + M_2.$$

To estimate M_1 we first define

$$S_{(m,n)} := \{(n_1, n_2, n_3) : n_1 \neq n_2, n_3,\ n = -n_1 + n_2 + n_3,\ m = -|n_1|^2 + |n_2|^2 + |n_3|^2\}.$$

By (6.3) we have $|S_{(m,n)}| \lesssim N_3^3 N_2^2$ since once n_3 is fixed we use $m = -|n_2 + n_3 - n|^2 + |n_2|^2 + |n_3|^2$ to count n_2 which lives on the intersection of a plane with a ball of radius N_2. Then as in (7.16), for ω outside a set of measure e^{-1/δ^r}, we have

$$M_1 \lesssim \sup_{n,m} \sum_{n_3} \Bigg| \sum_{\substack{n = -n_1 + n_2 + n_3, n_1 \neq n_2, n_3 \\ m = -|n_1|^2 + |n_2|^2 + |n_3|^2}} \frac{\overline{g}_{n_1}(\omega)}{|n_1|^{3/2}} \chi_C(n_1) \frac{g_{n_2}(\omega)}{|n_2|^{3/2}} \Bigg|^2$$
$$\lesssim \sup_{n,m} \delta^{-r} N_1^{-3} N_2^{-3} |S_{(m,n)}| \lesssim \delta^{-2r} N_1^{-3} N_2^{-3} N_3^3 N_2^2 \sim \delta^{-2r} N_1^{-3} N_2^{-1} N_3^3.$$

To estimate M_2 we first write

$$M_2^2 = \sum_{n \neq n'} \Big| \sum_{n_3} \sigma_{n,n_3} \overline{\sigma}_{n',n_3} \Big|^2 \sim \sum_{n \neq n'} \Big| \sum_{S_{(n,n',m)}} \frac{\overline{g}_{n_1}(\omega)}{|n_1|^{3/2}} \frac{g_{n_2}(\omega)}{|n_2|^{3/2}} \frac{g_{n_1'}(\omega)}{|n_1'|^{3/2}} \frac{\overline{g}_{n_2'}(\omega)}{|n_2'|^{3/2}} \Big|^2$$

where

$$S_{(n,n',m)} := \left\{ (n_3, n_1, n_2, n_1', n_2') : \begin{array}{l} n = -n_1 + n_2 + n_3, \; n' = -n_1' + n_2' + n_3, \\ n_1 \neq n_2, n_3, \; n_1' \neq n_2', n_3, \; n_1, n_1' \in C, \\ m = -|n_1|^2 + |n_2|^2 + |n_3|^2, \; m = -|n_1'|^2 + |n_2'|^2 + |n_3|^2 \end{array} \right\}.$$

Just as in the proof of (7.3), we need to organize the estimates according to whether some frequencies are the same or not; in all we have six cases.

- **Case β_1:** n_1, n_1', n_2, n_2' are all different.
- **Case β_2:** $n_1 = n_1'$; $n_2 \neq n_2'$.
- **Case β_3:** $n_1 \neq n_1'$; $n_2 = n_2'$.
- **Case β_4:** $n_1 \neq n_2'$; $n_2 = n_1'$.
- **Case β_5:** $n_1 = n_2'$; $n_2 \neq n_1'$.
- **Case β_6:** $n_1 = n_2'$; $n_2 = n_1'$.

Case β_1. In this case we let

$$S_{(m)} := \left\{ (n, n', n_3, n_1, n_2, n_1', n_2') : \begin{array}{l} n = -n_1 + n_2 + n_3, \; n' = -n_1' + n_2' - n_3, \\ n_1 \neq n_2, n_3, \; n_1' \neq n_2', n_3, \; n_1, n_1' \in C, \\ m = -|n_1|^2 + |n_2|^2 + |n_3|^2, \; m = -|n_1'|^2 + |n_2'|^2 + |n_3|^2 \end{array} \right\}$$

with $|S_{(m)}| \lesssim N_1^2 N_2^6 N_3^3$. As in the argument for (7.16), this implies that for ω outside a set of measure e^{-1/δ^r},

$$M_2^2 \lesssim \delta^{-4r} N_1^{-6} N_2^{-6} N_1^2 N_2^6 N_3^3 \sim \delta^{-4r} N_1^{-4} N_3^3.$$

Case β_2. In this case we define two sets. We start with

$$S_{(n,n',n_2,n_2',m)} := \left\{ (n_3, n_1) : \begin{array}{l} n = -n_1 + n_2 + n_3, \; n' = -n_1 + n_3 + n_2', \\ n_1 \neq n_2, n_2', n_3, \\ m = -|n_1|^2 + |n_2|^2 + |n_3|^2, \; m = -|n_1|^2 + |n_3|^2 + |n_2'|^2 \end{array} \right\}.$$

To compute $|S_{(n,n',n_2,n_2',m)}|$ we count n_3; then n_1 is determined. Since the n_3 sit on a plane, we see by (6.3) that $|S_{(n,n',n_2,n_2',m)}| \lesssim N_3^2$. Then we set

$$S_{(m)} := \left\{ (n, n', n_3, n_1, n_2, n_2') : \begin{array}{l} n = -n_1 + n_2 + n_3, \; n' = -n_1 + n_3 + n_2', \\ n_1 \neq n_2, n_2', n_3, \\ m = -|n_1|^2 + |n_2|^2 + |n_3|^2, \; m = -|n_1|^2 + |n_3|^2 + |n_2'|^2 \end{array} \right\}$$

for which $|S_{(m)}| \lesssim N_1 N_2^6 N_3^3$. Following the argument in (7.17)–(7.23) we have, for ω outside a set of measure e^{-1/δ^r},

$$M_2^2 \lesssim \delta^{-2r} N_1^{-6} N_2^{-6} \sum_{n\neq n'} \sum_{n_2,n_2'} |S_{(n,n',n_2,n_2',m)}|^2$$
$$\lesssim \delta^{-2r} N_1^{-6} N_2^{-6} N_3^2 \sum_{n\neq n'} \sum_{n_2,n_2'} |S_{(n,n',n_2,n_2',m)}| \lesssim \delta^{-2r} N_1^{-6} N_2^{-6} N_3^2 |S_{(m)}| \sim \delta^{-2r} N_1^{-5} N_3^5.$$

Case β_3. In this case we define first

$$S_{(n,n',n_1,n_1',m)} := \left\{ (n_2,n_3) : \begin{array}{l} n = -n_1+n_2+n_3,\ n' = -n_1'+n_2+n_3, \\ n_2, n_3 \neq n_1, n_1',\ n_1, n_1' \in C, \\ m = -|n_1|^2+|n_2|^2+|n_3|^2,\ m = -|n_1'|^2+|n_2|^2+|n_3|^2 \end{array} \right\}$$

with $|S_{(n,n',n_1,n_1',m)}| \lesssim N_3^2$, since n_2 is determined by n_3 and the latter lies on a sphere of radius at most N_1. On the other hand,

$$S_{(m)} := \left\{ (n,n',n_2,n_1,n_1',n_3) : \begin{array}{l} n = -n_1+n_2+n_3,\ n' = -n_1'+n_2+n_3, \\ n_2, n_3 \neq n_1, n_1',\ n_1, n_1' \in C, \\ m = -|n_1|^2+|n_2|^2+|n_3|^2,\ m = -|n_1'|^2+|n_2|^2+|n_3|^2 \end{array} \right\}$$

has $|S_{(m)}| \lesssim N_1^2 N_3^3 N_2^3$. Hence arguing as above we have

$$M_2^2 \lesssim \delta^{-2r} N_1^{-6} N_2^{-6} \sum_{n\neq n'} \sum_{n_1,n_1'} |S_{(n,n',n_1,n_1',m)}|^2$$
$$\lesssim \delta^{-2r} N_1^{-6} N_2^{-6} N_3^2 \sum_{n\neq n'} \sum_{n_1,n_1'} |S_{(n,n',n_1,n_1',m)}|$$
$$\lesssim \delta^{-2r} N_1^{-6} N_2^{-6} N_3^2 |S_{(m)}| \sim \delta^{-2r} N_1^{-4} N_2^{-3} N_3^5$$

for ω outside a set of measure e^{-1/δ^r}.

Case β_4. We define two sets:

$$S_{(n,n',n_1,n_2',m)} := \left\{ (n_2,n_3) : \begin{array}{l} n = -n_1+n_2+n_3,\ n' = -n_2+n_2'+n_3, \\ n_2, n_3 \neq n_1, n_2', \\ m = -|n_1|^2+|n_2|^2+|n_3|^2,\ m = -|n_2|^2+|n_2'|^2+|n_3|^2 \end{array} \right\},$$

for which, since n_3 lives on a sphere of radius at most N_1, we have $|S_{(n,n',n_1,n_2',m)}| \lesssim \min(N_1, N_3^2)$; and

$$S_{(m)} := \left\{ (n,n',n_3,n_1,n_2',n_2) : \begin{array}{l} n = -n_1+n_2+n_3,\ n' = -n_2+n_2'+n_3, \\ n_2, n_3 \neq n_1, n_2', \\ m = -|n_1|^2+|n_2|^2+|n_3|^2,\ m = -|n_2|^2+|n_2'|^2+|n_3|^2 \end{array} \right\}$$

with $|S_{(m)}| \lesssim N_1 N_3^3 N_2^6$. Then

$$\begin{aligned} M_2^2 &\lesssim \delta^{-2r} N_1^{-6} N_2^{-6} \sum_{n\neq n'} \sum_{n_1, n_2'} |S_{(n,n',n_1,n_2',m)}|^2 \\ &\lesssim \delta^{-2r} N_1^{-6} N_2^{-6} \min(N_1, N_3^2) \sum_{n\neq n'} \sum_{n_1,n_2'} |S_{(n,n',n_1,n_2',m)}| \\ &\lesssim \delta^{-2r} N_1^{-6} N_2^{-6} \min(N_1, N_3^2) |S_{(m)}| \sim \delta^{-2r} N_1^{-4} N_3^3 \end{aligned}$$

for ω outside a set of measure e^{-1/δ^r}.

Case β_5**.** This case is exactly the same as Case β_4.

We now estimate $\mathcal{T}$ in Cases β_1–β_5:

$$\begin{aligned} \mathcal{T} &\lesssim \|a_{n_2}\|_{\ell^2}^2 N_1 N_2 \|\mathcal{G}\mathcal{G}^*\| \lesssim \|a_{n_2}\|_{\ell^2}^2 N_1 N_2 (M_1 + M_2) \\ &\lesssim \|a_{n_2}\|_{\ell^2}^2 \delta^{-4r} [N_1 N_2 (N_1^{-5/2} N_3^{5/2} + N_1^{-2} N_3^{3/2})] \lesssim \delta^{-4r} [N_1^{-3/2} N_2 N_3^{5/2} \\ &\quad + N_1^{-1} N_2 N_3^{3/2}] \|a_{n_2}\|_{\ell^2}^2. \end{aligned}$$

Case β_6**.** In this case we set

$$S_{(n,n',m)} := \left\{ (n_3, n_1, n_2) : \begin{array}{l} n = -n_1 + n_2 + n_3,\ n' = -n_2 + n_1 + n_3, \\ n_1 \neq n_2, n_3,\ |n_1|^2 = |n_2|^2,\ m = |n_3|^2 \end{array} \right\}$$

so $N_1 \sim N_2$ and $\Delta m \lesssim N_3^2$. We have $|S_{(n,n',m)}| \lesssim N_2^3 N_3$ since n_3 sits on a sphere of radius at most N_3. As in (7.24), for ω outside a set of measure e^{-1/δ^r}, we have

$$M_2^2 \lesssim N_1^{-6+\varepsilon} N_2^{-6} N_2^3 N_3 |S_{(m)}|$$

where

$$S_{(m)} := \left\{ (n, n', n_3, n_1, n_2) : \begin{array}{l} n = -n_1 + n_2 + n_3,\ n' = -n_2 + n_1 + n_3, \\ n_1 \neq n_2, n_3,\ |n_1|^2 = |n_2|^2,\ m = |n_3|^2 \end{array} \right\}$$

and $|S_{(m)}| \lesssim N_3 N_2^3 N_2$ since again n_3 sits on a sphere of radius at most N_3 and for fixed n_2 we see that n_1 sits on a sphere of radius at most N_2. Hence $M_2 \lesssim N_1^{-5/2+\varepsilon} N_3$ and so

$$\mathcal{T} \lesssim \|a_{n_3}\|_{\ell^2}^2 N_3^3 N_1^{-5/2+\varepsilon}.$$

The proof of (7.8) proceeds very much like the one we have just presented. Actually when $n_1 = n_2$ the estimates may be made better since we will not have planes, but spheres involved in the counting. On the other hand, here $n_1 = n_2$ could be a possibility. In this case we set

$$\mathcal{T} := \sum_{m\in\mathbb{Z},\, n\in\mathbb{Z}^3} \Bigg| \sum_{\substack{n=-2n_1+n_3 \\ m=-2|n_1|^2+|n_3|^2}} \frac{(\overline{g}_{n_1}(\omega))^2}{|n_1|^3} a_{n_3} \Bigg|^2.$$

Let $S_{(m,n)} := \{(n_1, n_3) : n = -2n_1 + n_3, n_3 \neq n_1, m = -2|n_1|^2 + |n_3|^2\}$, and note that by Lemma 6.1, $|S_{(m,n)}| \lesssim \min(N_1, N_3^2)$. Then using (3.5), for ω outside a set of measure e^{-1/δ^r} we have

$$\mathcal{T} \lesssim \min(N_1, N_3^2) \sum_{m,n} \sum_{S_{(m,n)}} \frac{|g_{n_1}(\omega)|^4}{|n_1|^6} |a_{n-2n_1}|^2 \lesssim \min(N_1, N_3^2) N_1^{-3+\varepsilon} \|a_{n_3}\|_{l^2}^2. \qquad \square$$

Proposition 7.5. *Let D_j and R_k be as above and fix $N_1 \geq N_2 \geq N_3$, $r, \delta > 0$ and $C \in \mathcal{C}_{N_2}$. Then there exist $\mu, \varepsilon > 0$ and a set $\Omega_\delta \in A$ with $\mathbb{P}(\Omega_\delta^c) \leq e^{-1/\delta^r}$ such that for any $\omega \in \Omega_\delta$ we have (7.9) for any $0 \leq \theta \leq 1$, and (7.10) .*

Proof. We first handle (7.9). Without loss of generality we assume that $\tilde{D}_i = D_i$, $i = 2, 3$. We set

$$\mathcal{T} := \sum_{m \in \mathbb{Z},\, n \in \mathbb{Z}^3} \Big| \sum_{\substack{n = n_1 + n_2 + n_3 \\ m = |n_1|^2 + |n_2|^2 + |n_3|^2}} \chi_C(n_1) \frac{g_{n_1}(\omega)}{|n_1|^{3/2}} a_{n_2} a_{n_3} \Big|^2.$$

Then

$$\mathcal{T} \lesssim \sum_{m \in \mathbb{Z},\, n \in \tilde{C}} \Big| \sum_{n_2, n_3} \sigma_{n,n_2} a_{n_2} a_{n_3} \Big|^2$$

where $\tilde{C}$ is again a cube of sidelength approximately N_2 and

$$\sigma_{n,n_2} = \begin{cases} \dfrac{g_{n-n_2-n_3}(\omega)}{|n - n_2 - n_3|^{3/2}} & \text{if } m = |n - n_2 - n_3|^2 + |n_2|^2 + |n_3|^2, \\ 0 & \text{otherwise.} \end{cases}$$

Note that σ_{n,n_2} also depends on m and n_3 but we estimate it independently of m and n_3 and take the supremum over them. By Cauchy–Schwarz in n_3, the fact that $\Delta m \lesssim N_1 N_2$ and Lemma 6.3 we have

$$\mathcal{T} \lesssim \|a_{n_3}\|_{\ell^2}^2 \|a_{n_2}\|_{\ell^2}^2 N_1 N_2 N_3^3 \|\mathcal{G}\mathcal{G}^*\|;$$

and as usual by Lemma 6.3 we have

$$\|\mathcal{G}\mathcal{G}^*\| \lesssim \max_n \sum_{n_2} |\sigma_{n,n_2}|^2 + \Big(\sum_{n \neq n' \in \tilde{C}} \Big| \sum_{n_2} \sigma_{n,n_2} \overline{\sigma}_{n',n_2} \Big|^2 \Big)^{1/2} =: M_1 + M_2.$$

To estimate M_1 we will use the set $S(n, n_3, m) := \{n_2 : m = |n - n_2 - n_3|^2 + |n_2|^2 + |n_3|^2\}$, with cardinality $|S_{(n,n_3,m)}| \lesssim N_1$ since this set describes a sphere whose radius is at most N_1. Using (3.5) we estimate

$$M_1 \lesssim \sum_{n_2 \in S(n,n_3,m)} N_1^{-3+\varepsilon} \lesssim N_1^{-2+\varepsilon} \tag{7.29}$$

for ω outside a set of measure e^{-1/δ^r}.

To estimate M_2 we first define

$$S_{(n_3,m)} := \{(n, n', n_2) : m = |n - n_2 - n_3|^2 + |n_2|^2 + |n_3|^2,\ m = |n' - n_2 - n_3|^2 + |n_2|^2 + |n_3|^2\}$$

and note that $|S_{(n_3,m)}| \lesssim N_2^3 N_1^2$. Then using (3.4) and arguments similar to those for (7.17)–(7.23) we have

$$M_2^2 \lesssim \delta^{-2r} N_1^{-6} |S_{(n_3,m)}| \lesssim \delta^{-2r} N_1^{-6} N_2^3 N_1^2. \tag{7.30}$$

From the estimates of M_1 and M_2 we deduce that for ω outside a set of measure e^{-1/δ^r},

$$\mathcal{T} \lesssim \|a_{n_3}\|_{\ell^2}^2 \|a_{n_2}\|_{\ell^2}^2 \delta^{-r} N_1 N_2 N_3^3 N_1^{-2} N_2^{3/2} \lesssim \|a_{n_3}\|_{\ell^2}^2 \|a_{n_2}\|_{\ell^2}^2 \delta^{-r} N_1^{-1} N_2^{5/2} N_3^3. \tag{7.31}$$

We will interpolate this estimate with the one we obtain below:

$$\begin{aligned}
\mathcal{T} &\lesssim N_1 N_2 \sup_m \sum_{n\in\mathbb{Z}^3} \Big| \sum_{\substack{n=n_1+n_2+n_3\\ m=|n_1|^2+|n_2|^2+|n_3|^2}} \chi_C(n_1) \frac{g_{n_1}(\omega)}{|n_1|^{3/2}} a_{n_2} a_{n_3} \Big|^2 \\
&\lesssim N_1 N_2 \|a_{n_3}\|_{\ell^2}^2 \sup_m \sum_{n,n_3\in\mathbb{Z}^3} \Big| \sum_{\substack{n=n_1+n_2+n_3\\ m=|n_1|^2+|n_2|^2+|n_3|^2}} \chi_C(n_1) \frac{g_{n_1}(\omega)}{|n_1|^{3/2}} a_{n_2} \Big|^2 \\
&\lesssim N_1 N_2 \|a_{n_3}\|_{\ell^2}^2 N_1^{-3+\varepsilon} \sup_m \sum_{n,n_3\in\mathbb{Z}^3} |S_{(n,n_3,m)}| \sum_{S_{(n,n_3,m)}} |a_{n_2}|^2 \\
&\lesssim N_1 N_2 \|a_{n_3}\|_{\ell^2}^2 N_1^{-3+\varepsilon} \min(N_2^2, N_1) \sup_m \sum_{n_2} \sum_{S_{(n_2,m)}} |a_{n_2}|^2 \\
&\lesssim N_1 N_2 \|a_{n_3}\|_{\ell^2}^2 \|a_{n_2}\|_{\ell^2}^2 N_1^{-3+\varepsilon} \min(N_2^2, N_1) N_1 N_3^3 \\
&\sim N_1^{-1+\varepsilon} N_3^3 N_2 \min(N_2^2, N_1) \|a_{n_2}\|_{\ell^2}^2 \|a_{n_3}\|_{\ell^2}^2
\end{aligned} \tag{7.32}$$

where $S_{(n,n_3,m)} := \{(n_1, n_2) : n = n_1+n_2+n_3,\ n_1 \in C,\ m = |n_1|^2+|n_2|^2+|n_3|^2\}$ with $|S_{(n,n_3,m)}| \lesssim \min(N_2^2, N_1)$, $S_{(n_2,m)} := \{(n, n_1, n_3) : n = n_1 + n_2 + n_3,\ n_1 \in C,\ m = |n_1|^2 + |n_2|^2 + |n_3|^2\}$ with $|S_{(n_2,m)}| \lesssim N_1 N_3^3$, and we have used (3.5) for ω outside a set of measure e^{-1/δ^r}.

The estimate of (7.9) now follows by interpolating (7.32) with (7.31).

