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Topological classification of multiaxial $U(n)$-actions
(with an appendix by Jared Bass)

Abstract. This paper begins the classification of topological actions on manifolds by compact, connected, Lie groups beyond the circle group. It treats multiaxial topological actions of unitary and symplectic groups without the dimension restrictions used in earlier works by M. Davis and W. C. Hsiang on differentiable actions. The general results are applied to give detailed calculations for topological actions homotopically modeled on standard multiaxial representation spheres. In the present topological setting, Schubert calculus of complex Grassmannians surprisingly enters in the calculations, yielding a profusion of "fake" representation spheres compared with the paucity in the previously studied smooth setting.

Keywords. Transformation group, topological manifold, stratified space, multiaxial, surgery, assembly map

1. Introduction

In the last half century great progress has been made on both the differentiable and topological classification of finite group actions on spheres and more general manifolds. Deep, albeit indirect, connections of transformation groups to representation theory were discovered. For positive-dimensional groups beyond the case of the circle, essentially the only classification results obtained for differentiable actions are the classical results of M. Davis and W. C. Hsiang [D, DH] and their further development with J. Morgan [DHM] on concordance classes of multiaxial actions on homotopy spheres, in certain dimension ranges. On the other hand, certain topological phenomena, such as periodicity [WY] and the replacement of fixed points [CW2, CWY] showed that the topological classification of actions of positive-dimensional groups must be very different from the smooth case.

The present paper begins the classification of topological actions on manifolds by positive-dimensional groups beyond the case of the circle, by obtaining general results on multiaxial actions on topological manifolds. Here we will work with a more flexible notion of multiaxial (and without the dimension conditions) than had been considered for

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smooth actions. An action of a unitary group $U(n)$ on a manifold will be called *multiaxial* if all of its isotropy subgroups are unitary subgroups, the fixed sets are ANR homology manifolds, and embeddings of the strata in the orbit space are 1-LC (a standard weak kind of homotopy local flatness condition as regards fundamental groups, see Section 2). Our results will show that topological multiaxial actions are far more profuse and their classification is quite different from the smooth case, even when restricted to spheres. For example, the homology of complex Grassmannians enters into the classification of topological actions homotopically modeled on multiaxial representation spheres.

The connection of the theory of topological actions to representation theory is less direct than for smooth actions. This reflects the failure in the topological setting of some of the basic building blocks of the analogous smooth or PL theory of finite group actions. Whitehead torsion, a cornerstone of the classical theory of lens spaces (via Reidemeister torsion), plays in the general topological category a diminished role because of the absence of (canonical) tubular neighborhoods around fixed points [Q2, St] and more generally around subsets of given orbit types. Indeed, Milnor's counterexamples to the Hauptvermutung [Mi] showed that classical Whitehead torsion is not even always definable in the topological category for non-free actions. The divergence for actions of finite groups of the topological classification from the smooth or PL ones was strikingly reflected in the existence of non-linear similarities between some linearly inequivalent representations [CS].

On the other hand, key invariants defined in smooth or PL settings using the equivariant signature operator do remain well defined in topological settings [CSW, HP, MR] and play a major role there.

In this introduction, for simplicity and ease of exposition, we make the stronger assumption that $G = U(n)$ acts locally smoothly. In other words, every orbit has a neighborhood equivariantly homeomorphic to an open subset of an orthogonal representation of G. Moreover, we concentrate on the classical and more restrictive notion of multiaxial actions, for which the orthogonal representations are of the form $k\rho_n \oplus j\epsilon$, where ρ_n is the defining representation of $U(n)$ on $\mathbb{C}^n$ and ϵ is the trivial representation. While this allows for different choices of k and j at different locations in a manifold, the results presented in the introduction will assume the same k and the same j everywhere. In such a setting, we say the manifold is modeled on $k\rho_n \oplus j\epsilon$. Examples are the representation space $k\rho_n \oplus j\epsilon$ and the associated representation sphere.

The isotropy subgroups of a multiaxial $U(n)$-manifold M are conjugate to the specific unitary subgroups $U(i)$ of $U(n)$ that fix the subspaces $0 \oplus \mathbb{C}^{n-i}$ of $\mathbb{C}^n$. Then M is stratified by $M_{-i} = U(n)M^{U(i)}$, the set of points fixed by some conjugate of $U(i)$. Correspondingly, the orbit space $X = M/U(n)$ is stratified by $X_{-i} = M^{U(i)}/U(n-i)$ (see Lemma 2.1).

Our goal is to study the isovariant structure set $S_{U(n)}(M)$. Classically, the structure set $S(X)$ of a compact topological manifold X is the homeomorphism classes of topological manifolds equipped with a simple homotopy equivalence to X (with homotopy and homeomorphism defining the equivalence relation). The notion can be extended to the setting of a G-manifold M by letting $S_G(M)$ denote the equivariant homeomorphism classes of G-manifolds each equipped with an isovariant simple homotopy equivalence

to M. It can also be extended to $S(X)$ for stratified spaces X and stratified simple homotopy equivalences.

For $Y \subset X$, we use $S(X, \text{rel } Y)$ to denote the homeomorphism classes of simple homotopy equivalences that are already homeomorphisms on Y. On the other hand, $S(X, Y)$ denotes the homeomorphism classes of simple homotopy equivalences to the pair (X, Y).

We make use of some aspects of the theory of homotopically stratified spaces, as developed in [Q5, We], to which we refer. The strata in such spaces have "homotopy links": rather than having the local structure determined by fiber bundles, as in the theory of Whitney stratified spaces, the local structure is a fibration, and the homotopy links describe the homotopy fibers. We further allow the strata to be ANR homology manifolds that satisfy the DDP (Disjoint Disk Property) [BFMW]. The reader unfamiliar with these may imagine that all the strata are manifolds, but should note that these "manifolds" may have non-trivial 0-th Pontryagin class. If one wants to restrict attention just to genuine manifolds, then the calculated answers will be somewhat smaller: one has to compute at the end a set of "local signatures" [Q3, Q4] to see which structures have manifold strata. Likewise even if G acts locally smoothly on M, the structure set $S_G(M)$ may contain many multiaxial G-manifolds which are not locally smoothable.

When the orbit space M/G is homotopically stratified, we have $S_G(M) = S(M/G)$. The map $S_G(M) \to S(M/G)$ takes an isovariant homotopy equivalence $N \simeq_G M$ to a stratified homotopy equivalence $N/G \simeq M/G$. The inverse map $S(M/G) \to S_G(M)$ takes a stratified homotopy equivalence $X \simeq M/G$ and then constructs N as the pullback of $X \to M/G \leftarrow M$.

Classical surgery theory formulates $S(X)$ initially in terms of s-cobordism classes and then employs the s-cobordism theorem to reformulate this in terms of the more geometrically meaningful homeomorphism classification. The topological isovariant surgery theory of [We] similarly employs the stratified (and thus the equivariant) s-cobordism theorem of Quinn [Q2] and of Steinberger [St], which identifies stratified s-cobordisms with homeomorphisms when the strata are manifolds and no dimension 4 strata occur with "large fundamental groups" in the sense of [F, FQ].

Let X_α be the strata of a homotopically stratified space X. The pure strata

$$X^\alpha = X_\alpha - X_{<\alpha}, \quad X_{<\alpha} = \bigcup_{X_\beta \subsetneq X_\alpha} X_\beta,$$

are generally non-compact manifolds, and we have natural restriction maps

$$S(X) \to \bigoplus S^{\text{proper}}(X^\alpha).$$

Here S^{proper} denotes the proper homotopy equivalence version of the structure set. If we further know that all pure strata of links between strata of X are connected and simply connected (or more generally, the fundamental groups of these strata have trivial K-theory in low dimensions, according to Quinn [Q2]), then the complement $\bar{X}^\alpha$ of (the interior of) a regular neighborhood of $X_{<\alpha}$ in X_α is a topological manifold with boundary $\partial \bar{X}^\alpha$ and interior X^α, and the restriction maps natually factor through the structures of $(\bar{X}^\alpha, \partial \bar{X}^\alpha)$

$$S(X) \to \bigoplus S(\bar{X}^\alpha, \partial \bar{X}^\alpha) \to \bigoplus S^{\text{proper}}(X^\alpha).$$

The difference between the simple homotopy structure of $(\bar{X}^\alpha, \partial\bar{X}^\alpha)$ and the proper homotopy structure of X^α is captured by the finiteness obstruction at infinity and related Whitehead torsion considerations.

By the 1-LC assumption, the pure strata of links in our multiaxial $U(n)$-manifolds are indeed connected and simply connected. Our main result states that the stratified simple homotopy structure set of $X = M/U(n)$ is almost always determined by the restrictions to $S(\bar{X}^{-i}, \partial\bar{X}^{-i})$ using a particular half of the set of strata X_{-i}. More general versions are given by Theorems 5.1, 5.2, 5.3.

Theorem 1.1. *Suppose M is a multiaxial $U(n)$-manifold modeled on $k\rho_n \oplus j\epsilon$, and $X = M/U(n)$ is the orbit space.*

1. *If $k \geq n$ and $k - n$ is even, then we have a natural splitting*

$$S_{U(n)}(M) = \bigoplus_{i \geq 0} S(\bar{X}^{-2i}, \partial\bar{X}^{-2i}) = \bigoplus_{i \geq 0} S^{\mathrm{alg}}(X_{-2i}, X_{-2i-1}).$$

2. *If $k \geq n$, $k - n$ is odd and $M = W^{U(1)}$ for a multiaxial $U(n+1)$-manifold W modeled on $k\rho_{n+1} \oplus j\epsilon$, then we have a natural splitting*

$$S_{U(n)}(M) = S^{\mathrm{alg}}(X) \oplus \bigoplus_{i \geq 1} S(\bar{X}^{-2i+1}, \partial\bar{X}^{-2i+1})$$
$$= S^{\mathrm{alg}}(X) \oplus \bigoplus_{i \geq 1} S^{\mathrm{alg}}(X_{-2i+1}, X_{-2i}).$$

3. *If $k \leq n$, then $S_{U(n)}(M) = S_{U(k)}(M^{U(n-k)})$. Since $M^{U(n-k)}$ is a multiaxial $U(k)$-manifold modeled on $k\rho_k \oplus j\epsilon$, this case is reduced to $k = n$ treated in part 1.*

The reduction in the third case is due to $M/U(n) = M^{U(n-k)}/U(k)$ by Lemma 2.1, and $S_G(M) = S(M/G)$. More specifically, the map $S_{U(n)}(M) \to S_{U(k)}(M^{U(n-k)})$ simply takes the fixed parts by the subgroup $U(n - k)$. The inverse map $S_{U(k)}(M^{U(n-k)}) \to S_{U(n)}(M)$ takes the stratified homotopy equivalence of the orbit spaces and then takes the pullback construction.

The restriction to $k \leq n$ was critical in the works [D, DH, DHM] on differentiable multiaxial actions.

The *algebraic structure set S^{alg}* in the theorem denotes the following familiar homotopy functor [R].

Definition. For any (reasonable) topological space X, let $\mathbb{S}^{\mathrm{alg}}(X)$ be the homotopy fiber of the surgery assembly map $\mathbb{H}_*(X; \mathbb{L}) \to \mathbb{L}(\pi_1 X)$. Then $S^{\mathrm{alg}}(X) = \pi_{\dim X}\mathbb{S}^{\mathrm{alg}}(X)$.

In this definition, $\mathbb{L}(\pi)$ is the (simple) surgery obstruction spectrum for the fundamental group π, and $\mathbb{H}_*(X; \mathbb{L})$ is the homology theory associated to the spectrum $\mathbb{L} = \mathbb{L}(e)$. If X is a topological manifold of dimension ≥ 5 (or dimension 4 in case $\pi_1 X$ is not too intractable [F, FQ]), then $S^{\mathrm{alg}}(X)$ is the usual structure set that classifies topological (in fact, homology) manifolds simple homotopy equivalent to X. For a general topological

space X, however, $S^{\text{alg}}(X)$ no longer carries that geometrical meaning and is for the present purpose the result of some algebraic computation.

Notice that the expression in terms of $S^{\text{alg}}(X_{-i}, X_{-i-1})$ involves only objects that are a priori associated to the group action. However, the map from the left hand side to the right hand side, while related to the forgetful map to $S(\bar{X}^{-i}, \partial \bar{X}^{-i-1})$, is not quite obvious to define.

For a taste of what to expect when k and j are not assumed constant, the following is the simplest case of Theorem 5.2. The proof is given at the end of Section 5.

Theorem 1.2. *Suppose the circle S^1 acts semifreely and locally linearly on a topological manifold M. Let $M_0^{S^1}$ and $M_2^{S^1}$ be the unions of those connected components of M^{S^1} that are, respectively, of codimensions $0 \bmod 4$ and $2 \bmod 4$. Let N be the complement of (the interior of) an equivariant tube neighborhood of M^{S^1}, with boundaries $\partial_0 N$ and $\partial_2 N$ corresponding to the two parts of the fixed points. Then*

$$S_{S^1}(M) = S(M_0^{S^1}) \oplus S(N/S^1, \partial_2 N/S^1, \text{rel } \partial_0 N/S^1).$$

We note that N/S^1 is a manifold with boundary divided into two parts ∂_0 and ∂_2. The second summand means the homeomorphism classes of (homology) manifolds equipped with a simple homotopy equivalence to N/S^1 which restricts to a simple homotopy equivalence on ∂_2 and a homeomorphism on ∂_0.

We also note that it is a special feature of circle actions that the condition in Theorem 1.1 of the extendability of the $S^1 = U(1)$-action on M to a multiaxial $U(2)$-action on a manifold is never needed. It is an open question whether or not, in general, one can dispense with the extendability condition in part 2 of Theorem 1.1.

