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A new function space and applications

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Abstract. We define a new function space B, which contains in particular BMO, BV, and $W^{1/p,p}$, $1 < p < \infty$. We investigate its embedding into Lebesgue and Marcinkiewicz spaces. We present several inequalities involving L^p norms of integer-valued functions in B. We introduce a significant closed subspace, B_0, of B, containing in particular VMO and $W^{1/p,p}$, $1 \leq p < \infty$. The above mentioned estimates imply in particular that integer-valued functions belonging to B_0 are necessarily constant. This framework provides a "common roof" to various, seemingly unrelated, statements asserting that integer-valued functions satisfying some kind of regularity condition must be constant.

Keywords. BMO, VMO, BV, Sobolev spaces, integer-valued functions, constant function, isoperimetric inequality

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Let Ω be a connected domain in $\mathbb{R}^n$. Recall that if $f : \Omega \to \mathbb{Z}$ is a measurable function which satisfies one of the following regularity properties:

1. $f \in \mathrm{VMO}(\Omega)$;
2. $f \in W^{1,1}(\Omega)$;
3. $f \in W^{1/p,p}(\Omega)$ with $1 < p < \infty$,

then f is constant [3, Comment 2, pp. 223–224], [2, Theorem B.1]. The original motivation for this article was to provide a "common roof" to all these cases, and which yields in particular the following

Theorem 1. *Assume that $f : \Omega \to \mathbb{Z}$ is measurable and can be written as $f = f_1 + f_2 + f_3$ with $f_1 \in \mathrm{VMO}(\Omega; \mathbb{R})$, $f_2 \in W^{1,1}(\Omega; \mathbb{R})$ and $f_3 \in W^{1/p,p}(\Omega; \mathbb{R})$ for some $1 < p < \infty$. Then f is constant.*

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The proof of Theorem 1 relies heavily on the introduction of a new space of functions, which might be of interest well beyond the scope of Theorem 1.

In what follows we denote by Q the unit cube $(0,1)^n$. For $0<\varepsilon<1$, $Q_\varepsilon(a)$ is the ε-cube centered at a.

Given $f\in L^1(Q;\mathbb{R})$ and an ε-cube $Q_\varepsilon\subset Q$, we set

$$M(f,Q_\varepsilon)=\fint_{Q_\varepsilon}|f-f_{Q_\varepsilon}|,\quad \text{where}\quad f_{Q_\varepsilon}=\fint_{Q_\varepsilon} f, \tag{0.1}$$

$$M^*(f,Q_\varepsilon)=\fint_{Q_\varepsilon}\fint_{Q_\varepsilon}|f(y)-f(z)|\,dy\,dz. \tag{0.2}$$

Clearly, we have

$$M(f,Q_\varepsilon)\le M^*(f,Q_\varepsilon)\le 2M(f,Q_\varepsilon). \tag{0.3}$$

Note that if $f=\mathbb{1}_A$ with $A\subset Q$ measurable, then

$$M(f,Q_\varepsilon)=M^*(f,Q_\varepsilon)=\frac{2|A\cap Q_\varepsilon|(|Q_\varepsilon|-|A\cap Q_\varepsilon|)}{|Q_\varepsilon|^2}\le\frac{1}{2}. \tag{0.4}$$

The following quantity plays an important role:

$$[f]_\varepsilon=\sup_{\mathcal{F}}\left\{\varepsilon^{n-1}\sum_{j\in J}M(f,Q_\varepsilon(a_j))\right\}. \tag{0.5}$$

Here, $\mathcal{F}$ denotes a collection of mutually disjoint ε-cubes, $\mathcal{F}=(Q_\varepsilon(a_j))_{j\in J}$, such that $\#J=$ cardinality of $J\le 1/\varepsilon^{n-1}$ (instead of $\#J$ we sometimes write $\#\mathcal{F}$) and the sup in (0.5) is taken over all such collections.

We then introduce the space

$$B=\left\{f\in L^1(Q;\mathbb{R});\ \sup_{0<\varepsilon<1}[f]_\varepsilon<\infty\right\},$$

and the corresponding norm (modulo constants)

$$\|f\|_B=\sup_{0<\varepsilon<1}[f]_\varepsilon. \tag{0.6}$$

The definition of B is inspired by the celebrated BMO space of John–Nirenberg [4] equipped with the norm (modulo constants)

$$\|f\|_{\mathrm{BMO}}:=\sup_{0<\varepsilon<1}\sup_{a\in Q}\{M(f,Q_\varepsilon(a));\ Q_\varepsilon(a)\subset Q\}. \tag{0.7}$$

Here are several examples of functions in B.

Example 1. BMO $\subset B$ *with continuous injection.*

Indeed, using (0.7) we find that $\|f\|_B\le\|f\|_{\mathrm{BMO}}$.

When $n=1$, we clearly have $B=\mathrm{BMO}$; however, when $n\ge 2$, B is strictly bigger than BMO (see e.g. Example 2 below).

Example 2. BV $\subset B$ *with continuous injection.*

Indeed, by Poincaré's inequality

$$\fint_{Q_\varepsilon} |f - f_{Q_\varepsilon}| \le \frac{c_n}{\varepsilon^{n-1}} \int_{Q_\varepsilon} |\nabla f|,$$

so that

$$\sum_{j\in J} M(f, Q_\varepsilon(a_j)) \le \frac{c_n}{\varepsilon^{n-1}} \int_{\bigcup_{j\in J} Q_\varepsilon(a_j)} |\nabla f| \tag{0.8}$$

and

$$[f]_\varepsilon \le c_n \int_Q |\nabla f|. \tag{0.9}$$

Example 3. $W^{1/p,p} \subset B$, $1 < p < \infty$, *with continuous injection.*

Indeed, for every fixed $\alpha > 0$ we have

$$\int_{Q_\varepsilon}\int_{Q_\varepsilon} |f(y) - f(z)|\, dy\, dz \le n^{\alpha/2}\varepsilon^\alpha \int_{Q_\varepsilon}\int_{Q_\varepsilon} \frac{|f(y) - f(z)|}{|y - z|^\alpha}\, dy\, dz.$$

Choosing $\alpha = (n+1)/p$ and applying Hölder's inequality gives

$$M^*(f, Q_\varepsilon) \le \frac{c_n}{\varepsilon^{(n-1)/p}} \left[\int_{Q_\varepsilon}\int_{Q_\varepsilon} \frac{|f(y) - f(z)|^p}{|y - z|^{n+1}}\, dy\, dz\right]^{1/p} \quad \text{with} \quad c_n = n^{(n+1)/2},$$

and since $\#J \le 1/\varepsilon^{n-1}$ we obtain

$$\varepsilon^{n-1} \sum_{j\in J} M^*(f, Q_\varepsilon(a_j)) \le c_n \left[\sum_{j\in J} \int_{Q_\varepsilon(a_j)}\int_{Q_\varepsilon(a_j)} \frac{|f(y) - f(z)|^p}{|y - z|^{n+1}}\, dy\, dz\right]^{1/p}. \tag{0.10}$$

Therefore

$$[f]_\varepsilon \le c_n \|f\|_{W^{1/p,p}}.$$

An important quantity associated with B is defined by

$$[f] = \overline{\lim_{\varepsilon\to 0}}\, [f]_\varepsilon. \tag{0.11}$$

The subspace

$$B_0 = \{f \in B;\ [f] = 0\} \tag{0.12}$$

plays a key role in this article.

