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A strong maximum principle for the Paneitz operator and a non-local flow for the Q-curvature

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Abstract. In this paper we consider Riemannian manifolds (M^n, g) of dimension $n \geq 5$ with semi-positive Q-curvature and non-negative scalar curvature. Under these assumptions we prove (i) the Paneitz operator satisfies a strong maximum principle; (ii) the Paneitz operator is a positive operator; and (iii) its Green's function is strictly positive. We then introduce a non-local flow whose stationary points are metrics of constant positive Q-curvature. Modifying the test function construction of Esposito–Robert, we show that it is possible to choose an initial conformal metric so that the flow has a sequential limit which is smooth and positive, and defines a conformal metric of constant positive Q-curvature.

Keywords. Q-curvature, Paneitz operator, conformal geometry, non-local flow

1. Introduction

In 1983 S. Paneitz introduced a fourth-order conformally invariant differential operator acting on smooth functions, which is defined on any pseudo-Riemannian manifold [Pan08]. Subsequently, T. Branson [Bra85] recognized that this operator describes the conformal transformation of a curvature quantity which is fourth order in the metric.

To describe the operator and associated curvature quantity, let A denote the Schouten tensor

$$A = \frac{1}{n-2}\left(\mathrm{Ric} - \frac{1}{2(n-1)} R g\right),$$

where Ric is the Ricci tensor and R the scalar curvature, and $\sigma_k(A)$ denote the k^{th} symmetric function of the eigenvalues of A. Then the *Q-curvature* of Branson is defined by

$$Q = -\Delta\sigma_1(A) + 4\sigma_2(A) + \frac{n-4}{2}\sigma_1(A)^2, \tag{1.1}$$

and the eponymous operator of Paneitz is

$$P_g u = \Delta_g^2 u + \mathrm{div}_g\{(4A_g - (n-2)\sigma_1(A_g)g)(\nabla u, \cdot)\} + \frac{n-4}{2} Q_g u. \tag{1.2}$$

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The formula connecting P to Q is the following: if $n \neq 4$, suppose $\hat{g} = u^{4/(n-4)} g$ is a conformal metric; then the Q-curvature of $\hat{g}$ is given by

$$Q_{\hat{g}} = \frac{2}{n-4} u^{-\frac{n+4}{n-4}} P_g u. \tag{1.3}$$

When the dimension is four, one writes $\hat{g} = e^{2w} g$, and

$$Q_{\hat{g}} = e^{-4w}\big(-\tfrac{1}{2} P_g w + Q_g\big). \tag{1.4}$$

Branson pointed out that the formulas (1.3)–(1.4) naturally suggest a higher order version of the Yamabe problem: given (M^n, g), find a conformal metric of constant Q-curvature. In dimensions $n \neq 4$ this is equivalent to finding a positive solution of

$$P_g u = \lambda u^{\frac{n+4}{n-4}}, \tag{1.5}$$

where λ is a constant. In four dimensions the equation is

$$P_g w + 2Q_g = \lambda e^{4w}. \tag{1.6}$$

In both cases the sign of λ is determined by the conformal structure.

Considerable progress has been made on the existence problem for solutions of (1.6); see for example [CY95], [DM08], [LLL12], and references therein. Our interest in this paper is dimensions $n \geq 5$, where the lack of a maximum principle (since the equation is higher order) presents an obvious difficulty when seeking positive solutions of (1.3). Consequently, the existence theory is far less developed. Note that for (1.6) no sign condition on w is required.

There are some results in special geometric settings. Djadli–Hebey–Ledoux [DHL00] studied the optimal constant in the Sobolev embedding $W^{2,2} \hookrightarrow L^{2n/(n-4)}$ when $n \geq 5$. As a corollary of their analysis they proved some compactness results for solutions of (1.5) assuming a size condition on λ, and that P_g has constant coefficients (which holds, for example, if (M^n, g) is an Einstein metric). The assumption of constant coefficients allowed them to factor P into the product of two second order operators, then apply the standard maximum principle (see also [VdV93]). Esposito–Robert [ER02] were able to find solutions to the PDE

$$P_g u = \lambda |u|^{\frac{8}{n-4}} u$$

in dimension $n \geq 8$ for non-locally conformally flat manifolds, in the spirit of [Aub76], but with no information on their sign.

The first general existence result for (1.5) was due to Qing–Raske [QR06]. They considered locally conformally flat manifolds of positive scalar curvature, which allowed them to appeal to the work of Schoen–Yau [SY88] to lift the metric to a domain in the sphere via the developing map. Assuming the Poincaré exponent is less than $(n-4)/2$, they proved the existence of a positive solution to the Paneitz–Branson equation with $\lambda > 0$. Hebey–Robert [HR04] also considered the locally conformally flat case with positive scalar curvature, and assumed in addition that the Paneitz operator and its Green's function were positive. They showed that when the Green's function satisfies a positive mass theorem, then the space of solutions to (1.5) is compact. Later, Humbert–Raulot [HR09] verified the positive mass result (see Theorem 2.9 is Section 2.1). Collectively, the work of Hebey–Robert and Humbert–Raulot removed the topological assumption of Qing–Raske on the Poincaré exponent, but replaced it with strong positivity assumptions.

Our goal in this paper is to show that one can prove a maximum principle for P and existence of solutions to (1.5) under considerably weaker positivity assumptions. The conditions we impose are the following:

$$\begin{cases} Q_g \text{ is } \textit{semi-positive}\text{: } Q_g \geq 0, \text{ and } Q_g > 0 \text{ somewhere;} \\ \text{the scalar curvature } R_g \text{ is non-negative.} \end{cases} \tag{1.7}$$

The first main result of the paper is

Theorem A (see Theorem 2.2 below). *Let (M^n, g) be a closed Riemannian manifold of dimension $n \geq 5$ satisfying* (1.7). *If $u \in C^4$ satisfies*

$$P_g u \geq 0,$$

then either $u > 0$ or $u \equiv 0$ on M^n.

Theorem A is proved in Section 2.1, where we also show that (1.7) implies positivity of the Paneitz operator:

Proposition B (see Proposition 2.3 below). *Let (M^n, g) be a closed Riemannian manifold of dimension $n \geq 5$ satisfying* (1.7). *Then the Paneitz operator is positive:*

$$\int \phi P_g \phi \, dv \geq \mu(g) \int \phi^2 \, dv,$$

with $\mu(g) > 0$.

The proof is a simple extension of [Gur99], which considered the four-dimensional case. Since $P_g > 0$, given any $p \in M^n$ the Green's function with pole at p, denoted G_p, exists. As a corollary of Theorem A we have the positivity of G_p:

Proposition C (see Proposition 2.4). *Let (M^n, g) be a closed Riemannian manifold of dimension $n \geq 5$ satisfying* (1.7). *If G_p denotes the Green's function of the Paneitz operator with pole at $p \in M^n$, then $G_p > 0$ on $M^n \setminus \{p\}$.*

Armed with Theorem A, we then address the question of existence of solutions to (1.5). Given a Riemannian metric g_0 satisfying the positivity assumptions (1.7), we introduce a non-local flow whose stationary points are solutions of (1.5) with $\lambda > 0$:

$$\begin{cases} \dfrac{\partial u}{\partial t} = -u + \mu P_{g_0}^{-1}(|u|^{\frac{n+4}{n-4}}), \\ u(\cdot, 0) = 1, \end{cases} \qquad \mu = \frac{\int u P_{g_0} u \, dv_0}{\int |u|^{\frac{2n}{n-4}} dv_0}. \tag{1.8}$$

Using the strong maximum principle and some elementary integral estimates, we show in Section 3 that the flow (1.8) has a positive solution u for all time $t \geq 0$. We also show (see Section 3.2) that the flow has a variational structure. An important consequence of this fact is the monotonicity of the conformal volume:

$$\frac{d}{dt}\,\mathrm{Vol}(g) = \frac{d}{dt}\int u^{\frac{2n}{n-4}}\,dv_0 \ge 0.$$

This monotonicity property also implies the following space-time estimate:

$$\int_0^\infty \left(\int_{M^n} |-u + \mu P_{g_0}^{-1}(u^{\frac{n+4}{n-4}})|^{\frac{2n}{n-4}}\,dv_0 \right)^{\frac{n-4}{n}} dt < \infty.$$

Using these facts, it is possible to choose a sequence of times $t_j \nearrow \infty$ so that the sequence $u_j = u(\cdot, t_j)$ has a weak limit which is a solution of the Q-curvature equation.

To rule out trivial limits, in Sections 4–6 we show that it is possible to choose an initial metric in the conformal class of g_0 for which the solution of the flow satisfies

$$\int u^2\,dv_0 \ge \epsilon_0 > 0$$

for all time. The idea is to construct a test function whose Paneitz–Sobolev quotient is strictly less than the Euclidean value, and use this test function to define an initial conformal metric satisfying the positivity assumptions (i)–(ii) above.

When the dimension is $n = 5, 6$, or 7, or the manifold is locally conformally flat (LCF), the construction of initial data relies on a local expansion on the Green's function of the Paneitz operator. This is proved in Section 2, where we also prove a positive mass theorem. The positive mass result extends the version of Humbert–Raulot [HR09], which they proved in the LCF setting (see Proposition 2.5 and Theorem 2.9) below. When $n \ge 8$ and the metric is not locally conformally flat, we exploit instead some estimates of Esposito–Robert [ER02]. In all cases, we need to find positive test functions with semi-positive Q-curvature, and the strong maximum principle is crucial in this construction.

Finally, in Section 6 we show that the flow converges (up to choosing a suitable sequence of times) to a solution of the Q-curvature equation:

Theorem D (see Theorem 6.1). *Let (M^n, g) be a closed Riemannian manifold of dimension $n \ge 5$ satisfying (1.7). Then there is a conformal metric $h = u^{4/(n-4)}g$ with positive scalar curvature and constant positive Q-curvature.*

Remarks. 1. After a preliminary version of this manuscript was circulated, it was pointed out to us by E. Hebey and F. Robert that the maximum principle of Theorem A can be combined with compactness results in the literature, along with our positive mass result (Theorem 2.9), to give a proof of Theorem D by direct variational methods. When the dimension $n \ge 8$ and (M^n, g) is not locally conformally flat, one can use the expansions in Esposito–Robert [ER02] together with Proposition 4.1 and Theorem 5.2 in Robert's unpublished notes [Rob09] to obtain existence. When $n = 5, 6$ or 7, we construct the necessary test functions to deduce compactness in Proposition 5.1. When (M^n, g) is locally conformally flat, Theorem A and Propositions B and C imply that the Paneitz operator is "strongly positive" in the sense of Hebey–Robert [HR04], and their result provides the necessary compactness theory (see also the comment at the end of their paper regarding the subcritical equation). In particular, this implies that any conformal class of metrics which admits a metric with positive Q-curvature and positive scalar curvature also admits a minimizer of the total Q-curvature functional (with the same positivity conditions).

2. Since this paper was submitted, a number of preprints have appeared studying the Q-curvature in various settings; see [HYa], [HY14], [HYb], and [CC]. In particular, in [HY14], [HYb] the authors have improved our result by weakening the assumption on the scalar curvature; positive Yamabe invariant is sufficient.

We conclude the Introduction by explaining how the flow (1.8) is precisely the $W^{2,2}$-gradient flow for normalized total Q-curvature (up to a dimensional constant). We remark that Baird–Fardoun–Regbaoui [BFR06]considered a non-local flow for the Q-curvature in four dimensions. While their flow differs from ours, some of their ideas inspired our approach.

Given a Riemannian manifold of dimension $n \geq 5$, if the Paneitz operator satisfies $P_g > 0$ then as P is self-adjoint we can define the $W^{2,2}$ inner product by

$$\begin{aligned}\langle \phi, \psi \rangle_{W^{2,2}(g)} &= \int (P_g \phi)\psi \, dv_g \\ &= \int \left[(\Delta_g \phi)(\Delta_g \psi) - 4A_g(\nabla\phi, \nabla\psi) + (n-2)\sigma_1(A_g) g(\nabla\phi, \nabla\psi) + \frac{n-4}{2} Q_g \phi\psi \right] dv_g,\end{aligned}$$

which induces the $W^{2,2}$-norm. Denote the normalized total Q-curvature by

$$\mathcal{Q}[g] = \mathrm{Vol}(g)^{-\frac{n-4}{n}} \int Q_g \, dv_g .$$

By standard variational formulas, if $g' = \phi g$ is an infinitesimal conformal variation of a metric, then the variation of $\mathcal{Q}$ is given by

$$\mathcal{Q}'(g)\phi = \frac{n-4}{2} \int \phi (Q_g - \overline{Q}_g) \, dv_g ,$$

where $\overline{Q}_g$ is the mean value of Q. Since P_g is invertible,

$$\begin{aligned}\mathcal{Q}'(g)\phi &= \frac{n-4}{2} \int \phi P_g (P_g^{-1}(Q_g - \overline{Q}_g)) \, dv_g = \frac{n-4}{2} \int (P_g \phi)(P_g^{-1}(Q_g - \overline{Q}_g)) \, dv_g \\ &= \frac{n-4}{2} \langle \phi, P_g^{-1}(Q_g - \overline{Q}_g) \rangle_{W^{2,2}} .\end{aligned}$$

Therefore, the negative $W^{2,2}$-gradient flow for the total Q-curvature is

$$\frac{\partial}{\partial t} g = -\frac{n-4}{2} P_g^{-1}(Q_g - \overline{Q}_g) \cdot g . \tag{1.9}$$

To see that (1.9) is equivalent to our flow, write

$$g = u^{\frac{4}{n-4}} g_0 . \tag{1.10}$$

Using the conformal transformation law for the Q-curvature we find

$$\begin{aligned} Q_g &= \frac{2}{n-4} u^{\frac{n+4}{n-4}} P_{g_0} u, \\ \overline{Q}_g &= \frac{2}{n-4} \frac{\int u P_{g_0} u \, dv_0}{\int u^{\frac{2n}{n-4}} \, dv_0} = \frac{2}{n-4} \mu. \end{aligned} \tag{1.11}$$

Also, by the conformal covariance of the Paneitz operator, its inverse is also covariant:

$$P_g^{-1} = u^{-1} P_{g_0}^{-1} (u^{\frac{n+4}{n-4}} \cdot). \tag{1.12}$$

Therefore, using (1.10)–(1.12), we can rewrite (1.9) as

$$\frac{\partial}{\partial t} u = \frac{n-4}{4} \{-u + \mu P_{g_0}^{-1} (u^{\frac{n+4}{n-4}})\},$$

which only differs from our flow by the dimensional constant.

2. The Paneitz operator and its Green's function

In this section we prove various properties of the Paneitz operator and its Green's function that will be used throughout the paper.

