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Bistable traveling waves for monotone semiflows with applications

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Abstract. This paper is devoted to the study of traveling waves for monotone evolution systems of bistable type. In an abstract setting, we establish the existence of traveling waves for discrete time and continuous-time monotone semiflows in homogeneous and periodic habitats. The results are then extended to monotone semiflows with weak compactness. We also apply the theory to four classes of evolution systems.

Keywords. Monotone semiflows, traveling waves, bistable dynamics, periodic habitat

1. Introduction

In this paper, we study traveling waves for monotone (i.e., order preserving) semiflows $\{Q_t\}_{t\in\mathcal{T}}$ with the bistability structure on some subsets of the space $\mathcal{C} := C(\mathcal{H}, \mathcal{X})$ consisting of all continuous functions from the habitat $\mathcal{H}$ ($= \mathbb{R}$ or $\mathbb{Z}$) to the Banach lattice $\mathcal{X}$, where $\mathcal{T} = \mathbb{Z}^+$ or $\mathbb{R}^+$ is the set of evolution times. Here the bistability structure is generalized from a number of studies for various evolution equations. It means that the restricted semiflow on $\mathcal{X}$ admits two ordered stable equilibria, between which all others are unstable. We focus on the existence of traveling waves connecting these two stable equilibria, which we call bistable (traveling) waves. This setting allows us to study not only autonomous and time-periodic evolution systems in a homogeneous habitat (medium), but also those in a periodic habitat. Moreover, the results obtained can be extended to semiflows with weak compactness on some subsets of the space $\mathcal{M}$ of all monotone functions from $\mathbb{R}$ to $\mathcal{X}$.

To explain the concept of bistability structure, we recall some related works on typical evolution equations. Fife and McLeod [19, 20] proved the existence and global asymptotic stability of monotone traveling waves for the reaction-diffusion equation

$$u_t = u_{xx} + u(1-u)(u-a), \qquad x \in \mathbb{R},\ t > 0, \tag{1.1}$$

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where $a \in (0, 1)$. Clearly, the restriction of system (1.1) to $\mathcal{X} = \mathbb{R}$ is the ordinary differential equation $u' = u(1-u)(u-a)$, which admits a unique unstable equilibrium between two ordered and stable ones. The same property is shared by the nonlocal dispersal equation in [4, 16, 46] and the lattice equations in [3, 49, 50]. Chen [13] studied a general nonlocal evolution equation $u_t = \mathcal{A}(u(\cdot, t))$, which also possesses the above bistability structure. Some related investigations on discrete-time equations can be found in [27, 15]. For the time-periodic reaction-diffusion equation $u_t = u_{xx} + f(t, u)$, the spatially homogeneous system is a time-periodic ordinary differential equation. In this case, the equilibrium in the bistability structure should be understood as the time-periodic solution. Under such bistability assumption, Alikakos, Bates and Chen [1] obtained the existence of bistable time-periodic traveling waves. Recently, Yagisita [46] studied bistable traveling waves for discrete-time and continuous-time semiflows on the space consisting of all left-continuous and nondecreasing functions from $\mathbb{R}$ to $\mathcal{X} = \mathbb{R}$ under the assumption that there is exactly one intermediate unstable equilibrium. It should be mentioned that the result in [46] for continuous-time semiflows requires an assumption on the existence of a pair of upper and lower solutions.

Note that the restrictions to $\mathcal{X} = \mathbb{R}$ of the aforementioned systems are all scalar equations, and hence there is only one unstable equilibrium in between two stable ones. But in the case where $\mathcal{X} = \mathbb{R}^n$, there may be multiple unstable equilibria. This is one of the main reasons why some ideas and techniques developed for scalar equations cannot be easily extended to higher dimensional systems. Volpert [36] established the existence and stability of traveling waves for the bistable reaction-diffusion system $u_t = D\Delta u + f(u)$ by using topological methods, where D is a positive definite diagonal matrix. Fang and Zhao [18] further extended these results to the case where D is semi-positive definite via the vanishing viscosity approach.

Consider the following parabolic equation in a cylindrical domain $\Sigma = \mathbb{R} \times \Omega$:

$$\begin{cases} u_t = \Delta u + \alpha(y)u_x + f(u), & x \in \mathbb{R}, y = (y_1, \dots, y_{n-1}) \in \Omega,\ t > 0, \\ \partial u/\partial \nu = 0 & \text{on } \mathbb{R} \times \partial\Omega \times (0, \infty), \end{cases} \tag{1.2}$$

where f is of the same type as the nonlinearity in (1.1) and Ω is a bounded domain with smooth boundary in $\mathbb{R}^{n-1}$. Obviously, the restriction of the solution semiflow of (1.2) to $\mathcal{X} = C(\bar{\Omega}, \mathbb{R})$ gives rise to the following x-independent system:

$$\begin{cases} u_t = \Delta_y u + f(u), & y \in \Omega,\ t > 0, \\ \partial u/\partial \nu = 0 & \text{on } \partial\Omega \times (0, \infty). \end{cases} \tag{1.3}$$

One can see from Matano [30] (or Casten and Holland [12]) that any nonconstant steady state of (1.3) is linearly unstable when the domain Ω is convex. It follows that if Ω is convex, then (1.2) admits the bistability structure: its x-independent system has two (constant) linearly stable steady states, between which all others are linearly unstable. In such a case, Berestycki and Nirenberg [11] obtained the existence and uniqueness of bistable traveling waves. In the case where Ω is an appropriate dumbbell-shaped domain, Matano [30] constructed a counterexample to show that (1.3) has stable nonconstant steady states, and Berestycki and Hamel [6] also proved the nonexistence of traveling waves connecting two stable constant steady states. For bistable traveling waves in time-delayed reaction-diffusion equations, we refer to [31, 35, 29, 37]. For such an equation with time delay $\tau > 0$, one can choose $\mathcal{X} = C([-\tau, 0], \mathbb{R})$ so that its solution semiflow has the bistability structure.

