J. Eur. Math. Soc. 17, 1593–1627 

DOI 10.4171/JEMS/540

Dimitar Grantcharov · Ji Hye Jung · Seok-Jin Kang
Masaki Kashiwara · Myungho Kim

# Crystal bases for the quantum queer superalgebra

Received June 25, 2011

**Abstract.** We develop the crystal basis theory for the quantum queer superalgebra $U_q(\mathfrak{q}(n))$. We define the notion of crystal bases and prove the tensor product rule for $U_q(\mathfrak{q}(n))$-modules in the category $\mathcal{O}_{\text{int}}^{\geq 0}$. Our main theorem shows that every $U_q(\mathfrak{q}(n))$-module in the category $\mathcal{O}_{\text{int}}^{\geq 0}$ has a unique crystal basis.

**Keywords.** Quantum queer superalgebras, crystal bases, odd Kashiwara operators

## Introduction

For the past 30 years, one of the most striking and influential developments in combinatorial representation theory was the discovery of crystal bases for quantum groups and their representations [10, 11]. Right after that discovery, the crystal basis theory attracted a lot of attention and research activity because it has simple and explicit combinatorial features and has many significant applications to a wide variety of mathematical and physical theories. In particular, crystal bases have an extremely nice behavior with respect to tensor products, which leads to natural and exciting connections with combinatorics of Young tableaux and Young walls [6, 9, 14, 20, 22]. Moreover, inspired by the original works [10–13], many important and deep results have been established for crystal bases for quantum groups associated with symmetrizable Kac–Moody algebras (see, for example, [3, 4, 7, 8, 15–17, 21, 25]). In [18, 19], Lusztig provided a geometric approach to this subject.

D. Grantcharov: Department of Mathematics, University of Texas at Arlington, Arlington, TX 76021, USA; e-mail: grandim@uta.edu

J. H. Jung: Department of Mathematics, Sogang University, Seoul 121-742, Korea; e-mail: jhjung@sogang.ac.kr

S.-J. Kang: Department of Mathematical Sciences and Research Institute of Mathematics, Seoul National University, Seoul 151-747, Korea; e-mail: sjkang@math.snu.ac.kr

M. Kashiwara: Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606-8502, Japan; e-mail: masaki@kurims.kyoto-u.ac.jp

M. Kim: School of Mathematics, Korea Institute for Advanced Study, Seoul 130-722, Korea; e-mail: mhkim@kias.re.kr

*Mathematics Subject Classification (2010):* 17B37, 81R50

On the other hand, not much has been known about crystal bases for quantum groups corresponding to Lie superalgebras. A major difficulty one encounters in the superalgebra case is that the category of finite-dimensional representations is in general not semisimple. Fortunately, there is an interesting and natural category of finite-dimensional $U_q(\mathfrak{g})$-modules which is semisimple for the two super-analogues of the general linear Lie algebra $\mathfrak{gl}(n)$: $\mathfrak{g} = \mathfrak{gl}(m|n)$ and $\mathfrak{g} = \mathfrak{q}(n)$. This is the category $\mathcal{O}_{\text{int}}^{\geq 0}$ of so-called *tensor modules*, i.e., those that appear as submodules of tensor powers $\mathbf{V}^{\otimes N}$ of the natural $U_q(\mathfrak{g})$-module $\mathbf{V}$. The semisimplicity of $\mathcal{O}_{\text{int}}^{\geq 0}$ is verified in [1] for the general linear Lie superalgebra $\mathfrak{g} = \mathfrak{gl}(m|n)$ and in [2] for the queer Lie superalgebra $\mathfrak{g} = \mathfrak{q}(n)$.

Furthermore, the crystal basis theory of $\mathcal{O}_{\text{int}}^{\geq 0}$ for $\mathfrak{g} = \mathfrak{gl}(m|n)$ was developed in [1], while the foundations of the highest weight representation theory of $U_q(\mathfrak{q}(n))$ have been established in [2].

In this paper, we develop the crystal basis theory for $U_q(\mathfrak{q}(n))$-modules in the category $\mathcal{O}_{\text{int}}^{\geq 0}$. The (quantum) queer superalgebra is interesting not only as the remaining case for which $\mathcal{O}_{\text{int}}^{\geq 0}$ is semisimple, but also due to its remarkable combinatorial properties. An example of such properties is the queer analogue of the celebrated *Schur–Weyl duality*, often referred to as *Schur–Weyl–Sergeev duality*, which was obtained in [26] for $U(\mathfrak{q}(n))$ and in [23] for $U_q(\mathfrak{q}(n))$.

Being very interesting on the one hand, the representation theory of the (quantum) queer superalgebra faces numerous challenges on the other. The queer Lie superalgebra is the only classical Lie superalgebra whose Cartan subsuperalgebra has a nontrivial odd part. As a result, the highest weight space of any finite-dimensional $\mathfrak{q}(n)$-module has the structure of a Clifford module and the corresponding $\mathfrak{gl}(n)$-module appears with multiplicity higher than one (in fact, a power of 2). Also, as observed in [2], due to the different classification of Clifford modules over $\mathbb{C}$ and $\mathbb{C}(q)$, the classical limit of an irreducible highest weight $U_q(\mathfrak{q}(n))$-module is an irreducible highest weight $U(\mathfrak{q}(n))$-module or a direct sum of two irreducible highest weight $U(\mathfrak{q}(n))$-modules. On top of these and in contrast to the case of $\mathfrak{gl}(m|n)$, the odd root generators $e_{\bar{i}}$ and $f_{\bar{i}}$ of $U_q(\mathfrak{q}(n))$ are not nilpotent.

We overcome the challenges described above in several steps. First, we set the ground field to be the field $\mathbb{C}((q))$ of formal Laurent power series. By enlarging the base field, we obtain an equivalence of the two categories of Clifford modules, and in particular, establish a standard version of the classical limit theorem. As the next step, we introduce the *odd Kashiwara operators* $\tilde{e}_{\bar{1}}$, $\tilde{f}_{\bar{1}}$, and $\tilde{k}_{\bar{1}}$, where $\tilde{k}_{\bar{1}}$ corresponds to an odd element in the Cartan subsuperalgebra of $\mathfrak{q}(n)$. The definitions of $\tilde{e}_{\bar{1}}$, $\tilde{f}_{\bar{1}}$ are new in the sense that they are based solely on the comultiplication formulas for $e_{\bar{1}}$, $f_{\bar{1}}$ and lead to nilpotent operators on $L/qL$, where $L$ is a crystal lattice. Furthermore, from these definitions, we deduce a special *tensor product rule* for odd Kashiwara operators.

Our definition of a *crystal basis* for a $U_q(\mathfrak{q}(n))$-module $M$ in the category $\mathcal{O}_{\text{int}}^{\geq 0}$ is also new: such a basis is a triple $(L, B, (l_b)_{b\in B})$, where the crystal lattice $L$ is a free $\mathbb{C}[[q]]$-submodule of $M$, $B$ is a finite $\mathfrak{gl}(n)$-crystal, $(l_b)_{b\in B}$ is a family of nonzero vector spaces such that $L/qL = \bigoplus_{b\in B} l_b$, with a set of compatibility conditions for the action of the Kashiwara operators imposed in addition. The definition of crystal bases leads naturally to the notion of *abstract* $\mathfrak{q}(n)$*-crystals*, an example of which is the $\mathfrak{gl}(n)$-crystal $B$ in any crystal basis $(L, B, (l_b)_{b\in B})$. The modified notion of crystals allows us to consider the multiple occurrence of $\mathfrak{gl}(n)$-crystals corresponding to a highest weight $U_q(\mathfrak{q}(n))$-module $M$ in $\mathcal{O}_{\text{int}}^{\geq 0}$ as a single $\mathfrak{q}(n)$-crystal.

As a result of this new setting, the existence and uniqueness theorem for crystal bases is proved for any highest weight (not necessarily irreducible) module $M$ in the category $\mathcal{O}_{\text{int}}^{\geq 0}$. Moreover, the $\mathfrak{q}(n)$-crystal $B$ of $M$ depends only on the highest weight $\lambda$ of $M$ and hence we may write $B = B(\lambda)$. In addition to the existence and uniqueness theorem, the decompositions of the module $\mathbf{V} \otimes M$ and the crystal $\mathbf{B} \otimes B(\lambda)$ are established, where $\mathbf{B}$ is the crystal of $\mathbf{V}$. These decompositions are parametrized by the set of all $\lambda + \varepsilon_j$ such that $\lambda + \varepsilon_j$ is a strict partition $(j = 1, \ldots, n)$. One of key ingredients of the proof of our main theorem is the characterization of highest weight vectors in $\mathbf{B} \otimes B(\lambda)$ in terms of even Kashiwara operators and the highest weight vector of $B(\lambda)$. All these statements are verified simultaneously by a series of interlocking inductive arguments.

This paper is organized as follows. In Section 1, we recall some of the basic properties of $U_q(\mathfrak{q}(n))$-modules in the category $\mathcal{O}_{\text{int}}^{\geq 0}$. Section 2 is devoted to the definitions, examples, and some preparatory statements related to crystal bases. In particular, we prove the tensor product rule. In Section 3, we give algebraic and combinatorial characterizations of highest weight vectors in $\mathbf{B}^{\otimes N}$. In Section 4, we prove our main result: the existence and uniqueness theorem for crystal bases.

## 1. The quantum queer superalgebra

Let $\mathbf{F} = \mathbb{C}((q))$ be the field of formal Laurent series in an indeterminate $q$ and let $\mathbf{A} = \mathbb{C}[[q]]$ be the subring of $\mathbf{F}$ consisting of formal power series in $q$. For $k \in \mathbb{Z}_{\geq 0}$, we define

$$[k] = \frac{q^k - q^{-k}}{q - q^{-1}}, \quad [0]! = 1, \quad [k]! = [k][k-1]\cdots[2][1].$$

For an integer $n \geq 2$, let $P^\vee = \mathbb{Z}k_1 \oplus \cdots \oplus \mathbb{Z}k_n$ be a free abelian group of rank $n$ and let $\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} P^\vee$ be its complexification. Define the linear functionals $\epsilon_i \in \mathfrak{h}^*$ by $\epsilon_i(k_j) = \delta_{ij}$ $(i, j = 1, \ldots, n)$ and set $P = \mathbb{Z}\epsilon_1 \oplus \cdots \oplus \mathbb{Z}\epsilon_n$. We denote by $\alpha_i = \epsilon_i - \epsilon_{i+1}$ the *simple roots* and by $h_i = k_i - k_{i+1}$ the *simple coroots*.

**Definition 1.1.** The *quantum queer superalgebra* $U_q(\mathfrak{q}(n))$ is the superalgebra over $\mathbf{F}$ with 1 generated by the symbols $e_i$, $f_i$, $e_{\bar{i}}$, $f_{\bar{i}}$ $(i = 1, \ldots, n-1)$, $q^h$ $(h \in P^\vee)$, $k_{\bar{j}}$ $(j = 1, \ldots, n)$ with the following defining relations:

$$\begin{aligned}
&q^0 = 1, \quad q^{h_1}q^{h_2} = q^{h_1+h_2} \quad (h_1, h_2 \in P^\vee),\\
&q^h e_i q^{-h} = q^{\alpha_i(h)} e_i \quad (h \in P^\vee),\\
&q^h f_i q^{-h} = q^{-\alpha_i(h)} f_i \quad (h \in P^\vee),\\
&q^h k_{\bar{j}} = k_{\bar{j}} q^h,\\
&e_i f_j - f_j e_i = \delta_{ij} \frac{q^{k_i - k_{i+1}} - q^{-k_i + k_{i+1}}}{q - q^{-1}},
\end{aligned}$$

$$
\begin{aligned}
& e_i e_j - e_j e_i = f_i f_j - f_j f_i = 0 \qquad \text{if } |i-j|>1, \\
& e_i^2 e_j - (q+q^{-1}) e_i e_j e_i + e_j e_i^2 = 0 \qquad \text{if } |i-j|=1, \\
& f_i^2 f_j - (q+q^{-1}) f_i f_j f_i + f_j f_i^2 = 0 \qquad \text{if } |i-j|=1, \\
& k_{\bar{i}}^2 = \frac{q^{2k_i} - q^{-2k_i}}{q^2 - q^{-2}}, \\
& k_{\bar{i}} k_{\bar{j}} + k_{\bar{j}} k_{\bar{i}} = 0 \quad \text{if } i \neq j, \\
& k_{\bar{i}} e_i - q e_i k_{\bar{i}} = e_{\bar{i}} q^{-k_i}, \qquad q k_{\bar{i}} e_{i-1} - e_{i-1} k_{\bar{i}} = -q^{-k_i} e_{\overline{i-1}}, \\
& k_{\bar{i}} e_j - e_j k_{\bar{i}} = 0 \quad \text{if } j \neq i, i-1, \\
& k_{\bar{i}} f_i - q f_i k_{\bar{i}} = -f_{\bar{i}} q^{k_i}, \ q k_{\bar{i}} f_{i-1} - f_{i-1} k_{\bar{i}} = q^{k_i} f_{\overline{i-1}}, \\
& k_{\bar{i}} f_j - f_j k_{\bar{i}} = 0 \quad \text{if } j \neq i, i-1, \\
& e_i f_{\bar{j}} - f_{\bar{j}} e_i = \delta_{ij} (k_{\bar{i}} q^{-k_{i+1}} - k_{\overline{i+1}} q^{-k_i}), \\
& e_{\bar{i}} f_j - f_j e_{\bar{i}} = \delta_{ij} (k_{\bar{i}} q^{k_{i+1}} - k_{\overline{i+1}} q^{k_i}), \\
& e_i e_{\bar{i}} - e_{\bar{i}} e_i = f_i f_{\bar{i}} - f_{\bar{i}} f_i = 0, \\
& e_i e_{i+1} - q e_{i+1} e_i = e_{\bar{i}} e_{\overline{i+1}} + q e_{\overline{i+1}} e_{\bar{i}}, \\
& q f_{i+1} f_i - f_i f_{i+1} = f_{\bar{i}} f_{\overline{i+1}} + q f_{\overline{i+1}} f_{\bar{i}}, \\
& e_i^2 e_{\bar{j}} - (q+q^{-1}) e_i e_{\bar{j}} e_i + e_{\bar{j}} e_i^2 = 0 \qquad \text{if } |i-j|=1, \\
& f_i^2 f_{\bar{j}} - (q+q^{-1}) f_i f_{\bar{j}} f_i + f_{\bar{j}} f_i^2 = 0 \qquad \text{if } |i-j|=1.
\end{aligned} \tag{1.1}
$$

The generators $e_i$, $f_i$ $(i = 1, \dots, n-1)$, $q^h$ $(h \in P^\vee)$ are regarded as *even* and $e_{\bar{i}}$, $f_{\bar{i}}$ $(i = 1, \dots, n-1)$, $k_{\bar{j}}$ $(j = 1, \dots, n)$ are *odd*. From the defining relations, it is easy to see that the even generators together with $k_{\bar{1}}$ generate the whole algebra $U_q(\mathfrak{q}(n))$.

**Remark 1.2.** The generators in (1.1) are different from those in [2, Theorem 2.1]. The elements $e_i$, $f_i$, $e_{\bar{i}}$ and $f_{\bar{i}}$ in (1.1) correspond to $q^{k_{i+1}} e_i$, $f_i q^{-k_{i+1}}$, $q^{k_{i+1}} e_{\bar{i}}$ and $f_{\bar{i}} q^{-k_{i+1}}$ in [2, Theorem 2.1], respectively. We rewrite the whole defining relations in [2, Theorem 2.1] in terms of new generators and remove some relations which can be derived from the others.

The superalgebra $U_q(\mathfrak{q}(n))$ is a bialgebra with the comultiplication $\Delta\colon U_q(\mathfrak{q}(n)) \to U_q(\mathfrak{q}(n)) \otimes U_q(\mathfrak{q}(n))$ defined by

$$
\begin{aligned}
& \Delta(q^h) = q^h \otimes q^h \quad \text{for } h \in P^\vee, \\
& \Delta(e_i) = e_i \otimes q^{-k_i + k_{i+1}} + 1 \otimes e_i, \\
& \Delta(f_i) = f_i \otimes 1 + q^{k_i - k_{i+1}} \otimes f_i, \\
& \Delta(k_{\bar{1}}) = k_{\bar{1}} \otimes q^{k_1} + q^{-k_1} \otimes k_{\bar{1}}.
\end{aligned} \tag{1.2}
$$

Let $U^+$ (resp. $U^-$) be the subalgebra of $U_q(\mathfrak{q}(n))$ generated by $e_i$, $e_{\bar{i}}$ $(i = 1, \dots, n-1)$ (resp. $f_i$, $f_{\bar{i}}$ $(i = 1, \dots, n-1)$), and let $U^0$ be the subalgebra generated by $q^h$ $(h \in P^\vee)$

and $k_{\overline{j}}$ $(j = 1, \dots, n)$. In [2], it was shown that the algebra $U_q(\mathfrak{q}(n))$ has the *triangular decomposition*:

$$U^- \otimes U^0 \otimes U^+ \xrightarrow{\sim} U_q(\mathfrak{q}(n)). \tag{1.3}$$

Hereafter, a $U_q(\mathfrak{q}(n))$-module is understood as a $U_q(\mathfrak{q}(n))$-supermodule. A $U_q(\mathfrak{q}(n))$-module $M$ is called a *weight module* if $M$ has a weight space decomposition $M = \bigoplus_{\mu \in P} M_\mu$, where

$$M_\mu := \{m \in M;\ q^h m = q^{\mu(h)} m \text{ for all } h \in P^\vee\}.$$

The set of weights of $M$ is defined to be

$$\mathrm{wt}(M) = \{\mu \in P;\ M_\mu \neq 0\}.$$

**Definition 1.3.** A weight module $V$ is called a *highest weight module with highest weight* $\lambda \in P$ if $V_\lambda$ is finite-dimensional and satisfies the following conditions:

(a) $V$ is generated by $V_\lambda$,
(b) $e_i v = e_{\overline{i}} v = 0$ for all $v \in V_\lambda$, $i = 1, \dots, n-1$.

As seen in [2], there exists a unique irreducible highest weight module with highest weight $\lambda \in P$ up to parity change; it will be denoted by $V(\lambda)$.

Set

$$\begin{aligned} P^{\geq 0} &= \{\lambda = \lambda_1\epsilon_1 + \cdots + \lambda_n\epsilon_n \in P;\ \lambda_j \in \mathbb{Z}_{\geq 0} \text{ for all } j = 1, \dots, n\}, \\ \Lambda^+ &= \{\lambda = \lambda_1\epsilon_1 + \cdots + \lambda_n\epsilon_n \in P^{\geq 0};\ \lambda_i \geq \lambda_{i+1} \text{ and } \lambda_i = \lambda_{i+1} \text{ implies} \\ &\qquad \lambda_i = \lambda_{i+1} = 0 \text{ for all } i = 1, \dots, n-1\}. \end{aligned}$$

Note that each element $\lambda \in \Lambda^+$ corresponds to a *strict partition* $\lambda = (\lambda_1 > \cdots > \lambda_r > 0)$. Thus we will often call $\lambda \in \Lambda^+$ a strict partition. For the same reason, we call $\lambda = (\lambda_1, \dots, \lambda_n) \in P^{\geq 0}$ a *partition* if $\lambda_1 \geq \cdots \geq \lambda_r > \lambda_{r+1} = \cdots = \lambda_n = 0$. We denote $r$ by $\ell(\lambda)$.

**Example 1.4.** Let

$$\mathbf{V} = \bigoplus_{j=1}^{n} \mathbf{F} v_j \oplus \bigoplus_{j=1}^{n} \mathbf{F} v_{\overline{j}}$$

be the vector representation of $U_q(\mathfrak{q}(n))$. The action of $U_q(\mathfrak{q}(n))$ on $\mathbf{V}$ is given as follows:

$$\begin{aligned} &e_i v_j = \delta_{j,i+1} v_i, && e_i v_{\overline{j}} = \delta_{j,i+1} v_{\overline{i}}, && f_i v_j = \delta_{j,i} v_{i+1}, && f_i v_{\overline{j}} = \delta_{j,i} v_{\overline{i+1}}, \\ &e_{\overline{i}} v_j = \delta_{j,i+1} v_{\overline{i}}, && e_{\overline{i}} v_{\overline{j}} = \delta_{j,i+1} v_i, && f_{\overline{i}} v_j = \delta_{j,i} v_{\overline{i+1}}, && f_{\overline{i}} v_{\overline{j}} = \delta_{j,i} v_{i+1}, \\ &q^h v_j = q^{\epsilon_j(h)} v_j, && q^h v_{\overline{j}} = q^{\epsilon_j(h)} v_{\overline{j}}, && k_{\overline{i}} v_j = \delta_{j,i} v_{\overline{j}}, && k_{\overline{i}} v_{\overline{j}} = \delta_{j,i} v_j. \end{aligned} \tag{1.4}$$

Note that $\mathbf{V}$ is an irreducible highest weight module with highest weight $\epsilon_1$.

