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On the duality between p-modulus and probability measures

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Abstract. Motivated by recent developments on calculus in metric measure spaces $(X, \mathsf{d}, \mathfrak{m})$, we prove a general duality principle between Fuglede's notion [15] of p-modulus for families of finite Borel measures in (X, d) and probability measures with barycenter in $L^q(X, \mathfrak{m})$, with q the dual exponent of $p \in (1, \infty)$. We apply this general duality principle to study null sets for families of parametric and nonparametric curves in X. In the final part of the paper we provide a new proof, independent of optimal transportation, of the equivalence of notions of weak upper gradient based on p-modulus ([21], [23]) and suitable probability measures in the space of curves ([6], [7]).

Keywords. p-Modulus, capacity, duality, Sobolev functions

1. Introduction

The notion of p-modulus $\mathrm{Mod}_p(\Gamma)$ for a family Γ of curves was introduced by Ahlfors and Beurling [2] and then deeply studied by Fuglede [15], who realized its significance in real analysis and proved that Sobolev $W^{1,p}$ functions f in $\mathbb{R}^n$ have representatives $\tilde{f}$ that satisfy

$$\tilde{f}(\gamma_b) - \tilde{f}(\gamma_a) = \int_a^b \langle \nabla f(\gamma_t), \gamma_t' \rangle \, dt$$

for Mod_p-almost every absolutely continuous curve $\gamma : [a, b] \to \mathbb{R}^n$. Recall that if Γ is a family of absolutely continuous curves, $\mathrm{Mod}_p(\Gamma)$ is defined by

$$\mathrm{Mod}_p(\Gamma) := \inf\left\{ \int_{\mathbb{R}^n} f^p \, dx : \ f : \mathbb{R}^n \to [0, \infty] \text{ Borel}, \ \int_\gamma f \geq 1 \text{ for all } \gamma \in \Gamma \right\}. \quad (1.1)$$

It is obvious that this definition (as the notion of length) is parameter-free, because the curves are involved in the definition only through the curvilinear integral $\int_\gamma f$. Furthermore, if $\gamma : I \to X$, writing the curvilinear integral as $\int_I f(\gamma_t)|\dot{\gamma}_t| \, dt$, with $|\dot{\gamma}|$ equal to the metric derivative, one realizes immediately that this notion makes sense for absolutely

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continuous curves in a general metric space (X, d), if we add a reference measure $\mathfrak{m}$ to minimize the integral $\int f^p \, d\mathfrak{m}$. The notion, denoted by $\mathrm{Mod}_{p,\mathfrak{m}}(\cdot)$, actually extends to families of continuous curves with finite length which have a Lipschitz reparameterization. As in [15], one can even go a step further, realizing that the curvilinear integral in (1.1) can be written as

$$\int_X f \, dJ\gamma,$$

where $J\gamma$ is a positive finite measure in X, the image under γ of the measure $|\dot{\gamma}|\mathscr{L}^1 \llcorner I$, namely

$$J\gamma(B) = \int_{\gamma^{-1}(B)} |\dot{\gamma}_t| \, dt \qquad \forall B \in \mathscr{B}(X) \tag{1.2}$$

(here $\mathscr{L}^1 \llcorner I$ stands for the Lebesgue measure on I). It follows that one can define in a similar way the notion of p-modulus for families of measures in X.

In more recent times, Koskela–Mac Manus [21] and then Shanmugalingham [23] used the p-modulus to define the notion of p-weak upper gradient for a function f, namely Borel functions $g : X \to [0, \infty]$ such that the upper gradient inequality

$$|f(\gamma_b) - f(\gamma_a)| \leq \int_\gamma g \tag{1.3}$$

holds along $\mathrm{Mod}_{p,\mathfrak{m}}$-almost every absolutely continuous curve $\gamma : [a, b] \to X$. This approach leads to a very successful Sobolev space theory in metric measure spaces $(X, \mathsf{d}, \mathfrak{m})$; see for instance [17, 11] for a very nice account.

Even more recently, the first and third author and Nicola Gigli introduced (first in [6] for $p = 2$, and then in [7] for general p) another notion of weak upper gradient, based on suitable classes of probability measures on curves, described in more detail in the final section of this paper. Since the axiomatization in [6] is quite different and sensitive to parameterization, it is surprising that the two approaches lead essentially to the same Sobolev space theory (see [6, Remark 5.12] for a more detailed discussion, also in connection with Cheeger's approach [13], and Section 9 of this paper). We say "essentially" because, strictly speaking, the axiomatization of [6] is invariant (unlike Fuglede's approach) under modification of f on $\mathfrak{m}$-negligible sets and thus provides only Sobolev regularity and not absolute continuity along almost every curve; however, if we properly choose representatives in Lebesgue equivalence classes, the two Sobolev spaces can be identified.

Actually, as illustrated in [6], [8], [16] (see also the more recent work [10], in connection with Rademacher's theorem and Cheeger's Lipschitz charts), differential calculus and suitable notions of *tangent* bundle in metric measure spaces can be developed in a quite natural way using probability measures in the space of absolutely continuous curves.

With the goal of understanding deeper connections between the $\mathrm{Mod}_{p,\mathfrak{m}}$ and the probabilistic approaches, we show in this paper that the theory of p-modulus has a "dual" point of view, based on suitable probability measures $\boldsymbol{\pi}$ in the space of curves; the main difference from [6] is that, as it should be, the curves here are nonparametric, namely $\boldsymbol{\pi}$ should rather be thought of as measures in a quotient space of curves. Actually, this and other technical aspects (also relative to tightness, since much better compactness properties are available at the level of measures) are simplified if we consider the p-modulus of families of measures in $\mathcal{M}_+(X)$ (the space of all nonnegative and finite Borel measures on X), rather than the p-modulus of families of curves: if we have a family Γ of curves, we can consider the family $\Sigma = J(\Gamma)$ and derive a representation formula for $\mathrm{Mod}_{p,\mathfrak{m}}(\Gamma)$ (see Section 7). Correspondingly, $\boldsymbol{\pi}$ will be a measure on the Borel subsets of $\mathcal{M}_+(X)$.

For this reason, in Part I of this paper we investigate the duality at this level of generality, considering a family Σ of measures in $\mathcal{M}_+(X)$. Assuming only that (X,d) is complete and separable and $\mathfrak{m}$ is finite, we prove in Theorem 5.1 that for all Borel sets $\Sigma \subset \mathcal{M}_+(X)$ (and actually in the more general class of Suslin sets) the following duality formula holds:

$$\mathrm{Mod}_{p,\mathfrak{m}}(\Sigma)^{1/p} = \sup_{\boldsymbol{\eta}} \frac{\boldsymbol{\eta}(\Sigma)}{c_q(\boldsymbol{\eta})} = \sup_{\boldsymbol{\eta}(\Sigma)=1} \frac{1}{c_q(\boldsymbol{\eta})}, \qquad \frac{1}{p} + \frac{1}{q} = 1,\ p \in (1,\infty). \tag{1.4}$$

Here the suprema are taken over the class of Borel probability measures $\boldsymbol{\eta}$ in $\mathcal{M}_+(X)$ with barycenter in $L^q(X,\mathfrak{m})$, so that

$$\exists g \in L^q(X,\mathfrak{m})\ \forall A \in \mathscr{B}(X) \qquad \int \mu(A)\,d\boldsymbol{\eta}(\mu) = \int_A g\,d\mathfrak{m};$$

the constant $c_q(\boldsymbol{\eta})$ is then defined as the $L^q(X,\mathfrak{m})$ norm of the "barycenter" g. A byproduct of our proof is that $\mathrm{Mod}_{p,\mathfrak{m}}$ is a Choquet capacity in $\mathcal{M}_+(X)$ (see Theorem 5.1). In addition, we prove in Corollary 5.2 the existence of maximizers in (1.4) and obtain from this necessary and sufficient optimality conditions, both for $\boldsymbol{\eta}$ and for the minimal f involved in the definition of p-modulus analogous to (1.1). See also Remark 3.3 for a simple application of these optimality conditions involving pairs (μ, f) on which the constraint is saturated, that is, $\int_X f\,d\mu = 1$.

We are not aware of other representation formulas for $\mathrm{Mod}_{p,\mathfrak{m}}$, except in special cases: for instance in the case of the family Γ of curves connecting two disjoint compact sets K_0, K_1 of $\mathbb{R}^n$, the modulus in (1.1) equals (see [24] and also [20] for the extension to metric measure spaces, as well as [1] for related results) the capacity

$$C_p(K_0,K_1) := \inf\left\{\int_{\mathbb{R}^n} |\nabla u|^p\,dx : u \equiv 0 \text{ on } K_0,\ u \equiv 1 \text{ on } K_1\right\}.$$

In the conformal case $p = n$, it can also be proved that $C_n(K_0,K_1)^{-1/(n-1)}$ equals $\mathrm{Mod}_{n/(n-1)}(\Sigma)$, where Σ is the family of the Hausdorff measures $\mathscr{H}^{n-1} \llcorner S$ with S separating K_0 from K_1 (see [25]).

In the second part of the paper, after introducing in Section 6 the relevant space of curves $\mathrm{AC}^q([0,1];X)$ and a suitable quotient space $\mathscr{C}(X)$ of nonparametric nonconstant curves, we show how the basic duality result of Part I can be read in terms of measures and moduli in spaces of curves. For nonparametric curves this is accomplished in Section 7, by mapping curves in X to measures in X by means of the canonical map J in (1.2); in this case, the condition of having a barycenter in $L^q(X,\mathfrak{m})$ becomes

$$\left|\int\int_0^1 f(\gamma_t)|\dot\gamma_t|\,dt\,d\boldsymbol{\pi}(\gamma)\right| \le C\|f\|_{L^p(X,\mathfrak{m})} \qquad \forall f \in \mathrm{C}_b(X). \tag{1.5}$$

Section 8 is devoted instead to the case of parametric curves, where the relevant curves-to-measures map is

$$M\gamma(B) := \mathscr{L}^1(\gamma^{-1}(B)) \quad \forall B \in \mathscr{B}(X).$$

In this case the condition of having a *parametric* barycenter in $L^q(X, \mathfrak{m})$ becomes

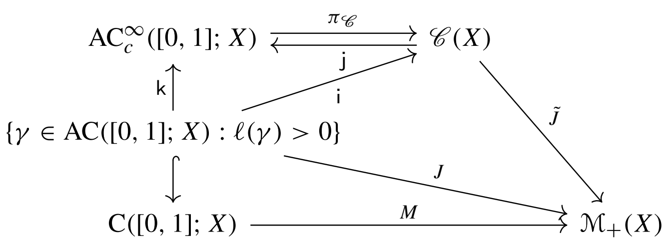
$$\left| \int \int_0^1 f(\gamma_t)\, dt\, d\boldsymbol{\pi}(\gamma) \right| \le C \|f\|_{L^p(X,\mathfrak{m})} \quad \forall f \in \mathrm{C}_b(X). \tag{1.6}$$

The parametric barycenter can of course be affected by reparameterizations; a key result, stated in Theorem 8.5, shows that suitable reparameterizations improve the parametric barycenter from $L^q(X, \mathfrak{m})$ to $L^\infty(X, \mathfrak{m})$. Then, in Section 9 we discuss the notion of null set of curves according to [6] and [7] (where (1.6) is strengthened by requiring $|\int f(\gamma_t)\, d\boldsymbol{\pi}(\gamma)| \le C\|f\|_{L^1(X,\mathfrak{m})}$ for all t, for some C independent of t) and, under suitable invariance and stability assumptions on the set of curves, we compare this notion with the one based on p-modulus. Eventually, in Section 10 we use these results to prove that if a Borel function $f : X \to \mathbb{R}$ has a continuous representative along a subcollection Γ of the set $\mathrm{AC}^\infty([0,1]; X)$ of Lipschitz parametric curves with $\mathrm{Mod}_{p,\mathfrak{m}}(M(\mathrm{AC}^\infty([0,1]; X) \setminus \Gamma)) = 0$, then it is possible to find a distinguished $\mathfrak{m}$-measurable representative $\tilde f$ such that $\mathfrak{m}(\{f \ne \tilde f\}) = 0$ and $\tilde f$ is absolutely continuous along $\mathrm{Mod}_{p,\mathfrak{m}}$-a.e. nonparametric curve. By using these results we provide a more direct proof of the equivalence of the two above mentioned notions of weak upper gradient, where different notions of null sets of curves are used to quantify exceptions to (1.3).

For the reader's convenience we collect in the next table and figure the main notation used, mostly in the second part of the paper.

Main notation

p, q	conjugate exponents in $[1, \infty]$, $p^{-1} + q^{-1} = 1$
$\mathcal{L}^p_+(X, \mathfrak{m})$	nonnegative Borel functions $f : X \to [0, \infty]$ with $\int_X f^p\, d\mathfrak{m} < \infty$
$L^p(X, \mathfrak{m})$	Lebesgue space of p-summable $\mathfrak{m}$-measurable functions
$\ell(\gamma)$	Length of a parametric curve γ
$\mathrm{AC}^q([0,1]; X)$	Space of parametric curves $\gamma : [0,1] \to X$ with q-integrable metric speed
$\mathrm{AC}_0([0,1]; X)$	Space of parametric curves with positive speed $\mathscr{L}^1$-a.e. in $(0,1)$
$\mathrm{AC}^\infty_c([0,1]; X)$	Space of parametric curves with positive and constant speed
k	Embedding of $\{\gamma \in \mathrm{AC}([0,1]; X) : \ell(\gamma) > 0\}$ into $\mathrm{AC}^\infty_c([0,1]; X)$
$\mathscr{C}(X)$	Space of nonparametric and nonconstant curves, see Definition 6.5
i	Embedding of $\{\gamma \in \mathrm{AC}([0,1]; X) : \ell(\gamma) > 0\}$ in $\mathscr{C}(X)$
j	Embedding of $\mathscr{C}(X)$ into $\mathrm{AC}^\infty_c([0,1]; X)$



Part I. Duality between modulus and content

2. Notation and preliminary notions

In a topological Hausdorff space (E, τ), we denote by $\mathscr{P}(E)$ the collection of all subsets of E, by $\mathscr{F}(E)$ (resp. $\mathscr{K}(E)$) the collection of all closed (resp. compact) sets of E, and by $\mathscr{B}(E)$ the σ-algebra of Borel sets of E. We denote by $\mathrm{C}_b(E)$ the space of bounded continuous functions on (E, τ), by $\mathcal{M}_+(E)$ the set of σ-additive measures $\mu : \mathscr{B}(E) \to [0, \infty)$, and by $\mathcal{P}(E)$ the subclass of probability measures. For a set $F \subset E$ and $\mu \in \mathcal{M}_+(E)$ we write $\chi_F : E \to \{0, 1\}$ for the characteristic function of F and $\mu \llcorner F$ for the measure $\chi_F \mu$, if F is μ-measurable. For a Borel map $L : E \to F$ we let $L_\sharp : \mathcal{M}_+(E) \to \mathcal{M}_+(F)$ be the induced push-forward operator between Borel measures, that is,

$$L_\sharp \mu(B) := \mu(L^{-1}(B)) \quad \forall \mu \in \mathcal{M}_+(E),\ B \in \mathscr{B}(F).$$

We denote by $\mathbb{N} = \{0, 1, \ldots\}$ the natural numbers, and by $\mathscr{L}^1$ the Lebesgue measure on the real line.

2.1. *Polish spaces*

Recall that (E, τ) is said to be *Polish* if there exists a distance ρ in E which induces the topology τ such that (E, ρ) is complete and separable. Notice that the inclusion of $\mathcal{M}_+(E)$ in $(\mathrm{C}_b(E))^*$ may be strict, because we are not making compactness or local compactness assumptions on (E, τ). Nevertheless, if (E, τ) is Polish we can always endow $\mathcal{M}_+(E)$ with a Polish topology w-$\mathrm{C}_b(E)$ whose convergent sequences are precisely the weakly convergent ones, i.e. sequences convergent in the duality with $\mathrm{C}_b(E)$. Obviously this Polish topology is unique. A possible choice, which can be easily adapted from the corresponding Kantorovich–Rubinstein distance on $\mathcal{P}(E)$ (see e.g. [12, §8.3] or [5, Section 7.1]), is to consider the duality with bounded and Lipschitz functions,

$$\rho_{KR}(\mu, \nu) := \sup\left\{ \left| \int_E f\, d\mu - \int_E f\, d\nu \right| : f \in \mathrm{Lip}_b(E),\ \sup_E |f| \le 1, \right.$$
$$\left. |f(x) - f(y)| \le \rho(x, y)\ \forall x, y \in E \right\}.$$

2.2. *Suslin, Lusin and analytic sets. Choquet theorem*

Denote by $\mathbb{N}^\infty$ the collection of all infinite sequences of natural numbers and by $\mathbb{N}_0^\infty$ the collection of all finite sequences $(n_0, \ldots, n_i)$ with $i \ge 0$ and n_i natural numbers. Let $\mathscr{A}$ be a subset of $\mathscr{P}(E)$ containing the empty set (typical examples are, in topological spaces (E, τ), the classes $\mathscr{F}(E)$, $\mathscr{K}(E)$, $\mathscr{B}(E)$). We define a *table of sets* in $\mathscr{A}$ to be a map C associating to each finite sequence $(n_0, \ldots, n_i) \in \mathbb{N}_0^\infty$ a set $C_{(n_0,\ldots,n_i)} \in \mathscr{A}$.

Definition 2.1 ($\mathscr{A}$-analytic sets). A set $S \subset E$ is said to be *$\mathscr{A}$-analytic* if there exists a table C of sets in $\mathscr{A}$ such that

$$S = \bigcup_{(n) \in \mathbb{N}^\infty} \bigcap_{i=0}^{\infty} C_{(n_0,\ldots,n_i)}.$$

Recall that, in a topological space (E, τ), $\mathscr{B}(E)$-analytic sets are *universally measurable* [12, Theorem 1.10.5], that is, σ-measurable for any $\sigma \in \mathcal{M}_+(E)$.

Definition 2.2 (Suslin and Lusin sets). Let (E, τ) be a Hausdorff topological space. Then $S \in \mathscr{P}(E)$ is said to be a *Suslin* (resp. *Lusin*) *set* if it is the image of a Polish space under a continuous (resp. continuous and injective) map.

Even though the Suslin and Lusin properties for subsets of a topological space are intrinsic, i.e. they depend only on the induced topology, we will often use the phrase "S is a Suslin subset of E" and similar to emphasize the ambient space; the Borel property, instead, is not intrinsic, since $S \in \mathscr{B}(S)$ if we endow S with the induced topology. Besides the obvious stability with respect to transformations under continuous (resp. continuous and injective) maps, the class of Suslin (resp. Lusin) sets enjoys nice properties, detailed below.