For $k \geq n$, the terms $S(\bar{X}^{-i}, \partial \bar{X}^{-i})$ in the decompositions of Theorem 1.1 could be reformulated in terms of the isovariant structure set

$$S(\bar{X}^{-i}, \partial \bar{X}^{-i}) = S_{U(n-i)}(M^{U(i)}, \text{rel } U(n-i)M^{U(i+2)}).$$

Here $M^{U(i)}$ is actually a multiaxial $U(n-i)$-manifold modeled on $k\rho_{n-i} \oplus j\epsilon$, and $U(n-i)M^{U(i+2)}$ is the stratum of the multiaxial $U(n-i)$-manifold two levels down. The right side classifies those $U(n-i)$-manifolds isovariantly simple homotopy equivalent to $M^{U(i)}$, such that the restrictions to the stratum two levels down are already equivariantly homeomorphic. The decomposition in Theorem 1.1 is then equivalent to the decomposition

$$S_{U(n)}(M) = S_{U(n)}(M, \text{rel } U(n)M^{U(i)}) \oplus S_{U(n-i)}(M^{U(i)}) \quad \text{for } k - n + i \text{ even.}$$

The map to the second summand is the obvious restriction. The fact that this restriction is onto has the following interpretation.

Theorem 1.3. *Suppose M is a multiaxial $U(n)$-manifold modeled on $k\rho_n \oplus j\epsilon$. Suppose $k \geq n > i$, $k - n + i$ is even, and additionally, when $k - n$ is odd, we have $M = W^{U(1)}$ for a multiaxial $U(n+1)$-manifold W modeled on $k\rho_{n+1} \oplus j\epsilon$. Then for any $U(n-i)$-isovariant simple homotopy equivalence $\phi \colon F \to M^{U(i)}$, there is a $U(n)$-isovariant simple homotopy equivalence $f \colon N \to M$ such that $\phi = f^{U(i)}$ is the restriction of f.*

The theorem means that half of the fixed point subsets can be homotopically replaced. The problem of homotopy replacement of the fixed point subset of a whole group was studied in [CW2, CWY]. Here equivariant replacement is achieved for the fixed point subsets of certain proper subgroups (and not others); this is the first appearance of such a phenomenon.

For $k \leq n$, by $S_{U(n)}(M) = S_{U(k)}(M^{U(n-k)})$, we may apply Theorem 1.3 to the k-axial $U(k)$-manifold $M^{U(n-k)}$ and get the following homotopy replacement result: For any even i with $i \leq k$ and $U(k-i)$-isovariant simple homotopy equivalence $\phi\colon F \to M^{U(n-k+i)}$, there is a $U(n)$-isovariant simple homotopy equivalence $f\colon N \to M$ such that $\phi = f^{U(n-k+i)}$ is the restriction of f.

Algebraically, the terms $S^{\mathrm{alg}}(X_{-i}, X_{-i-1})$ and $S^{\mathrm{alg}}(X)$ in the decompositions of Theorem 1.1 can be explicitly computed for the special case that M is the unit sphere of the representation $k\rho_n \oplus j\epsilon$. For $k \geq n$, let $A_{n,k}$ be the number of Schubert cells of dimensions 0 mod 4 in the complex Grassmannian $G(n, k)$, and let $B_{n,k}$ be the number of cells of dimensions 2 mod 4. Specifically, $A_{n,k}$ is the number of n-tuples $(\mu_1, \ldots, \mu_n)$ satisfying

$$0 \leq \mu_1 \leq \cdots \leq \mu_n \leq k - n, \quad \sum \mu_i \text{ is even,}$$

and $B_{n,k}$ is the similar number for the case $\sum \mu_i$ is odd. Then the following computation is carried out in Section 6.

Theorem 1.4. *Suppose $S(k\rho_n \oplus j\epsilon)$ is the unit sphere of the representation $k\rho_n \oplus j\epsilon$, $k \geq n$.*

1. *If $k - n$ is even, then*

$$S_{U(n)}(S(k\rho_n \oplus j\epsilon)) = \mathbb{Z}^{\sum_{0 \leq 2i < n} A_{n-2i,k}} \oplus \mathbb{Z}_2^{\sum_{0 \leq 2i < n} B_{n-2i,k}},$$

   with the only exception that there is one less copy of $\mathbb{Z}$ in case n is odd and $j = 0$.

2. *If $k - n$ is odd, then*

$$S_{U(n)}(S(k\rho_n \oplus j\epsilon)) = \mathbb{Z}^{A_{n,k-1} + \sum_{0 < 2i-1 < n} A_{n-2i+1,k}} \oplus \mathbb{Z}_2^{B_{n,k-1} + \sum_{0 < 2i-1 < n} B_{n-2i+1,k}},$$

   with the exceptions that there is one less copy of $\mathbb{Z}$ in case n is even and $j = 0$, and there is one more copy of $\mathbb{Z}_2$ in case n is odd and $j > 0$.

The computation generalizes the classical computation of the fake complex projective spaces [Su, Theorem 9], [Wa, Section 14C]. We also note that J. Levine [L] proved that smooth biaxial $U(1)$-actions on homotopy spheres with homotopy spheres as fixed sets could be identified with the set of diffeomorphism classes of codimension 3 knots. Since such codimension 3 knots are topologically unknotted, all such actions are topologically standard, which suggests $S_{U(1)}(2\rho_1 \oplus j\epsilon) = 0$. However, our computation gives $S_{U(1)}(2\rho_1 \oplus j\epsilon) = \mathbb{Z}^{A_{1,1}} = \mathbb{Z}$. The extra $\mathbb{Z}$ takes into account the existence of semifree $U(1)$-actions on the sphere such that the fixed set is non-resolvable, i.e., has non-trivial "local signature" [Q3, Q4]. In fact, even for fake complex projective spaces, our computation contains an extra copy of $\mathbb{Z}$ for the same reason.

If N is isovariant simple homotopy equivalent to the representation sphere $S(k\rho_n \oplus j\epsilon)$, then joining with the representation sphere $S(\rho_n)$ yields a manifold $N * S(\rho_n)$ isovariant simple homotopy equivalent to the representation sphere $S((k+1)\rho_n \oplus j\epsilon)$. This gives the suspension map

$$*S(\rho_n)\colon S_{U(n)}(S(k\rho_n \oplus j\epsilon)) \to S_{U(n)}(S((k+1)\rho_n \oplus j\epsilon)).$$

A consequence of the calculation in Theorem 1.4 is the following, proved in Section 7.

Theorem 1.5. *The suspension map is injective.*

For $S^1 = U(1)$, this followed from the classical computation for fake complex projective spaces.

Finally, in Section 8, we extend all the results to the similarly defined multiaxial $Sp(n)$-manifolds.

For $k - n$ odd, the proofs of both Theorems 1.4 and 1.5 depend on ingenious detailed calculations by Jared Bass, presented in an appendix written by him, of the homology of certain orbit spaces. In contrast, the other cases depend on the classical calculations of the cohomology of Grassmannians in terms of Schubert cells.

Some of the developments in this paper are applicable to the study of general stratified spaces and equivariant manifolds, e.g., the discussion in Section 3 of the fundamental groups of links and strata. Likewise, the stratified surgery results of Section 4 can be more naturally viewed outside of the specifically group action setting.

The paper is organized as follows.

2. Strata of multiaxial $U(n)$-manifolds.
3. Homotopy properties of multiaxial $U(n)$-manifolds.
4. General decomposition theorems.
5. Structure sets of multiaxial actions.
6. Structure sets of multiaxial representation spheres.
7. Suspensions of multiaxial representation spheres.
8. Multiaxial $Sp(n)$-manifolds.
9. Appendix (by Jared Bass): Homology of quotients of multiaxial representation spheres.

2. Strata of multiaxial $U(n)$-manifolds

Smooth multiaxial manifolds were introduced and studied in [D, DH, DHM], following earlier works on biaxial actions [Br1, Br2, Br3, HH, J], and multiaxial manifolds with three orbit types [EH]. As noted in the introduction, our definition of multiaxial actions in the topological category is more flexible, the actions are not assumed to be locally linear, and the local model may vary at different parts of the manifold.

Let $U(n)$ be the unitary group of linear transformations of $\mathbb{C}^n$ preserving the Euclidean norm. By a *unitary subgroup*, we mean the subgroup of unitary transformations fixing a linear subspace of $\mathbb{C}^n$. If the fixed subspace has complex dimension $n - i$, then

the unitary subgroup is conjugate to the specific unitary subgroup $U(i)$ of $U(n)$ that fixes the last $n - i$ coordinates.

The *normalizer* of the specific unitary subgroup is $NU(i) = U(i) \times U(n-i)$, where by an abuse of notation, $U(n-i)$ is the unitary subgroup that fixes the first i coordinates. Then the quotient group $NU(i)/U(i)$ may be naturally identified with $U(n-i)$. It is usually clear from the context when $U(k)$ is the specific unitary subgroup (fixing the last $n - k$ coordinates) and when it is the quotient group (fixing the first $n - k$ coordinates).

A $U(n)$-manifold is multiaxial if any isotropy group is a unitary subgroup, and some additional topological conditions are satisfied. The manifold M is stratified by $M_{-i} = U(n)M^{U(i)}$, the set of points fixed by some conjugate of $U(i)$. Correspondingly, the orbit space $X = M/U(n)$ is stratified by $X_{-i} = M_{-i}/U(n)$. In Lemma 2.1, we will show that $X_{-i} = M^{U(i)}/U(n-i)$, a special feature of multiaxial manifolds.

Definition. A topological $U(n)$-manifold M is *multiaxial* if any isotropy group is a unitary subgroup, the fixed sets $M^{U(i)}$ are ANR homology submanifolds, the orbit space $M/U(n)$ is homotopically stratified, and the strata of $M/U(n)$ are 1-LC embedded.

A group action is 1-LC if the fundamental group of the homotopy link between any two strata is isomorphic to the group of components of the isotropy group (of the smaller stratum). For the action by the connected group $U(n)$, the 1-LC condition means that the homotopy links of lower strata in higher strata are all simply connected. By Proposition 3.4, we only need the simple connectedness for the links of the lower *pure* strata in higher *pure* strata. For $U(n)$-manifolds locally modeled on $k\rho_n \oplus j\epsilon$, the pure strata of the links are actually homotopy equivalent to complex Grassmannians and are therefore simply connected. Such actions are thus multiaxial.

When the fixed sets are submanifolds, the 1-LC condition is slightly stronger than the local flatness of the fixed sets. For example, the biaxial $O(2)$-action on the join of a circle and a homology sphere, with the action all taking place on the circle, has the fixed set of $O(1) = \mathbb{Z}_2$ being locally flat, but the corresponding orbit may not be 1-LC.

We shall see in Section 3 that this definition of multiaxial $U(n)$-manifolds implies strong local homotopical information: The (homotopy) links between adjacent fixed sets in M are homotopy spheres, and the pure strata of the links of $M^{U(i)}$ in M are connected and simply connected (with the exception that the link of $M^{U(1)}$ in M can be the circle in the 1-axial case). Quinn [Q2] showed that such homotopy properties imply that the pure stratum $M^{-i} = M_{-i} - M_{-i-1}$ is an open manifold that can be completed into a manifold with boundary $U(n) \times_{U(n-i)} (\bar{M}^{U(i)}, \partial\bar{M}^{U(i)})$, by deleting (the interior of) regular neighborhoods of lower strata. The theory of [BFMW] shows that the same is true in the setting of homology manifolds, up to s-cobordism. Moreover, by the 1-LC condition, the pure stratum $X^{-i} = X_{-i} - X_{-i-1}$ is a homology manifold, and can also be completed into a homology manifold with boundary $(\bar{X}^{-i}, \partial\bar{X}^{-i})$.

For a multiaxial $U(n)$-manifold M, the fixed set $M^{U(i)}$ is a multiaxial $U(n-i)$-manifold, where $U(n-i) = NU(i)/U(i)$ is the quotient group. The following is a kind of "hereditary property" for multiaxial manifolds.

Lemma 2.1. *If M is a multiaxial $U(n)$-manifold, then $M_{-i}/U(n) = M^{U(i)}/U(n-i)$.*

The lemma shows that, as far as the orbit space is concerned, the study of a stratum of a multiaxial manifold is the same as the study of a "smaller" multiaxial manifold. In particular, if a multiaxial $U(n)$-manifold M does not have free points, then the minimal isotropy groups are conjugate to $U(m)$ for some $m > 0$, and the study of the $U(n)$-manifold M is the same as the study of the multiaxial $U(n-m)$-manifold $M^{U(m)}$. Since the $U(n-m)$-action on $M^{U(m)}$ has free points, we may thus always assume the existence of free points without loss of generality. For multiaxial manifolds modeled on $k\rho_n \oplus j\epsilon$, this means that we may always assume $k \geq n$. We remark that $k \leq n$ was always assumed in the study of differentiable multiaxial actions [D, DH, DHM].

Lemma 2.1 is a consequence of the following two propositions.

Proposition 2.2. *If $H \subset K \subset G = U(n)$ are unitary subgroups, then the NH-action on $(G/K)^H$ is transitive. In other words, if $H \subset K$ and $g^{-1}Hg \subset K$, then $g = \nu k$ for some $\nu \in NH$ and $k \in K$.*

Proof. The subgroups K and H consist of the unitary transformations of $\mathbb{C}^n$ that respectively fix some subspaces V_K and V_H. Then $H \subset K$ means $V_K \subset V_H$, and $g^{-1}Hg \subset K$ means $gV_K \subset V_H$. Therefore there is a unitary transformation ν that preserves V_H and restricts to g on V_K. Then $\nu^{-1}g$ fixes V_K, whence $\nu^{-1}g \in K$. Moreover, the fact that ν preserves V_H means that $\nu \in NH$.