**Definition 1.5.** We define $\mathcal{O}^{\geq 0}_{\mathrm{int}}$ to be the category of finite-dimensional weight modules $M$ satisfying the following conditions:

(a) $\mathrm{wt}(M) \subset P^{\geq 0}$,
(b) for any $\mu \in P^{\geq 0}$ and $i \in \{1, \ldots, n\}$ such that $\langle k_i, \mu \rangle = 0$, we have $k_{\bar{i}}|_{M_\mu} = 0$.

**Remark 1.6.** By Lemma 4.1 below, it is enough to assume $i = 1$ in (b). Note also that (b) is equivalent to saying that every weight space $M_\mu$ is completely reducible as a $U^0$-module.

The fundamental properties of the category $\mathcal{O}_{\mathrm{int}}^{\geq 0}$ are summarized in the following proposition.

**Proposition 1.7** ( [2]). (a) *Every $U_q(\mathfrak{q}(n))$-module in $\mathcal{O}_{\mathrm{int}}^{\geq 0}$ is completely reducible.*
(b) *Every irreducible object in $\mathcal{O}_{\mathrm{int}}^{\geq 0}$ has the form $V(\lambda)$ for some $\lambda \in \Lambda^+$.*
(c) *The category $\mathcal{O}_{\mathrm{int}}^{\geq 0}$ is stable under tensor products.*

In [2], we employed the rational function field $\mathbb{C}(q)$ as the base field of $U_q(\mathfrak{q}(n))$. But here, we employ $\mathbb{C}((q))$ instead of $\mathbb{C}(q)$ as the base field of $U_q(\mathfrak{q}(n))$. Note that when $m$ is a nonnegative integer, the $q$-integer $(q^{2m} - q^{-2m})/(q^2 - q^{-2})$ has a square root in $\mathbb{C}((q))$ but not in $\mathbb{C}(q)$. This difference gives the following two statements, which is simpler than the corresponding statements in [2].

**Proposition 1.8** (cf. [2, Corollary 3.9]). *Let $\mathrm{Cliff}_q(\lambda)$ be the associative superalgebra over $\mathbb{C}((q))$ generated by odd generators $\{t_{\bar{i}};\ i = 1, \ldots, n\}$ with the defining relations*

$$t_{\bar{i}} t_{\bar{j}} + t_{\bar{j}} t_{\bar{i}} = \delta_{ij} \frac{2(q^{2\lambda_i} - q^{-2\lambda_i})}{q^2 - q^{-2}}, \qquad i, j = 1, \ldots, n.$$

*Then $\mathrm{Cliff}_q(\lambda)$ has up to isomorphism*

(a) *two simple modules $E^q(\lambda)$ and $\Pi(E^q(\lambda))$ of dimension $2^{k-1}|2^{k-1}$ if $\ell(\lambda) = 2k$,*
(b) *one simple module $E^q(\lambda) \cong \Pi(E^q(\lambda))$ of dimension $2^k|2^k$ if $\ell(\lambda) = 2k + 1$.*

**Proposition 1.9** (cf. [2, Theorem 5.14]). *Let $V(\lambda)$ be an irreducible highest weight module with highest weight $\lambda \in \Lambda^+$. Then*

$$\mathrm{ch}\, V(\lambda) = \mathrm{ch}\, V_{\mathrm{cl}}(\lambda),$$

*where $V_{\mathrm{cl}}(\lambda)$ is an irreducible highest weight module over $\mathfrak{q}(n)$ with highest weight $\lambda$.*

In short, in contrast to [2], we have the same classification for the modules over $\mathrm{Cliff}_q(\lambda)$ as that for the modules over the Clifford algebra with the base field $\mathbb{C}$. Also we have the same characters of the irreducible modules over $U_q(\mathfrak{q}(n))$ as those of the irreducible modules over $\mathfrak{q}(n)$.

**Remark 1.10.** Define $\mathcal{O}_{\mathrm{int,cl}}^{\geq 0}$ to be the category of finite-dimensional weight modules $M$ over $\mathfrak{q}(n)$ such that (i) $\mathrm{wt}(M) \subset P^{\geq 0}$, (ii) $k_{\bar{i}}|_{M_\mu} = 0$ for $i \in \{1, \ldots, n\}$ and $\mu \in P^{\geq 0}$ satisfying $\langle k_i, \mu \rangle = 0$. Here $k_{\bar{i}}$ is the element of $\mathfrak{q}(n)$ given by $\begin{pmatrix} 0 & E_{i,i} \\ E_{i,i} & 0 \end{pmatrix}$, where $E_{i,i}$ is the $n \times n$-matrix having 1 in the $(i, i)$-position and 0 elsewhere. Let us denote the

Grothendieck rings of the categories by $K(\mathcal{O}_{\mathrm{int}}^{\geq 0})$ and $K(\mathcal{O}_{\mathrm{int,cl}}^{\geq 0})$, respectively. Since $\mathcal{O}_{\mathrm{int,cl}}^{\geq 0}$ and $\mathcal{O}_{\mathrm{int}}^{\geq 0}$ are semisimple categories, by taking the classical limit (i.e., taking the reduction at $q = 1$), we have a ring isomorphism

$$K(\mathcal{O}_{\mathrm{int}}^{\geq 0}) \xrightarrow{\sim} K(\mathcal{O}_{\mathrm{int,cl}}^{\geq 0})$$

which sends $V(\lambda) \mapsto V_{\mathrm{cl}}(\lambda)$.

Now we give a decomposition of the tensor product of the natural representation with a highest weight module.

**Theorem 1.11.** *Let $M$ be a highest weight $U_q(\mathfrak{q}(n))$-module in $\mathcal{O}_{\mathrm{int}}^{\geq 0}$ with highest weight $\lambda \in \Lambda^+$. Then*

$$\mathbf{V} \otimes M \simeq \bigoplus_{\substack{\lambda+\epsilon_j: \\ \text{strict partition}}} M_j,$$

*where $M_j$ is a highest weight $U_q(\mathfrak{q}(n))$-module in the category $\mathcal{O}_{\mathrm{int}}^{\geq 0}$ with highest weight $\lambda + \epsilon_j$ and* $\dim(M_j)_{\lambda+\epsilon_j} = 2 \dim M_\lambda$.

*Proof.* We will prove that our assertion holds for finite-dimensional highest weight modules over $\mathfrak{q}(n)$. Then, by Remark 1.10, our assertion also holds for finite-dimensional highest weight modules over $U_q(\mathfrak{q}(n))$.

Let $U(\mathfrak{q}(n))$ be the universal enveloping algebra of $\mathfrak{q}(n)$ and let $U^{\geq 0}$ be the universal enveloping algebra of the standard Borel subalgebra of $\mathfrak{q}(n)$. Let $M$ be a highest weight $U(\mathfrak{q}(n))$-module with highest weight $\lambda \in \Lambda^+$ and $\mathbf{V}_{\mathrm{cl}} = \bigoplus_{i=1}^n (\mathbb{C}v_i \oplus \mathbb{C}v_{\bar{i}})$ be the natural representation of $U(\mathfrak{q}(n))$. Consider a surjective homomorphism

$$U(\mathfrak{q}(n)) \otimes_{U^{\geq 0}} \mathbf{v}_\lambda \twoheadrightarrow M,$$

where $\mathbf{v}_\lambda \simeq M_\lambda$ as a $U^{\geq 0}$-module. Now we have

$$\mathbf{V}_{\mathrm{cl}} \otimes \big(U(\mathfrak{q}(n)) \otimes_{U^{\geq 0}} \mathbf{v}_\lambda\big) \simeq U(\mathfrak{q}(n)) \otimes_{U^{\geq 0}} (\mathbf{V}_{\mathrm{cl}} \otimes \mathbf{v}_\lambda).$$

Then $F_i(\mathbf{V}_{\mathrm{cl}} \otimes \mathbf{v}_\lambda) := \bigoplus_{j \leq i} (\mathbb{C}v_j \oplus \mathbb{C}v_{\bar{j}}) \otimes \mathbf{v}_\lambda$ is a $U^{\geq 0}$-module. We set

$$N := U(\mathfrak{q}(n)) \otimes_{U^{\geq 0}} (\mathbf{V}_{\mathrm{cl}} \otimes \mathbf{v}_\lambda), \quad F_i(N) := U(\mathfrak{q}(n)) \otimes_{U^{\geq 0}} F_i(\mathbf{V}_{\mathrm{cl}} \otimes \mathbf{v}_\lambda).$$

Since

$$F_i(\mathbf{V}_{\mathrm{cl}} \otimes \mathbf{v}_\lambda)/F_{i-1}(\mathbf{V}_{\mathrm{cl}} \otimes \mathbf{v}_\lambda) \simeq (\mathbb{C}v_i \oplus \mathbb{C}v_{\bar{i}}) \otimes \mathbf{v}_\lambda,$$

we see that

$$F_i(N)/F_{i-1}(N) \simeq U(\mathfrak{q}(n)) \otimes_{U^{\geq 0}} \big(F_i(\mathbf{V}_{\mathrm{cl}} \otimes \mathbf{v}_\lambda)/F_{i-1}(\mathbf{V}_{\mathrm{cl}} \otimes \mathbf{v}_\lambda)\big)$$

is a highest weight module with highest weight $\lambda + \varepsilon_i$.

Now we shall show

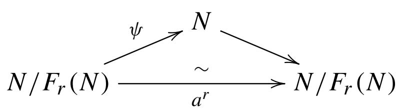
$$N \simeq \bigoplus_{k \le r} (F_k(N)/F_{k-1}(N)) \oplus N/F_r(N), \quad \text{where } r = \ell(\lambda). \tag{1.5}$$

First note that $F_i(N)/F_{i-1}(N)$ admits the central character

$$\chi_i := \chi_{\lambda+\varepsilon_i} : \mathcal{Z} \to \mathbb{C},$$

where $\mathcal{Z}$ is the center of $U(\mathfrak{q}(n))$ and $\chi_\mu$ is the central character afforded by the Weyl module $W(\mu)$ with highest weight $\mu$ (see [2, Section 1] for Weyl modules and central characters). From [2, Proposition 1.7], we know that $\chi_1, \dots, \chi_r, \chi_{r+1}$ are different from each other, and $\chi_{r+1} = \chi_{r+2} = \cdots = \chi_n$.

Choose $a \in \mathcal{Z}$ such that $\chi_1(a) = \cdots = \chi_r(a) = 0$ and $\chi_{r+1}(a) \neq 0$. Then $a|_{F_i(N)/F_{i-1}(N)} = 0$ and hence $aF_i(N) \subset F_{i-1}(N)$ for $i \le r$. It follows that $a^r F_r(N) = F_{-1}(N) = 0$. Hence $N \xrightarrow{a^r} N$ factors through $N \to N/F_r(N) \xrightarrow{\psi} N$. Since $a^r : N/F_r(N) \to N/F_r(N)$ is an isomorphism, we have the diagram



It follows that

$$N \simeq (N/F_r(N)) \oplus F_r(N).$$

Using a similar argument, we can conclude that

$$F_k(N) \simeq (F_k(N)/F_{k-1}(N)) \oplus F_{k-1}(N)$$

for $k \le r$. Hence we obtain (1.5).

By [2, Proposition 1.4(3)], we know that $F_i(N)/F_{i-1}(N)$ admits a finite-dimensional quotient if and only if $\lambda + \varepsilon_i$ is a strict partition, and $N/F_r(N)$ has only trivial finite-dimensional quotient. Since $\mathbf{V}_{\mathrm{cl}} \otimes M$ is a largest finite-dimensional quotient of $N$, we get the desired result. □

**Corollary 1.12.** *Any irreducible $U_q(\mathfrak{q}(n))$-module in $\mathcal{O}_{\mathrm{int}}^{\ge 0}$ appears as a direct summand of tensor products of* $\mathbf{V}$.

*Proof.* This follows immediately from Theorem 1.11. □

## 2. Crystal bases

Let $M$ be a $U_q(\mathfrak{q}(n))$-module in the category $\mathcal{O}_{\mathrm{int}}^{\ge 0}$. For $i = 1, \dots, n-1$, let $u \in M_\lambda$ ($\lambda \in P$) be a weight vector and consider the *$i$-string decomposition* of $u$:

$$u = \sum_{k \ge 0} f_i^{(k)} u_k,$$

where $e_i u_k = 0$ for all $k \geq 0$ and $f_i^{(k)} = f_i^k/[k]!$. We define the *even Kashiwara operators* $\tilde{e}_i, \tilde{f}_i$ $(i = 1, \ldots, n-1)$ by

$$\tilde{e}_i u = \sum_{k \geq 1} f_i^{(k-1)} u_k, \qquad \tilde{f}_i u = \sum_{k \geq 0} f_i^{(k+1)} u_k. \tag{2.1}$$

On the other hand, we define the *odd Kashiwara operators* $\tilde{k}_{\bar{1}}, \tilde{e}_{\bar{1}}, \tilde{f}_{\bar{1}}$ by

$$\begin{aligned} \tilde{k}_{\bar{1}} &= q^{k_1 - 1} k_{\bar{1}}, \\ \tilde{e}_{\bar{1}} &= -(e_1 k_{\bar{1}} - q k_{\bar{1}} e_1) q^{k_1 - 1}, \\ \tilde{f}_{\bar{1}} &= -(k_{\bar{1}} f_1 - q f_1 k_{\bar{1}}) q^{k_2 - 1}. \end{aligned} \tag{2.2}$$

The following lemma is obvious.

**Lemma 2.1.** *The operators* $\tilde{e}_{\bar{1}}$ *and* $\tilde{f}_{\bar{1}}$ *commute with* $\tilde{e}_i$ *and* $\tilde{f}_i$ $(3 \leq i \leq n-1)$.

Recall that an *abstract* $\mathfrak{gl}(n)$*-crystal* is a set $B$ together with the maps $\tilde{e}_i, \tilde{f}_i \colon B \to B \sqcup \{0\}$, $\varphi_i, \varepsilon_i \colon B \to \mathbb{Z} \sqcup \{-\infty\}$ $(i \in I = \{1, \ldots, n-1\})$, and $\mathrm{wt} \colon B \to P$ satisfying the following conditions (see [12]):

(i) $\mathrm{wt}(\tilde{e}_i b) = \mathrm{wt}(b) + \alpha_i$ if $i \in I$ and $\tilde{e}_i b \neq 0$,
(ii) $\mathrm{wt}(\tilde{f}_i b) = \mathrm{wt}(b) - \alpha_i$ if $i \in I$ and $\tilde{f}_i b \neq 0$,
(iii) for any $i \in I$ and $b \in B$, $\varphi_i(b) = \varepsilon_i(b) + \langle h_i, \mathrm{wt}(b) \rangle$,
(iv) for any $i \in I$ and $b, b' \in B$, $\tilde{f}_i b = b'$ if and only if $b = \tilde{e}_i b'$,
(v) for any $i \in I$ and $b \in B$ such that $\tilde{e}_i b \neq 0$, we have $\varepsilon_i(\tilde{e}_i b) = \varepsilon_i(b) - 1$, $\varphi_i(\tilde{e}_i b) = \varphi_i(b) + 1$,
(vi) for any $i \in I$ and $b \in B$ such that $\tilde{f}_i b \neq 0$, we have $\varepsilon_i(\tilde{f}_i b) = \varepsilon_i(b) + 1$, $\varphi_i(\tilde{f}_i b) = \varphi_i(b) - 1$,
(vii) for any $i \in I$ and $b \in B$ such that $\varphi_i(b) = -\infty$, we have $\tilde{e}_i b = \tilde{f}_i b = 0$.

In this paper, we say that an abstract $\mathfrak{gl}(n)$-crystal is a $\mathfrak{gl}(n)$*-crystal* if it is realized as a crystal basis of a finite-dimensional integrable $U_q(\mathfrak{gl}(n))$-module. In particular, for any $b$ in a $\mathfrak{gl}(n)$-crystal $B$, we have

$$\varepsilon_i(b) = \max\{n \in \mathbb{Z}_{\geq 0};\ \tilde{e}_i^n b \neq 0\}, \qquad \varphi_i(b) = \max\{n \in \mathbb{Z}_{\geq 0};\ \tilde{f}_i^n b \neq 0\}.$$

**Definition 2.2.** Let $M = \bigoplus_{\mu \in P^{\geq 0}} M_\mu$ be a $U_q(\mathfrak{q}(n))$-module in the category $\mathcal{O}_{\mathrm{int}}^{\geq 0}$. A *crystal basis* of $M$ is a triple $(L, B, l_B = (l_b)_{b \in B})$, where

(a) $L$ is a free $\mathbf{A}$-submodule of $M$ such that
  (i) $\mathbf{F} \otimes_{\mathbf{A}} L \xrightarrow{\sim} M$,
  (ii) $L = \bigoplus_{\mu \in P^{\geq 0}} L_\mu$, where $L_\mu = L \cap M_\mu$,
  (iii) $L$ is stable under the Kashiwara operators $\tilde{e}_i, \tilde{f}_i$ $(i = 1, \ldots, n-1)$, $\tilde{k}_{\bar{1}}, \tilde{e}_{\bar{1}}, \tilde{f}_{\bar{1}}$.
(b) $B$ is a $\mathfrak{gl}(n)$-crystal together with the maps $\tilde{e}_{\bar{1}}, \tilde{f}_{\bar{1}} \colon B \to B \sqcup \{0\}$ such that
  (i) $\mathrm{wt}(\tilde{e}_{\bar{1}} b) = \mathrm{wt}(b) + \alpha_1$, $\mathrm{wt}(\tilde{f}_{\bar{1}} b) = \mathrm{wt}(b) - \alpha_1$,
  (ii) for all $b, b' \in B$, $\tilde{f}_{\bar{1}} b = b'$ if and only if $b = \tilde{e}_{\bar{1}} b'$.

(c) $l_B = (l_b)_{b\in B}$ is a family of non-zero $\mathbb{C}$-vector subspaces of $L/qL$ such that
   (i) $l_b \subset (L/qL)_\mu$ for $b \in B_\mu$,
   (ii) $L/qL = \bigoplus_{b\in B} l_b$,
   (iii) $\tilde{k}_{\overline{1}} l_b \subset l_b$,
   (iv) for $i = 1, \dots, n-1, \overline{1}$, we have
      (1) if $\tilde{e}_i b = 0$ then $\tilde{e}_i l_b = 0$, and otherwise $\tilde{e}_i$ induces an isomorphism $l_b \xrightarrow{\sim} l_{\tilde{e}_i b}$,
      (2) if $\tilde{f}_i b = 0$ then $\tilde{f}_i l_b = 0$, and otherwise $\tilde{f}_i$ induces an isomorphism $l_b \xrightarrow{\sim} l_{\tilde{f}_i b}$.

**Proposition 2.3.** *Let $(L, B, l_B)$ be a crystal basis of a $U_q(\mathfrak{q}(n))$-module $M$. Then*

$$\tilde{e}_{\overline{1}}^2 = \tilde{f}_{\overline{1}}^2 = 0 \quad \textit{as endomorphisms on } L/qL.$$

*Proof.* Since every $u \in L_\lambda$ has a 1-string decomposition $u = \sum_{k=0}^{N} f_1^{(k)} u_k$ with $e_1 u_k = 0$ for $k = 0, \dots, N$, it suffices to show that $\tilde{e}_{\overline{1}}^2 u \equiv \tilde{f}_{\overline{1}}^2 u \equiv 0 \pmod{qL}$ for $u = f_1^{(s)} v$ with $e_1 v = 0$ and $\mathrm{wt}(v) = \mu$ $(s \geq 0)$.