Example 1′. VMO $\subset B_0$.

This is clear, since VMO functions (see [5]) are characterized by

$$\lim_{\varepsilon\to 0} \sup_{a\in Q} \{M(f, Q_\varepsilon(a));\ Q_\varepsilon(a) \subset Q\} = 0.$$

Moreover, VMO $= B_0$ when $n = 1$.

Example 2′. $W^{1,1} \subset B_0$.

This is clear from (0.8) and the fact that $|\bigcup_{j\in J} Q_\varepsilon(a_j)| \le \varepsilon$.

Example 3′. $W^{1/p,p} \subset B_0$, $1 < p < \infty$.

This is an immediate consequence of (0.10) and the fact that $|\bigcup_{j\in J} Q_\varepsilon(a_j) \times Q_\varepsilon(a_j)| \le \varepsilon^{n+1}$.

In particular we see that

$$\mathrm{VMO} + W^{1,1} + W^{1/p,p} \subset B_0. \tag{0.13}$$

1. Some properties of B

The main result of this section is

Theorem 2. *Let $n \ge 2$. Then $B \subset L^{n/(n-1),w}$, and*

$$\left\| f - \fint_Q f \right\|_{L^{n/(n-1),w}} \le C_n \|f\|_B, \qquad \forall f \in B. \tag{1.1}$$

In Theorem 2, the Marcinkiewicz space $L^{n/(n-1),w}$ cannot be replaced by $L^{n/(n-1)}$, as a consequence of the next result.

Proposition 3. *Let $n \ge 2$. There exists some $f \in B$ such that $f \notin L^{n/(n-1)}$.*

Proof of Theorem 2. We may assume that

$$\|f\|_B \le 1 \quad \text{and} \quad \fint_Q f = 0. \tag{1.2}$$

We also temporarily make the additional assumption that $f \in L^\infty$. Under these assumptions, we will prove that

$$\|f\|_{L^{n/(n-1),w}} \simeq \sup_{t>0} t |\{|f| > t\}|^{(n-1)/n} \le C_n. \tag{1.3}$$

For this purpose it suffices to consider, in (1.3), only $t \gtrsim 1$. We first note that, by (1.2), we have

$$\int_Q |f| \le 1. \tag{1.4}$$

In view of (1.4) we may consider, for $t > 1$, a Calderón–Zygmund decomposition at height t, i.e., we consider families $\mathcal{F}_j$ (with $j \ge 1$) of mutually disjoint cubes $Q_{2^{-j}} \subset Q$ of size 2^{-j} such that, if we set $\mathcal{F} = \bigcup_{j\ge 1} \mathcal{F}_j$, then

$$\fint_{Q_*} |f| \simeq t \quad \text{for every } Q_* \in \mathcal{F} \tag{1.5}$$

and

$$|f| \le t \quad \text{a.e. in } R := Q \setminus \bigcup_{Q_* \in \mathcal{F}} Q_*. \tag{1.6}$$

We next decompose $f = g + h$, with

$$g = f \mathbb{1}_R + \sum_{Q_* \in \mathcal{F}} \left(\fint_{Q_*} f \right) \mathbb{1}_{Q_*},$$

$$h = \sum_{j \ge 1} h_j, \quad h_j = \sum_{Q_* \in \mathcal{F}_j} \left(f - \fint_{Q_*} f \right) \mathbb{1}_{Q_*}.$$

By (1.5) and (1.6), we have

$$|g| \le Ct \quad \text{and thus} \quad \{|f| > 2Ct\} \subset \{|h| > Ct\}. \tag{1.7}$$

Using (1.7), we see that (1.3) amounts to

$$\sup_{t>1} t |\{|h| > Ct\}|^{(n-1)/n} \le c. \tag{1.8}$$

We now proceed with the proof of (1.8). Since $\|f\|_B = 1$, for every family $\mathcal{G} \subset \mathcal{F}_j$ such that $\#\mathcal{G} \le 1/(2^{-j})^{n-1} = 2^{j(n-1)}$ we have

$$2^{-j(n-1)} \sum_{Q_* \in \mathcal{G}} \fint_{Q_*} \left| f - \fint_{Q_*} f \right| \le 1.$$

By covering $\mathcal{F}_j$ with mutually disjoint sets $\mathcal{G}$ as above, we find that

$$\sum_{Q_* \in \mathcal{F}_j} \fint_{Q_*} \left| f - \fint_{Q_*} f \right| \le 2^{j(n-1)} + \#\mathcal{F}_j, \tag{1.9}$$

and thus

$$\|h_j\|_{L^1} \le 2^{-j} + 2^{-nj} \#\mathcal{F}_j. \tag{1.10}$$

On the other hand, we have (using (1.5))

$$1 \ge \|f\|_{L^1} \ge \sum_{j \ge 1} \sum_{Q_* \in \mathcal{F}_j} \int_{Q_*} |f| = \sum_{j \ge 1} \sum_{Q_* \in \mathcal{F}_j} 2^{-nj} \fint_{Q_*} |f| \gtrsim \sum_{j \ge 1} 2^{-nj} t \#\mathcal{F}_j. \tag{1.11}$$

From (1.10) and (1.11), we deduce that

$$\sum_{j \ge 1} \|h_j\|_{L^1} \lesssim \frac{1}{t} + \sum_{\mathcal{F}_j \ne \emptyset} 2^{-j}. \tag{1.12}$$

We next recall that

$$\|f\|_{L^{n/(n-1),w}} = \sup_{A \subset Q} |A|^{-1/n} \int_A |f|. \tag{1.13}$$

If $\mathcal{F}_j \neq \emptyset$ and $Q_* \in \mathcal{F}_j$, then (1.13) applied with $A = Q_*$, combined with (1.5), implies that

$$2^{-j} \lesssim (\|f\|_{L^{n/(n-1),w}}/t)^{1/(n-1)}. \tag{1.14}$$

By (1.12) and (1.14), we have

$$\|h\|_{L^1} \leq \sum_{j \geq 1} \|h_j\|_{L^1} \lesssim \frac{1}{t} + (\|f\|_{L^{n/(n-1),w}}/t)^{1/(n-1)}. \tag{1.15}$$

In turn, (1.15) implies that (with C as in (1.8))

$$|\{|h| > Ct\}| \leq \|h\|_{L^1}/(Ct) \lesssim 1/t^2 + (\|f\|_{L^{n/(n-1),w}}/t^n)^{1/(n-1)}, \tag{1.16}$$

and thus

$$t|\{|h| > Ct\}|^{(n-1)/n} \lesssim t^{(2-n)/n} + \|f\|^{1/n}_{L^{n/(n-1),w}} \lesssim 1 + \|f\|^{1/n}_{L^{n/(n-1),w}}. \tag{1.17}$$

By taking, in (1.17), the supremum over $t > 1$, we find that

$$\|f\|_{L^{n/(n-1),w}} \lesssim 1 + \|f\|^{1/n}_{L^{n/(n-1),w}},$$

and therefore $\|f\|_{L^{n/(n-1),w}} \lesssim 1$.