The transitivity of the NH-action on $(G/K)^H$ means that if $gK \in (G/K)^H$, then $gK = \nu K$ for some $\nu \in NH$. Since $gK \in (G/K)^H$ means $g^{-1}Hg \subset K$, and $gK = \nu K$ means $g = \nu k$ for some $k \in K$, we see that the transitivity is the same as the group-theoretic property above. $\square$

Proposition 2.3. *If G acts on a set M so that every pair of isotropy groups satisfy the property in Proposition 2.2, then $GM^H/G = M^H/NH$ for any isotropy group H.*

Proof. We always have the natural surjective map $M^H/NH \to GM^H/G$. Over a point in GM^H/G represented by $x \in M^H$, the fiber of the map is $(Gx)^H/NH$. Therefore the natural map is injective if and only if the fiber is a single point, which means that the action of NH on $(Gx)^H = (G/G_x)^H$ is transitive. $\square$

3. Homotopy properties of multiaxial $U(n)$-manifolds

Although our definition of multiaxial $U(n)$-manifold is more general than those in [D, DH, DHM] that are modeled on linear representations, many homotopy properties of the linear model are still preserved.

First we consider the (homotopy) link between adjacent strata of the orbit space $X = M/U(n)$ of a multiaxial $U(n)$-manifold M. By the link of X_{-j} in X_{-j+1}, we really mean the link of the pure stratum $X^{-j} = X_{-j} - X_{-j-1}$ in X_{-j+1} (same for the strata of M), and this link may be different along different connected components of X_{-j}. So for any $x \in X_{-j}$, we denote by X^x_{-j} the connected component of X_{-j} containing x. By the link of X^x_{-j} in X_{-j+1}, we really mean the link of $X^x_{-j} - X_{-j-1}$ in X_{-j+1}. We also denote by $M^{U(j),x}$ the corresponding connected component of $M^{U(j)}$, and hence $X^x_{-j} = M^{U(j),x}/U(n-j)$.

Lemma 3.1. *Suppose X is the orbit space of a multiaxial $U(n)$-manifold. Then for any $x \in X_{-i}$ and $1 \leq j \leq i$, the link of X^x_{-j} in X_{-j+1} is homotopy equivalent to $\mathbb{C}P^{r^x_j}$, and $r^x_j = r^x_{j-1} + 1$.*

The lemma paints the following picture of the strata of the links in a (connected) multiaxial $U(n)$-manifold. For any $x \in X^{-i}$, the stratification near x is given by

$$X^x_0 \supset X^x_{-1} \supset \cdots \supset X^x_{-i}, \qquad X^x_0 \text{ is a component of } X.$$

The *first gap* r^x_1 of x depends only on the connected component X^x_{-1} and determines the homotopy type $\mathbb{C}P^{r^x_1 + j - 1}$ of the link of X^x_{-j} in X_{-j+1}. Moreover, we have

$$\begin{aligned}
\dim M^{U(j-1),x} &- \dim M^{U(j),x} \\
&= \dim X^x_{-j+1} + \dim U(n-j+1) - \dim X^x_{-j} - \dim U(n-j) \\
&= \dim \mathbb{C}P^{r^x_1 + j - 1} + 1 + (n-j+1)^2 - (n-j)^2 = 2(r^x_1 + n).
\end{aligned}$$

The picture also shows that, near a point of M with isotropy group $gU(i)g^{-1}$, $gU(j)g^{-1}$ is the isotropy group of some nearby point for any j, $1 \leq j \leq i$.

If the multiaxial manifold is modeled on $k\rho_n \oplus j\epsilon$, then the first gap is independent of the connected component, and $r_1 = k - n$ in case $k \geq n$. On the other hand, multiaxial $U(1)$-manifolds are just semifree S^1-manifolds, for which any fixed point component has even codimension $2c$, and the first (and the only) gap of the component is $c - 1$.

Proof of Lemma 3.1. The link of X^x_{-j} in X_{-j+1} is the orbit space of the link of $M^{U(j)}$ in $M^{U(j-1)}$ by the free action of the quotient group $N_{U(j)}U(j-1)/U(j-1) = S^1$. The 1-LC condition implies that the link of $M^{U(j)}$ in $M^{U(j-1)}$ is homotopy equivalent to a sphere, and the orbit space of this homotopy sphere by the free S^1-action is simply connected. Therefore the orbit space is homotopy equivalent to a complex projective space $\mathbb{C}P^{r_j}$, where the superscript x is omitted from r_j.

Let $m_j = \dim M^{U(j),x}$ and $x_j = \dim X^x_{-j}$. From $X^x_{-j} = M^{U(j),x}/U(n-j)$, we have

$$x_j = \dim M^{U(j),x} - \dim U(n-j) = m_j - (n-j)^2.$$

Since the link of X^x_{-j} in X_{-j+1} is homotopy equivalent to $\mathbb{C}P^{r_j}$, we also have

$$x_{j-1} - x_j = 2r_j + 1.$$

Since all the isotropy groups are unitary subgroups, we know $M^{U(j)} = M^{T^j}$ for the maximal torus T^j of $U(j)$. Here T^j is the specific torus group acting by scalar multiplications on the first j coordinates of $\mathbb{C}^n$. Now we fix j and consider M as a T^j-manifold. By the multiaxial assumption, the isotropy groups of the T^j-manifold M are the tori that are in one-to-one correspondence with the choices of some coordinates from the first j coordinates of $\mathbb{C}^n$. The number j' of chosen coordinates is the rank of the isotropy torus. Since all the tori of the same rank j' are conjugate in $U(n)$ to the specific torus

group $T^{j'}$, their fixed point components containing $M^{U(j),x}$ have the same dimension, which is dim $M^{U(j'),x} = m_{j'}$.

For the case $j' = j - 1$ (corank 1 in T^j), there are j such isotropy tori. By a formula of Borel [Bo, Theorem XIII.4.3], we have

$$m_0 - m_j = j(m_{j-1} - m_j).$$

In terms of x_j, we have

$$x_0 + n^2 = j(x_{j-1} + (n - j + 1)^2) - (j - 1)(x_j + (n - j)^2),$$

or

$$(j - 1)^{-1}(x_{j-1} - x_0) - j^{-1}(x_j - x_0) = 1.$$

This gives $x_j - x_0 = j(a - j)$ and

$$x_{j-1} - x_j = 2j - 1 - a.$$

Combined with $x_{j-1} - x_j = 2r_j + 1$, this yields $r_j = r_{j-1} + 1$. $\square$

Next consider the links between any two (not necessarily adjacent) strata of a multiaxial manifold. The 1-LC assumption on the strata of $M/U(n)$ means (the connectedness and) the simple connectedness of the pure strata of the links in $M/U(n)$. We will need the following consequence of this 1-LC assumption.

Lemma 3.2. *Suppose X is a homotopically stratified space. If all pure strata of the links in X are connected and simply connected, then all strata of the links in X are also connected and simply connected. Moreover,*

$$\pi_1(X_{-i} - X_{-j}) = \pi_1(X_{-i}), \quad j > i,$$

and $\pi_1 X^{-i} = \pi_1 X_{-i}$ in particular.

Lemma 3.2 follows from Proposition 3.5 below. The proposition immediately implies the conclusion $\pi_1(X_{-i} - X_{-j}) = \pi_1(X_{-i})$. Then we note that the strata L_α of links in X are themselves homotopically stratified spaces, and the links in L_α are also links in X. Therefore we may apply the conclusion $\pi_1 X^{-i} = \pi_1 X_{-i}$ to L_α to prove the claim on the strata of links in the lemma.

The proof of Proposition 3.5 will be based on some well known general observations on the fundamental groups associated to homotopically stratified spaces. In a homotopically stratified space, the neighborhoods of strata are stratified systems of fibrations over the strata. The fundamental groups are related as follows.

Proposition 3.3. *Suppose $E \to X$ is a stratified system of fibrations over a homotopically stratified space X. If the fibers are non-empty and connected, then $\pi_1 E \to \pi_1 X$ is surjective. If the fibers are (non-empty and) connected and simply connected, then $\pi_1 E \to \pi_1 X$ is an isomorphism.*

Proof. If $E \to X$ is a genuine fibration, then the two claims follow from the exact sequence of homotopy groups associated to the fibration.

Inductively, we only need to consider $X = Z \cup_{\partial Z} Y$, where Y is the union of lower strata, Z is the complement of a regular neighborhood of Y, and ∂Z is the boundary of a regular neighborhood of Y as well as the boundary of Z. Correspondingly, we have $E = E_Z \cup_{E_{\partial Z}} E_Y$ where $E_Z \to Z$ is a fibration that restricts to the fibration $E_{\partial Z} \to \partial Z$, and $E_Y \to Y$ is a stratified system of fibrations. Then we consider the map

$$\pi_1 E = \pi_1 E_Z *_{\pi_1 E_{\partial Z}} \pi_1 E_Y \to \pi_1 X = \pi_1 Z *_{\pi_1 \partial Z} \pi_1 Y.$$

If the fibers of $E \to X$ are connected, then $\pi_1 E_Z \to \pi_1 Z$ and $\pi_1 E_{\partial Z} \to \pi_1 \partial Z$ are surjective by the genuine fibration case, and $\pi_1 E_Y \to \pi_1 Y$ is surjective by induction. Therefore the map $\pi_1 E \to \pi_1 X$ is surjective. If the fibers of $E \to Z$ are connected and simply connected, then all the maps are isomorphisms. Hence $\pi_1 E \to \pi_1 X$ is an isomorphism.

The proof makes use of van Kampen's theorem, which requires Y to be connected (which further implies that ∂Z is connected). In general, the argument can be carried out by successively adding connected components of Y to Z.   □

Proposition 3.4. *If X is a homotopically stratified space such that all pure strata are connected, and all links are not empty, then X is connected. Moreover, if all pure strata are connected and simply connected, and all links are connected, then X is simply connected.*

We remark that the link L of a stratum X_β in another stratum X_α is stratified, with strata L_γ corresponding to the strata X_γ satisfying $X_\beta \subsetneq X_\gamma \subset X_\alpha$. Moreover, the link of L_γ in $L_{\gamma'}$ is the same as the link of X_γ in $X_{\gamma'}$. The proposition implies that if the pure strata of the link between any two strata sandwiched between X_β and X_α are (non-empty and) connected and simply connected, then the link of X_β in X_α is simply connected.

Proof of Proposition 3.4. If the links are not empty, then any pure stratum is glued to higher pure strata. Therefore the connectedness of all pure strata implies the connectedness of the union, which is the whole space X.

Now assume that all pure strata are connected and simply connected, and all links are connected. Let Y be a minimum stratum. Then we have a decomposition $X = Z \cup_{\partial Z} Y$ similar to the one in the proof of Proposition 3.3. The complement Z of a regular neighborhood of Y is a stratified space, with the pure strata the same as the pure strata of X, except for the stratum Y. Moreover, the links in Z are the same as links in X. By induction, we may assume that Z (which has one less stratum than X) is simply connected. Moreover, Y is a pure stratum and is already assumed to be simply connected. If we know that ∂Z is connected, then we can apply van Kampen's theorem and conclude that $\pi_1 X = \pi_1 Z *_{\pi_1 \partial Z} \pi_1 Y$ is trivial.

To see that ∂Z is connected, we note that the base of the fibration $\partial Z \to Y$ is connected. So it is sufficient to show that the fiber L of the fibration is also connected. The fiber is the link L of Y in X, and is a stratified space with one less stratum than X. Moreover, the strata and links in L are links in X. Since all pure strata of links in X are connected, by the first part of the proposition, L is connected.   □

Proposition 3.5. *Suppose X is a homotopically stratified space, and Y is a closed union of strata of X. If for any link between strata of X, those pure strata of the link that are not contained in Y are connected and simply connected, then $\pi_1(X - Y) = \pi_1 X$.*

Proof. We have a decomposition $X = Z \cup_{\partial Z} Y$ similar to that in the proof of Proposition 3.3. The fiber of the stratified system of fibrations $\partial Z \to Y$ is a stratified space L_y depending on the location of the point $y \in Y$. If Y^y is the pure stratum containing y, then the pure strata of L_y are the pure strata of the link of Y^y in X that are not contained in Y. By Proposition 3.4 and the remark afterwards, the assumption of the proposition implies that L_y is connected and simply connected. Then we may apply Proposition 3.3 to get $\pi_1 \partial Z = \pi_1 Y$. Further application of van Kampen's theorem then yields $\pi_1 X = \pi_1 Z *_{\pi_1 \partial Z} \pi_1 Y = \pi_1 Z = \pi_1(X - Y)$. $\qquad\qquad\qquad\square$

4. General decomposition theorems

The homotopy properties of the last section will be used in producing a decomposition theorem for the structure sets of certain stratified spaces. We will use the spectra version of the surgery obstruction, homology and structure set. The equality of spectra really means homotopy equivalence.

Theorem 4.1. *Suppose $X = X_0 \supset X_{-1} \supset X_{-2} \supset \cdots$ is a homotopically stratified space satisfying the following properties:*

1. *The homotopy link of X_{-1} in X is homotopy equivalent to $\mathbb{C}P^r$ with r even.*
2. *The link fibration of X^{-1} in X is orientable, in the sense that the monodromy preserves the fundamental class of the fiber.*
3. *For any i, the top two pure strata of the link of X_{-i} in X are connected and simply connected.*

Then there is a natural homotopy equivalence of surgery obstructions

$$\mathbb{L}(X) = \mathbb{L}(X, \mathrm{rel}\, X_{-2}) \oplus \mathbb{L}(X_{-2}).$$

Moreover,

$$\mathbb{L}(X, \mathrm{rel}\, X_{-2}) = \mathbb{L}(\pi_1 X, \pi_1 X_{-1}),$$

and

$$\pi_1 X = \pi_1(X - X_{-1}) = \pi_1 \bar{X}^0, \quad \pi_1 X_{-1} = \pi_1 X^{-1} = \pi_1 \partial \bar{X}^0.$$

The surgery obstruction spectrum $\mathbb{L}$ for homotopically stratified space X is of the Browder–Quinn type [Q1]. Strictly speaking, the surgery obstruction of Browder–Quinn type is only defined for stratified spaces where the neighborhoods of strata have bundle structure. In the settings of this paper, the 1-LC assumption and the topological h-cobordism theory [Q2, St] implies that the neighborhoods of strata have block bundle structure. Thus the surgery obstruction is defined. In Theorem 4.1, the third condition is the 1-LC assumption on the top two pure strata, from which the surgery obstruction spectra in the theorem is also defined.