Proposition 2.3. *The following properties hold:*

(i) *In a Hausdorff topological space (E, τ), Suslin sets are $\mathscr{F}(E)$-analytic.*
(ii) *If (E, τ) is a Suslin space (in particular if it is a Polish or a Lusin space), the notions of Suslin and $\mathscr{F}(E)$-analytic sets concide and in this case Lusin sets are Borel and Borel sets are Suslin.*
(iii) *If E, F are Suslin spaces and $f : E \to F$ is a Borel injective map, then f^{-1} is Borel.*
(iv) *If E, F are Suslin spaces and $f : E \to F$ is a Borel map, then f maps Suslin sets to Suslin sets.*

Proof. We refer to [12] for all these statements: (i) is proved in Theorem 6.6.8 there; in connection with (ii), the equivalence between Suslin and $\mathscr{F}(E)$-analytic sets is proved in Theorem 6.7.2, the fact that Borel sets are Suslin in Corollary 6.6.7 and the fact that Lusin sets are Borel in Theorem 6.8.6; finally, (iii) and (iv) are proved in Theorem 6.7.3. □

Since in Polish spaces (E, τ) we have at the same time tightness of finite Borel measures and coincidence of Suslin and $\mathscr{F}(E)$-analytic sets, the measurability of $\mathscr{B}(E)$-analytic sets implies in particular that

$$\sigma(B) = \sup\{\sigma(K) : K \in \mathscr{K}(E),\ K \subset B\} \quad \text{for all } B \subset E \text{ Suslin}, \sigma \in \mathcal{M}_+(E). \tag{2.1}$$

We will need a property analogous to (2.1) for capacities [14], whose definition is recalled below.

Definition 2.4 (Capacity). A set function $\mathfrak{I} : \mathscr{P}(E) \to [0, \infty]$ is said to be a *capacity* if:

- $\mathfrak{I}$ is nondecreasing and, whenever $(A_n) \subset \mathscr{P}(E)$ is nondecreasing,

$$\lim_{n\to\infty} \mathfrak{I}(A_n) = \mathfrak{I}\Big(\bigcup_{n=0}^{\infty} A_n\Big);$$

- if $(K_n) \subset \mathscr{K}(E)$ is nonincreasing, then

$$\lim_{n\to\infty} \mathfrak{I}(K_n) = \mathfrak{I}\Big(\bigcap_{n=0}^{\infty} K_n\Big).$$

A set $B \subset E$ is said to be $\mathfrak{I}$-*capacitable* if $\mathfrak{I}(B) = \sup_{K\in\mathscr{K}(E),\, K\subset B} \mathfrak{I}(K)$.

Theorem 2.5 (Choquet, [14, Thm. 28.III]). *Every $\mathscr{K}(E)$-analytic set is capacitable.*

3. $(p, \mathfrak{m})$-modulus $\mathrm{Mod}_{p,\mathfrak{m}}$

In this section, (X, τ) is a topological space and $\mathfrak{m}$ is a fixed Borel and nonnegative reference measure, not necessarily finite or σ-finite.

Given a power $p \in [1, \infty)$, we set

$$\mathcal{L}^p_+(X, \mathfrak{m}) := \Big\{ f : X \to [0, \infty] : f \text{ Borel}, \int_X f^p \, d\mathfrak{m} < \infty \Big\}. \tag{3.1}$$

We stress that, unlike $L^p(X, \mathfrak{m})$, this space is not quotiented by any equivalence relation; however we will keep using the notation

$$\|f\|_p := \Big(\int_X |f|^p \, d\mathfrak{m}\Big)^{1/p}$$

as a seminorm on $\mathcal{L}^p_+(X, \mathfrak{m})$ and a norm in $L^p(X, \mathfrak{m})$.

Given $\Sigma \subset \mathcal{M}_+$ we define (with the usual convention $\inf \emptyset = \infty$)

$$\mathrm{Mod}_{p,\mathfrak{m}}(\Sigma) := \inf\Big\{ \int_X f^p \, d\mathfrak{m} : f \in \mathcal{L}^p_+(X, \mathfrak{m}), \int_X f \, d\mu \geq 1 \text{ for all } \mu \in \Sigma \Big\}, \tag{3.2}$$

$$\mathrm{Mod}_{p,\mathfrak{m},c}(\Sigma) := \inf\Big\{ \int_X f^p \, d\mathfrak{m} : f \in \mathrm{C}_b(X, [0, \infty)), \int_X f \, d\mu \geq 1 \text{ for all } \mu \in \Sigma \Big\}. \tag{3.3}$$

Equivalently, if $0 < \mathrm{Mod}_{p,\mathfrak{m}}(\Sigma) \leq \infty$, we can say that $\mathrm{Mod}_{p,\mathfrak{m}}(\Sigma)^{-1}$ is the least $\xi \in [0, \infty)$ such that

$$\Big(\inf_{\mu\in\Sigma} \int_X f \, d\mu\Big)^p \leq \xi \int_X f^p \, d\mathfrak{m} \quad \text{for all } f \in \mathcal{L}^p_+(X, \mathfrak{m}), \tag{3.4}$$

and similarly there is also an equivalent definition for $\mathrm{Mod}_{p,\mathfrak{m},c}(\Sigma)^{-1}$.

Notice that the infimum in (3.3) is unchanged if we restrict the minimization to nonnegative functions $f \in \mathrm{C}_b(X)$. As a consequence, since the finiteness of $\mathfrak{m}$ provides the inclusion of this class of functions in $\mathcal{L}^p_+(X, \mathfrak{m})$, we get $\mathrm{Mod}_{p,\mathfrak{m},c}(\Sigma) \geq \mathrm{Mod}_{p,\mathfrak{m}}(\Sigma)$ whenever $\mathfrak{m}$ is finite. Also, if Σ contains the null measure, then we have $\mathrm{Mod}_{p,\mathfrak{m},c}(\Sigma) \geq \mathrm{Mod}_{p,\mathfrak{m}}(\Sigma) = \infty$.

Definition 3.1 ($\mathrm{Mod}_{p,\mathfrak{m}}$-negligible sets). A set $\Sigma \subset \mathcal{M}_+(X)$ is said to be $\mathrm{Mod}_{p,\mathfrak{m}}$-*negligible* if $\mathrm{Mod}_{p,\mathfrak{m}}(\Sigma) = 0$.

A property P on $\mathcal{M}_+(X)$ is said to hold $\mathrm{Mod}_{p,\mathfrak{m}}$-a.e. if the set

$$\{\mu \in \mathcal{M}_+(X) : P(\mu) \text{ fails}\}$$

is $\mathrm{Mod}_{p,\mathfrak{m}}$-negligible. With this terminology, we can also write

$$\mathrm{Mod}_{p,\mathfrak{m}}(\Sigma) = \inf\left\{\int_X f^p \, d\mathfrak{m} : \int_X f \, d\mu \geq 1 \text{ for } \mathrm{Mod}_{p,\mathfrak{m}}\text{-a.e. } \mu \in \Sigma\right\}. \tag{3.5}$$

We now list some classical properties that will be useful in what follows; most of them are well known and easy to prove, but we provide complete proofs for the reader's convenience.

Proposition 3.2. *The set functions* $\mathcal{M}_+(X) \supset A \mapsto \mathrm{Mod}_{p,\mathfrak{m}}(A)$ *and* $\mathcal{M}_+(X) \supset A \mapsto \mathrm{Mod}_{p,\mathfrak{m},c}(A)$ *have the following properties:*

(i) *Both are monotone and their* $1/p$*-th powers are subadditive.*

(ii) *If* $g \in \mathcal{L}^p_+(X, \mathfrak{m})$ *then* $\int_X g \, d\mu < \infty$ *for* $\mathrm{Mod}_{p,\mathfrak{m}}$*-almost every* μ*; conversely, if* $\mathrm{Mod}_{p,\mathfrak{m}}(A) = 0$ *then there exists* $g \in \mathcal{L}^p_+(X, \mathfrak{m})$ *such that* $\int_X g \, d\mu = \infty$ *for every* $\mu \in A$.

(iii) *If* $(f_n) \subset \mathcal{L}^p_+(X, \mathfrak{m})$ *converges in* $L^p(X, \mathfrak{m})$ *seminorm to* $f \in \mathcal{L}^p_+(X, \mathfrak{m})$ *then there exists a subsequence* $(f_{n(k)})$ *such that*

$$\int_X f_{n(k)} \, d\mu \to \int_X f \, d\mu \qquad \mathrm{Mod}_{p,\mathfrak{m}}\text{-a.e. in } \mathcal{M}_+(X). \tag{3.6}$$

(iv) *If* $p > 1$ *then for every* $\Sigma \subset \mathcal{M}_+(X)$ *with* $\mathrm{Mod}_{p,\mathfrak{m}}(\Sigma) < \infty$ *there exists* $f \in \mathcal{L}^p_+(X, \mathfrak{m})$*, unique up to* $\mathfrak{m}$*-negligible sets, such that* $\int_X f \, d\mu \geq 1$ $\mathrm{Mod}_{p,\mathfrak{m}}$*-a.e. on* Σ *and* $\|f\|_p^p = \mathrm{Mod}_{p,\mathfrak{m}}(\Sigma)$*;*

(v) *If* $p > 1$ *and* A_n *are nondecreasing subsets of* $\mathcal{M}_+(X)$ *then*

$$\mathrm{Mod}_{p,\mathfrak{m}}(A_n) \uparrow \mathrm{Mod}_{p,\mathfrak{m}}\Big(\bigcup_n A_n\Big).$$

(vi) *If* K_n *are nonincreasing compact subsets of* $\mathcal{M}_+(X)$ *then*

$$\mathrm{Mod}_{p,\mathfrak{m},c}(K_n) \downarrow \mathrm{Mod}_{p,\mathfrak{m},c}\Big(\bigcap_n K_n\Big).$$

(vii) *Let* $A \subset \mathcal{M}_+(X)$*,* $F : A \to (0, \infty)$ *be a Borel map, and* $B = \{F(\mu)\mu : \mu \in A\}$*. If* $\mathrm{Mod}_{p,\mathfrak{m}}(A) = 0$ *then* $\mathrm{Mod}_{p,\mathfrak{m}}(B) = 0$ *as well.*

Proof. (i) Monotonicity is trivial. For subadditivity, if $\int_X f \, d\mu \geq 1$ on A and $\int_X g \, d\mu \geq 1$ on B, then $\int_X (f + g) \, d\mu \geq 1$ on $A \cup B$, hence $\mathrm{Mod}_{p,\mathfrak{m}}(A \cup B)^{1/p} \leq \|f + g\|_p \leq \|f\|_p + \|g\|_p$. Minimizing over f and g we get subadditivity.

(ii) Consider the set where the property fails:

$$\Sigma_g = \left\{\mu \in \mathcal{M}_+(X) : \int_X g \, d\mu = \infty\right\}.$$

Then it is clear that $\mathrm{Mod}_{p,\mathfrak{m}}(\Sigma_g) \le \|g\|_p^p$ but $\Sigma_g = \Sigma_{\lambda g}$ for every $\lambda > 0$, and so Σ_g is $\mathrm{Mod}_{p,\mathfrak{m}}$-negligible. Conversely, if $\mathrm{Mod}_{p,\mathfrak{m}}(A) = 0$ for every $n \in \mathbb{N}$, we can find $g_n \in \mathcal{L}_+^p(X, \mathfrak{m})$ with $\int_X g_n \, d\mu \ge 1$ for every $\mu \in A$ and $\int_X g_n^p \le 2^{-np}$. Thus $g := \sum_n g_n$ has the required properties.

(iii) Let $f_{n(k)}$ be a subsequence such that $\|f - f_{n(k)}\|_p \le 2^{-k}$ so that if we set

$$g(x) = \sum_{k=1}^{\infty} |f(x) - f_{n(k)}(x)|$$

then $g \in \mathcal{L}_+^p(X, \mathfrak{m})$ and $\|g\|_p \le 1$; in particular we see, from (ii) above, that $\int_X g \, d\mu$ is finite for $\mathrm{Mod}_{p,\mathfrak{m}}$-almost every μ. For those μ we have

$$\sum_{k=1}^{\infty} \int_X |f - f_{n(k)}| \, d\mu < \infty,$$

and thus we get (3.6).

(iv) Since we can use (3.5) to compute $\mathrm{Mod}_{p,\mathfrak{m}}(\Sigma)$, we deduce from (ii) and (iii) that the class of admissible functions f is a convex and closed subset of the Lebesgue space L^p. Hence, uniqueness follows by the strict convexity of the L^p norm.

(v) By monotonicity, it is clear that $\mathrm{Mod}_{p,\mathfrak{m}}(A_n)$ is an increasing sequence and $\mathrm{Mod}_{p,\mathfrak{m}}(\bigcup_n A_n) \ge \lim \mathrm{Mod}_{p,\mathfrak{m}}(A_n) =: C$. If $C = \infty$ there is nothing to prove; otherwise, we need to show that $\mathrm{Mod}_{p,\mathfrak{m}}(\bigcup_n A_n) \le C$. Let $(f_n) \subset \mathcal{L}_+^p(X, \mathfrak{m})$ be a sequence of functions such that $\int_X f_n \, d\mu \ge 1$ on A_n and $\|f_n\|_p^p \le \mathrm{Mod}_{p,\mathfrak{m}}(A_n) + 1/n$. In particular we get $\limsup_n \|f_n\|_p^p = C < \infty$, and so, possibly extracting a subsequence, we can assume that f_n weakly converges to some $f \in \mathcal{L}_+^p(X, \mathfrak{m})$. By Mazur's lemma we can find convex combinations

$$\hat{f}_n = \sum_{k=n}^{\infty} \lambda_{k,n} f_k$$

such that $\hat{f}_n$ converges strongly to f in $L^p(X, \mathfrak{m})$; furthermore $\int_X f_k \, d\mu \ge 1$ on A_n if $k \ge n$ and so

$$\int_X \hat{f}_n \, d\mu = \sum_{k=n}^{\infty} \lambda_{k,n} \int_X f_k \, d\mu \ge 1 \quad \text{on } A_n.$$

By (iii) we obtain a subsequence $n(k)$ and a $\mathrm{Mod}_{p,\mathfrak{m}}$-negligible set $\Sigma \subset \mathcal{M}_+(X)$ such that $\int_X \hat{f}_{n(k)} \, d\mu \to \int_X f \, d\mu$ outside Σ; in particular $\int_X f \, d\mu \ge 1$ on $\bigcup_n A_n \setminus \Sigma$. Then, by the very definition of $\mathrm{Mod}_{p,\mathfrak{m}}$-negligible set, for every $\varepsilon > 0$ we can find $g_\varepsilon \in \mathcal{L}_+^p(X, \mathfrak{m})$ such that $\|g_\varepsilon\|_p^p \le \varepsilon$ and $\int_X g_\varepsilon \, d\mu \ge 1$ on Σ, so that $\int_X (f + g_\varepsilon) \, d\mu \ge 1$ on $\bigcup_n A_n$ and

$$\mathrm{Mod}_{p,\mathfrak{m}}\Big(\bigcup_n A_n\Big)^{1/p} \le \|g_\varepsilon + f\|_p \le \|g_\varepsilon\|_p + \|f\|_p \le \varepsilon^{1/p} + \liminf \|f_n\|_p \le \varepsilon^{1/p} + C^{1/p}.$$

Letting $\varepsilon \to 0$ and taking the p-th power yields $\mathrm{Mod}_{p,\mathfrak{m}}(\bigcup_n A_n) \le \sup_n \mathrm{Mod}_{p,\mathfrak{m}}(A_n)$.

(vi) Let $K = \bigcap_n K_n$. As before, by monotonicity $\mathrm{Mod}_{p,\mathfrak{m},c}(K) \le \mathrm{Mod}_{p,\mathfrak{m},c}(K_n)$, and so letting C be the limit of $\mathrm{Mod}_{p,\mathfrak{m},c}(K_n)$ as $n \to \infty$, we only have to prove

$\mathrm{Mod}_{p,\mathfrak{m},c}(K) \geq C$. First, we deal with the case $\mathrm{Mod}_{p,\mathfrak{m},c}(K) > 0$. Using the equivalent definition, let $\phi_\varepsilon \in \mathrm{C}_b(X)$ be such that $\|\phi_\varepsilon\|_p = 1$ and

$$\inf_{\mu\in K} \int_X \phi_\varepsilon \, d\mu \geq \frac{1}{\mathrm{Mod}_{p,\mathfrak{m},c}(K)^{1/p}} - \varepsilon.$$

By the compactness of K and of K_n, it is clear that the infimum above is a minimum and $\min_{K_n} \int_X \phi_\varepsilon \, d\mu \to \min_K \int_X \phi_\varepsilon \, d\mu$, so that

$$\frac{1}{C^{1/p}} = \lim_{n\to\infty} \frac{1}{\mathrm{Mod}_{p,\mathfrak{m},c}(K_n)^{1/p}} \geq \lim_{n\to\infty} \min_{\mu\in K_n} \int_X \phi_\varepsilon \, d\mu \geq \frac{1}{\mathrm{Mod}_{p,\mathfrak{m},c}(K)^{1/p}} - \varepsilon.$$

The case $\mathrm{Mod}_{p,\mathfrak{m},c}(K) = 0$ is the same if we take $\phi_M \in \mathrm{C}_b(X)$ such that $\|\phi_M\|_p = 1$ and $\int_X \phi_M \, d\mu \geq M$ on K and then let $M \to \infty$.

(vii) Since $\mathrm{Mod}_{p,\mathfrak{m}}(A) = 0$, by (ii) we find $g \in \mathcal{L}^p_+(X, \mathfrak{m})$ such that $\int_X g \, d\mu = \infty$ for every $\mu \in A$; this yields $\int_X g \, d(F(\mu)\mu) = \infty$ for every $\mu \in A$, showing that $\mathrm{Mod}_{p,\mathfrak{m}}(B) = 0$. □

Remark 3.3. In connection with Proposition 3.2(iv), in general the constraint $\int_X f \, d\mu \geq 1$ is not saturated by the optimal f, namely the strict inequality can occur for a subset Σ_0 with positive $(p, \mathfrak{m})$-modulus. For instance, if $X = [0, 1]$ and $\mathfrak{m}$ is the Lebesgue measure, then

$$\mathrm{Mod}_{p,\mathfrak{m}}\big(\{\mathscr{L}^1 \llcorner [0, 1/2], \mathscr{L}^1 \llcorner [1/2, 1], \mathscr{L}^1 \llcorner [0, 1]\}\big) = 2^p \quad \text{and} \quad f \equiv 2,$$

but $\int_X f \, d\mathfrak{m} = 2$. However, we will prove using the duality formula $\mathrm{Mod}_{p,\mathfrak{m}} = C^p_{p,\mathfrak{m}}$ that one can always find a subset $\Sigma' \subset \Sigma$ (in the example above $\Sigma \setminus \Sigma' = \{\mathscr{L}^1 \llcorner [0, 1]\}$) with the same $(p, \mathfrak{m})$-modulus satisfying $\int_X f \, d\mu = 1$ for all $\mu \in \Sigma'$ (see the comment after Corollary 5.2).