We first show $\tilde{e}_{\overline{1}}^2 u \equiv 0 \pmod{qL}$. From the defining relations $k_{\overline{1}} e_1 - q e_1 k_{\overline{1}} = e_{\overline{1}} q^{-k_1}$ and $e_1 e_{\overline{1}} = e_{\overline{1}} e_1$, we obtain

$$e_1 k_{\overline{1}} e_1 - q e_1^2 k_{\overline{1}} = e_1 e_{\overline{1}} q^{-k_1} \quad \text{and} \quad k_{\overline{1}} e_1^2 - q e_1 k_{\overline{1}} e_1 = q^{-1} e_{\overline{1}} e_1 q^{-k_1} = q^{-1} e_1 e_{\overline{1}} q^{-k_1}.$$

Then

$$e_1 k_{\overline{1}} e_1 - q e_1^2 k_{\overline{1}} = q k_{\overline{1}} e_1^2 - q^2 e_1 k_{\overline{1}} e_1.$$

That is,

$$e_1 k_{\overline{1}} e_1 = e_1^{(2)} k_{\overline{1}} + k_{\overline{1}} e_1^{(2)}. \tag{2.3}$$

Using this formula, we obtain

$$\begin{aligned}
\tilde{e}_{\overline{1}}^2 &= (e_1 k_{\overline{1}} - q k_{\overline{1}} e_1)^2 q^{2k_1 - 1} \\
&= \big((e_1^{(2)} k_{\overline{1}} + k_{\overline{1}} e_1^{(2)}) k_{\overline{1}} - q e_1 k_{\overline{1}}^2 e_1 - q k_{\overline{1}} e_1^2 k_{\overline{1}} + q^2 k_{\overline{1}} (e_1^{(2)} k_{\overline{1}} + k_{\overline{1}} e_1^{(2)})\big) q^{2k_1 - 1} \\
&= \frac{q - q^{-1}}{q + q^{-1}} q^2 e_1^2 q^{4k_1}.
\end{aligned}$$

It follows that

$$\begin{aligned}
\tilde{e}_{\overline{1}}^2 u &= \frac{q - q^{-1}}{q + q^{-1}} q^{\langle 4k_1, \mu - s\alpha_1 \rangle + 2} e_1^2 f_1^{(s)} v \\
&= \frac{q - q^{-1}}{q + q^{-1}} q^{4\langle k_1, \mu \rangle - 4s + 2} [\langle k_1 - k_2, \mu \rangle - s + 1][\langle k_1 - k_2, \mu \rangle - s + 2] f_1^{(s-2)} v.
\end{aligned}$$

Note that $q^{2\langle k_1 - k_2, \mu \rangle - 2s + 1} [\langle k_1 - k_2, \mu \rangle - s + 1][\langle k_1 - k_2, \mu \rangle - s + 2] \equiv 1 \pmod{q\mathbf{A}}$. Since

$$4\langle k_1, \mu \rangle - 4s + 2 - (2\langle k_1 - k_2, \mu \rangle - 2s + 1) = 2(\langle k_1 - k_2, \mu \rangle - s) + 4\langle k_2, \mu \rangle + 1 \geq 1,$$

we have

$$q^{4\langle k_1,\mu\rangle-4s+2}[\langle k_1-k_2,\mu\rangle-s+1][\langle k_1-k_2,\mu\rangle-s+2]\in q\mathbf{A},$$

which implies $\tilde{e}_{\overline{1}}^2 u\equiv 0 \pmod{qL}$ as desired.

Now we show $\tilde{f}_{\overline{1}}^2 u\equiv 0 \pmod{qL}$. By a similar argument to the one above, we obtain

$$f_1 k_{\overline{1}} f_1 = f_1^{(2)} k_{\overline{1}} + k_{\overline{1}} f_1^{(2)}.$$

Then

$$\begin{aligned}\tilde{f}_{\overline{1}}^2 &= (k_{\overline{1}} f_1 - q f_1 k_{\overline{1}})^2 q^{2k_2-1}\\ &= \big(k_{\overline{1}}(f_1^{(2)}k_{\overline{1}}+k_{\overline{1}}f_1^{(2)}) - qk_{\overline{1}}f_1^2k_{\overline{1}} - qf_1k_{\overline{1}}^2f_1 + q^2(f_1^{(2)}k_{\overline{1}}+k_{\overline{1}}f_1^{(2)})k_{\overline{1}}\big)q^{2k_2-1}\\ &= \frac{q-q^{-1}}{q+q^{-1}} f_1^2 q^{2k_1+2k_2-2}.\end{aligned}$$

It follows that

$$\begin{aligned}\tilde{f}_{\overline{1}}^2 u &= \frac{q-q^{-1}}{q+q^{-1}} f_1^2 q^{\langle 2k_1+2k_2,\mu-s\alpha_1\rangle-2} f_1^{(s)} v\\ &= \frac{q-q^{-1}}{q+q^{-1}} q^{2\langle k_1+k_2,\mu\rangle-2}[s+2][s+1] f_1^{(s+2)} v.\end{aligned}$$

If $\langle k_1-k_2,\mu\rangle < s+2$, then $f_1^{(s+2)}v=0$, i.e., $\tilde{f}_{\overline{1}}^2 u\equiv 0 \pmod{qL}$. If $\langle k_1-k_2,\mu\rangle \geq s+2$, we have

$$2\langle k_1+k_2,\mu\rangle-2\geq 2\langle k_1-k_2,\mu\rangle-2\geq 2s+2.$$

Since $q^{2s+1}[s+2][s+1]\equiv 1 \bmod q\mathbf{A}$, we have

$$q^{\langle 2k_1+2k_2,\mu\rangle-2}[s+2][s+1]\in q\mathbf{A},$$

which proves our assertion. □

**Example 2.4.** Let $\mathbf{V}=\bigoplus_{j=1}^n \mathbf{F}v_j \oplus \bigoplus_{j=1}^n \mathbf{F}v_{\overline{j}}$ be the vector representation of $U_q(\mathfrak{q}(n))$. Set

$$\mathbf{L}=\bigoplus_{j=1}^n \mathbf{A}v_j \oplus \bigoplus_{j=1}^n \mathbf{A}v_{\overline{j}} \quad \text{and} \quad l_j=\mathbb{C}v_j\oplus\mathbb{C}v_{\overline{j}}\subset \mathbf{L}/q\mathbf{L},$$

and let $\mathbf{B}$ be the $\mathfrak{gl}(n)$-crystal with the $\overline{1}$-arrow given below.

$$\boxed{1}\ \underset{\overline{1}}{\overset{1}{\rightrightarrows}}\ \boxed{2}\xrightarrow{2}\boxed{3}\xrightarrow{3}\cdots\xrightarrow{n-1}\boxed{n}\ .$$

Here, the actions of $\tilde{f}_i$ $(i=1,\dots,n-1,\overline{1})$ are expressed by $i$-arrows. Then $(\mathbf{L},\mathbf{B},l_{\mathbf{B}}=(l_j)_{j=1}^n)$ is a crystal basis of $\mathbf{V}$.

**Remark 2.5.** Let $M$ be a $U_q(\mathfrak{q}(n))$-module in the category $\mathcal{O}_{\text{int}}^{\geq 0}$ with a crystal basis $(L, B, l_B)$, and let $B = \coprod_{k=1}^{s} B_k$ be the decomposition of $B$ into connected $\mathfrak{gl}(n)$-crystals. Then there exists a decomposition

$$M = \bigoplus_{k=1}^{s} \bigoplus_{j=1}^{m_k} M_{k,j}$$

of $M$ as a $U_q(\mathfrak{gl}(n))$-module, where

(a) $m_k = \dim l_b$ for some $b \in B_k$,
(b) $M_{k,j}$ has a $U_q(\mathfrak{gl}(n))$-crystal basis $(L_{k,j}, B_{k,j})$ such that

  (i) $L = \bigoplus_{k,j} L_{k,j}$,
  (ii) there exists a $\mathfrak{gl}(n)$-crystal isomorphism $\phi_{k,j} : B_k \xrightarrow{\sim} B_{k,j}$ such that the vectors $\phi_{k,j}(b)$ $(j = 1, \dots, m_k)$ form a basis of $l_b$ for each $b \in B_k$.

**Remark 2.6.** Let $M$ be a $U_q(\mathfrak{q}(n))$-module in the category $\mathcal{O}_{\text{int}}^{\geq 0}$ with a crystal basis $(L, B, l_B)$. For $i = 1, \dots, n-1, \overline{1}$ and $b, b' \in B$, if $b' = \tilde{f}_i b$, then we have isomorphisms $\tilde{f}_i : l_b \xrightarrow{\sim} l_{b'}$ and $\tilde{e}_i : l_{b'} \xrightarrow{\sim} l_b$. If $i = 1, \dots, n-1$, then they are inverses to each other by Remark 2.5. However, when $i = \overline{1}$, they are not inverses to each other in general.

The *tensor product rule* given in the following theorem is one of the most important features of crystal basis theory.

**Theorem 2.7.** *Let $M_j$ be a $U_q(\mathfrak{q}(n))$-module in $\mathcal{O}_{\text{int}}^{\geq 0}$ with a crystal basis $(L_j, B_j, l_{B_j})$ $(j = 1, 2)$. Set $B_1 \otimes B_2 = B_1 \times B_2$ and $l_{b_1 \otimes b_2} = l_{b_1} \otimes l_{b_2}$ for $b_1 \in B_1$ and $b_2 \in B_2$. Then $(L_1 \otimes_{\mathbf{A}} L_2, B_1 \otimes B_2, (l_b)_{b \in B_1 \otimes B_2})$ is a crystal basis of $M_1 \otimes_{\mathbf{F}} M_2$, where the action of the Kashiwara operators on $B_1 \otimes B_2$ is as follows:*

$$\begin{aligned} \tilde{e}_i(b_1 \otimes b_2) &= \begin{cases} \tilde{e}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) \geq \varepsilon_i(b_2), \\ b_1 \otimes \tilde{e}_i b_2 & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2), \end{cases} \\ \tilde{f}_i(b_1 \otimes b_2) &= \begin{cases} \tilde{f}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2), \\ b_1 \otimes \tilde{f}_i b_2 & \text{if } \varphi_i(b_1) \leq \varepsilon_i(b_2), \end{cases} \end{aligned} \tag{2.4}$$

$$\begin{aligned} \tilde{e}_{\overline{1}}(b_1 \otimes b_2) &= \begin{cases} \tilde{e}_{\overline{1}} b_1 \otimes b_2 & \text{if } \langle k_1, \mathrm{wt}(b_2) \rangle = \langle k_2, \mathrm{wt}(b_2) \rangle = 0, \\ b_1 \otimes \tilde{e}_{\overline{1}} b_2 & \text{otherwise,} \end{cases} \\ \tilde{f}_{\overline{1}}(b_1 \otimes b_2) &= \begin{cases} \tilde{f}_{\overline{1}} b_1 \otimes b_2 & \text{if } \langle k_1, \mathrm{wt}(b_2) \rangle = \langle k_2, \mathrm{wt}(b_2) \rangle = 0, \\ b_1 \otimes \tilde{f}_{\overline{1}} b_2 & \text{otherwise.} \end{cases} \end{aligned} \tag{2.5}$$

*Proof.* It is obvious that

$$(L_1 \otimes L_2)/q(L_1 \otimes L_2) = \bigoplus_{b_1 \in B_1,\, b_2 \in B_2} l_{b_1} \otimes l_{b_2},$$

$$l_{b_1} \otimes l_{b_2} \subset ((L_1 \otimes L_2)/q(L_1 \otimes L_2))_{\lambda+\mu} \quad \text{for } b_1 \in (B_1)_\lambda,\, b_2 \in (B_2)_\mu.$$

For $i = 1, \dots, n-1$, our assertions were already proved in [10, 11]. Let us show the $i = \overline{1}$ case. The following comultiplication formulas can be checked easily:

$$\begin{cases} \Delta(\tilde{k}_{\overline{1}}) = \tilde{k}_{\overline{1}} \otimes q^{2k_1} + 1 \otimes \tilde{k}_{\overline{1}}, \\ \Delta(\tilde{e}_{\overline{1}}) = \tilde{e}_{\overline{1}} \otimes q^{k_1+k_2} + 1 \otimes \tilde{e}_{\overline{1}} - (1-q^2)\tilde{k}_{\overline{1}} \otimes e_1 q^{2k_1}, \\ \Delta(\tilde{f}_{\overline{1}}) = \tilde{f}_{\overline{1}} \otimes q^{k_1+k_2} + 1 \otimes \tilde{f}_{\overline{1}} - (1-q^2)\tilde{k}_{\overline{1}} \otimes f_1 q^{k_1+k_2-1}. \end{cases}$$

Clearly, $L_1 \otimes L_2$ and $l_{b_1} \otimes l_{b_2}$ are stable under $\Delta(\tilde{k}_{\overline{1}})$ for all $b_1 \in B_1$, $b_2 \in B_2$.

We will show that $L_1 \otimes L_2$ is stable under $\Delta(\tilde{e}_{\overline{1}})$ and $\Delta(\tilde{f}_{\overline{1}})$. Let $u_1 \in L_1$ and $u_2 \in L_2$. Then the comultiplication formula implies

$$\Delta(\tilde{e}_{\overline{1}})(u_1 \otimes u_2) = \tilde{e}_{\overline{1}} u_1 \otimes q^{k_1+k_2} u_2 \pm u_1 \otimes \tilde{e}_{\overline{1}} u_2 - (1-q^2)\tilde{k}_{\overline{1}} u_1 \otimes e_1 q^{2k_1} u_2,$$

where $\pm$ is according to whether $u_1$ is even or odd. It is obvious that the first two terms belong to $L_1 \otimes L_2$. For the last term, we may assume that $u_2 = f_1^{(s)} v$ with $e_1 v = 0$. Then we have

$$\begin{aligned} e_1 q^{2k_1} u_2 &= e_1 q^{2k_1} f_1^{(s)} v = q^{2\langle k_1, \mathrm{wt}(v) - s\alpha_1 \rangle} [\langle k_1 - k_2, \mathrm{wt}(v) \rangle - s + 1] f_1^{(s-1)} v \\ &= q^{2\langle k_1, \mathrm{wt}(v) \rangle - 2s} [\langle k_1 - k_2, \mathrm{wt}(v) \rangle - s + 1] \tilde{e}_1 u_2 \\ &= \frac{q^{\langle 3k_1 - k_2, \mathrm{wt}(v) \rangle - 3s + 2} - q^{\langle k_1 + k_2, \mathrm{wt}(v) \rangle - s}}{q^2 - 1} \tilde{e}_1 u_2. \end{aligned}$$

Since

$$\begin{aligned} &\langle 3k_1 - k_2, \mathrm{wt}(v) \rangle - 3s + 2 = 3(\langle k_1 - k_2, \mathrm{wt}(v) \rangle - s) + 2\langle k_2, \mathrm{wt}(v) \rangle + 2 > 0, \\ &\langle k_1 + k_2, \mathrm{wt}(v) \rangle - s = \langle k_1, \mathrm{wt}(u_2) \rangle + \langle k_2, \mathrm{wt}(v) \rangle \geq \langle k_1, \mathrm{wt}(u_2) \rangle \geq 0, \end{aligned}$$

if $\langle k_1, \mathrm{wt}(u_2) \rangle = 0$, then $f_1 u_2 = 0$ and hence $s = -\langle h_1, \mathrm{wt}(u_2) \rangle = \langle k_2, \mathrm{wt}(u_2) \rangle$. Thus we conclude

$$\begin{aligned} &e_1 q^{2k_1} u_2 \equiv \tilde{e}_1 u_2 \pmod{L_2} && \text{if } \langle k_1, \mathrm{wt}(u_2) \rangle = 0, \\ &e_1 q^{2k_1} u_2 \in q L_2 && \text{if } \langle k_1, \mathrm{wt}(u_2) \rangle > 0. \end{aligned} \tag{2.6}$$

Hence $L_1 \otimes L_2$ is stable under $\Delta(\tilde{e}_{\overline{1}})$.

Similarly, one can show that $f_1 q^{k_1+k_2-1} L_2 \subset L_2$, which implies $L_1 \otimes L_2$ is stable under $\Delta(\tilde{f}_{\overline{1}})$. Thus we have shown that $L_1 \otimes L_2$ is stable under the Kashiwara operators.

We shall prove the tensor product rule. To prove the $\tilde{e}_{\overline{1}}$-case, let $u_1 \in l_{b_1}$, $u_2 \in l_{b_2}$, and consider the following three cases separately.

**Case 1:** $\langle k_1, \mathrm{wt}(b_2) \rangle = \langle k_2, \mathrm{wt}(b_2) \rangle = 0$. By the comultiplication formula, we have

$$\Delta(\tilde{e}_{\overline{1}})(u_1 \otimes u_2) = \tilde{e}_{\overline{1}} u_1 \otimes u_2 \pm u_1 \otimes \tilde{e}_{\overline{1}} u_2 - (1-q^2)\tilde{k}_{\overline{1}} u_1 \otimes e_1 u_2.$$

Since $\langle k_2, \mathrm{wt}(b_2) + \alpha_1 \rangle = \langle k_2, \mathrm{wt}(b_2) + \varepsilon_1 - \varepsilon_2 \rangle = -1 < 0$, we must have $\tilde{e}_{\overline{1}} u_2 = e_1 u_2 = 0$. Hence $\Delta(\tilde{e}_{\overline{1}})(u_1 \otimes u_2) = \tilde{e}_{\overline{1}} u_1 \otimes u_2$.

If $\tilde{e}_{\bar{1}} = 0$ on $l_{b_1}$, then $\tilde{e}_{\bar{1}} \otimes 1 = 0$ on $l_{b_1} \otimes l_{b_2}$. If $\tilde{e}_{\bar{1}} \colon l_{b_1} \to l_{\tilde{e}_{\bar{1}} b_1}$ is an isomorphism, then $\tilde{e}_{\bar{1}} \otimes 1 \colon l_{b_1} \otimes l_{b_2} \to l_{\tilde{e}_{\bar{1}} b_1} \otimes l_{b_2}$ is also an isomorphism as desired.

**Case 2:** $\langle k_1, \mathrm{wt}(b_2) \rangle > 0$. By the comultiplication formula and (2.6), we have

$$\begin{aligned}\Delta(\tilde{e}_{\bar{1}})(u_1 \otimes u_2) &= \tilde{e}_{\bar{1}} u_1 \otimes q^{\langle k_1 + k_2, \mathrm{wt}(b_2) \rangle} u_2 \pm u_1 \otimes \tilde{e}_{\bar{1}} u_2 - (1-q^2) \tilde{k}_{\bar{1}} u_1 \otimes e_1 q^{2k_1} u_2 \\ &\equiv \pm u_1 \otimes \tilde{e}_{\bar{1}} u_2 \pmod{q L_1 \otimes L_2}.\end{aligned}$$

**Case 3:** $\langle k_1, \mathrm{wt}(b_2) \rangle = 0$ and $\langle k_2, \mathrm{wt}(b_2) \rangle > 0$. The comultiplication formula and (2.6) yield

$$\begin{aligned}\Delta(\tilde{e}_{\bar{1}})(u_1 \otimes u_2) &= \tilde{e}_{\bar{1}} u_1 \otimes q^{\langle k_1 + k_2, \mathrm{wt}(b_2) \rangle} u_2 \pm u_1 \otimes \tilde{e}_{\bar{1}} u_2 - (1-q^2) \tilde{k}_{\bar{1}} u_1 \otimes e_1 q^{2k_1} u_2 \\ &\equiv \pm u_1 \otimes \tilde{e}_{\bar{1}} u_2 - \tilde{k}_{\bar{1}} u_1 \otimes e_1 u_2 \pmod{q L_1 \otimes L_2}.\end{aligned}$$

Since $\langle k_1, \mathrm{wt}(b_2) \rangle = 0$ and $\tilde{k}_{\bar{1}}^2 = (1-q^4)^{-1}(1-q^{4k_1})$, we have

$$k_{\bar{1}} u_2 = 0, \qquad \tilde{k}_{\bar{1}}^2 e_1 u_2 = \frac{1-q^{4k_1}}{1-q^4} e_1 u_2 = e_1 u_2.$$

It follows that

$$\tilde{e}_{\bar{1}} u_2 = -q^{-1}(e_1 k_{\bar{1}} - q k_{\bar{1}} e_1) q^{k_1} u_2 = k_{\bar{1}} e_1 q^{k_1} u_2 = k_{\bar{1}} q^{k_1 - 1} e_1 u_2 = \tilde{k}_{\bar{1}} e_1 u_2.$$

Hence we obtain

$$\tilde{k}_{\bar{1}} \tilde{e}_{\bar{1}} u_2 = \tilde{k}_{\bar{1}}^2 e_1 u_2 = e_1 u_2,$$

which implies

$$\Delta(\tilde{e}_{\bar{1}})(u_1 \otimes u_2) \equiv \pm u_1 \otimes \tilde{e}_{\bar{1}} u_2 - \tilde{k}_{\bar{1}} u_1 \otimes \tilde{k}_{\bar{1}} \tilde{e}_{\bar{1}} u_2 \equiv (1 - \tilde{k}_{\bar{1}} \otimes \tilde{k}_{\bar{1}})(1 \otimes \tilde{e}_{\bar{1}})(u_1 \otimes u_2).$$

The operator $1 - \tilde{k}_{\bar{1}} \otimes \tilde{k}_{\bar{1}}$ on $l_{b_1} \otimes l_{\tilde{e}_1 b_2}$ is invertible because $(\tilde{k}_{\bar{1}} \otimes \tilde{k}_{\bar{1}})^2 = -\tilde{k}_{\bar{1}}^2 \otimes \tilde{k}_{\bar{1}}^2 = -(1-q^4)^{-1}(1-q^{4k_1}) \otimes \mathrm{id}$ acts on $l_{b_1 \otimes \tilde{e}_{\bar{1}} b_2}$ as multiplication by a scalar different from 1. Hence the map $\Delta(\tilde{e}_{\bar{1}}) \colon l_{b_1} \otimes l_{b_2} \to l_{b_1} \otimes l_{\tilde{e}_{\bar{1}} b_2}$ is either 0 or an isomorphism according to whether $\tilde{e}_{\bar{1}} b_2 = 0$ or not.