To prove the theorem, we first establish the following result, which is essentially a reformulation of the periodicity for the classical surgery obstruction [Wa, Theorem 9.9] (geometrically reformulated in a relevant bundle setting in [CW1]).

Proposition 4.2. *Suppose X is a two-strata space such that the link fibration of X_{-1} in X is an orientable fibration with fiber homotopy equivalent to $\mathbb{C}P^r$ with even r. Then*

$$\mathbb{L}(X) = \mathbb{L}(\pi_1 X, \pi_1 X_{-1}), \quad \pi_1 X = \pi_1(X - X_{-1}).$$

Proof. Let Z be the complement of a regular neighborhood of X_{-1} in X. Let E be the boundary of Z as well as the boundary of the regular neighborhood. Then $X = Z \cup E \times [0, 1] \cup X_{-1}$, and $E \to X_{-1}$ is an orientable fibration with fiber homotopy equivalent to $\mathbb{C}P^r$.

The surgery obstruction $\mathbb{L}(X)$ of the two-strata space X fits into a fibration

$$\mathbb{L}(E \times [0, 1] \cup X_{-1}) \to \mathbb{L}(X) \to \mathbb{L}(Z, E),$$

where the mapping cylinder $E \times [0, 1] \cup X_{-1}$ is a regular neighborhood of X_{-1} in X and is a two-strata space with X_{-1} as the lower stratum. The surgery obstruction of the mapping cylinder further fits into a fibration

$$\mathbb{L}(E \times [0, 1] \cup X_{-1}) \xrightarrow{\text{res}} \mathbb{L}(X_{-1}) \xrightarrow{\text{trf}} \mathbb{L}(E),$$

given by the restriction and the transfer.

Since the fibration $E \to X_{-1}$ is orientable, and the fiber $\mathbb{C}P^r$ is simply connected, the surgery obstruction groups $\pi_* \mathbb{L}(X_{-1})$ and $\pi_* \mathbb{L}(E)$ are described in terms of the same quadratic forms (and formations). Moreover, the effect of the transfer map on surgery obstructions can be computed by the up-down formula of [LR, Theorem 2.1]. Specifically, the transfer of surgery obstructions is obtained by tensoring with the usual $\pi_1 X_{-1}$-equivariant intersection form on the middle homology $H_r \mathbb{C}P^r = \mathbb{Z}$, where the $\pi_1 X_{-1}$-module structure on $\mathbb{Z}$ comes from the monodromy. Since this tensoring operation induces an isomorphism on the surgery obstruction groups, we conclude that the transfer map is a homotopy equivalence.

We remark that our notion of orientability, as given by the second condition in Theorem 4.1, is weaker than the one in [LR]. Therefore Corollary 2.2 of [LR] cannot be directly applied.

Since the transfer map is a homotopy equivalence, the second fibration implies that $\mathbb{L}(E \times [0, 1] \cup X_{-1})$ is contractible. Then the first fibration further implies that $\mathbb{L}(X)$ and $\mathbb{L}(Z, E)$ are homotopy equivalent.

It remains to compute $\mathbb{L}(Z, E)$. The fibration $\mathbb{C}P^r \to E \to X_{-1}$ implies that $\pi_1 E = \pi_1 X_{-1}$. By van Kampen's theorem, $\pi_1 X = \pi_1 Z *_{\pi_1 E} \pi_1 X_{-1} = \pi_1 Z = \pi_1(X - X_{-1})$. ∎

Proof of Theorem 4.1. Let Z be the complement of a regular neighborhood of X_{-2} in X. Let E be the boundary of the regular neighborhood. Then Z and E are two-strata spaces with lower strata $Z_{-1} = Z \cap X_{-1}$ and $E_{-1} = E \cap X_{-1}$. Moreover, E is the boundary
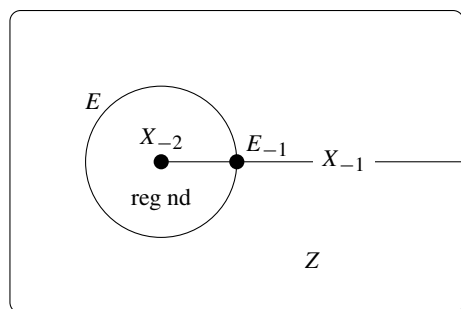
Fig. 1. Regular neighborhood of X_{-2} in X.

of Z in the sense that E has a collar neighborhood in Z. We will use Z and E to denote the two-strata spaces, and use (Z, E) to denote the space Z considered as a four-strata space, in which the two-strata of E are also counted.

Consider the commutative diagram of natural maps of surgery obstructions

$$\begin{array}{ccccc}
\mathbb{L}(Z) & \xrightarrow{\simeq} & \mathbb{L}(Z, E) & \longrightarrow & \mathbb{L}(E) \\
\downarrow{\simeq} & & \uparrow & & \\
\mathbb{L}(X, \text{rel } X_{-2}) & \longrightarrow & \mathbb{L}(X) & \longrightarrow & \mathbb{L}(X_{-2})
\end{array}$$

Both horizontal lines are fibrations of spectra. The vertical homotopy equivalence $\simeq$ is due to the fact that the inclusion $Z \to X - X_{-2}$ of two-strata spaces is a stratified homotopy equivalence. The horizontal homotopy equivalence $\simeq$ will be a consequence of the fact that $\mathbb{L}(E)$ is homotopically trivial. The two equivalences together give natural splitting to the map $\mathbb{L}(X, \text{rel } X_{-2}) \to \mathbb{L}(X)$. Then the bottom fibration implies $\mathbb{L}(X)$ is naturally homotopy equivalent to $\mathbb{L}(X, \text{rel } X_{-2}) \oplus \mathbb{L}(X_{-2})$.

To see the triviality of $\mathbb{L}(E)$, we note that the link of E_{-1} in E is the same as the link $\mathbb{C}P^r$ of X_{-1} in X. Therefore we may apply Proposition 4.2 to E and get

$$\mathbb{L}(E) = \mathbb{L}(\pi_1(E - E_{-1}), \pi_1 E_{-1}).$$

Let L be the link of X_{-i} in X. Then we have stratified systems of fibrations

$$L^0 = L - L_{-1} \to E - E_{-1} \to X_{-2}, \quad L^{-1} = L_{-1} - L_{-2} \to E_{-1} \to X_{-2}.$$

By the third condition, the fibers are always connected and simply connected, and we may apply Proposition 3.3 to get $\pi_1(E - E_{-1}) = \pi_1 E_{-1} = \pi_1 X_{-2}$. By the π-π theorem of classical surgery theory [Wa, Theorem 3.3], we conclude that $\mathbb{L}(E)$ is homotopically trivial.

Like E, the link of Z_{-1} in Z is also the same as the link $\mathbb{C}P^r$ of X_{-1} in X. Then Proposition 4.2 tells us

$$\mathbb{L}(X, \text{rel } X_{-2}) = \mathbb{L}(Z) = \mathbb{L}(\pi_1 Z, \pi_1 Z_{-1}).$$

From $Z \simeq X - X_{-2}$, $Z_{-1} \simeq X_{-1} - X_{-2} = X^{-1}$ and Lemma 3.2, we have

$$\pi_1 Z = \pi_1(X - X_{-2}) = \pi_1(X - X_{-1}) = \pi_1 X, \quad \pi_1 Z_{-1} = \pi_1 X^{-1} = \pi_1 X_{-1}.$$

From $X - X_{-1} \simeq \bar{X}^0$ and applying Proposition 3.3 to $\partial \bar{X}^0 \to X_{-1}$, which is a stratified system of fibrations with the top strata of the link of X_{-i} in X as fibers, we get

$$\pi_1 Z = \pi_1 \bar{X}^0, \quad \pi_1 Z_{-1} = \pi_1 \partial \bar{X}^0. \qquad \qquad \square$$

The natural splitting for the surgery obstruction in Theorem 4.1 induces a similar natural splitting for the structure set.

Theorem 4.3. *Suppose $X = X_0 \supset X_{-1} \supset X_{-2} \supset \cdots$ is a homotopically stratified space satisfying the following properties:*

1. *The homotopy link of X_{-1} in X is homotopy equivalent to $\mathbb{C}P^r$ with even r.*
2. *The link fibration of X^{-1} in X is orientable as in Theorem 4.1.*
3. *The pure strata of all links are connected and simply connected.*

Then there is a natural homotopy equivalence of structure sets

$$\mathbb{S}(X) = \mathbb{S}(X, \text{rel } X_{-2}) \oplus \mathbb{S}(X_{-2}).$$

Moreover,
$$\mathbb{S}(X, \text{rel } X_{-2}) = \mathbb{S}(\bar{X}^0, \partial \bar{X}^0) = \mathbb{S}^{\text{alg}}(X, X_{-1}).$$

As explained after Theorem 4.1, the third condition implies that the neighborhoods of strata have block bundle structure. The main result of [We] is a surgery theory for such homotopically stratified spaces (and the extension to the case when the pure strata of links may not be simply connected, where the topological K-theory of Quinn [Q2, Q5] and Steinberger [St] enters). The structure set $S(X)$ is the set of homotopically stratified spaces stratified simple homotopy equivalent to X up to s-cobordism, and fits into an exact sequence of abelian groups

$$\cdots \to L(X \times [0, 1], \text{rel } X \times \{0, 1\}) \to S(X) \to H(X; \mathbb{L}(\text{loc } X)) \to L(X).$$

The notation here differs slightly from that of [We], but we think the above is clearer. The homology group is a cosheaf homology group: The coefficient associates the surgery obstruction spectrum $\mathbb{L}(U, \text{rel } \infty) = \lim_{\text{compact } K \subset U} \mathbb{L}(U, \text{rel } U - K)$ to each open subset U. Thus for an n-dimensional orientable manifold X, the homology term is $H_n(X; L(e))$ because $L(\mathbb{R}^n)$ is just an $L(e)$ with a dimension shift. For not necessarily orientable X, the homology should be twisted in exactly the same way that the local identifications of $L(\text{loc } X)$ with $L(e)$ are. This twisted homology is isomorphic to $[X, \mathbb{Z} \times G/Top]$.

We will also use the modification of the above sequence when we work relative to a union Y of closed strata. We have an exact sequence

$$\cdots \to L(X \times [0, 1], \text{rel } Y \times [0, 1] \cup X \times \{0, 1\})$$
$$\to S(X, \text{rel } Y) \to H(X; \mathbb{L}(\text{loc } X, \text{rel } Y)) \to L(X, \text{rel } Y).$$

The cosheaf $\mathbb{L}(\mathrm{loc}\,X, \mathrm{rel}\,Y)$ assigns the surgery obstruction spectrum $\mathbb{L}(U, \mathrm{rel}\,U \cap Y$ and $\infty)$ to each open subset U. As a special case, when $X - Y$ is a manifold (i.e., Y contains all the singularities) and Y is 1-LC (i.e., has simply connected homotopy link), then we obtain the identification of the relative stratified structure set with the fiber of the assembly map

$$S(X, \mathrm{rel}\,Y) \cong S^{\mathrm{alg}}(X).$$

It is interesting to note in this formula that the left hand side is a geometrically defined object that has very few a priori defined functorialities, but the right hand side naturally has a rich algebraic and functorial nature. Such serendipities underlie the calculations in this paper.

Proof of Theorem 4.3. We will use the spectrum version of the stratified surgery theory outlined above. In other words, we have a surgery fibration

$$\mathbb{S}(X) \to \mathbb{H}(X; \mathbb{L}(\mathrm{loc}\,X)) \to \mathbb{L}(X).$$

By Theorem 4.1, we have a natural splitting of the surgery spectra

$$\mathbb{L}(X) = \mathbb{L}(X, \mathrm{rel}\,X_{-2}) \oplus \mathbb{L}(X_{-2}) = \mathbb{L}(\pi_1 X, \pi_1 X_{-1}) \oplus \mathbb{L}(X_{-2}).$$

Since the splitting is natural, it can be applied to the coefficient $\mathbb{L}(\mathrm{loc}\,X)$ in the homology and induces compatible assembly maps

$$\mathbb{H}(X; \mathbb{L}(\mathrm{loc}(X, \mathrm{rel}\,X_{-2}))) \to \mathbb{L}(X, \mathrm{rel}\,X_{-2}), \quad \mathbb{H}(X; \mathbb{L}(\mathrm{loc}\,X_{-2})) \to \mathbb{L}(X_{-2}).$$

The stratified surgery theory tells us that the homotopy fiber of the first assembly map is the structure set $\mathbb{S}(X, \mathrm{rel}\,X_{-2})$. Moreover, we have $\mathbb{H}(X; \mathbb{L}(\mathrm{loc}\,X_{-2})) = \mathbb{H}(X_{-2}; \mathbb{L}(\mathrm{loc}\,X_{-2}))$ because the coefficient spectrum $\mathbb{L}(\mathrm{loc}\,X_{-2})$ is concentrated on X_{-2}. Therefore the homotopy fiber of the second assembly map is the structure set $\mathbb{S}(X_{-2})$. Then we have the decomposition of $\mathbb{S}(X)$ as stated in the theorem.

It remains to compute $\mathbb{S}(X, \mathrm{rel}\,X_{-2})$. The coefficient $\mathbb{L}(\mathrm{loc}(X, \mathrm{rel}\,X_{-2}))$ of the homology depends on the location.