We complete the proof by removing the assumption that $f \in L^\infty$. Let

$$\Phi_N(s) = \begin{cases} s & \text{if } |s| \leq N, \\ N & \text{if } s > N, \\ -N & \text{if } s < -N, \end{cases}$$

and set $f_N := \Phi_N(f)$. By (0.3), we have $\|f_N\|_B \leq 2\|f\|_B$. In addition, f_N is bounded and thus satisfies (1.1), i.e.,

$$\left\| f_N - \fint_Q f_N \right\|_{L^{n/(n-1),w}} \leq 2C_n \|f\|_B. \tag{1.18}$$

Using (1.13) and letting $N \to \infty$ in (1.18) yields (1.1) for every $f \in B$. □

Proof of Proposition 3. Set

$$\varphi(x) = (1 - |x|)^+, \quad \forall x \in \mathbb{R}^n, \qquad N_m = 2^{2^m}, \quad \forall m \geq 1.$$

Consider a sequence $(b_m)_{m \geq 1}$ of points such that the open balls $B(b_m, 2/N_m)$ are contained in Q and mutually disjoint. (We may e.g. choose the points b_m on a line segment parallel to the x_1-axis.) Set

$$f_m(x) = N_m^{n-1} \varphi(N_m(x - b_m)), \qquad \forall m \geq 1, \tag{1.19}$$

$$f(x) = \sum_{m \geq 1} f_m(x). \tag{1.20}$$

We will prove that $f \in B$ and $f \notin L^{n/(n-1)}$. Note that

$$\operatorname{supp} f_m = \overline{B}(b_m, 1/N_m),$$

and the sets $\operatorname{supp} f_m$, $m \geq 1$, are mutually disjoint. Clearly,

$$\|f_m\|_{L^1(Q)} = C/N_m, \quad \forall m \geq 1, \tag{1.21}$$

and thus $f \in L^1(Q)$; here and in what follows we denote by C a generic constant depending only on n. We have

$$\|f_m\|^{n/(n-1)}_{L^{n/(n-1)}(Q)} = C, \quad \forall m \geq 1,$$

so that $f \notin L^{n/(n-1)}(Q)$.

Given $0 < \varepsilon < 1$ and integers $M_1 = M_1(\varepsilon) \geq 1$ and $M_2 = M_2(\varepsilon) > M_1(\varepsilon)$ to be defined later, we write

$$f = S_1 + S_2 + S_3, \tag{1.22}$$

with

$$S_1 = \sum_{m \leq M_1} f_m, \quad S_2 = \sum_{M_1 < m \leq M_2} f_m, \quad S_3 = \sum_{m > M_2} f_m. \tag{1.23}$$

We now estimate separately $[S_1]_\varepsilon$, $[S_2]_\varepsilon$ and $[S_3]_\varepsilon$.

Estimate of $[S_1]_\varepsilon$. Here we use the fact that if $h \in \operatorname{Lip}(Q)$ then

$$M(h, Q_\varepsilon(a)) \leq \sqrt{n}\,\varepsilon \|h\|_{\mathrm{Lip}}, \tag{1.24}$$

and thus $[h]_\varepsilon \leq \sqrt{n}\,\varepsilon \|h\|_{\mathrm{Lip}}$. In particular,

$$[f_m]_\varepsilon \leq C\varepsilon (N_m)^n. \tag{1.25}$$

Using (1.25) and the fact that $\sum_{i=1}^{p} X^i \leq \frac{X^{p+1}}{X-1}$ for all $X > 1$, we deduce that

$$[S_1]_\varepsilon \leq C\varepsilon 2^{n\,2^{M_1}}, \quad \forall \varepsilon \in (0,1). \tag{1.26}$$

Estimate of $[S_2]_\varepsilon$. Applying (0.9) to f_m yields

$$[f_m]_\varepsilon \leq C, \quad \forall m \geq 1,\ \forall \varepsilon \in (0,1),$$

and in particular

$$[S_2]_\varepsilon \leq C(M_2 - M_1), \quad \forall \varepsilon \in (0,1). \tag{1.27}$$

Estimate of $[S_3]_\varepsilon$. Clearly

$$[h]_\varepsilon \leq \frac{2}{\varepsilon}\|h\|_{L^1(Q)}, \quad \forall h \in L^1. \tag{1.28}$$

From (1.21) we deduce that

$$[f_m]_\varepsilon \leq \frac{C}{\varepsilon N_m}. \tag{1.29}$$

Using (1.29) and the fact that $\sum_{i=p}^{\infty} Y^i = \frac{Y^p}{1-Y}$ for all $Y \in [0,1)$, we see that

$$[S_3]_\varepsilon \le \frac{C}{\varepsilon 2^{2^{M_2}}}. \tag{1.30}$$

We now explain how to choose $M_1(\varepsilon)$ and $M_2(\varepsilon)$. Given $0 < \varepsilon < 1$, we denote by $M_1 = M_1(\varepsilon)$ the largest integer $\ell \ge 1$ such that

$$2^{n\,2^\ell} \le 2^{2n}/\varepsilon. \tag{1.31}$$

Equivalently, we have

$$2^{n\,2^{M_1}} \le 2^{2n}/\varepsilon, \tag{1.32}$$

$$2^{2n\,2^{M_1}} > 2^{2n}/\varepsilon. \tag{1.33}$$

Combining (1.26) and (1.32) yields

$$[S_1]_\varepsilon \le C, \quad \forall \varepsilon \in (0,1). \tag{1.34}$$

From (1.32) and (1.33) we obtain

$$|M_1(\varepsilon) - \log_2 \log_2(1/\varepsilon)| \le C, \quad \forall \varepsilon \in (0,1/2). \tag{1.35}$$

Next we denote by $M_2 = M_2(\varepsilon)$ the smallest integer $\ell \ge 1$ such that

$$2^{2^\ell} \ge 4/\varepsilon.$$

(Note that $M_2 > M_1$ since $2^{2^{M_1}} < 4/\varepsilon$.) Equivalently, we have

$$2^{2^{M_2}} \ge 4/\varepsilon, \tag{1.36}$$

$$2^{2^{M_2-1}} < 4/\varepsilon. \tag{1.37}$$

Combining (1.30) and (1.36) yields

$$[S_3]_\varepsilon \le C, \quad \forall \varepsilon \in (0,1). \tag{1.38}$$

From (1.36) and (1.37) we obtain

$$|M_2(\varepsilon) - \log_2 \log_2(1/\varepsilon)| \le C, \quad \forall \varepsilon \in (0,1/2). \tag{1.39}$$

Therefore,

$$|M_2(\varepsilon) - M_1(\varepsilon)| \le C, \quad \forall \varepsilon \in (0,1). \tag{1.40}$$

(Inequality (1.40) is deduced from (1.35) and (1.39) when $\varepsilon \in (0,1/2)$, and from (1.37) when $\varepsilon \in [1/2,1)$.)