The assertions on $\tilde{f}_{\bar{1}}$ can be verified in a similar manner. The remaining property (b)(ii) in Definition 2.2 follows immediately from formula (2.5). □

Motivated by the properties of crystal bases, we introduce the notion of abstract crystals.

**Definition 2.8.** An *abstract* $\mathfrak{q}(n)$-*crystal* is a $\mathfrak{gl}(n)$-crystal together with two maps $\tilde{e}_{\bar{1}}, \tilde{f}_{\bar{1}} \colon B \to B \sqcup \{0\}$ satisfying the following conditions:

(a) $\mathrm{wt}(B) \subset P^{\geq 0}$,
(b) $\mathrm{wt}(\tilde{e}_{\bar{1}} b) = \mathrm{wt}(b) + \alpha_1$, $\mathrm{wt}(\tilde{f}_{\bar{1}} b) = \mathrm{wt}(b) - \alpha_1$,
(c) for all $b, b' \in B$, $\tilde{f}_{\bar{1}} b = b'$ if and only if $b = \tilde{e}_{\bar{1}} b'$,
(d) if $3 \leq i \leq n-1$, then
  (i) the operators $\tilde{e}_{\bar{1}}$ and $\tilde{f}_{\bar{1}}$ commute with $\tilde{e}_i$ and $\tilde{f}_i$,
  (ii) if $\tilde{e}_{\bar{1}} b \in B$, then $\varepsilon_i(\tilde{e}_{\bar{1}} b) = \varepsilon_i(b)$ and $\varphi_i(\tilde{e}_{\bar{1}} b) = \varphi_i(b)$.

Note that any crystal basis of $U_q(\mathfrak{q}(n))$-modules in $\mathcal{O}_{\mathrm{int}}^{\geq 0}$ has property (d) by Lemma 2.1.

Let $B_1$ and $B_2$ be abstract $\mathfrak{q}(n)$-crystals. The *tensor product* $B_1 \otimes B_2$ of $B_1$ and $B_2$ is defined to be the $\mathfrak{gl}(n)$-crystal $B_1 \otimes B_2$ together with the maps $\tilde{e}_{\overline{1}}$, $\tilde{f}_{\overline{1}}$ defined by (2.5). Then it is an abstract $\mathfrak{q}(n)$-crystal.

The following associativity of the tensor product is easily checked.

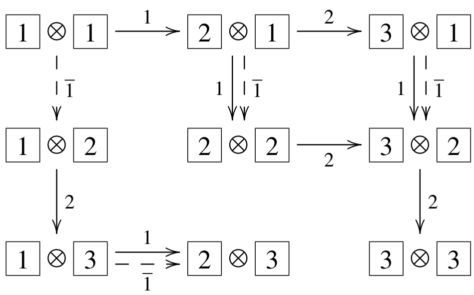
**Proposition 2.9.** *Let $B_1$, $B_2$ and $B_3$ be abstract $\mathfrak{q}(n)$-crystals. Then*

$$(B_1 \otimes B_2) \otimes B_3 \simeq B_1 \otimes (B_2 \otimes B_3).$$

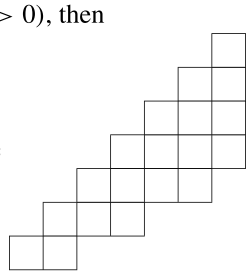
**Example 2.10.** (a) If $(L, B, l_B)$ is a crystal basis of a $U_q(\mathfrak{q}(n))$-module $M$ in the category $\mathcal{O}_{\mathrm{int}}^{\geq 0}$, then $B$ is an abstract $\mathfrak{q}(n)$-crystal.

(b) The crystal graph $\mathbf{B}$ of the vector representation $\mathbf{V}$ is an abstract $\mathfrak{q}(n)$-crystal.

(c) By the tensor product rule, $\mathbf{B}^{\otimes N}$ is an abstract $\mathfrak{q}(n)$-crystal. When $n = 3$, the $\mathfrak{q}(n)$-crystal structure of $\mathbf{B} \otimes \mathbf{B}$ is given below.



(d) For a strict partition $\lambda = (\lambda_1 > \cdots > \lambda_r > 0)$, let $Y_\lambda$ be the skew Young diagram having $\lambda_1$ boxes on the principal diagonal, $\lambda_2$ boxes on the second diagonal, etc. For example, if $\lambda = (7 > 6 > 4 > 2 > 0)$, then

$Y_\lambda =$  .

Let $\mathbf{B}(Y_\lambda)$ be the set of all semistandard tableaux of shape $Y_\lambda$ with entries from $1, \ldots, n$. Then by an *admissible reading* introduced in [1], $\mathbf{B}(Y_\lambda)$ can be embedded in $\mathbf{B}^{\otimes N}$, where $N = \lambda_1 + \cdots + \lambda_r$, and it is stable under the Kashiwara operators $\tilde{e}_i$, $\tilde{f}_i$ $(i = 1, \ldots, n-1, \overline{1})$. Hence it becomes an abstract $\mathfrak{q}(n)$-crystal. Moreover, the $\mathfrak{q}(n)$-crystal structure thus obtained does not depend on the choice of admissible reading.

Indeed, since $Y_\lambda$ is a skew Young diagram, it is stable under the even Kashiwara operators, and the $\mathfrak{gl}(n)$-crystal structure does not depend on the choice of admissible reading. Let $T$ be a semistandard tableau of shape $\lambda$ and let $\beta$ be the lowest box with entry 1 on the principal diagonal of $T$. Since a box with entry 1 must lie on the principal diagonal of $T$, every box with entry 1 except $\beta$ lies northeast of $\beta$. Let $\psi : \mathbf{B}(Y_\lambda) \to \mathbf{B}^{\otimes N}$

be an admissible reading. It follows that $\beta$ is the rightmost box with entry 1 in $\psi(T)$. If there is a box, say $\gamma$, with entry 2 southwest of $\beta$ in $T$, then $\gamma$ must appear after $\beta$ in $\psi(T)$. Thus we get $\tilde{f}_{\overline{1}}(\psi(T)) = 0$. If there is no box with entry 2 southwest of $\beta$ in $T$, then we know that every box with entry 2 must lie northeast of $\beta$ in $T$, and hence there is no box with entry 2 after $\beta$ in $\psi(T)$. Thus $\tilde{f}_{\overline{1}}$ acts on $\beta$. Since the entry of the right box of $\beta$ in $T$ is greater than or equal to 2, we have $\tilde{f}_{\overline{1}}(\psi(T)) = \psi(T')$, where $T'$ is the semistandard tableau of shape $\lambda$ obtained from $T$ by replacing the entry of $\beta$ from 1 to 2. It follows that $\mathbf{B}(Y_\lambda)$ is stable under the action of $\tilde{f}_{\overline{1}}$ and it does not depend on the choice of admissible reading.

Let $\delta$ be the leftmost box with entry 2 in $T$. If $\delta$ lies on the second diagonal, the entry of the box lying to the left of $\delta$ must be 1. Then, for any admissible reading $\psi$, $\tilde{e}_{\overline{1}}\psi(T) = 0$. Thus we may assume that $\delta$ lies on the principal diagonal of $T$, and our assertion on $\tilde{e}_{\overline{1}}$ follows from similar arguments to those above.

In Figure 1, we illustrate the crystal $\mathbf{B}(Y_\lambda)$ for $n = 3$ and $\lambda = (3 > 1 > 0)$. Note that it is connected. However, in general, $\mathbf{B}(Y_\lambda)$ is not connected.

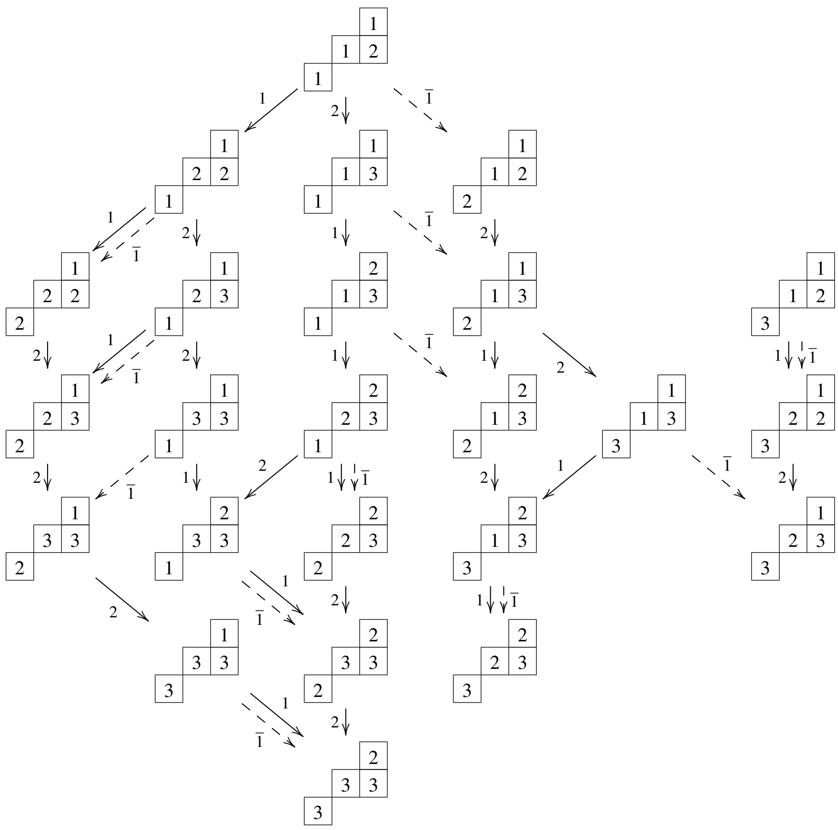


**Fig. 1.** $\mathbf{B}(Y_\lambda)$ for $n = 3$, $\lambda = (3 > 1 > 0)$.

Let $B$ be an abstract $\mathfrak{q}(n)$-crystal. For $i = 1, \ldots, n-1$, we define the automorphism $S_i$ on $B$ by

$$S_i b = \begin{cases} \tilde{f}_i^{\langle h_i, \mathrm{wt}(b)\rangle} b & \text{if } \langle h_i, \mathrm{wt}(b)\rangle \geq 0, \\ \tilde{e}_i^{-\langle h_i, \mathrm{wt}(b)\rangle} b & \text{if } \langle h_i, \mathrm{wt}(b)\rangle \leq 0. \end{cases} \tag{2.7}$$

Let $w$ be an element of the Weyl group $W$ of $\mathfrak{gl}(n)$. Then, as shown in [13], there exists a unique action $S_w \colon B \to B$ of $W$ on $B$ such that $S_{s_i} = S_i$ for $i = 1, \ldots, n-1$. Note that $\mathrm{wt}(S_w b) = w(\mathrm{wt}(b))$ for any $w \in W$ and $b \in B$.

For $i = 1, \ldots, n-1$, we set

$$w_i = s_2 \cdots s_i s_1 \cdots s_{i-1}. \tag{2.8}$$

Then $w_i$ is the shortest element in $W$ such that $w_i(\alpha_i) = \alpha_1$. We define the *odd Kashiwara operators* $\tilde{e}_{\bar{i}}$, $\tilde{f}_{\bar{i}}$ $(i = 2, \ldots, n-1)$ by

$$\tilde{e}_{\bar{i}} = S_{w_i^{-1}} \tilde{e}_{\bar{1}} S_{w_i}, \qquad \tilde{f}_{\bar{i}} = S_{w_i^{-1}} \tilde{f}_{\bar{1}} S_{w_i}.$$

We say that $b \in B$ is a *highest weight vector* if $\tilde{e}_i b = \tilde{e}_{\bar{i}} b = 0$ for all $i = 1, \ldots, n-1$.

**Remark 2.11.** These actions can be lifted to actions on $U_q(\mathfrak{q}(n))$-modules. Let $M$ be a $U_q(\mathfrak{q}(n))$-module in $\mathcal{O}_{\mathrm{int}}^{\geq 0}$. For each $i = 1, \ldots, n-1$, we have

$$M = \bigoplus_{\substack{\ell \geq k \geq 0 \\ \lambda \in P,\ \langle h_i, \lambda\rangle = \ell}} f_i^{(k)} (\mathrm{Ker}\,(e_i)_\lambda).$$

Hence we can define the endomorphism $S_i$ of $M$ by

$$S_i(f_i^{(k)} u) = f_i^{(\ell-k)} u \quad \text{for } u \in \mathrm{Ker}\,(e_i)_\lambda. \tag{2.9}$$

Then $S_i^2 = \mathrm{id}_M$ and $S_i(M_\lambda) = M_{s_i\lambda}$. If $(L, B, l_B)$ is a crystal basis of $M$, then $L$ is stable under $S_i$, and $S_i$ induces an action on $L$ and $L/qL$. Obviously, $S_i(l_b) = l_{S_i b}$ for $b \in B$, where $S_i b$ is defined in (2.7). We define the endomorphisms $\tilde{e}_{\bar{i}}$ and $\tilde{f}_{\bar{i}}$ of $M$ by

$$\begin{aligned} \tilde{e}_{\bar{i}} &= (S_2 \cdots S_i S_1 \cdots S_{i-1})^{-1} \circ \tilde{e}_{\bar{1}} \circ (S_2 \cdots S_i S_1 \cdots S_{i-1}), \\ \tilde{f}_{\bar{i}} &= (S_2 \cdots S_i S_1 \cdots S_{i-1})^{-1} \circ \tilde{f}_{\bar{1}} \circ (S_2 \cdots S_i S_1 \cdots S_{i-1}). \end{aligned} \tag{2.10}$$

Then

$$\tilde{e}_{\bar{i}} M_\mu \subset M_{\mu+\alpha_i} \quad \text{and} \quad \tilde{f}_{\bar{i}} M_\mu \subset M_{\mu-\alpha_i} \quad \text{for every } \mu \in P^{\geq 0}.$$

Let $(L, B, l_B)$ be a crystal basis of $M$. Then $L$ is stable under the action of $\tilde{e}_{\bar{i}}$, and $\tilde{e}_{\bar{i}}$ induces an action on $L/qL$, and we have

$$\begin{cases} \text{(i) if } \tilde{e}_{\bar{i}} b \neq 0, \text{ then } \tilde{e}_{\bar{i}} \text{ induces an isomorphism } l_b \xrightarrow{\sim} l_{\tilde{e}_{\bar{i}} b}, \\ \text{(ii) if } \tilde{e}_{\bar{i}} b = 0, \text{ then } \tilde{e}_{\bar{i}}(l_b) = 0. \end{cases}$$

Similar properties hold for $\tilde{f}_{\bar{i}}$. Note that

$$\begin{aligned} \mathrm{Ker}(\tilde{e}_{\bar{i}} \colon L/qL \to L/qL) &= \mathrm{Ker}(\tilde{e}_{\bar{1}} S_{w_i}) = S_{w_i}^{-1}(\mathrm{Ker}\,\tilde{e}_{\bar{1}}) = S_{w_i^{-1}}(\mathrm{Ker}\,\tilde{e}_{\bar{1}}) \\ &= S_{w_i^{-1}}\Big(\bigoplus_{\tilde{e}_{\bar{1}} b = 0} l_b\Big) = \bigoplus_{\tilde{e}_{\bar{1}} b = 0} l_{S_{w_i^{-1}} b} = \bigoplus_{\tilde{e}_{\bar{1}} S_{w_i} b = 0} l_b = \bigoplus_{\tilde{e}_{\bar{i}} b = 0} l_b. \end{aligned}$$

**Example 2.12.** Let $\lambda$ be a strict partition. Observe that $\mathbf{B}(Y_\lambda)$ has a unique element of weight $\lambda$, say $b_{Y_\lambda}$. Since $\lambda + \alpha_i \notin \mathrm{wt}(\mathbf{B}(Y_\lambda))$ for any $i = 1, \dots, n-1$, $b_{Y_\lambda}$ is a highest weight vector. Thus, for each admissible reading $\psi$, $\psi(b_{Y_\lambda})$ is a highest weight vector in $\mathbf{B}^{\otimes N}$.

**Lemma 2.13.** *Every abstract* $\mathfrak{q}(n)$*-crystal contains a highest weight vector.*

*Proof.* Recall that $\lambda \in \mathrm{wt}(B) := \{\mathrm{wt}(b);\ b \in B\}$ is called *maximal* if $\lambda + \alpha_i \notin \mathrm{wt}(B)$ for $i = 1, \dots, n-1$. Since $\mathrm{wt}(\tilde{e}_{\bar{i}} b) = \mathrm{wt}(b) + \alpha_i$, a vector in a crystal $B$ with a maximal weight is a highest weight vector. Because $\mathrm{wt}(B)$ is a finite set, there exists a maximal element $\lambda$ so that we have an element $b \in B$ with a maximal weight $\lambda$. $\square$

**Remark 2.14.** (a) Let $\lambda$ be a strict partition with $\ell(\lambda) = r$ and let $M$ be a highest weight module of highest weight $\lambda$ in $\mathcal{O}_{\mathrm{int}}^{\geq 0}$. Set $\tilde{k}_{\bar{i}} = q^{k_i - 1} k_{\bar{i}}$ for $i = 1, \dots, n$. Since $M \in \mathcal{O}_{\mathrm{int}}^{\geq 0}$, we have $\tilde{k}_{\bar{i}} = 0$ on $M_\lambda$ for $i > r$. Note that $\tilde{k}_{\bar{i}}^2 = \frac{1-q^{4\lambda_i}}{1-q^4}$ on $M_\lambda$ and $\left(\frac{1-q^{4\lambda_i}}{1-q^4}\right)^{-1/2} \in \mathbf{A} \subset \mathbf{F}$. Let

$$C_i := \left(\frac{1-q^{4\lambda_i}}{1-q^4}\right)^{-1/2} \tilde{k}_{\bar{i}}.$$

Then on $M_\lambda$, we have

$$C_i^2 = 1, \qquad C_i C_j + C_j C_i = 0 \quad (i \neq j). \tag{2.11}$$

Thus $M_\lambda$ can be regarded as a module over $\mathbf{F}[C_1, \dots, C_r]$, where $\mathbf{F}[C_1, \dots, C_r]$ is the associative $\mathbf{F}$-algebra generated by $\{C_i;\ i = 1, \dots, r\}$ with the defining relations (2.11).

(b) Let $\mathbb{C}[C_1, \dots, C_r]$ and $\mathbf{A}[C_1, \dots, C_r]$ be the associative $\mathbb{C}$-algebra and $\mathbf{A}$-algebra, respectively, generated by $\{C_i;\ i = 1, \dots, r\}$ with the defining relations (2.11). For a superring $R$, we define $\mathrm{Mod}(R)$ and $\mathrm{S\text{-}Mod}(R)$ to be the category of $R$-modules and the category of $R$-supermodules, respectively.

If $r$ is odd, then we have the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Mod}(\mathbf{A}) & \xrightarrow{\sim} & \mathrm{S\text{-}Mod}(\mathbf{A}[C_1, \dots, C_r]) \\ \Big\downarrow{\scriptstyle \mathbf{F}\otimes_{\mathbf{A}}(-)} & & \Big\downarrow{\scriptstyle \mathbf{F}\otimes_{\mathbf{A}}(-)} \\ \mathrm{Mod}(\mathbf{F}) & \xrightarrow{\sim} & \mathrm{S\text{-}Mod}(\mathbf{F}[C_1, \dots, C_r]) \end{array}$$

If $r$ is even, then we have the following commutative diagram:

$$\begin{array}{ccc} \mathrm{S\text{-}Mod}(\mathbf{A}) & \xrightarrow{\sim} & \mathrm{S\text{-}Mod}(\mathbf{A}[C_1, \dots, C_r]) \\ \Big\downarrow{\scriptstyle \mathbf{F}\otimes_{\mathbf{A}}(-)} & & \Big\downarrow{\scriptstyle \mathbf{F}\otimes_{\mathbf{A}}(-)} \\ \mathrm{S\text{-}Mod}(\mathbf{F}) & \xrightarrow{\sim} & \mathrm{S\text{-}Mod}(\mathbf{F}[C_1, \dots, C_r]) \end{array}$$

In both cases, the horizontal arrows are given by

$$K \mapsto V \otimes_{\mathbb{C}} K$$

for each module $K$ in the left hand side, where $V$ denotes an irreducible supermodule over $\mathbb{C}[C_1, \dots, C_r]$.