Recently, there is an increasing interest in reaction-diffusion equations in periodic habitats. A typical example is

$$u_t = (du_x)_x + f(u), \qquad x \in \mathbb{R},\ t > 0, \tag{1.4}$$

where $d \in C^1(\mathbb{R}, \mathbb{R})$ is a positive periodic function with period $r > 0$. Define $\mathcal{Y} := C([0, r], \mathbb{R})$ and $C_{\mathrm{per}}(\mathbb{R}, \mathbb{R}) := \{g \in C(\mathbb{R}, \mathbb{R}) : g(x + r) = g(x), \forall x \in \mathbb{R}\}$. It is easy to see that

$$C(\mathbb{R}, \mathbb{R}) = \{g \in C(r\mathbb{Z}, \mathcal{Y}) : g(ri)(r) = g(r(i + 1))(0),\ \forall i \in \mathbb{Z}\} =: \mathcal{K},$$

and that any element in $C_{\mathrm{per}}(\mathbb{R}, \mathbb{R})$ is a constant function in $\mathcal{K}$. Thus, the solution semiflow of (1.4) on $C(\mathbb{R}, \mathbb{R})$ can be regarded as a conjugate semiflow on $\mathcal{K}$, and hence the bistability structure should be understood as: the restriction of the solution semiflow of (1.4) on $C_{\mathrm{per}}(\mathbb{R}, \mathbb{R})$ has two ordered r-periodic steady states, between which all others are unstable. Assuming that the function f is of bistable type, Xin [43] obtained the existence of a spatially periodic (pulsating) traveling wave as long as d is sufficiently close to a positive constant in a certain sense (see also [44, 42]). However, whether the solution semiflow of (1.4) admits the bistability structure has remained an open problem. We will give an affirmative answer in Section 6.3 and further improve Xin's existence result. Meanwhile, a counterexample will be constructed to show that the solution semiflow of (1.4) has no bistability structure in the general case of varying $d(x)$. More recently, Chen, Guo and Wu [14] proved the existence, uniqueness and stability of spatially periodic traveling waves for one-dimensional lattice equations in a periodic habitat under the bistability assumption. There are also other types of bistable waves (see, e.g., [7, 33]). For monostable systems in periodic habitats, we refer to [5, 8, 9, 22, 23, 26] and references therein.

In general, there are multiple intermediate unstable equilibria in between two stable ones in the case where the space $\mathcal{X}$ is higher dimensional. Meanwhile, it is possible for the given autonomous system to have intermediate unstable time-periodic orbits in $\mathcal{X}$. These make the study of bistable semiflows more difficult than that of monostable ones, whose restrictions to $\mathcal{X}$ have only one unstable and one stable equilibrium. To overcome these difficulties, we will show that all these unstable equilibria and all points in these periodic orbits are unordered in $\mathcal{X}$ under some appropriate assumptions. With this in mind, a bistable system can be regarded as the union of two monostable systems, although such a union is not unique. From this point of view, we establish a link between monostable subsystems and the bistable system itself, which plays a vital role in the propagation of bistable traveling waves. This link is stated in terms of spreading speeds of monostable subsystems (see assumption (A6)). For spreading speeds of various monostable evolution systems, we refer to [2, 8, 22, 25, 26, 28, 38, 39, 48] and references therein.

Next we use a diagram to outline the proofs. For case (I), we combine the above observations for general bistable semiflows and Yagisita's [46] perturbation idea to prove

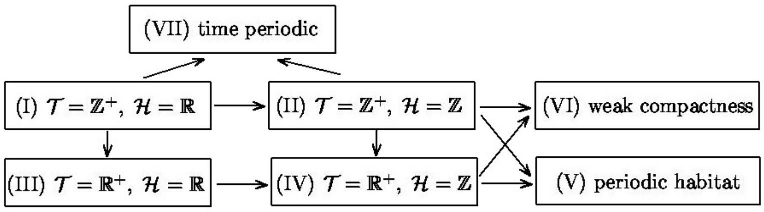

Fig. 1. Scheme of the proof.