It follows from (1.27) and (1.40) that

$$[S_2]_\varepsilon \le C, \quad \forall \varepsilon \in (0,1). \tag{1.41}$$

Putting together (1.34), (1.38) and (1.41) we conclude that $[f]_\varepsilon \le C$ for all $\varepsilon \in (0,1)$, and thus $f \in B$. $\square$

2. Some properties of B_0 and $[f]$

Our first result is

Theorem 4. *Let f be a $\mathbb{Z}$-valued function on Q such that $f \in B_0$. Then f is constant.*

Combining Theorem 4 with (0.13) we obtain Theorem 1.

When $n = 1$ we have $B_0 = \mathrm{VMO}$ and we may then invoke the fact that functions in $\mathrm{VMO}(Q; \mathbb{Z})$ are constant (for any $n \geq 1$); see [3, Comment 2, pp. 223–224]. Therefore it suffices to prove Theorem 4 when $n \geq 2$. Next, we observe that it suffices to prove Theorem 4 when $f = \mathbb{1}_A$ for some $A \subset Q$. Indeed, let $k \in \mathbb{Z}$ be such that $|f^{-1}(k)| > 0$. Set $A = f^{-1}(k)$ and $g = \mathbb{1}_A$. Clearly $M^*(f, Q_\varepsilon) \geq M^*(g, Q_\varepsilon)$ for every ε-cube Q_ε. Since $f \in B_0$, we deduce that $g \in B_0$. If Theorem 4 holds for g, then $g \equiv 1$, and thus $f \equiv k$.

Hence it remains to prove Theorem 4 when $n \geq 2$ and $f = \mathbb{1}_A$. In this case we have the following quantitative improvement of Theorem 4.

Theorem 5. *Let $n \geq 2$. There exists a constant C_n such that if $f = \mathbb{1}_A$ with $A \subset Q$ measurable, then*

$$\left\| f - \fint_Q f \right\|_{L^{n/(n-1)}(Q)} \leq C_n[f]. \tag{2.1}$$

Remark 6. A much more precise result (see [1]) asserts that there exist two constants $0 < \underline{c}_n \leq \overline{c}_n < \infty$ such that if $f = \mathbb{1}_A$, then

$$\underline{c}_n \min\left\{1, \int_Q |\nabla f|\right\} \leq [f] \leq \overline{c}_n \min\left\{1, \int_Q |\nabla f|\right\}, \tag{2.2}$$

with the convention that $\int_Q |\nabla f| = \infty$ if $f \notin \mathrm{BV}$. Note that

$$\left\| f - \fint_Q f \right\|_{L^{n/(n-1)}(Q)} \leq C \int_Q |\nabla f| \tag{2.3}$$

by the Sobolev embedding, and clearly

$$\left\| f - \fint_Q f \right\|_{L^{n/(n-1)}(Q)} \leq 2 \quad \text{when } f = \mathbb{1}_A. \tag{2.4}$$

Therefore

$$\left\| f - \fint_Q f \right\|_{L^{n/(n-1)}(Q)} \leq C \min\left\{1, \int_Q |\nabla f|\right\} \leq C'[f] \quad \text{by (2.2).}$$

In fact, using a variant of the definition (0.5) involving ε-cubes of general orientation, one obtains a quantity $[f]^*_\varepsilon$ satisfying

$$[f]_\varepsilon \leq [f]^*_\varepsilon \leq C_1[f]_{C_2\varepsilon}$$

for some constants $C_1, C_2 > 1$ depending only on n (see [1]). The main result in [1] asserts that if $f = \mathbb{1}_A$, then

$$\lim_{\varepsilon\to 0} [f]^*_\varepsilon = \frac{1}{2}\min\left\{1, \int_Q |\nabla f|\right\}; \tag{2.5}$$

the ingredients of the proof of (2.5) are much more sophisticated than the arguments presented below. We acknowledge that it was Theorem 5 which prompted one of us to conjecture that (2.5) holds.

The main tool in the proof of Theorem 5 is

Lemma 7. *Let $n \geq 2$. Let $U = \bigcup_{j\in J} Q_\varepsilon(a_j)$ be a union of ε-cubes. Then $Q \setminus U$ contains a connected set S of measure $\geq 1 - \alpha_n (\#J)^{n/(n-1)}\varepsilon^n$, for some positive constant α_n depending only on n.*

Here, the ε-cubes are not necessarily mutually disjoint, and we do not assume that they are completely contained in Q.

Remark 8. The conclusion of Lemma 7 is optimal. Indeed, consider a ball $B \subset Q$ of (small) radius R. We may cover the sphere $\Sigma = \partial B$ by a union of ε-cubes as above with $\#J\varepsilon^{n-1} \simeq R^{n-1}$. Then $|B| \simeq R^n \simeq (\#J)^{n/(n-1)}\varepsilon^n$.