To summarize, we obtain the following proposition.

**Proposition 2.15.** (a) *For a strict partition* $\lambda \in \Lambda^+$ *with* $\ell(\lambda) = r$, *let* $\mathrm{HT}(\lambda)$ *be the category of highest weight modules with highest weight* $\lambda$ *in* $\mathcal{O}^{\geq 0}_{\mathrm{int}}$. *Then* $\mathrm{HT}(\lambda)$ *is equivalent to* $\text{S-Mod}(\mathbf{F}[C_1, \dots, C_r])$, *where the equivalence is given by*

$$\mathrm{HT}(\lambda) \ni M \mapsto M_\lambda \in \text{S-Mod}(\mathbf{F}[C_1, \dots, C_r]).$$

*In particular, the homomorphism* $\mathrm{End}_{U_q(\mathfrak{q}(n))}(M) \to \mathrm{End}_{\mathbf{F}[C_1,\dots,C_r]}(M_\lambda)$ *is an isomorphism for any* $M \in \mathrm{HT}(\lambda)$.

(b) *For a* $U_q(\mathfrak{q}(n))$*-module* $M \in \mathrm{HT}(\lambda)$, *let* $L$, $L'$ *be finitely generated free* $\mathbf{A}$*-submodules of* $M_\lambda$ *such that*

(i) $L$ *and* $L'$ *are stable under* $\tilde{k}_{\bar{i}}$*'s* $(i = 1, \dots, n)$,
(ii) $\mathbf{F} \otimes_{\mathbf{A}} L \simeq \mathbf{F} \otimes_{\mathbf{A}} L' \simeq M_\lambda$.

*Then there exists a* $U_q(\mathfrak{q}(n))$*-module automorphism* $\varphi$ *of* $M$ *such that* $\varphi L = L'$.

## 3. Highest weight vectors in $\mathbf{B}^{\otimes N}$

In this section, we will give algebraic and combinatorial characterizations of highest weight vectors in the abstract $\mathfrak{q}(n)$-crystal $\mathbf{B}^{\otimes N}$.

**Definition 3.1.** Let $B$ be an abstract $\mathfrak{q}(n)$-crystal.

(i) An element $b \in B$ is called a $\mathfrak{gl}(a)$*-highest weight vector* if $\tilde{e}_i b = 0$ for $1 \leq i < a \leq n$.
(ii) An element $b \in B$ is called a $\mathfrak{q}(a)$*-highest weight vector* if $\tilde{e}_i b = \tilde{e}_{\bar{i}} b = 0$ for $1 \leq i < a \leq n$.

In particular, a highest weight vector in $B$ is a $\mathfrak{q}(n)$-highest weight vector.

From now on, we denote by $\bigotimes_{j \geq m \geq i} (r_1\, r_2 \cdots r_m)^{\otimes y_m}$ the following vector in $\mathbf{B}^{\otimes N}$:

$$\underbrace{(r_1 \otimes \cdots \otimes r_j) \otimes \cdots \otimes (r_1 \otimes \cdots \otimes r_j)}_{y_j \text{ times}} \otimes \underbrace{(r_1 \otimes \cdots \otimes r_{j-1}) \otimes \cdots \otimes (r_1 \otimes \cdots \otimes r_{j-1})}_{y_{j-1} \text{ times}} \otimes$$
$$\cdots \otimes \underbrace{(r_1 \otimes \cdots \otimes r_{i+1}) \otimes \cdots \otimes (r_1 \otimes \cdots \otimes r_{i+1})}_{y_{i+1} \text{ times}} \otimes \underbrace{(r_1 \otimes \cdots \otimes r_i) \otimes \cdots \otimes (r_1 \otimes \cdots \otimes r_i)}_{y_i \text{ times}},$$

where $N = \sum_{m=i}^{j} m y_m$.

Let $b$ be an element of a $\mathfrak{gl}(n)$-crystal $B$. We denote by $C(b)$ the connected component of $B$ containing $b$.

**Definition 3.2.** Let $B_i$ be a $\mathfrak{gl}(n)$-crystal and let $b_i \in B_i$ $(i = 1, 2)$. We say that $b_1$ is $\mathfrak{gl}(n)$*-crystal equivalent* to $b_2$ if there exists an isomorphism of $\mathfrak{gl}(n)$-crystals $C(b_1) \xrightarrow{\sim} C(b_2)$ sending $b_1$ to $b_2$.

Recall that $w_i = s_2 \cdots s_i s_1 \cdots s_{i-1}$.

**Lemma 3.3.** *Let $B$ be a $\mathfrak{gl}(n)$-crystal.*

(a) *A vector $b_0$ in $\mathbf{B} \otimes B$ is a $\mathfrak{gl}(n)$-highest weight vector if and only if $b_0 = 1 \otimes \tilde{f}_1 \cdots \tilde{f}_{j-1} b$ for some $j \in \{1, \dots, n\}$ and some $\mathfrak{gl}(n)$-highest weight vector $b \in B$ such that $\mathrm{wt}(b_0) = \mathrm{wt}(b) + \varepsilon_j$ is a partition.*

(b) *Let $b$ be a $\mathfrak{gl}(n)$-highest weight vector in $B$ and $j \in \{1, \dots, n\}$. If $\mathrm{wt}(b) + \varepsilon_j$ is a partition, then $b_0 = 1 \otimes \tilde{f}_1 \cdots \tilde{f}_{j-1} b$ is a $\mathfrak{gl}(n)$-highest weight vector in $\mathbf{B} \otimes B$ and we have*

$$S_{w_i} b_0 = \begin{cases} 3 \otimes \tilde{f}_3 \cdots \tilde{f}_{j+1} S_{w_i} b & \text{if } j+1 \le i < n, \\ 1 \otimes S_{w_i} b & \text{if } i = j, \\ 1 \otimes \tilde{f}_1 S_{w_i} b & \text{if } i = j-1, \end{cases}$$

*and*

$$S_{u_i} b_0 = 1 \otimes \tilde{f}_1 \tilde{f}_2 S_{u_i} b' \quad \text{if } i \le j-2,$$

*where $z_i = s_3 s_4 \cdots s_{i+1}$, $u_i = z_i w_i$ and $b' = \tilde{f}_{i+2} \cdots \tilde{f}_{j-1} b$.*

*Proof.* (a) For a partition $\lambda$, let us denote by $B_{\mathfrak{gl}(n)}(\lambda)$ the crystal graph of the highest weight $\mathfrak{gl}(n)$-module with highest weight $\lambda$. It is enough to show that the assertion holds for $B = B_{\mathfrak{gl}(n)}(\lambda)$ for any partition $\lambda$.

Let $b_0 = 1 \otimes \tilde{f}_1 \cdots \tilde{f}_{j-1} b$ for some $\mathfrak{gl}(n)$-highest weight vector $b \in B$ such that $\mathrm{wt}(b_0)$ is a partition. Since any two $\mathfrak{gl}(n)$-highest weight vectors with the same highest weight are $\mathfrak{gl}(n)$-crystal equivalent, by embedding $B$ into $\mathbf{B}^{\otimes N}$ for some $N$, we may assume that $b = \bigotimes_{n \ge m \ge 1} (12 \cdots m)^{\otimes x_m}$, where $x_m = \langle k_m - k_{m+1}, \mathrm{wt}(b) \rangle$ for $1 \le m \le n-1$. Since $\mathrm{wt}(b) + \varepsilon_j = \mathrm{wt}(b_0)$ is a partition, we have $x_{j-1} \ge 1$. Thus

$$\begin{aligned} & 1 \otimes \tilde{f}_1 \tilde{f}_2 \cdots \tilde{f}_{j-1} b \\ & \quad = 1 \otimes \bigotimes_{m \ge j} (1 \cdots m)^{\otimes x_m} \otimes (23 \cdots j) \otimes \bigotimes_{j-1 \ge m \ge 1} (1 \cdots m)^{\otimes (x_m - \delta_{m,j-1})}, \end{aligned} \tag{3.1}$$

which is a $\mathfrak{gl}(n)$-highest weight vector in $\mathbf{B} \otimes B$. Since

$$\mathbf{B} \otimes B \simeq \bigoplus_{\lambda + \varepsilon_j : \text{partition}} B_{\mathfrak{gl}(n)}(\lambda + \varepsilon_j),$$

the number of highest weight vectors in $\mathbf{B} \otimes B$ is the same as the number of vectors of the form in (3.1).

(b) We may assume that $b = \bigotimes_{n \ge m \ge 1} (12 \cdots m)^{\otimes x_m}$ as above. Then by (3.1),

$$b_0 = 1 \otimes \bigotimes_{m \ge j} (12 \cdots m)^{\otimes x_m} \otimes (23 \cdots j) \otimes \bigotimes_{j-1 \ge m \ge 1} (12 \cdots m)^{\otimes (x_m - \delta_{m,j-1})}. \tag{3.2}$$

We also have

$$S_{w_i} b = \bigotimes_{m \ge i+1} (12 \cdots m)^{\otimes x_m} \otimes (134 \cdots i+1)^{\otimes x_i} \otimes \bigotimes_{i-1 \ge m \ge 1} (34 \cdots m+2)^{\otimes x_m}. \tag{3.3}$$

Here we have used the following facts:

(1) For $w \in W$ and $\mathfrak{gl}(n)$-highest weight vectors $b_1$ and $b_2$,

$$S_w(b_1 \otimes b_2) = S_w b_1 \otimes S_w b_2.$$

(2) Suppose that $0 < a_1 < \cdots < a_r \le n$, $0 < x_1 < \cdots < x_r \le n$ and $w(\{a_1, \ldots a_r\}) = \{x_1, \ldots x_r\}$. Then

$$S_w(a_1 \otimes \cdots \otimes a_r) = x_1 \otimes \cdots \otimes x_r.$$

**Case 1:** $j + 1 \le i < n$. From (3.2), we have

$$\begin{aligned} S_{w_i} b_0 = 3 &\otimes \bigotimes_{m \ge i+1} (12 \cdots m)^{\otimes x_m} \otimes (134 \cdots i+1)^{\otimes x_i} \\ &\otimes \bigotimes_{i-1 \ge m \ge j} (34 \cdots m+2)^{\otimes x_m} \otimes (45 \cdots j+2) \\ &\otimes \bigotimes_{j-1 \ge m \ge 1} (3 \cdots m+2)^{\otimes (x_m - \delta_{m,j-1})}. \end{aligned}$$

On the other hand, from (3.3), we have

$$\begin{aligned} \tilde{f}_3 \cdots \tilde{f}_{j+1} S_{w_i} b = &\bigotimes_{m \ge i+1} (12 \cdots m)^{\otimes x_m} \otimes (134 \cdots i+1)^{\otimes x_i} \\ &\otimes \bigotimes_{i-1 \ge m \ge j} (34 \cdots m+2)^{\otimes x_m} \otimes (45 \cdots j+2) \\ &\otimes \bigotimes_{j-1 \ge m \ge 1} (34 \cdots m+2)^{\otimes (x_m - \delta_{m,j-1})}. \end{aligned}$$

Thus we get $S_{w_i} b_0 = 3 \otimes \tilde{f}_3 \cdots \tilde{f}_{j+1} S_{w_i} b$.

**Case 2:** $i = j$. From (3.2) and (3.3), we have

$$\begin{aligned} S_{w_i} b_0 &= S_{w_i} \Big( 1 \otimes \bigotimes_{m \ge j} (1 \cdots m)^{\otimes x_m} \otimes (2 \cdots j) \otimes \bigotimes_{j-1 \ge m \ge 1} (1 \cdots m)^{\otimes (x_m - \delta_{m,j-1})} \Big) \\ &= S_2 \cdots S_j \Big( 1 \otimes \bigotimes_{m \ge j} (1 \cdots m)^{\otimes x_m} \otimes \bigotimes_{j-1 \ge m \ge 1} (2 \cdots m+1)^{\otimes x_m} \Big) \\ &= 1 \otimes \bigotimes_{m \ge j+1} (1 \cdots m)^{\otimes x_m} \otimes (134 \cdots j+1)^{\otimes x_j} \otimes \bigotimes_{j-1 \ge m \ge 1} (3 \cdots m+2)^{\otimes x_m} \\ &= 1 \otimes S_{w_i} b. \end{aligned}$$

**Case 3:** $i = j - 1$. From (3.2), we have

$$\begin{aligned} S_{w_i} b_0 = S_2 \cdots S_{j-1} &\Big( 1 \otimes \bigotimes_{m \ge j} (1 \cdots m)^{\otimes x_m} \otimes (2 \cdots j) \otimes (12 \cdots j-1)^{\otimes (x_{j-1} - 1)} \\ &\otimes \bigotimes_{j-2 \ge m \ge 1} (2 \cdots m+1)^{\otimes x_m} \Big) \\ = 1 &\otimes \bigotimes_{m \ge j} (1 \cdots m)^{\otimes x_m} \otimes (2 \cdots j) \otimes (134 \cdots j)^{\otimes (x_{j-1} - 1)} \\ &\otimes \bigotimes_{j-2 \ge m \ge 1} (3 \cdots m+2)^{\otimes x_m}. \end{aligned}$$

On the other hand, from (3.3), we have

$$\begin{aligned}
&1 \otimes \tilde{f}_1 S_{w_i} b \\
&= 1 \otimes \tilde{f}_1 \Big( \bigotimes_{m \geq j} (1 \cdots m)^{\otimes x_m} \otimes (134 \cdots j)^{\otimes x_{j-1}} \otimes \bigotimes_{j-2 \geq m \geq 1} (3 \cdots m+2)^{\otimes x_m} \Big) \\
&= 1 \otimes \bigotimes_{m \geq j} (1 \cdots m)^{\otimes x_m} \otimes (2 \cdots j) \otimes (134 \cdots j)^{\otimes (x_{j-1}-1)} \otimes \bigotimes_{j-2 \geq m \geq 1} (3 \cdots m+2)^{\otimes x_m}.
\end{aligned}$$

Hence we get $S_{w_i} b_0 = 1 \otimes \tilde{f}_1 S_{w_i} b$.

**Case 4:** $i \leq j-2$. Note that

$$u_i(m) = \begin{cases} m+3, & 1 \leq m < i, \\ 1, & m = i, \\ 2, & m = i+1, \\ 3, & m = i+2, \\ m, & m \geq i+3. \end{cases}$$

We have

$$\begin{aligned}
S_{u_i} b_0 &= S_3 \cdots S_{i+1} \Big( 1 \otimes \bigotimes_{m \geq j} (1 \cdots m)^{\otimes x_m} \otimes (2 \cdots j) \otimes \bigotimes_{j-1 \geq m \geq i+1} (1 \cdots m)^{\otimes (x_m - \delta_{m,j-1})} \\
&\qquad \otimes (13 \cdots i+1)^{\otimes x_i} \otimes \bigotimes_{i-1 \geq m \geq 1} (3 \cdots m+2)^{\otimes x_m} \Big) \\
&= 1 \otimes \bigotimes_{m \geq j} (1 \cdots m)^{\otimes x_m} \otimes (2 \cdots j) \otimes \bigotimes_{j-1 \geq m \geq i+2} (1 \cdots m)^{\otimes (x_m - \delta_{m,j-1})} \\
&\quad \otimes (124 \cdots i+2)^{\otimes x_{i+1}} \otimes (14 \cdots i+2)^{\otimes x_i} \otimes \bigotimes_{i-1 \geq m \geq 1} (4 \cdots m+3)^{\otimes x_m}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\tilde{f}_1 \tilde{f}_2 (S_{u_i} b') &= \tilde{f}_1 \tilde{f}_2 S_{u_i} \Big( \bigotimes_{m \geq j} (1 \cdots m)^{\otimes x_m} \otimes (1 \cdots i+1\ i+3 \cdots j) \\
&\qquad \otimes \bigotimes_{j-1 \geq m \geq 1} (1 \cdots m)^{\otimes (x_m - \delta_{m,j-1})} \Big) \\
&= \tilde{f}_1 \tilde{f}_2 \Big( \bigotimes_{m \geq j} (1 \cdots m)^{\otimes x_m} \otimes (124 \cdots j) \otimes \bigotimes_{j-1 \geq m \geq i+2} (1 \cdots m)^{\otimes (x_m - \delta_{m,j-1})} \\
&\qquad \otimes (124 \cdots i+2)^{\otimes x_{i+1}} \otimes (14 \cdots i+2)^{\otimes x_i} \otimes \bigotimes_{i-1 \geq m \geq 1} (4 \cdots m+3)^{\otimes x_m} \Big) \\
&= \bigotimes_{m \geq j} (1 \cdots m)^{\otimes x_m} \otimes (234 \cdots j) \otimes \bigotimes_{j-1 \geq m \geq i+2} (1 \cdots m)^{\otimes (x_m - \delta_{m,j-1})} \\
&\quad \otimes (124 \cdots i+2)^{\otimes x_{i+1}} \otimes (14 \cdots i+2)^{\otimes x_i} \otimes \bigotimes_{i-1 \geq m \geq 1} (4 \cdots m+3)^{\otimes x_m}.
\end{aligned}$$

Thus, we obtain $S_{u_i} b_0 = 1 \otimes \tilde{f}_1 \tilde{f}_2 S_{u_i} b'$. □

**Lemma 3.4.** *Assume that* $b \in \mathbf{B}^{\otimes N}$ *satisfies* $\tilde{f}_1 b \neq 0$ *and* $\tilde{e}_{\bar{1}} \tilde{f}_1 b = 0$. *Then* $\tilde{e}_{\bar{1}} b = 0$.

*Proof.* If $b$ does not contain 2, then it is trivial. Assume that $b$ contains 2 and $\tilde{e}_{\bar{1}} b \neq 0$. Then we can write $b = b_1 \otimes 2 \otimes b_2$ such that $b_2$ contains neither 1 nor 2. Since $\tilde{f}_1 b \neq 0$, we have $\tilde{f}_1 b = (\tilde{f}_1 b_1) \otimes 2 \otimes b_2$ and $\tilde{f}_1 b_1 \neq 0$. Therefore, $\tilde{e}_{\bar{1}} \tilde{f}_1 b = (\tilde{f}_1 b_1) \otimes 1 \otimes b_2$ does not vanish, which is a contradiction. □

**Theorem 3.5.** *Suppose that* $b$ *is a* $\mathfrak{gl}(n)$*-highest weight vector in* $\mathbf{B}^{\otimes(N-1)}$ *and* $b_0 = 1 \otimes \tilde{f}_1 \cdots \tilde{f}_{j-1} b$ *is a highest weight vector in* $\mathbf{B}^{\otimes N}$. *Then* $b$ *is a highest weight vector in* $\mathbf{B}^{\otimes(N-1)}$.

*Proof.* We shall prove $\tilde{e}_{\bar{i}} b = 0$ for $1 \leq i < n$.

**Case 1:** $j + 1 \leq i < n$. By Lemma 3.3, we have $S_{w_i} b_0 = 3 \otimes \tilde{f}_3 \cdots \tilde{f}_{j+1} S_{w_i} b$. Since $0 = \tilde{e}_{\bar{1}} S_{w_i} b_0 = \tilde{e}_{\bar{1}} (3 \otimes \tilde{f}_3 \cdots \tilde{f}_{j+1} S_{w_i} b)$, we obtain $\tilde{e}_{\bar{1}} S_{w_i} b = 0$.

**Case 2:** $i = j$. We have $S_{w_i} b_0 = 1 \otimes S_{w_i} b$. Since $0 = \tilde{e}_{\bar{1}} S_{w_i} b_0 = \tilde{e}_{\bar{1}} (1 \otimes S_{w_i} b)$, we get $\tilde{e}_{\bar{1}} S_{w_i} b = 0$.

**Case 3:** $i = j - 1$. Since $S_{w_i} b_0 = 1 \otimes \tilde{f}_1 S_{w_i} b$, we have $\tilde{e}_{\bar{1}} \tilde{f}_1 S_{w_i} b = 0$. Hence Lemma 3.4 implies $\tilde{e}_{\bar{1}} S_{w_i} b = 0$.

**Case 4:** $i \leq j - 2$. Set $b' := \tilde{f}_{i+2} \cdots \tilde{f}_{j-1} b$. Then $\tilde{e}_k b' = 0$ for $k \leq i + 1$. Hence $b'$ is a $\mathfrak{gl}(i+2)$-highest weight vector. Since $u_i^{-1}(\alpha_1)$ and $u_i^{-1}(\alpha_2)$ are positive roots, $S_{u_i} b'$ is a $\mathfrak{gl}(3)$-highest weight vector. Here we have used the fact:

$$\text{if } b \text{ is a } \mathfrak{gl}(n)\text{-highest weight vector and } w^{-1}(\alpha_i) \text{ is a positive root for } w \in W \text{ and } i, \text{ then } \tilde{e}_i S_w b = 0. \tag{3.4}$$

For the same reason, $S_{u_i} b_0$ is a $\mathfrak{gl}(n)$-highest weight vector.