We now move to (7.10). Again without loss of generality we assume that $\tilde{D}_i = D_1$, $i = 1, 3$. We use duality and the change of variables $\zeta = m - |n_1|^2 = |n_2|^2 + |n_3|^3$ as in the proof of Proposition 7.2. We note that the variation of ζ is at most N_2^2 and that $n \in \tilde{C}$, a cube of sidelength approximately N_2. We use (3.5) for ω outside a set of measure e^{-1/δ^r} and Lemma 6.1 to reduce the bound for $\mathcal{T}$ to estimating

$$\begin{aligned}
&N_2^2 \sup_\zeta \sum_{n_1\in\mathbb{Z}^3} \Big| \sum_{\substack{n_1=n-n_2-n_3\\ \zeta=|n_2|^2+|n_3|^2}} \chi_{\tilde{C}}(n) k_n \frac{g_{n_2}(\omega)}{|n_2|^{3/2}} a_{n_3} \Big|^2 \|\chi_C a_{n_1}\|_{\ell^2}^2 \\
&\quad\lesssim N_2^2 \|\chi_C a_{n_1}\|_{\ell^2}^2 \|a_{n_3}\|_{\ell}^2 \sup_\zeta \sum_{n_1,n_3\in\mathbb{Z}^3} \Big| \sum_{\substack{n_1=n-n_2-n_3\\ \zeta=|n_2|^2+|n_3|^2}} \chi_{\tilde{C}}(n) k_n \frac{g_{n_2}(\omega)}{|n_2|^{3/2}} \Big|^2 \\
&\quad\lesssim N_2^2 \|\chi_C a_{n_1}\|_{\ell^2}^2 \|a_{n_3}\|_{\ell^2}^2 N_2^{-3+\varepsilon} \sup_\zeta \sum_{n_1,n_3\in\mathbb{Z}^3} |S_{(n_1,n_3,\zeta)}| \sum_{S_{(n_1,n_3,\zeta)}} \chi_{\tilde{C}}(n) |k_n|^2
\end{aligned}$$

$$\lesssim N_2^2 \|\chi_C a_{n_1}\|_{\ell^2}^2 \|a_{n_3}\|_{\ell^2}^2 N_2^{-3+\varepsilon} N_2 \sup_{\zeta} \sum_{n} \sum_{S_{(n,\zeta)}} |k_n|^2$$

$$\lesssim N_2^2 \|\chi_C a_{n_1}\|_{\ell^2}^2 \|a_{n_3}\|_{\ell^2}^2 N_2^{-3+\varepsilon} N_2 N_2 N_3^3 \|k_n\|_{\ell^2}^2 \sim N_2^{1+\varepsilon} N_3^3 \|\chi_C a_{n_1}\|_{\ell^2}^2 \|a_{n_3}\|_{\ell^2}^2 \|k_n\|_{\ell^2}^2$$

where $S_{(n_1,n_3,\zeta)} := \{(n, n_2) : n_1 = n - n_2 - n_3,\ n \in \tilde{C},\ \zeta = |n_2|^2 + |n_3|^2\}$ with $|S_{(n_1,n_3,\zeta)}| \lesssim N_2$, and $S_{(n,\zeta)} = \{(n_1, n_2, n_3) : n_1 = n - n_2 - n_3,\ n_1 \in C,\ \zeta = |n_2|^2 + |n_3|^2\}$ with $|S_{(n,\zeta)}| \lesssim N_2 N_3^3$. □

Proposition 7.6. *Let R_k be as above and fix $N_1 \geq N_2 \geq N_3$, $r, \delta > 0$ and $C \in \mathcal{C}_{N_2}$. Then there exists $\mu > 0$ and a set $\Omega_\delta \in A$ with $\mathbb{P}(\Omega_\delta^c) \leq e^{-1/\delta^r}$ such that for any $\omega \in \Omega_\delta$ we have* (7.11)–(7.13).

Proof. We start by estimating (7.11). We consider

$$\mathcal{T} := \sum_{m \in \mathbb{Z},\, n \in \mathbb{Z}^3} \Bigg| \sum_{\substack{n = n_1 + n_2 - n_3 \\ n_1, n_2 \neq n_3 \\ m = |n_1|^2 + |n_2|^2 - |n_3|^2}} \chi_C(n_1) \frac{\overline{g}_{n_1}(\omega)}{|n_1|^{3/2}} \frac{\overline{g}_{n_2}(\omega)}{|n_2|^{3/2}} \frac{g_{n_3}(\omega)}{|n_3|^{3/2}} \Bigg|^2. \tag{7.33}$$

Note that if $n_1 = n_2$ we get, say, $(\overline{g}_{n_1}(\omega))^2$ which are still independent and mean zero since the $g_{n_i}(\omega)$ are complex Gaussian random variables. Hence we are still within the framework of Lemma 3.4 and so this case does not require a separate argument.

We first remark that the variation Δm is $\sim N_1 N_2$. Then we use Lemma 3.4 to obtain, for ω outside a set of measure e^{-1/δ^r},

$$\mathcal{T} \lesssim \delta^{-3r/2} N_1 N_2 N_1^{-3} N_2^{-3} N_3^{-3} \sup_m |S_{(m)}| \lesssim \delta^{-3r/2} N_1^{-1} N_2$$

where $S_{(m)} := \{(n, n_1, n_2, n_3) : n = n_1 + n_2 - n_3,\ n_1 \in C,\ m = |n_1|^2 + |n_2|^2 - |n_3|^2\}$ and $|S_{(m)}| \lesssim N_3^3 N_2^3 N_1$.

To estimate (7.12) and (7.13) we proceed just as above. □

7.2. *Bilinear estimate*

We prove the following bilinear estimate which will be used in Section 8. We use the same notation as in Subsection 7.1.

Proposition 7.7. *Fix $N_1 \geq N_2 \geq N_3$ and $r, \delta > 0$. Assume also that C is a cube of sidelength N_2. Then there exist $\mu, \varepsilon > 0$ and a set $\Omega_\delta \in A$ with $\mathbb{P}(\Omega_\delta^c) \leq e^{-1/\delta^r}$ such that for any $\omega \in \Omega_\delta$ and $0 \leq \theta \leq 1$ we have*

$$\|P_C R_1 D_2\|_{L^2([0,1]\times\mathbb{T}^3)} \lesssim \delta^{-\mu r} N_1^{-1/2+\varepsilon} \min(N_1, N_2^2)^{(1-\theta)/2} N_2^{1/2+3\theta/4} \|D_2\|_{U^2_\Delta L^2_x}. \tag{7.34}$$

Proof. We follow the argument for (7.9) after applying Cauchy–Schwarz. In fact we have

$$\|P_C R_1 D_2\|_{L^2}^2 = \sum_{m \in \mathbb{Z},\, n \in \mathbb{Z}^3} \Bigg| \sum_{\substack{n = n_1 + n_2 \\ m = |n_1|^2 + |n_2|^2}} \chi_C(n_1) \frac{g_{n_1}(\omega)}{|n_1|^{3/2}} a_{n_2} \Bigg|^2.$$

Then

$$\|P_C R_1 D_2\|_{L^2}^2 \lesssim \sum_{m;n\in\tilde{C}} \Big|\sum_{n_2} \sigma_{n,n_2} a_{n_2}\Big|^2$$

where $\tilde{C}$ is a cube of sidelength approximately N_2 and

$$\sigma_{n,n_2} := \begin{cases} \dfrac{g_{n-n_2}(\omega)}{|n-n_2|^{3/2}} & \text{if } m = |n-n_2|^2 + |n_2|^2, \\ 0 & \text{otherwise.} \end{cases}$$

We then have

$$\|P_C R_1 D_2\|_{L^2}^2 \lesssim \|a_{n_2}\|_{\ell^2}^2 N_1 N_2 \|\mathcal{G}\mathcal{G}^*\|.$$

Then using the estimates (7.29) and (7.30) we obtain, for ω outside a set of measure e^{-1/δ^r},

$$\|P_C R_1 D_2\|_{L^2} \lesssim \|a_{n_2}\|_{\ell^2} \delta^{-r} N_1^{-1/2+\varepsilon} N_2^{5/4}. \tag{7.35}$$

We also use (7.32) to estimate (7.9). By repeating the argument to prove (7.32) in our bilinear setting, we obtain, for ω outside a set of measure e^{-1/δ^r},

$$\begin{aligned}
\|P_C R_1 D_2\|_{L^2}^2 &\lesssim N_1 N_2 \sup_m \sum_{n\in\mathbb{Z}^3} \Big| \sum_{\substack{n=n_1+n_2\\ m=|n_1|^2+|n_2|^2}} \chi_C(n_1) \frac{g_{n_1}(\omega)}{|n_1|^{3/2}} a_{n_2} \Big|^2 \\
&\lesssim N_1 N_2 \|a_{n_3}\|_{\ell^2}^2 \sup_m \sum_{n\in\mathbb{Z}^3} \Big| \sum_{\substack{n=n_1+n_2\\ m=|n_1|^2+|n_2|^2}} \chi_C(n_1) \frac{g_{n_1}(\omega)}{|n_1|^{3/2}} a_{n_2} \Big|^2 \\
&\lesssim N_1 N_2 N_1^{-3+\varepsilon} \sup_m \sum_{n\in\mathbb{Z}^3} |S_{(n,m)}| \sum_{S_{(n,m)}} |a_{n_2}|^2 \\
&\lesssim N_1 N_2 N_1^{-3+\varepsilon} \min(N_2^2, N_1) \sup_m \sum_{n_2} \sum_{S_{(n_2,m)}} |a_{n_2}|^2 \\
&\lesssim N_1 N_2 \|a_{n_2}\|_{\ell^2}^2 N_1^{-3+\varepsilon} \min(N_2^2, N_1) N_1 \\
&\sim N_1^{-1+\varepsilon} N_2 \min(N_2^2, N_1) \|a_{n_2}\|_{\ell^2}^2
\end{aligned}$$

where $S_{(n,m)} := \{(n_1, n_2) : n = n_1 + n_2,\ n_1 \in C,\ m = |n_1|^2 + |n_2|^2\}$ with $|S_{(n,m)}| \lesssim \min(N_2^2, N_1)$, $S_{(n_2,m)} := \{(n, n_1) : n = n_1 + n_2,\ n_1 \in C,\ m = |n_1|^2 + |n_2|^2\}$ with $|S_{(n_2,m)}| \lesssim N_1$, and we have used (3.5). Hence we also have

$$\|P_C R_2 D_1\|_{L^2} \lesssim \|a_{n_2}\|_{\ell^2} N_1^{-1/2+\varepsilon} N_2^{1/2} \min(N_2^2, N_1)^{1/2}. \tag{7.36}$$

By interpolating (7.35) and (7.36) we finally deduce the estimate (7.34). □

Remark 7.2. Later we only use (7.34) with $\theta = 1$ while estimating in the next section the term J_4 defined in (2.13).

8. Proof of Proposition 5.1

In this section we use notation similar to the one introduced at the beginning of Section 7 to indicate deterministic and random functions. The reader should pay attention though to the fact that the new functions we define in this section have a different normalization than the ones in Section 7, hence the slight change of notation.

If u_i is random, then we write

$$\widehat{P_{N_i}u_i}(n) = \chi_{\{|n|\sim N_i\}}(n)\frac{g_n(\omega)}{|n|^{5/2-\alpha}}e^{i|n|^2 t} \sim \widehat{\mathcal{R}}_i(n);$$

while if u_i is deterministic we write

$$\widehat{P_{N_i}u_i}(n) \sim \widehat{\mathcal{D}}_i(n)$$

where $\widehat{\mathcal{D}}_i(n)$ is supported in $\{|n| \sim N_i\}$. Below we will make heavy use of Proposition 7.1 with the functions $\mathcal{R}_i$ instead of R_i. This will not be explicitly mentioned every time, but the reader will notice that a normalization will take place at the appropriate places.

We first estimate the terms J_2–J_7, and then we turn to J_1.

8.1. Estimates involving the term J_2

We start by estimating the term J_2 as in (2.11). This reduces to analyzing the sum over $N_0, N_1, \ldots, N_3$ of quatrilinear forms

$$\left|\int_{\mathbb{T}}\int_{\mathbb{T}^3} T_\Upsilon(P_{N_1}u_1, P_{N_2}\overline{u}_2, P_{N_3}u_3)P_{N_0}\overline{h}\,dx\,dt\right| \tag{8.1}$$

where T_Υ is the multilinear operator defined in (7.2).

The general outline of the proof involves the use of Cauchy–Schwarz, cutting the top frequency window if necessary, the transfer principle Proposition 4.1 and suitably applying the trilinear estimates of Subsection 7.1. Without any loss of generality, we then fix the relative ordering $N_1 \geq N_2 \geq N_3$ above and consider the following cases where T_Υ acts on:

- **Case 1:** **(a)** $(\bar{\mathcal{R}}_1, \mathcal{R}_2, \mathcal{R}_3)$, **(b)** $(\mathcal{R}_1, \bar{\mathcal{R}}_2, \mathcal{R}_3)$, **(c)** $(\mathcal{R}_1, \mathcal{R}_2, \bar{\mathcal{R}}_3)$,
- **Case 2:** **(a)** $(\bar{\mathcal{D}}_1, \mathcal{R}_2, \mathcal{R}_3)$, **(b)** $(\mathcal{D}_1, \bar{\mathcal{R}}_2, \mathcal{R}_3)$, **(c)** $(\mathcal{D}_1, \mathcal{R}_2, \bar{\mathcal{R}}_3)$,
- **Case 3:** **(a)** $(\bar{\mathcal{R}}_1, \mathcal{R}_2, \mathcal{D}_3)$, **(b)** $(\mathcal{R}_1, \bar{\mathcal{R}}_2, \mathcal{D}_3)$, **(c)** $(\mathcal{R}_1, \mathcal{R}_2, \bar{\mathcal{D}}_3)$,
- **Case 4:** **(a)** $(\bar{\mathcal{R}}_1, \mathcal{D}_2, \mathcal{R}_3)$, **(b)** $(\mathcal{R}_1, \bar{\mathcal{D}}_2, \mathcal{R}_3)$, **(c)** $(\mathcal{R}_1, \mathcal{D}_2, \bar{\mathcal{R}}_3)$,
- **Case 5:** **(a)** $(\bar{\mathcal{D}}_1, \mathcal{R}_2, \mathcal{D}_3)$, **(b)** $(\mathcal{D}_1, \bar{\mathcal{R}}_2, \mathcal{D}_3)$, **(c)** $(\mathcal{D}_1, \mathcal{R}_2, \bar{\mathcal{D}}_3)$,
- **Case 6:** **(a)** $(\bar{\mathcal{R}}_1, \mathcal{D}_2, \mathcal{D}_3)$, **(b)** $(\mathcal{R}_1, \bar{\mathcal{D}}_2, \mathcal{D}_3)$, **(c)** $(\mathcal{R}_1, \mathcal{D}_2, \bar{\mathcal{D}}_3)$,
- **Case 7:** **(a)** $(\bar{\mathcal{D}}_1, \mathcal{D}_2, \mathcal{R}_3)$, **(b)** $(\mathcal{D}_1, \bar{\mathcal{D}}_2, \mathcal{R}_3)$, **(c)** $(\mathcal{D}_1, \mathcal{D}_2, \bar{\mathcal{R}}_3)$,
- **Case 8:** **(a)** $(\bar{\mathcal{D}}_1, \mathcal{D}_2, \mathcal{D}_3)$, **(b)** $(\mathcal{D}_1, \bar{\mathcal{D}}_2, \mathcal{D}_3)$, **(c)** $(\mathcal{D}_1, \mathcal{D}_2, \bar{\mathcal{D}}_3)$.

Case 1(a). If $N_1 \sim N_0$ we cut the support of $\widehat{h}$ and hence that of $\widehat{\mathcal{R}}_1$ with cubes C of sidelength N_2 and use Cauchy–Schwarz to get

$$(8.1) \lesssim \|P_C P_{N_0}h\|_{L^2_{x,t}} \|T_\Upsilon(P_C\bar{\mathcal{R}}_1, \mathcal{R}_2, \mathcal{R}_3)\|_{L^2_{x,t}}.$$

To estimate the second factor we use (7.13) and normalization and we obtain the bound

$$(8.1) \lesssim \delta^{-\mu r} N_1^{-3/2+\alpha} N_2^{-1/2+\alpha} N_3^{1-\alpha} \|P_C P_{N_0} h\|_{L^2_{x,t}}.$$

Renormalizing h and using the embedding (4.7) we obtain

$$\begin{aligned} \left| \int_0^\pi \int_{\mathbb{T}^3} T_\Upsilon(P_C P_{N_1} u_1, P_{N_2}\overline{u}_2, P_{N_3} u_3) P_C P_{N_0} \overline{h}\, dx\, dt \right| & \\ \lesssim \delta^{-\mu r} N_0^s N_1^{-3/2+\alpha} N_2^{-1/2+\alpha} N_3^{-1+\alpha} \|P_C P_{N_0} h\|_{Y^{-s}} & \\ \lesssim \delta^{-\mu r} N_1^{s+\alpha-3/2} \|P_C P_{N_0} h\|_{Y^{-s}}, & \end{aligned} \tag{8.2}$$

which suffices provided $s+\alpha < 3/2$ and $\alpha < 1/2$.

If $N_1 \sim N_2$ the cut with the cubes C above is not needed and the argument proceeds as above. The condition here is $s < 2 - 2\alpha$.

Cases 1(b), (c) are treated similarly replacing (7.13) respectively by (7.12) and (7.11).

Case 2(a). Assume that $N_0 \sim N_1$. We use the argument above. To estimate

$$\|T_\Upsilon(P_C \bar{\mathcal{D}}_1, \mathcal{R}_2, \mathcal{R}_3)\|_{L^2_{x,t}}$$

we use (7.6) and after taking derivatives and normalizing we obtain the bound

$$(8.2) \lesssim \delta^{-\mu r} N_2^{\alpha-1/4} \|P_{N_1} u_1\|_{U^2_\Delta H^s} \|P_C P_{N_0} h\|_{Y^{-s}},$$

which suffices provided $\alpha < 1/4$. A similar bound holds when $N_1 \sim N_2$ without cutting with cubes C.

Cases 2(b), (c) are treated similarly replacing (7.6) by (7.5).

Cases 3(a)–(c). We use the argument above with (7.7) and (7.8). If $N_1 \sim N_0$ we obtain a bound of the form

$$\begin{aligned} (8.2) &\lesssim \delta^{-\mu r} N_0^s [N_1^{\alpha-7/4} N_2^{-1/2+\alpha} N_3^{5/4-s} \\ &\quad + N_1^{\alpha-3/2} N_2^{-1/2+\alpha} N_3^{3/4-s}] \|P_{N_3} u_3\|_{U^2_\Delta H^s} \|P_C P_{N_0} h\|_{Y^{-s}} \\ &\lesssim \delta^{-\mu r} N_1^{-\beta(s,\alpha)} \|P_{N_3} u_3\|_{U^2_\Delta H^s} \|P_C P_{N_0} h\|_{Y^{-s}} \end{aligned}$$

provided $\alpha < 1/4$ and $s+\alpha < 3/2$. A similar bound holds when $N_1 \sim N_2$ without cutting with cubes C.

Cases 4(a)–(c). We use the argument above with (7.3) and (7.4). If $N_1 \sim N_0$ we obtain a bound of the form

$$(8.2) \lesssim \delta^{-\mu r} N_1^{-\beta(s,\alpha)} \|P_{N_2} u_2\|_{U^2_\Delta H^s} \|P_C P_{N_0} h\|_{Y^{-s}}$$

provided $\alpha < 1/4$ and $s+\alpha < 3/2$. A similar bound holds when $N_1 \sim N_2$ without cutting with cubes C.

Cases 5(a)–(c). We use the argument above with (7.10). If $N_1 \sim N_0$ we obtain a bound of the form

$$(8.2) \lesssim \delta^{-\mu r} N_2^{1+\alpha-s} \|P_{N_2} u_2\|_{U^2_\Delta H^s} \|P_{N_3} u_3\|_{U^2_\Delta H^s} \|P_C P_{N_0} h\|_{Y^{-s}},$$

which suffices provided $s > 1+\alpha$. A similar bound holds when $N_1 \sim N_2$ without cutting with cubes C.

Case 6(a). If $N_1 \sim N_0$ we proceed as above to bound

$$\left| \int_{\mathbb{T}} \int_{\mathbb{T}^3} T_\Upsilon(P_C P_{N_1} \bar{\mathcal{R}}_1, P_{N_2} \mathcal{D}_2, P_{N_3} \mathcal{D}_3) P_C P_{N_0} \bar{h}\, dx\, dt \right| \\ \lesssim \|T_\Upsilon(P_C P_{N_1} \bar{\mathcal{R}}_1, P_{N_2} \mathcal{D}_2, P_{N_3} \mathcal{D}_3)\|_{L^2_{xt}} \|P_C P_{N_0} h\|_{L^2_{xt}}.$$

Then we use (7.9), normalization and the embedding (4.7) to obtain the bound

$$(8.1) \lesssim \delta^{-\mu r} N_1^{s-3/2+\alpha+\varepsilon} N_2^{1/2+3\theta/4-s} \min(N_1, N_2^2)^{(1-\theta)/2} N_3^{-s+3/2} \\ \times \|P_{N_3} u_3\|_{U^2_\Delta H^s} \|P_{N_2} u_2\|_{U^2_\Delta H^s} \|P_C P_{N_0} h\|_{Y^{-s}}.$$

If $N_1 \geq N_2^2$ then

$$N_1^{s-3/2+\alpha+\varepsilon} N_2^{1/2+3\theta/4-s} \min(N_1, N_2^2)^{(1-\theta)/2} N_3^{-s+3/2} \leq N_1^{s-3/2+\alpha+\varepsilon} N_2^{3-2s-\theta/4} \\ \leq N_1^{\alpha+\varepsilon-\theta/8}$$

provided that $s < 3/2 - \theta/8$ which forces $\alpha < \theta/8$.