On the other hand, if the measures in Σ are nonatomic, using just the definition of p-modulus one can find instead a family Σ' of *smaller* measures with the same modulus as Σ on which the constraint is saturated: it suffices to find, for any $\mu \in \Sigma$, a smaller measure μ' (a subcurve, in the case of measures associated to curves) satisfying $\int_X f \, d\mu' = 1$. In the previous example the two constructions lead to the same result, but the two procedures are conceptually quite different.

Another important property is the tightness of $\mathrm{Mod}_{p,\mathfrak{m}}$ in $\mathcal{M}_+(X)$; it will play a crucial role in the proof of Theorem 5.1 to prove the inner regularity of $\mathrm{Mod}_{p,\mathfrak{m}}$ for arbitrary Suslin sets.

Lemma 3.4 (Tightness of $\mathrm{Mod}_{p,\mathfrak{m}}$). *If (X, τ) is Polish and $\mathfrak{m} \in \mathcal{M}_+(X)$, for every $\varepsilon > 0$ there exists $E_\varepsilon \subset \mathcal{M}_+(X)$ compact such that* $\mathrm{Mod}_{p,\mathfrak{m}}(E^c_\varepsilon) \leq \varepsilon$.

Proof. Since (X, τ) is Polish, by the Ulam theorem we can find a nondecreasing family of sets $K_n \in \mathscr{K}(X)$ such that $\mathfrak{m}(K^c_n) \to 0$. We claim the existence of $\delta_n \downarrow 0$ such that if we define

$$E_k = \{\mu \in \mathcal{M}_+(X) : \mu(X) \leq k \text{ and } \mu(K^c_n) \leq \delta_n \ \forall n \geq k\},$$

then E_k is compact and $\mathrm{Mod}_{p,\mathfrak{m}}(E_k^c) \to 0$ as $k \to \infty$. First of all it is easy to see that the family $\{E_k\}$ is compact by the Prokhorov theorem, because it is clearly tight.

To evaluate $\mathrm{Mod}_{p,\mathfrak{m}}(E_k^c)$ we have to build some functions. Let $m_n = \mathfrak{m}(K_n^c)$, assume with no loss of generality that $m_n > 0$ for all n, set $a_n = (\sqrt{m_n} + \sqrt{m_{n+1}})^{-1/p}$ and note that the latter sequence is nondecreasing and diverging to $+\infty$; finally, define

$$f_k(x) := \begin{cases} 0 & \text{if } x \in K_k, \\ a_n & \text{if } x \in K_{n+1} \setminus K_n \text{ and } n \geq k, \\ +\infty & \text{otherwise.} \end{cases}$$

Now we claim that if we set $\delta_n = a_n^{-1}$ in the definition of the E_k's we will have $\mathrm{Mod}_{p,\mathfrak{m}}(E_k^c) \to 0$. In fact, if $\mu \in E_k^c$ then either $\mu(X) > k$ or $\mu(K_n^c) > \delta_n$ for some $n \geq k$. In either case the integral of $f_k + 1/k$ with respect to μ is greater than or equal to 1:

- if $\mu(X) > k$ then
$$\int_X \left(f_k + \frac{1}{k} \right) d\mu \geq \int_X \frac{1}{k}\, d\mu \geq 1;$$
- if $\mu(K_n^c) > \delta_n$ for some $n \geq k$ then
$$\int_X \left(f_k + \frac{1}{k} \right) d\mu \geq \int_{K_n^c} f_k\, d\mu \geq \int_{K_n^c} a_n\, d\mu > \delta_n a_n = 1.$$

So $\mathrm{Mod}_{p,\mathfrak{m}}(E_k^c) \leq \|f_k + 1/k\|_p^p \leq (\|f_k\|_p + \|1/k\|_p)^p$. But

$$\int_X f_k^p\, d\mathfrak{m} = \sum_{n=k}^{\infty} \int_{K_{n+1}\setminus K_n} a_n^p\, d\mathfrak{m} = \sum_{n=k}^{\infty} \frac{m_n - m_{n+1}}{\sqrt{m_n} + \sqrt{m_{n+1}}}$$
$$= \sum_{n=k}^{\infty} (\sqrt{m_n} - \sqrt{m_{n+1}}) = \sqrt{m_k},$$

and so $\mathrm{Mod}_{p,\mathfrak{m}}(E_k^c) \leq (m_k^{1/(2p)} + \mathfrak{m}(X)^{1/p}/k)^p \to 0$. □

4. Plans with barycenter in $L^q(X, \mathfrak{m})$ and $(p, \mathfrak{m})$-capacity

In this section, (X, τ) is Polish and $\mathfrak{m} \in \mathcal{M}_+(X)$ is a fixed reference measure. We will endow $\mathcal{M}_+(X)$ with the Polish structure making the maps $\mu \mapsto \int_X f\, d\mu$, $f \in \mathrm{C}_b(X)$, continuous, as described in Section 2.

Definition 4.1 (Plans with barycenter in $L^q(X, \mathfrak{m})$). Let $q \in (1, \infty]$, $p = q'$. We say that a Borel probability measure $\boldsymbol{\eta}$ on $\mathcal{M}_+(X)$ is a *plan with barycenter in $L^q(X, \mathfrak{m})$* if there exists $c \in [0, \infty)$ such that

$$\int \int_X f\, d\mu\, d\boldsymbol{\eta}(\mu) \leq c \|f\|_p \quad \forall f \in \mathcal{L}_+^p(X, \mathfrak{m}). \tag{4.1}$$

If $\boldsymbol{\eta}$ is a plan with barycenter in $L^q(X, \mathfrak{m})$, we denote by $c_q(\boldsymbol{\eta})$ the minimal c in (4.1).

Notice that $c_q(\boldsymbol{\eta}) = 0$ iff $\boldsymbol{\eta}$ is the Dirac mass at the null measure in $\mathcal{M}_+(X)$. We have also used implicitly in (4.1) the fact that $\mu \mapsto \int_X f\,d\mu$ is Borel whenever $f \in \mathcal{L}^p_+(X, \mathfrak{m})$ (and we will use it later without further mention). The proof can be obtained by a standard monotone class argument.

An equivalent definition of the class of plans with barycenter in $L^q(X, \mathfrak{m})$, which also explains the terminology we adopted, is based on the requirement that the barycenter Borel measure

$$\underline{\mu} := \int \mu\,d\boldsymbol{\eta}(\mu) \tag{4.2}$$

is absolutely continuous with respect to $\mathfrak{m}$ and with a density ρ in $L^q(X, \mathfrak{m})$. Moreover,

$$c_q(\boldsymbol{\eta}) = \|\rho\|_q. \tag{4.3}$$

Indeed, choosing $f = \chi_A$ in (4.1) gives $\underline{\mu}(A) \le \mathfrak{m}(A)^{1/p}$, hence the Radon–Nikodym theorem provides the representation $\underline{\mu} = \rho\mathfrak{m}$ for some $\rho \in L^1(X, \mathfrak{m})$. Then (4.1) once more gives

$$\int_X \rho f\,d\mathfrak{m} \le c\|f\|_p \qquad \forall f \in L^p(X, \mathfrak{m})$$

and the duality of Lebesgue spaces gives $\rho \in L^q(X, \mathfrak{m})$ and $\|\rho\|_q \le c$. Conversely, if $\underline{\mu}$ has a density in $L^q(X, \mathfrak{m})$, Hölder's inequality shows that (4.1) holds with $c = \|\rho\|_q$.

Obviously, (4.1) still holds with $c = c_q(\boldsymbol{\eta})$ for all $f \in \mathrm{C}_b(X)$, not necessarily nonnegative, when $\boldsymbol{\eta}$ is a plan with good barycenter in $L^q(X, \mathfrak{m})$. Actually the next proposition shows that we need only check (4.1) for $f \in \mathrm{C}_b(X)$ nonnegative.

Proposition 4.2. *Let $\boldsymbol{\eta}$ be a probability measure on $\mathcal{M}_+(X)$ such that*

$$\int \int_X f\,d\mu\,d\boldsymbol{\eta}(\mu) \le c\|f\|_p \quad \textit{for all } f \in \mathrm{C}_b(X) \textit{ nonnegative} \tag{4.4}$$

for some $c \ge 0$. Then (4.4) holds, with the same constant c, also for every $f \in \mathcal{L}^p_+(X, \mathfrak{m})$.

Proof. It suffices to remark that (4.4) gives

$$\int_X f\,d\underline{\mu} \le c\|f\|_p \qquad \forall f \in \mathrm{C}_b(X),$$

with $\underline{\mu}$ defined in (4.2). Again the duality of Lebesgue spaces provides $\rho \in L^q(X, \mathfrak{m})$ with $\|\rho\|_q \le c$ satisfying $\int_X f\rho\,d\mathfrak{m} = \int_X f\,d\underline{\mu}$ for all $f \in \mathrm{C}_b(X)$, hence $\underline{\mu} = \rho\mathfrak{m}$. □

There is a simple duality inequality, involving the minimization in (3.2) and a maximization among all $\boldsymbol{\eta}$'s with barycenter in $L^q(X, \mathfrak{m})$. To see it, take $f \in \mathcal{L}^p_+(X, \mathfrak{m})$ such that $\int f\,d\mu \ge 1$ on $\Sigma \subset \mathcal{M}_+(X)$. Then, if Σ is universally measurable we may take any plan $\boldsymbol{\eta}$ with barycenter in $L^q(X, \mathfrak{m})$ to obtain

$$\boldsymbol{\eta}(\Sigma) \le \int \int_X f\,d\mu\,d\boldsymbol{\eta}(\mu) \le c_q(\boldsymbol{\eta})\|f\|_p. \tag{4.5}$$

In particular we have

$$\mathrm{Mod}_{p,\mathfrak{m}}(\Sigma) = 0 \;\Rightarrow\; \boldsymbol{\eta}(\Sigma) = 0 \text{ for all } \boldsymbol{\eta} \text{ with barycenter in } L^q(X, \mathfrak{m}). \tag{4.6}$$

In addition, taking in (4.5) the infimum over all the $f \in \mathcal{L}^p_+(X, \mathfrak{m})$ such that $\int f \, d\mu \geq 1$ on Σ and, at the same time, the supremum with respect to all plans $\boldsymbol{\eta}$ with barycenter in $L^q(X, \mathfrak{m})$ and $c_q(\boldsymbol{\eta}) > 0$, we find

$$\sup_{c(\boldsymbol{\eta})>0} \frac{\boldsymbol{\eta}(\Sigma)}{c_q(\boldsymbol{\eta})} \leq \mathrm{Mod}_{p,\mathfrak{m}}(\Sigma)^{1/p}. \tag{4.7}$$

The inequality (4.7) motivates the next definition.

Definition 4.3 ($(p, \mathfrak{m})$-content). If $\Sigma \subset \mathcal{M}_+(X)$ is a universally measurable set and $p \in [1, \infty)$, we define

$$C_{p,\mathfrak{m}}(\Sigma) := \sup_{c_q(\boldsymbol{\eta})>0} \frac{\boldsymbol{\eta}(\Sigma)}{c_q(\boldsymbol{\eta})}. \tag{4.8}$$

By convention, we set $C_{p,\mathfrak{m}}(\Sigma) = \infty$ if $0 \in \Sigma$.

A first important implication of (4.7) is that for any family $\mathcal{F}$ of plans $\boldsymbol{\eta}$ with barycenter in $L^q(X, \mathfrak{m})$,

$$C := \sup\{c_q(\boldsymbol{\eta}) : \boldsymbol{\eta} \in \mathcal{F}\} < \infty \;\Rightarrow\; \mathcal{F} \text{ is tight.} \tag{4.9}$$

Indeed, $\boldsymbol{\eta}(E^c_{\varepsilon^p}) \leq \varepsilon c_q(\boldsymbol{\eta}) \leq C\varepsilon$, where the $E_\varepsilon \subset \mathcal{M}_+(X)$ are the compact sets provided by Lemma 3.4. This allows us to prove existence of optimal $\boldsymbol{\eta}$'s in (4.8).

Lemma 4.4. *Let $\Sigma \subset \mathcal{M}_+(X)$ be a universally measurable set such that $C_{p,\mathfrak{m}}(\Sigma) > 0$. Then there exists an optimal plan $\boldsymbol{\eta}$ with barycenter in $L^q(X, \mathfrak{m})$ where the supremum in (4.8) is attained, and any optimal plan is concentrated on Σ. In particular*

$$C_{p,\mathfrak{m}}(\Sigma) = \frac{\boldsymbol{\eta}(\Sigma)}{c_q(\boldsymbol{\eta})} = \frac{1}{c_q(\boldsymbol{\eta})}.$$

Proof. First we claim that the supremum in (4.8) can be restricted to the plans with barycenter in $L^q(X, \mathfrak{m})$ concentrated on Σ. Indeed, given any admissible $\boldsymbol{\eta}$ with $\boldsymbol{\eta}(\Sigma) > 0$, defining $\boldsymbol{\eta}' = \boldsymbol{\eta}(\Sigma)^{-1}\chi_\Sigma \boldsymbol{\eta}$ we obtain another plan with barycenter in $L^q(X, \mathfrak{m})$ satisfying $\boldsymbol{\eta}'(\Sigma) = 1$ and

$$\begin{aligned}\int \int_X f \, d\mu \, d\boldsymbol{\eta}'(\mu) &= \frac{1}{\boldsymbol{\eta}(\Sigma)} \int_\Sigma \int_X f \, d\mu \, d\boldsymbol{\eta}(\mu) \leq \frac{1}{\boldsymbol{\eta}(\Sigma)} \int \int_X f \, d\mu \, d\boldsymbol{\eta}(\mu) \\ &\leq \frac{c_q(\boldsymbol{\eta})}{\boldsymbol{\eta}(\Sigma)} \|f\|_p\end{aligned}$$

for all $f \in \mathcal{L}^p_+(X, \mathfrak{m})$. In particular the definition of $c_q(\boldsymbol{\eta}')$ gives

$$c_q(\boldsymbol{\eta}') \leq \frac{c_q(\boldsymbol{\eta})}{\boldsymbol{\eta}(\Sigma)},$$

and proves our claim. The same argument proves that $\boldsymbol{\eta}' = \boldsymbol{\eta}$ whenever $\boldsymbol{\eta}$ is a maximizer. Now we know that

$$C_{p,\mathfrak{m}}(\Sigma) = \sup_{\boldsymbol{\eta}(\Sigma)=1} \frac{1}{c_q(\boldsymbol{\eta})},$$

where the supremum is taken over plans with barycenter in $L^q(X, \mathfrak{m})$. We take a maximizing sequence $(\boldsymbol{\eta}_k)$; for this sequence we have $c_q(\boldsymbol{\eta}_k) \le C$, so that $(\boldsymbol{\eta}_k)$ is tight by (4.9). Assume with no loss of generality that $\boldsymbol{\eta}_k$ weakly converges to some $\boldsymbol{\eta}$, which is clearly a probability measure in $\mathcal{M}_+(X)$. To see that $\boldsymbol{\eta}$ is a plan with barycenter in $L^q(X, \mathfrak{m})$ and that $c_q(\boldsymbol{\eta})$ is optimal, we notice that the continuity and nonnegativity of $\mu \mapsto \int_X f\, d\mu$ in $\mathcal{M}_+(X)$ for $f \in \mathrm{C}_b(X)$ nonnegative gives

$$\int\int_X f\, d\mu\, d\boldsymbol{\eta}(\mu) \le \liminf_{k\to\infty} \int\int_X f\, d\mu\, d\boldsymbol{\eta}_k(\mu) \le \lim_{k\to\infty} c_q(\boldsymbol{\eta}_k)\|f\|_p,$$

so that

$$\int\int_X f\, d\mu\, d\boldsymbol{\eta}(\mu) \le \frac{1}{C_{p,\mathfrak{m}}(\Sigma)}\|f\|_p \qquad \forall f \in \mathrm{C}_b(X).$$

The conclusion follows from Proposition 4.2. □

5. Equivalence between $C_{p,\mathfrak{m}}$ and $\mathrm{Mod}_{p,\mathfrak{m}}$

In the previous two sections, under the standing assumptions that (X, τ) is a Hausdorff topological space (Polish in the case of $C_{p,\mathfrak{m}}$), $\mu \in \mathcal{M}_+(X)$ and $p \in [1, \infty)$, we introduced a p-modulus $\mathrm{Mod}_{p,\mathfrak{m}}$ and a p-content $C_{p,\mathfrak{m}}$, proving the direct inequalities (see (4.7))

$$C^p_{p,\mathfrak{m}} \le \mathrm{Mod}_{p,\mathfrak{m}} \le \mathrm{Mod}_{p,\mathfrak{m},c} \qquad \text{on Suslin subsets of } \mathcal{M}_+(X).$$

Under the same assumptions on (X, τ) and $\mathfrak{m} \in \mathcal{M}_+(X)$, our goal in this section is the following result:

Theorem 5.1. *Let (X, τ) be a Polish topological space and $p \in (1, \infty)$. Then $\mathrm{Mod}_{p,\mathfrak{m}}$ is a Choquet capacity in $\mathcal{M}_+(X)$, and every Suslin set $\Sigma \subset \mathcal{M}_+(X)$ is capacitable and satisfies $\mathrm{Mod}_{p,\mathfrak{m}}(\Sigma)^{1/p} = C_{p,\mathfrak{m}}(\Sigma)$. If moreover Σ is also compact then $\mathrm{Mod}_{p,\mathfrak{m}}(\Sigma) = \mathrm{Mod}_{p,\mathfrak{m},c}(\Sigma)$.*

Proof. We split the proof into two steps:

- first, we prove that $\mathrm{Mod}_{p,\mathfrak{m},c}(\Sigma)^{1/p} \le C_{p,\mathfrak{m}}(\Sigma)$ if Σ is compact, so that in particular $\mathrm{Mod}^{1/p}_{p,\mathfrak{m}} = C_{p,\mathfrak{m}}$ on compact sets;
- then, we prove that $\mathrm{Mod}_{p,\mathfrak{m}}$ and $C_{p,\mathfrak{m}}$ are inner regular, and deduce that $\mathrm{Mod}^{1/p}_{p,\mathfrak{m}} = C_{p,\mathfrak{m}}$ on Suslin sets.

The two steps together yield $\mathrm{Mod}_{p,\mathfrak{m}} = \mathrm{Mod}_{p,\mathfrak{m},c}$ on compact sets, hence we can use Proposition 3.2(v,vi) to conclude that $\mathrm{Mod}_{p,\mathfrak{m}}$ is a Choquet capacity in $\mathcal{M}_+(X)$.

Step 1. Assume that $\Sigma \subset \mathcal{M}_+(X)$ is compact. In particular $\sup_\Sigma \mu(X)$ is finite and so the linear map $\Phi : \mathrm{C}_b(X) \to \mathrm{C}(\Sigma) = \mathrm{C}_b(\Sigma)$ given by

$$f \mapsto \Phi_f(\mu) := \int_X f \, d\mu$$

is a bounded linear operator.