the existence of traveling waves. For case (III), we use the candidates $\phi_{\pm}(x + c_{\pm,s}n)$ for bistable waves of the discrete-time semiflows $\{(Q_s)^n\}_{n\geq 0}$ to approximate that of the continuous-time semiflow $\{Q_t\}_{t\geq 0}$. This new approach heavily relies on an estimation of the boundedness of $(1/s)c_{\pm,s}$ as $s \to 0$, which is proved surprisingly by using the bistability structure of the semiflow (see inequalities (3.9) and (3.10)). It turns out that our result does not require the assumption on the existence of a pair of upper and lower solutions as in [46]. In case (II), both the evolution time $\mathcal{T}$ and the habitat $\mathcal{H}$ are discrete, a traveling wave $\psi(i + cn)$ of $\{Q^n\}_{n\geq 0}$ cannot be well defined in the usual way because the wave speed c, and hence the domain of ψ, is unknown. So we define it to be a traveling wave on $\mathbb{R}$ of an associated map $\tilde{Q}$. However, $\tilde{Q}$ has much weaker compactness than Q. To overcome this difficulty, we establish a variant of Helly's theorem for monotone functions from $\mathbb{R}$ to $\mathcal{X}$ in the Appendix, which is of independent interest. This discovery also enables us to study monotone semiflows in a periodic habitat and with weak compactness. Further, we can deal with case (IV) by a similar idea to that in case (III) because now traveling waves in case (II) are defined on $\mathbb{R}$. Traveling waves for a time-periodic system can be obtained with the help of the discrete-time semiflow generated by the associated Poincaré map. Motivated by the discussions in [26, Section 5], we can regard a semiflow in a periodic habitat as a conjugate semiflow in a homogeneous discrete habitat, and hence we can employ the arguments for cases (II) and (IV) to establish the existence of spatially periodic bistable traveling waves.

The rest of this paper is organized as follows. In Section 2, we present our main assumptions as well as their explanations. In Section 3, we deal with discrete-time, continuous-time, and time-periodic compact semiflows on some subsets of $\mathcal{C}$. In Section 4, we extend our results to compact semiflows in a periodic habitat. In Section 5, we further investigate semiflows with weak compactness. In Section 6, we apply the abstract results to four classes of evolution systems: a time-periodic reaction-diffusion system, a parabolic system in a cylinder, a parabolic equation with periodic diffusion, and a time-delayed reaction-diffusion equation. Finally, a short appendix section completes the paper.

2. Notation and assumptions

Throughout this paper, we assume that $\mathcal{X}$ is an ordered Banach space with the norm $\|\cdot\|_{\mathcal{X}}$ and the cone $\mathcal{X}^+$. Further, we assume that $\mathcal{X}$ is also a vector lattice with the following monotonicity condition:

$$|x|_{\mathcal{X}} \le |y|_{\mathcal{X}} \ \Rightarrow \ \|x\|_{\mathcal{X}} \le \|y\|_{\mathcal{X}},$$

where $|z|_{\mathcal{X}} := \sup\{z, -z\}$ denotes the least upper bound of z and $-z$. Such a Banach space is often called a *Banach lattice*. We use $C(M, \mathbb{R}^d)$ to denote the set of all continuous functions from the compact metric space M to the d-dimensional Euclidean space $\mathbb{R}^d$. We equip $C(M, \mathbb{R}^d)$ with the maximum norm and the standard positive cone consisting of all nonnegative functions. Then $C(M, \mathbb{R}^d)$ is a special Banach lattice, which will be used in this paper. For more general information about Banach lattices, we refer to the book [32].

Let the spatial habitat $\mathcal{H}$ be the real line $\mathbb{R}$ or the lattice

$$r\mathbb{Z} := \{\dots, -2r, -r, 0, r, 2r, \dots\}$$

for some positive number r. For simplicity, we let $r = 1$. We say a function $\phi : \mathcal{H} \to \mathcal{X}$ is bounded if the set $\{\|\phi(x)\|_{\mathcal{X}} : x \in \mathcal{H}\}$ is bounded. Throughout this paper, we always use $\mathcal{B}$ to denote the set of all bounded functions from $\mathbb{R}$ to $\mathcal{X}$, and $\mathcal{C}$ to denote the set of all bounded and continuous functions from $\mathcal{H}$ to $\mathcal{X}$. Moreover, any element in $\mathcal{X}$ can be regarded as a constant function in $\mathcal{B}$ and $\mathcal{C}$.

In this paper, we equip $\mathcal{C}$ with the compact-open topology, that is, a sequence ϕ_n converges to ϕ in $\mathcal{C}$ if and only if $\phi_n(x)$ converges to $\phi(x)$ in $\mathcal{X}$ uniformly for x in any compact subset of $\mathcal{H}$. The following norm on $\mathcal{C}$ induces this topology:

$$\|\phi\|_{\mathcal{C}} = \sum_{k=1}^{\infty} \frac{\max_{|x|\le k} \|\phi(x)\|_{\mathcal{X}}}{2^k}, \qquad \forall \phi \in \mathcal{C}. \tag{2.1}$$

Clearly, if $\mathcal{H} = \mathbb{Z}$, then $\phi_n \to \phi$ with respect to the compact-open topology if and only if $\phi_n(x) \to \phi(x)$ pointwise in $x \in \mathbb{Z}$.