On the other hand, if $N_1 < N_2^2$ we have

$$N_1^{s-3/2+\alpha+\varepsilon} N_2^{1/2+3\theta/4-s} \min(N_1, N_2^2)^{(1-\theta)/2} N_3^{-s+3/2} \leq N_1^{s-1+\alpha+\varepsilon-\theta/2} N_2^{2-2s+3\theta/4} \\ \leq N_2^{2\alpha+\varepsilon-\theta/4}$$

provided $s > 1 + \theta/2 - \alpha$. By letting, for example, $\theta = 10\alpha$ we obtain $1 + 4\alpha < s < 3/2 - 2\alpha$ in this case, while still satisfying the requirement that $\alpha < \theta/8$ from Case (a).

If $N_1 \sim N_2$ the argument is similar and easier. For Case 6(b), (c) we repeat the argument since (7.9) is not sensitive to conjugation on the random function.

Case 7(a). In this case we would like to use the Strichartz estimate (4.23). But since

$$T_\Upsilon(\bar{\mathcal{D}}_1, \mathcal{D}_2, \mathcal{R}_3) \neq \bar{\mathcal{D}}_1 \mathcal{D}_2 \mathcal{R}_3$$

we need to add back the frequencies that have been removed, i.e. allow for n_2 or n_3 to be equal to n_1. If we were working with spaces whose norms are based on the absolute value of the time-space Fourier coefficients, like the $X^{s,b}$ space, this would not be an issue, but since we are using $U^p L^2$ spaces we need to put back those missing frequencies. We show below that reintroducing these frequencies will not bring back the whole linear term that we have gauged away but only a part that has sufficient regularity to be controlled.

We start by assuming that the Fourier coefficient associated to $\mathcal{D}_1(t)$ is $a_{n_1}(t)$, to $\mathcal{D}_2(t)$ is $b_{n_2}(t)$ and to $\mathcal{R}_3(t)$ is $c_{n_3}(t)$. Then we write

$$\begin{aligned}\sum_{n=-n_1+n_2+n_3,\, n_2,n_3\neq n_1} \chi_{N_1} a_{n_1} \chi_{N_2} b_{n_2} \chi_{N_3} c_{n_3} &= -\chi_{N_3} c_n \Big(\sum_{n_1} \chi_{N_1} a_{n_1} \chi_{N_2} b_{n_1}\Big) \\ &- \chi_{N_2} b_n \Big(\sum_{n_3} \chi_{N_1} a_{n_3} \chi_{N_3} c_{n_3}\Big) + \chi_{N_1} a_n \chi_{N_2} b_n \chi_{N_3} c_n \\ &+ \sum_{n=-n_1+n_2+n_3} \chi_{N_1} a_{n_1} \chi_{N_2} b_{n_2} \chi_{N_3} c_{n_3} = A_1(n) + A_2(n) + A_3(n) + A_4(n).\end{aligned}$$

Then we have

$$(8.1) \lesssim \sum_{i=1}^{4} \left| \int_{\mathbb{T}} \int_{\mathbb{T}^3} \mathcal{F}^{-1}(A_i)(x,t) P_{N_0} \overline{h}(x,t)\, dx\, dt \right|.$$

We now start with the estimate of A_1. Using Plancherel and Cauchy–Schwarz we have

$$\left| \int_{\mathbb{T}} \int_{\mathbb{T}^3} \mathcal{F}^{-1}(A_1)(x,t) P_{N_0} \overline{h}(x,t)\, dx\, dt \right| \lesssim \|A_1(n)\|_{L^2(\mathbb{T},\ell^2)} \|P_{N_0} h(x,t)\|_{L^2_{x,t}}.$$

We first notice that A_1 is not zero only if $N_3 \sim N_1$. Then

$$\|A_1(n)\|_{L^2(\mathbb{T},\ell^2)} \lesssim \|\mathcal{D}_1\|_{L^\infty_t L^2_x} \|\mathcal{D}_2\|_{L^\infty_t L^2_x} \|\mathcal{R}_3\|_{L^2(\mathbb{T},L^2(\mathbb{T}^3))}.$$

By renormalizing and using the embedding (4.7) we obtain

$$\left| \int_{\mathbb{T}} \int_{\mathbb{T}^3} \mathcal{F}^{-1}(A_1)(x,t) P_{N_0} \overline{h}(x,t)\, dx\, dt \right| \lesssim N_2^{-s-1+\alpha} \|P_{N_1} u_1\|_{U^2_\Delta H^s} \|P_{N_2} u_2\|_{U^2_\Delta H^s} \|P_{N_0} h\|_{Y^{-s}}.$$

We now note that $A_2 = 0$ unless $N_0 \sim N_1 \sim N_2$, and

$$\|A_2(n)\|_{L^2([0,\pi],\ell^2)} \lesssim \|\mathcal{D}_2\|_{L^2(\mathbb{T},L^2(\mathbb{T}^3))} \|\mathcal{D}_1\|_{L^\infty_t L^2_x} \|\mathcal{R}_3\|_{L^\infty_t L^2_x}.$$

Also in this case we then have

$$\left| \int_{\mathbb{T}} \int_{\mathbb{T}^3} \mathcal{F}^{-1}(A_2)(x,t) P_{N_0} \overline{h}(x,t)\, dx\, dt \right| \lesssim N_2^{-s-1+\alpha} \|P_{N_1} u_1\|_{U^2_\Delta H^s} \|P_{N_2} u_2\|_{U^2_\Delta H^s} \|P_{N_0} h\|_{Y^{-s}}.$$

Now we note that $A_3 = 0$ unless $N_1 \sim N_2 \sim N_3$. Then

$$\|A_3(n)\|_{L^2(\mathbb{T},\ell^2)} \lesssim \|\mathcal{D}_1\|_{L^\infty_t L^2_x} \|\mathcal{D}_2\|_{L^\infty_t L^2_x} \|\mathcal{R}_3\|_{L^2(\mathbb{T},L^2(\mathbb{T}^3))},$$

where we have used $\|a_n\|_{\ell^\infty} \lesssim \|a_n\|_{\ell^2}$. Hence also in this case

$$\left| \int_{\mathbb{T}} \int_{\mathbb{T}^3} \mathcal{F}^{-1}(A_3)(x,t) P_{N_0} \overline{h}(x,t)\, dx\, dt \right| \lesssim N_2^{-s-1+\alpha} \|P_{N_1} u_1\|_{U^2_\Delta H^s} \|P_{N_2} u_2\|_{U^2_\Delta H^s} \|P_{N_0} h\|_{Y^{-s}}.$$

Finally we estimate the term involving A_4. Assume first that $N_0 \sim N_1$. Then we need to estimate

$$\left| \int_{\mathbb{T}} \int_{\mathbb{T}^3} \mathcal{F}^{-1}(A_4)(x,t) P_C P_{N_0} \overline{h}(x,t)\, dx\, dt \right| \tag{8.3}$$

where we cut with cubes C of size length N_2. We use Cauchy–Schwarz, (4.23), embedding (4.7) and normalization to obtain

$$\begin{aligned}(8.3) &\lesssim N_0^s N_1^{-s} N_2^{1-s} N_3^{\alpha} \|P_C P_{N_1} u_1\|_{U^4_\Delta H^s} \|P_{N_2} u_2\|_{U^4_\Delta H^s} \|P_C P_{N_0} h\|_{Y^{-s}} \\ &\lesssim N_2^{1-s+\alpha} \|P_C P_{N_1} u_1\|_{U^4_\Delta H^s} \|P_{N_2} u_2\|_{U^4_\Delta H^s} \|P_C P_{N_0} h\|_{Y^{-s}}.\end{aligned}$$

If $N_1 \sim N_2$ then the cutting with cubes C is automatic and a similar bound holds.

Cases (b) and (c) are similar since the argument presented above is not affected by complex conjugation.

Case 8. This case is similar and better than Case 7.

8.2. Estimates involving the term J_3

We start by noting that J_3 consists of terms of the form

$$\sum_{\Gamma(n)_{[123]},\, n_1, n_3 \neq n_2} \widehat{w}_{n_1}(t) \overline{a}_{n_2}(t) b_{n_3}(t), \tag{8.4}$$

where $\widehat{w}_{n_1}(t) = c_{n_1}(t) d_{n_1}(t) r_{n_1}(t)$. We note that in the worst case, i.e. when the three factors of $\widehat{w}$ correspond to random functions, we have $w(t) \in H^{3-3\alpha}$, hence w can always be thought of as a deterministic function. We estimate J_3 using the arguments presented for J_2 in Subsection 8.1, but for the reason just explained we do not have to consider Case 1 of that section. For Cases 2–6 we proceed by first applying the transfer principle to the *quintilinear* expression associated to (8.4) and then regroup into a single *deterministic* function those with the same frequency n_1. Then we apply the appropriate trilinear estimates of Proposition 7.1. The term involving the ℓ^2 norm of the product of the three coefficients in n_1 can be bounded by the product of the ℓ^2 norms of each. We transfer and normalize back as usual.

This same argument is also used to estimate the $A_i(x,t)$, $i = 1, 2, 3$, of Case 7. To estimate A_4 we use again the Strichartz inequality of Proposition 4.5 placing w in L^p with $p > 4$. Then we use (4.19).

8.3. Estimates involving the term J_4

Let w now be such that $\widehat{w}_{n_2}(t) = a_{n_2}(t) c_{n_2}(t) d_{n_2}(t) r_{n_2}(t)$ and v such that $\widehat{v}(n_1) = b_{n_1}$. To estimate the contribution of J_4 we need to estimate a term such as

$$\int_{\mathbb{T}} \int_{\mathbb{T}^3} P_{N_0}(wv) \overline{P_{N_0} h}\, dx\, dt = \int_{\mathbb{T}} \int_{\mathbb{T}^3} P_{N_0}\Big(\sum_{N_1, N_2} P_{N_1} v P_{N_2} w \Big) \overline{P_{N_0} h}\, dx\, dt.$$

Since $w \in H^{4-4\alpha}$, hence much smoother than v, the least advantageous situation is when $N_1 \sim N_0$ and $N_2 \ll N_1$ and this is the one we consider below. We cut the frequency support of $P_{N_0}h$ with cubes C of size N_2 and we write

$$\left(\int_{\mathbb{T}}\int_{\mathbb{T}^3} P_{N_0}P_{N_1}v P_{N_2}w\overline{P_{N_0}h}\,dx\,dt\right)^2 \lesssim \left(\sum_C \|P_C P_{N_0}h\|^2_{L^2_t L^2_x} \sup \|P_C P_{N_1}v P_{N_2}w\|_{L^2_t L^2_x}\right)^2.$$

We assume first that v is random. Then the remarks in Subsection 8.2 combined with the transfer principle and the bilinear estimate (7.34) with $\theta = 1$ give

$$\|P_C P_{N_1}v P_{N_2}w\|_{L^2_t L^2_x} \lesssim \delta^{-\mu r} N_1^{-1/2+\varepsilon} N_2^{5/4} \prod_{i \notin J} \|\mathcal{D}_i\|_{U^2_\Delta L^2}.$$

After normalizing we obtain the bound $N_1^{-3/2+s+\varepsilon+\alpha} N_2^{-11/4+4\alpha}$, which entails $s + \alpha < 3/2$.

If v is deterministic then we use the bilinear estimate (4.24) and after normalization we obtain the bound $N_2^{-7/2+4\alpha}$.

8.4. *Estimates involving the terms J_5, J_6 and J_7*

We work with the first term of J_5, the second term being analogous. Given a dual function h we define a new function k such that

$$\widehat{k}(n,t) = \chi_{N_0} a^1_n(t) a^2_n(t) \widehat{h}(n,t)$$

where the $a^i_n(t)$ are the Fourier coefficients of either a random or a deterministic function. Assume that $N_1 \sim N_0$. Then we cut the support of $\widehat{h}$ with cubes C of sidelength N_2. By Plancherel and Cauchy–Schwarz we need to bound

$$\|P_C k\|_{L^2_{xt}} \quad \text{and} \quad \Big\| \sum_{\Gamma(n)_{[1,2,3]}} \chi_C \chi_{N_1} b_{n_1} \chi_{N_2} \bar{c}_{n_2} \chi_{N_3} d_{n_3} \Big\|_{L^2_t \ell^2}.$$

Clearly

$$\|P_C k\|_{L^2_t L^2_x} \lesssim \|P_C P_{N_0} h\|_{L^\infty_t L^2_x} \prod_{i=1}^{2} \|\chi_{N_i} a^i_n\|^2_{L^\infty_t \ell^2}.$$

On the other hand, by (4.23) we find that

$$\Big\| \sum_{\Gamma(n)_{[1,2,3]}} \chi_C \chi_{N_1} b_{n_1} \chi_{N_2} \bar{c}_{n_2} \chi_{N_3} d_{n_3} \Big\|_{L^2_t \ell^2}$$

has a bound of $N_2 N_3$. By normalizing, assuming at worst that all functions are random, we obtain the bound $N_0^{s-2+2\alpha} N_1^{-1+3\alpha}$. If $N_1 \sim N_2$ the situation is similar.

To estimate J_6 we use Cauchy–Schwarz and (4.22), while for the two terms in J_7 we use respectively (4.21) and (4.20).

8.5. Estimates involving the term J_1

The term J_1 in (2.10) can be written as the sum over $N_0, N_1, \dots, N_5$ (dyadic numbers) of

$$\left| \int_{\mathbb{T}} \int_{\mathbb{T}^3} P_{N_0} T_\Upsilon (P_{N_1}\tilde{u}_1, P_{N_2}\tilde{u}_2, P_{N_3}\tilde{u}_3, P_{N_4}\tilde{u}_4, P_{N_5}\tilde{u}_5) P_{N_0}\overline{h}\, dx\, dt \right| \tag{8.5}$$

where T_Υ is the multilinear operator associated to the multiplier χ_Υ, the indicator function of the set Υ now defined by

$$\Upsilon(n,m) := \left\{ (n_1, m_1, \dots, n_5, m_5) : \begin{array}{l} n = (-1)^{\alpha_1} n_1 + \cdots + (-1)^{\alpha_5} n_5, \\ n_k \neq n_\ell \text{ whenever } \alpha_k \neq \alpha_\ell, \\ |n_j| \sim N_j,\ j = 1, \dots, 5, \\ m = (-1)^{\alpha_1} m_1 + \cdots + (-1)^{\alpha_5} m_5 \end{array} \right\} \tag{8.6}$$

where the α_j are 0 or 1 for $j = 1, \dots, 5$.

8.5.1. The all deterministic case DDDDD. Without loss of generality we assume that u_2 and u_4 are conjugated. Our goal is to use Strichartz estimates as in (4.23), but the operator $T_\Upsilon(P_{N_1}u_1, P_{N_2}\overline{u}_2, P_{N_3}u_3, P_{N_4}\overline{u}_4, P_{N_5}u_5)$ is not a product of the functions involved since in the convolution of the Fourier coefficients some frequencies have been removed. We need to add back the frequencies that have been removed, i.e. allow for n_2 or n_3 to be equal to n_1. If we were working with spaces whose norms are based on the absolute value of the time-space Fourier coefficients, like the $X^{s,b}$ space, this would not be an issue, but since we are using $U^p L^2$ spaces we need to put back those missing frequencies. We show below that reintroducing these frequencies will not bring back the whole linear term that we have gauged away but only a part that has sufficient regularity to be controlled. See also Subsection 8.1.

From (2.9) we see that

$$\begin{aligned} P_{N_0}(\mathcal{F}^{-1}J_1)(x,t) &= P_{N_0} T_\Upsilon (P_{N_1}u_1, P_{N_2}\overline{u}_2, P_{N_3}u_3, P_{N_4}\overline{u}_4, P_{N_5}u_5)(x,t) \\ &= P_{N_0}(P_{N_1}u_1 P_{N_2}\overline{u}_2 P_{N_3}u_3 P_{N_4}\overline{u}_4 P_{N_5}u_5)(x,t) \\ &\quad - \sum_{i=1}^{5} P_{N_0} P_{N_i} u_i(x,t) \int_{\mathbb{T}^3} \prod_{j\neq i,\ j\in\{1,2,3,4,5\}} P_{N_j}\tilde{u}_j(x,t)\, dx \\ &\quad - \sum_{i=2}^{7} c_i P_{N_0} \mathcal{F}^{-1} J_i (P_{N_1}u_1, P_{N_2}\overline{u}_2, P_{N_3}u_3, P_{N_4}\overline{u}_4, P_{N_5}u_5)(x,t), \end{aligned} \tag{8.7}$$

where the c_i are constants and we specified as an argument of $\mathcal{F}^{-1}J_1$ the functions involved in its definition. The last sum involving J_2–J_7 has already been estimated in Subsections 8.1–8.4 above. On the other hand, the first term, which is now a product of functions, can be estimated as in Proposition 4.4. Finally, we estimate

$$\left\| \sum_{i=1}^{5} P_{N_0} P_{N_i} \tilde{u}_i(x,t) \int_{\mathbb{T}^3} \prod_{j\neq i,\ j\in\{1,2,3,4,5\}} P_{N_j}\tilde{u}_j(y,t)\, dy \right\|_{L^2_{x,t}}. \tag{8.8}$$

We first note that each term of the sum is zero unless $N_i \sim N_0$ and that

$$(8.8) \lesssim \sum_{i=1}^{5} \|P_{N_0} P_{N_i} u_i\|_{L_t^\infty L_x^2} \prod_{j\neq i,\ j\in\{1,2,3,4,5\}} N_j^{1/4} \|P_{N_j} u_j(x,t)\|_{U_\Delta^4 L_x^2}, \tag{8.9}$$

and this is enough since all are deterministic.

8.5.2. The case DDDDR. In (8.5) we assume without any loss of generality that u_5 is random and $N_1 \geq N_2 \geq N_3 \geq N_4$. Also in the argument below one can check that the location of the complex conjugates does not affect the proof, hence here we assume that u_2 and u_4 are complex conjugate.

We consider the following cases:

- **Case (a):** $N_5 \sim N_0$ and $N_1 \leq N_5$.
- **Case (b):** $N_1 \sim N_0$ and $N_2 \leq N_5 \leq N_1$.
- **Case (c):** $N_1 \sim N_5$ and $N_0 \leq N_1$.
- **Case (d):** $N_1 \sim N_0$ and $N_5 \leq N_2$.
- **Case (e):** $N_1 \sim N_2$ and $N_5 \leq N_1$.

Case (a). Proceeding as in the trilinear estimates we first decompose the support of $\chi_{N_0}\widehat{h}$ using cubes C of sidelength N_1 in (8.5). By Cauchy–Schwarz, the transfer principle and Plancherel we are reduced to estimating

$$\sum_{(m,n)\in\mathbb{Z}\times\mathbb{Z}^3} \Bigg| \sum_{\substack{n=n_5-n_2+n_3-n_4+n_1 \\ n_1,n_3,n_5\neq n_2,n_4 \\ m=|n_5|^2-|n_2|^2+|n_3|^2-|n_4|^2+|n_1|^2}} \chi_C(n_5) \frac{g_{n_5}(\omega)}{|n_5|^{3/2}} a_{n_1}\overline{a}_{n_2} a_{n_3} \overline{a}_{n_4} \Bigg|^2. \tag{8.10}$$

We define

$$S_{(n_5,n,m)} := \left\{ (n_1,n_2,n_3,n_4) : \begin{array}{l} n = n_5 - n_2 + n_3 - n_4 + n_1, \\ n_1, n_3, n_5 \neq n_2, n_4,\ n_5 \in C, \\ m = |n_5|^2 - |n_2|^2 + |n_3|^2 - |n_4|^2 + |n_1|^2 \end{array} \right\}$$

and note that $|S_{(n_5,n,m)}| \lesssim N_4^3 N_3^3 N_2^2$. Also note that the variation of m is $\sim N_5 N_1$, therefore by Lemma 3.4, for ω outside a set of measure e^{-1/δ^r} we have

$$\begin{aligned}
(8.10) &\lesssim \delta^{-2\mu r} N_5 N_1 N_5^{-3} \sum_m \sum_{n_5} \Big| \sum_{S_{(n_5,n,m)}} a_{n_1}\overline{a}_{n_2} a_{n_3} \overline{a}_{n_4} \Big|^2 \\
&\lesssim \delta^{-2\mu r} N_5^{-2} N_1 \sup_m \sum_{n_5} \sum_{S_{(n_5,n,m)}} |S_{(n_5,n,m)}|\, |a_{n_1}|^2 |a_{n_2}|^2 |a_{n_3}|^2 |a_{n_4}|^2 \\
&\lesssim \delta^{-2\mu r} N_5^{-2} N_1 N_4^3 N_3^3 N_2^2 \sum_{n_1,n_2,n_3,n_4} |a_{n_1}|^2 |a_{n_2}|^2 |a_{n_3}|^2 |a_{n_4}|^2 |S_{(n_1,n_2,n_3,n_4,m)}| \\
&\lesssim \delta^{-2\mu r} N_5^{-1} N_1 N_4^3 N_3^3 N_2^2 \prod_{i=1}^{4} \|a_{n_i}\|_{\ell^2}^2
\end{aligned}$$

where

$$S_{(n_1,n_2,n_3,n_4,m)} := \left\{ (n_1, n_5) : \begin{array}{c} n = n_5 - n_2 + n_3 - n_4 + n_1, \ n_5 \in C, \\ m = |n_5|^2 - |n_2|^2 + |n_3|^2 - |n_4|^2 + |n_1|^2 \end{array} \right\}$$

and in the last inequality we have used $|S_{(n_1,n_2,n_3,n_4,m)}| \leq N_5$. After renormalizing and taking square roots we obtain the bound of $N_5^{-3s+\alpha+3}$, which entails $s > 1 + \alpha/3$.