2.1. Positivity of Paneitz operator and the Strong Maximum Principle

We begin with two results on the Paneitz operator: a comparison principle, and a coercivity estimate. We also prove a technical lemma; it shows that a metric with semi-positive Q-curvature and non-negative scalar curvature must have positive scalar curvature. The proof is a simple application of the maximum principle, and a similar idea will be used elsewhere in the paper. We first state the technical lemma:

Lemma 2.1. *Let (M^n, g) be a closed Riemannian manifold of dimension $n \geq 5$. Assume*

(i) *Q_g is semi-positive, i.e., $Q_g \geq 0$ and $Q_g > 0$ somewhere,*
(ii) *the scalar curvature R_g is non-negative.*

Then the scalar curvature is strictly positive: $R_g > 0$.

Proof. By (1.1) the Q-curvature can be expressed as

$$Q_g = -\frac{1}{2(n-1)} \Delta_g R_g + c_1(n) R_g^2 - c_2(n) |\mathrm{Ric}(g)|^2, \tag{2.1}$$

where $c_1(n), c_2(n) > 0$. Since Q_g is non-negative, it follows that

$$\frac{1}{2(n-1)} \Delta_g R_g \leq c_1(n) R_g^2.$$

By the strong maximum principle, either $R_g > 0$ or $R_g \equiv 0$. In the latter case, by (2.1) we would have $Q_g = -c_2(n) |\mathrm{Ric}(g)|^2 \leq 0$, which is a contradiction. □

We now prove Theorem A of the Introduction:

Theorem 2.2. *Let (M^n, g) be a closed Riemannian manifold of dimension $n \geq 5$. Assume*

(i) *Q_g is semi-positive,*
(ii) *$R_g \geq 0$.*

If $u \in C^4$ satisfies

$$P_g u \geq 0,$$

then either $u > 0$ or $u \equiv 0$ on M^n. Moreover, if $u > 0$, then $h = u^{4/(n-4)} g$ is a metric with non-negative Q-curvature and positive scalar curvature

Proof. For $\lambda \in [0, 1]$ we let

$$u_\lambda = (1 - \lambda) + \lambda u. \tag{2.2}$$

Then $u_0 \equiv 1$, while $u_1 = u$. Assume

$$\min_{M^n} u \leq 0. \tag{2.3}$$

Define $\lambda_0 \in (0, 1]$ by

$$\lambda_0 = \min\Big\{\lambda \in (0, 1] \; : \; \min_{M^n} u_\lambda = 0\Big\}. \tag{2.4}$$

Then for $0 < \lambda < \lambda_0$, it follows that $u_\lambda > 0$. Let

$$g_\lambda = u_\lambda^{\frac{4}{n-4}} g, \tag{2.5}$$

and let Q_λ denote the Q-curvature of g_λ. Note that for $0 < \lambda < \lambda_0$, we have

$$Q_\lambda \geq 0$$

and $Q_\lambda > 0$ somewhere. This follows from the transformation law for the Q-curvature:

$$\begin{aligned}
Q_\lambda &= \frac{2}{n-4} u_\lambda^{-\frac{n+4}{n-4}} P_g u_\lambda = \frac{2}{n-4} u_\lambda^{-\frac{n+4}{n-4}} \{P_g((1-\lambda) + \lambda u)\} \\
&= \frac{2}{n-4} u_\lambda^{-\frac{n+4}{n-4}} \{(1-\lambda) P_g(1) + \lambda P_g u\} = \frac{2}{n-4} u_\lambda^{-\frac{n+4}{n-4}} \left\{(1-\lambda)\frac{n-4}{2} Q_g + \lambda P_g u\right\} \\
&\geq (1-\lambda) Q_g u_\lambda^{-\frac{n+4}{n-4}}.
\end{aligned}$$

Since $\lambda < \lambda_0 \leq 1$ and Q_g is semi-positive, it follows that Q_λ is semi-positive.

Let R_λ denote the scalar curvature of g_λ. We also claim that for $0 \leq \lambda < \lambda_0$,

$$R_\lambda > 0. \tag{2.6}$$

This certainly holds for $\lambda = 0$; but if there were a $\lambda_1 \in (0, \lambda_0)$ with $\min R_{\lambda_1} = 0$, then this would contradict Lemma 2.1.

By the formula for the transformation of the scalar curvature under a conformal change of metric,

$$R_\lambda = u_\lambda^{-\frac{n}{n-4}} \left\{ -\frac{4(n-1)}{n-4} \Delta_g u_\lambda - \frac{8(n-1)}{(n-4)^2} \frac{|\nabla_g u_\lambda|^2}{u_\lambda} + R_g u_\lambda \right\}. \tag{2.7}$$

Since $R_\lambda > 0$, this implies u_λ satisfies the differential inequality

$$\Delta_g u_\lambda \leq \frac{n-4}{4(n-1)} R_g u_\lambda. \tag{2.8}$$

Taking the limit as $\lambda \nearrow \lambda_0$, this also holds for $\lambda = \lambda_0$. By the strong maximum principle, (2.4) and (2.8) imply $u_{\lambda_0} \equiv 0$. If $\lambda_0 = 1$, then we are done. Therefore, assume $\lambda_0 \in (0, 1)$. It follows from (2.2) that $u = -(1-\lambda_0)/\lambda_0$, hence

$$P_g u = -\frac{n-4}{2} \frac{1-\lambda_0}{\lambda_0} Q_g.$$

Since by assumption $Q_g > 0$ somewhere, this contradicts $P_g u \geq 0$. We conclude that $u \equiv 0$ or $u > 0$.

If $u > 0$, then the metric $h = u^{4/(n-4)} g$ is well defined and has non-negative Q-curvature. Once again, we can define the family of functions $\{u_\lambda\}$ as in (2.2) and the metrics g_λ as in (2.5). Then the scalar curvature of g_λ satisfies (2.7), and by the strong maximum principle it follows that either $R_\lambda > 0$ or $R_\lambda \equiv 0$. Recall by Lemma 2.1 that $R_g > 0$. Therefore, we cannot have $R_\lambda \equiv 0$, since a conformal class which admits a metric of positive scalar curvature cannot admit a scalar-flat metric. It follows that $R_\lambda > 0$ for all $\lambda \in [0, 1]$. $\square$

We now show that the positivity assumptions of the preceding theorem imply the positivity of the Paneitz operator. This is easy to prove in dimensions $n \geq 6$, but for $n = 5$ we need to adapt the idea of the $n = 4$ case appearing in [Gur99].

Proposition 2.3. *Under the assumptions of Theorem 2.2 the Paneitz operator is positive: there exists $\mu(g) > 0$ such that*

$$\int \phi P_g \phi \, dv \geq \mu(g) \int \phi^2 dv.$$

Consequently, the Paneitz–Sobolev constant is also positive:

$$q_0(M^n, g) \equiv \inf_{\phi \in W^{2,2} \setminus \{0\}} \frac{\int \phi P_g \phi \, dv}{\left(\int |\phi|^{\frac{2n}{n-4}} dv\right)^{\frac{n-4}{n}}} > 0.$$

Proof. By (1.2),

$$\int \phi P \phi \, dv = \int \left\{ (\Delta\phi)^2 - 4A(\nabla\phi, \nabla\phi) + (n-2)\sigma_1(A)|\nabla\phi|^2 + \frac{n-4}{2} Q\phi^2 \right\} dv, \tag{2.9}$$

where we have omitted the subscript g. There are two cases to consider: $n = 5$ and $n \geq 6$. In the latter case we use the integrated Bochner formula

$$\begin{aligned}\int (\Delta\phi)^2\,dv &= \int |\nabla^2\phi|^2\,dv + \int \mathrm{Ric}(\nabla\phi, \nabla\phi)\,dv \\ &= \int |\nabla^2\phi|^2\,dv + (n-2)\int A(\nabla\phi, \nabla\phi)\,dv + \int \sigma_1(A)|\nabla\phi|^2\,dv,\end{aligned}$$

which gives

$$\int -4A(\nabla\phi, \nabla\phi)\,dv = \int \left\{-\frac{4}{n-2}(\Delta\phi)^2 + \frac{4}{n-2}|\nabla\phi|^2 + \frac{4}{n-2}\sigma_1(A)|\nabla\phi|^2\right\}dv.$$

Substituting this into (2.9) we find

$$\begin{aligned}&\int \phi P\phi\,dv \\ &= \int \left\{\frac{n-6}{n-2}(\Delta\phi)^2 + \frac{4}{n-2}|\nabla^2\phi|^2 + \frac{(n-2)^2+4}{n-2}\sigma_1(A)|\nabla\phi|^2 + \frac{n-4}{2}Q\phi^2\right\}dv.\end{aligned}$$

Consequently, when $n \geq 6$ the positivity of P follows.

When $n = 5$ we need to adapt the argument for the four-dimensional case in [Gur99]. First, when $n = 5$ we note that

$$\int \phi P\phi\,dv = \int (\Delta\phi)^2\,dv - 4\int A(\nabla\phi, \nabla\phi)\,dv + 3\int \sigma_1(A)|\nabla\phi|^2\,dv + \frac{1}{2}\int Q\phi^2\,dv, \tag{2.10}$$

while the Q-curvature is given by

$$0 \leq Q = -\Delta\sigma_1(A) - 2|A|^2 + \tfrac{5}{2}\sigma_1(A)^2. \tag{2.11}$$

Consider the second term on the right-hand side of (2.10). Since by Lemma 2.1 the scalar curvature is positive, using the arithmetic/geometric mean inequality (AGM) we estimate

$$4A(\nabla\phi, \nabla\phi) \leq 2\frac{|A|^2}{\sigma_1(A)}|\nabla\phi|^2 + 2\sigma_1(A)|\nabla\phi|^2.$$

By (2.11),

$$2\frac{|A|^2}{\sigma_1(A)}|\nabla\phi|^2 \leq -\frac{\Delta\sigma_1(A)}{\sigma_1(A)}|\nabla\phi|^2 + \frac{5}{2}\sigma_1(A)|\nabla\phi|^2,$$

hence

$$4\int A(\nabla\phi, \nabla\phi) \leq -\int \frac{\Delta\sigma_1(A)}{\sigma_1(A)}|\nabla\phi|^2\,dv + \frac{9}{2}\int \sigma_1(A)|\nabla\phi|^2\,dv. \tag{2.12}$$

For the first term on the right, we integrate by parts and use the AGM inequality to get

$$\begin{aligned}
-\int \frac{\Delta\sigma_1(A)}{\sigma_1(A)}|\nabla\phi|^2\,dv &= \int\left\{-\frac{|\nabla\sigma_1(A)|^2}{\sigma_1(A)^2}|\nabla\phi|^2 + \left\langle\frac{\nabla\sigma_1(A)}{\sigma_1(A)}, \nabla|\nabla\phi|^2\right\rangle\right\}dv \\
&= \int\left\{-\frac{|\nabla\sigma_1(A)|^2}{\sigma_1(A)^2}|\nabla\phi|^2 + 2\nabla^2\phi\left(\frac{\nabla\sigma_1(A)}{\sigma_1(A)}, \nabla\phi\right)\right\}dv \\
&\le \int\left\{-\frac{|\nabla\sigma_1(A)|^2}{\sigma_1(A)^2}|\nabla\phi|^2 + 2|\nabla^2\phi|\frac{|\nabla\sigma_1(A)|}{\sigma_1(A)}|\nabla\phi|\right\}dv \\
&\le \int\left\{-\frac{|\nabla\sigma_1(A)|^2}{\sigma_1(A)^2}|\nabla\phi|^2 + \frac{|\nabla\sigma_1(A)|^2}{\sigma_1(A)^2}|\nabla\phi|^2 + |\nabla^2\phi|^2\right\}dv \\
&= \int |\nabla^2\phi|^2\,dv.
\end{aligned}$$

Substituting this back into (2.12) gives

$$4\int A(\nabla\phi,\nabla\phi) \le \int |\nabla^2\phi|^2\,dv + \frac{9}{2}\int \sigma_1(A)|\nabla\phi|^2\,dv. \tag{2.13}$$

In dimension five the Bochner formula gives

$$\int |\nabla^2\phi|^2\,dv = \int (\Delta\phi)^2\,dv - 3\int A(\nabla\phi,\nabla\phi)\,dv - \int \sigma_1(A)|\nabla\phi|^2\,dv,$$

and substituting this into (2.13) we arrive at

$$4\int A(\nabla\phi,\nabla\phi) \le \int (\Delta\phi)^2\,dv - 3\int A(\nabla\phi,\nabla\phi)\,dv + \frac{7}{2}\int \sigma_1(A)|\nabla\phi|^2\,dv.$$

Combining the Schouten tensor terms we have

$$7\int A(\nabla\phi,\nabla\phi) \le \int (\Delta\phi)^2\,dv + \frac{7}{2}\int \sigma_1(A)|\nabla\phi|^2\,dv,$$

hence

$$4\int A(\nabla\phi,\nabla\phi) \le \frac{4}{7}\int (\Delta\phi)^2\,dv + 2\int \sigma_1(A)|\nabla\phi|^2\,dv,$$

or

$$-4\int A(\nabla\phi,\nabla\phi) \ge -\frac{4}{7}\int (\Delta\phi)^2\,dv - 2\int \sigma_1(A)|\nabla\phi|^2\,dv.$$

Finally, substituting this into (2.10) gives

$$\begin{aligned}
\int \phi P\phi\,dv &= \int (\Delta\phi)^2\,dv - 4\int A(\nabla\phi,\nabla\phi)\,dv + 3\int \sigma_1(A)|\nabla\phi|^2\,dv + \frac{1}{2}\int Q\phi^2\,dv \\
&\ge \frac{3}{7}\int (\Delta\phi)^2\,dv + \int \sigma_1(A)|\nabla\phi|^2\,dv + \frac{1}{2}\int Q\phi^2\,dv,
\end{aligned}$$

and the positivity of P follows. □

From Proposition 2.3 we conclude that under the assumptions of Lemma 2.1, for any $p \in M^n$ the Green's function G_p of the Paneitz operator exists, satisfying $P_g G_p = \delta_p$, where δ_p is the Dirac mass at p. We now prove Proposition C of the Introduction:

Proposition 2.4. *Suppose (M^n, g) satisfies the assumptions of Theorem 2.2. If G_p denotes the Green's function of the Paneitz operator with pole at $p \in M^n$, then $G_p > 0$ on $M^n \setminus \{p\}$.*

Proof. Consider a sequence of continuous functions f_j on M which are non-negative, whose supports shrink to $\{p\}$, and such that $\int_M f_j\, dv = 1$ for all j. Then $f_j \rightharpoonup \delta_p$ in the sense of distributions. If G_j is the solution to $P_g G_j = f_j$, it is easy to show that

$$G_j \to G_p \quad \text{in } C^4_{\text{loc}}(M^n \setminus \{p\}).$$

By Theorem 2.2 one has $G_j > 0$ on M^n, which immediately implies that

$$G_p \geq 0 \quad \text{on } M^n \setminus \{p\}.$$

Suppose there exists $x_0 \neq p$ such that $G_p(x_0) = 0$, and consider the sequence of conformal metrics $g_j = G_j^{4/(n-4)} g$. By construction $P_g G_j \geq 0$, hence by Theorem 2.2 the metrics g_j have positive scalar curvature and semi-positive Q-curvature. It follows that the scalar curvature of g_j satisfies

$$\frac{1}{2(n-1)} \Delta_{g_j} R_{g_j} \leq c_1(n) R_{g_j}^2.$$

Also, arguing as in the proof of Lemma 2.1 (see (2.8)), we find that G_j satisfies the differential inequality

$$\Delta_g G_j \leq \frac{n-4}{4(n-1)} R_g G_j \quad \text{on } M^n.$$

Passing to the limit $j \to \infty$ on $M^n \setminus \{p\}$ we have

$$\Delta_g G_p \leq \frac{n-4}{4(n-1)} R_g G_p.$$

By the strong maximum principle, $G_p(x_0) = 0$ implies $G_p \equiv 0$, a contradiction. □

2.2. *Regularity of the Green's function*

Our next results concern the behavior of the Green's function near the pole. We will show that if the dimension is 5, 6 or 7, or if the manifold is locally conformally flat, then in conformal normal coordinates the Green's function of the Paneitz operator is equal to the sum of the fundamental solution of the bi-Laplace equation and a weighted Lipschitz function:

Proposition 2.5. *Let (M^n, g) be a closed Riemannian manifold, satisfying the assumptions of Lemma 2.1:*

(i) *Q_g is semi-positive,*
(ii) $R_g \geq 0$.