We assume that $\mathrm{Int}(\mathcal{X}^+)$ is not empty. For any $u, v \in \mathcal{X}$, we write $u \ge v$ provided that $u - v \in \mathcal{X}^+$, $u > v$ provided $u \ge v$ but $u \ne v$, and $u \gg v$ provided that $u - v \in \mathrm{Int}(\mathcal{X}^+)$. A subset in $\mathcal{X}$ is said to be *totally unordered* if no two elements are ordered. For any $\phi, \psi \in \mathcal{C}$, $\phi \ge \psi$ provided that $\phi(x) \ge \psi(x)$ for all $x \in \mathcal{H}$, $\phi > \psi$ provided that $\phi \ge \psi$ but $\phi \ne \psi$, and $\phi \gg \psi$ provided that $\phi(x) \gg \psi(x)$ for all $x \in \mathcal{H}$. For any $\gamma \in \mathcal{X}$ with $\gamma > 0$, we define $\mathcal{X}_\gamma := \{u \in \mathcal{X} : \gamma \ge u \ge 0\}$, $\mathcal{C}_\gamma := \{\phi \in \mathcal{C} : \gamma \ge \phi \ge 0\}$ and $\mathcal{B}_\gamma := \{\phi \in \mathcal{B} : \gamma \ge \phi \ge 0\}$. For any $\phi, \psi \in \mathcal{C}$, the interval $[\phi, \psi]_{\mathcal{C}}$ is the set $\{w \in \mathcal{C} : \phi \le w \le \psi\}$, $[[\phi, \psi]]_{\mathcal{C}}$ is the set $\{w \in \mathcal{C} : \phi \ll w \ll \psi\}$, and similarly for $[\phi, \psi]]_{\mathcal{C}}$ and $[[\phi, \psi]_{\mathcal{C}}$. For any $u \le v$ in $\mathcal{X}$, we define $[u, v]_{\mathcal{X}}$, $[[u, v]]_{\mathcal{X}}$, $[[u, v]_{\mathcal{X}}$ and $[u, v]]_{\mathcal{X}}$ in a similar way.

Let $\beta \in \mathrm{Int}(\mathcal{X}^+)$ and Q be a map from $\mathcal{C}_\beta$ to $\mathcal{C}_\beta$. Let E be the set of all fixed points of Q restricted to $\mathcal{X}_\beta$.

Definition 2.1. For the map $Q : \mathcal{X}_\beta \to \mathcal{X}_\beta$, a fixed point $\alpha \in E$ is said to be *strongly stable from below* if there exist a number $\delta > 0$ and a unit vector $e \in \mathrm{Int}(\mathcal{X}^+)$ such that

$$Q[\alpha - \eta e] \gg \alpha - \eta e \qquad \text{for any } \eta \in (0, \delta]. \tag{2.2}$$

Strong instability from below is defined by reversing the inequality in (2.2). Similarly, we can define strong stability (and instability) from above.

Given $y \in \mathcal{H}$, define the translation operator T_y on $\mathcal{B}$ by $T_y[\phi](x) = \phi(x - y)$. Assume that 0 and β are in E. We impose the following hypotheses on Q:

(A1) (*Translation invariance*) $T_y \circ Q[\phi] = Q \circ T_y[\phi]$ for all $\phi \in \mathcal{C}_\beta$ and $y \in \mathcal{H}$.
(A2) (*Continuity*) $Q : \mathcal{C}_\beta \to \mathcal{C}_\beta$ is continuous in the compact-open topology.
(A3) (*Monotonicity*) Q is order preserving in the sense that $Q[\phi] \geq Q[\psi]$ whenever $\phi \geq \psi$ in $\mathcal{C}_\beta$.
(A4) (*Compactness*) $Q : \mathcal{C}_\beta \to \mathcal{C}_\beta$ is compact in the compact-open topology.
(A5) (*Bistability*) The fixed points 0 and β are strongly stable from above and below, respectively, for the map $Q : \mathcal{X}_\beta \to \mathcal{X}_\beta$, and the set $E \setminus \{0, \beta\}$ is totally unordered.

Note that the above bistability assumption is imposed on the spatially homogeneous map $Q : \mathcal{X}_\beta \to \mathcal{X}_\beta$. We allow the existence of other fixed points on the boundary of $\mathcal{X}_\beta$ so that the theory is applicable to competitive evolution models. The unordering property of $E \setminus \{0, \beta\}$ can be obtained by the strong instability of all fixed points in this set if the semiflow is eventually strongly monotone. More precisely, a sufficient condition for hypothesis (A5) to hold is:

(A5$'$) (*Bistability*) $Q : \mathcal{X}_\beta \to \mathcal{X}_\beta$ is eventually strongly monotone in the sense that there exists $m_1 \in \mathbb{Z}_+$ such that $Q^m[u] \gg Q^m[v]$ for all $m \geq m_1$ whenever $u > v$ in $\mathcal{X}_\beta$. Further, for the map $Q : \mathcal{X}_\beta \to \mathcal{X}_\beta$, the fixed points 0 and β are strongly stable from above and below, respectively, and each $\alpha \in E \setminus \{0, \beta\}$ is strongly unstable from both below and above.

Figure 2 illustrates the bistability structures in (A5) and (A5$'$). Next we show that (A5$'$) implies (A5). In applications, however, one may find other sufficient conditions for (A5) to hold, weaker than (A5$'$).