Case (b). We will proceed by duality and a change of variables $\zeta = m - |n_1|^2$ as in the proof of Proposition 7.2 (in particular see (7.15)). We also cut the N_1 window with cubes C of sidelength N_5. We have to bound

$$\|\gamma\|_{\ell^2_\zeta}^2 \|\chi_C a_{n_1}\|_{\ell^2}^2 \sum_{(\zeta,n_1)\in\mathbb{Z}\times\mathbb{Z}^3} \Bigg| \sum_{\substack{n=n_5-n_2+n_3-n_4+n_1 \\ n_1,n_3,n_5\neq n_2,n_4 \\ \zeta=|n_5|^2-|n_2|^2+|n_3|^2-|n_4|^2}} \frac{g_{n_5}(\omega)}{|n_5|^{3/2}} \chi_{\tilde{C}}(n) k_n \overline{a}_{n_2} a_{n_3} \overline{a}_{n_4} \Bigg|^2 \tag{8.11}$$

where $\tilde{C}$ is of size approximately N_5. We define

$$S_{(n_5,n_1,\zeta)} := \left\{ (n, n_2, n_3, n_4) : \begin{array}{c} n = n_5 - n_2 + n_3 - n_4 + n_1, \\ n_1, n_3, n_5 \neq n_2, n_4, \ n \in \tilde{C}, \\ \zeta = |n_5|^2 - |n_2|^2 + |n_3|^2 - |n_4|^2 \end{array} \right\}$$

and note that $|S_{(n_5,n_1,\zeta)}| \lesssim N_4^3 N_3^3 N_2^2$. Note also that $\Delta\zeta \lesssim N_5^2$, hence we can continue for ω outside a set of measure e^{-1/δ^r} with

$$\begin{aligned}
(8.11) &\lesssim \delta^{-2\mu} \|\gamma\|_{\ell^2_\zeta}^2 \|\chi_C a_{n_1}\|_{\ell^2}^2 N_5^2 N_5^{-3} \sup_\zeta \sum_{n_1,n_5} \Bigg| \sum_{S_{(n_5,n_1,\zeta)}} \chi_{\tilde{C}}(n) k_n \overline{a}_{n_2} a_{n_3} \overline{a}_{n_4} \Bigg|^2 \\
&\lesssim \delta^{-2\mu} \|\gamma\|_{\ell^2_\zeta}^2 \|\chi_C a_{n_1}\|_{\ell^2}^2 N_5^{-1} \\
&\quad \times \sup_\zeta \sum_{n_1,n_5} \sum_{S_{(n_5,n_1,\zeta)}} |S_{(n_5,n_1,\zeta)}| \, |a_{n_2}|^2 |a_{n_3}|^2 |a_{n_4}|^2 |\chi_{\tilde{C}}(n) k_n|^2 \\
&\lesssim \delta^{-2\mu} \|\gamma\|_{\ell^2_\zeta}^2 \|\chi_C a_{n_1}\|_{\ell^2}^2 N_5^{-1} N_4^3 N_3^3 N_2^2 \\
&\quad \times \sum_{n,n_2,n_3,n_4} |a_{n_2}|^2 |a_{n_3}|^2 |a_{n_4}|^2 |\chi_{\tilde{C}}(n) k_n|^2 |S_{(n,n_2,n_3,n_4,\zeta)}| \\
&\lesssim \delta^{-2\mu} \|\chi_C a_{n_1}\|_{\ell^2}^2 N_4^3 N_3^3 N_2^2 \prod_{i=2}^4 \|a_{n_i}\|_{\ell^2}^2 \|k_n\|_{\ell^2}^2 \|\gamma\|_{\ell^2_\zeta}^2
\end{aligned}$$

where

$$S_{(n,n_2,n_3,n_4,\zeta)} := \left\{ (n, n_5) : \begin{array}{c} n = n_5 - n_2 + n_3 - n_4 + n_1, \ n_5 \in C, \\ \zeta = |n_5|^2 - |n_2|^2 + |n_3|^2 - |n_4|^2 \end{array} \right\}$$

and in the last inequality we have used $|S_{(n,n_2,n_3,n_4,\zeta)}| \leq N_5$. After renormalizing and taking square roots we obtain a bound of $N_5^{-3s+\alpha+3}$, which entails $s > 1 + \alpha/3$.

Case (c). This is like Case (b), but now we do not need to cut the support of the N_1 window with N_5.

Case (d). In this case we proceed as in Subsection 8.5.1, the only difference being in the treatment of the terms in (8.7). More precisely, here we show how to estimate the random term in (8.9). For v_0^ω as in (5.1) we have

$$N_0^s N_3^{-1+\alpha} \| P_{N_0} P_{N_3} v_0^\omega \|_{L_t^\infty H_x^{1-\alpha}} \prod_{j=1}^{4} N_j^{1/4-s} \| P_{N_j} u_j(x,t) \|_{U_\Delta^4 H^s}, \tag{8.12}$$

where we notice that $N_3 \sim N_0$, otherwise the contribution would be null. This is enough to obtain the desired bound.

Case (e). This is like Case (d), but now we do not need to cut the support of the N_1 window with N_2.

8.5.3. The DDDRR case. To estimate the expression in (8.5) we will assume without any loss of generality that u_4, u_5 are random and $N_4 \geq N_5$. We can also assume that $N_1 \geq N_2 \geq N_3$. We have two different scenarios: Case 1: $u_4 u_5$ or Case 2: $\overline{u}_4 u_5$, the other cases being obtained by complex conjugation since we do not care about bars on deterministic functions. The only difference between Cases 1 and 2 is that in Case 2 we automatically have $n_4 \neq n_5$ which allows us to use Proposition 3.1, and hence the same argument as in Case 1 applies. We discuss Case 1 within the context of the following cases (Case 2 being analogous after appropriately rewriting the corresponding constraints):

- **Case (a):**
  - (i) $N_4 \sim N_5 \geq N_0, N_1$.
  - (ii) $N_4 \sim N_1 \geq N_0$.
- **Case (b):** $N_4 \sim N_0$ and
  - (i) $N_5 \geq N_1$.
  - (ii) $N_4 \geq N_1 \geq N_5 \geq N_2$.
  - (iii) $N_4 \geq N_1$ and $N_2 \geq N_5 \geq N_3$.
  - (iv) $N_4 \geq N_1$ and $N_3 \geq N_5$.
- **Case (c):** $N_1 \sim N_0$ and
  - (i) $N_1 \geq N_4, N_5 \geq N_2$.
  - (ii) $N_1 \geq N_4 \geq N_2 \geq N_5 \geq N_3$.
  - (iii) $N_1 \geq N_4 \geq N_2 \geq N_3 \geq N_5$.
  - (iv) $N_2 \geq N_4, N_5 \geq N_3$.
  - (v) $N_2 \geq N_4 \geq N_3 \geq N_5$.
  - (vi) $N_3 \geq N_4$.
- **Case (d):** $N_1 \sim N_2 \geq N_0, N_4$.

Below we always treat Case 1 and without any loss of generality we may assume $\tilde{u}_1 = u_1$, $\tilde{u}_j = \overline{u}_j$, $j = 2, 3$.

Case (a)(i). In this case, $N_4 \sim N_5 \geq N_0, N_1$. By Cauchy–Schwarz, the transfer principle and Plancherel we are reduced to estimating

$$\sum_{(m,n)\in\mathbb{Z}\times\mathbb{Z}^3}\Bigg|\sum_{n_4,n_5}\Bigg[\sum_{S_{(n_4,n_5,n,m)}} a_{n_1}\overline{a}_{n_2}\overline{a}_{n_3}\frac{1}{|n_4|^{3/2}}\frac{1}{|n_5|^{3/2}}\Bigg]g_{n_4}(\omega)g_{n_5}(\omega)\Bigg|^2 \tag{8.13}$$

where

$$S_{(n_4,n_5,n,m)} := \left\{(n_1,n_2,n_3) : \begin{array}{l} n = n_4+n_5+n_1-n_2-n_3, \\ n_2,n_3 \neq n_1,n_4,n_5, \\ m = |n_4|^2+|n_5|^2+|n_1|^2-|n_2|^2-|n_3|^2 \end{array}\right\}$$

with $|S_{(n_4,n_5,n,m)}| \lesssim N_3^3 N_2^2$ and $\Delta m \sim N_4^2$. We then have, for ω outside a set of measure e^{-1/δ^r},

$$\begin{aligned}
(8.13) &\lesssim \delta^{-2\mu r} N_4^2 N_4^{-3} N_5^{-3} \sup_m \sum_{n_4,n_5,n}\Bigg|\sum_{S_{(n_4,n_5,n,m)}} a_{n_1}\overline{a}_{n_2}\overline{a}_{n_3}\Bigg|^2 \\
&\lesssim \delta^{-2\mu r} N_4^{-1} N_5^{-3} \sup_m \sum_{n_4,n_5,n}\sum_{S_{(n_4,n_5,n,m)}} |S_{(n_4,n_5,n,m)}||a_{n_1}|^2|a_{n_2}|^2|a_{n_3}|^2 \\
&\lesssim \delta^{-2\mu r} N_4^{-1} N_5^{-3} N_2^2 N_3^3 \sup_m \sum_{n_1,n_2,n_3} |a_{n_1}|^2|a_{n_2}|^2|a_{n_3}|^2|S_{(n_1,n_2,n_3,m)}| \\
&\lesssim \delta^{-2\mu r} N_4^{-1} N_5^{-3} N_2^2 N_3^3 N_4 N_5^3 \sum_{n_1,n_2,n_3} |a_{n_1}|^2|a_{n_2}|^2|a_{n_3}|^2 \\
&\lesssim \delta^{-2\mu r} N_2^2 N_3^3 \prod_{i=1}^{3}\|a_{n_i}\|_{\ell^2}^2
\end{aligned} \tag{8.14}$$

where

$$S_{(n_1,n_2,n_3,m)} := \left\{(n,n_4,n_5) : \begin{array}{l} n = n_4+n_5+n_1-n_2-n_3, \\ n_2,n_3 \neq n_1,n_4,n_5, \\ m = |n_4|^2+|n_5|^2+|n_1|^2-|n_2|^2-|n_3|^2 \end{array}\right\}$$

with $|S_{(n_1,n_2,n_3,m)}| \lesssim N_5^3 N_4$. Taking square roots and normalizing we then obtain the bound $N_4^{s-2+2\alpha}$, which requires $s < 2-2\alpha$.

Case (a)(ii). In this case $N_4 \sim N_1 \geq N_0$, we repeat the argument in Case (a)(i), but in this case after taking square roots and normalizing we obtain the bound $N_4^{3/2-2s+\alpha}$.

Case (b)(i). In this case, $N_4 \sim N_0$ and $N_5 \geq N_1$. From (8.5), we first decompose the support of $\chi_{N_0}\widehat{h}$ by taking cubes C of sidelength N_5 and then apply Cauchy–Schwarz, the transfer principle and Plancherel. We are thus reduced to estimating an expression just as in (8.13) but where now $\Delta m \sim N_4 N_5$ and thus we obtain instead of (8.14) the estimate $\delta^{-2\mu r} N_4^{-1} N_5 N_2^2 N_3^3 \prod_{i=1}^{3}\|a_{n_i}\|_{\ell^2}^2$. Taking square roots and normalizing we obtain the bound $N_4^{s-3/2+\alpha}$ provided $\alpha < 1/2$, which in turn entails $1 \leq s < 3/2-\alpha$.

Case (b)(ii). In this case we have $N_4 \sim N_0$ and $N_1 \geq N_5$. The proof follows that of Case (b)(i) except that now we first decompose the support of $\chi_{N_0}\widehat{h}$ (and hence the N_4 Fourier window) with cubes C of sidelength N_1. We then have $\Delta m \sim N_4 N_1$ and instead of (8.14) we obtain the estimate $\delta^{-2\mu r} N_4^{-1} N_1 N_2^2 N_3^3 \prod_{i=1}^3 \|a_{n_i}\|_{\ell^2}^2$. Taking square roots and normalizing we obtain the bound $N_4^{s-3/2+\alpha}$ as before.

Cases (b)(iii), (iv) are analogous to Case (b)(ii).

Case (c)(i). In this case we have $N_0 \sim N_1 \geq N_4, N_5 \geq N_2$. We will proceed by duality and the change of variables $\zeta = m - |n_1|^2$ as in the proof of Proposition 7.2, (7.15) and also as in (8.11). We also cut the N_1 window with cubes C of sidelength N_4. We have to bound

$$\|\chi_C a_{n_1}\|_{\ell^2}^2 \|\gamma\|_{\ell^2_\zeta}^2 \sum_{\zeta \in \mathbb{Z}, n_1 \in \mathbb{Z}^3} \Bigg| \sum_{\substack{n = n_1 + n_4 + n_5 - n_2 - n_3 \\ n_2, n_3 \neq n_1, n_4, n_5 \\ \zeta = |n_4|^2 + |n_5|^2 - |n_2|^2 - |n_3|^2}} \frac{g_{n_4}(\omega)}{|n_4|^{3/2}} \frac{g_{n_5}(\omega)}{|n_5|^{3/2}} \chi_{\tilde{C}}(n) k_n \overline{a}_{n_2} \overline{a}_{n_3} \Bigg|^2, \tag{8.15}$$

where $\tilde{C}$ is of size approximately N_4. Let us now define

$$\sigma_{n_1,n_2} := \sum_{\substack{n = n_1 + n_4 + n_5 - n_2 - n_3 \\ n_2, n_3 \neq n_1, n_4, n_5 \\ \zeta = |n_4|^2 + |n_5|^2 - |n_2|^2 - |n_3|^2}} \chi_{\tilde{C}}(n) k_n \overline{a}_{n_3} \frac{g_{n_4}(\omega)}{|n_4|^{3/2}} \frac{g_{n_5}(\omega)}{|n_5|^{3/2}}, \tag{8.16}$$

and note that then $\Delta\zeta \sim N_4^2$. Then

$$\begin{aligned}
(8.15) &\lesssim \|\chi_C a_{n_1}\|_{\ell^2}^2 \|\gamma\|_{\ell^2_\zeta}^2 N_4^2 \sup_\zeta \sum_{n_1 \in C} |\sigma_{n_1,n_2} \overline{a}_{n_2}|^2 \\
&\leq N_4^2 \|\chi_C a_{n_1}\|_{\ell^2}^2 \|\gamma\|_{\ell^2_\zeta}^2 \|a_{n_2}\|_{\ell^2}^2 \sup_\zeta \|\mathcal{G}\mathcal{G}^*\|.
\end{aligned} \tag{8.17}$$

As in Section 7, we write

$$\|\mathcal{G}\mathcal{G}^*\| \lesssim \max_{n_1} \sum_{n_2, n_2 \neq n_1} |\sigma_{n_1,n_2}|^2 + \Bigg(\sum_{n_1 \neq n_1'} \Bigg| \sum_{n_2} \sigma_{n_1,n_2} \overline{\sigma}_{n_1',n_2} \Bigg|^2 \Bigg)^{1/2} =: M_1 + M_2, \tag{8.18}$$

and estimate each term separately. For M_1 we proceed as follows:

$$\begin{aligned}
M_1 &= \sup_{n_1} \sum_{n_2, n_2 \neq n_1} \Bigg| \sum_{n_4, n_5} \Bigg[\sum_{S_{(n_1,n_2,n_4,n_5,\zeta)}} \chi_{\tilde{C}}(n) k_n \overline{a}_{n_3} \frac{1}{|n_4|^{3/2}} \frac{1}{|n_5|^{3/2}} \Bigg] g_{n_4}(\omega) g_{n_5}(\omega) \Bigg|^2, \\
&\lesssim \delta^{-2\mu r} \sup_{n_1} \sum_{n_2 \neq n_1, n_4, n_5} N_4^{-3} N_5^{-3} \Bigg| \sum_{S_{(n_1,n_2,n_4,n_5,\zeta)}} \chi_{\tilde{C}}(n) k_n \overline{a}_{n_3} \Bigg|^2 \\
&\lesssim \delta^{-2\mu r} \sup_{n_1} \sum_{n_2, n_4, n_5} N_4^{-3} N_5^{-3} |S_{(n_1,n_2,n_4,n_5,\zeta)}| \sum_{S_{(n_1,n_2,n_4,n_5,\zeta)}} |\chi_C(n) k_n|^2 |\overline{a}_{n_3}|^2
\end{aligned} \tag{8.19}$$

for ω outside a set of measure e^{-1/δ^r}, where

$$S_{(n_1,n_2,n_4,n_5,\zeta)} := \left\{ (n, n_3) : \begin{array}{c} n = n_1 + n_4 + n_5 - n_2 - n_3, \\ n_2, n_3 \neq n_1, n_4, n_5,\ n \in \tilde{C}, \\ \zeta = |n_4|^2 + |n_5|^2 - |n_2|^2 - |n_3|^2 \end{array} \right\}$$

with $|S_{(n_1,n_2,n_4,n_5,\zeta)}| \lesssim N_3^2$. Hence for

$$S_{(n,n_3,\zeta)} := \left\{ (n_2, n_4, n_5) : \begin{array}{c} n = n_1 + n_4 + n_5 - n_2 - n_3, \\ n_2, n_3 \neq n_1, n_4, n_5,\ n \in \tilde{C}, \\ \zeta = |n_4|^2 + |n_5|^2 - |n_2|^2 - |n_3|^2 \end{array} \right\}$$

we have

$$\begin{aligned}
(8.19) &\lesssim \delta^{-2\mu r} N_4^{-3} N_5^{-3} N_3^2 \sum_{n,n_3} |\chi_{\tilde{C}}(n) k_n|^2 |\overline{a}_{n_3}|^2 |S_{(n,n_3,\zeta)}| \\
&\lesssim \delta^{-2\mu r} N_4^{-3} N_5^{-3} N_3^2 N_2^3 N_5^3 N_4 \|\chi_{\tilde{C}}(n) k_n\|_{\ell^2}^2 \|a_{n_3}\|_{\ell^2}^2 \\
&\lesssim \delta^{-2\mu r} N_4^{-2} N_2^3 N_3^2 \|\chi_{\tilde{C}}(n) k_n\|_{\ell^2}^2 \|a_{n_3}\|_{\ell^2}^2.
\end{aligned}$$

Hence the contribution of M_1 to (8.17) is

$$\delta^{-2\mu r} N_4^2 N_4^{-2} N_2^3 N_3^2 \|\chi_C a_{n_1}\|_{\ell^2}^2 \|a_{n_2}\|_{\ell^2}^2 \|a_{n_3}\|_{\ell^2}^2 \|\gamma\|_{\ell^2_\zeta}^2 \|\chi_{\tilde{C}}(n) k_n\|_{\ell^2}^2.$$

After taking square roots and normalizing we obtain a bound of $N_4^{-1+\alpha} N_5^{-s+1/2+\alpha}$, which suffices provided $s > 1/2 + \alpha$.