If Σ contains the null measure there is nothing to prove, because $\mathrm{Mod}_{p,\mathfrak{m},c}(\Sigma) = \infty$ by definition and $C_{p,\mathfrak{m}}(\Sigma) = \infty$ by convention. If not, by compactness, we find that $\varepsilon := \inf_\Sigma \mu(X) > 0$, so that taking $f \equiv \varepsilon^{-1}$ in (3.3) we obtain $\mathrm{Mod}_{p,\mathfrak{m},c}(\Sigma) < \infty$. We can also assume that $\mathrm{Mod}_{p,\mathfrak{m},c}(\Sigma) > 0$, otherwise there is nothing to prove.

Our first step is the construction of a plan $\boldsymbol{\eta}$ with barycenter in $L^q(X, \mathfrak{m})$ concentrated on Σ. By the equivalent definition analogous to (3.4) for $\mathrm{Mod}_{p,\mathfrak{m},c}$, the constant $\xi = \mathrm{Mod}_{p,\mathfrak{m},c}(\Sigma)^{-1/p}$ satisfies

$$\inf_{\mu \in \Sigma} \Phi_f(\mu) \le \xi \|f\|_p \qquad \forall f \in \mathrm{C}(X). \tag{5.1}$$

Denoting by $v = v(\mu)$ a generic element of $\mathrm{C}(\Sigma)$, we will now consider two functions on $\mathrm{C}(\Sigma)$:

$$F_1(v) = \inf\{\|f\|_p : f \in \mathrm{C}_b(X),\ \Phi_f \ge v \text{ on } \Sigma\},$$
$$F_2(v) = \min\{v(\mu) : \mu \in \Sigma\}.$$

The following properties are immediate to check, using the linearity of $f \mapsto \Phi_f$ for the first one and (5.1) for the third one:

- F_1 is convex;
- F_2 is continuous and concave;
- $F_2 \le \xi \cdot F_1$.

With these properties, standard Banach theory gives a continuous linear functional $L \in (\mathrm{C}(\Sigma))^*$ such that

$$F_2(v) \le L(v) \le \xi \cdot F_1(v) \qquad \forall v \in \mathrm{C}(\Sigma). \tag{5.2}$$

For the reader's convenience we detail the argument: first we apply the geometric form of the Hahn–Banach theorem in the space $\mathrm{C}(\Sigma) \times \mathbb{R}$ to the convex sets $A = \{F_2(v) > t\}$ and $B = \{F_1(v) \le t/\xi\}$, where the former is also open, to obtain a continuous linear functional G on $\mathrm{C}(\Sigma) \times \mathbb{R}$ such that

$$G(v, t) < G(w, s) \quad \text{ whenever } F_2(v) > t,\ F_1(w) \le s/\xi.$$

If we represent $G(v, t)$ as $H(v) + \beta t$ for some $H \in (\mathrm{C}(\Sigma))^*$ and $\beta \in \mathbb{R}$, the inequality reads

$$H(v) + \beta t < H(w) + \beta s \quad \text{ whenever } F_2(v) > t,\ F_1(w) \le s/\xi.$$

Since F_1 and F_2 are real-valued, we have $\beta > 0$; we immediately get $F_2 \le (\gamma - H)/\beta \le \xi F_1$, with $\gamma := \sup H(v) + \beta F_2(v)$. On the other hand, $F_1(0) = F_2(0) = 0$ implies $\gamma = 0$, so that we can take $L = -H/\beta$ in (5.2).

In particular from (5.2) we infer that if $v \geq 0$ then $L(v) \geq F_2(v) \geq 0$, and so, since Σ is compact, we can apply the Riesz theorem to obtain a nonnegative measure $\boldsymbol{\eta}$ in Σ representing L:

$$L(v) = \int_\Sigma v(\mu)\, d\boldsymbol{\eta} \qquad \forall v \in \mathrm{C}(\Sigma).$$

Furthermore this measure cannot be null since (here $\mathbb{1}$ is the function identically equal to 1)

$$\boldsymbol{\eta}(\Sigma) = L(\mathbb{1}) \geq F_2(\mathbb{1}) = 1,$$

and so $\boldsymbol{\eta}(\Sigma) \geq 1$. Now we claim that $\boldsymbol{\eta}$ is a plan with barycenter in $L^q(X, \mathfrak{m})$; first we prove that $\boldsymbol{\eta}(\Sigma) \leq 1$, so that $\boldsymbol{\eta}$ will be a probability measure. In fact, we know $F_2(v)\boldsymbol{\eta}(\Sigma) \leq L(v)$ because $v \geq F_2(v)$ on Σ, and so

$$F_2(v)\boldsymbol{\eta}(\Sigma) \leq \xi F_1(v).$$

In particular, inserting in this inequality $v = \Phi_\phi$ with $\phi \in \mathrm{C}_b(X)$, we obtain

$$\inf_\Sigma \Phi_\phi \leq \frac{\xi}{\boldsymbol{\eta}(\Sigma)} \|\phi\|_p,$$

and so $\mathrm{Mod}_{p,\mathfrak{m},c}(\Sigma) \geq (\boldsymbol{\eta}(\Sigma)/\xi)^p = \boldsymbol{\eta}(\Sigma)^p\, \mathrm{Mod}_{p,\mathfrak{m},c}(\Sigma)$, which implies $\boldsymbol{\eta}(\Sigma) \leq 1$. Now we have

$$\int_\Sigma \left(\int_X f\, d\mu \right) d\boldsymbol{\eta} = L(\Phi_f) \leq \xi \cdot F_1(\Phi_f) \leq \xi \cdot \|f\|_p \qquad \forall f \in \mathrm{C}_b(X), \tag{5.3}$$

and so, by Proposition 4.2, this inequality is true for every $f \in \mathcal{L}^p_+(X, \mathfrak{m})$, showing that $\boldsymbol{\eta}$ is a plan with barycenter in $L^q(X, \mathfrak{m})$; as a byproduct we also find that $c_q(\boldsymbol{\eta}) \leq \xi$, which gives $C_{p,\mathfrak{m}}(\Sigma) \geq \mathrm{Mod}_{p,\mathfrak{m},c}(\Sigma)^{1/p}$, thus proving that

$$C_{p,\mathfrak{m}}(\Sigma) = \mathrm{Mod}_{p,\mathfrak{m}}(\Sigma)^{1/p} = \mathrm{Mod}_{p,\mathfrak{m},c}(\Sigma)^{1/p}.$$

Step 2. Now we will prove that $\mathrm{Mod}_{p,\mathfrak{m}}$ and $C_{p,\mathfrak{m}}$ are both inner regular, that is, their value on Suslin sets is the supremum of their values on compact subsets. Inner regularity and equality on compact sets yield $C_{p,\mathfrak{m}}(B) = \mathrm{Mod}_{p,\mathfrak{m}}(B)^{1/p}$ on every Suslin subset B of $\mathcal{M}_+(X)$.

$\mathrm{Mod}_{p,\mathfrak{m}}$ **is inner regular.** Proposition 3.2(v,vi) and the fact that $\mathrm{Mod}_{p,\mathfrak{m},c} = \mathrm{Mod}_{p,\mathfrak{m}}$ if the set is compact show that $\mathrm{Mod}_{p,\mathfrak{m}}$ is a capacity. For any set $L \subset \mathcal{M}_+(X)$ we have $\mathrm{Mod}_{p,\mathfrak{m}}(L) = \sup_\varepsilon \mathrm{Mod}_{p,\mathfrak{m}}(L \cap E_\varepsilon)$, where E_ε are the compact sets given by Lemma 3.4. Therefore, it suffices to show inner regularity for a Suslin set B contained in E_ε for some ε. Since E_ε is compact, B is a Suslin compact set and from Choquet Theorem 2.5 it follows that for every $\delta > 0$ there is a compact set $K \subset B$ such that $\mathrm{Mod}_{p,\mathfrak{m}}(K) \geq \mathrm{Mod}_{p,\mathfrak{m}}(B) - \delta$.

$C_{p,\mathfrak{m}}$ is inner regular. Since Suslin sets are universally measurable and $\mathcal{M}_+(X)$ is Polish, we can apply (2.1) to any Suslin set B with $\sigma = \boldsymbol{\eta}$ to get

$$\sup_{K\subset B} C_{p,\mathfrak{m}}(K) = \sup_{K\subset B}\sup_{c_q(\boldsymbol{\eta})>0}\frac{\boldsymbol{\eta}(K)}{c_q(\boldsymbol{\eta})} = \sup_{c_q(\boldsymbol{\eta})>0}\sup_{K\subset B}\frac{\boldsymbol{\eta}(K)}{c_q(\boldsymbol{\eta})} = \sup_{c_q(\boldsymbol{\eta})>0}\frac{\boldsymbol{\eta}(B)}{c_q(\boldsymbol{\eta})}$$
$$= C_{p,\mathfrak{m}}(B). \qquad \square$$

The duality formula and the existence of maximizers and minimizers provide the following result.

Corollary 5.2 (Necessary and sufficient optimality conditions). *Let $p \in (1,\infty)$, and let $\Sigma \subset \mathcal{M}_+(X)$ be a Suslin set such that $\mathrm{Mod}_{p,\mathfrak{m}}(\Sigma) > 0$. Then:*

(a) *There exists $f \in \mathcal{L}^p_+(X,\mathfrak{m})$, unique up to $\mathfrak{m}$-negligible sets, such that $\int_X f\,d\mu \geq 1$ for $\mathrm{Mod}_{p,\mathfrak{m}}$-a.e. $\mu \in \Sigma$ and $\|f\|_p^p = \mathrm{Mod}_{p,\mathfrak{m}}(\Sigma)$.*
(b) *There exists a plan $\boldsymbol{\eta}$ with barycenter in $L^q(X,\mathfrak{m})$ concentrated on Σ such that $C_{p,\mathfrak{m}}(\Sigma) = 1/c_q(\boldsymbol{\eta})$.*
(c) *For the function f in* (a) *and any $\boldsymbol{\eta}$ in* (b)*, we have*

$$\int_X f\,d\mu = 1 \quad \textit{for } \boldsymbol{\eta}\textit{-a.e. } \mu \quad \textit{and} \quad \int_X \mu\,d\boldsymbol{\eta}(\mu) = \frac{f^{p-1}}{\|f\|_p^p}\mathfrak{m}. \tag{5.4}$$

Finally, if $f \in \mathcal{L}^p_+(X,\mathfrak{m})$ is optimal in (3.2)*, then any plan $\boldsymbol{\eta}$ with barycenter in $L^q(X,\mathfrak{m})$ concentrated on Σ such that $c_q(\boldsymbol{\eta}) = \|f\|_p^{-1}$ is optimal in* (4.8)*. Conversely, if $\boldsymbol{\eta}$ is optimal in* (4.8)*, and $f \in \mathcal{L}^p_+(X,\mathfrak{m})$ and $\int_X f\,d\mu = 1$ for $\boldsymbol{\eta}$-a.e. μ, then f is optimal in* (3.2).

Proof. The existence of f follows from Proposition 3.2(iv). The existence of a maximizer $\boldsymbol{\eta}$ in the duality formula, concentrated on Σ and satisfying $C_{p,\mathfrak{m}}(\Sigma) = 1/c_q(\boldsymbol{\eta})$, follows from Lemma 4.4. Since (4.6) gives $\int_X f\,d\mu \geq 1$ for $\boldsymbol{\eta}$-a.e. $\mu \in \Sigma$, we can still derive the inequality (4.5) and deduce from Theorem 5.1 that all inequalities are equalities. Hence, $\int_X f\,d\mu = 1$ for $\boldsymbol{\eta}$-a.e. $\mu \in \mathcal{M}_+(X)$. Now, setting $\underline{\mu} := \int \mu\,d\boldsymbol{\eta}(\mu)$, from (4.3) we get $\underline{\mu} = g\mathfrak{m}$ with $\|g\|_q = c_q(\boldsymbol{\eta})$. This, in combination with

$$\int_X fg\,d\mathfrak{m} = \int\int_X f\,d\mu\,d\boldsymbol{\eta}(\mu) = c_q(\boldsymbol{\eta})\|f\|_p = \|g\|_q\|f\|_p,$$

gives $g = f^{p-1}/\|f\|_p^p$.

Finally, the last statements follow directly from (4.5) and Theorem 5.1. $\square$

In particular, choosing $\boldsymbol{\eta}$ as in (b) and defining

$$\Sigma' := \left\{\mu \in \mathcal{M}_+(X) : \int_X f\,d\mu = 1\right\},$$

since $\boldsymbol{\eta}(\Sigma) = \boldsymbol{\eta}(\Sigma')$ we obtain a subfamily with the same p-modulus on which the constraint is saturated.

Part II. Modulus of families of curves and weak gradients

6. Absolutely continuous curves

If (X, d) is a metric space and $I \subset \mathbb{R}$ is an interval, we denote by $\mathrm{C}(I; X)$ the class of continuous maps (often called parametric curves) from I to X. We will use the notation γ_t for the value of the map at time t, and $\mathrm{e}_t : \mathrm{C}(I; X) \to X$ for the evaluation map at time t; occasionally, in order to avoid double subscripts, we will also use the notation $\gamma(t)$. The subclass $\mathrm{AC}(I; X)$ is defined by the property

$$\mathsf{d}(\gamma_s, \gamma_t) \leq \int_s^t g(r)\, dr, \qquad s, t \in I,\ s \leq t,$$

for some (nonnegative) $g \in L^1(I)$. The least, up to $\mathscr{L}^1$-negligible sets, function g with this property is called the *metric derivative* (or *metric speed*)

$$|\dot{\gamma}_t| := \lim_{h \to 0} \frac{\mathsf{d}(\gamma_{t+h}, \gamma_t)}{|h|}$$

(see [9]). The classes $\mathrm{AC}^p(I; X)$, $1 \leq p \leq \infty$, are defined analogously, by requiring that $|\dot{\gamma}| \in L^p(I)$. The *p-energy* of a curve is then defined as

$$\mathcal{E}_p(\gamma) := \begin{cases} \int_I |\dot{\gamma}_t|^p\, dt & \text{if } \gamma \in \mathrm{AC}^p(I; X), \\ +\infty & \text{otherwise}, \end{cases} \tag{6.1}$$

and $\mathcal{E}_1(\gamma) = \ell(\gamma)$, the length of γ, when $p = 1$. Notice that $\mathrm{AC}^1 = \mathrm{AC}$ and that $\mathrm{AC}^\infty(I; X)$ coincides with the class of d-Lipschitz functions.

If (X, d) is complete the interval I can be taken closed with no loss of generality, because absolutely continuous functions extend continuously to the closure of the interval. In addition, if (X, d) is complete and separable then $\mathrm{C}(I; X)$ is a Polish space, and $\mathrm{AC}^p(I; X)$, $1 \leq p \leq \infty$, are Borel subsets of $\mathrm{C}(I; X)$ (see for instance [6]). We will use $\mathcal{M}_+(\mathrm{AC}^p(I; X))$ to denote the finite Borel measures in $\mathrm{C}(I; X)$ concentrated on $\mathrm{AC}^p(I; X)$.

6.1. Reparameterization

In the next proposition we collect a few properties which are well-known in a smooth setting, but still valid in general metric spaces. We introduce the notation

$$\mathrm{AC}^\infty_c([0,1]; X) := \{\sigma \in \mathrm{AC}^\infty([0,1]; X) : |\dot{\sigma}| = \ell(\sigma) > 0\ \mathscr{L}^1\text{-a.e. on } (0,1)\} \tag{6.2}$$

for the subset of $\mathrm{AC}([0,1]; X)$ consisting of all nonconstant curves with constant speed. It is easy to check that $\mathrm{AC}^\infty_c([0,1]; X)$ is a Borel subset of $\mathrm{C}([0,1]; X)$, since it can also be characterized by

$$\gamma \in \mathrm{AC}^\infty_c([0,1]; X) \ \Leftrightarrow\ 0 < \mathrm{Lip}(\gamma) \leq \ell(\gamma), \tag{6.3}$$

and the maps $\gamma \mapsto \mathrm{Lip}(\gamma)$ and $\gamma \mapsto \ell(\gamma)$ are lower semicontinuous.

Proposition 6.1 (Constant speed reparameterization). *For any $\gamma \in \mathrm{AC}([0,1];X)$ with $\ell(\gamma) > 0$, setting*

$$\mathsf{s}(t) := \frac{1}{\ell(\gamma)} \int_0^t |\dot\gamma_r|\,dr, \tag{6.4}$$

there exists a unique $\eta \in \mathrm{AC}_c^\infty([0,1];X)$ such that $\gamma = \eta \circ \mathsf{s}$. Furthermore, $\eta = \gamma \circ \mathsf{s}^{-1}$ where s^{-1} is any right inverse of s. We shall denote by

$$\mathsf{k} : \{\gamma \in \mathrm{AC}([0,1];X) : \ell(\gamma) > 0\} \to \mathrm{AC}_c^\infty([0,1];X), \qquad \gamma \mapsto \eta = \gamma \circ \mathsf{s}^{-1}, \tag{6.5}$$

the corresponding map.

Proof. We prove existence only, the proof of uniqueness being analogous. Let us now define a right inverse, denoted by s^{-1}, of s (i.e. $\mathsf{s} \circ \mathsf{s}^{-1}$ is equal to the identity): we define in the obvious way s^{-1} at points $y \in [0,1]$ such that $\mathsf{s}^{-1}(y)$ is a singleton; since, by construction, γ is constant in all (maximal) intervals $[c,d]$ where s is constant, at points y such that $\{y\} = \mathsf{s}([c,d])$ we define $\mathsf{s}^{-1}(y)$ by choosing any element of $[c,d]$, so that $\gamma \circ \mathsf{s}^{-1} \circ \mathsf{s} = \gamma$ (even though it could be that $\mathsf{s}^{-1} \circ \mathsf{s}$ is not the identity). Therefore, if we define $\eta = \gamma \circ \mathsf{s}^{-1}$, we find that $\gamma = \eta \circ \mathsf{s}$ and that η is independent of the right inverse chosen.