In addition, assume one of the following holds:

- *$n = 5, 6,$ or 7; or*
- *(M^n, g) is locally conformally flat and $n \geq 5$.*

For $p \in M$, consider the conformal normal coordinates centered at p constructed in [LP87] with conformal metric $\tilde{g}$. If $G_p(x)$ is the Green's function for the Paneitz operator with pole at p, then there exists a constant α such that in conformal normal coordinates,

$$G_p(x) = \frac{c_n}{d_{\tilde{g}}(x, p)^{n-4}} + \alpha + O^{(4)}(r), \tag{2.14}$$

where $c_n = \frac{1}{(n-2)(n-4)\omega_{n-1}}$, $\omega_{n-1} = |S^{n-1}|$, and $O^{(k)}(r^m)$ denotes any quantity f satisfying

$$|\nabla^j f(x)| \le C_j r^{m-j} \quad \textit{for } 1 \le j \le k,$$

where $r = |x| = d_{\tilde{g}}(x, p)$.

Proof. In the locally conformally flat case, one can conformally change and use Euclidean coordinates near p, and the expansion (2.14) appears in [HR09]. For the non-LCF cases we will use the classical parametrix method; namely we start with functions which properly approximate G_p and then use elliptic regularity theory. We begin with some preliminary lemmas.

Lemma 2.6. *In conformal normal coordinates, if u is a radial function then one has the following expansions:*

$$\nabla_i \nabla_j u = \frac{x_i x_j}{r^2} u'' - \frac{x_i x_j}{r^3} u' + \frac{\delta_{ij}}{r} u' + O(r)|u'|, \tag{2.15}$$

$$\Delta_{\tilde{g}} u = u'' + \frac{n-1}{r} u' + O''(r^{N-1}) u', \tag{2.16}$$

$$\Delta_{\tilde{g}}^2 u = \Delta_0^2 u + O(r^{N-1}) u''' + O(r^{N-2}) u'' + O(r^{N-3}) u', \tag{2.17}$$

where $N \ge 5$ and Δ_0 denotes the Euclidean Laplacian.

Proof. Let $\{x^i\}$ denote conformal normal coordinates associated with the metric $\tilde{g}$, and let $\{r, \vartheta^\alpha\}$ denote the corresponding polar coordinates, where $r = |x|$ and $\{\vartheta^\alpha\}$ are coordinates on the unit sphere. We let $\tilde{g} = \tilde{g}_{ij}$ denote the matrix of components of $\tilde{g}$ with respect to the $\{x^i\}$ coordinates, and $\tilde{g}' = \tilde{g}'_{\alpha\beta}$ the components of $\tilde{g}$ the polar coordinate system. It follows that

$$\sqrt{\det \tilde{g}'} = r^{n-1} \sqrt{\det \tilde{g}}.$$

If u is radial, then

$$\begin{aligned}
\Delta u(r) &= \frac{1}{\sqrt{\det \tilde{g}'}} \partial_r \left(\sqrt{\det \tilde{g}'}\, \partial_r u \right) = u'' + \partial_r \left(\log \sqrt{\det \tilde{g}'} \right) u' \\
&= u'' + \partial_r \left(\log r^{n-1} \sqrt{\det \tilde{g}} \right) u' = u'' + \frac{n-1}{r} u' + u' \partial_r \log \sqrt{\det \tilde{g}} \\
&= \Delta_0 u + u' \partial_r \log \sqrt{\det \tilde{g}}.
\end{aligned} \tag{2.18}$$

In conformal normal coordinates (see [LP87, Theorem 5.1]) the determinant of $\tilde{g}$ approaches 1 smoothly at the origin at order N, where $N \ge 5$, and in particular

$$\det \tilde{g} = 1 + O^{(3)}(r^N). \tag{2.19}$$

Therefore,

$$\partial_r \log \sqrt{\det \tilde{g}} = O''(r^{N-1}).$$

Substituting into (2.18), we arrive at (2.16). The formula (2.17) for the bi-Laplacian follows immediately.

Recall that, in normal coordinates,

$$\tilde{g}_{ij} = \delta_{ij} - \tfrac{1}{3} R_{i\alpha j\beta} x^\alpha x^\beta + O^{(4)}(r^3),$$

where R_* denotes the curvature tensor (with respect to $\tilde{g}$) evaluated at p. As

$$\Gamma_{ij}^k = \tfrac{1}{2} g^{km} [\partial_i \tilde{g}_{jm} + \partial_j \tilde{g}_{im} - \partial_m \tilde{g}_{ij}],$$

we deduce that $|\Gamma_{ij}^k(x)| = O'''(r)$ and $|\partial_l \Gamma_{ij}^k(x)| = O''(1)$. This implies that

$$\nabla_i \nabla_j u = \partial_i \partial_j u + O(r)|u'|.$$

As u is radial, we obtain the conclusion. □

Remark. In the estimates that follow we will only need the order of flatness in (2.19) to be $N = 5$. Therefore, we will assume from now on that $N \geq 5$ is fixed.

Lemma 2.7. *In conformal normal coordinates one has the following expansions for the Schouten tensor $A_{ij} = \frac{1}{n-2}\big[R_{ij} - \frac{1}{2(n-1)} R_{\tilde{g}} \tilde{g}_{ij}\big]$ and for the Q-curvature:*

$$\begin{aligned}
&A_{ij}(0) = 0, \qquad (\nabla_k A_{ij} + \nabla_i A_{jk} + \nabla_j A_{ik})(0) = 0,\\
&\nabla_k \nabla_l A_{ij}(0) x^k x^l x^i x^j = -\frac{r^2}{n-2} \nabla_k \nabla_l \sigma_1(0) x^k x^l,\\
&Q = -\frac{1}{2(n-1)} \big[-\tfrac{1}{6}|W|^2(0) + O(r)\big].
\end{aligned}$$

Proof. Recall that in conformal normal coordinates one has

$$\begin{aligned}
&R_{ij}(0) = 0, \qquad (\nabla_k R_{ij} + \nabla_i R_{jk} + \nabla_j R_{ik})(0) = 0,\\
&(\nabla_k \nabla_l R_{ij} + \nabla_l \nabla_i R_{jk} + \nabla_i \nabla_j R_{kl} + \nabla_j \nabla_k R_{li})(0) = 0,\\
&R(0) = 0, \qquad \nabla_{\tilde{g}} R(0) = 0, \qquad \Delta_{\tilde{g}} R(0) = \tfrac{1}{6}|W|^2(0).
\end{aligned}$$

Then the conclusion follows immediately from the definition of A_{ij} and Q. □

Lemma 2.8. *If u is a radial function, then in conformal normal coordinates and conformal metric $\tilde{g}$ one has*

$$\begin{aligned}
P_{\tilde{g}} u = \Delta_0^2 u &+ \nabla_k \nabla_l \sigma_1(0) x^k x^l \mathfrak{Q}(u) + \frac{n-4}{24(n-1)} |W|^2(0)\, u + O(r^3)|u''|\\
&+ O(r^2)|u'| + O(r)u + O(r^{N-1})u''' + O(r^{N-2})u'' + O(r^{N-3})u',
\end{aligned}$$

where

$$\mathfrak{Q}(u) = \frac{u'}{r}\left(\frac{2(n-1)}{n-2} - \frac{(n-1)(n-2)}{2} + 6 - n\right) - u''\left(\frac{n-2}{2} + \frac{2}{n-2}\right).$$

Proof. Recall that

$$P_g u = \Delta_{\tilde g}^2 u + \mathrm{div}_{\tilde g}\left\{\left(4A_{\tilde g} - (n-2)\sigma_1(A_{\tilde g})\tilde g\right)(\nabla u, \cdot)\right\} + \frac{n-4}{2} Q_{\tilde g} u.$$

We consider the term

$$\mathrm{div}_{\tilde g}\left\{\left(4A_{\tilde g} - (n-2)\sigma_1(A_{\tilde g})\tilde g\right)(\nabla u, \cdot)\right\} = 4A_{ij}\nabla_i\nabla_j u - (n-2)\sigma_1\Delta_{\tilde g} u + (6-n)\langle\nabla\sigma_1, \nabla u\rangle.$$

Using Lemma 2.6 for the Hessian of u and Lemma 2.7 for the vanishing of $A_{ij}(0)$ we find that

$$\begin{aligned} A_{ij}\nabla_i\nabla_j u &= \left(A_{ij}(0) + \nabla_k A_{ij}(0)x^k + \tfrac{1}{2}\nabla_k\nabla_l A_{ij}(0)x^k x^l + O(r^3)\right)(\partial^2_{ij} u + O(r)|u'|) \\ &= \left(\nabla_k A_{ij}(0)x^k + \tfrac{1}{2}\nabla_k\nabla_l A_{ij}(0)x^k x^l + O(r^3)\right) \\ &\quad \times \left[\left(\frac{\delta^{ij}}{r^2} - \frac{x^i x^j}{r^3}\right)u' + \frac{x^i x^j}{r^2}u'' + O(r)|u'|\right] = I + II + III + IV + V, \end{aligned}$$

where

$$\begin{aligned} I &= \nabla_k A_{ij}(0)x^k \frac{\delta^{ij}}{r^2}u', \qquad II = \nabla_k A_{ij}(0)x^k\left(\frac{x^i x^j}{r^2}u'' - \frac{x^i x^j}{r^3}u'\right), \\ III &= \tfrac{1}{2}\nabla_k\nabla_l A_{ij}(0)x^k x^l\frac{\delta^{ij}}{r^2}u', \quad IV = \tfrac{1}{2}\nabla_k\nabla_l A_{ij}(0)x^k x^l\left(\frac{x^i x^j}{r^2}u'' - \frac{x^i x^j}{r^3}u'\right), \\ V &= O(r^3)(\partial^2_{ij}u + O(r)|u'|) + \left(\nabla_k A_{ij}(0)x^k + \tfrac{1}{2}\nabla_k\nabla_l A_{ij}(0)x^k x^l\right) \times O(r)|u'|. \end{aligned}$$

As the scalar curvature vanishes to first order at p we find immediately that $I = 0$. Also, since II stays unchanged after permutation of the indices i, j, k, by the second statement of Lemma 2.7 we find that also $II = 0$. Turning to III, we have

$$III = \tfrac{1}{2}\nabla_k\nabla_l\sigma_1 x^k x^l\frac{u'}{r}.$$

Concerning IV instead, using the third identity in Lemma 2.7 we find that

$$IV = -\frac{1}{2(n-2)}\nabla_k\nabla_l\sigma_1 x^k x^l\left(u'' - \frac{u'}{r}\right).$$

Expanding then also V one finds

$$4A_{ij}\nabla_i\nabla_j u = \frac{2}{n-2}\nabla_k\nabla_l\sigma_1(0)x^k x^l\left[(n-1)\frac{u'}{r} - u''\right] + O(r^3)|u''| + O(r^2)|u'|.$$

Similarly, using the second assertion of Lemma 2.6 and a Taylor expansion of the scalar curvature one finds

$$-(n-2)\sigma_1\Delta u = -\frac{n-2}{2}\nabla_k\nabla_l\sigma_1(0)x^k x^l\left[(n-1)\frac{u'}{r} + u''\right] + O(r^3)|u''| + O(r^2)|u'|.$$

Furthermore

$$(6-n)\langle\nabla\sigma_1, \nabla u\rangle = (6-n)\nabla_k\nabla_l\sigma_1(0)x^k x^l \frac{u'}{r} + O(r^2)|u'|.$$

By the third assertion of Lemma 2.6 and summing all the above terms in $P_{\tilde{g}}u$ (taking into account the expression of $Q_{\tilde{g}}$ in Lemma 2.7) one gets the conclusion. □

Using the preceding technical lemmas, we can now compute $P_{\tilde{g}}(r^{4-n})$. By Lemma 2.8, one has

$$P_{\tilde{g}}(r^{4-n}) = \mathfrak{A}_n\delta_p + \nabla_k\nabla_l\sigma_1(0)x^k x^l \mathfrak{Q}_n r^{2-n} + \frac{n-4}{24(n-1)}|W|^2(0)\, r^{4-n} + O(r^{5-n}), \tag{2.20}$$

where

$$\mathfrak{A}_n = 2(n-2)(n-4)|S^{n-1}|,$$
$$\mathfrak{Q}_n = (4-n)\left[\left(\frac{2(n-1)}{n-2} - \frac{(n-1)(n-2)}{2} + 6 - n\right) - (3-n)\frac{(n-2)^2+4}{2(n-2)}\right].$$

It follows from (2.20) that

$$P_{\tilde{g}}\left(G_p - \frac{1}{\mathfrak{A}_n}r^{4-n}\right) = O(r^{4-n}). \tag{2.21}$$

By elliptic regularity, if we can show that the right-hand side of (2.21) is in L^p for some $p > n/3$, then we would conclude

$$G_p - \frac{1}{\mathfrak{A}_n}r^{4-n} \in W^{4,p} \hookrightarrow C^{1,\alpha}$$

with $\alpha > 0$, and (2.14) would follow. However, $r^{4-n} \in L^p$ for $p < n/(n-4)$, hence we need p to satisfy

$$\frac{n}{3} < p < \frac{n}{n-4}.$$

This can only hold if $n = 5$ or $n = 6$; when $n = 7$ we have equality, so this is the borderline case.