By Lemma 3.3, we have

$$S_{u_i} b_0 = 1 \otimes \tilde{f}_1 \tilde{f}_2 S_{u_i} b'.$$

Since $\tilde{e}_{\bar{1}}$ commutes with $S_3, \ldots S_{n-1}$, $\tilde{e}_{\bar{1}}$ commutes with $S_{z_i}$. Hence

$$\tilde{e}_{\bar{1}} S_{u_i} b_0 = \tilde{e}_{\bar{1}} S_{z_i} S_{w_i} b_0 = S_{z_i} \tilde{e}_{\bar{1}} S_{w_i} b_0 = 0.$$

Since $w_2 u_i = z_i w_{i+1}$, we also have

$$\tilde{e}_{\bar{1}} S_{w_2} S_{u_i} b_0 = \tilde{e}_{\bar{1}} S_{z_i} S_{w_{i+1}} b_0 = S_{z_i} \tilde{e}_{\bar{1}} S_{w_{i+1}} b_0 = 0.$$

Thus $S_{u_i} b_0$ is a $\mathfrak{q}(3)$-highest weight vector. By Lemma 3.6 below, we have $\tilde{e}_{\bar{1}} S_{u_i} b' = 0$. Since $\tilde{e}_{\bar{1}}$ commutes with $S_{z_i}$, we get $\tilde{e}_{\bar{1}} S_{z_i} S_{w_i} b' = S_{z_i} \tilde{e}_{\bar{1}} S_{w_i} b'$, and hence we conclude $\tilde{e}_{\bar{i}} b' = 0$.

On the other hand, $\tilde{e}_{\bar{i}}$ commutes with $\tilde{e}_{j-1} \cdots \tilde{e}_{i+2}$, because $\tilde{e}_k$ ($k \geq i + 2$) commutes with $S_1, \ldots, S_i$ and $\tilde{e}_{\bar{1}}$. Hence $\tilde{e}_{j-1} \cdots \tilde{e}_{i+2}$ commutes with $\tilde{e}_{\bar{i}}$. Since $b = \tilde{e}_{j-1} \cdots \tilde{e}_{i+2} b'$, we obtain $\tilde{e}_{\bar{i}} b = \tilde{e}_{\bar{i}} \tilde{e}_{j-1} \cdots \tilde{e}_{i+2} b' = \tilde{e}_{j-1} \cdots \tilde{e}_{i+2} \tilde{e}_{\bar{i}} b' = 0$. □

**Lemma 3.6.** *Suppose that $b$ is a $\mathfrak{gl}(3)$-highest weight vector in $\mathbf{B}^{\otimes(N-1)}$ and $b_0 = 1 \otimes \tilde{f}_1 \tilde{f}_2 b$ is a $\mathfrak{q}(3)$-highest weight vector in $\mathbf{B}^{\otimes N}$. Then $\tilde{e}_{\bar{1}} b = 0$.*

*Proof.* If $\tilde{e}_{\bar{1}} b \neq 0$, then $b = b_1 \otimes 2 \otimes b_2$, where $b_2$ contains neither 1 nor 2. Since $\tilde{e}_{\bar{1}} b_0 = 0$, we have $\tilde{e}_{\bar{1}} \tilde{f}_1 \tilde{f}_2 b = 0$ and hence Lemma 3.4 implies $\tilde{e}_{\bar{1}} \tilde{f}_2 b = 0$. It follows that $\tilde{f}_2 b = b_1 \otimes 3 \otimes b_2$. Hence $\tilde{e}_{\bar{1}} \tilde{f}_2 b = 0$ implies $\tilde{e}_{\bar{1}} b_1 = 0$. Moreover, $\tilde{f}_2(b_1 \otimes 2 \otimes b_2) = b_1 \otimes 3 \otimes b_2$ implies that $\varphi_2(b_1) = 0$ and $b_2$ does not contain 3. Since $\varepsilon_2(b_1) = 0$, we conclude that $b_1$ is $\mathfrak{gl}(3)$-crystal equivalent to $1^{\otimes x}$ for some positive integer $x$. Thus

$$\begin{aligned} S_2 S_1 b_0 &= S_2 S_1 (1 \otimes \tilde{f}_1 \tilde{f}_2 b) = S_2 S_1 (1 \otimes \tilde{f}_1 b_1 \otimes 3 \otimes b_2) \\ &= S_2 (1 \otimes S_1 b_1 \otimes 3 \otimes b_2) = 1 \otimes \tilde{e}_2 S_2 S_1 b_1 \otimes 3 \otimes b_2. \end{aligned} \tag{3.5}$$

Here the third equality follows from

$$S_1(1 \otimes \tilde{f}_1(1^{\otimes x})) = S_1(1 \otimes 2 \otimes 1^{\otimes(x-1)}) = 1 \otimes 2 \otimes 2^{\otimes(x-1)} = 1 \otimes S_1(1^{\otimes x}),$$

and the last equality follows from

$$S_2(1 \otimes S_1(1^{\otimes x}) \otimes 3) = S_2(1 \otimes 2^{\otimes x} \otimes 3) = 1 \otimes 3^{\otimes(x-1)} \otimes 2 \otimes 3 = 1 \otimes \tilde{e}_2 S_2 S_1(1^{\otimes x}) \otimes 3.$$

Since $\tilde{e}_{\bar{2}} b_0 = 0$ by assumption, we have $\tilde{e}_{\bar{1}} S_2 S_1 b_0 = 0$, and (3.5) implies

$$\tilde{e}_{\bar{1}} \tilde{e}_2 S_2 S_1 b_1 = 0.$$

On the other hand, $\tilde{f}_1(b_1 \otimes 3 \otimes b_2) = \tilde{f}_1 \tilde{f}_2 b \neq 0$ implies $\tilde{f}_1 b_1 \neq 0$. Hence $b_1$ contains 1, and $\tilde{e}_{\bar{1}} b_1 = 0$ implies that $b_1 = b_3 \otimes 1 \otimes b_4$ where $b_4$ contains neither 1 nor 2. Since $b_3$ is a $\mathfrak{gl}(3)$-highest weight vector, we have $S_1 b_1 = S_1 b_3 \otimes 2 \otimes b_4$. Since $\tilde{e}_2 S_1 b_1 = 0$ by (3.4), we have $\tilde{e}_2 S_1 b_3 = 0$. Consequently, $\tilde{e}_2 S_2 S_1 b_1 = b_5 \otimes 2 \otimes b_4$ for some $b_5$, because $\tilde{e}_2 S_2(2^{\otimes y} \otimes 2 \otimes 3^{\otimes z}) = \tilde{e}_2(3^{\otimes(y+1-z)} \otimes 2^{\otimes z} \otimes 3^{\otimes z}) = 3^{\otimes(y-z)} \otimes 2^{\otimes z} \otimes 2 \otimes 3^{\otimes z}$. This contradicts $\tilde{e}_{\bar{1}} \tilde{e}_2 S_2 S_1 b_1 = 0$. Hence we get the desired result $\tilde{e}_{\bar{1}} b = 0$. □

**Lemma 3.7.** *If $\varepsilon_1(b) = 0$ and $\langle k_1, \mathrm{wt}(b) \rangle = \langle k_2, \mathrm{wt}(b) \rangle > 0$, then $\tilde{e}_{\bar{1}} b \neq 0$.*

*Proof.* Assume that $\tilde{e}_{\bar{1}} b = 0$. Then $b = b_1 \otimes 1 \otimes b_2$ for some $b_1$ and $b_2$, where $b_2$ contains neither 1 nor 2. Since $\varepsilon_1(b_1) = 0$, we have

$$\langle k_1, \mathrm{wt}(b) \rangle = 1 + \langle k_1, \mathrm{wt}(b_1) \rangle \geq 1 + \langle k_2, \mathrm{wt}(b_1) \rangle = \langle k_2, \mathrm{wt}(b) \rangle + 1,$$

which is a contradiction. □

**Proposition 3.8.** *If $b$ is a highest weight vector in $\mathbf{B}^{\otimes N}$, then $\mathrm{wt}(b)$ is a strict partition.*

*Proof.* Assuming that $\langle k_i, \mathrm{wt}(b) \rangle = \langle k_{i+1}, \mathrm{wt}(b) \rangle > 0$, we shall derive a contradiction. Set $b' := S_{w_i} b$. Since $w_i^{-1}(\alpha_1) = \alpha_i$, (3.4) implies $\tilde{e}_1 b' = 0$. Hence Lemma 3.7 implies $\tilde{e}_{\bar{1}} b' \neq 0$, which is a contradiction. □

**Lemma 3.9.** *Let $b$ be a vector in $\mathbf{B}^{\otimes N}$.*

(a) *If $\tilde{e}_{\bar{1}} b = \tilde{e}_1 b = 0$ and $\langle k_1, \mathrm{wt}(b) \rangle \geq \langle k_2, \mathrm{wt}(b) \rangle + 2$, then $\tilde{e}_{\bar{1}}(1 \otimes \tilde{f}_1 b) = 0$.*
(b) *If $\tilde{e}_{\bar{1}} b = \tilde{e}_1 b = \tilde{e}_2 b = 0$ and $\langle k_2, \mathrm{wt}(b) \rangle > \langle k_3, \mathrm{wt}(b) \rangle$, then $\tilde{e}_{\bar{1}}(1 \otimes \tilde{f}_1 \tilde{f}_2 b) = 0$.*

*Proof.* (a) Since $\langle k_1, \mathrm{wt}(b)\rangle > 0$ and $\tilde{e}_{\bar{1}} b = 0$, we can write $b = b_1 \otimes 1 \otimes b_2$ for some $b_1$ and $b_2$ such that $b_2$ contains neither 1 nor 2. Then we get

$$2 \leq \langle k_1, \mathrm{wt}(b)\rangle - \langle k_2, \mathrm{wt}(b)\rangle = \langle k_1, \mathrm{wt}(b_1)\rangle - \langle k_2, \mathrm{wt}(b_1)\rangle + 1 = \varphi_1(b_1) - \varepsilon_1(b_1) + 1.$$

Thus $\varphi_1(b_1) > 0 = \varepsilon_1(1)$ and hence $\tilde{f}_1 b = \tilde{f}_1 b_1 \otimes 1 \otimes b_2$. It follows that $\tilde{e}_{\bar{1}}(1 \otimes \tilde{f}_1 b) = 0$.

(b) Since $\langle k_1, \mathrm{wt}(b)\rangle \geq \langle k_2, \mathrm{wt}(b)\rangle > 0$ and $\tilde{e}_{\bar{1}} b = 0$, we can write $b = b_1 \otimes 1 \otimes b_2$ for some $b_1$ and $b_2$ such that $b_2$ contains neither 1 nor 2. It follows that $\varepsilon_2(b_1) = \varphi_2(b_2) = 0$. Observe that

$$\varphi_2(b_1) = \langle k_2, \mathrm{wt}(b_1)\rangle - \langle k_3, \mathrm{wt}(b_1)\rangle > \langle k_3, \mathrm{wt}(b_2)\rangle - \langle k_2, \mathrm{wt}(b_2)\rangle = \varepsilon_2(b_2).$$

Hence $\tilde{f}_2 b = \tilde{f}_2 b_1 \otimes 1 \otimes b_2$. Since $\varepsilon_1(\tilde{f}_2 b_1) = 0$, we deduce that

$$\varphi_1(\tilde{f}_2 b_1) = \langle k_1 - k_2, \mathrm{wt}(\tilde{f}_2 b_1)\rangle = \langle k_1 - k_2, \mathrm{wt}(b_1)\rangle + 1 = \varphi_1(b_1) + 1 > 0 = \varepsilon_1(1 \otimes b_2),$$

and hence $\tilde{f}_1 \tilde{f}_2 b = \tilde{f}_1 \tilde{f}_2 b_1 \otimes 1 \otimes b_2$. Therefore $\tilde{e}_{\bar{1}}(1 \otimes \tilde{f}_1 \tilde{f}_2 b) = 0$. □

**Proposition 3.10.** *If $b \in \mathbf{B}^{\otimes(N-1)}$ is a highest weight vector with $\langle k_{j-1}, \mathrm{wt}(b)\rangle \geq \langle k_j, \mathrm{wt}(b)\rangle + 2$, then $b_0 = 1 \otimes \tilde{f}_1 \cdots \tilde{f}_{j-1} b$ is a highest weight vector in $\mathbf{B}^{\otimes N}$.*

*Proof.* We will show $\tilde{e}_{\bar{i}} b_0 = 0$ for $i = 1, \dots, n-1$.

**Case 1:** $i \geq j+1$. By Lemma 3.3, we have $S_{w_i} b_0 = 3 \otimes \tilde{f}_3 \cdots \tilde{f}_{j+1} S_{w_i} b$. Thus we obtain $\tilde{e}_{\bar{1}} S_{w_i} b_0 = 3 \otimes \tilde{f}_3 \cdots \tilde{f}_{j+1} \tilde{e}_{\bar{1}} S_{w_i} b = 0$.

**Case 2:** $i = j$. Since $S_{w_i} b_0 = 1 \otimes S_{w_i} b$, we have $\tilde{e}_{\bar{1}} S_{w_i} b_0 = 0$.

**Case 3:** $i = j-1$. We have $S_{w_i} b_0 = 1 \otimes \tilde{f}_1 S_{w_i} b$ and

$$\langle k_1, \mathrm{wt}(S_{w_i} b)\rangle = \langle k_{j-1}, \mathrm{wt}(b)\rangle \geq \langle k_j, \mathrm{wt}(b)\rangle + 2 = \langle k_2, \mathrm{wt}(S_{w_i} b)\rangle + 2.$$

By Lemma 3.9(a), we obtain $\tilde{e}_{\bar{1}} S_{w_i} b_0 = 0$.

**Case 4:** $i \leq j-2$. Set $b' := \tilde{f}_{i+2} \cdots \tilde{f}_{j-1} b$. Here we understand $b' = b$ if $i = j-2$. Then $b'$ is a $\mathfrak{gl}(i+2)$-highest weight vector and $\tilde{e}_{\bar{1}} b' = 0$. Because $\tilde{e}_{\bar{1}}$ commutes with $S_{u_i}$, we have $\tilde{e}_{\bar{1}} S_{u_i} b' = 0$. Since $u_i^{-1}(\alpha_1)$ and $u_i^{-1}(\alpha_2)$ are positive roots, $S_{u_i} b'$ is a $\mathfrak{gl}(3)$-highest weight vector by (3.4).

By Lemma 3.3, we have $S_{u_i} b_0 = 1 \otimes \tilde{f}_1 \tilde{f}_2 S_{u_i} b'$. Observe that

$$\begin{aligned}\langle k_2, \mathrm{wt}(S_{u_i} b')\rangle - \langle k_3, \mathrm{wt}(S_{u_i} b')\rangle &= \langle k_{i+1}, \mathrm{wt}(b')\rangle - \langle k_{i+2}, \mathrm{wt}(b')\rangle \\ &= \langle k_{i+1} - k_{i+2}, \mathrm{wt}(b) - \varepsilon_{i+2} + \varepsilon_j\rangle \\ &= \langle k_{i+1} - k_{i+2}, \mathrm{wt}(b)\rangle + 1 - \delta_{j,i+2} \geq 1.\end{aligned}$$

By Lemma 3.9(b), we get $\tilde{e}_{\bar{1}} S_{u_i} b_0 = 0$. Since $S_{u_i} = S_{z_i} S_{w_i}$ and $\tilde{e}_{\bar{1}}$ commutes with $S_{z_i}$, we obtain $\tilde{e}_{\bar{1}} S_{w_i} b_0 = 0$. □

**Theorem 3.11.** *Assume that $b$ is a $\mathfrak{gl}(n)$-highest weight vector in $\mathbf{B}^{\otimes(N-1)}$ and $b_0 := 1 \otimes \tilde{f}_1 \cdots \tilde{f}_{j-1} b$ is a $\mathfrak{gl}(n)$-highest weight vector in $\mathbf{B}^{\otimes N}$. Then $b_0$ is a highest weight vector if and only if $b$ is a highest weight vector and $\mathrm{wt}(b_0) = \mathrm{wt}(b) + \varepsilon_j$ is a strict partition.*

*Proof.* Note that $\mathrm{wt}(b)$ and $\mathrm{wt}(b_0)$ are partitions.

If $b_0$ is a highest weight vector, then by Theorem 3.5 and Proposition 3.8, $b$ is a highest weight vector and $\mathrm{wt}(b_0)$ is a strict partition.

Conversely, if $b$ is a highest weight vector such that $\mathrm{wt}(b)$ is a strict partition and $\mathrm{wt}(b) + \varepsilon_j$ is still a strict partition, then $\langle k_{j-1} - k_j, \mathrm{wt}(b) \rangle \geq 2$ and hence, by Proposition 3.10, $b_0$ is a highest weight vector. □

## 4. Existence and uniqueness

In this section, we state and prove the main result of our paper: the existence and uniqueness theorem for crystal bases. We first prove several lemmas that are needed in the proof of our main theorem.

We set

$$\tilde{k}_{\bar{i}} = q^{k_i - 1} k_{\bar{i}} \qquad \text{for all } i = 1, \dots, n. \tag{4.1}$$

**Lemma 4.1.** *Let $M$ be a $U_q(\mathfrak{q}(n))$-module in $\mathcal{O}_{\mathrm{int}}^{\geq 0}$.*

(a) *For $\mu \in \mathrm{wt}(M)$ and $i \in \{1, \dots, n-1\}$ such that $\mu + \alpha_i \notin \mathrm{wt}(M)$, we have*

$$\tilde{k}_{\overline{i+1}} = S_i \circ \tilde{k}_{\bar{i}} \circ S_i \qquad \textit{as endomorphisms of } M_\mu,$$

*where $S_i$ is defined in Remark 2.11.*

(b) *Assume that $\lambda \in \mathrm{wt}(M)$ satisfies $\lambda + \alpha_i \notin \mathrm{wt}(M)$ for all $i = 1, \dots, n-1$. If $(L, B, l_B)$ is a crystal basis of $M$, then $L_\lambda$ is invariant under $\tilde{k}_{\bar{i}}$ for all $i = 1, \dots, n$.*

*Proof.* (a) Set $\ell := \langle h_i, \mu \rangle \geq 0$. Then $S_i : M_\mu \xrightarrow{\sim} M_{s_i \mu}$ is given by $f_i^{(\ell)}$, and its inverse is given by $e_i^{(\ell)}$. Note that $e_i M_\mu = 0$.