To estimate M_2 we first write

$$\begin{aligned}
M_2^2 &= \sum_{n_1 \neq n_1'} \Big| \sum_{n_2} \sigma_{n_1,n_2} \overline{\sigma}_{n_1',n_2} \Big|^2 \\
&\sim \sum_{n_1 \neq n_1'} \Big| \sum_{S_{(n_1,n_1',\zeta)}} \chi_{\tilde{C}}(n) k_n \chi_{\tilde{C}}(n') k_{n'} \overline{a}_{n_3} a_{n_3'} \frac{g_{n_4}(\omega)}{|n_4|^{3/2}} \frac{g_{n_5}(\omega)}{|n_5|^{3/2}} \frac{\overline{g}_{n_4'}(\omega)}{|n_4'|^{3/2}} \frac{\overline{g}_{n_5'}(\omega)}{|n_5'|^{3/2}} \Big|^2 \qquad (8.20)
\end{aligned}$$

where

$$S_{(n_1,n_1',\zeta)} := \left\{ (n, n_2, n_3, n_3', n_4, n_4', n_5, n_5') : \begin{array}{c} n = n_1 + n_4 + n_5 - n_2 - n_3, \\ n' = n_1' + n_4' + n_5' - n_2 - n_3', \\ n_2, n_3 \neq n_1, n_4, n_5;\ n_2', n_3' \neq n_1', n_4', n_5';\ n, n' \in \tilde{C}, \\ \zeta = |n_4|^2 + |n_5|^2 - |n_2|^2 - |n_3|^2, \\ \zeta = |n_4'|^2 + |n_5'|^2 - |n_2|^2 - |n_3'|^2 \end{array} \right\}. \qquad (8.21)$$

To streamline the exposition let

$$\mathscr{C} := \begin{cases} n = n_1 + n_4 + n_5 - n_2 - n_3,\ n' = n_1' + n_4' + n_5' - n_2 - n_3'; \\ \zeta = |n_4|^2 + |n_5|^2 - |n_2|^2 - |n_3|^2,\ \zeta = |n_4'|^2 + |n_5'|^2 - |n_2|^2 - |n_3'|^2; \\ n_2, n_3 \neq n_1, n_4, n_5;\ n_2', n_3' \neq n_1', n_4', n_5';\ n, n' \in \tilde{C}. \end{cases}$$

We need to organize the estimates according to whether some frequencies are the same or not; in all we have seven cases:

- **Case** β_1**:** $n_4, n_5 \neq n_4', n_5'$.
- **Case** β_2**:** $n_4 = n_4'$; $n_5 \neq n_5'$.
- **Case** β_3**:** $n_4 \neq n_4'$; $n_5 = n_5'$.
- **Case** β_4**:** $n_4 \neq n_5'$; $n_5 = n_4'$.
- **Case** β_5**:** $n_4 = n_5'$; $n_5 \neq n_4'$.
- **Case** β_6**:** $n_4 = n_5'$; $n_5 = n_4'$.
- **Case** β_7**:** $n_4 = n_4'$; $n_5 = n_5'$.

Case β_1**.** To estimate the contribution of M_2, we first define

$$S_{(n_1,n_1',n_4,n_4',n_5,n_5',\zeta)} := \{(n, n', n_2, n_3, n_3') \text{ satisfying } \mathscr{C}\},$$

with $|S_{(n_1,n_1',n_4,n_4',n_5,n_5',\zeta)}| \lesssim N_3^6 N_2^2$. Next, for ω outside a set of measure e^{-1/δ^r}, we estimate M_2^2 as follows:

$$\begin{aligned}
(8.20) &\lesssim \delta^{-4\mu r} \sum_{n_1 \neq n_1'} N_4^{-6} N_5^{-6} \sum_{n_4 \neq n_4', n_5 \neq n_5'} \Big[\sum_{S_{(n_1,n_1',n_4,n_4',n_5,n_5',\zeta)}} \chi_C(n) k_n \chi_C(n') k_{n'} \overline{a}_{n_3} a_{n_3'} \Big]^2 \\
&\lesssim \delta^{-4\mu r} \\
&\quad \times \sum_{n_1 \neq n_1'} N_4^{-6} N_5^{-6} N_3^6 N_2^2 \sum_{n_4, n_4', n_5, n_5'} \sum_{S_{(n_1,n_1',n_4,n_4',n_5,n_5',\zeta)}} |\chi_{\tilde{C}}(n) k_n|^2 |\chi_{\tilde{C}}(n') k_{n'}|^2 |\overline{a}_{n_3}|^2 |a_{n_3'}|^2 \\
&\lesssim \delta^{-4\mu r} N_4^{-6} N_5^{-6} N_3^6 N_2^2 \sum_{n, n', n_3, n_3'} |S_{(n,n',n_3,n_3',\zeta)}| \, |\chi_{\tilde{C}}(n) k_n|^2 |\chi_{\tilde{C}}(n') k_{n'}|^2 |\overline{a}_{n_3}|^2 |a_{n_3'}|^2 \\
&\lesssim \delta^{-4\mu r} N_4^{-6} N_5^{-6} N_3^6 N_2^2 N_2^3 N_5^6 N_4^2 \|\chi_{\tilde{C}}(n) k_n\|_{\ell^2}^2 \|\chi_{\tilde{C}}(n') k_{n'}\|_{\ell^2}^2 \|a_{n_3}\|_{\ell^2}^2 \|a_{n_3'}\|_{\ell^2}^2 \\
&\lesssim \delta^{-4\mu r} N_4^{-4} N_3^6 N_2^5 \|\chi_{\tilde{C}}(n) k_n\|_{\ell^2}^2 \|\chi_{\tilde{C}}(n') k_{n'}\|_{\ell^2}^2 \|a_{n_3}\|_{\ell^2}^2 \|a_{n_3'}\|_{\ell^2}^2,
\end{aligned}$$

where we have used the fact that $S_{(n,n',n_3,n_3',\zeta)} := \{(n_1, n_1', n_2, n_4, n_4', n_5, n_5')$ satisfying $\mathscr{C}\}$ has cardinality less than or equal to $N_2^3 N_5^6 N_4^2$.

All in all, the contribution of $\Delta\zeta M_2$ is bounded by $N_3^3 N_2^{5/2}$. Taking square roots and normalizing we finally obtain the bound $N_4^{-1+\alpha} N_5^{7/4-2s+\alpha}$, which suffices provided $s > 7/8 + \alpha/2$.

Case β_2. Now we have $n_4 = n_4'$ while $n_5 \neq n_5'$, rendering (8.20) equal to

$$\sum_{n_1\neq n_1'}\Big|\sum_{S_{(n_1,n_1',\zeta)}} \chi_{\tilde{C}}(n)k_n\chi_{\tilde{C}}(n')k_{n'}\overline{a}_{n_3}a_{n_3'}\frac{|g_{n_4}(\omega)|^2}{|n_4|^3}\frac{g_{n_5}(\omega)}{|n_5|^{3/2}}\frac{\overline{g}_{n_5'}(\omega)}{|n_5'|^{3/2}}\Big|^2. \tag{8.22}$$

We proceed in a similar fashion to (7.17)–(7.23) and define

$$\mathcal{Q}_1 := \sum_{n_1\neq n_1'}\Big|\sum_{S_{(n_1,n_1',\zeta)}} k_n^{\tilde{C}}k_{n'}^{\tilde{C}}\overline{a}_{n_3}a_{n_3'}\frac{|g_{n_4}(\omega)|^2-1}{|n_4|^3}\frac{g_{n_5}(\omega)}{|n_5|^{3/2}}\frac{\overline{g}_{n_5'}(\omega)}{|n_5'|^{3/2}}\Big|^2, \tag{8.23}$$

$$\mathcal{Q}_2 := \sum_{n_1\neq n_1'}\Big|\sum_{S_{(n_1,n_1',\zeta)}} k_n^{\tilde{C}}k_{n'}^{\tilde{C}}\overline{a}_{n_3}a_{n_3'}\frac{1}{|n_4|^3}\frac{g_{n_5}(\omega)}{|n_5|^{3/2}}\frac{\overline{g}_{n_5'}(\omega)}{|n_5'|^{3/2}}\Big|^2, \tag{8.24}$$

where we have denoted $k_n^{\tilde{C}} := \chi_{\tilde{C}}(n)k_n$ and similarly for $k_{n'}^{\tilde{C}}$.

To estimate $\mathcal{Q}_2$ define

$$S_{(n_1,n_1',n_5,n_5',\zeta)} := \{(n,n',n_2,n_4,n_3,n_3') \text{ satisfying } \mathscr{C}\},$$

with $|S_{(n_1,n_1',n_5,n_5',\zeta)}| \lesssim N_3^6N_2^3N_4$. Then for ω outside a set of measure e^{-1/δ^r},

$$\begin{aligned}
(8.24) &\lesssim \delta^{-4\mu r}\sum_{n_1\neq n_1'}N_4^{-6}N_5^{-6}\sum_{n_5\neq n_5'}\Big[\sum_{S_{(n_1,n_1',n_5,n_5',\zeta)}}k_n^{\tilde{C}}k_{n'}^{\tilde{C}}\overline{a}_{n_3}a_{n_3'}\Big]^2\\
&\lesssim \delta^{-4\mu r}\sum_{n_1\neq n_1'}N_4^{-6}N_5^{-6}N_3^6N_2^3N_4\sum_{n_5,n_5'}\sum_{S_{(n_1,n_1',n_5,n_5',\zeta)}}|k_n^{\tilde{C}}|^2|k_{n'}^{\tilde{C}}|^2|\overline{a}_{n_3}|^2|a_{n_3'}|^2\\
&\lesssim \delta^{-4\mu r}N_4^{-6}N_5^{-6}N_3^6N_2^3N_4\sum_{n,n',n_3,n_3'}|S_{(n,n',n_3,n_3',\zeta)}|\,|k_n^{\tilde{C}}|^2|k_{n'}^{\tilde{C}}|^2|\overline{a}_{n_3}|^2|a_{n_3'}|^2\\
&\lesssim \delta^{-4\mu r}N_4^{-6}N_5^{-6}N_3^6N_2^3N_4N_2^3N_5^6N_4\|\chi_{\tilde{C}}(n)k_n\|_{\ell^2}^2\|\chi_{\tilde{C}}(n')k_{n'}\|_{\ell^2}^2\|a_{n_3}\|_{\ell^2}^2\|a_{n_3'}\|_{\ell^2}^2\\
&\lesssim \delta^{-4\mu r}N_4^{-4}N_3^6N_2^6\|\chi_{\tilde{C}}(n)k_n\|_{\ell^2}^2\|\chi_{\tilde{C}}(n')k_{n'}\|_{\ell^2}^2\|a_{n_3}\|_{\ell^2}^2\|a_{n_3'}\|_{\ell^2}^2
\end{aligned}$$

where we have used the fact that $S_{(n,n',n_3,n_3',\zeta)} := \{(n_1,n_1',n_2,n_4,n_5,n_5') \text{ satisfying } \mathscr{C}\}$ has cardinality less than or equal to $N_2^3N_5^6N_4$.

The bound for $\mathcal{Q}_1$ is smaller, just as in the proof of Proposition 7.2, (7.17)–(7.23). We omit the details.

Thus the contribution of $\Delta\zeta M_2$ is bounded by $N_3^3N_2^3$, which after taking square roots and normalizing gives a bound of $N_4^{-1+\alpha}N_5^{2-2s+\alpha}$, which suffices provided $s > 1+\alpha/2$.

Case β_3. Now $n_4 \neq n_4'$ while $n_5 = n_5'$, rendering (8.20) equal to

$$\sum_{n_1\neq n_1'}\Big|\sum_{S_{(n_1,n_1',\zeta)}} k_n^{\tilde{C}}k_{n'}^{\tilde{C}}\overline{a}_{n_3}a_{n_3'}\frac{|g_{n_5}(\omega)|^2}{|n_5|^3}\frac{g_{n_4}(\omega)}{|n_4|^{3/2}}\frac{\overline{g}_{n_4'}(\omega)}{|n_4'|^{3/2}}\Big|^2. \tag{8.25}$$

We proceed as above, defining analogous $\mathcal{Q}_1$ and $\mathcal{Q}_2$ terms bounding (8.25) in this case.

To estimate $\mathcal{Q}_2$ we define

$$S_{(n_1,n_1',n_4,n_4',\zeta)} := \{(n, n', n_2, n_5, n_3, n_3') \text{ satisfying } \mathscr{C}\}$$

with $|S_{(n_1,n_1',n_4,n_4',\zeta)}| \lesssim N_3^6 N_2^3 \min(N_5^2, N_4) \le N_3^6 N_2^3 N_5^2$. Then for ω outside a set of measure e^{-1/δ^r},

$$\begin{aligned}
\mathcal{Q}_2 &\lesssim \delta^{-4\mu r} \sum_{n_1 \neq n_1'} N_4^{-6} N_5^{-6} \sum_{n_4 \neq n_4'} \Big[\sum_{S_{(n_1,n_1',n_4,n_4',\zeta)}} k_n^{\tilde{C}} k_{n'}^{\tilde{C}} \overline{a}_{n_3} a_{n_3'} \Big]^2 \\
&\lesssim \delta^{-4\mu r} \sum_{n_1 \neq n_1'} N_4^{-6} N_5^{-6} N_3^6 N_2^3 N_5^2 \sum_{n_4, n_4'} \sum_{S_{(n_1,n_1',n_4,n_4',\zeta)}} |k_n^{\tilde{C}}|^2 |k_{n'}^{\tilde{C}}|^2 |\overline{a}_{n_3}|^2 |a_{n_3'}|^2 \\
&\lesssim \delta^{-4\mu r} N_4^{-6} N_5^{-6} N_3^6 N_2^3 N_5^2 \sum_{n,n',n_3,n_3'} |S_{(n,n',n_3,n_3',\zeta)}| \, |k_n^{\tilde{C}}|^2 |k_{n'}^{\tilde{C}}|^2 |\overline{a}_{n_3}|^2 |a_{n_3'}|^2 \\
&\lesssim \delta^{-4\mu r} N_4^{-6} N_5^{-6} N_3^6 N_2^3 N_5^2 N_2^3 N_5^3 N_4^2 \|k_n^{\tilde{C}}\|_{\ell^2}^2 \|k_{n'}^{\tilde{C}}\|_{\ell^2}^2 \|a_{n_3}\|_{\ell^2}^2 \|a_{n_3'}\|_{\ell^2}^2 \\
&\lesssim \delta^{-4\mu r} N_4^{-4} N_5^{-1} N_3^6 N_2^6 \|\chi_{\tilde{C}}(n) k_n\|_{\ell^2}^2 \|\chi_{\tilde{C}}(n') k_{n'}\|_{\ell^2}^2 \|a_{n_3}\|_{\ell^2}^2 \|a_{n_3'}\|_{\ell^2}^2
\end{aligned} \tag{8.26}$$

where we have now used the fact that $S_{(n,n',n_3,n_3',\zeta)} := \{(n_1, n_1', n_2, n_4, n_4', n_5)$ satisfying $\mathscr{C}\}$ has cardinality less than or equal to $N_2^3 N_5^3 N_4^2$. Note this is a better bound than that obtained in Case β_2.

Since just as before, the bound for $\mathcal{Q}_1$ is smaller, the contribution of $\Delta\zeta M_2$ is bounded by $N_3^3 N_2^3$. After taking square roots and normalizing, the latter gives the same bound as in Case β_2.

Case β_4**.** In this case $n_4 \neq n_5'$ while $n_5 = n_4'$, rendering (8.20) equal to

$$\sum_{n_1 \neq n_1'} \Big| \sum_{S_{(n_1,n_1',\zeta)}} k_n^{\tilde{C}} k_{n'}^{\tilde{C}} \overline{a}_{n_3} a_{n_3'} \frac{|g_{n_5}(\omega)|^2}{|n_5|^3} \frac{g_{n_4}(\omega)}{|n_4|^{3/2}} \frac{\overline{g}_{n_5'}(\omega)}{|n_5'|^{3/2}} \Big|^2. \tag{8.27}$$

Once again, we proceed by defining the corresponding $\mathcal{Q}_1$ and $\mathcal{Q}_2$ terms bounding (8.27) and note the estimate for $\mathcal{Q}_1$ is better than that for $\mathcal{Q}_2$. In the latter case, we proceed as in (8.26) in Case β_3, but now

$$S_{(n_1,n_1',n_4,n_5',\zeta)} := \{(n, n', n_2, n_5, n_3, n_3') \text{ satisfying } \mathscr{C}\}$$

has $|S_{(n_1,n_1',n_4,n_5',\zeta)}| \lesssim N_3^6 N_2^3 N_4$. Furthermore, since $n_5 = n_4'$, we have $\Delta\zeta \lesssim N_5^2$ from the definition of $\mathscr{C}$. Thus for ω outside a set of measure e^{-1/δ^r},

$$\begin{aligned}
\mathcal{Q}_2 &\lesssim \delta^{-4\mu r} \sum_{n_1 \neq n_1'} N_4^{-3} N_5^{-9} \sum_{n_4 \neq n_5'} \Big[\sum_{S_{(n_1,n_1',n_4,n_5',\zeta)}} k_n^{\tilde{C}} k_{n'}^{\tilde{C}} \overline{a}_{n_3} a_{n_3'} \Big]^2 \\
&\lesssim \delta^{-4\mu r} \sum_{n_1 \neq n_1'} N_4^{-3} N_5^{-9} N_3^6 N_2^3 N_4 \sum_{n_4, n_5'} \sum_{S_{(n_1,n_1',n_4,n_5',\zeta)}} |k_n^{\tilde{C}}|^2 |k_{n'}^{\tilde{C}}|^2 |\overline{a}_{n_3}|^2 |a_{n_3'}|^2
\end{aligned}$$

$$
\begin{aligned}
&\lesssim N_4^{-3}N_5^{-9}N_3^6N_2^3N_4 \sum_{n,n',n_3,n_3'} |S_{(n,n',n_3,n_3',\zeta)}|\,|k_n^{\tilde{C}}|^2|k_{n'}^{\tilde{C}}|^2|\overline{a}_{n_3}|^2|a_{n_3'}|^2\\
&\lesssim \delta^{-4\mu r}N_4^{-3}N_5^{-9}N_3^6N_2^3N_4N_2^3N_5^5N_4\|k_n^{\tilde{C}}\|_{\ell^2}^2\|k_{n'}^{\tilde{C}}\|_{\ell^2}^2\|a_{n_3}\|_{\ell^2}^2\|a_{n_3'}\|_{\ell^2}^2\\
&\lesssim \delta^{-4\mu r}N_4^{-1}N_5^{-4}N_3^6N_2^6\|\chi_{\tilde{C}}(n)k_n\|_{\ell^2}^2\|\chi_{\tilde{C}}(n')k_{n'}\|_{\ell^2}^2\|a_{n_3}\|_{\ell^2}^2\|a_{n_3'}\|_{\ell^2}^2 \qquad (8.28)
\end{aligned}
$$

where now $S_{(n,n',n_3,n_3',\zeta)} := \{(n_1,n_1',n_2,n_4,n_5',n_5)$ satisfying $\mathscr{C}\}$ has cardinality less than or equal to $N_2^3N_5^3N_4$. Thus

$$
\Delta\zeta M_2 \lesssim \delta^{-4\mu r}N_5^2N_4^{-1/2}N_5^{-2}N_3^3N_2^3\|\chi_{\tilde{C}}(n)k_n\|_{\ell^2}\|\chi_{\tilde{C}}(n')k_{n'}\|_{\ell^2}\|a_{n_3}\|_{\ell^2}\|a_{n_3'}\|_{\ell^2},
$$

whence after taking square roots and normalizing we obtain a bound of $N_4^{-5/4+\alpha}N_5^{2-2s+\alpha}$, which suffices provided $s > 1+\alpha/2$.

Case β_5. In this case $n_4 = n_5'$ while $n_5 \neq n_4'$, rendering (8.20) equal to

$$
\sum_{n_1\neq n_1'}\left|\sum_{S_{(n_1,n_1',\zeta)}} k_n^{\tilde{C}}k_{n'}^{\tilde{C}}\overline{a}_{n_3}a_{n_3'}\frac{|g_{n_4}(\omega)|^2}{|n_4|^3}\frac{g_{n_4'}(\omega)}{|n_4'|^{3/2}}\frac{\overline{g}_{n_5}(\omega)}{|n_5|^{3/2}}\right|^2. \qquad (8.29)
$$

Once again, we define the corresponding $\mathcal{Q}_1$ and $\mathcal{Q}_2$ terms bounding (8.29). We treat $\mathcal{Q}_2$ as in (8.26) in Case β_3 but with

$$
S_{(n_1,n_1',n_4',n_5,\zeta)} := \{(n,n',n_2,n_4,n_3,n_3') \text{ satisfying } \mathscr{C}\}
$$

having $|S_{(n_1,n_1',n_4',n_5,\zeta)}| \lesssim N_3^6N_2^3N_4$. Then for ω outside a set of measure e^{-1/δ^r},

$$
\begin{aligned}
\mathcal{Q}_2 &\lesssim \delta^{-4\mu r}\sum_{n_1\neq n_1'}N_4^{-9}N_5^{-3}\sum_{n_4'\neq n_5}\Big[\sum_{S_{(n_1,n_1',n_4',n_5,\zeta)}} k_n^{\tilde{C}}k_{n'}^{\tilde{C}}\overline{a}_{n_3}a_{n_3'}\Big]^2\\
&\lesssim \delta^{-4\mu r}\sum_{n_1\neq n_1'}N_4^{-9}N_5^{-3}N_3^6N_2^3N_4\sum_{n_4',n_5}\sum_{S_{(n_1,n_1',n_4',n_5,\zeta)}}|k_n^{\tilde{C}}|^2|k_{n'}^{\tilde{C}}|^2|\overline{a}_{n_3}|^2|a_{n_3'}|^2\\
&\lesssim N_4^{-9}N_5^{-3}N_3^6N_2^3N_4\sum_{n,n',n_3,n_3'}|S_{(n,n',n_3,n_3',\zeta)}|\,|k_n^{\tilde{C}}|^2|k_{n'}^{\tilde{C}}|^2|\overline{a}_{n_3}|^2|a_{n_3'}|^2\\
&\lesssim \delta^{-4\mu r}N_4^{-9}N_5^{-3}N_3^6N_2^3N_4N_2^3N_5^3N_4^2\|k_n^{\tilde{C}}\|_{\ell^2}^2\|k_{n'}^{\tilde{C}}\|_{\ell^2}^2\|a_{n_3}\|_{\ell^2}^2\|a_{n_3'}\|_{\ell^2}^2\\
&\lesssim \delta^{-4\mu r}N_4^{-6}N_3^6N_2^6\|\chi_{\tilde{C}}(n)k_n\|_{\ell^2}^2\|\chi_{\tilde{C}}(n')k_{n'}\|_{\ell^2}^2\|a_{n_3}\|_{\ell^2}^2\|a_{n_3'}\|_{\ell^2}^2
\end{aligned}
$$

where now $S_{(n,n',n_3,n_3',\zeta)} := \{(n_1,n_1',n_2,n_4,n_5',n_5)$ satisfying $\mathscr{C}\}$ has cardinality less than or equal to $N_2^3N_5^3N_4^2$.