In order to prove that $\eta \in \mathrm{AC}_c^\infty([0,1];X)$ we define $\ell_k := \ell(\gamma) + 1/k$ and we approximate uniformly in $[0,1]$ the map s by the maps $\mathsf{s}_k(t) := \ell_k^{-1} \int_0^t (k^{-1} + |\dot\gamma_r|)\,dr$, whose inverses $\mathsf{s}_k^{-1} : [0,1] \to I$ are Lipschitz. By Helly's theorem and passing to the limit as $k \to \infty$ in $\mathsf{s}_k \circ \mathsf{s}_k^{-1}(y) = y$, we can assume that a subsequence $\mathsf{s}_{k(p)}^{-1}$ pointwise converges to a right inverse s^{-1} as $p \to \infty$; the curves $\eta^p := \gamma \circ \mathsf{s}_{k(p)}^{-1}$ are absolutely continuous, pointwise converge to $\eta := \gamma \circ \mathsf{s}^{-1}$ and

$$|\eta^p(t)'| = \frac{|\gamma'(\mathsf{s}_{k(p)}^{-1}(t))|}{\mathsf{s}_{k(p)}'(\mathsf{s}_{k(p)}^{-1}(t))} \le \ell_{k(p)} \qquad \text{for } \mathscr{L}^1\text{-a.e. in } t \in (0,1).$$

It follows that η is absolutely continuous and $|\dot\eta| \le \ell(\gamma)\ \mathscr{L}^1$-a.e. in $(0,1)$. If strict inequality occurs in a set of positive Lebesgue measure, the inequality $\ell(\eta) < \ell(\gamma)$ provides a contradiction. □

6.2. *Equivalence relation in* AC([0, 1]; X)

We can identify curves $\gamma, \tilde\gamma \in \mathrm{AC}([0,1];X)$ if there exists $\varphi : [0,1] \to [0,1]$ increasing with $\varphi, \varphi^{-1} \in \mathrm{AC}([0,1];[0,1])$ such that $\gamma = \tilde\gamma \circ \varphi$. In this case we write $\gamma \sim \tilde\gamma$. Thanks to the following lemma, the absolute continuity of φ^{-1} is equivalent to $\varphi' > 0\ \mathscr{L}^1$-a.e. in $(0,1)$.

Lemma 6.2 (Absolute continuity criterion). *Let I, $\tilde I$ be compact intervals in $\mathbb{R}$ and let $\varphi : I \to \tilde I$ be an absolutely continuous homeomorphism with $\varphi' > 0\ \mathscr{L}^1$-a.e. in I. Then $\varphi^{-1} : \tilde I \to I$ is absolutely continuous.*

Proof. Let $\psi = \varphi^{-1}$; it is a continuous function of bounded variation whose distributional derivative we shall denote by μ. Since $\mu([a, b]) = \psi(b) - \psi(a)$ for all $0 \le a \le b \le 1$, we need to show that $\mu \ll \mathscr{L}^1$. It is a general property of BV functions (see for instance [4, Proposition 3.92]) that $\mu(\psi^{-1}(B)) = 0$ for all Borel and $\mathscr{L}^1$-negligible sets $B \subset \mathbb{R}$. Choosing $B = \psi(E)$, where E is an $\mathscr{L}^1$-negligible set where the singular part μ^s of μ is concentrated, the area formula gives

$$\int_B \varphi'(s)\, ds = \mathscr{L}^1(E) = 0,$$

so that the positivity of φ' gives $\mathscr{L}^1(B) = 0$. It follows that $\mu^s = 0$. □

Definition 6.3 (The map J). For any $\gamma \in \mathrm{AC}([0, 1]; X)$ we denote by $J\gamma \in \mathcal{M}_+(X)$ the push forward under γ of the measure $|\dot\gamma|\mathscr{L}^1 \llcorner [0, 1]$, namely

$$\int_X g\, dJ\gamma = \int_0^1 g(\gamma_t)|\dot\gamma_t|\, dt \qquad \text{for all } g : X \to [0, \infty] \text{ Borel.} \tag{6.6}$$

In particular, $J\gamma = J\eta$ whenever $\gamma \sim \eta$, and $J\gamma = J\mathsf{k}\gamma$.

Although this will not play a role in the following, for completeness we provide an intrinsic description of the measure $J\gamma$. We denote by $\mathscr{H}^1$ the 1-dimensional Hausdorff measure of a subset B of X, that is, $\mathscr{H}^1(B) = \lim_{\delta\downarrow 0} \mathscr{H}^1_\delta(B)$, where

$$\mathscr{H}^1_\delta(B) := \inf\Big\{\sum_{i=0}^{\infty} \mathrm{diam}(B_i) : B \subset \bigcup_{i=0}^{\infty} B_i,\ \mathrm{diam}(B_i) < \delta\Big\}$$

(with the convention $\mathrm{diam}(\emptyset) = 0$).

Proposition 6.4 (Area formula). *If $\gamma \in \mathrm{AC}([0, 1]; X)$, then for all $g : X \to [0, \infty]$ Borel the area formula holds:*

$$\int_0^1 g(\gamma_t)|\dot\gamma_t|\, dt = \int_X g(x)N(\gamma, x)\, d\mathscr{H}^1(x), \tag{6.7}$$

where $N(\gamma, x) := \mathrm{card}(\gamma^{-1}(x))$ is the multiplicity function of γ. Equivalently,

$$J\gamma = N(\gamma, \cdot)\mathscr{H}^1. \tag{6.8}$$

Proof. For an elementary proof of (6.7), see for instance [9, Theorem 3.4.6]. □

6.3. Nonparametric curves

We can now introduce the class of nonparametric curves; notice that we are conventionally excluding from this class the constant curves. We introduce the notation

$$\mathrm{AC}_0([0, 1]; X) := \{\gamma \in \mathrm{AC}([0, 1]; X) : |\dot\gamma| > 0\ \mathscr{L}^1\text{-a.e. on } (0, 1)\}.$$

It is not difficult to show that $\mathrm{AC}_0([0,1];X)$ is a Borel subset of $\mathrm{C}([0,1];X)$. In addition, Lemma 6.2 shows that for any $\gamma \in \mathrm{AC}_0([0,1];X)$ the curve $\mathsf{k}\gamma \in \mathrm{AC}^\infty_c([0,1];X)$ is equivalent to γ.

Definition 6.5 (The class $\mathscr{C}(X)$ of nonparametric curves). The class $\mathscr{C}(X)$ is defined as

$$\mathscr{C}(X) := \mathrm{AC}_0([0,1];X)/\!\sim, \tag{6.9}$$

endowed with the quotient topology $\tau_{\mathscr{C}}$ and the canonical projection $\pi_{\mathscr{C}(X)}$.

We shall denote the typical element of $\mathscr{C}(X)$ either by $\underline{\gamma}$ or by $[\gamma]$, to mark a distinction with the notation used for parametric curves. We will use $\underline{\gamma}_{\mathrm{ini}}$ and $\underline{\gamma}_{\mathrm{fin}}$ for the initial and final point of the curve $\underline{\gamma} \in \mathscr{C}(X)$, respectively.

Definition 6.6 (Canonical maps). We denote:

(a) by $\mathsf{i} := \pi_{\mathscr{C}} \circ \mathsf{k} : \{\gamma \in \mathrm{AC}([0,1];X) : \ell(\gamma) > 0\} \to \mathscr{C}(X)$ the projection provided by Proposition 6.1, which coincides with the canonical projection $\pi_{\mathscr{C}(X)}$ on the quotient when restricted to $\mathrm{AC}_0([0,1];X)$;
(b) by $\mathsf{j} := \mathsf{k} \circ \pi_{\mathscr{C}}^{-1} : \mathscr{C}(X) \to \mathrm{AC}^\infty_c([0,1];X)$ the canonical representation of a nonparametric curve by a parameterization in $[0,1]$ with constant velocity.
(c) by $\tilde{J} : \mathscr{C}(X) \to \mathcal{M}_+(X) \setminus \{0\}$ the quotient of the map J in (6.6), defined by

$$\tilde{J}[\gamma] := J\gamma. \tag{6.10}$$

Lemma 6.7 (Measurable structure of $\mathscr{C}(X)$). *If (X,d) is complete and separable, the space $(\mathscr{C}(X),\tau_{\mathscr{C}})$ is a Lusin Hausdorff space and the restriction of the map i to $\mathrm{AC}^\infty_c([0,1];X)$ is a Borel isomorphism. In particular, a collection of curves $\Gamma \subset \mathscr{C}(X)$ is Borel if and only if $\mathsf{j}(\Gamma)$ is Borel in $\mathrm{C}([0,1];X)$. Analogously, $\Gamma \subset \mathscr{C}(X)$ is Suslin if and only if $\mathsf{j}(\Gamma)$ is Suslin in $\mathrm{C}([0,1];X)$.*

Proof. Let us first show that $(\mathscr{C}(X),\tau_{\mathscr{C}})$ is Hausdorff. For contradiction, suppose that there exist curves $\mathsf{i}(\sigma_i) \in \mathscr{C}(X)$ with $\sigma_i \in \mathrm{AC}^\infty_c([0,1];X)$, $i = 1, 2$, and a sequence of parameterizations $\mathsf{s}^n_i \in \mathrm{AC}([0,1];[0,1])$ with $(\mathsf{s}^n_i)' > 0$ $\mathscr{L}^1$-a.e. in $(0,1)$, such that

$$\lim_{n\to\infty} \sup_{t\in[0,1]} \mathsf{d}(\sigma_1(\mathsf{s}^n_1(t)), \sigma_2(\mathsf{s}^n_2(t))) = 0.$$

Denoting $\mathsf{r}^n_1(t) := \mathsf{s}^n_1 \circ (\mathsf{s}^n_2)^{-1}$ and $\mathsf{r}^n_2(t) := \mathsf{s}^n_2 \circ (\mathsf{s}^n_1)^{-1}$, we get

$$\lim_{n\to\infty} \sup_{t\in[0,1]} \mathsf{d}(\sigma_1(t), \sigma_2(\mathsf{r}^n_2(t))) = 0, \qquad \lim_{n\to\infty} \sup_{t\in[0,1]} \mathsf{d}(\sigma_1(\mathsf{r}^n_1(t)), \sigma_2(t)) = 0.$$

The lower semicontinuity of length with respect to uniform convergence yields $\ell := \ell(\sigma_1) = \ell(\sigma_2)$, and therefore for every $0 \le t' < t'' \le 1$,

$$\ell \liminf_{n\to\infty}(r^n_2(t'') - r^n_2(t')) = \lim_{n\to\infty} \int_{t'}^{t''} |(\sigma_2 \circ \mathsf{r}^n_2)'|\,dt \ge \int_{t'}^{t''} |\sigma_1'|\,dt = \ell(t''-t').$$

Choosing first $t' = t$ and $t'' = 1$ and then $t' = 0$ and $t'' = t$ we conclude that $\lim_n r_2^n(t) = t$ for every $t \in [0,1]$ and therefore $\sigma_1 = \sigma_2$.

Notice that $\mathrm{AC}_c^\infty([0,1];X)$ is a Lusin space, since $\mathrm{AC}_c^\infty([0,1];X)$ is a Borel subset of $\mathrm{C}([0,1];X)$. The restriction of i to $\mathrm{AC}_c^\infty([0,1];X)$ is thus a continuous and injective map from the Lusin space $\mathrm{AC}_c^\infty([0,1];X)$ to the Hausdorff space $(\mathscr{C}(X), \tau_{\mathscr{C}})$ (notice that the topology $\tau_{\mathscr{C}}$ is a priori weaker than the one induced by the restriction of i to $\mathrm{AC}_c^\infty([0,1];X)$). It follows by definition that $\mathscr{C}(X)$ is Lusin. Now, Proposition 2.3(iii) shows that the restriction of i is a Borel isomorphism. □

Lemma 6.8 (Borel regularity of J and $\tilde{J}$). *The map* $J : \mathrm{AC}([0,1];X) \to \mathcal{M}_+(X)$ *is Borel, where* $\mathrm{AC}([0,1];X)$ *is endowed with the* $\mathrm{C}([0,1];X)$ *topology. In particular, if* (X,d) *is complete and separable then the map* $\tilde{J} : \mathscr{C}(X) \to \mathcal{M}_+(X) \setminus \{0\}$ *is Borel and* $\tilde{J}(\Gamma)$ *is Suslin in* $\mathcal{M}_+(X)$ *whenever* Γ *is Suslin in* $\mathscr{C}(X)$.

Proof. It is easy to check, using the formula $J\gamma = \gamma_\sharp(|\dot\gamma|\mathscr{L}^1 \llcorner [0,1])$, that

$$J\gamma = \lim_{n\to\infty} \sum_{i=0}^{n-1} \mathsf{d}(\gamma_{(i+1)/n}, \gamma_{i/n})\delta_{\gamma_{i/n}} \qquad \text{weakly in } \mathcal{M}_+(X)$$

for all $\gamma \in \mathrm{AC}([0,1];X)$ (the simple details are left to the reader). Since the approximating maps are continuous, we conclude that J is Borel. The Borel regularity of $\tilde{J}$ follows from Lemma 6.7 and the identity $\tilde{J} = J \circ \mathrm{j}$. Since $\tilde{J}$ is Borel, we can apply Proposition 2.3(iv) to conclude that $\tilde{J}$ maps Suslin sets to Suslin sets. □

7. Modulus of families of nonparametric curves

In this section we assume that (X,d) is a complete and separable metric space and that $\mathfrak{m} \in \mathcal{M}_+(X)$.

In order to apply the results of the previous sections (with the topology τ induced by d) to families of nonparametric curves, we consider the canonical map $\tilde{J} : \mathscr{C}(X) \to \mathcal{M}_+(X)\setminus\{0\}$ of Definition 6.6(c). For simplicity, we will not distinguish between J and $\tilde{J}$, writing $J\underline{\gamma}$ or $J[\gamma] = J\gamma$ (this is not a big abuse of notation, since $\tilde{J}$ is a quotient map).

Now we discuss the notion of $(p,\mathfrak{m})$-modulus for $p \in [1,\infty)$. The $(p,\mathfrak{m})$-modulus for families $\Gamma \subset \mathscr{C}(X)$ of nonparametric curves is given by

$$\mathrm{Mod}_{p,\mathfrak{m}}(\Gamma) := \inf\left\{ \int_X g^p \, d\mathfrak{m} : g \in \mathcal{L}_+^p(X,\mathfrak{m}),\ \int_{\underline{\gamma}} g \ge 1 \text{ for all } \underline{\gamma} \in \Gamma \right\}. \tag{7.1}$$

We adopted the same notation $\mathrm{Mod}_{p,\mathfrak{m}}$ because the identity $\int_{\underline{\gamma}} g = \int_X g \, dJ\underline{\gamma}$ immediately gives

$$\mathrm{Mod}_{p,\mathfrak{m}}(\Gamma) = \mathrm{Mod}_{p,\mathfrak{m}}(J(\Gamma)). \tag{7.2}$$

In a similar vein, setting $q = p'$, in the space $\mathscr{C}(X)$ we can define plans with barycenter in $L^q(X,\mathfrak{m})$ as Borel probability measures $\boldsymbol{\pi}$ in $\mathscr{C}(X)$ satisfying

$$\int_{\mathscr{C}(X)} J\underline{\gamma} \, d\boldsymbol{\pi}(\underline{\gamma}) = g\mathfrak{m} \qquad \text{for some } g \in L^q(X,\mathfrak{m}).$$

Notice that the integral makes sense because the Borel regularity of J easily implies that $\underline{\gamma} \mapsto J\underline{\gamma}(A)$ is Borel in $\mathscr{C}(X)$ for all $A \in \mathscr{B}(X)$. We define, exactly as in (4.3), $c_q(\boldsymbol{\pi})$ to be the $L^q(X, \mathfrak{m})$ norm of the barycenter g. Then the same argument leading to (4.5) gives

$$\frac{\boldsymbol{\pi}(\Gamma)}{c_q(\boldsymbol{\pi})} \leq \mathrm{Mod}_{p,\mathfrak{m}}(\Gamma)^{1/p} \quad \text{for all } \boldsymbol{\pi} \in \mathcal{P}(\mathscr{C}(X)) \text{ with barycenter in } L^q(X, \mathfrak{m}) \tag{7.3}$$

for every universally measurable set Γ in $\mathscr{C}(X)$.

Remark 7.1 (Democratic plans). In more explicit terms, Borel probability measures $\boldsymbol{\pi}$ in $\mathscr{C}(X)$ with barycenter in $L^q(X, \mathfrak{m})$ satisfy

$$\int_0^1 (\mathrm{e}_t)_\sharp(|\dot{\gamma}_t|\boldsymbol{\pi})\, dt = g\mathfrak{m} \quad \text{for some } g \in L^q(X, \mathfrak{m}) \tag{7.4}$$

when we view them as measures on nonconstant curves $\gamma \in \mathrm{AC}([0, 1]; X)$. For instance, in the particular case when $\boldsymbol{\pi}$ is concentrated on the family of geodesics parameterized with constant speed and with length uniformly bounded from below, the case $q = \infty$ corresponds to the class of democratic plans considered in [22].

Defining $C_{p,\mathfrak{m}}(\Gamma)$ as the supremum of the left hand side of (7.3), we can now use Theorem 5.1 to show that even in this case there is no duality gap.

Theorem 7.2. *For every* $p \in (1, \infty)$ *and every Suslin set* $\Gamma \subset \mathscr{C}(X)$ *with* $\mathrm{Mod}_{p,\mathfrak{m}}(\Gamma) > 0$ *there exists a* $\boldsymbol{\pi} \in \mathcal{P}(\mathscr{C}(X))$ *with barycenter in* $L^q(X, \mathfrak{m})$*, concentrated on* Γ *and satisfying* $c_q(\boldsymbol{\pi}) = \mathrm{Mod}_{p,\mathfrak{m}}(\Gamma)^{-1/p}$.

Proof. From Theorem 5.1 and its Corollary 5.2 we deduce the existence of $\boldsymbol{\eta} \in \mathcal{P}(\mathcal{M}_+(X))$ with barycenter in $L^q(X, \mathfrak{m})$ concentrated on the Suslin set $J(\Gamma)$ and satisfying

$$\frac{1}{c_q(\boldsymbol{\eta})} = \mathrm{Mod}_{p,\mathfrak{m}}(J(\Gamma))^{1/p} = \mathrm{Mod}_{p,\mathfrak{m}}(\Gamma)^{1/p}.$$

By a measurable selection theorem [12, Theorem 6.9.1] we can find an $\boldsymbol{\eta}$-measurable map $f : J(\Gamma) \to \mathscr{C}(X)$ such that $f(\mu) \in \Gamma \cap J^{-1}(\mu)$ for all $\mu \in J(\Gamma)$. The measure $\boldsymbol{\pi} := f_\sharp \boldsymbol{\eta}$ is concentrated on Γ, and the equality between the barycenters

$$\int_{\mathscr{C}(X)} J\underline{\gamma}\, d\boldsymbol{\pi}(\underline{\gamma}) = \int \mu\, d\boldsymbol{\eta}(\mu)$$

gives $c_q(\boldsymbol{\pi}) = c_q(\boldsymbol{\eta})$. □

8. Modulus of families of parametric curves

In this section we still assume that (X, d) is a complete and separable metric space and that $\mathfrak{m} \in \mathcal{M}_+(X)$. We consider a notion of p-modulus for *parametric* curves, enforcing the condition (7.4) (at least when Lipschitz curves are considered), and we compare it with the nonparametric counterpart. To this end, we introduce the continuous map

$$M : \mathrm{C}([0, 1]; X) \to \mathcal{P}(X), \qquad M(\gamma) := \gamma_\sharp(\mathscr{L}^1 \llcorner [0, 1]). \tag{8.1}$$

Indeed, replacing $J\gamma = \gamma_\sharp(|\dot\gamma|\mathscr{L}^1 \llcorner [0,1])$ with M we can consider a "parametric" modulus of a family of curves $\Sigma \subset \mathrm{C}([0,1];X)$ just by evaluating $\mathrm{Mod}_{p,\mathfrak{m}}(M(\Sigma))$. By Proposition 3.2(vii), if $\Sigma \subset \mathrm{AC}^\infty_c([0,1];X)$ then

$$\mathrm{Mod}_{p,\mathfrak{m}}(M(\Sigma)) = 0 \ \Leftrightarrow\ \mathrm{Mod}_{p,\mathfrak{m}}(J(\Sigma)) = 0. \tag{8.2}$$

On the other hand, things are more subtle when the speed is not constant.