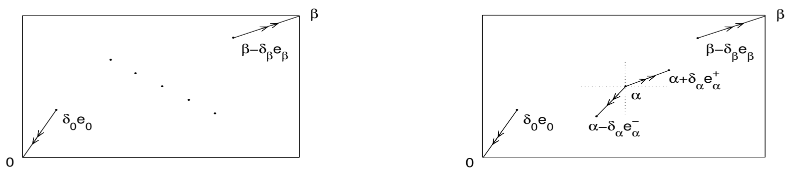

Fig. 2. Left: the set E satisfying (A5). Right: the set E satisfying (A5$'$).

Proposition 2.1. *If* (A5$'$) *holds, then for any* $\alpha_1, \alpha_2 \in E \setminus \{0, \beta\}$*, we have* $\alpha_1 \not< \alpha_2$ *and* $\alpha_2 \not< \alpha_1$.

Proof. Without loss of generality, we only show $\alpha_1 \not< \alpha_2$. Assume for contradiction that $\alpha_1 < \alpha_2$. Then $\alpha_1 = Q^{m_1}[\alpha_1] \ll Q^{m_1}[\alpha_2] = \alpha_2$. Since α_1 is strongly unstable from above, there exist $\delta_{\alpha_1} > 0$ and $e_{\alpha_1} \in \mathrm{Int}(\mathcal{X}^+)$ such that $u_0 := \alpha_1 + \delta_{\alpha_1} e_{\alpha_1} \in [[\alpha_1, \alpha_2]]_{\mathcal{X}}$ and $Q[u_0] \gg u_0$. Define the recursion $u_{n+1} = Q[u_n], n \geq 0$. Then u_n is convergent to some $\alpha \in \mathcal{X}$ with $\alpha_1 \ll \alpha \leq \alpha_2$ due to hypothesis (A4). By the eventual strong monotonicity of Q, we see that

$$u_n = Q^{m_1}[u_{n-m_1}] \ll Q^{m_1}[u_{n+1-m_1}] = u_{n+1} \ll Q^{m_1}[\alpha] = \alpha, \quad \forall n \geq m_1.$$

Since α is strongly unstable from below, we can find $\delta_\alpha > 0$ and $e_\alpha \in \mathrm{Int}(\mathcal{X}^+)$ such that $Q[\alpha - \delta e_\alpha] \ll \alpha - \delta e_\alpha$ for all $\delta \in (0, \delta_\alpha]$. Choose $n_1 \geq m_1$ such that $u_{n_1} \geq \alpha - \delta_\alpha e_\alpha$. Define $\eta := \sup\{\delta \in (0, \delta_\alpha] : u_{n_1} \leq \alpha - \delta e_\alpha\}$. Thus, $u_{n_1} \not\ll \alpha - \eta\delta_\alpha$. On the other hand,

$$u_{n_1} \ll u_{n_1+1} = Q[u_{n_1}] \leq Q[\alpha - \eta e_\alpha] \ll \alpha - \eta e_\alpha,$$

a contradiction. □

Due to assumption (A5), for any given $\alpha \in E \setminus \{0, \beta\}$, we have two monostable subsystems: $\{Q^n\}_{n\geq 0}$ restricted to $[0, \alpha]_{\mathcal{C}}$ and $[\alpha, \beta]_{\mathcal{C}}$, respectively. With this in mind, we next construct an initial function ϕ_α^- so that we can define the leftward asymptotic speed of propagation of ϕ_α^-, and then present our last assumption.

Note that in (A5) we do not require $\alpha \gg 0$ or $\alpha \ll \beta$ in $\mathcal{X}$. But assumption (A5) is sufficient to guarantee that α and β can be separated by two neighborhoods in $[\alpha, \beta]_{\mathcal{X}}$, and a similar claim is valid for 0 and α (see Lemma 3.1). In view of assumption (A5), we can find $\delta_\beta > 0$ and a unit vector $e_\beta \in \mathrm{Int}(\mathcal{X}^+)$ such that

$$Q[\beta - \eta e_\beta] \gg \beta - \eta e_\beta, \quad \forall \eta \in (0, \delta_\beta]. \tag{2.3}$$

Define

$$v_\alpha^- := \sup\{\alpha, \beta - \delta_\beta e_\beta\}. \tag{2.4}$$

Hence, $Q[v_\alpha^-] \geq Q[\beta - \delta_\beta e_\beta] \gg \beta - \delta_\beta e_\beta$. This, together with the definition of v_α^- in (2.4), implies that there exists a neighborhood $\mathcal{N}$ of $Q[v_\alpha^-]$ in $[\alpha, \beta]_{\mathcal{X}}$ such that $v_\alpha^- < \gamma$ for all $\gamma \in \mathcal{N}$. Choose a nondecreasing initial function $\phi_\alpha^- \in \mathcal{C}_\beta$ with

$$\phi_\alpha^-(x) = \alpha, \ \forall x \leq -1, \quad \text{and} \quad \phi_\alpha^-(x) = v_\alpha^-, \ \forall x \geq 0. \tag{2.5}$$

It then follows from assumptions (A1), (A2) and (A5) that

$$\lim_{x\to\infty} Q[\phi_\alpha^-](x) = Q[\phi_\alpha^-(\infty)](0) = Q[v_\alpha^-],$$

and hence there exists $\sigma > 0$ such that

$$Q[\phi_\alpha^-](x) \gg \beta - \delta_\beta e_\beta \quad \text{and} \quad Q[\phi_\alpha^-](x) \geq v_\alpha^-, \quad \forall x \geq \sigma - 1.$$