Thus the contribution of $\Delta\zeta M_2$ is bounded by $N_4^{-1}N_5^{-2}N_3^3N_2^3$. After taking square roots and normalizing we obtain a bound of $N_4^{-3/2+\alpha}N_5^{2-2s+\alpha}$, which suffices provided $s > 1+\alpha/2$.

Case β_6. In this case $n_4 = n_5'$ and $n_5 = n_4'$, and (8.20) has enough decay to use Lemma 3.4. We define

$$S_{(n_1,n_1',\zeta)} := \{(n, n', n_2, n_3, n_3', n_4, n_4') \text{ satisfying } \mathscr{C}\}$$

with $|S_{(n_1,n_1',\zeta)}| \lesssim N_3^6 N_2^3 N_4^4$ and proceed as follows for ω outside a set of measure e^{-1/δ^r}:

$$\begin{aligned}
&\sum_{n_1\neq n_1'} \Big| \sum_{S_{(n_1,n_1',\zeta)}} k_n^{\tilde{C}} k_{n'}^{\tilde{C}} \overline{a}_{n_3} a_{n_3'} \frac{|g_{n_4}(\omega)|^2}{|n_4|^3} \frac{|g_{n_4'}(\omega)|^2}{|n_4'|^3} \Big|^2 \\
&\quad \lesssim N_4^{-12+\varepsilon} \sum_{n_1\neq n_1'} |S_{(n_1,n_1',\zeta)}| \sum_{S_{(n_1,n_1',\zeta)}} |k_n^{\tilde{C}}|^2 |k_{n'}^{\tilde{C}}|^2 |\overline{a}_{n_3}|^2 |a_{n_3'}|^2, \\
&\quad \lesssim N_4^{-12+\varepsilon} N_3^6 N_2^3 N_4^4 N_2^3 N_4^4 \sum_{n,n',n_3,n_3'} |S_{(n,n',n_3,n_3',\zeta)}| \, |k_n^{\tilde{C}}|^2 |k_{n'}^{\tilde{C}}|^2 |\overline{a}_{n_3}|^2 |a_{n_3'}|^2 \\
&\quad \lesssim N_4^{-4+\varepsilon} N_3^6 N_2^6 \|\chi_{\tilde{C}}(n) k_n\|_{\ell^2}^2 \|\chi_{\tilde{C}}(n') k_{n'}\|_{\ell^2}^2 \|a_{n_3}\|_{\ell^2}^2 \|a_{n_3'}\|_{\ell^2}^2 \qquad (8.30)
\end{aligned}$$

where we have used the fact that $S_{(n,n',n_3,n_3',\zeta)} := \{(n_1, n_1', n_2, n_4, n_4') \text{ satisfying } \mathscr{C}\}$ has cardinality less than or equal to $N_2^3 N_4^4$.

Thus the contribution of $\Delta\zeta M_2$ is bounded by $N_4^{\varepsilon} N_3^3 N_2^3$. After taking square roots and normalizing we obtain a bound of $N_4^{-1+\alpha+\varepsilon} N_5^{2-2s+\alpha}$, which suffices provided $s > 1 + \alpha/2$.

Case β_7. In this case $n_4 = n_4'$ and $n_5 = n_5'$, and once again (8.20) has enough decay to use Lemma 3.4. Define

$$S_{(n_1,n_1',\zeta)} := \{(n, n', n_2, n_3, n_3', n_4, n_5) \text{ satisfying } \mathscr{C}\}$$

with $|S_{(n_1,n_1',\zeta)}| \lesssim N_3^6 N_2^3 N_5^3 N_4$ and proceed as follows for ω outside a set of measure e^{-1/δ^r}:

$$\begin{aligned}
&\sum_{n_1\neq n_1'} \Big| \sum_{S_{(n_1,n_1',\zeta)}} k_n^{\tilde{C}} k_{n'}^{\tilde{C}} \overline{a}_{n_3} a_{n_3'} \frac{|g_{n_4}(\omega)|^2}{|n_4|^3} \frac{|g_{n_5}(\omega)|^2}{|n_5|^3} \Big|^2 \\
&\quad \lesssim N_4^{-6+\varepsilon} N_5^{-6} \sum_{n_1\neq n_1'} |S_{(n_1,n_1',\zeta)}| \sum_{S_{(n_1,n_1',\zeta)}} |k_n^{\tilde{C}}|^2 |k_{n'}^{\tilde{C}}|^2 |\overline{a}_{n_3}|^2 |a_{n_3'}|^2, \\
&\quad \lesssim N_4^{-6+\varepsilon} N_5^{-6} N_3^6 N_2^3 N_5^3 N_4 \sum_{n,n',n_3,n_3'} |S_{(n,n',n_3,n_3',\zeta)}| \, |k_n^{\tilde{C}}|^2 |k_{n'}^{\tilde{C}}|^2 |a_{n_3}|^2 |a_{n_3'}|^2 \\
&\quad \lesssim N_4^{-4+\varepsilon} N_3^6 N_2^6 \|\chi_{\tilde{C}}(n) k_n\|_{\ell^2}^2 \|\chi_{\tilde{C}}(n') k_{n'}\|_{\ell^2}^2 \|a_{n_3}\|_{\ell^2}^2 \|a_{n_3'}\|_{\ell^2}^2
\end{aligned}$$

where we have used the fact that

$$S_{(n,n',n_3,n_3',\zeta)} := \{(n_1, n_1', n_2, n_4, n_5) \text{ satisfying } \mathscr{C} \text{ for fixed } (n, n', n_3, n_3', \zeta)\}$$

has cardinality less than or equal to $N_2^3 N_5^3 N_4$.

Thus the contribution of $\Delta\zeta M_2$ is bounded by $N_4^{\varepsilon}N_3^3N_2^3$. After taking square roots and normalizing we obtain a bound of $N_4^{-1+\alpha+\varepsilon}N_5^{2-2s+\alpha}$, which suffices provided $s > 1+\alpha/2$.

Case (c)(ii). In this case we have $N_0 \sim N_1 \geq N_4 \geq N_2 \geq N_5 \geq N_3$. As in Case (c)(i) after duality, changing variables $\zeta := m - |n_1|^2$ and cutting the N_1 window with cubes C of sidelength N_4 we have to estimate expression (8.15). Since $\Delta\zeta \sim N_4^2$ we once again bound (8.15) by

$$\|\chi_C a_{n_1}\|_{\ell^2}^2 \|\gamma\|_{\ell^2_\zeta}^2 N_4^2 \sup_\zeta \sum_{n_1 \in C} |\sigma_{n_1,n_2}\overline{a}_{n_2}|^2 \leq N_4^2 \|\chi_C a_{n_1}\|_{\ell^2}^2 \|\gamma\|_{\ell^2_\zeta}^2 \|a_{n_2}\|_{\ell^2}^2 \sup_\zeta \|\mathcal{G}\mathcal{G}^*\|$$

where σ_{n_1,n_2} is defined as in (8.16) and $\mathcal{G}$ denotes, as usual, the matrix of entries σ_{n_1,n_2}. Just as in Case (c)(i) we are then reduced to estimating M_1 and M_2 as defined in (8.18).

To estimate M_1 we proceed just as in (8.19) to obtain for ω outside a set of measure e^{-1/δ^r} the same bound

$$M_1 \lesssim \delta^{-2\mu r} N_4^{-2} N_2^3 N_3^2 \|\chi_{\tilde{C}}(n) k_n\|_{\ell^2}^2 \|a_{n_3}\|_{\ell^2}^2. \tag{8.31}$$

Hence $\Delta\zeta M_1$ is bounded once again by

$$\delta^{-2\mu r} N_2^3 N_3^2 \|\chi_C a_{n_1}\|_{\ell^2}^2 \|a_{n_2}\|_{\ell^2}^2 \|a_{n_3}\|_{\ell^2}^2 \|\gamma\|_{\ell^2_\zeta}^2 \|\chi_{\tilde{C}}(n) k_n\|_{\ell^2}^2,$$

which after taking square roots and normalizing gives the bound $N_4^{-s+1/2+\alpha}$, which suffices provided $s > 1/2+\alpha$.

The estimate for M_2 proceeds as in Case (c)(i) by analyzing Cases β_1–β_7 as stated there, yielding the same bounds for $\Delta\zeta M_2$. We do not repeat the arguments but rather indicate the bound we obtain in each case after taking square roots and normalizing since now $N_2 \geq N_5 \geq N_3$, so we need to trade the slower decay of the random term $\tilde{u}_5$ for the better regularity of the deterministic function $\tilde{u}_2$.

Case β_1. In this case the contribution of $\Delta\zeta M_2$ is bounded by $N_3^3 N_2^{5/2}$. Taking square roots and normalizing we obtain the bound $N_4^{-1+\alpha} N_2^{5/4-s} N_5^{1/2-s+\alpha}$, which suffices provided $s > 1/2+\alpha$ and $\alpha < 1$.

Cases β_2 and β_3. In these cases the contribution of $\Delta\zeta M_2$ is bounded by $N_3^3 N_2^3$. Taking square roots and normalizing we obtain the bound $N_4^{1/2-s+\alpha} N_5^{1/2-s+\alpha}$, which suffices provided $s > 1/2+\alpha$.

Case β_4. In this case the contribution of $\Delta\zeta M_2$ is bounded by $N_4^{-1/2} N_2^3 N_3^3$. Taking square roots and normalizing gives the bound $N_4^{1/4-s+\alpha} N_5^{1/2-s+\alpha}$, which suffices provided $s > 1/2+\alpha$.

Case β_5. In this case the contribution of $\Delta\zeta M_2$ is bounded by $N_4^{-1} N_5^{-2} N_2^3 N_3^3$, which is smaller than the bound in Case β_4.

Cases β_6 and β_7. In these cases the contribution of $\Delta\zeta M_2$ is bounded by $N_4^{\varepsilon} N_2^3 N_3^3$. After taking square roots and normalizing we get the bound $N_4^{1/2-s+\alpha+\varepsilon} N_5^{1/2-s+\alpha}$, which once again suffices provided $s > 1/2 + \alpha$.

Case (c)(iii). In this case $N_0 \sim N_1 \geq N_4 \geq N_2 \geq N_3 \geq N_5$. Since $\Delta\zeta M_1$ is bounded by

$$\delta^{-2\mu r} N_2^3 N_3^2 \|\chi_C a_{n_1}\|_{\ell^2}^2 \|a_{n_2}\|_{\ell^2}^2 \|a_{n_3}\|_{\ell^2}^2 \|\gamma\|_{\ell^2_\zeta}^2 \|\chi_{\tilde{C}}(n) k_n\|_{\ell^2}^2,$$

after taking square roots and normalizing we have the bound $N_4^{-s+1/2+\alpha}$ just as before. The latter suffices provided $s > 1/2 + \alpha$. For M_2, following the scheme presented above for Case (c)(ii) we now have:

Case β_1. Since the contribution of $\Delta\zeta M_2$ is bounded by $N_3^3 N_2^{5/2}$, after taking square roots and normalizing we obtain the bound $N_4^{7/4-2s+\alpha}$, which suffices provided $s > 7/8 + \alpha/2$.

Cases β_2 and β_3. In these cases the contribution of $\Delta\zeta M_2$ is bounded by $N_3^3 N_2^3$. Taking square roots and normalizing we obtain the bound $N_4^{2-2s+\alpha}$, which suffices provided $s > 1 + \alpha/2$.

Case β_4. In this case the contribution of $\Delta\zeta M_2$ is bounded by $N_4^{-1/2} N_2^3 N_3^3$, which is smaller than the bound in Cases β_2, β_3.

Case β_5. In this case the contribution of $\Delta\zeta M_2$ is bounded by $N_4^{-1} N_5^{-2} N_2^3 N_3^3$, which is smaller than the bound in Case β_4.

Cases β_6 and β_7. In these cases the contribution of $\Delta\zeta M_2$ is bounded by $N_4^{\varepsilon} N_2^3 N_3^3$. After taking square roots and normalizing we get the bound $N_4^{2-2s+\alpha+\varepsilon}$, which once again suffices provided $s > 1 + \alpha/2$.

Cases (c)(iv)–(vi) and (d). In these cases we proceed as in Subsection 8.5.1. Assume $N_0 \sim N_1 \geq N_2 \geq N_4$, Case (d) having similar or better bounds. The estimates of the trilinear expressions will give after normalization

$$N_0^s N_1^{-s} N_2^{-s+1} N_3^{1-s} N_4^{\alpha} N_5^{\alpha}$$

and we assume that $s > 1 + \alpha$.

One also needs to estimate the terms in (8.7). Here we show how to estimate the term involving the random function at frequency N_4 in (8.9). We first observe that in order for this term not to be zero it must be that $N_4 \sim N_0$. Then for v_0^{ω} in (5.1), after normalization we have the bound

$$N_0^s N_1^{-s} N_2^{-s} N_3^{-s} N_4^{-1+\alpha} N_5^{-1+\alpha} \\ \times \|P_{N_0} P_{N_4} v_0^{\omega}\|_{L_t^{\infty} H_x^{1-\alpha}} N_5^{1/4} \|D^{1-\alpha}(P_{N_0} P_{N_5} v_0^{\omega})\|_{L_t^4 L_x^4} \prod_{j=1,2,3} N_j^{1/4} \|P_{N_j} u_j(x,t)\|_{U_\Delta^4 H^s}.$$

The latter together with the Strichartz estimate (4.16) are enough to obtain the desired bound since for $\alpha < 3/4$, we have

$$N_0^s N_0^{-s+1/4-1+\alpha} \sim N_0^{-3/4+\alpha} < 1.$$

8.5.4. The DDRRR case. To estimate the expression in (8.5) we first observe that in terms of bars we need to estimate only the following cases: Case 1: u_1, u_3, u_5 are random, that is, none of the random functions are conjugated, and Case 2: only one of these functions is conjugated; the other cases are obtained by conjugating the whole expression in (8.5). We will remark later on how the estimates change depending on these two cases.

We now assume that the first three functions are random and the last two are deterministic. We also assume that $N_1 \geq N_2 \geq N_3$ and $N_4 \geq N_5$. We then have the following subcases:

- **Case (a):** $N_4 = \max(N_1, N_4)$ and
  - (i) $N_2 \leq N_5 \leq N_4$.
  - (ii) $N_2 \leq N_5 \leq N_1 \leq N_4$.
  - (iii) $N_3 \leq N_5 \leq N_2 \leq N_1 \leq N_4$.
  - (iv) $N_5 \leq N_3 \leq N_2 \leq N_1 \leq N_4$.
- **Case (b):** $N_1 = \max(N_1, N_4)$, $N_2 \geq N_4$ and
  - (i) $N_3 \geq N_4$.
  - (ii) $N_5 \leq N_3 \leq N_4 \leq N_2$.
  - (iii) $N_3 \leq N_5 \leq N_4 \leq N_2$.
- **Case (c):** $N_1 = \max(N_1, N_4)$, $N_4 \geq N_2$ and
  - (i) $N_2 \leq N_5 \leq N_4 \leq N_1$.
  - (ii) $N_3 \leq N_5 \leq N_2 \leq N_4 \leq N_1$.
  - (iii) $N_5 \leq N_3 \leq N_2 \leq N_4 \leq N_1$.

Case (a)(i). In this case we proceed as in Subsection 8.5.1. Assume for simplicity that $N_0 \sim N_4$; the other cases are smoother. The estimates of the trilinear expressions will give after normalization

$$N_0^s N_4^{-s} N_5^{-s+1} N_3^{\alpha} N_2^{\alpha} N_1^{\alpha},$$

and we assume that $s > 1 + 3\alpha$. One also needs to estimate the terms in (8.7). Here we show how to estimate the factor involving the random term at frequency N_1 in (8.9). We have, for v_0^{ω} of (5.1),

$$N_0^s N_1^{-1+\alpha} N_2^{-1+\alpha} N_3^{-1+\alpha} \| P_{N_0} P_{N_3} v_0^{\omega} \|_{L_t^{\infty} H_x^{1-\alpha}} \prod_{j=4,5} N_j^{1/4-s} \| P_{N_j} u_j(x,t) \|_{U_{\Delta}^4 H^s}, \quad (8.32)$$

where we notice that $N_1 \sim N_0$ since otherwise the contribution would be null. This is enough to obtain the desired bound since

$$N_0^s N_0^{-1+\alpha} N_4^{1/4-s} \sim N_4^{-3/4+\alpha}.$$

Also note that this case is not affected by conjugation, hence it is the same in Case 1 and Case 2.

Case (a)(ii). We also assume that $N_4 \sim N_0$, this is the least favorable situation. We proceed by duality and a change of variables $\zeta = m \pm |n_4|^2$ as in the proof of Proposition 7.2 (in particular see (7.15)).

We have to bound

$$\|\gamma\|_{\ell^2_\zeta}^2 \|a_{n_4}\|_{\ell^2}^2 \sum_{(\zeta,n_4)\in\mathbb{Z}\times\mathbb{Z}^3} \Bigg| \sum_{\substack{n=\pm n_1\pm n_2\pm n_3\pm n_4\pm n_5\\ n_i,n_j,n_k\neq n_r,n_p\\ \zeta=\pm|n_1|^2\pm|n_2|^2\pm|n_3|^2\pm|n_5|^2}} \frac{\tilde g_{n_1}(\omega)}{|n_1|^{3/2}} \frac{\tilde g_{n_2}(\omega)}{|n_2|^{3/2}} \frac{\tilde g_{n_3}(\omega)}{|n_3|^{3/2}} k_n \tilde a_{n_5}\Bigg|^2. \tag{8.33}$$

We now consider two cases:

- **Case** A_0**:** n_1, n_2, n_3 are all different from each other.
- **Case** A_1**:** At least two of the frequencies n_1, n_2, n_3 are equal.

Case A_0**.** We define the set

$$S_{(\zeta,n_4,n_1,n_2,n_3)} := \left\{ (n,n_5) : \begin{array}{l} n = \pm n_1 \pm n_2 \pm n_3 \pm n_4 \pm n_5, \\ n_i, n_j, n_k \neq n_r, n_p, \\ \zeta = \pm|n_1|^2 \pm |n_2|^2 \pm |n_3|^2 \pm |n_5|^2 \end{array} \right\}$$

with $|S_{(\zeta,n_4,n_1,n_2,n_3)}| \lesssim N_5^2$ and we write

$$\begin{aligned}(8.33) &\lesssim \|\gamma\|_{\ell^2_\zeta}^2 \|a_{n_4}\|_{\ell^2}^2 \\ &\quad\times \sum_{(\zeta,n_4)\in\mathbb{Z}\times\mathbb{Z}^3} N_1^{-3}N_2^{-3}N_3^{-3} \Bigg| \sum_{n_1,n_2,n_3} \tilde g_{n_1}(\omega)\tilde g_{n_2}(\omega)\tilde g_{n_3}(\omega) \sum_{S_{(\zeta,n_4,n_1,n_2,n_3)}} k_n \tilde a_{n_5}\Bigg|^2.\end{aligned}$$

By using Lemma 3.4 we can continue, for ω outside a set of measure e^{-1/δ^r}, with

$$\begin{aligned}&\lesssim \delta^{-2\mu r} \|\gamma\|_{\ell^2_\zeta}^2 \|a_{n_4}\|_{\ell^2}^2 \\ &\quad\times \sum_{(\zeta,n_4)\in\mathbb{Z}\times\mathbb{Z}^3} N_1^{-3}N_2^{-3}N_3^{-3} \sum_{n_1,n_2,n_3} \sum_{S_{(\zeta,n_4,n_1,n_2,n_3)}} |S_{(\zeta,n_4,n_1,n_2,n_3)}|\,|k_n|^2|a_{n_5}|^2 \\ &\lesssim \delta^{-2\mu r} \|\gamma\|_{\ell^2_\zeta}^2 \|a_{n_4}\|_{\ell^2}^2 N_1^{-3}N_2^{-3}N_3^{-3}N_5^2 \sum_{n,n_5} |k_n|^2|a_{n_5}|^2|S_{(n_5,n)}|\end{aligned}$$

where

$$S_{(n,n_5)} := \left\{ (\zeta,n_4,n_1,n_2,n_3) : \begin{array}{l} n = \pm n_1 \pm n_2 \pm n_3 \pm n_4 \pm n_5, \\ n_i, n_j, n_k \neq n_r, n_p, \\ \zeta = \pm|n_1|^2 \pm |n_2|^2 \pm |n_3|^2 \pm |n_5|^2 \end{array} \right\}$$

and $|S_{(n,n_5)}| \lesssim N_1^3N_2^3N_3^3$, where we have used $\Delta\zeta \lesssim N_1^2$. Hence we can continue with

$$\lesssim \delta^{-2\mu r} \|\gamma\|_{\ell^2_\zeta}^2 \|a_{n_4}\|_{\ell^2}^2 N_5^2 \|k_n\|_{\ell^2}^2 \|a_{n_5}\|_{\ell^2}^2$$

and after taking square roots and normalizing we obtain the bound

$$N_5^{1-s}N_1^{-1+\alpha}N_2^{-1+\alpha}N_3^{-1+\alpha}.$$

We note that this case is the same in Case 1 and Case 2.