From the defining relation it follows that

$$\begin{aligned}
e_{\bar{i}} f_i - f_i e_{\bar{i}} &= e_{\bar{i}} q^{-k_i} q_i^k f_i - f_i e_{\bar{i}} q^{-k_i} q^{k_i} \\
&= (k_{\bar{i}} e_i - q e_i k_{\bar{i}}) q^{k_i} f_i - f_i (k_{\bar{i}} e_i - q e_i k_{\bar{i}}) q^{k_i} \\
&= (k_{\bar{i}} e_i - q e_i k_{\bar{i}}) f_i q^{k_i - 1} - f_i (k_{\bar{i}} e_i - q e_i k_{\bar{i}}) q^{k_i} \\
&= k_{\bar{i}} e_i f_i q^{k_i - 1} - q e_i k_{\bar{i}} f_i q^{k_i - 1} - f_i k_{\bar{i}} e_i q^{k_i} + q f_i e_i k_{\bar{i}} q^{k_i} \\
&= k_{\bar{i}} \left( f_i e_i + \frac{q^{h_i} - q^{-h_i}}{q - q^{-1}} \right) q^{k_i - 1} - q e_i k_{\bar{i}} f_i q^{k_i - 1} - f_i k_{\bar{i}} e_i q^{k_i} + q f_i e_i k_{\bar{i}} q^{k_i}.
\end{aligned}$$

Thus on $M_\mu$, we have

$$e_{\bar{i}} f_i - f_i e_{\bar{i}} = k_{\bar{i}} \frac{q^{h_i} - q^{-h_i}}{q - q^{-1}} q^{k_i - 1} - e_i k_{\bar{i}} f_i q^{k_i},$$

which yields

$$\begin{aligned}
k_{\overline{i+1}} &= q^{-h_i} k_{\bar{i}} - (e_{\bar{i}} f_i - f_i e_{\bar{i}}) q^{-k_i} \\
&= q^{-h_i} k_{\bar{i}} - \left( k_{\bar{i}} \frac{q^{h_i} - q^{-h_i}}{q - q^{-1}} q^{k_i - 1} - e_i k_{\bar{i}} f_i q^{k_i} \right) q^{-k_i} \\
&= q^{-h_i} k_{\bar{i}} - k_{\bar{i}} \frac{q^{h_i} - q^{-h_i}}{q - q^{-1}} q^{-1} + e_i k_{\bar{i}} f_i \\
&= -k_{\bar{i}} \left( \frac{q^{h_i - 1} - q^{-h_i + 1}}{q - q^{-1}} \right) + e_i k_{\bar{i}} f_i = -[\ell - 1] k_{\bar{i}} + e_i k_{\bar{i}} f_i. \qquad (4.2)
\end{aligned}$$

On the other hand, we have, similarly to (2.3),

$$e_i^{(2)} k_{\bar{i}} = e_i k_{\bar{i}} e_i - k_{\bar{i}} e_i^{(2)}.$$

By induction on $s$, we obtain

$$e_i^{(s)} k_{\bar{i}} = e_i k_{\bar{i}} e_i^{(s-1)} - [s-1] k_{\bar{i}} e_i^{(s)} (s \geq 1).$$

If $\ell > 0$, on $M_\mu$ we have

$$\begin{aligned}
e_i^{(\ell)} q^{k_i - 1} k_{\bar{i}} f_i^{(\ell)} &= q^{-\ell} e_i^{(\ell)} k_{\bar{i}} f_i^{(\ell)} q^{k_i - 1} \\
&= q^{-\ell} (e_i k_{\bar{i}} e_i^{(\ell-1)} - [\ell - 1] k_{\bar{i}} e_i^{(\ell)}) f_i^{(\ell)} q^{k_i - 1} \\
&= q^{-\ell} (e_i k_{\bar{i}} f_i - [\ell - 1] k_{\bar{i}}) q^{k_i - 1} \\
&= (e_i k_{\bar{i}} f_i - [\ell - 1] k_{\bar{i}}) q^{k_{i+1} - 1} = k_{\overline{i+1}} q^{k_{i+1} - 1} = \tilde{k}_{\overline{i+1}}.
\end{aligned}$$

If $\ell = 0$, then $f_i M_\mu = 0$, and hence (4.2) implies $k_{\overline{i+1}} = k_{\bar{i}}$. Therefore $\tilde{k}_{\overline{i+1}} = k_{\overline{i+1}} q^{k_{i+1}-1} = k_{\bar{i}} q^{k_i - 1} = \tilde{k}_{\bar{i}}$ on $M_\mu$. In both cases, $\tilde{k}_{\overline{i+1}} = S_i^{-1} \tilde{k}_{\bar{i}} S_i$ on $M_\mu$.

(b) Let $M' = U_q(\mathfrak{q}(n)) M_\lambda \subset M$, and let $L' = L \cap M'$. Set $\mu_j := s_j \cdots s_{i-1} \lambda$ for $j = 1, \ldots, i$. Then $\langle h_j, \mu_{j+1} \rangle \geq 0$, $s_j \mu_{j+1} = \mu_j$, and $\mu_{j+1} + \alpha_j \notin \mathrm{wt}(M')$. From (a) it follows that $\tilde{k}_{\overline{j+1}}|_{M'_{\mu_{j+1}}} = S_j \circ \tilde{k}_{\bar{j}} \circ S_j|_{M'_{\mu_{j+1}}}$. Hence, if $L'_{\mu_j}$ is stable under $\tilde{k}_{\bar{j}}$, then $L'_{\mu_{j+1}}$ is stable under $\tilde{k}_{\overline{j+1}}$. Since $L'_{\mu_1}$ is stable under $\tilde{k}_{\bar{1}}$, $L'_{\mu_j}$ is stable under $\tilde{k}_{\bar{j}}$ for all $j = 1, \ldots, i$ by induction. In particular, $L_\lambda = L'_\lambda$ is stable under $\tilde{k}_{\bar{i}}$. □

**Lemma 4.2.** *Let $M$ be a $U_q(\mathfrak{q}(n))$-module in $\mathcal{O}_{\mathrm{int}}^{\geq 0}$, and $\lambda \in P^{\geq 0}$. Let $(L, B, l_B)$ be a crystal basis of $M$ such that any connected component of $B$ intersects $B_\lambda$. Let $L'$ be an* **A***-submodule of $M$ with the weight space decomposition $L' = \bigoplus_{\mu \in P^{\geq 0}} (L' \cap M_\mu)$, which is stable under $\tilde{e}_i$, $\tilde{f}_i$ $(i = 1, \ldots, n-1, \bar{1})$. Then*

(a) *$L'_\lambda \subset L_\lambda$ implies $L' \subset L$,*
(b) *$L'_\lambda \supset L_\lambda$ implies $L' \supset L$.*

*Proof.* (a) Assume that $L'_\lambda \subset L_\lambda$. Set $S := (L \cap qL')/(qL \cap qL')$. Then $S \subset L/qL$ and $S$ is stable under $\tilde{e}_i, \tilde{f}_i$ $(i = 1, \dots, n-1, \overline{1})$. Note that

$$S_\lambda = S \cap (L/qL)_\lambda = (L_\lambda \cap qL'_\lambda)/(qL_\lambda \cap qL'_\lambda) = 0.$$

For each $b \in B$, let $P_b \colon L/qL \twoheadrightarrow l_b$ be the canonical projection. Since $S_\lambda = 0$, we have $P_b(S) = 0$ for any $b \in B_\lambda$. If $\tilde{e}_i b \neq 0$ for some $i = 1, \dots, n-1, \overline{1}$, then $\tilde{e}_i \circ P_b = P_{\tilde{e}_i b} \circ \tilde{e}_i$ implies $\tilde{e}_i P_b(S) = P_{\tilde{e}_i b} \tilde{e}_i(S) \subset P_{\tilde{e}_i b}(S)$. Therefore, if $P_{\tilde{e}_i b}(S) = 0$, then $P_b(S) = 0$. The same property holds for $\tilde{f}_i$.

Since any $b \in B$ can be connected to an element of weight $\lambda$ by a sequence of operators in $\tilde{e}_i, \tilde{f}_i$ $(i = 1, \dots, n-1, \overline{1})$, we have $P_b(S) = 0$ for all $b \in B$. It follows that $S = 0$ and hence $L \cap qL' \subset qL$.

Since

$$L' \cap q^{-m} L \subset q^{-(m-1)}(L' \cap q^{-1} L) \subset q^{-(m-1)} L$$

for all $m \geq 1$, we have $L' \cap q^{-m} L \subset L' \cap q^{-(m-1)} L$. Hence we obtain $L' \cap q^{-m} L \subset L$. It follows that $L' \subset L$ as desired.

(b) Assume that $L'_\lambda \supset L_\lambda$. Set $S := (L' \cap L)/(L' \cap qL)$. Then $S \subset L/qL$ and $S$ is stable under $\tilde{e}_i, \tilde{f}_i$. Note that $l_b \subset S$ for any $b \in B_\lambda$. If $\tilde{e}_i b \neq 0$ and $l_b \subset S$, then $l_{\tilde{e}_i b} = \tilde{e}_i l_b \subset \tilde{e}_i S \subset S$.

The same is true for $\tilde{f}_i$. Thus $L/qL = \bigoplus_{b \in B} l_b \subset S$. By Nakayama's lemma, we have $L' \cap L = L$. □

**Lemma 4.3.** *Let $M$ be a highest weight $U_q(\mathfrak{q}(n))$-module with highest weight $\lambda \in \Lambda^+$ in the category $\mathcal{O}^{\geq 0}_{\mathrm{int}}$. Suppose that $M$ has a crystal basis $(L, B, l_B)$ such that $B_\lambda = \{b_\lambda\}$ and $B$ is connected. Let $L_\lambda = \bigoplus_{j=1}^{s} E_j$ be a decomposition into indecomposable modules over $\mathbf{A}[C_1, \dots, C_r]$ (see Remark 2.14), where $r = \ell(\lambda)$, and let*

$$M_j := U_q(\mathfrak{q}(n)) E_j, \quad L_j := M_j \cap L, \quad l_b^j := l_b \cap (L_j/qL_j).$$

*Then*

(a) *$M_j$ is irreducible over $U_q(\mathfrak{q}(n))$,*
(b) *$M = \bigoplus_{j=1}^{s} M_j$, $L = \bigoplus_{j=1}^{s} L_j$ and $l_b = \bigoplus_{j=1}^{s} l_b^j$,*
(c) *$(L_j, B, (l_b^j)_{b \in B})$ is a crystal basis of $M_j$.*

*Proof.* By Remark 2.14, we see that $(M_j)_\lambda \simeq \mathbf{F} \otimes_{\mathbf{A}} E_j$ is an irreducible module over $\mathbf{F}[C_1, \dots, C_r]$ for each $j = 1, \dots, s$. Hence, Proposition 2.15(a) implies that $M_j$ is irreducible over $U_q(\mathfrak{q}(n))$ and $M = \bigoplus_{j=1}^{s} M_j$. Note that

$$L_j/qL_j \subset L/qL \quad (j = 1, \dots, s).$$

Since $\bigoplus_{j=1}^{s} (L_j)_\lambda = \bigoplus_{j=1}^{s} (M_j \cap L_\lambda) = \bigoplus_{j=1}^{s} E_j = L_\lambda$, we have

$$l_{b_\lambda} = L_\lambda/qL_\lambda = \bigoplus_{j=1}^{s} \big((L_j)_\lambda/q(L_j)_\lambda\big) = \bigoplus_{j=1}^{s} l_{b_\lambda}^j.$$

Consider $b_1, b_2 \in B$ such that $b_2 = \tilde{e}_i b_1$ (equivalently, $b_1 = \tilde{f}_i b_2$) for some $i = 1, \ldots, n-1, \overline{1}$. Then we have injective maps

$$\tilde{e}_i|_{l^j_{b_1}} : l^j_{b_1} \rightarrowtail l^j_{b_2}, \quad \tilde{f}_i|_{l^j_{b_2}} : l^j_{b_2} \rightarrowtail l^j_{b_1}.$$

Hence comparing the dimensions, we conclude that

$$\tilde{e}_i : l^j_{b_1} \xrightarrow{\sim} l^j_{b_2} \quad \text{and} \quad \tilde{f}_i : l^j_{b_2} \xrightarrow{\sim} l^j_{b_1} \quad \text{for all } j = 1, \ldots, s.$$

Therefore $l_{b_1} = \bigoplus_{j=1}^s l^j_{b_1}$ if and only if $l_{b_2} = \bigoplus_{j=1}^s l^j_{b_2}$. Since $B$ is connected, $\bigoplus_{j=1}^s l^j_b = l_b$ for all $b \in B$. Since

$$L/qL = \bigoplus_{b\in B} l_b = \bigoplus_{j=1}^{s} \bigoplus_{b\in B} l^j_b \subset \bigoplus_{j=1}^{s} L_j/qL_j,$$

Nakayama's lemma implies that $L = \bigoplus_{j=1}^s L_j$, and $(L_j, B, (l^j_b)_{b\in B})$ is a crystal basis of $M_j$. □

**Lemma 4.4.** *Let $M$ be a $U_q(\mathfrak{q}(n))$-module in the category $\mathcal{O}^{\geq 0}_{\text{int}}$ and let $(L_1, B_1, l^1_{B_1})$, $(L_2, B_2, l^2_{B_2})$ be two crystal bases of $M$ such that $L_1 = L_2$. If $B_1$ is a connected abstract $\mathfrak{q}(n)$-crystal and there exist $b_1 \in B_1$ and $b_2 \in B_2$ such that $l^1_{b_1} = l^2_{b_2}$, then there exists a bijection $\varphi\colon B_1 \to B_2$ which commutes with the Kashiwara operators and $l^1_b = l^2_{\varphi(b)}$ for all $b \in B_1$.*

*Proof.* Let us set $S = \{b \in B_1;$ there exists $b' \in B_2$ such that $l^1_b = l^2_{b'}\}$. Then it is easy to see that it is stable under the Kashiwara operators and it contains $b_1$. Hence $S$ coincides with $B_1$. Therefore for every $b \in B_1$, there exists a $b' \in B_2$ such that $l^1_b = l^2_{b'}$. Such a $b'$ is unique and we can define $\varphi$ by $\varphi(b) = b'$. Since $L_1/qL_1 = \bigoplus_{b\in B_1} l^1_b = \bigoplus_{b\in B_2} l^2_b$, $\varphi\colon B_1 \to B_2$ is bijective. □

**Lemma 4.5.** *Let $\lambda \in \Lambda^+$ and assume that $V(\lambda)$ has a crystal basis $(L_0, B_0, l_{B_0})$ such that $B_0$ is connected and $(B_0)_\lambda = \{b_\lambda\}$. Let $M \in \mathcal{O}^{\geq 0}_{\text{int}}$ be a highest weight $U_q(\mathfrak{q}(n))$-module with highest weight $\lambda \in \Lambda^+$. If $E$ is a free* **A***-submodule of $M_\lambda$, which is stable under $\tilde{k}_{\overline{i}}$ for $i = 1, \ldots, n$ and generates $M_\lambda$ over* **F***, then there exists a unique crystal basis $(L, B, l_B)$ such that*

(a) *$L_\lambda = E$,*
(b) *$B \simeq B_0$ as an abstract $\mathfrak{q}(n)$-crystal.*

*Proof.* By Lemma 4.1 and Proposition 2.15, there exists a finitely generated free **A**-module $K$ such that $M \simeq K \otimes_{\mathbf{A}} V(\lambda)$ and $E \simeq K \otimes_{\mathbf{A}} (L_0)_\lambda$. Then $(K \otimes_{\mathbf{A}} (L_0), B_0, (K \otimes l_b)_{b\in B_0})$ is a crystal basis for $M$. Uniqueness follows from Lemmas 4.2 and 4.4. □

For a weight $\lambda = \lambda_1\epsilon_1 + \cdots + \lambda_n\epsilon_n \in P$, define $|\lambda| = \sum_{i=1}^n \lambda_i$. Now we are ready to state our main theorem.

**Theorem 4.6.**

(a) *Let $M$ be an irreducible highest weight $U_q(\mathfrak{q}(n))$-module with highest weight $\lambda \in \Lambda^+$. Then there exists a crystal basis $(L, B, l_B)$ of $M$ such that*

(i) $B_\lambda = \{b_\lambda\}$,

(ii) *$B$ is connected.*

*Moreover, such a crystal basis is unique up to an automorphism of $M$. In particular, $B$ depends only on $\lambda$ as an abstract $\mathfrak{q}(n)$-crystal and we write $B = B(\lambda)$.*

(b) *The $\mathfrak{q}(n)$-crystal $B(\lambda)$ has a unique highest weight vector $b_\lambda$.*

(c) *A vector $b \in \mathbf{B} \otimes B(\lambda)$ is a highest weight vector if and only if*

$$b = 1 \otimes \tilde{f}_1 \cdots \tilde{f}_{j-1} b_\lambda$$

*for some $j$ such that $\lambda + \epsilon_j$ is a strict partition.*

(d) *Let $M$ be a finite-dimensional highest weight $U_q(\mathfrak{q}(n))$-module with highest weight $\lambda \in \Lambda^+$. Assume that $M$ has a crystal basis $(L, B(\lambda), l_{B(\lambda)})$ such that $L_\lambda / q L_\lambda = l_{b_\lambda}$. Then*

(i) $\mathbf{V} \otimes M = \bigoplus_{\lambda+\epsilon_j:\, strict} M_j$, *where $M_j$ is a highest weight $U_q(\mathfrak{q}(n))$-module with highest weight $\lambda + \epsilon_j$ and* $\dim (M_j)_{\lambda+\epsilon_j} = 2 \dim M_\lambda$,

(ii) *if we set $L_j = (\mathbf{L} \otimes L) \cap M_j$ and $B_j = \{b \in \mathbf{B} \otimes B(\lambda);\ l_b \subset L_j / q L_j\}$, then* $\mathbf{B} \otimes B(\lambda) = \coprod_{\lambda+\epsilon_j:\, strict} B_j$ *and* $L_j / q L_j = \bigoplus_{b \in B_j} l_b$,

(iii) *$M_j$ has a crystal basis $(L_j, B_j, l_{B_j})$,*

(iv) *$B_j \simeq B(\lambda + \epsilon_j)$ as an abstract $\mathfrak{q}(n)$-crystal.*

*Proof.* We shall argue by induction on $|\lambda|$.

For a positive integer $k$, we denote by $(\mathrm{a})_k$, $(\mathrm{b})_k$, $(\mathrm{c})_k$ and $(\mathrm{d})_k$ the assertions (a), (b), (c) and (d) for $\lambda$ with $|\lambda| = k$, respectively.

It is straightforward to check $(\mathrm{a})_1$ and $(\mathrm{b})_1$. Assuming the assertions $(\mathrm{a})_k$, $(\mathrm{b})_k$ for $k \le N$ and the assertions $(\mathrm{c})_k$, $(\mathrm{d})_k$ for $k < N$, let us show $(\mathrm{a})_{N+1}$, $(\mathrm{b})_{N+1}$, $(\mathrm{c})_N$ and $(\mathrm{d})_N$.

**Step 1:** We shall prove $(\mathrm{c})_N$. Let $\lambda$ be a strict partition with $|\lambda| = N$. By choosing a sequence of strict partitions $\varepsilon_1 = \lambda_1, \lambda_2, \ldots, \lambda_N = \lambda$ such that $\lambda_{k+1} = \lambda_k + \varepsilon_{j_k}$ for some $j_k$ and applying $(\mathrm{d})_k$ on each $\lambda_k$ successively for $k < N$, we can embed $B(\lambda)$ into $\mathbf{B}^{\otimes N}$. It follows that $\mathbf{B} \otimes B(\lambda) \subset \mathbf{B}^{\otimes (N+1)}$. By $(\mathrm{b})_N$, we know that there exists a unique highest weight vector, say $b_\lambda$, in $B(\lambda)$. By Theorem 3.11, an element $b \in \mathbf{B} \otimes B(\lambda)$ is a highest weight vector if and only if

$$b = 1 \otimes \tilde{f}_1 \cdots \tilde{f}_{j-1} b_\lambda$$

for some $j$ such that $\lambda + \varepsilon_j$ is a strict partition. So $(\mathrm{c})_N$ holds.

**Step 2:** We shall show that $(\mathrm{d})_N$ holds except (iv). Let $M$ be a finite-dimensional highest weight module with highest weight $\lambda \in \Lambda^+$ with $|\lambda| = N$ and let $(L, B(\lambda), l_{B(\lambda)})$ be a crystal basis of $M$. By Theorem 1.11, we have a decomposition $\mathbf{V} \otimes M = \bigoplus_{\lambda+\epsilon_j:\,\mathrm{strict}} M_j$, where $M_j$ is a highest weight $U_q(\mathfrak{q}(n))$-module with highest weight $\lambda+\epsilon_j$ and $\dim (M_j)_{\lambda+\epsilon_j} = 2 \dim M_\lambda$.

By Theorem 2.7, $\mathbf{V}\otimes M$ admits a crystal basis $(\tilde{L}, \mathbf{B}\otimes B(\lambda), l_{\mathbf{B}\otimes B(\lambda)})$ where $\tilde{L}:=\mathbf{L}\otimes L$. Set $L_j := M_j \cap \tilde{L}$. Note that

$$\mathbf{F}\otimes_{\mathbf{A}} L_j \xrightarrow{\sim} M_j \quad \text{and} \quad L_j = \bigoplus_{\mu\in P^{\geq 0}} L_j \cap (M_j)_\mu.$$

Then

$$L_j/qL_j \subset \tilde{L}/q\tilde{L} = \bigoplus_{b\in\mathbf{B}\otimes B(\lambda)} l_b.$$

Since $\tilde{e}_i(M_j)_{\lambda+\epsilon_j} = \tilde{e}_{\bar{i}}(M_j)_{\lambda+\epsilon_j} = 0$ for any $i = 1,\dots,n-1$ (see Remark 2.11), we have, as subspaces of $\tilde{L}/q\tilde{L}$,

$$(L_j)_{\lambda+\epsilon_j}/q(L_j)_{\lambda+\epsilon_j} \subset \Big(\bigcap_{i=1}^{n-1}\operatorname{Ker}\tilde{e}_i \cap \bigcap_{i=1}^{n-1}\operatorname{Ker}\tilde{e}_{\bar{i}}\Big)_{\lambda+\epsilon_j} = \bigoplus_{\substack{\mathrm{wt}(b)=\lambda+\epsilon_j\\ \tilde{e}_i b=\tilde{e}_{\bar{i}}b=0}} l_b = l_{b_j},$$

where $b_j = 1\otimes \tilde{f}_1\cdots\tilde{f}_{j-1}b_\lambda$ in $\mathbf{B}\otimes B(\lambda)$. Here, the last equality follows from $(\mathrm{c})_N$.