Define a sequence $a_{n,\sigma}$ of points in $\mathcal{X}$ as follows:

$$a_{n,\sigma} = Q^n[\phi_\alpha^-](\sigma n), \quad n \geq 1.$$

Then

$$a_{2,\sigma} = Q^2[\phi_\alpha^-](2\sigma) = Q[Q[\phi_\alpha^-](\cdot + \sigma)](\sigma) \geq Q[\phi_\alpha^-](\sigma) = a_{1,\sigma}.$$

By induction, we see that $a_{n,\sigma}$ is nondecreasing in n. Thus, assumption (A4) implies that $a_{n,\sigma}$ tends to a fixed point e with $e \geq a_{1,\sigma} \gg \beta - \delta_\beta e_\beta$. Therefore, $e = \beta$.

By the above observation, we have

$$\beta \geq \lim_{n\to\infty,\, x\geq\sigma n} Q^n[\phi_\alpha^-](x) \geq \lim_{n\to\infty} Q^n[\phi_\alpha^-](\sigma n) = \lim_{n\to\infty} a_{n,\sigma} = \beta,$$

and hence

$$(-\infty,-\sigma]\subset\Lambda(\phi_\alpha^-):=\Big\{c\in\mathbb{R}:\lim_{n\to\infty,\,x\geq -cn}Q^n[\phi_\alpha^-](x)=\beta\Big\}.$$

Define

$$c_-^*(\alpha,\beta):=\sup\Lambda(\phi_\alpha^-). \tag{2.6}$$

Clearly, $c_-^*(\alpha,\beta)\in[-\sigma,\infty]$ and $(-\infty,c_-^*(\alpha,\beta))\subset\Lambda(\phi_\alpha^-)$. We further claim that $c_-^*(\alpha,\beta)$ is independent of the choice of ϕ_α^- as long as ϕ_α^- has the property (2.5). Indeed, for any given ϕ with the property (2.5), we have

$$\phi_\alpha^-(x-1)\leq\phi(x)\leq\phi_\alpha^-(x+1),\qquad\forall x\in\mathcal{H}.$$

It then follows that for any $c\in\Lambda(\phi_\alpha^-)$ and $\epsilon>0$,

$$\begin{aligned}\beta&=\lim_{n\to\infty,\,x\geq -cn}Q^n[\phi_\alpha^-](x)=\lim_{n\to\infty,\,x\geq-(c-\epsilon)n}Q^n[\phi_\alpha^-](x-1)\\&\leq\lim_{n\to\infty,\,x\geq-(c-\epsilon)n}Q^n[\phi](x)\leq\lim_{n\to\infty,\,x\geq-(c-\epsilon)n}Q^n[\phi_\alpha^-](x+1)\\&=\lim_{n\to\infty,\,x\geq -cn}Q^n[\phi_\alpha^-](x)=\beta,\end{aligned}$$

which implies that $c-\epsilon\in\Lambda(\phi)$, and hence $\sup\Lambda(\phi_\alpha^-)=\sup\Lambda(\phi)$. For convenience, we may call $c_-^*(\alpha,\beta)$ the *leftward asymptotic speed of propagation* of ϕ_α^-.

Following the above procedure, we can find $\delta_0>0$ and $e_0\in\mathrm{Int}(\mathcal{X}^+)$ such that

$$Q[\eta e_0]\ll\eta e_0,\qquad\forall\eta\in(0,\delta_0]. \tag{2.7}$$

Here we emphasize that δ_0, e_0 above and δ_β, e_β will play a vital role in the whole paper because they describe the local stability of the fixed points 0 and β. Similarly, we can define

$$v_\alpha^+:=\inf\{\alpha,\delta_0 e_0\}.$$

Let $\phi_\alpha^+\in\mathcal{C}_\beta$ be a nondecreasing initial function with

$$\phi_\alpha^+(x)=\alpha,\ \forall x\geq 1,\quad\text{and}\quad\phi_\alpha^+(x)=v_\alpha^+,\ \forall x\leq 0.$$

For the same reason, we define

$$c_+^*(0,\alpha):=\sup\Big\{c\in\mathbb{R}:\lim_{n\to\infty,\,x\leq cn}Q^n[\phi_\alpha^+](x)=0\Big\}, \tag{2.8}$$

which is called the *rightward asymptotic speed of propagation* of ϕ_α^+. As showed above, these two speeds are bounded below, but may be plus infinity. To better understand these two spreading speeds, see Figure 3 (left).

Now we are ready to state our last assumption on Q:

(A6) (*Counter-propagation*) For each $\alpha\in E\setminus\{0,\beta\}$, $c_-^*(\alpha,\beta)+c_+^*(0,\alpha)>0$.

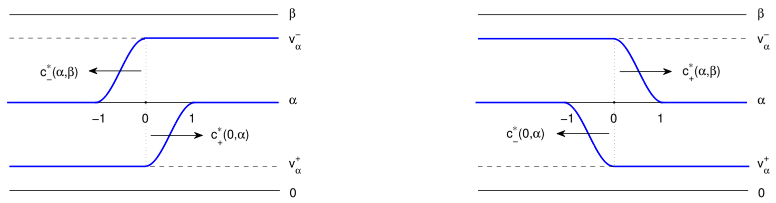

Fig. 3. Left: $c_-^*(\alpha,\beta)$ and $c_+^*(0,\alpha)$. Right: $c_+^*(\alpha,\beta)$ and $c_-^*(0,\alpha)$.