Definition 8.1 (q-energy and parametric barycenter). Let $\boldsymbol{\rho} \in \mathscr{P}(\mathrm{C}([0,1];X))$ and $q \in [1,\infty)$. We say that $\boldsymbol{\rho}$ has *finite q-energy* if $\boldsymbol{\rho}$ is concentrated on $\mathrm{AC}^q([0,1];X)$ and

$$\int \int_0^1 |\dot\gamma_t|^q \, dt \, d\boldsymbol{\rho}(\gamma) < \infty. \tag{8.3}$$

We say that $\boldsymbol{\rho}$ has *parametric barycenter* $h \in L^q(X,\mathfrak{m})$ if

$$\int \int_0^1 f(\gamma_t)\, dt\, d\boldsymbol{\rho}(\gamma) = \int_X f\, h\, d\mathfrak{m} \quad \forall f \in \mathrm{C}_b(X). \tag{8.4}$$

The finiteness condition (8.3) and the concentration on $\mathrm{AC}^q([0,1];X)$ can also be written, recalling the definition (6.1) of $\mathcal{E}_q$, as follows:

$$\int \mathcal{E}_q(\gamma)\, d\boldsymbol{\rho}(\gamma) < \infty.$$

Notice also that the definition (8.1) of M implies that (8.4) is equivalent to requiring the existence of a constant $C \geq 0$ such that

$$\int \int_X f\, dM\gamma\, d\boldsymbol{\rho}(\gamma) \leq C\left(\int_X f^p\, d\mathfrak{m}\right)^{1/p} \quad \forall f \in \mathrm{C}_b(X),\ f \geq 0. \tag{8.5}$$

In this case the best constant C in (8.5) corresponds to $\|h\|_{L^q(X,\mathfrak{m})}$ for h as in (8.4).

Remark 8.2. It is not difficult to check that a Borel probability measure $\boldsymbol{\rho}$ concentrated on a set $\Gamma \subset \mathrm{AC}^\infty([0,1];X)$ with $\boldsymbol{\rho}$-essentially bounded Lipschitz constants and parametric barycenter in $L^q(X,\mathfrak{m})$ also has (nonparametric) barycenter in $L^q(X,\mathfrak{m})$. Conversely, if $\boldsymbol{\pi} \in \mathscr{P}(\mathscr{C}(X))$ with barycenter in $L^q(X,\mathfrak{m})$ and $\boldsymbol{\pi}$-essentially bounded length $\ell(\gamma)$, then $\mathrm{j}_\sharp\boldsymbol{\pi}$ has parametric barycenter in $L^q(X,\mathfrak{m})$.

Now, arguing as in the proof of Theorem 7.2 (which provided existence of plans $\boldsymbol{\pi}$ in $\mathscr{C}(X)$) we can use a measurable selection theorem to deduce from our basic duality Theorem 5.1 the following result.

Theorem 8.3. *For every $p \in (1,\infty)$ and every Suslin set $\Sigma \subset \mathrm{C}([0,1];X)$, the inequality $\mathrm{Mod}_{p,\mathfrak{m}}(M(\Sigma)) > 0$ is equivalent to the existence of $\boldsymbol{\rho} \in \mathscr{P}(\mathrm{C}([0,1];X))$ concentrated on Σ with parametric barycenter in $L^q(X,\mathfrak{m})$.*

Our next goal is to use reparameterizations to improve the parametric barycenter from $L^q(X,\mathfrak{m})$ to $L^\infty(X,\mathfrak{m})$. To this end, we begin by proving the Borel regularity of some parameterization maps. Let $h : X \to (0,\infty)$ be a Borel map with $\sup_X h < \infty$ and for every $\sigma \in \mathrm{C}([0,1];X)$ set

$$G(\sigma) := \int_0^1 h(\sigma_r)\,dr, \qquad \mathsf{t}_\sigma(s) := \frac{1}{G(\sigma)}\int_0^s h(\sigma_r)\,dr : [0,1] \to [0,1]. \tag{8.6}$$

Since t_σ is Lipschitz and $\mathsf{t}'_\sigma > 0$ $\mathscr{L}^1$-a.e. in $(0,1)$, its inverse $\mathsf{s}_\sigma : [0,1] \to [0,1]$ is absolutely continuous and we can define

$$H : \mathrm{AC}([0,1];X) \to \mathrm{AC}([0,1];X), \qquad H\sigma(t) := \sigma(\mathsf{s}_\sigma(t)). \tag{8.7}$$

Notice that $H(\mathrm{AC}^\infty_c([0,1];X)) \subset \mathrm{AC}_0([0,1];X)$.

Lemma 8.4. *If $h : X \to \mathbb{R}$ is a bounded Borel function, the map G in (8.6) is Borel. If we assume, in addition, that $h > 0$ in X, then also t_σ in (8.6) is Borel and the map H in (8.7) is Borel and injective.*

Proof. Let us prove first that the map

$$\sigma \mapsto \tilde{\mathsf{t}}_\sigma(t) = \int_0^t h(\sigma_r)\,dr$$

is Borel from $\mathrm{C}([0,1];X)$ to $\mathrm{C}([0,1])$ for any bounded Borel function $h : X \to \mathbb{R}$. This follows by a monotone class argument (see for instance [12, Theorem 2.12.9(iii)]), since the class of functions h for which the statement is true is a vector space containing all bounded continuous functions and stable under equibounded pointwise limits. By the continuity of the integral operator, the map G is Borel as well.

Now we turn to H, assuming that $h > 0$. By Proposition 2.3(iii) it will be sufficient to show that the inverse of H, namely the map $\sigma \mapsto \sigma \circ \mathsf{t}_\sigma$, is Borel. Since the map $(\sigma, \mathsf{t}) \mapsto \sigma \circ \mathsf{t}$ is continuous from $\mathrm{C}([0,1];X) \times \mathrm{C}([0,1])$ to $\mathrm{C}([0,1];X)$, the Borel regularity of the inverse of H follows from the Borel regularity of $\sigma \mapsto \mathsf{t}_\sigma$. □

Theorem 8.5. *Let $q \in (1,\infty)$ and $p = q'$. If $\boldsymbol{\rho} \in \mathcal{P}(\mathrm{C}([0,1];X))$ has finite q-energy and parametric barycenter $h \in L^\infty(X,\mathfrak{m})$, then $\boldsymbol{\pi} = \mathsf{i}_\sharp\boldsymbol{\rho}$ has barycenter in $L^q(X,\mathfrak{m})$, and*

$$c_q(\boldsymbol{\pi}) \le \left(\int \mathcal{E}_q(\gamma)\,d\boldsymbol{\rho}(\gamma)\right)^{1/q} \|h\|^{1/p}_{L^\infty(X,\mathfrak{m})}. \tag{8.8}$$

Conversely, if $\boldsymbol{\pi} \in \mathcal{P}(\mathscr{C}(X))$ has barycenter in $L^q(X,\mathfrak{m})$ and $\boldsymbol{\pi}$-essentially bounded length $\ell(\gamma)$, concentrated on a Suslin set $\Gamma \subset \mathscr{C}(X)$, then there exists $\boldsymbol{\rho} \in \mathcal{P}(\mathrm{C}([0,1];X))$ with finite q-energy and parametric barycenter in $L^\infty(X,\mathfrak{m})$ concentrated on a Suslin set contained in $[\mathsf{j}(\Gamma)]$.

More generally, let $\boldsymbol{\sigma} \in \mathcal{P}(\mathrm{C}([0,1];X))$ be concentrated on a Suslin set $\Gamma \subset \mathrm{AC}^\infty([0,1];X)$, with parametric barycenter in $L^q(X,\mathfrak{m})$ and with $\boldsymbol{\sigma}$-essentially bounded Lipschitz constants. Then there exists $\boldsymbol{\rho} \in \mathcal{P}(\mathrm{C}([0,1];X))$ with finite q-energy and parametric barycenter in $L^\infty(X,\mathfrak{m})$ concentrated on a Suslin set contained in $[\Gamma]$.

Proof. Notice that for every nonnegative Borel f we have

$$\begin{aligned}\iint_{\underline{\gamma}} f\,d\boldsymbol{\pi}(\underline{\gamma}) &= \iint_0^1 f(\gamma_t)\,|\dot\gamma_t|\,dt\,d\boldsymbol{\rho}(\gamma)\\ &\le \left(\int \mathcal{E}_q\,d\boldsymbol{\rho}\right)^{1/q}\left(\iiint_0^1 f(\gamma_t)^p\,dt\,d\boldsymbol{\rho}(\gamma)\right)^{1/p} \le \left(\int \mathcal{E}_q\,d\boldsymbol{\rho}\right)^{1/q}\left(\int_X f^p\,h\,d\mathfrak{m}\right)^{1/p}\\ &\le \left(\int \mathcal{E}_q\,d\boldsymbol{\rho}\right)^{1/q}\|h\|^{1/p}_{L^\infty(X,\mathfrak{m})}\|f\|_{L^p(X,\mathfrak{m})},\end{aligned}$$

so that (8.8) holds.

Let us now prove the last statement from $\boldsymbol{\sigma}$ to $\boldsymbol{\rho}$, since the "converse" statement from $\boldsymbol{\pi}$ to $\boldsymbol{\rho}$ simply follows by applying the last statement to $\boldsymbol{\sigma} := \mathsf{j}_\sharp\boldsymbol{\pi}$ and recalling Remark 8.2. Let $g \in L^q(X,\mathfrak{m})$ be the parametric barycenter of $\boldsymbol{\sigma}$ and set $h := 1/(\varepsilon \vee g)$, with $\varepsilon > 0$ fixed. Up to a modification of g on an $\mathfrak{m}$-negligible set, it is not restrictive to assume that h is Borel and with values in $(0, 1/\varepsilon]$, so that the corresponding maps G and H defined as in (8.6) and (8.7) are Borel.

We set $\hat{\boldsymbol{\rho}} := z^{-1}G(\cdot)\boldsymbol{\sigma}$, where $z \in (0, 1/\varepsilon]$ is the normalization constant $\int G(\gamma)\,d\boldsymbol{\sigma}(\gamma)$. Consider the inverse $\mathsf{s}_\sigma : [0,1] \to [0,1]$ of the map t_σ in (8.6), which is absolutely continuous for every σ, and the corresponding transformation $H\sigma$ in (8.7). We denote by L the $\boldsymbol{\sigma}$-essential supremum of the Lipschitz constants of the curves in Γ. Notice that for $\boldsymbol{\sigma}$-a.e. σ,

$$|(H\sigma)'|(t) \le L\mathsf{s}'_\sigma(t) = \frac{L\,G(\sigma)}{h(H\sigma(t))} \qquad \mathscr{L}^1\text{-a.e. in } (0,1), \tag{8.9}$$

and that for every nonnegative Borel function f one has

$$\begin{aligned}\int_0^1 f(H\sigma(t))\,dt = \int_0^1 f(\sigma(\mathsf{s}_\sigma(t)))\,dt = \int_0^1 f(\sigma(s))\mathsf{t}'_\sigma(s)\,ds\\ = \frac{1}{G(\sigma)}\int_0^1 f(\sigma(s))h(\sigma(s))\,ds,\end{aligned}$$

so that choosing $f = h^{-q}$ and using the inequality $G \le 1/\varepsilon$ yields

$$\mathcal{E}_q(H\sigma) \le L^q G(\sigma)^q \int_0^1 h(H\sigma(t))^{-q}\,dt \le \frac{L^q}{\varepsilon^{q-1}}\int_0^1 h(\sigma(s))^{1-q}\,ds. \tag{8.10}$$

Now we set $\boldsymbol{\rho} := H_\sharp\hat{\boldsymbol{\rho}}$ and notice that, by construction, $\boldsymbol{\rho}$ is concentrated on the Suslin set $H(\Gamma) \subset [\Gamma]$. Integrating the q-energy with respect to $\boldsymbol{\rho}$ we obtain

$$\begin{aligned}\int \mathcal{E}_q(\theta)\,d\boldsymbol{\rho}(\theta) = \int \mathcal{E}_q(H\sigma)\,d\hat{\boldsymbol{\rho}}(\sigma) \le \frac{L^q}{z\varepsilon^{q-1}}\int G(\sigma)\int_0^1 h(\sigma(s))^{1-q}\,ds\,d\boldsymbol{\sigma}(\sigma)\\ \le \frac{L^q}{z\varepsilon^q}\int_X g h^{1-q}\,d\mathfrak{m} = \frac{L^q}{z\varepsilon^q}\int_X g(\varepsilon\vee g)^{q-1}\,d\mathfrak{m} < \infty,\end{aligned}$$

thus proving that $\boldsymbol{\rho}$ has finite q-energy. Similarly

$$\int\int_0^1 f(\theta(t))\,dt\,d\boldsymbol{\rho}(\theta) = \int\int_0^1 f(H\sigma(t))\,dt\,d\hat{\boldsymbol{\rho}}(\sigma)$$
$$= \frac{1}{z}\int\int_0^1 f(\sigma(s))h(\sigma(s))\,ds\,d\boldsymbol{\sigma}(\sigma) = \frac{1}{z}\int_X fgh\,d\mathfrak{m}.$$

Since $gh \le 1$, this shows that $\boldsymbol{\rho}$ has parametric barycenter in $L^\infty(X,\mathfrak{m})$. □

In the next corollary, in order to avoid further measurability issues, we state our result with the inner measure

$$\mu_*(E) := \sup\{\mu(B) : B \text{ Borel}, B \subset E\}.$$

This formulation is sufficient for our purposes.

Corollary 8.6. *A Suslin set $\Gamma \subset \mathscr{C}(X)$ is $\mathrm{Mod}_{p,\mathfrak{m}}$-negligible, $p \in (1,\infty)$, if and only if $\boldsymbol{\rho}_*([\mathrm{j}\Gamma]) = 0$ for every $\boldsymbol{\rho} \in \mathcal{P}(\mathrm{C}([0,1];X))$ concentrated on $\mathrm{AC}^q([0,1];X)$ and with parametric barycenter in $L^\infty(X,\mathfrak{m})$.*

Proof. First suppose that Γ is $\mathrm{Mod}_{p,\mathfrak{m}}$-negligible and denote by $h \in L^\infty(X,\mathfrak{m})$ the parametric barycenter of $\boldsymbol{\rho}$ and let us prove that $\boldsymbol{\rho}_*([\mathrm{j}\Gamma]) = 0$. Since $\boldsymbol{\rho}$ is concentrated on $\mathrm{AC}^q([0,1];X)$ we can assume with no loss of generality (possibly restricting $\boldsymbol{\rho}$ to the class of curves σ with $\mathcal{E}_q(\sigma) \le n$ and normalizing) that $\boldsymbol{\rho}$ has finite q-energy. We observe that if $\sigma \in \mathrm{AC}([0,1];X)$ and $f : X \to [0,\infty]$ is Borel, then

$$\int_0^1 f(\sigma(t))|\dot{\sigma}(t)|\,dt \le \left(\int_0^1 f(\sigma(t))^p\,dt\right)^{1/p} \mathcal{E}_q(\sigma)^{1/q}. \tag{8.11}$$

If

$$\int_{\underline{\gamma}} f \ge 1 \qquad \forall \underline{\gamma} \in \Gamma$$

we obtain $\int_\sigma f \ge 1$ for all $\sigma \in [\mathrm{j}\Gamma]$. We can now integrate with respect to $\boldsymbol{\rho}$ and use (8.11) to get

$$\begin{aligned}\boldsymbol{\rho}_*([\mathrm{j}\Gamma]) &\le \left(\int\int_0^1 f(\sigma(t))^p\,dt\,d\boldsymbol{\rho}(\sigma)\right)^{1/p}\left(\int \mathcal{E}_q(\sigma)\,d\boldsymbol{\rho}(\sigma)\right)^{1/q}\\ &= \left(\int_X f^p\,h\,d\mathfrak{m}\right)^{1/p}\left(\int \mathcal{E}_q(\sigma)\,d\boldsymbol{\rho}(\sigma)\right)^{1/q}\\ &\le \|f\|_p\|h\|_\infty^{1/p}\left(\int \mathcal{E}_q(\sigma)\,d\boldsymbol{\rho}(\sigma)\right)^{1/q}.\end{aligned} \tag{8.12}$$

By minimizing with respect to f we see that $\boldsymbol{\rho}_*([\mathrm{j}\Gamma]) = 0$.

Conversely, suppose that $\mathrm{Mod}_{p,\mathfrak{m}}(\Gamma) > 0$; possibly passing to a smaller set, by the countable subadditivity of $\mathrm{Mod}_{p,\mathfrak{m}}$ we can assume that ℓ is bounded on Γ; then by Theorem 7.2 there exists $\boldsymbol{\pi} \in \mathcal{P}(\mathscr{C}(X))$ with barycenter in $L^q(X,\mathfrak{m})$ concentrated on Γ, and therefore the boundedness of ℓ allows us to apply the final statement of Theorem 8.5 to obtain $\boldsymbol{\rho} \in \mathcal{P}(\mathrm{C}([0,1];X))$ with finite q-energy, parametric barycenter in $L^\infty(X,\mathfrak{m})$ and concentrated on a Suslin subset of $[\mathrm{j}\Gamma]$. □

Corollary 8.7. *Let $\Gamma \subset \mathrm{AC}^\infty([0,1];X)$ be a Suslin set such that $\boldsymbol{\rho}_*([\Gamma]) = 0$ for every plan $\boldsymbol{\rho} \in \mathcal{P}(\mathrm{C}([0,1];X))$ concentrated on $\mathrm{AC}^q([0,1];X)$, $q \in (1,\infty)$, and with parametric barycenter in $L^\infty(X,\mathfrak{m})$. Then $M(\Gamma)$ is $\mathrm{Mod}_{p,\mathfrak{m}}$-negligible.*

Proof. Suppose for contradiction that $\mathrm{Mod}_{p,\mathfrak{m}}(M(\Gamma)) > 0$; possibly passing to a smaller set, by the countable subadditivity of $\mathrm{Mod}_{p,\mathfrak{m}}$ we can assume that Lip is bounded on Γ. By Theorem 8.3 there exists $\boldsymbol{\pi} \in \mathcal{P}(\mathrm{C}([0,1];X))$ with parametric barycenter in $L^q(X,\mathfrak{m})$ concentrated on Γ. The boundedness of Lip on Γ allows us to apply the second part of Theorem 8.5 to obtain $\boldsymbol{\rho} \in \mathcal{P}(\mathrm{C}([0,1];X))$ with parametric barycenter in $L^\infty(X,\mathfrak{m})$, finite q-energy and concentrated on a Suslin subset of $[\Gamma]$. □

9. Test plans and their null sets

In this section we will assume that (X,d) is a complete and separable metric space and $\mathfrak{m} \in \mathcal{M}_+(X)$. The following notions have already been used in [6] ($q=2$) and [7] (in connection with the Sobolev spaces with gradient in $L^p(X,\mathfrak{m})$, with $q=p'$; see also [3] in connection with the BV theory).