Granted Lemma 7, we turn to

Proof of Theorem 5. Let $f = \mathbb{1}_A$, with $A \subset Q$. Fix any $\lambda \in (0, 1/2)$, e.g. $\lambda = 1/4$. In view of (2.4), we may assume that

$$0 \leq [f] < 2\lambda(1-\lambda), \tag{2.6}$$

for otherwise the conclusion is clear with $C_n = 1/(\lambda(1-\lambda))$. Note that, by (0.4),

$$M(f, Q_\varepsilon) = 2 f_{Q_\varepsilon}(1 - f_{Q_\varepsilon}).$$

Therefore,

$$M(f, Q_\varepsilon) < 2\lambda(1-\lambda) \;\Rightarrow\; \text{either } f_{Q_\varepsilon} < \lambda, \text{ or } f_{Q_\varepsilon} > 1-\lambda. \tag{2.7}$$

With ε small and $\widetilde{Q} = (\varepsilon, 1-\varepsilon)^n$, consider a maximal family $J = J_\varepsilon$ of points $a \in \widetilde{Q}$ such that the cubes $Q_\varepsilon(a)$ are mutually disjoint and satisfy

$$M(f, Q_\varepsilon(a)) \geq 2\lambda(1-\lambda), \qquad \forall a \in J. \tag{2.8}$$

Let $\nu > 0$ (to be chosen arbitrarily small later). We claim that for ε sufficiently small (depending on ν) we have

$$\#J \leq \delta/\varepsilon^{n-1} \quad \text{with} \quad \delta = \frac{[f]+\nu}{2\lambda(1-\lambda)}. \tag{2.9}$$

Indeed, we first see that, for ε sufficiently small,

$$\#J \leq 1/\varepsilon^{n-1}. \tag{2.10}$$

Otherwise, we may choose a subfamily $\widetilde{J}$ such that $\#\widetilde{J} = I(1/\varepsilon^{n-1})$, where $I(t)$ denotes the integer part of t. Then

$$[f]_\varepsilon \geq \varepsilon^{n-1}(\#\widetilde{J})2\lambda(1-\lambda) \geq \varepsilon^{n-1}\left(\frac{1}{\varepsilon^{n-1}} - 1\right)2\lambda(1-\lambda),$$

which, by (2.6), is impossible for ε small. From (2.10) and the definition of $[f]_\varepsilon$ we have

$$[f]_\varepsilon \geq \varepsilon^{n-1}(\#J)2\lambda(1-\lambda),$$

which yields (2.9) for ε sufficiently small.

Set $U := \bigcup_{a\in J} Q_{2\varepsilon}(a)$. By Lemma 7 and a scaling argument, $\widetilde{Q} \setminus U$ contains a connected set $S = S_\varepsilon$ such that

$$|S_\varepsilon| \geq (1-2\varepsilon)^n - \alpha_n' \delta^{n/(n-1)}, \tag{2.11}$$

where $\alpha_n' = 2^n \alpha_n$. We next note that (by the maximality of J) U contains the set

$$V = V_\varepsilon := \{b \in \widetilde{Q};\ M(f, Q_\varepsilon(b)) \geq 2\lambda(1-\lambda)\}, \tag{2.12}$$

and thus $S \subset \widetilde{Q} \setminus V$. We consider the continuous function

$$f_\varepsilon : \widetilde{Q} \to \mathbb{R}, \qquad f_\varepsilon(a) = f_{Q_\varepsilon(a)}.$$

By (2.7) and (2.12), on the set $\widetilde{Q} \setminus V$ the function f_ε takes values in $[0, \lambda) \cup (1-\lambda, 1]$. $S \subset \widetilde{Q} \setminus V$ being connected, we find that either $f_\varepsilon < \lambda$, or $f_\varepsilon > 1-\lambda$ in S.

We assume e.g. that $f_\varepsilon < \lambda$ in S_ε along a sequence $\varepsilon_m \to 0$. Clearly,

$$\int_{A\cap\widetilde{Q}} |1 - f_\varepsilon| \to 0 \quad \text{as } \varepsilon \to 0,$$

and thus

$$(1-\lambda)|S_{\varepsilon_m} \cap A| \leq \int_{S_{\varepsilon_m}\cap A} (1 - f_{\varepsilon_m}) \to 0 \quad \text{as } m \to \infty. \tag{2.13}$$

On the other hand, by (2.11) and (2.13) we have

$$|A| = |S_{\varepsilon_m} \cap A| + |(\widetilde{Q} \setminus S_{\varepsilon_m}) \cap A| + |(Q \setminus \widetilde{Q}) \cap A| \leq \alpha_n' \delta^{n/(n-1)} + o(1) \quad \text{as } m \to \infty,$$

and thus $|A| \leq \alpha_n' \delta^{n/(n-1)}$, so that

$$|A|^{(n-1)/n} \leq \alpha_n'' \delta = \alpha_n'' \frac{[f] + \nu}{2\lambda(1-\lambda)} \quad \text{with} \quad \alpha_n'' = (\alpha_n')^{(n-1)/n}.$$

Since $\nu > 0$ can be chosen arbitrarily small, we deduce that

$$|A|^{(n-1)/n} \leq \frac{\alpha_n''[f]}{2\lambda(1-\lambda)}. \tag{2.14}$$

Finally, we note that

$$\left\| f - \fint f \right\|_{L^{n/(n-1)}} = \left(|A|(1-|A|)^{n/(n-1)} + (1-|A|)|A|^{n/(n-1)}\right)^{(n-1)/n}$$
$$\leq 2\min\{|A|^{(n-1)/n}, |A^c|^{(n-1)/n}\}. \tag{2.15}$$

Combining (2.14) and (2.15) yields (2.1). □

For further use, let us note that the proof of Theorem 5 leads to the following result.

Lemma 9. *Let $n \geq 2$ and $\lambda \in (0, 1/2)$. Let $A \subset Q$ be measurable and set $f := \mathbb{1}_A$. Assume that there exists a sequence $\varepsilon_m \to 0$ and families*

$$J_m \subset \widetilde{Q}^m := (3\varepsilon_m, 1 - 3\varepsilon_m)^n$$

of points a with the following property: If $b \in \widetilde{Q}^m \setminus \bigcup_{a \in J_m} Q_{2\varepsilon_m}(a)$, then $M(f, Q_{\varepsilon_m}(b)) < 2\lambda(1-\lambda)$. Let

$$\delta := \varliminf_{m\to\infty} (\varepsilon_m)^{n-1} \# J_m.$$

Then either $|A| \geq 1 - \widetilde{c}_n \delta^{n/(n-1)}$, or $|A^c| \geq 1 - \widetilde{c}_n \delta^{n/(n-1)}$.

Proof of Lemma 7. Recall a standard "relative" isoperimetric inequality. Let $B \subset Q$ satisfy $|B| \leq 1/2$. By (2.3) (applied with $f = \mathbb{1}_B$) and (2.15), we have

$$|B| \leq c_n \left(\int_Q |\nabla \mathbb{1}_B| \right)^{n/(n-1)} = c_n [P(B)]^{n/(n-1)}, \tag{2.16}$$

where $P(B)$ represents the perimeter of B relative to Q. When B is a Lipschitz domain (which will be the case in what follows), $P(B)$ is the (surface) measure of $\partial B \cap Q$.