1. At $x \in X^0 = X - X_{-1}$, the coefficient is $\mathbb{L}(D^p) = \mathbb{L}(e)$, where D^p is a ball neighborhood of x in the manifold pure stratum X^0.
2. At $x \in X^{-1}$, the coefficient is $\mathbb{L}(c\mathbb{C}P^r \times D^p)$, where $c\mathbb{C}P^r$ is the cone on the link of X_{-1} in X, and D^p is a ball neighborhood of x in the manifold pure stratum X^{-1}. Since r is even, the surgery obstruction $\mathbb{L}(c\mathbb{C}P^r \times D^p)$ is contractible by Proposition 4.2.
3. At $x \in X_{-2}$, we have $x \in X^{-i}$ for some $i \geq 2$. Let L be the link of X_{-i} in X, and let D^p be a ball neighborhood of x in the manifold pure stratum X^{-i}. Then the coefficient is

$$\begin{aligned}
\mathbb{L}(cL \times D^p, \mathrm{rel}\,cL_{-2} \times D^p) &= \mathbb{L}(cL \times D^p - c \times D^p, \mathrm{rel}\,cL_{-2} \times D^p - c \times D^p) \\
&= \mathbb{L}(L \times [0,1] \times D^p, \mathrm{rel}\,L_{-2} \times [0,1] \times D^p) \\
&= \Omega^{p+1} \mathbb{L}(L, \mathrm{rel}\,L_{-2}).
\end{aligned}$$

We may apply Theorem 4.1 to get $\mathbb{L}(L, \mathrm{rel}\,L_{-2}) = \mathbb{L}(\pi_1 L^0, \pi_1 L^{-1})$. By the third condition, the pure strata L^0 and L^{-1} are connected and simply connected. Therefore the surgery obstruction spectrum is contractible.

Thus the coefficient is the surgery obstruction spectrum $\mathbb{L} = \mathbb{L}(e)$ on the top pure stratum $X^0 = X - X_{-1}$ and is trivial on X_{-1}. Therefore the homology is

$$\mathbb{H}(X; \mathbb{L}(\mathrm{loc}(X, \mathrm{rel}\, X_{-2}))) = \mathbb{H}(X, X_{-1}; \mathbb{L}).$$

Moreover, Theorem 4.1 tells us

$$\mathbb{L}(X, \mathrm{rel}\, X_{-2}) = \mathbb{L}(\pi_1 X, \pi_1 X_{-1}).$$

Therefore the homotopy fiber of the assembly map is $\mathbb{S}^{\mathrm{alg}}(X, X_{-1})$.

By excision, we have $\mathbb{H}(X, X_{-1}; \mathbb{L}) = \mathbb{H}(\bar{X}^0, \partial \bar{X}^0; \mathbb{L})$. By Theorem 4.1, we also know $\mathbb{L}(\pi_1 X, \pi_1 X_{-1}) = \mathbb{L}(\pi_1 \bar{X}^0, \pi_1 \partial \bar{X}^0)$. Therefore the homotopy fiber of the assembly map is also the structure spectrum $\mathbb{S}(\bar{X}^0, \partial \bar{X}^0)$ of the manifold $(\bar{X}^0, \partial \bar{X}^0)$. $\square$

We note that, in the setup of Theorem 4.3, the restriction to X_{-2} factors through X_{-1}. Then the fact that the restriction $\mathbb{S}(X) \to \mathbb{S}(X_{-2})$ is naturally split surjective implies that the restriction $\mathbb{S}(X_{-1}) \to \mathbb{S}(X_{-2})$ is also naturally split surjective, and we get

$$\mathbb{S}(X_{-1}) = \mathbb{S}(X_{-1}, \mathrm{rel}\, X_{-2}) \oplus \mathbb{S}(X_{-2}).$$

Another way of looking at this is that if a stratified space X is the singular part of a stratified space Y satisfying the conditions of Theorem 4.3, i.e., $X = Y_{-1}$, then we have the natural splitting

$$\mathbb{S}(X) = \mathbb{S}(X, \mathrm{rel}\, X_{-1}) \oplus \mathbb{S}(X_{-1}).$$

The following computes $\mathbb{S}(X, \mathrm{rel}\, X_{-1})$ for the case relevant to multiaxial manifolds.

Theorem 4.4. *Suppose $X = X_0 \supset X_{-1} \supset X_{-2} \supset \cdots$ is a homotopically stratified space such that for any i, the top pure stratum of the link of X_{-i} in X is connected and simply connected. Then*

$$\mathbb{S}(X, \mathrm{rel}\, X_{-1}) = \mathbb{S}^{\mathrm{alg}}(X).$$

Proof. Similar to the proof of Theorem 4.3, the simple connectedness assumption implies that the structure set $\mathbb{S}(X, \mathrm{rel}\, X_{-1})$ is the homotopy fiber of the assembly map

$$\mathbb{H}(X; \mathbb{L}(\mathrm{loc}(X, \mathrm{rel}\, X_{-1}))) \to \mathbb{L}(X, \mathrm{rel}\, X_{-1}),$$

and the coefficient $\mathbb{L}(\mathrm{loc}(X, \mathrm{rel}\, X_{-1})) = \mathbb{L}$. We also get $\pi_1(X - X_{-1}) = \pi_1 X$ from Proposition 3.5. Therefore the assembly map is $\mathbb{H}(X; \mathbb{L}) \to \mathbb{L}(\pi_1 X)$, and its homotopy fiber is $\mathbb{S}^{\mathrm{alg}}(X)$. $\square$

5. Structure sets of nultiaxial actions

Let M be a multiaxial $U(n)$-manifold. By Lemma 3.2, the pure strata of links in the orbit space $X = M/U(n)$ are all connected and simply connected. To apply the theorems of Section 4 to X, we also need to know the orientability of the link fibration. Since

the monodromy map on the fiber $\mathbb{C}P^r$ comes from the S^1-equivariant monodromy map on the homotopy link sphere between the adjacent strata, the monodromy map must be homotopic to the identity. Therefore the link fibration has trivial monodromy and is, in particular, orientable.

Recall the concept of the first gap defined after the statement of Lemma 3.1. The number $r = r_1^x$ depends only on the connected component of the singular part X_{-1}. For any connected component X_{-1}^x, the number is characterized as the link of X_{-1}^x in X being homotopy equivalent to $\mathbb{C}P^r$. This number is also characterized by the equality $\dim M^{U(j-1),x} - \dim M^{U(j),x} = 2(r+n)$.

It is easy to see that Theorem 4.3 remains true in case X_{-1} has several connected components, and perhaps with different $\mathbb{C}P^r$ for different components, as long as all r are even. Therefore if all the first gaps of a multiaxial $U(n)$-manifold M are even, then we have a natural splitting

$$\mathbb{S}_{U(n)}(M) = \mathbb{S}_{U(n)}(M, \operatorname{rel} M_{-2}) \oplus \mathbb{S}_{U(n)}(M_{-2}).$$

By the computation in Theorem 4.3, we have

$$\mathbb{S}_{U(n)}(M, \operatorname{rel} M_{-2}) = \mathbb{S}(\bar{X}^0, \partial \bar{X}^0) = \mathbb{S}^{\mathrm{alg}}(X, X_{-1}).$$

By deleting an equivariant regular neighborhood of $M_{-1} = U(n)M^{U(1)}$ from M, we get a free $U(n)$-manifold with boundary $(\bar{M}^0, \partial \bar{M}^0)$, and

$$\mathbb{S}(\bar{X}^0, \partial \bar{X}^0) = \mathbb{S}_{U(n)}(\bar{M}^0, \partial \bar{M}^0).$$

On the other hand, by Lemma 2.1, we have $\mathbb{S}_{U(n)}(M_{-2}) = \mathbb{S}_{U(n-2)}(M^{U(2)})$, where $M^{U(2)}$ is a multiaxial $U(n-2)$-manifold. Moreover, Lemma 3.1 further tells us that, for $x \in M^{U(i)}$, $i > 2$, the first gap of x in $M^{U(2)}$ is $r_3^x = r_1^x + 2$, where r_1^x is the first gap of x in M. This can also be seen from

$$\dim (M^{U(2)})^{U(j-1),x} - \dim (M^{U(2)})^{U(j),x} = \dim M^{U(j+1),x} - \dim M^{U(j+2),x}$$
$$= 2(r_1^x + n) = 2(r_3^x + (n-2)),$$

where we use $n-2$ on the right because $M^{U(2)}$ is a multiaxial $U(n-2)$-manifold. The upshot of this is that all the first gaps of $M^{U(2)}$ remain even, and we have a further natural splitting

$$\mathbb{S}_{U(n)}(M_{-2}) = \mathbb{S}_{U(n-2)}(M^{U(2)}) = \mathbb{S}_{U(n-2)}(M^{U(2)}, \operatorname{rel} M_{-2}^{U(2)}) \oplus \mathbb{S}_{U(n-2)}(M_{-2}^{U(2)}).$$

Moreover, we have

$$\mathbb{S}_{U(n-2)}(M^{U(2)}, \operatorname{rel} M_{-2}^{U(2)}) = \mathbb{S}_{U(n-2)}(\bar{M}^{U(2)}, \partial \bar{M}^{U(2)}) = \mathbb{S}^{\mathrm{alg}}(X_{-2}, X_{-3}),$$

and

$$\mathbb{S}_{U(n-2)}(M_{-2}^{U(2)}) = \mathbb{S}_{U(n-4)}(M^{U(4)}).$$

The splitting continues and gives us the general version of part 1 of Theorem 1.1 in the introduction. The mod 4 condition on the codimensions is a rephrasing of the even first gap.

Theorem 5.1. *Suppose M is a multiaxial $U(n)$-manifold such that the connected components of $M^{U(1)}$ have codimensions $2n$ mod 4. Then we have a natural splitting*

$$\mathbb{S}_{U(n)}(M) = \bigoplus_{i \geq 0} \mathbb{S}_{U(n-2i)}(\bar{M}^{U(2i)}, \partial \bar{M}^{U(2i)}) = \bigoplus_{i \geq 0} \mathbb{S}^{\mathrm{alg}}(X_{-2i}, X_{-2i-1}).$$

In general, a multiaxial manifold may have even as well as odd first gaps. Denote by $M_{\mathrm{even}}^{U(1)}$ the union of the connected components of $M^{U(1)}$ of dimension dim $M - 2n$ mod 4. Denote by $M_{\mathrm{odd}}^{U(1)}$ the union of the connected components of $M^{U(1)}$ of dimension dim $M - 2(n + 1)$ mod 4. Then we have

$$M^{U(i)} = M_{\mathrm{even}}^{U(i)} \cup M_{\mathrm{odd}}^{U(i)}, \quad M_{\mathrm{even}}^{U(i)} = M^{U(i)} \cap M_{\mathrm{even}}^{U(1)}, \quad M_{\mathrm{odd}}^{U(i)} = M^{U(i)} \cap M_{\mathrm{odd}}^{U(1)},$$

so that the components in $M_{\mathrm{even}}^{U(i)}$ have even first gaps, and the components in $M_{\mathrm{odd}}^{U(i)}$ have odd first gaps. This leads to

$$M_{-i,\mathrm{even}} = U(n) M_{\mathrm{even}}^{U(i)}, \quad M_{-i,\mathrm{odd}} = U(n) M_{\mathrm{odd}}^{U(i)}.$$

We also have the corresponding decompositions

$$X_{-i} = X_{-i,\mathrm{even}} \cup X_{-i,\mathrm{odd}}, \quad \bar{M}^{U(i)} = \bar{M}_{\mathrm{even}}^{U(i)} \cup \bar{M}_{\mathrm{odd}}^{U(i)}.$$

By the same proof as Theorem 5.1, we get the same natural splitting for those with even first gaps

$$\mathbb{S}_{U(n)}(M) = \mathbb{S}^{\mathrm{alg}}(X, \mathrm{rel}\, X_{-2,\mathrm{even}}) \oplus \mathbb{S}^{\mathrm{alg}}(X_{-2,\mathrm{even}}).$$

Here the multiaxial $U(n - 2)$-manifold $M_{\mathrm{even}}^{U(2)}$ satisfies the condition of Theorem 5.1. Hence the second summand can be further split,

$$\mathbb{S}^{\mathrm{alg}}(X_{-2,\mathrm{even}}) = \bigoplus_{i \geq 1} \mathbb{S}^{\mathrm{alg}}(X_{-2i,\mathrm{even}}, X_{-2i-1,\mathrm{even}}).$$

In terms of the multiaxial manifold, this splitting is

$$\mathbb{S}_{U(n-2)}(M_{\mathrm{even}}^{U(2)}) = \bigoplus_{i \geq 1} \mathbb{S}_{U(n-2i)}(\bar{M}_{\mathrm{even}}^{U(2i)}, \partial \bar{M}_{\mathrm{even}}^{U(2i)}).$$

On the other hand, the first summand is

$$\mathbb{S}^{\mathrm{alg}}(X, \mathrm{rel}\, X_{-2,\mathrm{even}}) = \mathbb{S}_{U(n)}(M, \mathrm{rel}\, M_{-2,\mathrm{even}}).$$

Let N_{even} and N_{odd} be equivariant neighborhoods of $M_{-1,\mathrm{even}}$ and $M_{-1,\mathrm{odd}}$. Then by the same proof as Theorem 5.1, we have

$$\mathbb{S}_{U(n)}(M, \mathrm{rel}\, M_{-2,\mathrm{even}}) = \mathbb{S}_{U(n)}(\overline{M - N_{\mathrm{even}}}, \partial N_{\mathrm{even}}).$$

Combining everything, we get the following decomposition.

Theorem 5.2. *Suppose M is a multiaxial $U(n)$-manifold. Then we have a natural splitting*

$$\mathbb{S}_{U(n)}(M) = \mathbb{S}^{\mathrm{alg}}(X, \mathrm{rel}\, X_{-2,\mathrm{even}}) \oplus \bigoplus_{i \geq 1} \mathbb{S}^{\mathrm{alg}}(X_{-2i,\mathrm{even}}, X_{-2i-1,\mathrm{even}}).$$

Moreover,

$$\mathbb{S}^{\mathrm{alg}}(X, \mathrm{rel}\, X_{-2,\mathrm{even}}) = \mathbb{S}_{U(n)}(\overline{M - N_{\mathrm{even}}}, \partial N_{\mathrm{even}}),$$

$$\mathbb{S}^{\mathrm{alg}}(X_{-2i,\mathrm{even}}, X_{-2i-1,\mathrm{even}}) = \mathbb{S}_{U(n-2i)}(\bar{M}_{\mathrm{even}}^{U(2i)}, \partial \bar{M}_{\mathrm{even}}^{U(2i)}).$$

In the theorem above, $U(n-2i)$ acts freely on $\bar{M}_{\mathrm{even}}^{U(2i)}$, and the structure set is about the ordinary manifold $\bar{M}_{\mathrm{even}}^{U(2i)}/U(n-2i)$.