Case A_1. We first assume that only two frequencies are equal. The important remark is that we have removed the frequencies that would give rise to $|g_n(\omega)|^2$ so in (8.33) we would see either $(\tilde{g}_{n_1})^2(\omega)\tilde{g}_{n_3}(\omega)$ or $\tilde{g}_{n_1}(\omega)(\tilde{g}_{n_2})^2(\omega)$. In both cases we can still use Lemma 3.4 and proceed as above to obtain in fact better estimates, since the cardinalities of the sets involved are smaller due to the collapse of the frequencies that are equal.

If all three frequencies are equal, and this can happen only in Case 2, then $N_1 \sim N_2 \sim N_3$ and

$$(8.33) \lesssim \|\gamma\|^2_{\ell^2_\zeta} \|a_{n_4}\|^2_{\ell^2} \sum_{(\zeta,n_4)\in\mathbb{Z}\times\mathbb{Z}^3} N_1^{-12} \sum_{n_3} |g_{n_3}(\omega)|^3 \Big| \sum_{S_{(\zeta,n_4,n_3)}} k_n \tilde{a}_{n_5}\Big|^2$$

where

$$S_{(\zeta,n_4,n_3)} := \{(n,n_5) : n = \pm 3n_3 \pm n_4 \pm n_5,\ \zeta = \pm 3|n_3|^2 \pm |n_5|^2\}.$$

Then by using Lemma 3.4 we can continue, for ω outside a set of measure e^{-1/δ^r}, with

$$\lesssim \|\gamma\|^2_{\ell^2_\zeta} \|a_{n_4}\|^2_{\ell^2} \sum_{(\zeta,n_4)\in\mathbb{Z}\times\mathbb{Z}^3} N_1^{-12+\varepsilon} \sum_{S_{(\zeta,n_4)}} |k_n|^2 |a_{n_5}|^2 |S_{(\zeta,n_4)}|$$

where

$$S_{(\zeta,n_4)} := \{(n,n_3,n_5) : n = \pm 3n_3 \pm n_4 \pm n_5,\ \zeta = \pm 3|n_3|^2 \pm |n_5|^2\}$$

with $|S_{(\zeta,n_4)}| \lesssim N_5^2 N_3$, and we continue with

$$\lesssim \|\gamma\|^2_{\ell^2_\zeta} \|a_{n_4}\|^2_{\ell^2} N_1^{-12+\varepsilon} N_5^2 N_3 \sum_{n,n_5} |k_n|^2 |a_{n_5}|^2 |S_{(n,n_5)}|$$

where

$$S_{(n,n_5)} := \{(\zeta,n,n_3,n_4) : n = \pm 3n_3 \pm n_4 \pm n_5,\ \zeta = \pm 3|n_3|^2 \pm |n_5|^2\}$$

with $|S_{(n,n_5)}| \lesssim N_3^3$. We obtain the bound $N_1^{-6+\varepsilon}$, which clearly suffices without any further restriction when we take square roots and normalize.

We now observe that Cases (a)(iii), (iv) can be analyzed just like Case (a)(i) since N_4 and N_1 are still the top frequencies and the order of the rest is not relevant.

Case (b)(i). We assume first that $N_1 \sim N_0$. We cut the N_0 and N_1 frequency windows with cubes C of sidelength N_2. After using Cauchy–Schwarz we need to estimate

$$\sum_{m\in\mathbb{Z},n\in C} \Big| \sum_{n_1,n_2,n_3;n_1\in C} \tilde{g}_{n_1}(\omega)\tilde{g}_{n_2}(\omega)\tilde{g}_{n_3}(\omega) \Big[\sum_{S_{(m,n,n_1,n_2,n_3)}} \frac{1}{|n_1|^{3/2}} \frac{1}{|n_2|^{3/2}} \frac{1}{|n_3|^{3/2}} \tilde{a}_{n_5}\tilde{a}_{n_4}\Big]\Big|^2 \tag{8.34}$$

where

$$S_{(m,n,n_1,n_2,n_3)} := \left\{ (n_4,n_5) : \begin{array}{l} n = \pm n_1 \pm n_2 \pm n_3 \pm n_4 \pm n_5, \\ n_i,n_j,n_k \neq n_r,n_p,\ m = \pm|n_1|^2 \pm |n_2|^2 \pm |n_3|^2 \pm |n_5|^2 \pm |n_4|^2 \end{array} \right\}$$

with $|S_{(m,n,n_1,n_2,n_3)}| \lesssim N_5^2$. We now consider two cases:

- **Case** A_0**:** n_1, n_2, n_3 are all distinct.
- **Case** A_1**:** At least two of the frequencies n_1, n_2, n_3 are equal.

Case A_0**.** We use Lemma 3.4 and, for ω outside a set of measure e^{-1/δ^r}, we have

$$
\begin{aligned}
(8.34) &\lesssim \delta^{-2\mu r} \sum_{m\in\mathbb{Z}, n\in C} \sum_{n_1,n_2,n_3; n_1\in C} \left[\sum_{S_{(m,n,n_1,n_2,n_3)}} \frac{1}{|n_1|^{3/2}} \frac{1}{|n_2|^{3/2}} \frac{1}{|n_3|^{3/2}} \tilde{a}_{n_5} \tilde{a}_{n_4} \right]^2 \\
&\lesssim \delta^{-2\mu r} N_1^{-3} N_2^{-3} N_3^{-3} \sum_{m\in\mathbb{Z},\, n\in C} \sum_{S_{(m,n)}} \# S_{(m,n,n_1,n_2,n_3)} |a_{n_5}|^2 |a_{n_4}|^2 \\
&\lesssim \delta^{-2\mu r} N_1^{-3} N_2^{-3} N_3^{-3} N_5^2 \sum_{n_4,n_5} |a_{n_5}|^2 |a_{n_4}|^2 |S_{(n_4,n_5)}|
\end{aligned}
$$

where

$$
S_{(n_4,n_5)} := \left\{ (m, n, n_1, n_2, n_3) : \begin{array}{l} n = \pm n_1 \pm n_2 \pm n_3 \pm n_4 \pm n_5, \\ m = \pm |n_1|^2 \pm |n_2|^2 \pm |n_3|^2 \pm |n_5|^2 \pm |n_4|^2 \end{array} \right\}
$$

with $|S_{(n_4,n_5)}| \lesssim N_1 N_2 N_1 N_2^3 N_3^3$, which finally gives

$$
(8.34) \lesssim \delta^{-2\mu r} N_1^{-1} N_2 \|a_{n_5}\|_{\ell^2}^2 \|a_{n_4}\|_{\ell^2}^2.
$$

By taking square roots and normalizing we require that

$$
N_0^s N_1^{-3/2+\alpha} N_2^{-1/2+\alpha} N_3^{-1+\alpha} N_4^{-s} N_5^{-s} \lesssim N_1^{-\beta},
$$

and this follows from assuming $s < 3/2 - \alpha$.

Case A_1**.** We proceed just as in the same case for Case (a)(ii). Here we only work out the details for the case when all frequencies are equal; again this can happen only in Case 2. We have $N_1 \sim N_2 \sim N_3$ and

$$
(8.34) \lesssim \sum_{(m,n)\in\mathbb{Z}\times\mathbb{Z}^3} N_1^{-12} \left| \sum_{n_3} |g_{n_3}(\omega)|^3 \sum_{S_{(m,n,n_3)}} \tilde{a}_{n_4} \tilde{a}_{n_5} \right|^2
$$

where

$$
S_{(m,n,n_3)} := \{(n_4, n_5) : n = \pm 3n_3 \pm n_4 \pm n_5,\ m = \pm 3|n_3|^2 \pm |n_4|^2 \pm |n_5|^2\}.
$$

Then by using Lemma 3.4, for ω outside a set of measure e^{-1/δ^r}, we can continue with

$$
\lesssim \sum_{(m,n)\in\mathbb{Z}\times\mathbb{Z}^3} N_1^{-12+\varepsilon} \sum_{S_{(m,n)}} |a_{n_4}|^2 |a_{n_5}|^2 |S_{(m,n)}|
$$

where

$$
S_{(m,n)} := \{(n_3, n_4, n_5) : n = \pm 3n_3 \pm n_4 \pm n_5,\ m = \pm 3|n_3|^2 \pm |n_4|^2 \pm |n_5|^2\}
$$

with $|S_{(m,n)}| \lesssim N_5^2 N_3$, and we continue with

$$
\lesssim N_1^{-12+\varepsilon} N_5^2 N_3 \sum_{n_4,n_5} |a_{n_4}|^2 |a_{n_5}|^2 |S_{(n_4,n_5)}|
$$

where

$$S_{(n_4,n_5)} := \{(m, n, n_3) : n = \pm 3n_3 \pm n_4 \pm n_5,\ m = \pm 3|n_3|^2 \pm |n_4|^2 \pm |n_5|^2\}$$

with $|S_{(n_4,n_5)}| \lesssim N_1^2 N_3$. We obtain the bound $N_1^{-6+\varepsilon}$, which clearly suffices without any further restriction when we take square roots and normalize.

Now assume that $N_1 \sim N_2$. Here we do not need to cut with cubes C, but the argument and the estimates are similar to the ones we have just analyzed.

Cases (b)(ii), (iii). These cases are estimated just like the case we have just analyzed since the two highest frequencies are still N_1 and N_2 and the order of the others is not relevant.

Case (c)(i). Assume first $N_0 \sim N_1$. This case is handled like Case (b)(i) above. Here we cut with cubes C of sidelength N_4. This gives in particular $\Delta m \lesssim N_1 N_4$.

Case A_0. Just as in Case (b)(i) we have, for ω outside a set of measure e^{-1/δ^r},

$$(8.34) \lesssim \delta^{-2\mu r} N_1^{-3} N_2^{-3} N_3^{-3} N_5^2 \sum_{n_4,n_5} |a_{n_5}|^2 |a_{n_4}|^2 |S_{(n_4,n_5)}|$$

where now $|S_{(n_4,n_5)}| \lesssim N_1 N_4 N_1 N_2^3 N_3^3$ since $\Delta m \lesssim N_1 N_4$. This finally gives

$$(8.34) \lesssim \delta^{-2\mu r} N_1^{-1} N_4 \|a_{n_5}\|_{\ell^2}^2 \|a_{n_4}\|_{\ell^2}^2.$$

By taking square roots and normalizing we require that

$$N_0^s N_1^{-3/2+\alpha} N_2^{-1+\alpha} N_3^{-1+\alpha} N_4^{-s+1/2} N_5^{-s} \lesssim N_1^{-\beta},$$

and this follows from assuming again $s < 3/2 - \alpha$.

Case A_1: This is like the same case for Case (b)(i).

Case (c)(i). Now assume $N_4 \sim N_1$. Here we do not need to cut, and the same estimates as before hold.

Cases (c)(ii), (iii). These cases are estimated just like the case we have just analyzed since the two highest frequencies are still N_1 and N_4 and the order of the others is not relevant.

8.5.5. The DRRRR case. To estimate the expression in (8.5) we assume without any loss of generality that u_5 is the deterministic function and it is not conjugated. By Cauchy–Schwarz and Proposition 4.1 we are reduced to estimating

$$\sum_{m\in\mathbb{Z},\, n\in C} \Bigg| \sum_{\substack{n_1,n_2,n_3,n_4 \\ n_1,n_3\neq n_2,n_4}} \Bigg[\sum_{\substack{n_5=-n_1+n_2-n_3+n_4-n \\ n_5\neq n_2,n_4 \\ m=|n_1|^2-|n_2|^2+|n_3|^2-|n_4|^2+|n_5|^2}} a_{n_5} \Bigg] \frac{g_{n_1}(\omega)}{|n_1|^{3/2}} \frac{\overline{g}_{n_2}(\omega)}{|n_2|^{3/2}} \frac{g_{n_3}(\omega)}{|n_3|^{3/2}} \frac{\overline{g}_{n_4}(\omega)}{|n_4|^{3/2}} \Bigg|^2 \tag{8.35}$$

where we have assumed that $\widehat{u}_5(n_5, t) = e^{it|n_5|^2} a_{n_5}$ and C is a cube of sidelength to be determined later.

Since we have removed the frequencies $n_1, n_3 = n_2$ or $n_1, n_3 = n_4$, which would give rise to terms of the form $|g_i(\omega)|^2$, we can invoke Lemma 3.4 and proceed by further considering the following subcases, for $i, j \in \{1, 2, 3, 4\}$:

- **Case (a):** There exists j such that $N_0 \sim N_j$, $N_5 \lesssim N_j$.
- **Case (b):** There exist $j \neq i$ such that $N_i \sim N_j$ and $N_5, N_0 \lesssim N_i$.
- **Case (c):** $N_0 \sim N_5$ and $N_j \lesssim N_5$.
- **Case (d):** There exist $j \neq i$ such that $N_5 \sim N_j$ and $N_0, N_i \lesssim N_j$.

Case (a). Assume $N_k, k \in \{1,2,3,4,5\}$, $k \neq j$, is the second largest frequency. Then let C be of sidelength N_k and let

$$S_{(n,m,n_1,n_2,n_3,n_4)} := \left\{ n_5 : \begin{array}{l} n_5 = -n_1 + n_2 - n_3 + n_4 - n, \\ n_5 \neq n_2, n_4,\ n_j \in C, \\ m = |n_1|^2 - |n_2|^2 + |n_3|^2 - |n_4|^2 + |n_5|^2 \end{array} \right\}.$$

By Lemma 3.4, for ω outside a set of measure e^{-1/δ^r}, we have

$$\begin{aligned}(8.35) &\lesssim \delta^{-2\mu r} \sum_{m\in\mathbb{Z},\, n\in C} N_1^{-3}N_2^{-3}N_3^{-3}N_4^{-3} \sum_{n_1,n_2,n_3,n_4} \Big| \sum_{S_{(n,m,n_1,n_2,n_3,n_4)}} a_{n_5} \Big|^2 \\ &\lesssim \delta^{-2\mu r} \sum_{m\in\mathbb{Z},\, n\in C} N_1^{-3}N_2^{-3}N_3^{-3}N_4^{-3} \sum_{S_{(n,m)}} |a_{n_5}|^2\end{aligned}$$

where

$$S_{(n,m)} := \left\{ (n_1,n_2,n_3,n_4,n_5) : \begin{array}{l} n = n_1 - n_2 + n_3 - n_4 + n_5,\ n_j \in C, \\ m = |n_1|^2 - |n_2|^2 + |n_3|^2 - |n_4|^2 + |n_5|^2 \end{array} \right\}.$$

We now define the set

$$S_{(n_5)} := \left\{ (m,n,n_1,n_2,n_3,n_4) : \begin{array}{l} n = n_1 - n_2 + n_3 - n_4 + n_5,\ n_j \in C, \\ m = |n_1|^2 - |n_2|^2 + |n_3|^2 - |n_4|^2 + |n_5|^2 \end{array} \right\}$$

with $|S_{(n_5)}| \lesssim N_j^2 N_k^4 N_p^3 N_q^3$. Then we continue with

$$(8.35) \lesssim \delta^{-2\mu r} N_1^{-3}N_2^{-3}N_3^{-3}N_4^{-3} \sum_{n_5} |a_{n_5}|^2 |S_{(n_5)}| \lesssim \delta^{-2\mu r} N_j^{-1} N_k \|a_{n_5}\|_{\ell^2}^2,$$

By taking square roots and normalizing we obtain the bound $N_j^{s+\alpha-3/2} N_k^{-1/2+\alpha}$, which entails $s+\alpha < 3/2$ and $\alpha < 1/2$.

Case (b). We go back to (8.35) and we let C be of its natural sidelength N_0. We then repeat the argument above with the role of N_k played by N_j and we count the set

$$S_{(n_5)} := \left\{ (m,n,n_1,n_2,n_3,n_4) : \begin{array}{l} n = n_1 - n_2 + n_3 - n_4 + n_5, \\ m = |n_1|^2 - |n_2|^2 + |n_3|^2 - |n_4|^2 + |n_5|^2 \end{array} \right\}$$

obtaining $|S_{(n_5)}| \lesssim N_j^3 N_i^3 N_p^3 N_q^3$. By taking square roots and normalizing we obtain the bound $N_j^{s+2\alpha-2}$, which entails $s+2\alpha < 2$.

Case (c). We proceed as in Case (b) of Subsection 8.5.2; more precisely we use duality and the change of variables $\zeta = m - |n_5|^2$ as in the proof of Proposition 7.2 (in particular see (7.15)). Here we let C be of its natural sidelength N_0. Let also N_k, $k \in \{1, \dots, 4\}$, be the second largest frequency. We have to bound

$$\|\gamma\|_{\ell^2_\zeta}^2 \|\chi_C a_{n_5}\|_{\ell^2}^2 \times \sum_{(\zeta,n_5)\in\mathbb{Z}\times\mathbb{Z}^3} \Bigg| \sum_{\substack{n=n_5-n_2+n_3-n_4+n_1 \\ n_1,n_3,n_5\neq n_2,n_4 \\ \zeta=|n_1|^2-|n_2|^2+|n_3|^2-|n_4|^2}} \chi_C(n) k_n \frac{g_{n_1}(\omega)}{|n_1|^{3/2}} \frac{\overline{g}_{n_2}(\omega)}{|n_2|^{3/2}} \frac{g_{n_3}(\omega)}{|n_3|^{3/2}} \frac{\overline{g}_{n_4}(\omega)}{|n_4|^{3/2}} \Bigg|^2. \tag{8.36}$$

We proceed again as above where now we have to replace $S_{(n_5)}$ by

$$S_{(n)} := \left\{ (\zeta, n_1, n_2, n_3, n_4, n_5) : \begin{array}{l} n = n_1 - n_2 + n_3 - n_4 + n_5, \\ \zeta = |n_1|^2 - |n_2|^2 + |n_3|^2 - |n_4|^2 \end{array} \right\}$$

with $|S_{(n)}| \lesssim N_k^3 N_i^3 N_p^3 N_q^3$, where we have used $\Delta\zeta \lesssim N_k^2$. By taking square roots and normalizing we obtain the bound $N_k^{\alpha-1}$.

Case (d). This case is analogous to Case (c).

8.5.6. The all random $R\overline{R}R\overline{R}R$ case. Since we have removed the frequencies with $n_1, n_3 = n_2$ or $n_1, n_3 = n_4$, which would give rise to terms of the form $|g_i(\omega)|^2$, we can invoke Lemma 3.4 and proceed to estimate the expression in (8.5) by further considering the following two subcases,

- **Case (a):** $N_0 \sim N_i$ for some $i = 1, \dots, 5$.
- **Case (b):** $N_i \sim N_j$ for $i, j \neq 0$.