We now turn to the proof of the lemma. Set $\delta = (\# J)\varepsilon^{n-1}$. Let $(A_i)_{i \in I}$ be the connected components of the open set $Q \setminus \bigcup_{j \in J} \overline{Q}_\varepsilon(a_j)$. Note that each A_i is Lipschitz, and

$$\bigcup_{i \in I} (\partial A_i \cap Q) \subset \bigcup_{j \in J} (\partial Q_\varepsilon(a_j) \cap Q). \tag{2.17}$$

Let

$$G_j := \{x \in \partial Q_\varepsilon(a_j) \cap Q;\ x \text{ does not belong to the } (n-2)\text{-skeleton of } \partial Q_\varepsilon(a_j)\}.$$

Note that $[\bigcup_{i \in I} (\partial A_i \cap Q)] \setminus [\bigcup_{j \in J} G_j]$ has zero $\mathcal{H}^{n-1}$-measure. Since a point $x \in G_j$ belongs to at most one ∂A_i, we find, using (2.17), that

$$\sum_{i \in I} P(A_i) \leq \sum_{j \in J} P(Q_\varepsilon(a_j)) \leq c'_n \delta. \tag{2.18}$$

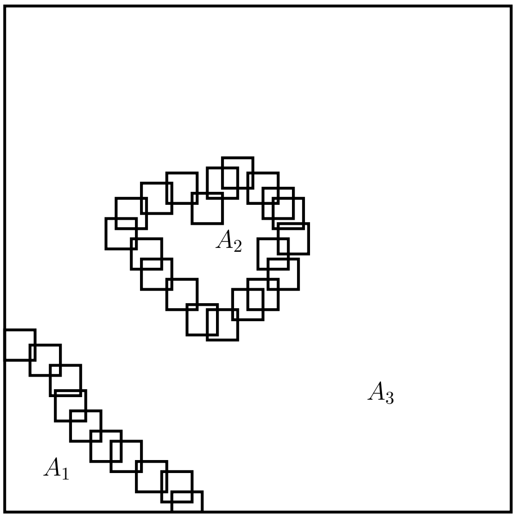

Fig. 1. The components of $Q \setminus \bigcup_{j\in J} \overline{Q}_\varepsilon(a_j)$.

We claim that if $\delta < \delta_n$ (a positive number depending only on n), then there exists some $i_0 \in I$ such that $|A_{i_0}| > 1/2$. Indeed, assume that $|A_i| \le 1/2$ for all $i \in I$. By (2.16) and (2.18), we have

$$1 - |U| = |Q \setminus U| = \sum_{i\in I} |A_i| \le c_n \sum_{i\in I} [P(A_i)]^{n/(n-1)} \le c_n \Big[\sum_{i\in I} P(A_i)\Big]^{n/(n-1)}$$
$$\le c_n (c_n'\delta)^{n/(n-1)} = c_n''\delta^{n/(n-1)}. \tag{2.19}$$

On the other hand

$$|U| \le (\#J)\varepsilon^n = \delta\varepsilon < \delta. \tag{2.20}$$

Combining (2.19) and (2.20) we obtain

$$1 \le \delta + c_n''\delta^{n/(n-1)};$$

this is impossible when $\delta < \delta_n$, where δ_n is the solution of $1 = \delta_n + c_n''(\delta_n)^{n/(n-1)}$, and thus the claim is established when $\delta < \delta_n$.

Set $S = A_{i_0}$, which is clearly connected and contained in $Q \setminus U$. Applying (2.16) to $B = S^c$ we find (using (2.18))

$$1 - |S| \le c_n[P(S^c)]^{n/(n-1)} = c_n[P(S)]^{n/(n-1)} \le c_n''\delta^{n/(n-1)},$$

which is the desired conclusion when $\delta < \delta_n$.

Finally, we observe that

$$1 - \frac{1}{(\delta_n)^{n/(n-1)}}\delta^{n/(n-1)} \le 0$$

when $\delta \ge \delta_n$, and therefore Lemma 7 holds with $\alpha_n = \max\{c_n'', 1/(\delta_n)^{n/(n-1)}\}$. $\square$

3. An extension of Theorem 5 to $\mathbb{Z}$-valued functions

Our main result in this section is

Theorem 10. *Let $n \geq 2$. There exists a positive constant c (independent of n) such that if f is a $\mathbb{Z}$-valued function in B and $[f] < c$, then $f \in L^{n/(n-1)}(Q)$ and*

$$\left\| f - \fint_Q f \right\|_{L^{n/(n-1)}(Q)} \leq C_n[f], \tag{3.1}$$

for some constant C_n depending only on n.

Theorem 5 can be deduced from Theorem 10. Indeed, let $f = \mathbb{1}_A$. Then either $[f] \leq c$, and Theorem 10 applies, or $[f] > c$, and then

$$\left\| f - \fint_Q f \right\|_{L^{n/(n-1)}(Q)} \leq 2 \leq (2/c)[f].$$

The smallness condition on $[f]$ in Theorem 10 is essential, as shown by the following improvement of Proposition 3.

Proposition 11. *Let $n \geq 2$. There exists a $\mathbb{Z}$-valued function $f \in B$ such that $f \notin L^{n/(n-1)}(Q)$.*

Proof of Theorem 10. The proof is divided into several steps.

Step 1: Decomposition of f as a sum of characteristic functions. We temporarily assume that $f \geq 0$. Then f is a sum of characteristic functions. Indeed, set

$$A_k := \{x \in Q;\ f(x) \geq k\}, \qquad \forall k \in \mathbb{N}^*,$$

and let $g_k := \mathbb{1}_{A_k}$. Then we claim that

$$f = \sum_{k>0} g_k \tag{3.2}$$

and

$$|f(x) - f(y)| = \sum_{k>0} |g_k(x) - g_k(y)|, \qquad \forall x, y \in Q. \tag{3.3}$$

Indeed, on the one hand (3.2) follows from

$$\sum_{k>0} g_k(x) = \sum_{0<k\leq f(x)} 1 = f(x).$$

On the other hand, assuming e.g. that $f(x) \geq f(y)$, we have $g_k(x) = g_k(y)$ provided either $k \leq f(y)$ or $k > f(x)$, and thus

$$\begin{aligned}\sum_{k>0} |g_k(x) - g_k(y)| &= \sum_{f(y)<k\leq f(x)} |g_k(x) - g_k(y)| = \sum_{f(y)<k\leq f(x)} 1 = f(x) - f(y)\\ &= |f(x) - f(y)|;\end{aligned}$$

that is, (3.3) holds.

We next note that (3.3) implies

$$M^*(f, Q_\varepsilon) = \sum_{k>0} M^*(g_k, Q_\varepsilon), \tag{3.4}$$

and in particular

$$M(g_k, Q_\varepsilon) \le M^*(f, Q_\varepsilon), \quad \forall k > 0. \tag{3.5}$$

Step 2: Construction of maximal families of "bad" cubes. Fix some $\lambda \in (0, 1/2)$ and consider a sequence $\varepsilon_m \to 0$. Let $\widetilde{Q}^m := (3\varepsilon_m, 1 - 3\varepsilon_m)^n$. Let J_m be a maximal family of points $a \in \widetilde{Q}^m$ such that the cubes $Q_{\varepsilon_m}(a)$, $a \in J_m$, are mutually disjoint and satisfy $M^*(f, Q_{\varepsilon_m}(a)) \ge 2\lambda(1-\lambda)$.