Assumption (A6) ensures that the two initial functions in the left part of Figure 3 will eventually propagate in opposite directions although one of these two speeds may be negative. It is interesting to note that assumption (A6) is nearly necessary for the propagation of a bistable traveling wave. Indeed, if a monotone evolution system admits a bistable traveling wave, then it is usually unique (up to translation) and globally attractive (see, e.g., Remark 6.2). This implies that the solution starting from the initial data $\frac{1}{2}(\phi_\alpha^+ + \phi_\alpha^-)$ converges to a phase shift of the bistable wave. If $c_-^*(\alpha,\beta) + c_+^*(0,\alpha) < 0$, then the comparison principle would force the solutions starting from $\phi_\alpha^\pm$ to split the bistable wave.

Comparing this with the definition of spreading speeds (short for asymptotic speeds of spread/propagation) for monostable semiflows (see, e.g., [2, 26]), one can find that the leftward spreading speed of the monostable subsystem $\{Q^n\}_{n\geq 0}$ restricted to $[\alpha,\beta]_{\mathcal{C}}$ is shared by a large class of initial functions, and in many applications, it equals $c_-^*(\alpha,\beta)$. A similar observation holds for $c_+^*(0,\alpha)$. Thus, for a specific bistable system, assumption (A6) can be verified by using the properties of spreading speeds for monostable subsystems.

Remark 2.1. If we consider the nonincreasing traveling waves, then we can similarly define the numbers $c_+^*(\alpha,\beta)$ and $c_-^*(0,\alpha)$ (see Figure 3 (right)). Then (A6) should be stated as $c_+^*(\alpha,\beta) + c_-^*(0,\alpha) > 0$.

3. Semiflows in a homogeneous habitat

We say a habitat is *homogeneous* for the semiflow $\{Q_t\}_{t\in\mathcal{T}}$ on a metric space $\mathcal{E} \subset \mathcal{C}$ if

$$Q_t[\phi](x-y) = Q_t[\phi(\cdot - y)](x), \qquad \forall \phi \in \mathcal{E},\ x, y \in \mathcal{H},\ t \in \mathcal{T}.$$

In this section, we will establish the existence of bistable traveling waves for the semiflow $\{Q_t\}_{t\in\mathcal{T}}$ on $\mathcal{E}$ in the following order: discrete-time semiflows in a continuous habitat, discrete-time semiflows in a discrete habitat, time-periodic semiflows, continuous-time semiflows in a continuous habitat, and continuous-time semiflows in a discrete habitat.

3.1. Discrete-time semiflows in a continuous habitat

In this case, time $\mathcal{T}$ is discrete and habitat $\mathcal{H}$ is continuous: $\mathcal{T} = \mathbb{Z}^+$ and $\mathcal{H} = \mathbb{R}$. For convenience, we use Q to denote Q_1, and consider the semiflow $\{Q^n\}_{n\geq 0}$, where Q^n is the n-th iteration of Q.

Definition 3.1. $\psi(x + cn)$ with $\psi \in \mathcal{C}$ is said to be a *traveling wave with speed* $c \in \mathbb{R}$ of the discrete semiflow $\{Q^n\}_{n\geq 0}$ if $Q^n[\psi](x) = \psi(x + cn)$ for all $x \in \mathbb{R}$ and $n \geq 0$. We say that ψ *connects* 0 *to* β if $\psi(-\infty) := \lim_{x\to-\infty} \psi(x) = 0$ and $\psi(\infty) := \lim_{x\to\infty} \psi(x) = \beta$.

We first show that 0 and β are two isolated fixed points of Q in $\mathcal{X}_\beta$ if (A5) holds.

Lemma 3.1. *Let* δ_0, e_0 *and* δ_β, e_β *be chosen so that* (2.7) *and* (2.3) *hold, respectively. Then* $E \cap \mathcal{X}_{\delta_0 e_0} = \{0\}$ *and* $E \cap [\beta - \delta_\beta e_\beta, \beta]_{\mathcal{X}} = \{\beta\}$.

Proof. Assume for contradiction that $0 \neq \alpha \in E \cap \mathcal{X}_{\delta_0 e_0}$. Define

$$\bar{\delta} := \inf\{\delta \in (0, \delta_0] : \alpha \in [0, \delta e_0]_{\mathcal{X}}\}.$$

Then $\alpha \leq \bar{\delta} e_0$ but $\alpha \notin [0, \bar{\delta} e_0]]_{\mathcal{X}}$. However, by the monotonicity of Q and the fact that 0 is strongly stable,

$$\alpha = Q[\alpha] \leq Q[\bar{\delta} e_0] \ll \bar{\delta} e_0.$$

This contradicts $\alpha \notin [0, \bar{\delta} e_0]]_{\mathcal{X}}$. Hence, $E \cap \mathcal{X}_{\delta_0 e_0} = \{0\}$. Similarly, $E \cap [\beta - \delta_\beta e_\beta, \beta]_{\mathcal{X}} = \{\beta\}$. □