When $n = 7$ we can add a further correction term to study the asymptotics of G_p. We begin by writing the trailing terms in (2.20) as

$$\nabla_k\nabla_l\sigma_1(0)x^k x^l \mathfrak{Q}_n|x|^{2-n} + \frac{n-4}{24(n-1)}|W|^2(0)\,|x|^{4-n} = \mathfrak{B}_0|x|^{-3} + \mathfrak{B}_2(\theta)|x|^{-3},$$

where $\mathfrak{B}_0$ is a constant and $\mathfrak{B}_2(\theta)$ is a second spherical harmonic function (with zero average) on S^6, with θ denoting the spherical coordinates. As the second eigenvalue of the Laplace–Beltrami operator on S^6 is equal to 14, using polar coordinates one can easily check that

$$\Delta_0^2|x| = -\frac{24}{|x|^3}, \qquad \Delta_0^2(\mathfrak{B}_2(\theta)|x|) = \frac{172}{|x|^3}.$$

Therefore in conformal normal coordinates one finds that

$$\Delta\left(-\frac{1}{24}\mathfrak{B}_0|x| + \frac{\mathfrak{B}_2(\theta)}{172}|x|\right) = \mathfrak{B}_0|x|^{-3} + \mathfrak{B}_2(\theta)|x|^{-3} + O(r^{-2}),$$

which implies that

$$P_{\tilde{g}}\left(G_p - \frac{1}{\mathfrak{A}_n}|x|^{4-n} + \frac{1}{24}\mathfrak{B}_0|x| - \frac{\mathfrak{B}_2(\theta)}{172}|x|\right) = O(r^{-2}).$$

By elliptic regularity theory and by Morrey's embedding theorems we then deduce that the function

$$G_p - \frac{1}{\mathfrak{A}_n}|x|^{4-n} + \frac{1}{24}\mathfrak{B}_0|x| - \frac{\mathfrak{B}_2(\theta)}{172}|x|$$

possesses Hölder continuous derivatives, which, taking Schauder's estimates into account, implies the conclusion when $n = 7$. $\square$

2.3. A positive mass theorem for the Paneitz operator

We conclude this section by proving an inequality for the constant α in the expansion for the Green's function in Proposition 2.5. In the locally conformally flat case, this was proved by Humbert–Raulot [HR09]. In fact, their proof is easily adapted to the non-LCF case when the dimension is 5, 6, or 7.

Theorem 2.9. *Under the assumptions of Proposition 2.5, the constant α in the expansion (2.14) satisfies $\alpha \geq 0$, with equality if and only if (M^n, g) is conformally equivalent to the round sphere.*

Proof. Let Γ_p denote the Green's function for the conformal Laplacian $L = -\Delta + \frac{n-2}{4(n-1)}R$ with pole at p. As in [HR09], we consider the conformal blow-up of g defined by

$$\hat{g} = \Gamma_p^{\frac{4}{n-2}} g.$$

This defines an asymptotically flat, scalar-flat metric on $X^n = M^n \setminus \{p\}$. Let

$$\Phi = \Gamma_p^{-\frac{n-4}{n-2}} G_p.$$

By the conformal covariance of the Paneitz operator, on X^n we have

$$P_{\hat{g}}\Phi = P_{\Gamma_p^{4/(n-2)}g}(\Gamma_p^{-\frac{n-4}{n-2}} G_p) = \Gamma_p^{-\frac{n+4}{n-2}} P_g(G_p) = 0.$$

Also, since $\hat{g}$ is scalar flat, its Q-curvature is given by

$$Q_{\hat{g}} = -2|A(\hat{g})|^2,$$

where A is the Schouten tensor. By the formula for the Paneitz operator (1.2),

$$0 = P_{\hat{g}}\Phi = \Delta_{\hat{g}}^2\Phi + \mathrm{div}_{\hat{g}}\{4A_{\hat{g}}(\nabla\Phi, \cdot)\} - (n-4)|A(\hat{g})|^2\Phi. \tag{2.22}$$

Fix $\delta > 0$ small and let B_δ once again denote the geodesic ball centered at p of radius $\delta > 0$ (as measured in the metric g, not $\hat{g}$). As in [HR09], we integrate (2.22) over $M^n \setminus B_\delta$ and apply the divergence theorem:

$$\begin{aligned}
0 &= \int_{M^n \setminus B_\delta} P_{\hat{g}} \Phi \, dv_{\hat{g}} \\
&= \int_{M^n \setminus B_\delta} \left\{ \Delta_{\hat{g}}^2 \Phi + \operatorname{div}_{\hat{g}}\{4A_{\hat{g}}(\nabla \Phi, \cdot)\} - (n-4)|A(\hat{g})|^2 \Phi \right\} dv_{\hat{g}} \\
&= \oint_{\partial B_\delta} \left\{ \frac{\partial}{\partial \nu}(\Delta_{\hat{g}} \Phi) + 4A_{\hat{g}}(\nabla \Phi, \nu) \right\} dS_{\hat{g}} - (n-4) \int_{M^n \setminus B_\delta} |A(\hat{g})|^2 \Phi \, dv_{\hat{g}}, \qquad (2.23)
\end{aligned}$$

where ν is the (outward) normal to ∂B_δ in the metric $\hat{g}$.

Considering the boundary integrals, we first note that since $\hat{g}$ is scalar-flat,

$$\frac{\partial}{\partial \nu}(\Delta_{\hat{g}} \Phi) = -\frac{\partial}{\partial \nu}(L_{\hat{g}} \Phi).$$

Using the covariance of the conformal Laplacian and the definition of Φ, we find that

$$L_{\hat{g}} \Phi = \Gamma_p^{-\frac{n+2}{n-2}} L_g (\Gamma_p^{\frac{2}{n-2}} G_p).$$

Let $r(x) = d_g(x, p)$ denote the distance function from p in the metric g. By Lemma 6.4 of [LP87], we can normalize Γ_p so that

$$\Gamma_p^{\frac{2}{n-2}} = \begin{cases} r^{-2} + O(r) & \text{if } n = 5, \\ r^{-2} + O(r^2 \log r) & \text{if } n = 6, \\ r^{-2} + O(r^2) & \text{if } n = 7. \end{cases} \qquad (2.24)$$

Combining this with Proposition 2.5, for $n = 5, 6, 7$ we have

$$\Gamma_p^{\frac{2}{n-2}} G_p = c_n r^{2-n} + \alpha r^{-2} + O(r^{-1}). \qquad (2.25)$$

Using Lemma 2.6 and the fact that $R_g = O(r^2)$ in conformal normal coordinates, we get

$$\begin{aligned}
L_g(\Gamma_p^{\frac{2}{n-2}} G_p) &= -\Delta_g(\Gamma_p^{\frac{2}{n-2}} G_p) + \frac{n-2}{4(n-1)} R_g \Gamma_p^{\frac{2}{n-2}} G_p \\
&= 2(n-4)\alpha r^{-4} + O(r^{4-n}) \\
&= 2(n-4)\alpha r^{-4} + O(r^{-3}) \quad \text{if } 5 \leq n \leq 7.
\end{aligned}$$

Note that in dimensions $n \geq 8$ the second term is no longer lower order. By (2.24),

$$\Gamma_p^{-\frac{n+2}{n-2}} = r^{n+2} + O(r^{n+3}),$$

hence

$$L_{\hat{g}} \Phi = \Gamma_p^{-\frac{n+2}{n-2}} L_g(\Gamma_p^{\frac{2}{n-2}} G_p) = 2(n-4)\alpha r^{n-2} + O(r^{n-1}). \qquad (2.26)$$

It is easy to verify that

$$\frac{\partial}{\partial \nu} = -\Gamma_p^{-\frac{2}{n-2}} \frac{\partial}{\partial r},$$

so combining (2.25) and (2.26) we find

$$\frac{\partial}{\partial \nu}(L_{\hat{g}}\Phi)|_{\partial B_\delta} = -2(n-2)(n-4)\alpha\delta^{n-1} + O(\delta^n).$$

Also, the surface measure transforms by

$$\oint_{\partial B_\delta} dS_{\hat{g}} = \oint_{\partial B_\delta} \Gamma_p^{\frac{2(n-1)}{(n-2)}} dS_g = \omega_{n-1}\delta^{1-n} + O(\delta^{2-n}).$$

Consequently, the leading boundary term in (2.23) is

$$\oint_{\partial B_\delta} \frac{\partial}{\partial \nu}(\Delta_{\hat{g}}\Phi)\, dS_{\hat{g}} = 2(n-2)(n-4)\omega_{n-1}\alpha + o(1).$$

We can argue as in [HR09] to show that the second boundary integral in (2.23) satisfies

$$\oint_{\partial B_\delta} 4A_{\hat{g}}(\nabla\Phi, \nu) dS_{\hat{g}} = o(1),$$

hence

$$2(n-2)(n-4)\omega_{n-1}\alpha = (n-4)\int_{M^n \setminus B_\delta} |A(\hat{g})|^2 \Phi\, dv_{\hat{g}} + o(1). \tag{2.27}$$

It follows that $\alpha \geq 0$. Moreover, if $\alpha = 0$ then $\hat{g}$ is Ricci-flat, which implies $(X^n, \hat{g})$ is isometric to flat Euclidean space (see, for example, [Sch84, Proposition 2, p. 492]). This completes the proof. □

3. The flow

3.1. The initial assumptions

In the following, we assume (M^n, g_0) is a closed Riemannian manifold of dimension $n \geq 5$ with

$$Q_{g_0} \text{ is semi-positive, and } R_{g_0} \geq 0. \tag{3.1}$$

Note that by Lemma 2.1, the assumption on the Q-curvature implies $R_{g_0} > 0$. Also, by Proposition 2.3, P_{g_0} is invertible. Therefore, we can consider the flow

$$\begin{cases} \dfrac{\partial u}{\partial t} = -u + \mu P_{g_0}^{-1}(|u|^{\frac{n+4}{n-4}}), \\ u(\cdot, 0) = 1, \end{cases} \tag{3.2}$$

where

$$\mu = \frac{\int u P_{g_0} u\, dv_0}{\int |u|^{\frac{2n}{n-4}} dv_0}.$$

Lemma 3.1. *The flow* (3.2) *has a smooth solution for* $0 \le t < T$, *where* $0 < T \le \infty$.

Proof. Consider the flow

$$\begin{cases} \dfrac{\partial v}{\partial t} = -v + P_{g_0}^{-1}(|v|^{\frac{n+4}{n-4}}), \\ v(\cdot, 0) = 1, \end{cases} \tag{3.3}$$

which differs from (3.2) by the normalizing term μ. In fact, these flows just differ by a rescaling in space-time. To see this, suppose $v \in C^{4,\alpha}(M^n \times [0, T))$ is a solution of (3.3), and define

$$\nu = \nu(t) = \frac{\int v P_{g_0} v \, dv_0}{\int |v|^{\frac{2n}{n-4}} \, dv_0}, \qquad s(t) = \int_0^t \nu(\tau) \, d\tau.$$

Let

$$u(x, t) = e^{s(t)-t} v(x, s(t)).$$

It is easy to see that u satisfies (3.2) on some time interval $[0, \tilde{T})$.

Short-time existence for the flow (3.3) follows from the Picard–Lindelöf theorem on Banach spaces; if we denote $X_\epsilon = C^{4,\alpha}(M^n \times [0, \epsilon])$, then the mapping

$$v \mapsto \Psi(v)(x, t) = 1 - \int_0^t v(x, \tau) \, d\tau + \int_0^t P_{g_0}^{-1}(|v|^{\frac{n+4}{n-4}})(x, \tau) \, d\tau$$

is a contraction on a small neighborhood of $v_0 \equiv 1$ in X_ϵ for $\epsilon > 0$ small. A fixed point of Ψ solves (3.3).

Note that as (3.3) is a non-local ODE in $C^{4,\alpha}(M)$, there is in general no gain of (spatial) derivatives. □

Proposition 3.2. *For all* $0 \le t < T$,

$$u(t, x) > 0.$$

Proof. By (3.2),

$$\frac{\partial}{\partial t} P_{g_0} u = P_{g_0}\left(\frac{\partial}{\partial t} u\right) = -P_{g_0} u + \mu |u|^{\frac{n+4}{n-4}}, \tag{3.4}$$

hence

$$\frac{\partial}{\partial t} P_{g_0} u \ge -P_{g_0} u.$$

Integrating this inequality we get

$$P_{g_0} u(t, x) \ge e^{-t} P_{g_0} u(0, x) = e^{-t} P_{g_0}(1) = \frac{n-4}{2} e^{-t} Q_{g_0}(x).$$

It follows that $P_{g_0} u \ge 0$, and $P_{g_0} u > 0$ somewhere (namely, where the Q-curvature is initially positive). By the strong maximum principle of Theorem 2.2 it follows that $u > 0$ for $t \in [0, T)$. □

Remark 3.3. It follows from the proof of Lemma 3.2 that $Q_g > 0$ for all $t \in (0, T)$: since $u > 0$ for all time, from (3.4) we have

$$\frac{\partial}{\partial t} P_{g_0} u \ge -P_{g_0} u + \mu u^{\frac{n+4}{n-4}},$$

and integrating this we see that $P_{g_0} u > 0$ for $t \in (0, T)$.