By the maximality of J_m and by (3.5), we have

$$M(g_k, Q_{\varepsilon_m}(b)) \le M^*(f, Q_{\varepsilon_m}(b)) < 2\lambda(1-\lambda), \quad \forall b \in \widetilde{Q}_m \setminus \bigcup_{a \in J_m} Q_{2\varepsilon_m}(a). \tag{3.6}$$

We next associate to each k an appropriate subfamily extracted from J_m. More specifically, let

$$J_m^k := \{a \in J_m;\ 3^{2n} M^*(g_k, Q_{3\varepsilon_m}(a)) \ge 2\lambda(1-\lambda)\}. \tag{3.7}$$

We claim that

$$M(g_k, Q_{\varepsilon_m}(b)) < 2\lambda(1-\lambda), \quad \forall b \in \widetilde{Q}_m \setminus \bigcup_{a \in J_m^k} Q_{2\varepsilon_m}(a). \tag{3.8}$$

Indeed, (3.6) implies that (3.8) holds for $b \in \widetilde{Q}_m \setminus \bigcup_{a \in J_m} Q_{2\varepsilon_m}(a)$.

It remains to establish (3.8) when $b \in Q_{2\varepsilon_m}(a)$ for some $a \in J_m \setminus J_m^k$. In this case, we have $Q_{\varepsilon_m}(b) \subset Q_{3\varepsilon_m}(a)$ and thus

$$M^*(g_k, Q_{\varepsilon_m}(b)) \le 3^{2n} M^*(g_k, Q_{3\varepsilon_m}(a)) < 2\lambda(1-\lambda).$$

This completes the proof of (3.8).

Step 3: A first estimate of $\|f - \fint_Q f\|_{L^{n/(n-1)}}$. By (2.15), (3.8), and Lemma 9, we have

$$\left\| g_k - \fint_Q g_k \right\|_{L^{n/(n-1)}} \le 2(\widetilde{c}_n)^{(n-1)/n} \varliminf_{m\to\infty} (\varepsilon_m)^{n-1} \# J_m^k. \tag{3.9}$$

Thus

$$\sum_{k>0} \left\| g_k - \fint_Q g_k \right\|_{L^{n/(n-1)}} \le 2(\widetilde{c}_n)^{(n-1)/n} \varliminf_{m\to\infty} (\varepsilon_m)^{n-1} \sum_{k>0} \# J_m^k, \tag{3.10}$$

and therefore

$$\left\| f - \fint_Q f \right\|_{L^{n/(n-1)}} \le 2(\widetilde{c}_n)^{(n-1)/n} \varliminf_{m\to\infty} (\varepsilon_m)^{n-1} \sum_{k>0} \# J_m^k. \tag{3.11}$$

Step 4: A second estimate of $\|f - \fint_Q f\|_{L^{n/(n-1)}}$. In this step, we assume that

$$[f] < d := \lambda(1-\lambda) \quad \text{with } \lambda \text{ chosen as in Step 2.} \tag{3.12}$$

Under this assumption, we will prove that

$$c_n' \varliminf_{m\to\infty} (\varepsilon_m)^{n-1} \sum_{k>0} \# J_m^k \le [f] \quad \text{for some constant } c_n' > 0. \tag{3.13}$$

Granted this estimate, we obtain (using (3.11))

$$\left\| f - \fint_Q f \right\|_{L^{n/(n-1)}} \le \widetilde{C}_n[f] \quad \text{with} \quad \widetilde{C}_n = 2(\widetilde{c}_n)^{(n-1)/n}/c_n'. \tag{3.14}$$

We now proceed to the proof of (3.13). We first note that (by (0.3)) we have

$$M(f, Q_{\varepsilon_m}(a)) \ge \lambda(1-\lambda), \qquad \forall a \in J_m. \tag{3.15}$$

Repeating the proof of (2.10) (and using (3.12) and (3.15)), for large m we have

$$\# J_m \le 1/(\varepsilon_m)^{n-1}. \tag{3.16}$$

We next rely on the following lemma, well-known to experts, whose proof is omitted.

Lemma 12. *Let $\{Q_\varepsilon(a);\ a \in J\}$ be a family of mutually disjoint ε-cubes. Then there exists a constant $N = N(n)$ such that*

1. $J = J^1 \cup \cdots \cup J^N$.
2. *For every j, the cubes $Q_{3\varepsilon}(a)$, $a \in J^j$, are mutually disjoint.*
3. *For every j, we have $\# J^j \le \# J/3^{n-1}$.*

By Lemma 12, for every family of mutually disjoint ε-cubes $Q_\varepsilon(a)$, $a \in J \subset (3\varepsilon, 1-3\varepsilon)^n$, such that $\# J \le 1/\varepsilon^{n-1}$, we have

$$(3\varepsilon)^{n-1} \sum_{a \in J} M(h, Q_{3\varepsilon}(a)) \le N[h]_{3\varepsilon}, \qquad \forall h : Q \to \mathbb{R}. \tag{3.17}$$

In particular, for large m we have (using (3.16) and (3.17))

$$(\varepsilon_m)^{n-1} \sum_{a \in J_m} M(f, Q_{3\varepsilon_m}(a)) \le (N/3^{n-1})[f]_{3\varepsilon_m}. \tag{3.18}$$

Combining (3.18) with (0.3), we see that

$$(\varepsilon_m)^{n-1} \sum_{a \in J_m} M^*(f, Q_{3\varepsilon_m}(a)) \le 2(N/3^{n-1})[f]_{3\varepsilon_m}. \tag{3.19}$$

We now use successively (3.19), (3.4) and (3.7) to obtain

$$\begin{aligned} [f]_{3\varepsilon_m} &\ge \frac{3^{n-1}}{2N}(\varepsilon_m)^{n-1} \sum_{a \in J_m} M^*(f, Q_{3\varepsilon_m}(a)) \\ &= \frac{3^{n-1}}{2N}(\varepsilon_m)^{n-1} \sum_{a \in J_m} \sum_{k>0} M^*(g_k, Q_{3\varepsilon_m}(a)) \\ &\ge \frac{3^{n-1}}{2N}(\varepsilon_m)^{n-1} \sum_{k>0} \sum_{a \in J_m^k} M^*(g_k, Q_{3\varepsilon_m}(a)) \\ &\ge \frac{\lambda(1-\lambda)}{3^{n+1}N}(\varepsilon_m)^{n-1} \sum_{k>0} \# J_m^k = c_n'(\varepsilon_m)^{n-1} \sum_{k>0} \# J_m^k, \end{aligned} \tag{3.20}$$

with $c_n' := \lambda(1-\lambda)/(3^{n+1}N)$. We derive (3.13) by letting $m \to \infty$ in (3.20).