In the first summand $\mathbb{S}_{U(n)}(\overline{M - N_{\mathrm{even}}}, \partial N_{\mathrm{even}})$, all the gaps in the multiaxial $U(n)$-manifold $\overline{M - N_{\mathrm{even}}}$ are odd. This leads to the study of multiaxial $U(n)$-manifolds M such that all first gaps are odd. We may use the idea presented before Theorem 4.4. Suppose $M = W^{U(1)}$ for a multiaxial $U(n+1)$-manifold W. Let $Y = W/U(n+1)$ be the orbit space of W. Then $X_{-i} = Y_{-i-1}$. By Lemma 3.1, for any $x \in X_{-1} = Y_{-2}$, the first gap of x in Y is one less than the first gap of x in X. Therefore the first gap of x in Y is even, and the natural splitting of $\mathbb{S}(Y)$ induces the natural splitting

$$\mathbb{S}(X) = \mathbb{S}(X, \mathrm{rel}\, X_{-1}) \oplus \mathbb{S}(X_{-1}).$$

Since the first gap in the $U(n-1)$-manifold $M^{U(1)}$ is one more than the first gap in M and is therefore also even, we may apply Theorem 5.1 to get a further natural splitting

$$\mathbb{S}(X_{-1}) = \bigoplus_{i \geq 1} \mathbb{S}^{\mathrm{alg}}(X_{-2i+1}, X_{-2i}).$$

On the other hand, by the computation in Theorem 4.4, the first summand is

$$\mathbb{S}(X, \mathrm{rel}\, X_{-1}) = \mathbb{S}^{\mathrm{alg}}(X).$$

Then we get the general version of part 2 of Theorem 1.1 in the introduction.

Theorem 5.3. *Suppose M is a multiaxial $U(n)$-manifold such that the connected components of $M^{U(1)}$ have codimensions $2(n+1) \bmod 4$. If $M = W^{U(1)}$ for a multiaxial $U(n+1)$-manifold W, then we have a natural splitting*

$$\mathbb{S}_{U(n)}(M) = \mathbb{S}^{\mathrm{alg}}(X) \oplus \bigoplus_{i \geq 1} \mathbb{S}^{\mathrm{alg}}(X_{-2i+1}, X_{-2i}).$$

Moreover,

$$\mathbb{S}^{\mathrm{alg}}(X_{-2i+1}, X_{-2i}) = \mathbb{S}_{U(n-2i+1)}(\bar{M}^{U(2i-1)}, \partial \bar{M}^{U(2i-1)}).$$

We remark that if $M = W^{U(1)}$ and M is connected, then there is only one first gap r in M, uniquely determined by

$$\dim W - \dim M = 2(r + n + 1).$$

In case r is odd, there is actually no $M_{\mathrm{even}}^{U(1)}$.

Theorem 1.2 in the introduction gives another case where $\mathbb{S}(X_{-1})$ splits off from $\mathbb{S}(X)$ under the assumption that all the first gaps are odd (but not necessarily equal). The theorem deals with semifree S^1-manifolds, which are the same as multiaxial $U(1)$-manifolds.

Proof of Theorem 1.2. The codimensions of $M_0^{S^1}$ and $M_2^{S^1}$ mean that their first gaps are respectively odd and even. Let N_0 and N_2 be their respective equivariant neighborhoods. Then Theorem 5.2 implies

$$\mathbb{S}_{S^1}(M) = \mathbb{S}_{S^1}(\overline{M - N_2}, \partial N_2).$$

Next we want to split off the structure set of the fixed point set, $\overline{M - N_2}^{S^1} = M_0^{S^1}$.

As argued at the beginning of this section, the link fibration of $M_0^{S^1}/S^1 = M_0^{S^1}$ in M/S^1 has trivial monodromy. Moreover, the fiber of this link fibration is homotopy equivalent to $\mathbb{C}P^r$ for odd (first gap) r. By [LR, Mo], crossing with $\mathbb{C}P^r$ kills the surgery obstruction. Then the homotopy replacement argument in [CW2, CWY] can be applied to show that the natural map

$$\mathbb{S}_{S^1}(\overline{M - N_2}, \partial N_2) \to \mathbb{S}_{S^1}(\overline{M - N_2}^{S^1}) = \mathbb{S}_{S^1}(M_0^{S^1})$$

is split surjective. Since $\mathbb{S}_{S^1}(\overline{M - N_2}, \partial N_2, \text{rel } \partial N_0)$ is the kernel of the natural map, the theorem is proved. $\square$

6. Structure sets of multiaxial representation spheres

Let ρ_n be the defining representation of $U(n)$. Let ϵ be the real 1-dimensional trivial representation. Then for any integers $k > 0$ and $j \geq 0$, the unit sphere

$$M = S(k\rho_n \oplus j\epsilon) = S(k\rho_n) * S^{j-1}$$

of the representation $k\rho_n \oplus j\epsilon$ is a multiaxial $U(n)$-manifold. In this section, we compute the structure set of this representation sphere.

If $k < n$, then $M = U(n)S(k\rho_k \oplus j\epsilon)$, and the problem is reduced to the $U(k)$-representation sphere $S(k\rho_k \oplus j\epsilon)$. Without loss of generality, therefore, we will always assume $k \geq n$ in the subsequent discussion.

The fixed point subsets are

$$M^{U(i)} = S(k\rho_n^{U(i)} \oplus j\epsilon) = S(k\rho_{n-i} \oplus j\epsilon) = S(k\rho_{n-i}) * S^{j-1}.$$

We have

$$\dim M^{U(i)} = 2k(n-i) - 1 + j, \quad \dim M^{U(i-1)} - \dim M^{U(i)} = 2k.$$

Therefore the first gap is $k - n$. If $k - n$ is even, then we can use Theorem 5.1 to compute the structure set. If $k - n$ is odd, then we may use $M = S(k\rho_{n+1} \oplus j\epsilon)^{U(1)}$, where $k - (n+1)$ is even, whence Theorem 5.3 can be applied.

We first assume $k - n$ is even and compute the top summand $S^{\text{alg}}(X, X_{-1})$ in the decomposition for $S(X) = S_{U(n)}(S(k\rho_n \oplus j\epsilon))$ in Theorem 5.1. Since the representation sphere is the link of the origin in the representation space $k\rho_n \oplus j\epsilon = \mathbb{C}^{kn} \oplus \mathbb{R}^j$, by Lemma 3.2, both X and X_{-1} are connected and simply connected. If the action is neither trivial nor free, then $X_{-1} \neq \emptyset$, and the surgery obstruction $\mathbb{L}(\pi_1 X, \pi_1 X_{-1}) = \mathbb{L}(e, e)$ is trivial. Therefore the top summand is the same as the homology

$$\mathbb{S}^{\text{alg}}(X, X_{-1}) = \mathbb{H}(X, X_{-1}; \mathbb{L}).$$

Let

$$Z = S(k\rho_n)/U(n), \quad d = \dim Z = 2kn - 1 - n^2.$$

Then

$$(X, X_{-1}) = (Z, Z_{-1}) * S^{j-1}, \quad \dim X = d + j,$$

and

$$S^{\text{alg}}(X, X_{-1}) = \pi_{d+j} \mathbb{S}^{\text{alg}}(X, X_{-1}) = \pi_{d+j} \mathbb{H}(X, X_{-1}; \mathbb{L}) = H_d(Z, Z_{-1}; \mathbb{L}).$$

Proposition 6.1. *If $k \geq n$, then for $Z = S(k\rho_n)/U(n)$, we have*

$$H_{\dim Z}(Z, Z_{-1}; \mathbb{L}) = \mathbb{Z}^{A_{n,k}} \oplus \mathbb{Z}_2^{B_{n,k}},$$

where $A_{n,k}$ is the number of n-tuples $(\mu_1, \ldots, \mu_n)$ satisfying

$$0 \leq \mu_1 \leq \cdots \leq \mu_n \leq k - n, \quad \sum \mu_i \text{ is even},$$

and $B_{n,k}$ is the number of n-tuples satisfying

$$0 \leq \mu_1 \leq \cdots \leq \mu_n \leq k - n, \quad \sum \mu_i \text{ is odd}.$$

Proof. The homology can be computed by a spectral sequence

$$E_{p,q}^2 = H_p(Z, Z_{-1}; \pi_q \mathbb{L}(e)) = \begin{cases} H_p(Z, Z_{-1}; \mathbb{Z}) & \text{if } q = 0 \text{ mod } 4, \\ H_p(Z, Z_{-1}; \mathbb{Z}_2) & \text{if } q = 2 \text{ mod } 4, \\ 0 & \text{if } q \text{ is odd}. \end{cases}$$

Since the top pure stratum $Z - Z_{-1}$ is a manifold, by the Poincaré duality we have $H_p(Z, Z_{-1}; R) = H^{d-p}(Z - Z_{-1}; R)$. The homotopy type of $Z - Z_{-1}$ is well known to be the complex Grassmannian $G(n, k)$. Therefore

$$E_{p,q}^2 = \begin{cases} H^{d-p}(G(n, k); \mathbb{Z}) & \text{if } q = 0 \text{ mod } 4, \\ H^{d-p}(G(n, k); \mathbb{Z}_2) & \text{if } q = 2 \text{ mod } 4, \\ 0 & \text{if } q \text{ is odd}. \end{cases}$$

Using the CW structure of $G(n, k)$ given by the Schubert cells, which are all even-dimensional, we see that $E_{p,q}^2$ vanishes when either q or $d - p$ is odd. This implies

that all the differentials in $E_{p,q}^2$ vanish. Therefore the spectral sequence collapses, and we get

$$H_d(Z, Z_{-1}; \mathbb{L}) = \left(\bigoplus_{q \leq d,\, q=0\,(4)} H^q(G(n, k); \mathbb{Z}) \right) \oplus \left(\bigoplus_{q \leq d,\, q=2\,(4)} H^q(G(n, k); \mathbb{Z}_2) \right).$$

Since $G(n, k)$ is a closed manifold, we always have $q \leq \dim G(n, k) \leq \dim Z = d$. Of course this is nothing but $q \leq 2n(k - n) \leq 2kn - 1 - n^2 = d$. Therefore the requirement $q \leq d$ is automatically satisfied in the summation above, and we have

$$H_d(Z, Z_{-1}; \mathbb{L}) = \mathbb{Z}^{A_{n,k}} \oplus \mathbb{Z}_2^{B_{n,k}},$$

where $A_{n,k}$ is the number of Schubert cells in $G(n, k)$ of dimension 0 mod 4, and $B_{n,k}$ is the number of Schubert cells of dimension 2 mod 4. The description of $A_{n,k}$ and $B_{n,k}$ in the proposition is the well known number of such Schubert cells. $\quad\square$

The unitary group $U(n)$ acts trivially on $S(k\rho_n \oplus j\epsilon)$ only when $n = 0$ and $j > 0$. In this case, we have $S^{\mathrm{alg}}(X, X_{-1}) = S^{\mathrm{alg}}(X) = S(S^{j-1})$. (Here the first S in $S(S^{j-1})$ denotes the structure set, and the second the sphere.) By the Poincaré conjecture, the structure set of the sphere is trivial. This means that we should require $n > 0$ in the notation $\mathbb{Z}^{A_{n,k}} \oplus \mathbb{Z}_2^{B_{n,k}}$.

The action is free only when $n = 1$ and $j = 0$. In this case, we have $S^{\mathrm{alg}}(X, X_{-1}) = S^{\mathrm{alg}}(X) = S(\mathbb{C}P^{k-1})$. The homology is still $\mathbb{Z}^{A_{1,k}} \oplus \mathbb{Z}_2^{B_{1,k}}$. But the surgery obstruction is $L_{2(k-1)}(\pi_1 X, \pi_1 X_{-1}) = L_{2(k-1)}(\pi_1 X) = L_{2(k-1)}(e) = \mathbb{Z}$. Here we recall that $k - 1 = k - n$ is assumed even. Since this piece of the surgery obstruction is simply the summand $H^0(G(1, k); \mathbb{Z})$ in the computation of the homology, this reduces the number of copies of $\mathbb{Z}$ by 1. The computation is exactly that of fake complex projective spaces studied in [Su, Theorem 9], [Wa, Section 14C].

If $k - n$ is even, then Proposition 6.1 and the subsequent discussion about the exceptions can be applied to the summands $S^{\mathrm{alg}}(X_{-2i}, X_{-2i-1})$ in the decomposition for $S(X) = S_{U(n)}(S(k\rho_n \oplus j\epsilon))$ in Theorem 5.1, simply by replacing n with $n - 2i$. The exception is that, in case n is odd and $j = 0$, the $U(1)$-action on $M^{U(n-1)}$ is free, which implies $X_{-n} = \emptyset$. The exception happens to the last summand $S^{\mathrm{alg}}(X_{-n+1}, X_{-n}) = S^{\mathrm{alg}}(X_{-n+1}) = S^{\mathrm{alg}}(\mathbb{C}P^{k-1})$, and the number of copies of $\mathbb{Z}$ is reduced by 1. This concludes the proof of part 1 of Theorem 1.4.