3.2. Variational properties

Since $u > 0$ for as long as the flow exists, we can rewrite (3.2) as

$$\frac{\partial}{\partial t}u = -u + \mu P_{g_0}(u^{\frac{n+4}{n-4}}) \tag{3.5}$$

with

$$\mu = \frac{\int u P_{g_0} u \, dv_0}{\int u^{\frac{2n}{n-4}} \, dv_0}. \tag{3.6}$$

Lemma 3.4.

$$\frac{d}{dt}\int u P_{g_0} u \, dv_0 = 0.$$

Proof. From (3.5) and (3.6),

$$\begin{aligned}
\frac{d}{dt}\int u P_{g_0} u \, dv_0 &= \int \left\{ \frac{\partial u}{\partial t} P_{g_0} u + u P_{g_0}\left(\frac{\partial u}{\partial t}\right) \right\} dv_0 = 2\int u P_{g_0}\left(\frac{\partial u}{\partial t}\right) dv_0 \\
&= 2\int u P_{g_0}(-u + \mu P_{g_0}^{-1}(u^{\frac{n+4}{n-4}})) \, dv_0 \\
&= 2\int (-u P_{g_0} u + \mu u P_{g_0} P_{g_0}^{-1}(u^{\frac{n+4}{n-4}})) \, dv_0 \\
&= \int \{-2u P_{g_0} u + 2\mu u^{\frac{2n}{n-4}}\} \, dv_0 \\
&= -2\int u P_{g_0} u \, dv_0 + 2\left(\frac{\int u P_{g_0} u \, dv_0}{\int u^{\frac{2n}{n-4}} \, dv_0} \right) \int u^{\frac{2n}{n-4}} \, dv_0 = 0. \qquad \square
\end{aligned}$$

To state the next lemma, we denote

$$f = -u + \mu P_{g_0}^{-1}(u^{\frac{n+4}{n-4}}).$$

Lemma 3.5. *The conformal volume satisfies*

$$\frac{d}{dt}V = \frac{d}{dt}\int u^{\frac{2n}{n-4}} \, dv_0 = \frac{2n}{n-4}\frac{1}{\mu}\int f P_{g_0} f \, dv_0 \geq 0. \tag{3.7}$$

In particular, the volume is increasing along the flow, while μ and the Paneitz–Sobolev quotient are both decreasing:

$$\frac{d}{dt}\mu = \frac{d}{dt}\left(\frac{\int u P_{g_0} u \, dv_0}{V} \right) \leq 0, \qquad \frac{d}{dt}\mathcal{F}_{g_0}[u] = \frac{d}{dt}\left(\frac{\int u P_{g_0} u \, dv_0}{V^{\frac{n-4}{n}}} \right) \leq 0.$$

Finally, the volume is bounded above:

$$V \leq C_0(g_0).$$

Proof. To prove the lemma, we differentiate:

$$\begin{aligned}\frac{d}{dt}\int u^{\frac{2n}{n-4}}\,dv_0 &= \frac{2n}{n-4}\int u^{\frac{n+4}{n-4}}\frac{\partial u}{\partial t}\,dv_0 = \frac{2n}{n-4}\int u^{\frac{n+4}{n-4}}\{-u+\mu P_{g_0}^{-1}(u^{\frac{n+4}{n-4}})\}\,dv_0\\ &= \frac{2n}{n-4}\int\{-u^{\frac{2n}{n-4}}+\mu u^{\frac{n+4}{n-4}}P_{g_0}^{-1}(u^{\frac{n+4}{n-4}})\}\,dv_0.\end{aligned} \tag{3.8}$$

Note that

$$\begin{aligned}\int fP_{g_0}f\,dv_0 &= \int\{-u+\mu P_{g_0}^{-1}(u^{\frac{n+4}{n-4}})\}\{-P_{g_0}u+\mu u^{\frac{n+4}{n-4}}\}\,dv_0\\ &= \int\{uP_{g_0}u-\mu u^{\frac{2n}{n-4}}-\mu P_{g_0}^{-1}(u^{\frac{n+4}{n-4}})P_{g_0}u+\mu^2u^{\frac{n+4}{n-4}}P_{g_0}^{-1}(u^{\frac{n+4}{n-4}})\}\,dv_0\\ &= \int\{-\mu u^{\frac{2n}{n-4}}+\mu^2u^{\frac{n+4}{n-4}}P_{g_0}^{-1}(u^{\frac{n+4}{n-4}})\}\,dv_0.\end{aligned} \tag{3.9}$$

Comparing (3.8) and (3.9), we arrive at (3.7).

To see that the volume is bounded above, we use the fact that the Paneitz–Sobolev constant is positive:

$$\begin{aligned}0<q_0\le\mathcal{F}_{g_0}[u] &= V^{-\frac{n-4}{n}}\int uP_{g_0}u\,dv_0 = V^{-\frac{n-4}{n}}\int u_0P_{g_0}u_0\,dv_0\\ &= \frac{n-4}{2}V^{-\frac{n-4}{n}}\int Q_{g_0}\,dv_0,\end{aligned}$$

hence $V\le C(g_0)$. □

Corollary 3.6. *We have the space-time estimates*

$$\int_0^T\|f\|_{W^{2,2}}\,dt\le C_1(g_0),\qquad \int_0^T\left(\int|f|^{\frac{2n}{n-4}}\,dv_0\right)^{\frac{n-4}{n}}dt\le C_2(g_0). \tag{3.10}$$

Proof. From the upper bound on volume we have

$$\int_0^T\left(\int_{M^n}fP_{g_0}f\,dv_0\right)dt\le C_1(g_0).$$

Since P_{g_0} is positive,

$$\|\phi\|_{W^{2,2}}\approx\int\phi P_{g_0}\phi\,dv_0,$$

and the first estimate in (3.10) follows. The second estimate follows from the lower bound on the Paneitz–Sobolev quotient. □

3.3. Long time existence

Proposition 3.7. *The flow* (3.2) *has a smooth solution for all time. Moreover,*

$$u \le C' e^{Ct}, \tag{3.11}$$

where $C, C' > 0$ *are constants depending on* g_0 *and the initial datum.*

Proof. Let $s > 1$. Since $u > 0$ and $P_{g_0} u > 0$ for as long as the flow exists, by (3.4) we have

$$\begin{aligned}\frac{d}{dt}\int (P_{g_0}u)^s\, dv_0 &= s\int (P_{g_0}u)^{s-1}\frac{\partial}{\partial t}(P_{g_0}u)\, dv_0\\ &= s\int (P_{g_0}u)^{s-1}\{-P_{g_0}u + \mu u^{\frac{n+4}{n-4}}\}\, dv_0\\ &= -s\int (P_{g_0}u)^s + s\mu \int (P_{g_0}u)^{s-1} u^{\frac{n+4}{n-4}}\, dv_0. \end{aligned}\tag{3.12}$$

For the second integral above we use Hölder's inequality to write

$$\int (P_{g_0}u)^{s-1} u^{\frac{n+4}{n-4}}\, dv_0 \le \left(\int (P_{g_0}u)^s\, dv_0\right)^{\frac{s-1}{s}} \left(\int u^{\frac{n+4}{n-4}s}\, dv_0\right)^{\frac{1}{s}} \tag{3.13}$$

Assume

$$\frac{2n}{n+4} < s < \frac{n}{4}. \tag{3.14}$$

Then we can apply Hölder's inequality again to get

$$\left(\int u^{\frac{n+4}{n-4}s}\, dv_0\right)^{\frac{1}{s}} \le \left(\int u^{\frac{ns}{n-4s}}\, dv_0\right)^{\frac{n-4s}{ns}} \left(\int u^{\frac{2n}{n-4}}\, dv_0\right)^{\frac{4}{n}}. \tag{3.15}$$

By the Sobolev embedding theorem $W^{s,4} \hookrightarrow L^{ns/(n-4s)}$ for $1 < s < n/4$. Also, since $P_{g_0} > 0$ we have $\|u\|_{W^{s,4}} \approx \|P_{g_0}u\|_{L^s}$. Therefore,

$$\left(\int u^{\frac{ns}{n-4s}}\, dv_0\right)^{\frac{n-4s}{ns}} \le C_s \left(\int (P_{g_0}u)^s\, dv_0\right)^{\frac{1}{s}}$$

for s in the range given by (3.14). Substituting this into (3.15) and using the conformal volume bound of Lemma 3.5 we have

$$\int (P_{g_0}u)^{s-1} u^{\frac{n+4}{n-4}}\, dv_0 \le C_s \int (P_{g_0}u)^s\, dv_0.$$

Substituting this into (3.12) gives

$$\frac{d}{dt}\int (P_{g_0}u)^s\, dv_0 \le C_s \int (P_{g_0}u)^s\, dv_0, \qquad \frac{2n}{n+4} < s < \frac{n}{4}.$$

Integrating this we get

$$\int (P_{g_0}u)^s\, dv_0 \le C_0 e^{C_s t}, \qquad 0 \le t < T.$$

By the Sobolev embedding, this implies

$$\|u\|_{L^{\frac{ns}{n-4s}}} \le C_1 e^{C_s' t}.$$

By choosing s sufficiently close to $n/4$, we conclude that $\|u\|_{L^p} \le C_3 e^{C_p t}$, for any $p > 1$.

Now fix $s > n/4$; say $s = n/4 + 1$. Returning to (3.13), we have

$$\begin{aligned}
\int (P_{g_0} u)^{s-1} u^{\frac{n+4}{n-4}}\, dv_0 &\le \left(\int (P_{g_0} u\, dv_0)^s \right)^{\frac{s-1}{s}} \left(\int u^{\frac{n+4}{n-4}s}\, dv_0 \right)^{\frac{1}{s}} \\
&\le \left(\int (P_{g_0} u)^s\, dv_0 \right)^{\frac{s-1}{s}} (C_3 e^{C_n t})^{\frac{1}{s}} \le C_4 e^{C_5 t} \left(\int (P_{g_0} u)^s\, dv_0 \right)^{\frac{s-1}{s}} \\
&\le \int (P_{g_0} u)^s\, dv_0 + C_6 e^{C_7 t}.
\end{aligned}$$

Substituting this into (3.12) gives

$$\frac{d}{dt} \int (P_{g_0} u)^s\, dv_0 \le C' e^{Ct}, \qquad s = \frac{n}{4} + 1.$$

Integrating this and using the Sobolev–Kondrashov theorem we conclude that $\|u\|_{C^\alpha} \le C' e^{Ct}$ for some $\alpha \in (0, 1)$. This implies (3.11) and, via (3.4), that the C^α-norm of $P_{g_0} u$ grows at most exponentially fast. It follows that the $C^{4,\alpha}$-norm of u grows at most exponentially fast, so we cannot have blow-up in finite time. □

4. Constructing the initial data, part I: $n \ge 8$

To prove the convergence of the flow we will show that it is possible to construct initial data satisfying the positivity conditions (3.1) and with energy below the Euclidean value. By a standard argument (see Section 6) the latter fact will imply that the flow has a non-zero weak limit which defines a metric of constant Q-curvature.

Our first result in this direction considers the case where the dimension is large (i.e., $n \ge 8$) and the underlying manifold is not locally conformally flat:

Proposition 4.1. *Let $(M^n, \bar{g})$ be a closed Riemannian manifold of dimension $n \ge 8$. Assume*

(i) *$Q_{\bar{g}}$ is semi-positive,*
(ii) *$R_{\bar{g}} \ge 0$,*
(iii) *$(M^n, \bar{g})$ is not locally conformally flat.*

If at $x_0 \in M$ the Weyl tensor $W(x_0)$ is non-zero, then for $\varepsilon > 0$ small there exists a function $\psi_\varepsilon \in C^\infty$ and a dimensional constant c_n such that

$$\mathcal{F}_{\bar{g}}(\psi_\varepsilon) \le \begin{cases} S_n - c_n \varepsilon^4 |\log \varepsilon|\, |W(x_0)|^2 & \text{if } n = 8, \\ S_n - c_n \varepsilon^4 |W(x_0)|^2 & \text{if } n \ge 9, \end{cases}$$

where S_n is the Euclidean Paneitz–Sobolev constant:

$$S_n = \inf_{\varphi \in C_0^\infty(\mathbb{R}^n)} \frac{\int (\Delta_0 \varphi)^2 \, dx}{\left(\int |\varphi|^{\frac{2n}{n-4}} \, dx\right)^{\frac{n-4}{n}}}.$$

Moreover, ψ_ε is positive and induces a conformal metric $h = \psi_\varepsilon^{4/(n-4)} \bar{g}$ with the following properties:

(i′) *Q_h is semi-positive,*
(ii′) *$R_h > 0$,*
(iii′)

$$\mathcal{F}_h(1) \le \begin{cases} S_n - c_n \varepsilon^4 |\log \varepsilon| \, |W(x_0)|^2 & \text{if } n = 8, \\ S_n - c_n \varepsilon^4 |W(x_0)|^2 & \text{if } n \ge 9. \end{cases} \tag{4.1}$$

Proof. Let $\tilde{g} = \varphi^{4/(n-4)} \bar{g}$ denote the metric satisfying the conformal normal coordinate conditions of [LP87] at x_0 (we assume φ is globally defined). Consider the test function in [ER02, Section 6] defined by

$$\tilde{u}_\varepsilon(x) = \frac{\eta(x)\varphi(x)}{(\varepsilon^2 + d_{\tilde{g}}(x, x_0)^2)^{\frac{n-4}{2}}},$$

where $\eta(x)$ is a cut-off function with support in a ball $B_{2\delta}(x_0)$, identically equal to 1 in $B_\delta(x_0)$.

In [ER02, Section 7] it was shown that, for $\varepsilon > 0$ small one has the estimates

$$\mathcal{F}_{\tilde{g}}(\tilde{u}_\varepsilon) \le \begin{cases} S_n - C(n)\varepsilon^4 |\log \varepsilon| \, |W(x_0)|^2 & \text{if } n = 8, \\ S_n - C(n)\varepsilon^4 |W(x_0)|^2 & \text{if } n \ge 9. \end{cases}$$

We will show that it is possible to modify these test functions in order to produce a strictly positive conformal factor which defines a metric with semi-positive Q and positive scalar curvatures, while preserving the property of the Paneitz–Sobolev quotient being below the Euclidean value. We begin with the following lemma:

Lemma 4.2. *If $\tilde{g}$ is as above, if we set*

$$u_\varepsilon(x) = \frac{\eta(x)}{(\varepsilon^2 + d_{\tilde{g}}(x, x_0)^2)^{\frac{n-4}{2}}},$$

then

$$P_{\tilde{g}}(u_\varepsilon) = \frac{n(n-4)(n^2-4)\varepsilon^4}{(\varepsilon^2 + |x|^2)^{\frac{n+4}{2}}} + \frac{O(1)}{(\varepsilon^2 + r^2)^{\frac{n-4}{2}}} \quad \text{in } B_{2\delta}(x_0). \tag{4.2}$$

Notice that by the conformal covariance of the Paneitz operator we have $\mathcal{F}_{\tilde{g}}(\tilde{u}_\varepsilon) = \mathcal{F}_{\tilde{g}}(u_\varepsilon)$. From now on we will work in the metric $\tilde{g}$.