Case (a). Let N_j be the second largest frequency size after N_i. We cut the N_0 window with cubes C of sidelength N_j. By Cauchy–Schwarz and Plancherel we estimate

$$\sum_{m\in\mathbb{Z},n\in\mathbb{Z}^3} \Bigg| \sum_{\substack{n=n_1-n_2+n_3-n_4+n_5 \\ n_1,n_3,n_5\neq n_2,n_4 \\ m=|n_1|^2-|n_2|^2+|n_3|^2-|n_4|^2+|n_5|^2}} \frac{g_{n_1}(\omega)}{|n_1|^{3/2}} \frac{\overline{g}_{n_2}(\omega)}{|n_2|^{3/2}} \frac{g_{n_3}(\omega)}{|n_3|^{3/2}} \frac{\overline{g}_{n_4}(\omega)}{|n_4|^{3/2}} \frac{g_{n_5}(\omega)}{|n_5|^{3/2}} \Bigg|^2. \tag{8.37}$$

By Lemma 3.4 we have, for ω outside a set of measure e^{-1/δ^r},

$$\begin{aligned} (8.37) &\lesssim \delta^{-2\mu r} \sum_{m\in\mathbb{Z},n\in\mathbb{Z}^3} \sum_{\substack{n=n_1-n_2+n_3-n_4+n_5 \\ n_1,n_3,n_5\neq n_2,n_4,n_i\in C \\ m=|n_1|^2-|n_2|^2+|n_3|^2-|n_4|^2+|n_5|^2}} \frac{1}{|n_1|^3} \frac{1}{|n_2|^3} \frac{1}{|n_3|^3} \frac{1}{|n_4|^3} \frac{1}{|n_5|^3} \\ &\lesssim \delta^{-2\mu r} |S| \prod_{k=1}^{5} N_k^{-3} \lesssim N_i^{-1} N_j \end{aligned}$$

where

$$S := \left\{ (m, n, n_1, \dots, n_5) : \begin{array}{l} n = n_1 - n_2 + n_3 - n_4 + n_5,\ n_i \in C, \\ m = |n_1|^2 - |n_2|^2 + |n_3|^2 - |n_4|^2 + |n_5|^2 \end{array} \right\}$$

with $|S| \lesssim N_i^2 N_j^4 \prod_{k \neq i,j,0} N_k^3$. Taking square roots and normalizing we obtain the bound

$$N_i^{s+\alpha-3/2} N_j^{-1/2+\alpha} \prod_{k \neq i,j,0} N_k^{-1+\alpha},$$

which suffices provided $s + \alpha < 3/2$ and $\alpha < 1/2$.

Case (b). This is like Case (a), but now we do not need to cut the support of the N_0 window with N_j.

8.5.7. The $U^4_\Delta L^2$ estimates. Assume that N_i are dyadic numbers and without loss of generality $N_1 \geq \cdots \geq N_5$. We start by rewriting

$$\begin{aligned} \int_0^{2\pi} \int_{\mathbb{T}^3} D^s(\mathcal{N}(P_{N_i}(w + v_0^\omega)))\overline{P_{N_0} h}\, dx\, dt &= \int_0^{2\pi} \int_{\mathbb{T}^3} D^s(\mathcal{N}(P_{N_i} w))\overline{P_{N_0} h}\, dx\, dt \\ &+ \int_0^{2\pi} \int_{\mathbb{T}^3} D^s(\mathcal{N}(P_{N_i} v_0^\omega))\overline{P_{N_0} h}\, dx\, dt \\ &+ \int_0^{2\pi} \int_{\mathbb{T}^3} D^s(\mathcal{N}(P_{N_i} w, P_{N_i} v_0^\omega))\overline{P_{N_0} h}\, dx\, dt, \\ &= \mathcal{T}_1 + \mathcal{T}_2 + \mathcal{T}_3 \end{aligned}$$

where in $\mathcal{T}_3$ we include all the nonlinear expressions with both random and deterministic terms. Our goal is to obtain an estimate for the first and last term with the $U^4_\Delta L^2$ norms of w on the right hand side possibly paying the prize of N_2^γ, with $\gamma > 0$. Then using the interpolation Proposition 4.2, we combine this estimate with the ones involving the $U^2_\Delta L^2$ norms of previous sections and the embeddings (4.5) and (4.7) to finally conclude the proof of Proposition 5.1.

Clearly we do not need to estimate $\mathcal{T}_2$ that involves purely random terms. For the other two we have

$$\mathcal{T}_1 + \mathcal{T}_3 \lesssim [\|\mathcal{N}(P_{N_i} w)\|_{L^4_t L^2_x} + \|\mathcal{N}(P_{N_i} w, P_{N_i} v_0^\omega)\|_{L^4_t L^2_x}] \|P_{N_0} h\|_{L^\infty_t L^2_x},$$

and from (2.9), with a certain abuse of notation,

$$\begin{aligned} &\|\mathcal{N}(P_{N_i} w)\|_{L^4_t L^2_x} + \|\mathcal{N}(P_{N_i} w, P_{N_i} v_0^\omega)\|_{L^4_t L^2_x} \\ &\lesssim \sum_{i=1}^{9} \|\mathcal{F}^{-1} J_i(w)\|_{L^4_t L^2_x} + \sum_{j=1}^{9} \|\mathcal{F}^{-1} J_i(P_{N_i} w, P_{N_i} v_0^\omega)\|_{L^4_t L^2_x} = \sum_{i=1}^{9} (\mathcal{S}_1^i + \mathcal{S}_2^i) \end{aligned}$$

where $J_i(w, v_0^\omega)$ indicates that the functions involved could be both w and v_0^ω. To estimate $\mathcal{S}_1^i$ and $\mathcal{S}_2^i$ we use the transfer principle of Proposition 4.1 and we assume that $\widehat{w}(t, n) = e^{it|n|^2} b_n(t)$. Below we write $a^j_{n_j}$ to indicate b_n or the Fourier coefficients of v_0^ω or their conjugates. Now define

$$\Phi_i(n,t) := \Big| \sum_{\substack{n=\sum_{i=1}^5 \pm n_i, n_i\sim N_i \\ W_i(n_1,n_2,n_3,n_4,n_5)}} a^1_{n_1} e^{\pm it|n_1|^2} a^2_{n_2} e^{\pm it|n_2|^2} a^3_{n_3} e^{\pm it|n_3|^2} a^4_{n_4} e^{\pm it|n_4|^2} a^5_{n_5} e^{\pm it|n_5|^2} \Big|^2$$

where $W_i(n_1, n_2, n_3, n_4, n_5)$ indicates the constraints among the five frequencies in J_i. Then for $i = 1, \ldots, 9$ and $k = 1, 2$,

$$(\mathcal{S}^i_k)^2 \lesssim \sup_{t\in[0,2\pi]} \sum_n \Phi_i(n,t) \lesssim \sup_{t\in[0,2\pi]} \sum_n \Big| \sum_{S_{(n)}} |a^1_{n_1}|\,|a^2_{n_2}|\,|a^3_{n_3}|\,|a^4_{n_4}|\,|a^5_{n_5}| \Big|^2$$

where

$$S_{(n)} = \Big\{ (n_1, n_2, n_3, n_4, n_5) : n = \sum_{j=1}^5 \pm n_j,\ n_j \sim N_j \Big\}.$$

Assume now that N_1, the highest frequency, is such that $N_1 \sim N_0$, which is in fact the least favorable situation. Then $|S_{(n)}| \lesssim N_2^3 N_3^3 N_4^3 N_5^3$ and by Cauchy–Schwarz

$$\Big| \sum_{S_{(n)}} |a^1_{n_1}|\,|a^2_{n_2}|\,|a^3_{n_3}|\,|a^4_{n_4}|\,|a^5_{n_5}| \Big|^2 \lesssim N_2^3 N_3^3 N_4^3 N_5^3 \|a^1_{n_1}\|^2_{\ell^2} \prod_{j=2}^5 \|a^j_{n_j}\|^2_{\ell^2}. \tag{8.38}$$

We then have

$$\mathcal{S}^i_1 \lesssim N_2^{6-4s} \|P_{N_1} w\|_{U^4_\Delta H^s} \prod_{j=2}^5 \|P_{N_j} w\|_{U^4_\Delta H^s}. \tag{8.39}$$

We observe that a similar estimate holds for $\mathcal{S}^i_2$ when the function associated to frequency N_1 is also deterministic. In fact, in this case we have

$$\mathcal{S}^i_2 \lesssim N_2^{2+4\alpha} \|P_{N_1} w\|_{U^4_\Delta H^s} \prod_{j\notin J, j\neq 1} \|P_{N_j} w\|_{U^4_\Delta H^s}. \tag{8.40}$$

Finally, if the function associated to frequency N_1 is random, then

$$\mathcal{S}^i_2 \lesssim N_1^{s-1+\alpha} N_2^{2+4\alpha} \prod_{j\notin J} \|P_{N_j} w\|_{U^4_\Delta H^s}. \tag{8.41}$$

We conclude by using the interpolation Proposition 4.2. Note here that in both (8.39) and (8.40) the interpolation at most introduces a factor of N_2^ε which can be easily absorbed by the negative power of N_2 in the estimates involving norms $U^2_\Delta L^2$ (see previous subsections). On the other hand, (8.41) and interpolation introduce a factor of N_1^ε. But this too can be absorbed thanks to the presence of a negative power of N_1 in the estimates involving $U^2_\Delta L^2$ norms in those cases in which the highest frequency is associated to a random function.

This concludes the proof of Proposition 5.1.

9. Proof of Proposition 5.2

We first give an improved version of Proposition 5.1: if $r > 0$ is small enough then there exists $\theta > 0$ such that for $\omega \in \Omega_\delta$ we have: if $N_1 \gg N_0$ or $P_{N_1} w = P_{N_1} v_0^\omega$ then

$$\left| \int_0^{2\pi} \int_{\mathbb{T}^3} D^s \big(\psi_\delta(t) \mathcal{N}(P_{N_i}(w + v_0^\omega))\big) \overline{P_{N_0} h} \, dx \, dt \right| \lesssim \delta^\theta N_1^{-\varepsilon} \| P_{N_0} h \|_{Y^{-s}} \Big(1 + \prod_{i \notin J} \| \psi_\delta P_{N_i} w \|_{X^s} \Big), \tag{9.1}$$

and if $N_1 \sim N_0$ and $P_{N_1} w \neq P_{N_1} v_0^\omega$ then

$$\left| \int_0^{2\pi} \int_{\mathbb{T}^3} D^s \big(\psi_\delta(t) \mathcal{N}(P_{N_i}(w + v_0^\omega))\big) \overline{P_{N_0} h} \, dx \, dt \right| \lesssim \delta^\theta N_2^{-\varepsilon} \| P_{N_0} h \|_{Y^{-s}} \| \psi_\delta P_{N_1} w \|_{X^s} \Big(1 + \prod_{i \notin J, i \neq 1} \| \psi_\delta P_{N_i} w \|_{X^s} \Big), \tag{9.2}$$

for some small $\varepsilon > 0$.

To prove (9.1) and (9.2) we first observe that in the proof of Proposition 5.1, in particular the estimates involving the terms $J_2, \dots, J_7$ in (5.3), we always have the factor $\| P_{N_0} h \|_{L_t^2 L_x^2}$ on the right hand side. We can then replace this by

$$\| \psi_\delta(t) P_{N_0} h \|_{L_t^2 L_x^2} \lesssim \delta^{1/2} \| \psi_\delta(t) P_{N_0} h \|_{L_t^\infty L_x^2} \lesssim \delta^{1/2} \| \psi_\delta(t) P_{N_0} h \|_{Y^0}, \tag{9.3}$$

where we have used (4.7), and obtain the proof of Proposition 5.2 for the nonlinear terms involving $J_2, \dots, J_7$.

To estimate the term involving J_1 we go back to Subsections 8.5.1–8.5.6. We recall that except when the top frequencies, say N_1 and N_2, are associated to two deterministic functions, also in this case we have $\| P_{N_0} h \|_{L_t^2 L_x^2}$ on the right hand side, and (9.3) can be used again.

We are then reduced to estimating the term involving J_1 where the top frequencies N_1 and N_2 are associated to two deterministic functions. So we consider

$$\left| \int_0^{2\pi} \int_{\mathbb{T}^3} \mathcal{F}^{-1} J_1(\psi_\delta(t) P_{N_i}(u_i)) \psi_\delta(t) \overline{P_{N_0} h} \, dx \, dt \right| \tag{9.4}$$

where without loss of generality $N_1 \geq \cdots \geq N_5$ and u_1 and u_2 are deterministic functions, while u_{N_i}, $i = 3, 4, 5$, represents either w or v_0^ω. We consider two cases, for $\sigma > 0$ to be determined later:

- **Case 1:** $\delta^{-\sigma} > N_2$.
- **Case 2:** $\delta^{-\sigma} \leq N_2$.

Case 1. We observe that the estimate of (9.4) can be reduced to analyzing an expression such as

$$\left| \int_0^{\delta} \int_{\mathbb{T}^3} \tilde{u}_{N_1} \tilde{u}_{N_2} \tilde{u}_{N_3} \tilde{u}_{N_4} \tilde{u}_{N_5} \tilde{h}_{N_0} \, dx \, dt \right| \tag{9.5}$$

where u_{N_i} are as above. In fact to obtain the full product as in (9.5) one needs to put back some frequencies, and hence some terms (see for example (8.7) in Subsection 8.5.1). But these terms are similar to those involved in $J_2, \ldots, J_7$ and again the gain on δ is guaranteed by (9.3).

We then go back to (9.5) and we further assume that $N_1 \sim N_0$, which is the least favorable situation. We cut the N_0, and hence N_1, frequency window with cubes C of sidelength N_2 and we obtain the bound

$$\left|\int_0^\delta \int_{\mathbb{T}^3} \tilde{u}_{N_1}\tilde{u}_{N_2}\tilde{u}_{N_3}\tilde{u}_{N_4}\tilde{u}_{N_5}\tilde{h}_{N_0}\,dx\,dt\right|^2 \lesssim \delta \sum_C \|P_C\tilde{u}_{N_1}\|^2_{L^{12}_t L^{12}_x}\|P_C\tilde{h}_{N_0}\|^2_{L^{12}_t L^{12}_x} \times \|\tilde{u}_{N_2}\|^2_{L^{12}_t L^{12}_x}\|\tilde{u}_{N_3}\|^2_{L^{12}_t L^{12}_x}\|\tilde{u}_{N_4}\|^2_{L^{12}_t L^{12}_x}\|\tilde{u}_{N_5}\|^2_{L^{12}_t L^{12}_x},$$

and from (4.16)–(4.18) we can continue with

$$\lesssim \delta N_2^{m(\alpha,s)} \sum_C \|P_C u_{N_1}\|^2_{U^{12}_\Delta L^2}\|P_C h_{N_0}\|^2_{U^{12}_\Delta L^2} \prod_{i\notin J, i\neq 1} \|u_{N_i}\|^2_{U^{12}_\Delta L^2} \tag{9.6}$$

where $J \subset \{2,3,4,5\}$ is the set of indices corresponding to random linear solutions.

Then normalizing, interpolating through Proposition 4.2, and using the embedding (4.7) combined with (4.6), we have

$$\begin{aligned}&\left|\int_0^\delta \int_{\mathbb{T}^3} \tilde{u}_{N_1}\tilde{u}_{N_2}\tilde{u}_{N_3}\tilde{u}_{N_4}\tilde{u}_{N_5}\tilde{h}_{N_0}\,dx\,dt\right| \\ &\quad\lesssim \delta^{1/2} N_2^{m(\alpha,s)} \|P_{N_0}h\|_{Y^{-s}}\|\psi_\delta P_{N_1}w\|_{X^s}\Big(1+\prod_{i\notin J,i\neq 1}\|\psi_\delta P_{N_i}w\|_{X^s}\Big) \\ &\quad\lesssim \delta^{1/3}\|P_{N_0}h\|_{Y^{-s}}\|\psi_\delta P_{N_1}w\|_{X^s} N_2^{-\varepsilon}\Big(1+\prod_{i\notin J,i\neq 1}\|\psi_\delta P_{N_i}w\|_{X^s}\Big)\end{aligned}$$

if we take $\sigma < 1/(100m(\alpha,s))$.

Case 2. Here we go back to (5.4) and (5.5). We recall that $P_{N_1}u_1$ is deterministic and again we assume that $N_1 \sim N_0$; the other cases can be treated similarly. Then we use (5.4) and we have

$$\begin{aligned}&\left|\int_0^{2\pi} \int_{\mathbb{T}^3} \mathcal{F}^{-1}J_1(\psi_\delta(t)P_{N_i}(u_i))\psi_\delta(t)\overline{P_{N_0}h}\,dx\,dt\right| \\ &\quad\lesssim \delta^\gamma \delta^{-\gamma-\mu r} N_2^{-\rho(\alpha,s)}\|P_{N_0}h\|_{Y^{-s}}\|\psi_\delta P_{N_1}w\|_{X^s}\prod_{i\notin J,i\neq 1}\|\psi_\delta P_{N_i}w\|_{X^s} \\ &\quad\lesssim \delta^\gamma N_2^{-\varepsilon}\|P_{N_0}h\|_{Y^{-s}}\|\psi_\delta P_{N_1}w\|_{X^s}\Big(1+\prod_{i\notin J,\, i\neq 1}\|\psi_\delta P_{N_i}w\|_{X^s}\Big)\end{aligned}$$

provided $\sigma \geq (\gamma+\mu r)/(\rho(\alpha,s))$, which is satisfied for γ, r small enough.

To finish the proof we now need to sum the dyadic blocks just as in [23]. In (9.1) we have enough decay in the highest frequency N_1 that we can use Cauchy–Schwarz in all the smaller frequency terms and just pay with an $N_1^{-\varepsilon/2}$. In (9.2) instead we use Cauchy–Schwarz for the lower frequencies N_5, N_4, N_3 and pay with an $N_2^{-\varepsilon/2}$ that can be absorbed and use Cauchy–Schwarz on $N_0 \sim N_1$. □

10. Long time large data infinite energy solutions

In this section we show that by inspecting the proof of the local result above we are able to prove the following long time and large data result.

Theorem 10.1. *For fixed $T > 0$ there exists a set Σ_T with $\mathbb{P}(\Sigma_T) > 0$ such that for $\alpha > 0$ small and every $\omega \in \Sigma_T$,*

$$\phi^\omega(x) := \sum_{n\in\mathbb{Z}^3} \frac{g_n(\omega)}{\langle n\rangle^{5/2-\alpha}} e^{in\cdot x} \in H^\gamma(\mathbb{T}^3), \qquad \gamma = \gamma(\alpha) < 1,$$

evolves up to time T into a solution $u(t)$ of the initial value problem (1.1). Moreover $u(t) - e^{it\Delta}\phi^\omega \in X^s([0,T))_d$, $s = s(\alpha) > 1$, as in Theorem 1.3.

This theorem in particular shows that we find a nontrivial set of large data which gives rise to long time solutions below the critical space $H^1(\mathbb{T}^3)$, that is, in the supercritical scaling regime.

Proof. We follow the same steps as in the proof of Theorem 1.3. In these steps, whenever we apply Proposition 3.1 we replace δ^{-r} by ρ^β, for $\beta > 0$. This will determine a set of ω that we call Σ_ρ such that $\mathbb{P}(\Sigma_\rho^c) < e^{-\rho^\beta}$. Therefore the bounds on the $g_n(\omega)$ from Lemma 3.4 hold on Σ_ρ.

We consider the initial value problem (2.17) and set up the fixed point argument for the difference equation (5.2). We repeat the estimates leading to the proof of Proposition 5.2 and obtain a set Σ_ρ such that for $\omega \in \Sigma_\rho$,

$$\|\mathcal{I}(\mathcal{N}(w + v_0^\omega))\|_{X^s([0,2\pi])} \lesssim C(\rho + \|w\|_{X^s([0,2\pi])})^5 \tag{10.1}$$

where $\mathcal{I}$ denotes the Duhamel operator as in (4.13) and $\mathcal{N}(\cdot)$ was defined in (2.18).

We want to prove that we can find $\rho = \rho(T)$ small enough such that for any $\omega \in \Sigma_{\rho(T)}$ we can iterate the argument up to time T. To this end we perform a continuity argument to obtain a uniform bound for $w(t)$ in H^s for all $t \in [0,T]$.

We have

$$\|w\|_{X^s([0,2\pi])} \le C(\rho + \|w\|_{X^s([0,2\pi])})^5,$$

and also since the estimates are subcritical we have, for $\delta \ll 1$,

$$\|w\|_{X^s([0,\delta])} \le C\delta^\gamma(\rho + \|w\|_{X^s([0,\delta])})^5. \tag{10.2}$$

We now study the function

$$f_1(x) = C(\rho + x)^5 - x$$

where $x = x(t) = \|w\|_{X^s([0,t])}$. We easily find that $f_1(0) = C\rho^5$, the value $x_0 = \left(\frac{1}{5C}\right)^{1/4}\rho$ is a minimum point, and for $x_1 = 2C\rho^5 < x_0$ we have $f_1(x_1) < 0$ and $x(0) \le x_1$ thanks to (10.2). As a consequence of the fact that $x(t)$ is continuous in time we have

$$\|w\|_{X^s([0,2\pi])} \le 2C\rho^5 \quad \text{and} \quad \|w(t)\|_{H^s} \le 2C\rho^5, \quad t \in [0,2\pi].$$

A similar argument at step m with

$$f_m(x) = 2(m-1)C\rho^5 + C(\rho + x)^5 - x, \qquad x_m = 2Cm\rho^5,$$

gives

$$\|w\|_{X^s([0,m2\pi])} \le 2Cm\rho^5 \quad \text{and} \quad \|w(t)\|_{H^s} \le 2Cm\rho^5, \quad t \in [0, m2\pi],$$

and in order for this process to be continued to step $m+1$ we need to guarantee that

$$\sum_{i=0}^{4} \alpha_i (2mC)^{5-i} \rho^{25-4i} < \frac{\rho^5}{100},$$

where the α_i are the binomial coefficients. At the final step T we then pick $\rho = \rho(T)$ small enough and finish the proof. □

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