If $k - n$ is odd, then Proposition 6.1 can be applied to all summands except the top one in the decomposition for $S(X)$ in Theorem 5.3, simply by replacing n with $n - 2i + 1$. The exception is that, in case n is even and $j = 0$, the last summand is $S^{\mathrm{alg}}(X_{-n+1}) = S^{\mathrm{alg}}(\mathbb{C}P^{k-1})$, and the number of copies of $\mathbb{Z}$ should be reduced by 1. The top summand $S^{\mathrm{alg}}(X)$ may be computed by the surgery fibration

$$\mathbb{S}^{\mathrm{alg}}(X) \to \mathbb{H}(X; \mathbb{L}) \to \mathbb{L}(\pi_1 X).$$

Since X is simply connected, $\mathbb{L}(\pi_1 X)$ is the usual surgery spectrum $\mathbb{L}$, and the assembly map is induced by the map from X to a single point. Therefore

$$S^{\mathrm{alg}}(X) = \tilde{H}_{d+j}(X; \mathbb{L}) = \begin{cases} H_d(Z; \mathbb{L}) & \text{if } j > 0, \\ \tilde{H}_d(Z; \mathbb{L}) & \text{if } j = 0. \end{cases}$$

The reduced homology is given by Proposition 9.1 of the appendix by Jared Bass. Since $k - n$ is odd, we have

$$\tilde{H}_d(Z; \mathbb{L}) = \mathbb{Z}^{A_{n,k-1}} \oplus \mathbb{Z}_2^{B_{n,k-1}}.$$

The unreduced homology is modified from the reduced one according to Corollary 9.2 of the appendix. This yields part 2 of Theorem 1.4.

7. Suspensions of multiaxial representation spheres

The suspension map is natural with respect to the restrictions to fixed points of unitary subgroups. In other words, the following diagram is commutative:

$$
\begin{array}{ccc}
S_{U(n)}(S(k\rho_n \oplus j\epsilon)) & \xrightarrow{\ *S(\rho_n)\ } & S_{U(n)}(S((k+1)\rho_n \oplus j\epsilon)) \\
\Big\downarrow{\scriptstyle \mathrm{res}} & & \Big\downarrow{\scriptstyle \mathrm{res}} \\
S_{U(n-i)}(S(k\rho_{n-i} \oplus j\epsilon)) & \xrightarrow{\ *S(\rho_{n-i})\ } & S_{U(n-i)}(S((k+1)\rho_{n-i} \oplus j\epsilon))
\end{array}
$$

Since the decomposition of the structure sets of multiaxial manifolds in Section 5 is obtained from such restrictions, it is tempting to break the suspension map into a direct sum of suspension maps between direct summands. However, such a decomposition is not immediately clear because the parity requirements on $k - n + i$ for the split surjectivity of the restriction maps on the left and right sides are different.

So instead of the (single) suspension, we consider the commutative diagram of the double suspension map:

$$
\begin{array}{ccc}
S_{U(n)}(S(k\rho_n \oplus j\epsilon)) & \xrightarrow{\ *S(2\rho_n)\ } & S_{U(n)}(S((k+2)\rho_n \oplus j\epsilon)) \\
\Big\downarrow{\scriptstyle \mathrm{res}} & & \Big\downarrow{\scriptstyle \mathrm{res}} \\
S_{U(n-i)}(S(k\rho_{n-i} \oplus j\epsilon)) & \xrightarrow{\ *S(2\rho_{n-i})\ } & S_{U(n-i)}(S((k+2)\rho_{n-i} \oplus j\epsilon))
\end{array}
$$

Since the parity requirements for the split surjectivity are the same on both sides, the double suspension map is indeed a direct sum of double suspension maps between direct summands of the respective decompositions of the structure sets.

We will argue that the double suspension maps between direct summands are injective. This implies that the whole double suspension map is also injective. Since the double suspension map is the composition of two (single) suspension maps, the suspension map is also injective.

To simplify the notation in the discussion, we assume $j = 0$. Let

$$X = S(k\rho_n)/U(n), \quad Y = S((k+2)\rho_n)/U(n).$$

We have

$$S((k+2)\rho_n) = S(k\rho_n) \times D(2\rho_n) \cup S(2\rho_n), \quad Y = (S(k\rho_n) \times D(2\rho_n))/U(n) \cup D^3,$$

and a stratified system of fibrations

$$p \colon (S(k\rho_n) \times D(2\rho_n))/U(n) \to X = S(k\rho_n)/U(n).$$

An element of $S_{U(n)}(S(k\rho_n))$ may be interpreted as a stratified simple homotopy equivalence $f\colon X' \to X$. The pullback of p along f gives a stratified simple homotopy equivalence $Y' \to Y$, which after adding the extra D^3 further gives the suspension element of f in $S_{U(n)}(S((k+2)\rho_n))$.

Suppose $k - n$ is even. Then the double suspension map $*S(2\rho_n)$ decomposes into suspension maps between the direct summands

$$\sigma_i \colon S^{\mathrm{alg}}(X_{-2i}, X_{-2i-1}) \to S^{\mathrm{alg}}(Y_{-2i}, Y_{-2i-1}).$$

By the computation in Section 6, with one exception, the direct summands are the same as the corresponding normal invariants. Therefore we consider the suspension maps on the normal invariants

$$\sigma_i \colon H_{\dim X_{-2i}}(X_{-2i}, X_{-2i-1}; \mathbb{L}) \to H_{\dim Y_{-2i}}(Y_{-2i}, Y_{-2i-1}; \mathbb{L}).$$

The interpretation of the suspension as the pullback of p implies that the suspension of the normal invariants is simply given by the transfer along p. On the strata we are interested in, the projection

$$Y_{-2i} \supset (S(k\rho_{n-2i}) \times D(2\rho_{n-2i}))/U(n-2i) \xrightarrow{p} X_{-2i} = S(k\rho_{n-2i})/U(n-2i)$$

takes (the orbit of) a $(k+2)$-tuple $\xi = (v_1, \ldots, v_k, v_{k+1}, v_{k+2})$ of vectors in ρ_{n-2i} and drops the last two vectors v_{k+1} and v_{k+2}. Note that ξ is mapped into the pure stratum X^{-2i} if and only if the k-tuple $p(\xi) = (v_1, \ldots, v_k)$ already spans the whole vector space ρ_{n-2i}. This implies that $p^{-1}X^{-2i} \to X^{-2i}$ is a trivial bundle with fiber $D^{4(n-i)}$ (given by the choices $(v_{k+1}, v_{k+2}) \in D(2\rho_{n-2i})$). Since

$$p^{-1}X^{-2i} = p^{-1}X_{-2i} - p^{-1}X_{-2i-1} = Y_{-2i} - p^{-1}X_{-2i-1} \cup D^3,$$
$$Y_{-2i-1} \subset p^{-1}X_{-2i-1} \cup D^3,$$

up to excision, the pair $(Y_{-2i}, p^{-1}X_{-2i-1} \cup D^3)$ is the same as the Thom space of the trivial disk bundle $p^{-1}X^{-2i} \to X^{-2i}$. The transfer of the normal invariants along this bundle may be identified with the homological Thom isomorphism, and we have a commutative diagram

$$
\begin{array}{ccc}
H_{\dim X_{-2i}}(X_{-2i}, X_{-2i-1}; \mathbb{L}) & \xrightarrow{\sigma_i} & H_{\dim Y_{-2i}}(Y_{-2i}, Y_{-2i-1}; \mathbb{L}) \\
\big\| & & \downarrow{\scriptstyle \mathrm{incl}} \\
H_{\dim X_{-2i}}(X_{-2i}, X_{-2i-1}; \mathbb{L}) & \xrightarrow[\mathrm{Thom}\ \cong]{} & H_{\dim Y_{-2i}}(Y_{-2i}, p^{-1}X_{-2i-1} \cup D^3; \mathbb{L})
\end{array}
$$

The commutative diagram shows that the suspension map σ_i is injective.

There is only one exception to the discussion above. In case n is odd (so k is also odd) and $j = 0$, the last summand in the decomposition of $S_{U(n)}(S(k\rho_n))$ is $S(\mathbb{C}P^{k-1})$. The double suspension is the usual double suspension map $S(\mathbb{C}P^{k-1}) \to S(\mathbb{C}P^{k+1})$, which is well known to be injective. In fact, the structure sets also embed into the corresponding normal invariants, and the injectivity still follows from the Thom isomorphism. This completes the proof of the injectivity of the suspension for the case of $k - n$ even.

Now we turn to the case of $k - n$ odd. The double suspension map $*S(2\rho_n)$ decomposes into suspension maps

$$\sigma_i \colon S^{\mathrm{alg}}(X_{-2i+1}, X_{-2i}) \to S^{\mathrm{alg}}(Y_{-2i+1}, Y_{-2i}), \quad i \geq 1,$$

and

$$\sigma_0 \colon S^{\mathrm{alg}}(X) \to S^{\mathrm{alg}}(Y).$$

The argument for the injectivity of σ_i for $i \geq 1$ is the same as in the case of $k - n$ even. By the computation in Section 6, the top summands are the same as the reduced homologies

$$\sigma_0 \colon \tilde{H}_{\dim X}(X; \mathbb{L}) \to \tilde{H}_{\dim Y}(Y; \mathbb{L}).$$

Let $X'^0 \subset X^0 \subset X$ be those points represented by k-tuples of vectors in ρ_n such that the first $k - 1$ vectors already span the whole vector space ρ_n. (This means $r = n$ and $m_n > 1$ in Bass' terminology.) Then by the computation of Jared Bass, the map $\tilde{H}_{\dim X}(X; \mathbb{L}) \to \tilde{H}_{\dim X}(X, X - X'^0; \mathbb{L})$ is injective. On the other hand, the preimage $Y'^0 = p^{-1}(X'^0) \subset Y^0 \subset Y$ consists of those $(k+2)$-tuples in ρ_n such that the first $k - 1$ vectors already span the whole vector space ρ_n. (This means $r = n$ and $m_n > 3$ in Bass' terminology.) Since Y'^0 is obtained by adding two vectors $(v_{k+1}, v_{k+2}) \in D(2\rho_n)$ to the representatives of points in X'^0, the projection $Y'^0 \to X'^0$ is a trivial bundle with D^{4n} as fiber. The transfer of the normal invariants along this bundle may be identified with the homological Thom isomorphism, and we have a commutative diagram

$$
\begin{array}{ccc}
\tilde{H}_{\dim X}(X; \mathbb{L}) & \xrightarrow{\;\sigma_0\;} & \tilde{H}_{\dim Y}(Y; \mathbb{L}) \\
\Big\downarrow{\scriptstyle\mathrm{inj}} & & \Big\downarrow{\scriptstyle\mathrm{incl}} \\
H_{\dim X}(X, X - X'^0; \mathbb{L}) & \xrightarrow[\mathrm{Thom}\,\cong]{} & H_{\dim Y}(Y, Y - Y'^0; \mathbb{L})
\end{array}
$$

The diagram shows that the suspension map σ_0 is injective.

Again there is only one exception to the discussion. In case n is even (so k is odd), and $j = 0$, the last summand in the decomposition of $S_{U(n)}(S(k\rho_n))$ is $S(\mathbb{C}P^{k-1})$. The double suspension on this summand is injective, just like the exceptional case when $k - n$ is even. This completes the proof of the injectivity of the suspension for the case of $k - n$ odd.

8. Multiaxial $Sp(n)$-manifolds

The symplectic group $Sp(n)$ consists of $n \times n$ quaternionic matrices that preserve the standard hermitian form on $\mathbb{H}^n$,

$$\langle x, y \rangle = \bar{x}_1 y_1 + \bar{x}_2 y_2 + \cdots + \bar{x}_n y_n.$$

A symplectic subgroup associated to a quaternionic subspace of $\mathbb{H}^n$ consists of the quaternionic matrices that preserve the standard hermitian form and fix the quaternionic subspace. Any symplectic subgroup is conjugate to a specific symplectic subgroup $Sp(i)$ associated to the specific subspace $0 \oplus \mathbb{H}^{n-i}$.

We call an $Sp(n)$-manifold *multiaxial* if any isotropy group is a symplectic subgroup, $M^{Sp(i)}$ is an ANR homology manifold, the orbit space $M/Sp(n)$ is homotopically stratified, and the strata of $M/Sp(n)$ are 1-LC embedded. All our discussion about multiaxial $U(n)$-manifolds can be extended to multiaxial $Sp(n)$-manifolds.

The role played by $U(1) = S^1$ is replaced by $Sp(1) = S^3$, the group of quaternions of unit length. If S^3 acts freely on a sphere, then the dimension of the sphere is 3 mod 4, and the quotient is homotopy equivalent to $\mathbb{H}P^r$. In analogy to the unitary case, we have $M^{Sp(j)} = M^{T^j}$ for the maximal torus T^j of $Sp(j)$, and all such maximal tori for the given j are conjugate in $Sp(n)$. Hence the proof of Lemma 3.1 using the Borel formula remains valid, and we get a quaternionic version of the formula for the first gap,

$$\dim M^{Sp(j-1),x} - \dim M^{Sp(j),x} = 4(r_1^x + n).$$

Since $\mathbb{H}P^r$ is always connected and simply connected, Lemma 3.2 can also be applied to multiaxial $Sp(n)$-manifolds.

The key reasons behind the results in Section 4 is that for even r, $\mathbb{C}P^r$ is a manifold of signature one, which makes the surgery transfer an equivalence, even after taking account of the monodromy. This remains valid with $\mathbb{H}P^r$ in place of $\mathbb{C}P^r$, so that all the results in Section 4 still hold.

The key reason that we can apply the results in Section 4 to multiaxial $U(n)$-manifolds is that the fibers of the link fibration between adjacent strata are homotopy equivalent to $\mathbb{C}P^r$, and the link fibration has trivial monodromy and is therefore orientable. Since the same reasoning remains valid for multiaxial $Sp(n)$-manifolds, the splitting theorems in Section 5 can be extended.