Proof. The estimate is trivial in $B_{2\delta}(x_0) \setminus B_\delta(x_0)$ (where the second term on the r.h.s. of (4.2) dominates the first one). It is therefore sufficient to prove it in $B_\delta(x_0)$, where η is identically equal to 1, and hence here it is enough to estimate

$$P_{\tilde{g}}\big((\varepsilon^2 + r^2)^{\frac{4-n}{2}}\big).$$

Let us first consider the bi-Laplacian term: for a radial function $f(r)$ in conformal normal coordinates we have

$$\Delta f(r) = \frac{1}{\sqrt{\det g}} \partial_r (\sqrt{\det g}\, \partial_r f) = f'' + \frac{n-1}{r} f' + O(r^{N-1}) f',$$

where $N \geq 5$. Therefore, if Δ_0 denotes the Euclidean Laplacian, then

$$\Delta^2 f(r) = \Delta_0^2 f + O(r^{N-1}) f''' + O(r^{N-2}) f'' + O(r^{N-3}) f'.$$

By an explicit computation we find that, if $f(r) = (\varepsilon^2 + r^2)^{(4-n)/2}$, then

$$\Delta_0^2 f = n(n-4)(n^2-4) \frac{\varepsilon^4}{(\varepsilon^2 + r^2)^{\frac{n+4}{2}}} =: b_n \frac{\varepsilon^4}{(\varepsilon^2 + r^2)^{\frac{n+4}{2}}},$$

and (for a dimensional constant a_n)

$$|f'| \leq \frac{a_n r}{(\varepsilon^2 + r^2)^{\frac{n-2}{2}}}, \qquad |f''| \leq \frac{a_n}{(\varepsilon^2 + r^2)^{\frac{n-2}{2}}}, \qquad |f'''| \leq \frac{a_n r}{(\varepsilon^2 + r^2)^{\frac{n}{2}}}.$$

Therefore we obtain

$$\Delta^2 f(r) = b_n \frac{\varepsilon^4}{(\varepsilon^2 + r^2)^{\frac{n+4}{2}}} + \frac{O(r^{N-2})}{(\varepsilon^2 + r^2)^{\frac{n-2}{2}}} = \frac{b_n \varepsilon^4 + O(r^{N-2} (\varepsilon^2 + r^2)^3)}{(\varepsilon^2 + r^2)^{\frac{n+4}{2}}}.$$

Next, we check the lower order terms of the Paneitz operator. Recall

$$Pf = \Delta^2 f + c_1 R_{ij} \nabla_i \nabla_j f + c_2 R \Delta f + c_3 \langle \nabla R, \nabla f \rangle + c_4 Q f,$$

where R_{ij} are the components of the Ricci tensor, and the c_i's are dimensional constants. In conformal normal coordinates,

$$\mathrm{Ric}\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right) = O(r^2), \qquad R = O(r^2), \qquad |\nabla R| = O(r), \qquad |Q| = O(1).$$

Therefore, the terms in Pf involving first and second derivatives of f are of the order

$$r f' + r^2 f'',$$

which are bounded by

$$\frac{O(r^2)}{(\varepsilon^2 + r^2)^{\frac{n-2}{2}}} = \frac{O(1)}{(\varepsilon^2 + r^2)^{\frac{n-4}{2}}}.$$

The term $Q_{\tilde{g}} u_\varepsilon$ is bounded by a constant times f, namely

$$\frac{O(1)}{(\varepsilon^2 + r^2)^{\frac{n-4}{2}}}.$$

In conclusion we find that

$$P_{\tilde{g}}(u_\varepsilon) = \frac{b_n\,\varepsilon^4}{(\varepsilon^2 + |x|^2)^{\frac{n+4}{2}}} + \frac{O(r^{N-2}(\varepsilon^2 + r^2)^3)}{(\varepsilon^2 + r^2)^{\frac{n+4}{2}}} + \frac{O(1)}{(\varepsilon^2 + r^2)^{\frac{n-4}{2}}}.$$

For N sufficiently large the second term on the r.h.s. can be absorbed into the third, so we obtain the desired estimate. $\square$

Recalling the invertibility of P from Proposition 2.3, we consider next the function $\hat{u}_\varepsilon$ defined by the equation

$$P_{\tilde{g}} \hat{u}_\varepsilon = \eta(x) \frac{b_n\,\varepsilon^4}{(\varepsilon^2 + |x|^2)^{\frac{n+4}{2}}}. \tag{4.3}$$

We aim to estimate the difference between this new function and u_ε.

Lemma 4.3. *If $\hat{u}_\varepsilon$ is as above, set*

$$v_\varepsilon = \hat{u}_\varepsilon - u_\varepsilon.$$

Then there exists $C > 0$ such that in $B_{2\delta}(x_0)$ we have the estimates

$$|v_\varepsilon| \leq \begin{cases} C(\varepsilon^2 + |x|^2)^{\frac{8-n}{2}} & \text{if } n > 8, \\ C \log\left(\dfrac{1}{\varepsilon^2 + |x|^2}\right) & \text{if } n = 8. \end{cases}$$

On $M \setminus B_{2\delta}(x_0)$ we have simply

$$|v_\varepsilon| \leq C.$$

Proof. We notice that, by Lemma 4.2,

$$P_{\tilde{g}}(v_\varepsilon) = P_{\tilde{g}}(\hat{u}_\varepsilon - u_\varepsilon) = \eta(x) \frac{b_n\,\varepsilon^4}{(\varepsilon^2 + |x|^2)^{\frac{n+4}{2}}} - P_{\tilde{g}} u_\varepsilon = \frac{O(1)}{(\varepsilon^2 + r^2)^{\frac{n-4}{2}}}.$$

Recall also that the r.h.s. is supported in $B_{2\delta}(x_0)$ as η and u_ε are. We estimate now the convolution of the r.h.s. with the Green's function of the Paneitz operator, which is bounded above by $O(1)/d_{\tilde{g}}(x, y)^{n-4}$.

For $n = 8$ we can divide between the regime $|x| = O(\varepsilon)$ and $|x| \geq C_0 \varepsilon$ for a large constant C_0. When $n = 8$ and $|x| = O(\varepsilon)$ the convolution is bounded by

$$C \int_{|y| \leq 1} \frac{1}{|x - y|^4} \frac{dy}{(\varepsilon^2 + |y|^2)^{\frac{n-4}{2}}}.$$

By a change of variables ($y = \varepsilon w$) one finds that this integral can be controlled by

$$C \int_{|w| \leq 1/\varepsilon} \frac{1}{|\overline{x} - w|^4} \frac{dw}{(1 + |w|^2)^{\frac{n-4}{2}}},$$

where $|\overline{x}| = O(1)$. One can easily see that the latter integral is of order $\log(1/\varepsilon)$. On the other hand, for $|x| \geq C_0 \varepsilon$ we can write

$$\int_{|y| \leq 1} \frac{1}{|x - y|^4} \frac{dy}{(\varepsilon^2 + |y|^2)^{\frac{n-4}{2}}} = \int_{|y| \leq \frac{1}{|x|}} \frac{dw}{\left|\frac{x}{|x|} - w\right|^4 \left(\frac{\varepsilon^2}{|x|^2} + |w|^2\right)^2} \leq \log \frac{1}{|x|}.$$

In conclusion for $n = 8$ we get

$$|v_\varepsilon|(x) \leq \log\left(\frac{1}{\varepsilon^2 + |x|^2}\right), \qquad x \in B_{2\delta}(x_0).$$

The estimate on v_ε outside $B_{2\delta}(x_0)$ is immediate.

Let us consider now the case $n \geq 9$. We distinguish again between $|x| = O(\varepsilon)$ and $|x| \geq C_0 \varepsilon$. In the former case we get, similarly to before

$$C \int_{|y| \leq 1} \frac{1}{|x - y|^{n-4}} \frac{dy}{(\varepsilon^2 + |y|^2)^{\frac{n-4}{2}}} = C \varepsilon^{8-n} \int_{|w| \leq 1/\varepsilon} \frac{1}{|\overline{x} - w|^{n-4}} \frac{dw}{(1 + |w|^2)^{\frac{n-4}{2}}},$$

with $|\overline{x}| = O(1)$. The last integral is uniformly bounded for $n > 9$.

If the case $|x| \geq C_0 \varepsilon$ we write

$$\begin{aligned}\int_{|y| \leq 1} \frac{1}{|x - y|^{n-4}} \frac{dy}{(\varepsilon^2 + |y|^2)^{\frac{n-4}{2}}} &= |x|^{8-n} \int_{|y| \leq \frac{1}{|x|}} \frac{dw}{\left|\frac{x}{|x|} - w\right|^{n-4} \left(\frac{\varepsilon^2}{|x|^2} + |w|^2\right)^{\frac{n-4}{2}}} \\ &\leq C|x|^{8-n}.\end{aligned}$$

In conclusion for $n > 8$ we get

$$|v_\varepsilon|(x) \leq C(\varepsilon^2 + |x|^2)^{\frac{8-n}{2}}, \qquad x \in B_{2\delta}(x_0).$$

The estimate on v_ε outside $B_{2\delta}(x_0)$ is again quite easy.

This concludes the proof. □

We check next the effect of the correction v_ε on the Paneitz–Sobolev quotient, and in particular how much it deviates from the Euclidean one.

Lemma 4.4. *One has*

$$\mathcal{F}_{\tilde{g}}(\hat{u}_\varepsilon) = \begin{cases} \mathcal{F}_{\tilde{g}}(u_\varepsilon) + o(\varepsilon^4 |\log \varepsilon|) & \textit{for } n = 8, \\ \mathcal{F}_{\tilde{g}}(u_\varepsilon) + o(\varepsilon^4) & \textit{for } n \geq 9. \end{cases}$$

Proof. Denoting by $\mathcal{N}$ and $\mathcal{D}$ the numerator and the denominator in the quotient, we have

$$\mathcal{N}(\hat{u}_\varepsilon) = \int_M \hat{u}_\varepsilon P_{\tilde{g}} \hat{u}_\varepsilon \, dv_{\tilde{g}} = \int_M u_\varepsilon P_{\tilde{g}} u_\varepsilon \, dv_{\tilde{g}} + 2 \int_M v_\varepsilon P_{\tilde{g}} u_\varepsilon \, dv_{\tilde{g}} + \int_M v_\varepsilon P_{\tilde{g}} v_\varepsilon \, dv_{\tilde{g}}.$$

The second term by Lemma 4.2 can be estimated by

$$2 \int_M v_\varepsilon \left(\frac{b_n \varepsilon^4}{(\varepsilon^2 + |x|^2)^{\frac{n+4}{2}}} + \frac{O(1)}{(\varepsilon^2 + |x|^2)^{\frac{n-4}{2}}} \right) dv_{\tilde{g}}.$$

By Lemma 4.3 we can write

$$\int_M v_\varepsilon \frac{O(1)}{(\varepsilon^2 + |x|^2)^{\frac{n-4}{2}}} \, dv_{\tilde{g}} \leq C \int_{B_1(0)} \log\left(\frac{1}{\varepsilon^2 + |x|^2} \right) \frac{dx}{(\varepsilon^2 + |x|^2)^{\frac{n-4}{2}}}$$

for $n = 8$, and

$$\int_M v_\varepsilon \frac{O(1)}{(\varepsilon^2 + |x|^2)^{\frac{n-4}{2}}} \, dv_{\tilde{g}} \leq C \int_{B_1(0)} \frac{dx}{(\varepsilon^2 + |x|^2)^{\frac{n-4}{2} + \frac{n-8}{2}}} \tag{4.4}$$

for $n \geq 9$. In the former case, using the change of variables $s = \varepsilon^2 + |x|^2$ we can write

$$\begin{aligned} \int_{B_1(0)} \log\left(\frac{1}{\varepsilon^2 + |x|^2} \right) \frac{dx}{(\varepsilon^2 + |x|^2)^{\frac{8-4}{2}}} &\leq C \int_0^1 \log\left(\frac{1}{\varepsilon^2 + |x|^2} \right) (\varepsilon^2 + |x|^2)^3 |x| \frac{d|x|}{(\varepsilon^2 + |x|^2)^2} \\ &\leq C \int_0^1 \log\left(\frac{1}{s} \right) s \, ds \leq C. \end{aligned} \tag{4.5}$$

In the latter case, one can also easily check boundedness of the l.h.s. of (4.4) using a change of variables. In either case we can write that

$$2 \int_M v_\varepsilon P_{\tilde{g}} u_\varepsilon \, dv_{\tilde{g}} = 2 \int_M v_\varepsilon \frac{b_n \varepsilon^4}{(\varepsilon^2 + |x|^2)^{\frac{n+4}{2}}} \, dv_{\tilde{g}} + O(1).$$

In conclusion we get

$$\mathcal{N}(\hat{u}_\varepsilon) = \int_M u_\varepsilon P_{\tilde{g}} u_\varepsilon \, dv_{\tilde{g}} + 2 b_n \, \varepsilon^4 \int_M u_\varepsilon^{\frac{n+4}{n-4}} v_\varepsilon \, dv_{\tilde{g}} + O(1).$$

We turn next to the denominator $\mathcal{D}$, for which we have

$$\mathcal{D}(\hat{u}_\varepsilon) = \left(\int_M |u_\varepsilon + v_\varepsilon|^{\frac{2n}{n-4}} \, dv_{\tilde{g}} \right)^{\frac{n-4}{n}}.$$

In $B_\delta(x_0)$, by Lemma 4.3 and the explicit expression of u_ε, we have $|v_\varepsilon| \leq C|u_\varepsilon|$, so a Taylor expansion gives

$$\left| |u_\varepsilon + v_\varepsilon|^{\frac{2n}{n-4}} - u_\varepsilon^{\frac{2n}{n-4}} - \frac{2n}{n-4} u_\varepsilon^{\frac{n+4}{n-4}} v_\varepsilon \right| \leq C u_\varepsilon^{\frac{8}{n-4}} v_\varepsilon^2 \quad \text{in } B_\delta(x_0).$$

Hence, using again Lemma 4.3 and the explicit expression of u_ε we can write

$$\begin{aligned}\int_M |u_\varepsilon+v_\varepsilon|^{\frac{2n}{n-4}}\,dv_{\tilde g} &= \int_{B_\delta(x_0)} |u_\varepsilon+v_\varepsilon|^{\frac{2n}{n-4}}\,dv_{\tilde g} + \int_{M\setminus B_\delta(x_0)} |u_\varepsilon+v_\varepsilon|^{\frac{2n}{n-4}}\,dv_{\tilde g}\\ &= \int_{B_\delta(x_0)} \left(u_\varepsilon^{\frac{2n}{n-4}} + \frac{2n}{n-4}u_\varepsilon^{\frac{n+4}{n-4}}v_\varepsilon + O(u_\varepsilon^{\frac{8}{n-4}}v_\varepsilon^2)\right)dv_{\tilde g} + O(1)\\ &= \int_M u_\varepsilon^{\frac{2n}{n-4}}\,dv_{\tilde g} + \frac{2n}{n-4}\int_M u_\varepsilon^{\frac{n+4}{n-4}}v_\varepsilon\,dv_{\tilde g} + \int_{B_\delta(x_0)} O(u_\varepsilon^{\frac{8}{n-4}}v_\varepsilon^2)\,dv_{\tilde g} + O(1).\end{aligned}$$