Because $\operatorname{rank}_{\mathbf{A}}(L_j)_\mu = \dim_{\mathbf{F}}(M_j)_\mu$ for any $\mu\in\mathrm{wt}(M_j)$, we have

$$\begin{aligned}\dim_{\mathbb{C}}\big((L_j)_{\lambda+\epsilon_j}/q(L_j)_{\lambda+\epsilon_j}\big) &= \operatorname{rank}_{\mathbf{A}}(L_j)_{\lambda+\epsilon_j} = \dim_{\mathbf{F}}(M_j)_{\lambda+\epsilon_j}\\ &= 2\dim_{\mathbf{F}} M_\lambda = \dim_{\mathbb{C}} l_{b_j}\end{aligned}$$

and hence

$$(L_j/qL_j)_{\lambda+\epsilon_j} = (L_j)_{\lambda+\epsilon_j}/q(L_j)_{\lambda+\epsilon_j} = l_{b_j}.$$

Let $B_j$ be the connected component containing $b_j$ in $\mathbf{B}\otimes B(\lambda)$. By $(\mathrm{c})_N$ and Lemma 2.13, we obtain $\bigcup_j B_j = \mathbf{B}\otimes B(\lambda)$. Since $L_j$ is stable under $\tilde{e}_i, \tilde{f}_i, \tilde{e}_{\bar{1}}$ and $\tilde{f}_{\bar{1}}$, we have

$$\bigoplus_{b\in B_j} l_b \subset L_j/qL_j.$$

It follows that

$$\tilde{L}/q\tilde{L} = \bigoplus_{b\in\mathbf{B}\otimes B(\lambda)} l_b = \bigoplus_{b\in\bigcup_j B_j} l_b \subset \sum_j (L_j/qL_j).$$

By Nakayama's lemma, we get

$$\tilde{L} = \sum_j L_j. \tag{4.3}$$

Since $\sum_j L_j = \bigoplus_j L_j$, we obtain $\tilde{L} = \bigoplus_j L_j$ and

$$\bigoplus_{b\in\bigcup_j B_j} l_b = \tilde{L}/q\tilde{L} \simeq \bigoplus_j (L_j/qL_j) \supseteq \bigoplus_j \bigoplus_{b\in B_j} l_{b_j}.$$

Therefore,

$$L_j/qL_j = \bigoplus_{b\in B_j} l_b \quad \text{and} \quad \mathbf{B}\otimes B(\lambda) = \coprod_j B_j.$$

Thus $(L_j, B_j, l_{B_j} = (l_b)_{b\in B_j})$ is a crystal basis of $M_j$. Note that each $B_j$ has a unique highest weight vector $b_j$ and that $B_j$ is connected.

**Step 3:** We will show $(a)_{N+1}$. Since an irreducible highest weight module is uniquely determined up to parity change, and since the crystal structure does not vary under the parity change functor, it is enough to show that there exists an irreducible highest weight module with a crystal basis which satisfies (i) and (ii) of (a).

Let $\lambda$ be a strict partition with $|\lambda| = N + 1$. Choose a strict partition $\mu$ and $\ell$ in $\{1, \dots, n\}$ such that $\lambda = \mu + \epsilon_\ell$. By $(a)_N$, there exists an irreducible highest weight $U_q(\mathfrak{q}(n))$-module $M$ of highest weight $\mu$ which has a crystal basis $(L, B(\mu), l_{B(\mu)})$. Then

$$\mathbf{V} \otimes M = \bigoplus_{\mu+\epsilon_j:\,\text{strict}} M_j,$$

and each $M_j$ has a crystal basis as in Step 2. Therefore there exists a finite-dimensional highest weight $U_q(\mathfrak{q}(n))$-module $M$ with highest weight $\lambda$ which has a crystal basis $(L, B, l_B)$ such that $B$ is connected and $B_\lambda = \{b_\ell\}$. Moreover we see that in Step 2, $B$ has a unique highest weight vector. By Lemma 4.3, we conclude that each irreducible summand of $M$ admits a crystal basis with the abstract crystal $B$ which satisfies (i) and (ii) of (a) and $B$ has a unique highest weight vector.

Uniqueness in $(a)_{N+1}$ immediately follows from Lemma 4.1, Lemma 4.4 and Proposition 2.15. Property $(b)_{N+1}$ is obvious. The remaining (iv) of $(d)_N$ follows from Lemma 4.5. □

**Corollary 4.7.** (a) *Every $U_q(\mathfrak{q}(n))$-module in the category $\mathcal{O}_{\text{int}}^{\geq 0}$ has a crystal basis.*

(b) *If $M$ is a $U_q(\mathfrak{q}(n))$-module in the category $\mathcal{O}_{\text{int}}^{\geq 0}$ and $(L, B, l_B)$ is a crystal basis of $M$, then there exist decompositions $M = \bigoplus_{a\in A} M_a$ as a $U_q(\mathfrak{q}(n))$-module, $L = \bigoplus_{a\in A} L_a$ as an $\mathbf{A}$-module, $B = \coprod_{a\in A} B_a$ as a $\mathfrak{q}(n)$-crystal, parametrized by a set $A$ such that the following conditions are satisfied for any $a \in A$:*

(i) *$M_a$ is a highest weight module with highest weight $\lambda_a$ and $B_a \simeq B(\lambda_a)$ for some strict partition $\lambda_a$,*
(ii) *$L_a = L \cap M_a$, $L_a/qL_a = \bigoplus_{b\in B_a} l_b$,*
(iii) *$(L_a, B_a, l_{B_a})$ is a crystal basis of $M_a$.*

*Proof.* (a) Our assertion follows from the semisimplicity of the category $\mathcal{O}_{\text{int}}^{\geq 0}$. Indeed, if $M = \bigoplus_\nu M_\nu$ is a decomposition of $M$ into irreducible $U_q(\mathfrak{q}(n))$-modules, then each $M_\nu$ is an irreducible highest weight module, and hence it admits a crystal basis $(L_\nu, B_\nu, l_{B_\nu})$ by Theorem 4.6. Set

$$L := \bigoplus_\nu L_\nu,\; B := \coprod_\nu B_\nu,\; l_B := (l_b)_{b\in B}.$$

Then $(L, B, l_B)$ is a crystal basis of $M$.

(b) Let $\lambda$ be a maximal element in $\mathrm{wt}(B) = \mathrm{wt}(M)$. Note that if $\ell(\lambda) = r$ is odd, then we have the following commutative diagram (see Remark 2.14 for notation):

$$\begin{array}{ccc} \mathrm{Mod}(\mathbf{A}) & \xrightarrow{\sim} & \text{S-Mod}(\mathbf{A}[C_1, \dots, C_r]) \\ \downarrow{\scriptstyle \mathbb{C}\otimes_{\mathbf{A}/q\mathbf{A}}(-)} & & \downarrow{\scriptstyle \mathbb{C}\otimes_{\mathbf{A}/q\mathbf{A}}(-)} \\ \mathrm{Mod}(\mathbb{C}) & \xrightarrow{\sim} & \text{S-Mod}(\mathbb{C}[C_1, \dots, C_r]) \end{array}$$

and if $r$ is even, then we have the following commutative diagram:

$$\begin{array}{ccc} \text{S-Mod}(\mathbf{A}) & \xrightarrow{\sim} & \text{S-Mod}(\mathbf{A}[C_1, \dots, C_r]) \\ \big\downarrow{\scriptstyle \mathbb{C}\otimes_{\mathbf{A}/q\mathbf{A}}(-)} & & \big\downarrow{\scriptstyle \mathbb{C}\otimes_{\mathbf{A}/q\mathbf{A}}(-)} \\ \text{S-Mod}(\mathbb{C}) & \xrightarrow{\sim} & \text{S-Mod}(\mathbb{C}[C_1, \dots, C_r]) \end{array}$$

The horizontal arrows are given by $K \mapsto V \otimes_{\mathbb{C}} K$ for each module $K$ in the left hand side, where $V$ denotes an irreducible supermodule over $\mathbb{C}[C_1, \dots, C_r]$.

Let $M^{(\lambda)} := U_q(\mathfrak{q}(n))M_\lambda$ be the isotypic component of $M$ that is a highest weight module of highest weight $\lambda$. Let $B_\lambda = \{b^\nu;\ \nu = 1, \dots, s\}$. Then $L_\lambda/qL_\lambda = \bigoplus_{\nu=1}^{s} l_{b^\nu}$. Hence one can find an $\mathbf{A}[C_1, \dots, C_r]$-submodule $E_\nu$ of $L_\lambda$ for each $\nu = 1, \dots, s$ such that

$$E_\nu/qE_\nu = l_{b^\nu} \quad \text{and} \quad L_\lambda = \bigoplus_{\nu=1}^{s} E_\nu.$$

Setting $M^\nu := U_q(\mathfrak{q}(n))E_\nu$, we have

$$M^{(\lambda)} = \bigoplus_{\nu=1}^{s} M^\nu.$$

By Lemma 4.5, $M^\nu$ has a crystal basis

$$(L(M^\nu), B(\lambda), (l_b^\nu)_{b\in B(\lambda)})$$

such that $L(M^\nu)_\lambda = E_\nu$. Hence the direct sum $\bigoplus_{\nu=1}^{s}(L(M^\nu), B(\lambda), (l_b^\nu)_{b\in B(\lambda)})$ is a crystal basis of $M^{(\lambda)}$. Set $L(M^{(\lambda)}) := M^{(\lambda)} \cap L$. Since $L(M^{(\lambda)})_\lambda = L_\lambda = \sum_\nu E_\nu = (\sum_\nu L(M^\nu))_\lambda$, Lemma 4.2 implies $L(M^{(\lambda)}) = \bigoplus_{\nu=1}^{s} L(M^\nu)$. In particular, $L(M^\nu) = L \cap M^\nu$, and we can regard $L(M^\nu)/qL(M^\nu)$ as a subspace of $L/qL$.

The set $\{b \in B(\lambda);\ l_b^\nu = l_{b'} \text{ for some } b' \in B\}$ is stable under the Kashiwara operators and contains $b_\lambda$, and hence it coincides with $B(\lambda)$. Therefore the map $\phi_\nu\colon B(\lambda) \to B$ given by $l_b^\nu = l_{\phi_\nu(b)}$ $(b \in B(\lambda))$ is injective and commutes with the Kashiwara operators. Its image $C_\nu$ is thus the connected component of $b_\nu$ and we obtain

$$L(M^\nu)/qL(M^\nu) = \bigoplus_{b\in C_\nu} l_b.$$

Write $B = B_1 \sqcup B_2$, where $B_1 = \coprod_{\nu=1}^{s} C_\nu$. Then $(L(M^{(\lambda)}), B_1, l_{B_1})$ is a crystal basis of $M^{(\lambda)}$ and coincides with the direct sum of the crystal bases $(L(M^\nu), B(\lambda), l_{B(\lambda)}^\nu)$ of $M^\nu$.

Let $M = M^{(\lambda)} \oplus \widetilde{M}$ be the decomposition as a $U_q(\mathfrak{q}(n))$-module, and set $\widetilde{L} := L \cap \widetilde{M}$.

Set $S := q^{-1}L(M^{(\lambda)}) \cap (q^{-1}\widetilde{L} + L(M^{(\lambda)}))$. Then $S$ is invariant under the Kashiwara operators and $S_\lambda = L(M^{(\lambda)})_\lambda$. Hence by Lemma 4.2, we have $S = L(M^{(\lambda)})$, which implies $L(M^{(\lambda)}) \cap (\widetilde{L} + qL(M^{(\lambda)})) = qL(M^{(\lambda)})$. Hence

$$(L(M^{(\lambda)})/qL(M^{(\lambda)})) \cap (\widetilde{L}/q\widetilde{L}) = 0 \quad \text{as a subspace of } L/qL. \tag{4.4}$$

By comparing dimensions, we have

$$L/qL = (L(M^{(\lambda)})/qL(M^{(\lambda)})) \oplus (\widetilde{L}/q\widetilde{L}).$$

Therefore, by Nakayama's lemma,

$$L = L(M^{(\lambda)}) + \widetilde{L} = L(M^{(\lambda)}) \oplus \widetilde{L}. \tag{4.5}$$

Now, we shall show

$$\widetilde{L}/q\widetilde{L} = \bigoplus_{b \in B_2} l_b. \tag{4.6}$$

For $b \in B$, let $P_b \colon L/qL \twoheadrightarrow l_b$ be the canonical projection. Then, for $i = 1, \dots, n-1, \overline{1}$ satisfying $\tilde{e}_i b \in B$, we have a commutative diagram

$$\begin{array}{ccc} L/qL & \xrightarrow{\tilde{e}_i} & L/qL \\ \Big\downarrow{\scriptstyle P_b} & & \Big\downarrow{\scriptstyle P_{\tilde{e}_i b}} \\ l_b & \underset{\tilde{e}_i}{\xrightarrow{\sim}} & l_{\tilde{e}_i b} \end{array}$$

Hence $P_{\tilde{e}_i b}(\widetilde{L}/q\widetilde{L}) = 0$ implies $P_b(\widetilde{L}/q\widetilde{L}) = 0$. Similarly, $P_{\tilde{f}_i b}(\widetilde{L}/q\widetilde{L}) = 0$ implies $P_b(\widetilde{L}/q\widetilde{L}) = 0$. Hence $S := \{b \in B_1;\ P_b(\widetilde{L}/q\widetilde{L}) = 0\}$ is stable under the Kashiwara operators. Since $S_\lambda = B_\lambda$, we obtain $S = B_1$. Hence $\widetilde{L}/q\widetilde{L} \subset \bigoplus_{b \in B_2} l_b$. Then (4.5) implies the desired result $\widetilde{L}/q\widetilde{L} = \bigoplus_{b \in B_2} l_b$.

Therefore $(\widetilde{L}, B_2, (l_b)_{b \in B_2})$ is a crystal basis of $\widetilde{M}$. Hence the crystal basis $(L, B, l_B)$ of $M$ is the direct sum of a crystal basis of $\widetilde{M}$ and crystal bases of $M^\nu$ ($\nu = 1, \dots, s$). Since $\dim \widetilde{M} < \dim M$, our assertion follows by induction on $\dim M$. □

*Acknowledgments.* The work of D. Grantcharov was partially supported by NSA grant H98230-10-1-0207 and by Max Planck Institute for Mathematics, Bonn.

The work of J. H. Jung was partially supported by BK21 Mathematical Sciences Division and by NRF Grant # 2010-0010753.

The work of S.-J. Kang was partially supported by KRF Grant # 2007-341-C00001 and by National Institute for Mathematical Sciences (2010 Thematic Program, TP1004).

The work of M. Kashiwara was partially supported by Grant-in-Aid for Scientific Research (B) 23340005, Japan Society for the Promotion of Science.

The work of M. Kim was partially supported by KRF Grant # 2007-341-C00001 and by NRF Grant # 2010-0019516.

## References

[1] Benkart, G., Kang, S.-J., Kashiwara, M.: Crystal bases for the quantum superalgebra $U_q(\mathfrak{gl}(m,n))$. J. Amer. Math. Soc. **13**, 295–331 (2000) Zbl 0963.17010 MR 1694051

[2] Grantcharov, D., Jung, J. H., Kang, S.-J., Kim, M.: Highest weight modules over quantum queer superalgebra $U_q(\mathfrak{q}(n))$. Comm. Math. Phys. **296**, 827–860 (2010) Zbl 1268.17015 MR 2628823

[3] Hong, J., Kang, S.-J.: Introduction to Quantum Groups and Crystal Bases. Grad. Stud. Math. 42, Amer. Math. Soc. (2002) Zbl 1134.17007 MR 1881971

[4] Jimbo, M., Misra, K. C., Miwa, T., Okado, M.: Combinatorics of representations of $U_q(\widehat{\mathfrak{sl}}(n))$ at $q = 0$. Comm. Math. Phys. **136**, 543–566 (1991) Zbl 0749.17015 MR 1099695

[5] Kac, V.: Lie superalgebras. Adv. Math. **26**, 8–96 (1977) Zbl 0366.17012 MR 0486011

[6] Kang, S.-J.: Crystal bases for quantum affine algebras and combinatorics of Young walls. Proc. London Math. Soc. **(3) 86**, 29–69 (2003) Zbl 1030.17013 MR 1971463

[7] Kang, S.-J., Kashiwara, M., Misra, K. C., Miwa, T., Nakashima, T., Nakayashiki, A.: Affine crystals and vertex models. In: Infinite Analysis, Part A, B (Kyoto, 1991), Adv. Ser. Math. Phys. 16, World Sci., 449–484 (1992) Zbl 0925.17005 MR 1187560

[8] Kang, S.-J., Kashiwara, M., Misra, K. C., Miwa, T., Nakashima, T., Nakayashiki, A.: Perfect crystals of quantum affine Lie algebras. Duke Math. J. **68**, 499–607 (1992) Zbl 0774.17017 MR 1194953

[9] Kang, S.-J., Kwon, J.-H.: Fock space representations of quantum affine algebras and generalized Lascoux–Leclerc–Thibon algorithm. J. Korean Math. Soc. **45**, 1135–1202 (2008) Zbl 1226.17010 MR 2422732

[10] Kashiwara, M.: Crystalizing the $q$-analogue of universal enveloping algebras. Comm. Math. Phys. **133**, 249–260 (1990) Zbl 0724.17009 MR 1090425

[11] Kashiwara, M.: On crystal bases of the $q$-analogue of universal enveloping algebras. Duke Math. J. **63**, 465–516 (1991) Zbl 0739.17005 MR 1115118

[12] Kashiwara, M.: Crystal base and Littelmann's refined Demazure character formula. Duke Math. J. **71**, 839–858 (1993) Zbl 0794.17008 MR 1240605

[13] Kashiwara, M.: Crystal bases of modified quantized enveloping algebra. Duke Math. J. **73**, 383–413 (1994) Zbl 0794.17009 MR 1262212

[14] Kashiwara, M., Nakashima, T.: Crystal graphs for representations of the $q$-analogue of classical Lie algebras. J. Algebra **165**, 295–345 (1994) Zbl 0808.17005 MR 1273277

[15] Kashiwara, M., Saito, Y.: Geometric construction of crystal bases. Duke Math. J. **89**, 9–36 (1997) Zbl 0901.17006 MR 1458969

[16] Littelmann, P.: A Littlewood–Richardson rule for symmetrizable Kac–Moody algebras. Invent. Math. **116**, 329–346 (1994) Zbl 0805.17019 MR 1253196

[17] Littelmann, P.: Paths and root operators in representation theory. Ann. of Math. (2) **142**, 499–525 (1995) Zbl 0858.17023 MR 1356780

[18] Lusztig, G.: Canonical bases arising from quantized enveloping algebras. J. Amer. Math. Soc. **3**, 447–498 (1990) Zbl 0703.17008 MR 1035415

[19] Lusztig, G.: Quivers, perverse sheaves, and quantized enveloping algebras. J. Amer. Math. Soc. **4**, 365–421 (1991) Zbl 0738.17011 MR 1088333

[20] Misra, K., Miwa, T.: Crystal base for the basic representation of $U_q(\widehat{\mathfrak{sl}}(n))$. Comm. Math. Phys. **134**, 79–88 (1990) Zbl 0724.17010 MR 1079801

[21] Nakajima, H.: Quiver varieties and finite-dimensional representations of quantum affine algebras. J. Amer. Math. Soc. **14**, 145–238 (2001) Zbl 0981.17016 MR 1808477

[22] Nakashima, T.: Crystal base and a generalization of the Littlewood–Richardson rule for the classical Lie algebras. Comm. Math. Phys. **154**, 215–243 (1993) Zbl 0795.17016 MR 1224078

[23] Olshanski, G. I.: Quantized universal enveloping superalgebra of type $Q$ and a super-extension of the Hecke algebra. Lett. Math. Phys. **24**, 93–102 (1992) Zbl 0766.17017 MR 1163061

[24] Penkov, I., Serganova, V.: Characters of finite-dimensional irreducible $q(n)$-modules. Lett. Math. Phys. **40**, 147–158 (1997) Zbl 0892.17006 MR 1463616

[25] Saito, Y.: Crystal bases and quiver varieties. Math. Ann. **324**, 675–688 (2002) Zbl 1041.17018 MR 1942245

[26] Sergeev, A.: The tensor algebra of the tautological representation as a module over the Lie superalgebras $\mathfrak{gl}(n, m)$ and $Q(n)$. Mat. Sb. **123**, 422–430 (1984) (in Russian) Zbl 0573.17002 MR 0735715