Theorem 8.1. *Suppose M is a multiaxial Sp(n)-manifold such that the connected components of $M^{Sp(1)}$ have codimensions 4n mod 8. Then we have a natural splitting*

$$\mathbb{S}_{Sp(n)}(M) = \bigoplus_{i \geq 0} \mathbb{S}_{Sp(n-2i)}(\bar{M}^{Sp(2i)}, \partial \bar{M}^{Sp(2i)}) = \bigoplus_{i \geq 0} \mathbb{S}^{\mathrm{alg}}(X_{-2i}, X_{-2i-1}).$$

Theorem 8.2. *Suppose M is a multiaxial Sp(n)-manifold such that the connected components of $M^{Sp(1)}$ have codimensions $4(n + 1)$ mod 8. If $M = W^{Sp(1)}$ for a multiaxial $Sp(n + 1)$-manifold W, then we have a natural splitting*

$$\mathbb{S}_{Sp(n)}(M) = \mathbb{S}^{\mathrm{alg}}(X) \oplus \bigoplus_{i \geq 1} \mathbb{S}^{\mathrm{alg}}(X_{-2i+1}, X_{-2i}).$$

Moreover,

$$\mathbb{S}^{\mathrm{alg}}(X_{-2i+1}, X_{-2i}) = \mathbb{S}_{Sp(n-2i+1)}(\bar{M}^{Sp(2i-1)}, \partial \bar{M}^{Sp(2i-1)}).$$

Theorem 5.2 can be extended. The proof of Theorem 1.2 at the end of Section 5 can also be extended, in view of the fact that the signature of $\mathbb{H}P^r$ is zero for odd r. So we have the quaternonic version of Theorem 1.2.

Theorem 8.3. *Suppose the quaternionic sphere S^3 acts semifreely on a topological manifold M. Let $M_0^{S^3}$ and $M_4^{S^3}$ be the unions of those connected components of M^{S^3} that are, respectively, of codimensions $0 \bmod 8$ and $4 \bmod 8$. Let N be the complement of (the interior of) an equivariant tube neighborhood of M^{S^3}, with boundaries $\partial_0 N$ and $\partial_4 N$ corresponding to the two parts of the fixed points. Then*

$$S_{S^3}(M) = S(M_0^{S^3}) \oplus S(N/S^3, \partial_4 N/S^3, \text{rel } \partial_0 N/S^3).$$

We can also compute the structure sets of multiaxial $Sp(n)$-representation spheres. The dimensions of the Schubert cells of quaternionic Grassmannians $G_{\mathbb{H}}(n, k)$ are multiples of 4. Therefore the analogue of Proposition 6.1 gives copies of $L_{4i}(e) = \mathbb{Z}$, regardless of the parity. Since the total number of Schubert cells in $G_{\mathbb{H}}(n, k)$ is $A_{n,k} + B_{n,k} = \binom{k}{n}$, we have

$$H_d(S(k\rho_n)/Sp(n), S(k\rho_n)_{-1}/Sp(n); \mathbb{L}) = \mathbb{Z}^{\binom{k}{n}}, \quad k \geq n,$$

where

$$d = \dim S(k\rho_n)/Sp(n) = 4kn - 1 - n(2n + 1).$$

On the other hand, the CW structure used by Jared Bass can also be applied to the orbit space $S(k\rho_n)/Sp(n)$. The reason is that for complex matrices, the unique representative by row echelon form is a consequence of the fact that $GL(n, \mathbb{C}) = U(n)N$, where $U(n)$ is the maximal compact subgroup of the semisimple Lie group $SL(n, \mathbb{C})$ and N is the group of upper triangular matrices with positive diagonal entries. This is a special example of the Iwasawa decomposition. When the decomposition is applied to the semisimple Lie group $SL(n, \mathbb{H})$, for which $Sp(n)$ is the maximal compact subgroup, we get $GL(n, \mathbb{H}) = Sp(n)N$. Therefore the orbit space $S(k\rho_n)/Sp(n)$ has cells $B(m_1, \ldots, m_r)$ similar to the orbit space in the complex case, except that

$$\dim B(m_1, \ldots, m_r) = 4(m_1 + \cdots + m_r) - 3r - 1.$$

This leads to the analogue of Proposition 9.1,

$$\tilde{H}_d(S(k\rho_n)/Sp(n); \mathbb{L}) = \mathbb{Z}^{\binom{k-1}{n}}, \quad k \geq n.$$

For $k - n$ odd, this is the top summand

$$S^{\mathrm{alg}}(S(k\rho_n)/Sp(n)) = \tilde{H}_d(S(k\rho_n)/Sp(n); \mathbb{L})$$

in the decomposition of the structure set $S_{Sp(n)}(S(k\rho_n))$. If $k - n$ is odd and $j > 0$, then the top summand is

$$S^{\mathrm{alg}}((k\rho_n \oplus j\epsilon)/Sp(n)) = \tilde{H}_{d+j}(X; \mathbb{L}) = H_d(S(k\rho_n)/Sp(n); \mathbb{L})$$
$$= \tilde{H}_d(S(k\rho_n)/Sp(n); \mathbb{L}) \oplus H_0(Z; \pi_d \mathbb{L}).$$

The extra homology at the base point is

$$H_0(Z; \pi_{4kn-1-n(2n+1)} \mathbb{L}) = L_{4kn-1-n(2n+1)}(e) = \begin{cases} \mathbb{Z} & \text{if } n = 1 \bmod 4, \\ \mathbb{Z}_2 & \text{if } n = 3 \bmod 4, \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

Finally, we need to consider the case in which the last summand in the decomposition is $S(\mathbb{H}P^{k-1})$, which happens when k, n are odd and $j = 0$, or k is odd, n even and $j = 0$. In this case, the number of copies of $\mathbb{Z}$ should be reduced by 1.

In summary, the quaternionic analogue of Theorem 1.4 is the following.

Theorem 8.4. *Suppose $k \geq n$ and ρ_n is the canonical representation of $Sp(n)$.*

1. *If $k - n$ is even, then*

$$S_{Sp(n)}(S(k\rho_n \oplus j\epsilon)) = \mathbb{Z}^{\sum_{0 \leq 2i < n} \binom{k}{n-2i}},$$

with the only exception that there is one less $\mathbb{Z}$ in case n is odd and $j = 0$.

2. *If $k - n$ is odd, then*

$$S_{Sp(n)}(S(k\rho_n \oplus j\epsilon)) = \mathbb{Z}^{\binom{k-1}{n} + \sum_{0 \leq 2i-1 < n} \binom{k}{n-2i+1}},$$

with the following exceptions: (i) there is one less $\mathbb{Z}$ in case n is even and $j = 0$; (ii) there is one more $\mathbb{Z}$ in case $n = 1 \bmod 4$ and $j > 0$; (iii) there is one more $\mathbb{Z}_2$ in case $n = 3 \bmod 4$ and $j > 0$.

Finally, the discussion on the suspension

$$*S(\rho_n)\colon S_{Sp(n)}(S(k\rho_n \oplus j\epsilon)) \to S_{Sp(n)}(S((k+1)\rho_n \oplus j\epsilon))$$

can be carried out just as in Section 7 and we deduce the injectivity of the suspension in the symplectic setting as well.

9. Appendix (by Jared Bass): Homology of quotients of multiaxial representation spheres

Following earlier notation, we write

$$Z = S(k\rho_n)/U(n), \quad d = \dim Z = 2kn - 1 - n^2.$$

Through an explicit CW decomposition, we will compute the reduced homology $\tilde{H}_d(Z; \mathbb{L})$.

Proposition 9.1. *If $k \geq n$, then for $Z = S(k\rho_n)/U(n)$, we have*

$$\tilde{H}_{\dim Z}(Z; \mathbb{L}) = \mathbb{Z}^{a_{n,k}} \oplus \mathbb{Z}_2^{b_{n,k}},$$

where $a_{n,k}$ is the number of n-tuples $(\mu_1, \ldots, \mu_n)$ satisfying

$$0 \leq \mu_1 \leq \cdots \leq \mu_n \leq k - n - 1, \quad \sum \mu_i + kn \text{ is even,}$$

and $b_{n,k}$ is the number of n-tuples satisfying

$$0 \leq \mu_1 \leq \cdots \leq \mu_n \leq k - n - 1, \quad \sum \mu_i + kn \text{ is odd.}$$

In case $k - n$ is odd, which is what we are really interested in, we note that $\sum \mu_i + kn$ and $\sum \mu_i$ have the same parity. Therefore $a_{n,k} = A_{n,k-1}$ and $b_{n,k} = B_{n,k-1}$ from Proposition 6.1. In case $k - n$ is even, $\sum \mu_i + kn$ and $\sum \mu_i + n$ have the same parity.

Proof of Proposition 9.1. An element in $S(k\rho_n)$ is a k-tuple $\xi = (v_1, \ldots, v_k)$ of vectors in ρ_n satisfying $\|\xi\|^2 = \|v_1\|^2 + \cdots + \|v_k\|^2 = 1$, with the $U(n)$-action $g\xi = (gv_1, \ldots, gv_k)$. We may regard ξ as a complex $n \times k$ matrix. We claim that we can find a unique representative for the orbit of ξ in the row echelon form

$$
\bar{\xi} = \begin{bmatrix}
\lambda_1 & \cdots & * & \cdots & * & \cdots & * & \cdots \\
 & & \lambda_2 & \cdots & * & \cdots & * & \cdots \\
 & & & & \lambda_3 & \cdots & * & \cdots \\
 & & & & & \ddots & \vdots & \cdots \\
 & & & & & & \lambda_r & \cdots \\
 & & & & & & & \\
\end{bmatrix},
$$

where the empty spaces are occupied by 0, $*$ and vertical and horizontal dots mean complex numbers, $\lambda_i > 0$, and the total length of all the entries is 1, as it was for ξ. To get $\bar{\xi}$, apply the Gram–Schmidt process to the columns of ξ to obtain an orthonormal basis for $\mathbb{C}^n$ (adding extra vectors if necessary). If we then apply to ξ the unitary matrix taking this new basis to the standard basis, we get $\bar{\xi}$ as desired. The orbit space Z is the collection of all matrices $\bar{\xi}$ of the above form.

If λ_j is m_j places from the right end of the matrix (i.e., λ_j lies in the $k - m_j + 1$-th column), then we say that the matrix has *shape* $(m_1, \ldots, m_r)$. Note that r is the rank of the matrix ξ. For any $r \le n$, $k \ge m_1 > \cdots > m_r > 0$, all $\bar{\xi}$ of the shape $(m_1, \ldots, m_r)$ form a cell $B(m_1, \ldots, m_r)$ of dimension

$$
\dim B(m_1, \ldots, m_r) = 2(m_1 + \cdots + m_r) - r - 1.
$$

Geometrically, the cell is the subset of a sphere of the above dimension determined by r coordinates being non-negative. The boundary of this cell consists of those shapes $(m'_1, \ldots, m'_{r'})$ satisfying $r' \le r$ and $m'_i \le m_i$, with at least one inequality being strict. In homological computation, only those shapes of one dimension less matter. This only occurs when

$$
m_r = 1, \quad r' = r - 1, \quad m'_i = m_i \text{ for } 1 \le i < r.
$$

Therefore, the only non-trivial boundary map of the cellular chain complex is

$$
\partial B(m_1, \ldots, m_{r-1}, 1) = B(m_1, \ldots, m_{r-1}).
$$

The homology is then freely generated by the shapes that are neither $(m_1, \ldots, m_{r-1}, 1)$ nor $(m_1, \ldots, m_{r-1})$ in the equality above. These are exactly the shapes satisfying $r = n$ (meaning ξ has full rank) and $m_n > 1$, and the shape (1) (meaning $r = 1$ and $m_1 = 1$). The shape (1) is the base point of Z.

The reduced homology $\tilde{H}_*(Z; \mathbb{L})$ is the limit of a spectral sequence with

$$
E_2^{p,q} = \tilde{H}_p(Z; \pi_q \mathbb{L}) = \begin{cases}
\tilde{H}_p(Z; \mathbb{Z}) & \text{if } q = 0 \bmod 4, \\
\tilde{H}_p(Z; \mathbb{Z}_2) & \text{if } q = 2 \bmod 4, \\
0 & \text{if } q \text{ is odd.}
\end{cases}
$$

Note that the reduced homology $\tilde{H}_p Z$ is freely generated by shapes satisfying $r = n$ and $m_n > 1$. Since the dimensions of such cells have the same parity as $n + 1$, $\tilde{H}_p Z$ is non-trivial only if p has the same parity as $n + 1$. This implies that $E_2^{p,q}$ already collapses and

$$\tilde{H}_d(Z; \mathbb{L}) = \Big(\bigoplus_{q=0\,(4)} \tilde{H}_{d-q}(Z; \mathbb{Z}) \Big) \oplus \Big(\bigoplus_{q=2\,(4)} \tilde{H}_{d-q}(Z; \mathbb{Z}_2) \Big).$$

We have

$$\bigoplus_{q=0\,(4)} \tilde{H}_{d-q}(Z; \mathbb{Z}) = \mathbb{Z}^{a_{n,k}},$$

where $a_{n,k}$ is the number of shapes $(m_1, \ldots, m_n)$ satisfying

$$m_n > 1, \quad 2(m_1 + \cdots + m_n) - n - 1 = d = 2kn - 1 - n^2 \bmod 4.$$

Let $\mu_i = m_{n-i+1} - (i + 1)$, so this condition can be interpreted in terms of the non-decreasing sequence of non-negative integers $(\mu_1, \ldots, \mu_n)$, as in the statement of the proposition. Through a similar computation we get the description of $b_{n,k}$ for $q = 2$ mod 4. $\qquad\square$

For the unreduced homology $H_d(Z; \mathbb{L})$, we also need to consider the basepoint. So we need to further take the direct sum with the homology at the base, $H_0(Z; \pi_d \mathbb{L}) = L_d(e)$. In our case of interest, when $k - n$ is odd, we have $d = -1 - n^2$ mod 4. This yields the following.

Corollary 9.2. *For $k - n$ odd, the unreduced homology $H_{\dim Z}(Z; \mathbb{L})$ is given by Proposition 9.1 with an additional summand of*

$$H_0(Z; \pi_d \mathbb{L}) = L_d(e) = \begin{cases} \mathbb{Z}_2 & \text{if } n \text{ is odd}, \\ 0 & \text{if } n \text{ is even}. \end{cases}$$

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