Similarly to (4.5) for $n=8$ and with a change of variables for $n\geq 9$ we obtain

$$\int_{B_\delta(x_0)} O(u_\varepsilon^{\frac{8}{n-4}}v_\varepsilon^2)\,dv_{\tilde g} = O(1),$$

and hence we find

$$\mathcal{D}(\hat u_\varepsilon) = \left(\int_M u_\varepsilon^{\frac{2n}{n-4}}\,dv_{\tilde g} + \frac{2n}{n-4}\int_M u_\varepsilon^{\frac{n+4}{n-4}}v_\varepsilon\,dv_{\tilde g} + O(1)\right)^{\frac{n-4}{n}}.$$

In conclusion we deduce

$$\mathcal{F}_{\tilde g}(\hat u_\varepsilon) = \frac{\int_M u_\varepsilon P_{\tilde g}u_\varepsilon\,dv_{\tilde g} + 2b_n\varepsilon^4\int_M u_\varepsilon^{\frac{n+4}{n-4}}v_\varepsilon\,dv_{\tilde g} + O(1)}{(\int_M u_\varepsilon^{\frac{2n}{n-4}}\,dv_{\tilde g} + \frac{2n}{n-4}\int_M u_\varepsilon^{\frac{n+4}{n-4}}v_\varepsilon\,dv_{\tilde g} + O(1))^{\frac{n-4}{n}}},$$

which means

$$\mathcal{F}_{\tilde g}(\hat u_\varepsilon) = \frac{\mathcal{N}(u_\varepsilon)}{\mathcal{D}(u_\varepsilon)}\,\frac{1+\frac{2b_n\varepsilon^4\int_M u_\varepsilon^{\frac{n+4}{n-4}}v_\varepsilon\,dv_{\tilde g}}{\int_M u_\varepsilon P_{\tilde g}u_\varepsilon\,dv_{\tilde g}} + \frac{O(1)}{\int_M u_\varepsilon P_{\tilde g}u_\varepsilon\,dv_{\tilde g}}}{\left(1+\frac{2n}{n-4}\frac{\int_M u_\varepsilon^{\frac{n+4}{n-4}}v_\varepsilon\,dv_{\tilde g}}{\int_M u_\varepsilon^{\frac{2n}{n-4}}\,dv_{\tilde g}} + \frac{O(1)}{\int_M u_\varepsilon^{\frac{2n}{n-4}}\,dv_{\tilde g}}\right)^{\frac{n-4}{n}}}.$$

Notice that by Lemma 4.2,

$$\int_M u_\varepsilon P_{\tilde g}u_\varepsilon\,dv_{\tilde g} = b_n\varepsilon^4(1+o_\varepsilon(1))\int_M u_\varepsilon^{\frac{2n}{n-4}}\,dv_{\tilde g},$$

which implies

$$\mathcal{F}_{\tilde g}(\hat u_\varepsilon) = \frac{\mathcal{N}(u_\varepsilon)}{\mathcal{D}(u_\varepsilon)}\,\frac{1+\frac{2\int_M u_\varepsilon^{\frac{n+4}{n-4}}v_\varepsilon\,dv_{\tilde g}}{(1+o_\varepsilon(1))\int_M u_\varepsilon^{\frac{2n}{n-4}}\,dv_{\tilde g}} + \frac{O(1)}{\int_M u_\varepsilon^{\frac{2n}{n-4}}\,dv_{\tilde g}}}{\left(1+\frac{2n}{n-4}\frac{\int_M u_\varepsilon^{\frac{n+4}{n-4}}v_\varepsilon\,dv_{\tilde g}}{\int_M u_\varepsilon^{\frac{2n}{n-4}}\,dv_{\tilde g}} + \frac{O(1)}{\int_M u_\varepsilon^{\frac{2n}{n-4}}\,dv_{\tilde g}}\right)^{\frac{n-4}{n}}}. \tag{4.6}$$

Now notice that we have the asymptotics

$$\int_M u_\varepsilon^{\frac{2n}{n-4}}\, dv_{\tilde g} \simeq \varepsilon^{-n} \quad \text{and} \quad \int_M u_\varepsilon^{\frac{n+4}{n-4}} v_\varepsilon\, dv_{\tilde g} = \begin{cases} O(\varepsilon^{-4}|\log\varepsilon|) & \text{for } n = 8, \\ O(\varepsilon^{4-n}) & \text{for } n \geq 9. \end{cases}$$

Hence from a Taylor expansion of the denominator in (4.6) we find that

$$\mathcal{F}_{\tilde g}(\hat u_\varepsilon) = \begin{cases} (1 + o(\varepsilon^4|\log\varepsilon|))\mathcal{F}_{\tilde g}(u_\varepsilon) & \text{for } n = 8, \\ (1 + o(\varepsilon^4))\mathcal{F}_{\tilde g}(u_\varepsilon) & \text{for } n \geq 9. \end{cases}$$

This concludes the proof. □

Lemma 4.5. *$\hat u_\varepsilon$ is positive.*

Proof. By the defining equation for $\hat u_\varepsilon$ and the conformal covariance of the Paneitz operator,

$$P_{\bar g}(\varphi \hat u_\varepsilon) = \varphi^{\frac{n+4}{n-4}} P_{\tilde g}(\hat u_\varepsilon) = \varphi(x)^{\frac{n+4}{n-4}} \eta(x) \frac{n(n-4)(n^2-4)\varepsilon^4}{(\varepsilon^2 + |x|^2)^{\frac{n+4}{2}}} \geq 0.$$

Since $Q_{\bar g} \geq 0$ and $R_{\bar g} > 0$, by the strong maximum principle of Theorem 2.2 it follows that $\hat u_\varepsilon > 0$. □

Let

$$\psi_\varepsilon = \varphi \hat u_\varepsilon, \qquad h = \psi_\varepsilon^{\frac{4}{n-4}} \bar g = \hat u_\varepsilon^{\frac{4}{n-4}} \tilde g.$$

Lemma 4.6. *The scalar curvature of the metric h is positive.*

Proof. For $0 \leq s \leq 1$ let

$$w_s = (1-s)\varphi^{-1} + s\hat u_\varepsilon, \qquad h_s = w_s^{\frac{4}{n-4}} \tilde g.$$

Then

$$h_0 = \varphi^{-\frac{4}{n-4}} \tilde g = \varphi^{-\frac{4}{n-4}} \{\varphi^{\frac{4}{n-4}} g\} = \bar g,$$

and $h_1 = h$. Observe that the Q-curvature of h_s is semi-positive; this follows from the fact that

$$\begin{aligned} P_{\tilde g}(w_s) &= (1-s)P_{\tilde g}(\varphi^{-1}) + sP_{\tilde g}\hat u_\varepsilon = (1-s)P_{\varphi^{4/(n-4)}\bar g}(\varphi^{-1}) + sP_{\tilde g}\hat u_\varepsilon \\ &= (1-s)\varphi^{-\frac{n+4}{n-4}} P_{\bar g}(1) + sP_{\tilde g}\hat u_\varepsilon = (1-s)\frac{n-4}{2}\varphi^{-\frac{n+4}{n-4}} Q_{\bar g} + sP_{\tilde g}\hat u_\varepsilon \geq 0, \end{aligned}$$

and clearly $P_{\tilde g}(w_s) > 0$ somewhere. Also, note that $R_{h_0} = R_{\bar g} > 0$. Therefore, if there were an $s_1 \in (0, 1]$ such that $\min R_{h_{s_0}} = 0$, then this would contradict Lemma 2.1. It follows that $R_h > 0$. □

To conclude the proof of Proposition 4.1, we point out that the defining equation for $\hat u_\varepsilon$ clearly implies that $P_{\tilde g}\hat u_\varepsilon \geq 0$, with $P_{\tilde g}\hat u_\varepsilon > 0$ near x_0. In particular, the Q-curvature of h is non-negative everywhere, and positive near x_0. We conclude that (i′) and (ii′) both hold. Finally, (4.1) follows from Lemma 4.4 and the conformal invariance of $\mathcal{F}$. □

5. Constructing the initial data, part II: $n = 5, 6, 7$ or $\bar{g}$ locally conformally flat

In low dimensions (i.e., $n = 5, 6, 7$) or in the locally conformally flat case, the Green's function plays a role in the Paneitz–Sobolev quotient expansion, just as for Yamabe's problem in Schoen's work [Sch84]. Using Theorem 2.9, we will prove

Proposition 5.1. *Let $(M^n, \bar{g})$ be a closed Riemannian manifold of dimension n, with $n = 5, 6$, or 7; or let $(M^n, \bar{g})$ be locally conformally flat of dimension $n \geq 5$. Assume*

(i) *$Q_{\bar{g}}$ is semi-positive,*
(ii) *$R_{\bar{g}} \geq 0$.*

If $(M^n, \bar{g})$ is not conformally equivalent to the standard sphere, then for $\varepsilon > 0$ small and every $x_0 \in M$, there exists a function $\psi_\varepsilon \in C^\infty$ and a constant $c_{x_0} > 0$ such that

$$\mathcal{F}_{\bar{g}}(\psi_\varepsilon) \leq S_n - c_{x_0}\varepsilon^{n-4}. \tag{5.1}$$

Moreover, ψ_ε is positive and induces a conformal metric $h = \psi_\varepsilon^{4/(n-4)}\bar{g}$ with the following properties:

(i′) *Q_h is semi-positive,*
(ii′) *$R_h > 0$,*
(iii′) *$\mathcal{F}_h(1) \leq S_n - c_{x_0}\varepsilon^{n-4}$.*

Proof. If $n = 5, 6$ or 7, we let φ be as in the proof of Proposition 4.1. If $\bar{g}$ is locally conformally flat, we choose φ so that $\tilde{g} = \varphi^{4/(n-4)}\bar{g}$ is flat near x_0. We still consider the functions $\hat{u}_\varepsilon$ as in (4.3), with base point x_0, and we try to deduce estimates by evaluating the Paneitz operator on an approximation.

We consider a cut-off function $\tilde{\chi}_{\tilde{\delta}}(x) = \tilde{\chi}(x/\tilde{\delta})$, where $\tilde{\chi}$ is a cut-off function equal to 1 in B_1 and equal to zero outside B_2. We then define an approximate solution $\check{u}_\varepsilon$ by

$$\check{u}_\varepsilon := \tilde{\chi}_{\tilde{\delta}}(u_\varepsilon + \beta) + (1 - \tilde{\chi}_{\tilde{\delta}})\bar{g}_{x_0},$$

where $\beta = \beta_{x_0} = (1/c_n)\alpha_{x_0} > 0$, α_{x_0} appears in the expansion of G_{x_0} in (2.14), and $\bar{g}_{x_0} = (1/c_n)G_{x_0}$ with $\tilde{\delta} \ll \delta$. Notice that by the positivity of the Green's function (see Section 2), the function $\check{u}_\varepsilon$ is positive on M. The Paneitz operator on u_ε was already estimated in the previous section. We have the following estimate of $P_{\tilde{g}}\check{u}_\varepsilon$ in $B_{2\tilde{\delta}} \setminus B_{\tilde{\delta}}$:

Lemma 5.2. *There exists a constant $C > 0$ such that*

$$|P_{\tilde{g}}\check{u}_\varepsilon| \leq C\tilde{\delta}^{-3} \quad \text{in } B_{2\tilde{\delta}} \setminus B_{\tilde{\delta}}.$$

Proof. We can write $\check{u}_\varepsilon = \bar{g}_{x_0} + \tilde{\chi}_{\tilde{\delta}}(u_\varepsilon + \beta - \bar{g}_{x_0})$, and hence, in $B_{2\tilde{\delta}} \setminus B_{\tilde{\delta}}$,

$$\begin{aligned}|P_{\tilde{g}}\check{u}_\varepsilon| \leq{}& |\nabla^4\tilde{\chi}_{\tilde{\delta}}|\,|u_\varepsilon + \beta - \bar{g}_{x_0}| + |\nabla^3\tilde{\chi}_{\tilde{\delta}}|\,|\nabla(u_\varepsilon + \beta - \bar{g}_{x_0})| \\ &+ |\nabla^2\tilde{\chi}_{\tilde{\delta}}|\,|\nabla^2(u_\varepsilon + \beta - \bar{g}_{x_0})| \\ &+ |\nabla\tilde{\chi}_{\tilde{\delta}}|\,|\nabla^3(u_\varepsilon + \beta - \bar{g}_{x_0})| + |P_{\tilde{g}}(u_\varepsilon + \beta - \bar{g}_{x_0})|.\end{aligned}$$

As $\chi_{\tilde{\delta}}$ satisfies the estimates

$$|\nabla \tilde{\chi}_{\tilde{\delta}}| \le C/\tilde{\delta}, \quad |\nabla^2 \tilde{\chi}_{\tilde{\delta}}| \le C/\tilde{\delta}^2, \quad |\nabla^3 \tilde{\chi}_{\tilde{\delta}}| \le C/\tilde{\delta}^3, \quad |\nabla^4 \tilde{\chi}_{\tilde{\delta}}| \le C/\tilde{\delta}^4,$$

it will be sufficient to show that in $B_{2\tilde{\delta}} \setminus B_{\tilde{\delta}}$,

$$|u_\varepsilon + \beta - \bar{g}_{x_0}| \le C\tilde{\delta}, \quad |\nabla(u_\varepsilon + \beta - \bar{g}_{x_0})| \le C, \quad |\nabla^2(u_\varepsilon + \beta - \bar{g}_{x_0})| \le C/\tilde{\delta},$$
$$|\nabla^3(u_\varepsilon + \beta - \bar{g}_{x_0})| \le C/\tilde{\delta}^2, \quad |P_{\tilde{g}}(u_\varepsilon + \beta - \bar{g}_{x_0})| \le C/\tilde{\delta}^3.$$

We begin with the last inequality: we have

$$P_{\tilde{g}}(u_\varepsilon + \beta - \bar{g}_{x_0}) = P_{\tilde{g}}(u_\varepsilon + \beta) = P_{\tilde{g}} u_\varepsilon + \beta\, Q_{\tilde{g}} = O(\tilde{\delta}^{-3}),$$

where we have used Lemma 4.2 (in the locally conformally flat case it is obvious). To prove the remaining estimates we use the fact that in $B_{2\tilde{\delta}}$,

$$u_\varepsilon + \beta - \bar{g}_{x_0} = (\varepsilon^2 + |x|^2)^{\frac{4-n}{2}} - |x|^{4-n} + O_p(1),$$

by Proposition 2.5. We remark that in the locally conformally flat case the above estimate simply follows from the fact that, in the metric $\tilde{g}$, $\bar{g}_{x_0}(\cdot) - \beta - d_{\tilde{g}}(x_0, \cdot)$ is a smooth bi-harmonic function.