Definition 9.1 (q-test plans and negligible sets). Let $\boldsymbol{\rho} \in \mathcal{P}(\mathrm{C}([0,1];X))$ and $q \in [1,\infty]$. We say that $\boldsymbol{\rho}$ is a *q-test plan* if

(i) $\boldsymbol{\rho}$ is concentrated on $\mathrm{AC}^q([0,1];X)$;
(ii) there exists a constant $C = C(\boldsymbol{\rho}) > 0$ satisfying $(\mathrm{e}_t)_\sharp\boldsymbol{\rho} \le C\mathfrak{m}$ for all $t \in [0,1]$.

We say that a universally measurable set $\Gamma \subset \mathrm{C}([0,1];X)$ is *q-negligible* if $\boldsymbol{\rho}(\Gamma) = 0$ for all q-test plans $\boldsymbol{\rho}$.

Notice that, by definition, $\mathrm{C}([0,1];X) \setminus \mathrm{AC}^q([0,1];X)$ is q-negligible. The lack of invariance of these concepts, even under bi-Lipschitz reparameterizations, is due to condition (ii), which is imposed at any given time and with no averaging (and no dependence on speed as well). Since condition (ii) is more restrictive compared for instance to the notion of democratic test plan of [22] (see Remark 7.1), this means that sets of curves have higher chances of being negligible with respect to this notion, as the next elementary example shows.

We now want to relate null sets according to Definition 9.1 to null sets in the sense of p-modulus. Notice first that in the definition of q-negligible set we might consider only plans $\boldsymbol{\rho}$ satisfying the stronger condition

$$\operatorname{ess\,sup} \mathcal{E}_q(\sigma) < \infty \tag{9.1}$$

because any q-test plan can be monotonically approximated by q-test plans satisfying this condition. Arguing as in the proof of (8.12) we easily see that

$$\Gamma \subset \mathscr{C}(X)\ \mathrm{Mod}_{p,\mathfrak{m}}\text{-negligible} \ \Rightarrow\ \mathrm{i}^{-1}(\Gamma)\ q\text{-negligible}. \tag{9.2}$$

The following simple example shows that the implication cannot be reversed, namely sets whose images under i^{-1} are q-negligible need not be $\mathrm{Mod}_{p,\mathfrak{m}}$-null.

Example 9.2. Let $X = \mathbb{R}^2$, d the Euclidean distance, $\mathfrak{m} = \mathscr{L}^2$. The family of parametric segments

$$\Sigma = \{\gamma^x : x \in [0, 1]\} \subset \mathrm{AC}([0, 1]; \mathbb{R}^2)$$

with $\gamma^x_t = (x, t)$ is q-negligible for any q, but $\mathsf{i}(\Sigma)$ has p-modulus equal to 1.

In the previous example the implication fails because the trajectories γ^x fall, at any given time t, into an $\mathfrak{m}$-negligible set, and actually the same would be true if this concentration property held at some fixed time. It is tempting to imagine that the implication is restored if we add to the initial family of curves all their reparameterizations (an operation that leaves the p-modulus invariant). However, since any reparameterization fixes the endpoints, even this fails. Nonetheless, in the following, we will see that the implication

$$\Gamma \ q\text{-negligible} \ \Rightarrow \ \mathrm{Mod}_{p,\mathfrak{m}}(\mathsf{i}(\Gamma)) = 0$$

can be restored if we add some structural assumptions on Γ (in particular a "stability" condition); the collections of curves we are mainly interested in are those connected with the theory of Sobolev spaces in [6], [7], and we will find a new proof of the fact that if we define weak upper gradients according to the two notions, the Sobolev spaces are eventually the same.

We now fix some additional notation: for $I = [a, b] \subset [0, 1]$ we define the "stretching" map $\mathsf{s}_I : \mathrm{AC}([0, 1]; X) \to \mathrm{AC}([0, 1]; X)$, mapping γ to $\gamma \circ s_I$, where $s_I : [0, 1] \to [a, b]$ is the affine map with $s_I(0) = a$ and $s_I(1) = b$. Notice that this map also acts in all the other spaces AC^q, AC_0, AC^∞_c of parametric curves we are considering.

Definition 9.3 (Stable and invariant sets of curves).

(i) We say that $\Gamma \subset \{\gamma \in \mathrm{AC}([0, 1]; X) : \ell(\gamma) > 0\}$ is *invariant under constant speed reparameterization* if $\mathsf{k}\gamma \in \Gamma$ for all $\gamma \in \Gamma$.

(ii) We say that $\Gamma \subset \mathrm{AC}([0, 1]; X)$ is $\sim$-*invariant* if $[\gamma] \subset \Gamma$ for all $\gamma \in \Gamma$.

(iii) We say that $\Gamma \subset \mathrm{AC}([0, 1]; X)$ is *stable* if for every $\gamma \in \Gamma$ there exists $\varepsilon \in (0, 1/2)$ such that $s_I\gamma \in \Gamma$ whenever $I = [a, b] \subset [0, 1]$ and $|a| + |1 - b| \le \varepsilon$.

The following theorem provides key connections between q-negligibility and $\mathrm{Mod}_{p,\mathfrak{m}}$-negligibility, both in the nonparametric sense (statement (i)) and in the parametric case (statement (ii)), for stable sets of curves.

Theorem 9.4. *Let $\Gamma \subset \mathrm{AC}([0, 1]; X)$ be a Suslin and stable set of curves, and let $p, q \in (1, \infty)$.*

(i) *If, in addition, $\ell(\gamma) > 0$ for all $\gamma \in \Gamma$, and Γ is both $\sim$-invariant and invariant under constant speed reparameterization, then Γ is q-negligible if and only if $J(\Gamma)$ is $\mathrm{Mod}_{p,\mathfrak{m}}$-negligible in $\mathcal{M}_+(X)$ (equivalently, $\mathsf{i}(\Gamma)$ is $\mathrm{Mod}_{p,\mathfrak{m}}$-negligible in $\mathscr{C}(X)$).*

(ii) *If Γ is q-negligible and $[\Gamma \cap \mathrm{AC}^\infty([0, 1]; X)] \subset \Gamma$, then $M(\Gamma \cap \mathrm{AC}^\infty([0, 1]; X))$ is $\mathrm{Mod}_{p,\mathfrak{m}}$-negligible in $\mathcal{M}_+(X)$. If Γ is also $\sim$-invariant then the converse holds, too.*

Proof. (i) The proof of the nontrivial implication, from positivity of $\mathrm{Mod}_{p,\mathfrak{m}}(J(\Gamma))$ to Γ being non-q-negligible, is completely analogous to the proof of (ii), given below, by applying Corollary 8.6 to $\mathrm{i}(\Gamma)$ in place of Corollary 8.7 applied to $\Gamma \cap \mathrm{AC}^\infty([0,1];X)$, and by using the same rescaling technique. Since we will only need (ii), we only give a detailed proof of (ii).

(ii) Let us prove that the positivity of $\mathrm{Mod}_{p,\mathfrak{m}}(M(\Gamma \cap \mathrm{AC}^\infty([0,1];X)))$ implies that Γ is not q-negligible. Since $\Gamma \cap \mathrm{AC}^\infty([0,1];X)$ is stable, we can assume the existence of $\varepsilon \in (0,1/2)$ such that $s_I\gamma \in \Gamma$ whenever $I=[a,b] \subset [0,1]$ and $|a|+|1-b| \le \varepsilon$.

By applying Corollary 8.7 to $\Gamma \cap \mathrm{AC}^\infty([0,1];X)$ we obtain the existence of $\boldsymbol{\rho} \in \mathcal{P}(\mathrm{AC}^q([0,1];X))$ concentrated on a Suslin subset of $[\Gamma \cap \mathrm{AC}^\infty([0,1];X)]$, and then on Γ, with L^∞ parametric barycenter, i.e. such that

$$\int_0^1 (\mathrm{e}_t)_\sharp \boldsymbol{\rho}\, dt \le C\mathfrak{m} \quad \text{for some } C>0. \tag{9.3}$$

Define a family of reparameterization maps $F_\varepsilon^\tau : \mathrm{AC}^q([0,1];X) \to \mathrm{AC}^q([0,1];X)$ by

$$F_\varepsilon^\tau \gamma(t) = \gamma\left(\frac{t+\tau}{1+\varepsilon}\right), \qquad t \in [0,1], \qquad \forall \gamma \in \mathrm{AC}^q([0,1];X),\ \forall \tau \in [0,\varepsilon]. \tag{9.4}$$

Consider now the measure

$$\boldsymbol{\rho}_\varepsilon = \frac{1}{\varepsilon}\int_0^\varepsilon (F_\varepsilon^\tau)_\sharp \boldsymbol{\rho}\, d\tau.$$

We claim that $\boldsymbol{\rho}_\varepsilon$ is a q-plan: it is clear that $\boldsymbol{\rho}_\varepsilon$ is a probability measure on $\mathrm{AC}^q([0,1];X)$, and so we have to check only the marginals at every time:

$$\begin{aligned}(\mathrm{e}_t)_\sharp \boldsymbol{\rho}_\varepsilon &= \frac{1}{\varepsilon}\int_0^\varepsilon (\mathrm{e}_t)_\sharp\big((F_\varepsilon^\tau)_\sharp \boldsymbol{\rho}\big)\, d\tau = \frac{1}{\varepsilon}\int_0^\varepsilon (\mathrm{e}_{\frac{t+\tau}{1+\varepsilon}})_\sharp \boldsymbol{\rho}\, d\tau \\ &= \frac{1+\varepsilon}{\varepsilon}\int_{\frac{t}{1+\varepsilon}}^{\frac{t+\varepsilon}{1+\varepsilon}} (\mathrm{e}_s)_\sharp \boldsymbol{\rho}\, ds \le \frac{1+\varepsilon}{\varepsilon}\int_0^1 (\mathrm{e}_s)_\sharp \boldsymbol{\rho}\, ds \le C\frac{1+\varepsilon}{\varepsilon}\mathfrak{m} \quad \text{for all } t \in [0,1].\end{aligned}$$

Now we reach the absurd if we show that $\boldsymbol{\rho}_\varepsilon$ is concentrated on Γ; in order to do so it is sufficient to notice that $F_\varepsilon^\tau = s_I$ with $I = I_\varepsilon^\tau = \left[\frac{\tau}{1+\varepsilon}, \frac{1+\tau}{1+\varepsilon}\right]$ and $\tau \in [0,\varepsilon]$.

Now, if we assume also that $[\Gamma] \subset \Gamma$, then we know that for all $\gamma \in \Gamma$ the curve $\eta := \gamma \circ \mathsf{s}_1^{-1}$ belongs to $\Gamma \cap \mathrm{AC}^\infty([0,1];X)$, where $\mathsf{s}_1 : [0,1] \to [0,1]$ is the parameterization defined in the proof of Proposition 6.1. We recall that by definition we have $(1+\ell(\gamma))\mathsf{s}_1'(t) = 1+|\dot\gamma_t|$ for $\mathscr{L}^1$-a.e. t; in particular, the change of variables formula gives

$$\int_0^1 (1+|\dot\gamma_t|)g(\gamma_t)\, dt = (1+\ell(\gamma))\int_0^1 g(\eta_s)\, ds \quad \forall g : X \to [0,\infty] \text{ Borel}. \tag{9.5}$$

We suppose that $M(\Gamma \cap \mathrm{AC}^\infty([0,1];X))$ is $\mathrm{Mod}_{p,\mathfrak{m}}$-negligible; this gives $f \in \mathcal{L}_+^p(X,\mathfrak{m})$ such that

$$\int_0^1 f(\eta_s)\, ds = \infty \quad \forall \eta \in \Gamma \cap \mathrm{AC}^\infty([0,1];X). \tag{9.6}$$

Now given any q-plan $\boldsymbol{\pi}$ we have

$$\begin{aligned}\int_\Gamma \int_0^1 &(|\dot\gamma_t|+1) f(\gamma_t)\,dt\,d\boldsymbol{\pi}(\gamma)\\ &\le \left(\int\int_0^1 (|\dot\gamma_t|+1)^q\,dt\,d\boldsymbol{\pi}(\gamma)\right)^{1/q}\left(\int\int_0^1 f(\gamma_t)^p\,dt\,d\boldsymbol{\pi}(\gamma)\right)^{1/p}\\ &\le \left(\left(\int \mathcal{E}_q\,d\boldsymbol{\pi}\right)^{1/q}+1\right)\left(C(\boldsymbol{\pi})\cdot\int_X f^p\,d\mathfrak{m}\right)^{1/p} < \infty. \qquad (9.7)\end{aligned}$$

Now, (9.6) and (9.5) with $g = f$ shows $\int_0^1 (|\dot\gamma_t|+1) f(\gamma_t)\,dt = \infty$ for all $\gamma \in \Gamma$, so that (9.7) gives $\boldsymbol{\pi}(\Gamma) = 0$. Since $\boldsymbol{\pi}$ is arbitrary, Γ is q-negligible. □

Remark 9.5. The proof shows that if Γ is $\sim$-invariant and $M(\Gamma \cap \mathrm{AC}^\infty([0,1];X))$ is $\mathrm{Mod}_{p,\mathfrak{m}}$-negligible in $\mathcal{M}_+(X)$, then Γ is q-negligible, independently of the stability assumption that we used in the converse implication.

10. Weak upper gradients

As in the previous sections, (X, d) will be a complete and separable metric space and $\mathfrak{m} \in \mathcal{M}_+(X)$.

Recall that a Borel function $g : X \to [0,\infty]$ is an *upper gradient* of $f : X \to \mathbb{R}$ if

$$|f(\underline{\gamma}_{\mathrm{fin}}) - f(\underline{\gamma}_{\mathrm{ini}})| \le \int_{\underline{\gamma}} g \qquad (10.1)$$

for all $\underline{\gamma} \in \mathscr{C}(X)$. Here, the curvilinear integral $\int_{\underline{\gamma}} g$ is given by $\int_J g(\gamma_t)|\dot\gamma_t|\,dt$, where $\gamma : J \to X$ is any parameterization of the curve $\underline{\gamma}$ (i.e., $\underline{\gamma} = \mathrm{i}\gamma$, and one can canonically take $\gamma = \mathrm{j}\underline{\gamma}$). It follows from Proposition 6.4 that the upper gradient property can be equivalently written in the form

$$|f(\underline{\gamma}_{\mathrm{fin}}) - f(\underline{\gamma}_{\mathrm{ini}})| \le \int_X g\,dJ\underline{\gamma}.$$

Now we introduce two different notions of Sobolev function and a corresponding notion of p-weak gradient; the first one was first given in [23] while the second one in [6] for $p = 2$ and in [7] for general exponents. When discussing the corresponding notions of (minimal) weak gradient we will follow the terminology of [7].

Definition 10.1 ($N^{1,p}$ and p-upper gradient). Let $p \in (1,\infty)$, and let f be an $\mathfrak{m}$-measurable and p-integrable function on X. We say that f belongs to the space $N^{1,p}(X,\mathsf{d},\mathfrak{m})$ if there exists $g \in \mathcal{L}^p_+(X,\mathfrak{m})$ such that (10.1) is satisfied for $\mathrm{Mod}_{p,\mathfrak{m}}$-a.e. curve $\underline{\gamma}$.

Functions in $N^{1,p}$ have the important Beppo-Levi property of being absolutely continuous along $\mathrm{Mod}_{p,\mathfrak{m}}$-a.e. curve $\underline{\gamma}$ (more precisely, this means $f \circ \mathrm{j}\underline{\gamma} \in \mathrm{AC}([0,1])$), see [23, Proposition 3.1]. Because of the implication (9.2), functions in $N^{1,p}(X,\mathsf{d},\mathfrak{m})$ belong to the Sobolev space defined below (see [6], [7]) where (10.1) is required for q-a.e. curve γ.

Definition 10.2 ($W^{1,p}$ and p-weak upper gradient). Let $p \in (1, \infty)$, and let f be an $\mathfrak{m}$-measurable and p-integrable function on X. We say that f belongs to the space $W^{1,p}(X, \mathsf{d}, \mathfrak{m})$ if there exists $g \in \mathcal{L}^p_+(X, \mathfrak{m})$ such that

$$|f(\gamma_1) - f(\gamma_0)| \leq \int_0^1 g(\gamma_t)|\dot{\gamma}_t|\,dt$$

for q-a.e. curve $\gamma \in \mathrm{AC}^q([0, 1]; X)$.

We remark that there is an important difference between the two definitions, namely the first one is a priori not invariant if we change the function f on an $\mathfrak{m}$-negligible set, while the second one has this kind of invariance, because for any q-test plan $\boldsymbol{\rho}$, any $\mathfrak{m}$-negligible Borel set N and any $t \in [0, 1]$ the set $\{\gamma : \gamma_t \in N\}$ is $\boldsymbol{\rho}$-negligible. Associated to these two notions are the minimal p-upper gradient and the minimal p-weak upper gradient, both uniquely determined up to $\mathfrak{m}$-negligible sets (for a more detailed discussion, see [7, 23]).

As an application of Theorem 9.4, we show that these two notions are essentially equivalent modulo the choice of a representative in the equivalence class: more precisely, for any $f \in W^{1,p}(X, \mathsf{d}, \mathfrak{m})$ there exists an $\mathfrak{m}$-measurable representative $\tilde{f}$ of f which belongs to $N^{1,p}(X, \mathsf{d}, \mathfrak{m})$. This result is not new, because in [6] and [7] the equivalence has already been shown. On the other hand, the proof of the equivalence in [6] and [7] is by no means elementary, it passes through the use of tools from the theory of gradient flows and optimal transport theory and it provides the equivalence with another relevant notion of "relaxed" gradient based on approximation by Lipschitz functions. We provide a totally different proof, using the results proved in this paper about negligibility of sets of curves.

In the following theorem we provide, first, the existence of a "good representative" of f. Notice that the standard theory of Sobolev spaces provides the existence of this representative via approximation by Lipschitz functions.