Step 5: Removing the assumption $f \geq 0$. We note that $f = f^+ - f^-$, and

$$|f^{\pm}(x) - f^{\pm}(y)| \leq |f(x) - f(y)|, \quad \forall x, y \in Q. \tag{3.21}$$

By (0.3) and (3.21), we have

$$M^*(f^{\pm}, Q_\varepsilon) \leq M^*(f, Q_\varepsilon) \leq 2M(f, Q_\varepsilon),$$

and thus $[f^{\pm}] \leq 2[f]$. By the first part of the proof of this theorem, we have

$$\left\| f^{\pm} - \fint_Q f^{\pm} \right\|_{L^{n/(n-1)}} \leq \widetilde{C}_n[f^{\pm}] \leq 2\widetilde{C}_n[f], \tag{3.22}$$

provided $[f] < c := d/2$. Finally, (3.22) implies that

$$\left\| f - \fint_Q f \right\|_{L^{n/(n-1)}} \leq C_n[f] \quad \text{provided } [f] < c,$$

with $C_n := 4\widetilde{C}_n$.

The proof of Theorem 10 is complete. □

Proof of Proposition 11. We use the same notation and the same strategy as in the proof of Proposition 3, with some minor modifications. Set

$$g_m(x) = I(f_m(x)), \quad \forall m \geq 1, \quad g(x) = \sum_{m \geq 1} g_m(x)$$

(recall that $I(t)$ denotes the integer part of t). Clearly,

$$\|g_m\|_{L^1(Q)} \leq \|f_m\|_{L^1(Q)} = C/N_m \tag{3.23}$$

(by (1.21)), so that $g \in L^1(Q)$. On the other hand

$$\|g_m\|^{n/(n-1)}_{L^{n/(n-1)}(Q)} \geq \|f_m - 1\|^{n/(n-1)}_{L^{n/(n-1)}([f_m > 1])} \geq \alpha > 0, \quad \forall m \geq 1,$$

and thus $g \notin L^{n/(n-1)}(Q)$.

We will now prove that $g \in B$. Write

$$g = T_1 + T_2 + T_3, \quad T_1 = \sum_{m \leq M_1} g_m, \quad T_2 = \sum_{M_1 < m \leq M_2} g_m, \quad T_3 = \sum_{m > M_2} g_m,$$

where $M_1 = M_1(\varepsilon)$ and $M_2 = M_2(\varepsilon)$ are defined exactly as in the proof of Proposition 3.

Estimate of $[T_1]_\varepsilon$. Since $g_m \notin \mathrm{Lip}(Q)$, we need to modify the argument. We claim that, for sufficiently small ε (depending only on n), given any cube $Q_\varepsilon(a)$ there exists at most one integer $m \leq M_1(\varepsilon)$ such that

$$Q_\varepsilon(a) \cap (\operatorname{supp} g_m) \neq \emptyset. \tag{3.24}$$

Indeed, if (3.24) holds, then $Q_\varepsilon(a) \cap B(b_m, 1/N_m) \neq \emptyset$, and thus

$$Q_\varepsilon(a) \subset B(b_m, 2/N_m) \tag{3.25}$$

provided

$$1/N_m + \sqrt{n}\,\varepsilon \leq 2/N_m, \qquad \forall m \leq M_1. \tag{3.26}$$

On the other hand, (1.32) implies that $N_{M_1} \leq 4/\varepsilon^{1/n}$, and thus (3.26) holds when

$$\varepsilon \leq \varepsilon_0 := \frac{1}{4^{n/(n-1)} n^{n/[2(n-1)]}}.$$

We deduce the claim using (3.25) and the fact that the balls $B(b_m, 2/N_m)$ are mutually disjoint.

Therefore, for $\varepsilon \leq \varepsilon_0$ we have

$$M(T_1, Q_\varepsilon(a)) \leq ⨍_{Q_\varepsilon(a)} ⨍_{Q_\varepsilon(a)} |g_m(y) - g_m(z)|\, dy\, dz \tag{3.27}$$

for some $m \leq M_1(\varepsilon)$.

If $y, z \in Q_\varepsilon(a)$, we have

$$|f_m(y) - f_m(z)| \leq |y - z|\, \|f_m\|_{\mathrm{Lip}} \leq (N_m)^n \sqrt{n}\,\varepsilon \leq C$$

(by (1.32)). Hence

$$|g_m(y) - g_m(z)| \leq C, \tag{3.28}$$

since $|I(t) - I(s)| \leq |t - s| + 1$ for all t, s.

Combining (3.27) and (3.28) yields $M(T_1, Q_\varepsilon(a)) \leq C$ and therefore

$$[T_1]_\varepsilon \leq C, \qquad \forall \varepsilon \in (0, \varepsilon_0). \tag{3.29}$$

For $\varepsilon \in [\varepsilon_0, 1)$, we use (1.28) to assert that

$$[T_1]_\varepsilon \leq \frac{2}{\varepsilon_0} \|T_1\|_{L^1(Q)} \leq \frac{2}{\varepsilon_0} \|g\|_{L^1(Q)}. \tag{3.30}$$

Combining (3.29) with (3.30) we deduce that

$$[T_1]_\varepsilon \leq C, \qquad \forall \varepsilon \in (0, 1). \tag{3.31}$$

Estimate of $[T_2]_\varepsilon$. We claim that

$$\int_Q |\nabla g_m| \leq C, \qquad \forall m \geq 1, \tag{3.32}$$

and this implies via (0.9) that

$$[g_m]_\varepsilon \leq C, \qquad \forall m \geq 1,\ \forall \varepsilon \in (0, 1),$$

so that

$$[T_2]_\varepsilon \leq C(M_2 - M_1) \leq C, \qquad \forall \varepsilon \in (0, 1) \tag{3.33}$$

(by (1.40)). In order to prove (3.32), note that

$$\int_Q |\nabla g_m| = \sum_{k=1}^{(N_m)^{n-1}} \mathcal{H}^{n-1}([f_m = k]) = C \sum_{k=1}^{(N_m)^{n-1}-1} \left(1 - \frac{k}{(N_m)^{n-1}}\right)^{n-1} \frac{1}{(N_m)^{n-1}}$$

$$= C \sum_{\ell=1}^{(N_m)^{n-1}-1} \left(\frac{\ell}{(N_m)^{n-1}}\right)^{n-1} \frac{1}{(N_m)^{n-1}} \le C.$$

Estimate of $[T_3]_\varepsilon$. The technique for estimating $[S_3]_\varepsilon$ in the proof of Proposition 3 gives

$$[T_3]_\varepsilon \le C, \qquad \forall \varepsilon \in (0, 1). \tag{3.34}$$

Combining (3.31), (3.33) and (3.34) yields $g \in B$. □

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