From a Taylor expansion of $(\varepsilon^2 + |x|^2)^{(4-n)/2}$ one easily finds that

$$u_\varepsilon + \beta - \bar{g}_{x_0} = O(\varepsilon^2 |x|^{2-n}) + O_p(1).$$

This implies the conclusion. □

Combining the estimates of Lemmas 4.2 and 5.2 we find

$$\left| P_{\tilde{g}} \check{u}_\varepsilon - \frac{n(n-4)(n^2-4)\varepsilon^4}{(\varepsilon^2 + |x|^2)^{\frac{n+4}{2}}} \right| \le \begin{cases} O(1)/(\varepsilon^2 + r^2)^{\frac{n-4}{2}} & \text{for } |x| \le \tilde{\delta}, \\ O(\tilde{\delta}^{-3}) & \text{for } \tilde{\delta} \le |x| \le 2\tilde{\delta}. \end{cases}$$

We can use the latter estimate to control the difference between u_ε and $\check{u}_\varepsilon$ by convolving with the Green's function.

Lemma 5.3. *The following estimate holds, for some constant $C > 0$:*

$$|\hat{u}_\varepsilon - \check{u}_\varepsilon| \le o(1) + C\tilde{\delta}^{n-3} \min\{|x|^{4-n}, \delta^{4-n}\} = o(1), \qquad \tilde{\delta} \to 0.$$

Proof. By the formula before the lemma we can write $|\hat{u}_\varepsilon - \check{u}_\varepsilon| \le u_1 + u_2$, where

$$u_1(x) = \int_{B_{\tilde{\delta}}(0)} G_x(y) \frac{dy}{(\varepsilon^2 + |y|^2)^{\frac{n-4}{2}}}, \qquad u_2(x) = \delta^{-3} \int_{\tilde{\delta} \le |y| \le 2\tilde{\delta}} G_x(y)\, dy.$$

To estimate u_1 we reason as in the proof of Lemma 4.3: we divide again into the cases $|x| = O(\varepsilon)$ and $|x| \ge C_0\varepsilon$. In the former case we get

$$C \int_{|y| \le \tilde{\delta}} \frac{1}{|x-y|^{n-4}} \frac{dy}{(\varepsilon^2 + |y|^2)^{\frac{n-4}{2}}} = C\varepsilon^{8-n} \int_{|w| \le \tilde{\delta}/\varepsilon} \frac{1}{|\overline{x} - w|^{n-4}} \frac{dw}{(1 + |w|^2)^{\frac{n-4}{2}}},$$

with $|\overline{x}| = O(1)$. The last integral is uniformly bounded by $\tilde{\delta}^{n-8}\varepsilon^{n-8}$, so we get a quantity of order $o(1)$ as $\tilde{\delta} \to 0$.

In the case $|x| \geq C_0\varepsilon$ we write

$$\int_{|y|\leq\tilde{\delta}} \frac{1}{|x-y|^{n-4}} \frac{dy}{(\varepsilon^2+|y|^2)^{\frac{n-4}{2}}} = |x|^{8-n} \int_{|y|\leq\tilde{\delta}/|x|} \frac{dw}{\left|\frac{x}{|x|}-w\right|^{n-4}\left(\frac{\varepsilon^2}{|x|^2}+|w|^2\right)^{\frac{n-4}{2}}} \leq \tilde{\delta}^{n-8}.$$

Therefore we get a uniform bound on u_1 of order $o(1)$ as $\tilde{\delta} \to 0$.

Turning to u_2, one can distinguish the cases $|x| \leq 2\tilde{\delta}$ and $|x| > 2\tilde{\delta}$. In the former one finds $|u_2(x)| \leq C\tilde{\delta}$, in the latter $|u_2(x)| \leq C\tilde{\delta}^{n-3}|x|^{4-n}$. The bounds on u_1 and u_2 yield the conclusion. $\square$

To estimate the quotient of $\hat{u}_\varepsilon$, we have, by definition of $\hat{u}_\varepsilon$,

$$\begin{aligned}\int_M \hat{u}_\varepsilon P_{\tilde{g}} \hat{u}_\varepsilon \, dv_{\tilde{g}} &= \int_M \hat{u}_\varepsilon \frac{\eta(x) b_n \varepsilon^4}{(\varepsilon^2+|x|^2)^{\frac{n+4}{2}}} \, dv_{\tilde{g}} \\ &= \int_M \big(\tilde{\chi}_{\tilde{\delta}}(u_\varepsilon+\beta) + (1-\tilde{\chi}_{\tilde{\delta}})\bar{g}_{x_0} + (\hat{u}_\varepsilon - \check{u}_\varepsilon)\big) \frac{\eta(x) b_n \varepsilon^4}{(\varepsilon^2+|x|^2)^{\frac{n+4}{2}}} \, dv_{\tilde{g}} \\ &=: I_1 + I_2 + I_3.\end{aligned}$$

We next estimate each of these three terms. Concerning I_1 we have

$$I_1 = b_n\varepsilon^4\bigg(\int_M u_\varepsilon^{\frac{2n}{n-4}} \, dv_{\tilde{g}} + \beta \int_M u_\varepsilon^{\frac{n+4}{n-4}} \, dv_{\tilde{g}} + O(1)\bigg), \qquad \tilde{\delta} \to 0.$$

For I_2, since $1-\tilde{\chi}_{\tilde{\delta}}$ vanishes in a $\tilde{\delta}$-neighborhood of p we simply have

$$I_2 = \varepsilon^4 O(1).$$

For I_3 we can use Lemma 5.3 to find that

$$|I_3| \leq Co(1) \int_M \frac{\eta(x) b_n \varepsilon^4}{(\varepsilon^2+|x|^2)^{\frac{n+4}{2}}} \, dv_{\tilde{g}}.$$

Therefore, we obtain

$$\mathcal{N}(\hat{u}_\varepsilon) = b_n\varepsilon^4\bigg(\int_M u_\varepsilon^{\frac{2n}{n-4}} \, dv_{\tilde{g}} + \beta(1+o(1)) \int_M u_\varepsilon^{\frac{n+4}{n-4}} \, dv_{\tilde{g}} + O(1)\bigg). \tag{5.2}$$

On the other hand for the denominator we have

$$\int_M \hat{u}_\varepsilon^{\frac{2n}{n-4}} \, dv_{\tilde{g}} = \int_{B_{\tilde{\delta}}(x_0)} (u_\varepsilon+\beta)^{\frac{2n}{n-4}} \, dv_{\tilde{g}} + O(1).$$

Since in $B_{\tilde{\delta}}(x_0)$, β is bounded by u_ε, we have

$$\begin{aligned}\int_M \hat{u}_\varepsilon^{\frac{2n}{n-4}}\, dv_{\tilde{g}} &= \int_{B_{\tilde{\delta}}(x_0)} u_\varepsilon^{\frac{2n}{n-4}}\, dv_{\tilde{g}} + \frac{2n}{n-4}\beta \int_{B_{\tilde{\delta}}(x_0)} u_\varepsilon^{\frac{n+4}{n-4}}\, dv_{\tilde{g}} \\ &\quad + \beta^2 \int_{B_\delta(x_0)} O(u_\varepsilon^{\frac{8}{n-4}})\, dv_{\tilde{g}} + O(1) \\ &= \int_M u_\varepsilon^{\frac{2n}{n-4}}\, dv_{\tilde{g}} + \frac{2n}{n-4}\beta \int_M u_\varepsilon^{\frac{n+4}{n-4}}\, dv_{\tilde{g}} + \beta^2 \int_{B_{\tilde{\delta}}(x_0)} O(u_\varepsilon^{\frac{8}{n-4}})\, dv_{\tilde{g}} + O(1).\end{aligned}$$

Therefore one finds that

$$\mathcal{F}_{\tilde{g}}(\hat{u}_\varepsilon) = \frac{b_n \varepsilon^4 \big(\int_M u_\varepsilon^{\frac{2n}{n-4}}\, dv_{\tilde{g}} + \beta(1 + o_{\tilde{\delta}}(1)) \int_M u_\varepsilon^{\frac{n+4}{n-4}}\, dv_{\tilde{g}} + O_{\tilde{\delta}}(1)\big)}{\big(\int_M u_\varepsilon^{\frac{2n}{n-4}}\, dv_{\tilde{g}} + \frac{2n}{n-4}\beta \int_M u_\varepsilon^{\frac{n+4}{n-4}}\, dv_{\tilde{g}} + \beta^2 \int_{B_{\tilde{\delta}}(x_0)} O(u_\varepsilon^{\frac{8}{n-4}})\, dv_{\tilde{g}} + O_{\tilde{\delta}}(1)\big)^{\frac{n-4}{n}}}.$$

We now notice that the following asymptotics hold:

$$\int_M u_\varepsilon^{\frac{2n}{n-4}}\, dv_{\tilde{g}} \simeq \varepsilon^{-n}, \qquad \int_M u_\varepsilon^{\frac{n+4}{n-4}}\, dv_{\tilde{g}} \simeq \varepsilon^{-4}, \qquad \int_{B_{\tilde{\delta}}(x_0)} O(u_\varepsilon^{\frac{8}{n-4}})\, dv_{\tilde{g}} \simeq \varepsilon^{n-8}.$$

These and a Taylor expansion of the denominator in $\mathcal{F}_{\tilde{g}}(\hat{u}_\varepsilon)$ imply

$$\mathcal{F}_{\tilde{g}}(\hat{u}_\varepsilon) = S_n \left(1 - \beta(1 + o(1)) \frac{\int_M u_\varepsilon^{\frac{n+4}{n-4}}\, dv_{\tilde{g}}}{\int_M u_\varepsilon^{\frac{2n}{n-4}}\, dv_{\tilde{g}}} \right).$$

This completes the proof of (5.1). The proof of (i′)–(iii′) is the same as in the proof of Proposition 4.1. $\square$

6. Sequential convergence of the flow

In this section we prove the main existence result: under the assumptions of Proposition 4.1 or 5.1, we show the flow converges (up to choosing a suitable sequence of times) to a solution of the Q-curvature equation.

Theorem 6.1. *Let $(M^n, \bar{g})$ be a closed Riemannian manifold of dimension $n \geq 5$ which is not conformally equivalent to the standard sphere. Suppose that*

(i) *$Q_{\bar{g}}$ is semi-positive,*
(ii) *$R_{\bar{g}} \geq 0$.*

Let $g_0 = h$, where h is the metric constructed in Proposition 5.1 (when $5 \leq n \leq 7$, or $\bar{g}$ is locally conformally flat and $n \geq 5$) or Proposition 4.1 (when $n \geq 8$ and $\bar{g}$ is not locally conformally flat). Then the flow (3.2) has a solution for all time satisfying

$$\int u^2\, dv_0 \geq C_0$$

for some constant $C_0 > 0$. Moreover, it is possible to choose a sequence of times $t_j \nearrow \infty$ such that $u_j = u_j(t_j, \cdot)$ converges weakly in $W^{2,2}(M^n)$ to a smooth solution $u > 0$ of

$$P_{g_0} u = \bar{\mu} u^{\frac{n+4}{n-4}},$$

where $\bar{\mu} > 0$. In particular, $g_\infty = u^{4/(n-4)} g_0$ defines a metric with positive scalar curvature and constant positive Q-curvature.

Proof. If we take our initial metric g_0 to be the metric in the conclusion of Proposition 4.1 or 5.1, then by Proposition 3.7 the flow (3.2) exists for all time. In addition, by the same propositions we know

$$\mathcal{F}_{g_0}[u_0] \le S_n - \epsilon_0,$$

where $u_0 \equiv 1$ is our initial datum for the flow and $\epsilon_0 > 0$. It follows from Lemma 3.5 that

$$\mathcal{F}_{g_0}[u] = \frac{\int u(P_{g_0} u)\, dv_0}{\left(\int u^{\frac{2n}{n-4}}\, dv_0\right)^{\frac{n-4}{n}}} \le S_n - \epsilon_0 \tag{6.1}$$

for all times.

Recall the Euclidean Paneitz–Sobolev constant is

$$S_n = \inf_{\varphi \in C_0^\infty(\mathbb{R}^n)} \frac{\int (\Delta_0 \varphi)^2\, dx}{\left(\int |\varphi|^{\frac{2n}{n-4}}\, dx\right)^{\frac{n-4}{n}}}.$$

On the compact Riemannian manifold (M, g_0), given $\delta > 0$ we can use a cut-and-paste argument to prove that

$$\left(\int |\varphi|^{\frac{2n}{n-4}}\, dv_0\right)^{\frac{n-4}{n}} \le (S_n^{-1} + \delta) \int (\Delta_{g_0} \varphi)^2\, dv_0 + C_\delta \int \varphi^2\, dv_0,$$

which implies

$$\left(\int |\varphi|^{\frac{2n}{n-4}}\, dv_0\right)^{\frac{n-4}{n}} \le (S_n^{-1} + 2\delta) \int \varphi(P_{g_0} \varphi)\, dv_0 + C'_\delta \int \varphi^2\, dv_0. \tag{6.2}$$

Plugging (6.1) into (6.2) gives

$$\begin{aligned}\left(\int u^{\frac{2n}{n-4}}\, dv_0\right)^{\frac{n-4}{n}} &\le (S_n^{-1} + 2\delta) \int u(P_{g_0} u)\, dv_0 + C'_\delta \int u^2\, dv_0 \\ &\le (S_n^{-1} + 2\delta)(S_n - \epsilon_0)\left(\int u^{\frac{2n}{n-4}}\, dv_0\right)^{\frac{n-4}{n}} + C'_\delta \int u^2\, dv_0.\end{aligned}$$

If we take $\delta = \epsilon_0/10$, then the first term on the right-hand side can be absorbed into the left-hand side, and we get

$$\left(\int u^{\frac{2n}{n-4}}\, dv_0\right)^{\frac{n-4}{n}} \le C(\epsilon_0) \int u^2\, dv_0.$$

Since the l.h.s. is just a power of the conformal volume (which is non-decreasing), we conclude that $\int u^2 \, dv_0 \geq C_0 > 0$ for all time, as claimed.

By Lemma 3.5 and Corollary 3.6 we can choose a sequence of times $t_j \nearrow \infty$ such that $u_j = u(t_j, \cdot)$ and $\mu_j = \mu(t_j)$ satisfy

$$\begin{aligned} &\mu_j \nearrow \bar{\mu}, \\ &u_j \rightharpoonup u \quad \text{weakly in } W^{2,2}(M^n), \\ &u_j \to u \quad \text{strongly in } L^2(M^n), \\ &f_j = -u_j + \mu_j P_{g_0}^{-1}(u_j^{\frac{n+4}{n-4}}) \to 0 \quad \text{strongly in } W^{2,2}(M^n). \end{aligned}$$

It follows that $u \geq 0$ satisfies

$$u = \bar{\mu} P_{g_0}^{-1}(u^{\frac{n+4}{n-4}}),$$

and by elliptic regularity u is a strong solution of

$$P_{g_0} u = \bar{\mu} u^{\frac{n+4}{n-4}}.$$

By the strong maximum principle (Theorem 2.2), in fact $u > 0$. This completes the proof of the theorem. □

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