Theorem 10.3 (Good representative). *Let $f : X \to \mathbb{R}$ be a Borel function and set*

$$\Gamma = \{\gamma \in \mathrm{AC}^\infty([0, 1]; X) : f \circ \gamma \text{ has a continuous representative } f_\gamma : [0, 1] \to \mathbb{R}\}.$$

If $\mathrm{Mod}_{p,\mathfrak{m}}(M(\mathrm{AC}^\infty([0, 1]; X) \setminus \Gamma)) = 0$ and $p \in (1, \infty)$, there exists an $\mathfrak{m}$-measurable representative $\tilde{f} : X \to \mathbb{R}$ of f satisfying

$$\mathrm{Mod}_{p,\mathfrak{m}}(M(\{\gamma \in \Gamma : \tilde{f} \circ \gamma \not\equiv f_\gamma\})) = 0. \tag{10.2}$$

In particular

(i) $\tilde{f} \circ \gamma \equiv f_\gamma$ *for q-a.e. curve γ;*
(ii) $\tilde{f} \circ \mathrm{j}\underline{\gamma} \equiv f_{\mathrm{j}\underline{\gamma}}$ *for $\mathrm{Mod}_{p,\mathfrak{m}}$-a.e. curve $\underline{\gamma}$.*

Proof. Set $\tilde{\Gamma} := \mathrm{AC}^\infty([0, 1]; X) \setminus \Gamma$, so that our assumption reads $\mathrm{Mod}_{p,\mathfrak{m}}(M(\tilde{\Gamma})) = 0$. Notice first that (ii) makes sense because $f_{\mathrm{j}\underline{\gamma}}$ exists for $\mathrm{Mod}_{p,\mathfrak{m}}$-a.e. curve $\underline{\gamma}$ thanks to (8.2) and $\mathrm{Mod}_{p,\mathfrak{m}}(M(\tilde{\Gamma} \cap \mathrm{AC}^\infty_c([0, 1]; X))) = 0$ (also, constant curves are all contained in Γ). Moreover, (i) makes sense thanks to Remark 9.5 and to the fact that the defining property of Γ is $\sim$-invariant.

Step 1 (Construction of a good set Γ_g of curves). Since $\mathrm{Mod}_{p,\mathfrak{m}}(M(\tilde{\Gamma})) = 0$, there exists $h \in \mathcal{L}^p_+(X, \mathfrak{m})$ such that $\int_0^1 h \circ \sigma = \infty$ for every $\sigma \in \tilde{\Gamma}$. Starting from Γ and h, we can define the set $\Gamma_g = \{\eta \in \Gamma : \int_0^1 h \circ \eta < \infty\}$ of "good" curves, satisfying the following three conditions:

(a) $f \circ \eta$ has a continuous representative for all $\eta \in \Gamma_g$;
(b) $\int_0^1 h \circ \eta < \infty$ for all $\eta \in \Gamma_g$;
(c) $M(\mathrm{AC}^\infty([0, 1]; X) \setminus \Gamma_g)$ is $\mathrm{Mod}_{p,\mathfrak{m}}$-negligible.

Indeed, properties (a) and (b) follow easily by definition, while (c) follows from the inclusion

$$M(\mathrm{AC}^\infty([0, 1]; X) \setminus \Gamma_g) \subset M(\mathrm{AC}^\infty([0, 1]; X) \setminus \Gamma) \cup \left\{\mu : \int_X h\, d\mu = \infty\right\}.$$

Step 2 (Construction of $\tilde{f}$). For every point $x \in X$ we set

$$\Theta_x = \{(\eta, t) \in \Gamma_g \times [0, 1] : \eta(t) = x\}.$$

Thanks to property (a) of Γ_g, we can partition this set according to the value of the continuous representative f_η at t:

$$\Theta_x = \bigcup_{r\in\mathbb{R}} \Theta_x^r \quad \text{with} \quad \Theta_x^r = \{(\eta, t) \in \Theta_x : f_\eta(t) = r\}.$$

Now, the key point is that for every $x \in X$ there exists at most one r such Θ_x^r is not empty. Indeed, suppose that $r_1 \neq r_2$ and that there exist $(\eta_1, t_1) \in \Theta_x^{r_1}$ and $(\eta_2, t_2) \in \Theta_x^{r_2}$ such that $r_1 = f_{\eta_1}(t_1) \neq f_{\eta_2}(t_2) = r_2$; since $\eta_1, \eta_2 \in \Gamma_g$, property (b) of Γ_g gives

$$\int_0^1 h \circ \eta_1\, dt + \int_0^1 h \circ \eta_2\, dt < \infty. \tag{10.3}$$

Suppose to fix ideas that $t_1 > 0$ and $t_2 < 1$ (otherwise we reverse time for one curve, or both, in the following argument). Now we create a new curve $\eta_3 \in \mathrm{AC}^\infty([0, 1]; X)$ by concatenation:

$$\eta_3(s) := \begin{cases} \eta_1(2st_1) & \text{if } s \in [0, 1/2], \\ \eta_2(1 - 2(1 - s)(1 - t_2)) & \text{if } s \in [1/2, 1]. \end{cases}$$

This curve is clearly absolutely continuous and it follows first η_1 for half of the time, and then η_2; it is clear that since $f \circ \eta_3$ coincides $\mathscr{L}^1$-a.e. in $(0, 1)$ with the function

$$a(s) := \begin{cases} f_{\eta_1}(2st_1) & \text{if } s \in [0, 1/2], \\ f_{\eta_2}(1 - 2(1 - s)(1 - t_2)) & \text{if } s \in [1/2, 1], \end{cases}$$

which has a jump discontinuity at $s = 1/2$, $f \circ \eta_3$ has no continuous representative. Consequently, η_3 belongs to $\tilde{\Gamma}$ and therefore $\int_0^1 h \circ \eta_3 = \infty$. But, since

$$\frac{1}{2t_1}\int_0^1 h \circ \eta_1 \, dt + \frac{1}{2(1-t_2)}\int_0^1 h \circ \eta_2 \, dt \geq \int_0^1 h \circ \eta_3 \, dt,$$

we get a contradiction with (10.3).

Now we define

$$\tilde{f}(x) := \begin{cases} f_\eta(t) & \text{if } (\eta, t) \in \Theta_x \text{ for some } \eta \in \Gamma_g, t \in [0,1], \\ f(x) & \text{otherwise.} \end{cases}$$

By construction, $\tilde{f}(\eta(t)) = f_\eta(t)$ for all $t \in [0,1]$ and $\eta \in \Gamma_g$, so that property (c) of Γ_g shows (10.2). Using Remark 9.5 and the fact that $\{\gamma \in \mathrm{AC}([0,1]; X) : \tilde{f} \circ \gamma \equiv f_\gamma\}$ is clearly $\sim$-invariant, we obtain (i) from (10.2). Moreover, from (10.2) we get in particular

$$\mathrm{Mod}_{p,\mathfrak{m}}(M(\{\gamma \in \Gamma \cap \mathrm{AC}_c^\infty([0,1]; X) : \tilde{f} \circ \gamma \not\equiv f_\gamma\})) = 0. \tag{10.4}$$

Recalling (8.2) and the fact that j is a Borel isomorphism, we can rewrite (10.4) as

$$\mathrm{Mod}_{p,\mathfrak{m}}(J(\{\underline{\gamma} \in \mathscr{C}(X) : \tilde{f} \circ \mathrm{j}\underline{\gamma} \not\equiv f_{\mathrm{j}\underline{\gamma}}\})) = 0,$$

and so we have proved also (ii).

Step 3 (The set $F := \{f \neq \tilde{f}\}$ is $\mathfrak{m}$-negligible). Let γ^x be the curve identically equal to x, that is, $\gamma^x_t = x$ for all $t \in [0,1]$. It is clear that γ^x belongs to Γ for every $x \in X$; in particular $f_{\gamma^x}(t) = f(x)$ for every $t \in [0,1]$. The basic observation is that if we consider the set $\tilde{\Gamma}_c$ of constant curves γ^x satisfying $\tilde{f} \circ \gamma^x \not\equiv f_{\gamma^x}$, then $f(x) \neq \tilde{f}(x)$ for every such curve, hence $\tilde{\Gamma}_c = \{\gamma^x : x \in F\}$. In particular, $M(\tilde{\Gamma}_c) = \{\delta_x : x \in F\}$. Now, from (10.2), we know that $\mathrm{Mod}_{p,\mathfrak{m}}(M(\tilde{\Gamma}_c)) = 0$; this provides the existence of $g \in \mathcal{L}^p_+(X, \mathfrak{m})$ such that $g(x) = \infty$ for every $x \in F$, and so F is contained in an $\mathfrak{m}$-negligible set. □

The following simple example shows that, in Theorem 10.3, the "nonparametric" assumption that $J(\mathrm{AC}([0,1]; X) \setminus \Gamma)$ is $\mathrm{Mod}_{p,\mathfrak{m}}$-negligible is not sufficient to conclude that $\tilde{f} = f$ $\mathfrak{m}$-a.e. in X.

Example 10.4. Let $X = [0,1]$, d the Euclidean distance, $\mathfrak{m} = \mathscr{L}^1 + \delta_{1/2}$, $p \in [1,\infty)$. The function f identically 0 on $X \setminus \{1/2\}$ and equal to 1 at $x = 1/2$ has a continuous (actually, identically equal to 0) representative $f_{\mathrm{j}\underline{\gamma}}$ for $\mathrm{Mod}_{p,\mathfrak{m}}$-a.e. curve $\underline{\gamma}$, but any function $\tilde{f}$ such that $\tilde{f} \circ \mathrm{j}\underline{\gamma} \equiv f_{\mathrm{j}\underline{\gamma}}$ for $\mathrm{Mod}_{p,\mathfrak{m}}$-a.e. $\underline{\gamma}$ should be 0 also at $x = 1/2$, so that $\mathfrak{m}(\{f \neq \tilde{f}\}) = 1$.

Now, we are going to apply Theorem 10.3 to the problem of equivalence of Sobolev spaces. We begin with a few preliminary results and definitions.

Let $f : X \to \mathbb{R}$ and $g : X \to [0, \infty]$ be Borel functions. We consider the sets

$$\mathcal{I}(g) := \left\{ \gamma \in \mathrm{AC}([0,1]; X) : \int_\gamma g < \infty \right\}, \tag{10.5}$$

$$\mathcal{B}(f,g) := \left\{ \gamma \in \mathcal{I}(g) : f\circ\gamma \in W^{1,1}(0,1),\ \left|\frac{d}{dt}(f\circ\gamma)\right| \le |\dot\gamma| g\circ\gamma\ \mathscr{L}^1\text{-a.e. in } (0,1) \right\}. \tag{10.6}$$

We will need the following simple measure-theoretic lemma, which says that integration in one variable maps Borel functions to Borel functions. Its proof is an elementary consequence of a monotone class argument (see for instance [12, Theorem 2.12.9(iii)]) and of the fact that the statement is true for F bounded and continuous.

Lemma 10.5. *Let (Y, d_Y) be a metric space and let $F : [0,1] \times Y \to [0, \infty]$ be Borel. Then the function $\mathfrak{I}_F : Y \to [0, \infty]$ defined by $y \mapsto \int_0^1 F(t, y)\, dt$ is a Borel function.*

Lemma 10.6. *Let $f : X \to \mathbb{R}$ and $g : X \to [0, \infty]$ be Borel functions. Then $\mathcal{I}(g) \setminus \mathcal{B}(f,g)$ is a Borel set, stable and $\sim$-invariant.*

Proof. Stability is simple to check: if, for contradiction, there were $\gamma \in \mathcal{I}(g) \setminus \mathcal{B}(f,g)$ and $\mathsf{s}_{[a_n,b_n]}\gamma \in \mathcal{B}(f,g)$ with $a_n \downarrow 0$ and $b_n \uparrow 1$, we would get $f\circ\gamma \in W^{1,1}(a_n,b_n)$ and $\left|\frac{d}{dt} f\circ\gamma\right| \le |\dot\gamma| g\circ\gamma \in L^1(0,1)$ $\mathscr{L}^1$-a.e. in (a_n,b_n). Taking limits, we would obtain $\gamma \in \mathcal{B}(f,g)$, a contradiction.

For the proof of $\sim$-invariance we note that, first of all, Lemma 10.5 with $F(t,\gamma) := g(\gamma_t)|\dot\gamma_t|$ guarantees that $\mathcal{I}(g)$ is a $\sim$-invariant Borel set, provided we define F using a Borel representative of $|\dot\gamma|$; this can be achieved, for instance, using the lim inf of the metric difference quotients. Analogously, the set

$$\mathsf{L} := \left\{ \gamma \in \mathrm{AC}([0,1]; X) : \int_0^1 |f(\gamma_t)|\, dt < \infty \right\}$$

is Borel. Now, $\gamma \in \mathcal{B}(f,g)$ if and only if $\gamma \in \mathcal{I}(g) \cap \mathsf{L}$ and

$$\left| \int_0^1 \phi'(t) f(\gamma_t)\, dt \right| \le \int_0^1 |\phi(t)| g(\gamma_t) |\dot\gamma_t|\, dt \qquad \text{for all } \phi \in W \tag{10.7}$$

with $W = \{\phi \in \mathrm{AC}([0,1];[0,1]) : \phi(0) = \phi(1) = 0\}$. Now, if both s and s^{-1} are absolutely continuous from $[0,1]$ to $[0,1]$, then setting $\eta := \gamma\circ\mathsf{s}$, we can use the change of variables formula to deduce that $(\phi\circ\mathsf{s})' f\circ\eta \in L^1(0,1)$ for all $\phi \in W$ and that

$$\left| \int_0^1 (\phi\circ\mathsf{s})'(r) f(\eta_r)\, dr \right| \le \int_0^1 |\phi\circ\mathsf{s}(r)| g(\eta_r) |\dot\eta_r|\, dr \qquad \text{for all } \phi \in W.$$

Since $W\circ\mathsf{s} = W$ we eventually obtain $\phi' f\circ\eta \in L^1(0,1)$ for all $\phi \in W$ (so that $f\circ\eta$ is locally integrable in $(0,1)$) and

$$\left| \int_0^1 \phi'(r) f(\eta_r)\, dr \right| \le \int_0^1 |\phi(r)| g(\eta_r) |\dot\eta_r|\, dr \qquad \text{for all } \phi \in W. \tag{10.8}$$

It is easy to check that these two conditions, in combination with $\int_\eta g < \infty$, imply that $\eta \in \mathsf{L}$, therefore $f\circ\eta$ belongs to $\mathcal{B}(f,g)$ and $\sim$-invariance is proved.

In order to prove that $\mathcal{B}(f, g)$ is Borel we follow a similar path: we already know that both $\mathcal{I}(g)$ and L are Borel, and then in the class $\mathcal{I}(g) \cap \mathsf{L}$ the condition (10.7), now with W replaced by a countable dense subset of $\mathrm{C}^1_c(0, 1)$ for the C^1 norm, provides a characterization of $\mathcal{B}(f, g)$. Since for $\phi \in \mathrm{C}^1_c(0, 1)$ fixed the maps

$$\eta \in \mathsf{L} \mapsto \int_0^1 \phi'(r) f(\eta_r)\, dr, \qquad \eta \mapsto \int_0^1 |\phi(r)| g(\eta_r) |\dot\eta_r|\, dr$$

are easily seen to be Borel in $\mathrm{AC}([0, 1]; X)$ (as a consequence of Lemma 10.5, splitting the first integral into the positive and negative parts, and once more using a Borel representative of $|\dot\eta|$ in the second integral), we conclude that $\mathcal{B}(f, g)$ is Borel. □

Theorem 10.7 (Equivalence theorem). *Let $p \in (1, \infty)$. Any $f \in N^{1,p}(X, \mathsf{d}, \mathfrak{m})$ belongs to $W^{1,p}(X, \mathsf{d}, \mathfrak{m})$. Conversely, for any $f \in W^{1,p}(X, \mathsf{d}, \mathfrak{m})$ there exists an $\mathfrak{m}$-measurable representative $\tilde f$ that belongs to $N^{1,p}(X, \mathsf{d}, \mathfrak{m})$. More precisely, $\tilde f$ satisfies:*

(i) $\tilde f \circ \gamma \in \mathrm{AC}([0, 1])$ *for q-a.e. curve* $\gamma \in \mathrm{AC}([0, 1]; X)$*;*
(ii) $\tilde f \circ \mathrm{j}\underline{\gamma} \in \mathrm{AC}([0, 1])$ *for* $\mathrm{Mod}_{p,\mathfrak{m}}$*-a.e. curve* $\underline{\gamma}$*.*

Proof. We have already discussed the easy implication from $N^{1,p}$ to $W^{1,p}$, so let us focus on the converse one. Fix $f \in W^{1,p}(X, \mathsf{d}, \mathfrak{m})$ and a p-weak upper gradient g. By Fubini's theorem, it is easily seen that the space $W^{1,p}(X, \mathsf{d}, \mathfrak{m})$ is invariant under modifications on $\mathfrak{m}$-negligible sets; as a consequence, since the Borel σ-algebra is countably generated, we can assume with no loss of generality that f is Borel. Another simple application of Fubini's theorem (see [7, Remark 4.10]) shows that for q-a.e. curve γ there exists an absolutely continuous function $f_\gamma : [0, 1] \to \mathbb{R}$ such that $f_\gamma = f \circ \gamma$ $\mathscr{L}^1$-a.e. in $(0, 1)$ and $\left|\frac{d}{dt} f_\gamma\right| \le |\dot\gamma| g \circ \gamma$ $\mathscr{L}^1$-a.e. in $(0, 1)$. Since the L^p integrability of g implies that the complement of $\mathcal{I}(g)$ is q-negligible, we can use Lemma 10.6 and Theorem 9.4(ii) to infer that $\Sigma = \mathcal{I}(g) \setminus \mathcal{B}(f, g)$ satisfies $\mathrm{Mod}_{p,\mathfrak{m}}(M(\Sigma \cap AC^\infty([0, 1]; X))) = 0$.

By Theorem 10.3 we obtain an $\mathfrak{m}$-measurable representative $\tilde f$ of f such that $\tilde f \circ \gamma \equiv f_\gamma$ for q-a.e. curve γ and $\tilde f \circ \mathrm{j}\underline{\gamma} \equiv f_{\mathrm{j}\underline{\gamma}}$ for $\mathrm{Mod}_{p,\mathfrak{m}}$-a.e. $\underline{\gamma}$. Hence, the fundamental theorem of calculus for absolutely continuous functions gives

$$|\tilde f(\underline{\gamma}_{\mathrm{fin}}) - \tilde f(\underline{\gamma}_{\mathrm{ini}})| = |f_{\mathrm{j}\underline{\gamma}}(1) - f_{\mathrm{j}\underline{\gamma}}(0)| \le \int_0^1 g((\mathrm{j}\underline{\gamma})_t) |(\dot{\mathrm{j}\underline{\gamma}})_t|\, dt = \int_{\underline{\gamma}} g$$

for $\mathrm{Mod}_{p,\mathfrak{m}}$-a.e. $\underline{\gamma}$. □

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