Choose $\delta > 0$ such that

$$\delta < \min\{\delta_0, \delta_\beta\} \quad \text{and} \quad \delta e_0 \ll \beta - \delta e_\beta. \tag{3.1}$$

Assume that $\underline{\psi}$ and $\bar{\psi}$ are nondecreasing functions in $C(\mathbb{R}, \mathcal{X}_\beta)$ with

$$\underline{\psi}(x) = \begin{cases} 0, & x \leq 0, \\ \beta - \delta e_\beta, & x \geq 1, \end{cases} \quad \text{and} \quad \bar{\psi}(x) = \begin{cases} \delta e_0, & x \leq -1, \\ \beta, & x \geq 0. \end{cases}$$

Clearly $\underline{\psi} \leq \bar{\psi}$. We have the following observation.

Lemma 3.2. *Assume that* Q *satisfies* (A1)–(A3) *and* (A5). *Then there exists a positive rational number* $\bar{c}$ *such that for any* $c \geq \bar{c}$, *we have*

$$Q[\underline{\psi}](x) \geq \underline{\psi}(x - c) \quad \textit{and} \quad Q[\bar{\psi}](x) \leq \bar{\psi}(x + c) \quad \textit{for any } x \in \mathbb{R}.$$

Proof. Assume that $x_n \to \infty$ is an increasing sequence in $\mathbb{R}$. Then $\psi_n := \underline{\psi}(\cdot + x_n)$ converges to $\beta - \delta e_\beta$ in $\mathcal{C}_\beta$ since $\underline{\psi}(x) = \beta - \delta e_\beta$ for all $x \geq 1$. It then follows from (A1)–(A2) and (A5) that

$$Q[\underline{\psi}](\infty) = \lim_{n\to\infty} Q[\underline{\psi}](x_n) = \lim_{n\to\infty} Q[\underline{\psi}(\cdot + x_n)](0) = Q[\beta - \delta e_\beta] \gg \beta - \delta e_\beta.$$

Therefore, there exists $y_0 > 0$ such that $Q[\underline{\psi}](y_0) \geq \beta - \delta e_\beta$. Note that $Q[\underline{\psi}](x)$ is nondecreasing in x. Then for any $c \geq y_0$ we have

$$Q[\underline{\psi}](x) \geq Q[\underline{\psi}](y_0) \geq \beta - \delta e_\beta \geq \underline{\psi}(x - c), \quad \forall x \geq y_0,$$

and

$$Q[\underline{\psi}](x) \geq 0 = \underline{\psi}(0) \geq \underline{\psi}(x - y_0) \geq \underline{\psi}(x - c), \quad \forall x < y_0,$$

which means $Q[\underline{\psi}](x) \geq \underline{\psi}(x-c)$ for all $c \geq y_0$. Similarly,

$$Q[\bar{\psi}](-\infty) = \lim_{n\to\infty} Q[\underline{\psi}](-x_n) = \lim_{n\to\infty} Q[\bar{\psi}(\cdot - x_n)](0) = Q[\delta e_0] \ll \delta e_0 = \bar{\psi}(-\infty),$$

and hence there exists $z_0 > 0$ such that $Q[\bar{\psi}](x) \leq \bar{\psi}(x+c)$ for all $c \geq z_0$. Choosing $\bar{c} = \max\{y_0, z_0\}$ completes the proof. □

Let $\kappa_n := (n+\bar{c})/n$. Clearly, κ_n is a rational number for all $n \geq 1$. For any $\xi \in \mathbb{R}$, define a map $A_\xi : \mathcal{B} \to \mathcal{B}$ by $A_\xi[\phi](x) = \phi(\xi x)$ for all $x \in \mathbb{R}$. Define $\underline{\psi}_n, \bar{\psi}_n \in \mathcal{C}_\beta$ by

$$\underline{\psi}_n(x) = \underline{\psi}(x-(n+\bar{c})) \quad \text{and} \quad \bar{\psi}_n(x) = \bar{\psi}(x+(n+\bar{c})).$$

Lemma 3.3. *Assume that Q satisfies* (A1)–(A5). *Then for each $n \in \mathbb{N}$, $G_n := Q \circ A_{\kappa_n}$ has a fixed point ϕ_n in $\mathcal{C}_\beta$ such that ϕ_n is nondecreasing and $\underline{\psi}_n \leq \phi_n \leq \bar{\psi}_n$.*

Proof. We first show that $\underline{\psi}_n \leq G_n[\underline{\psi}_n]$. Indeed, when $x < n$ we have

$$\underline{\psi}_n(x+\bar{c}) \leq \underline{\psi}_n(n+\bar{c}) = \underline{\psi}(0) = 0 \leq A_{\kappa_n}[\underline{\psi}_n](x);$$

when $x \geq n$ we have

$$A_{\kappa_n}[\underline{\psi}_n](x) = \underline{\psi}_n(\kappa_n x) = \underline{\psi}_n\left(x + \frac{\bar{c}}{n}x\right) \geq \underline{\psi}_n(x+\bar{c}),$$

and hence $\underline{\psi}_n(x+\bar{c}) \leq A_{\kappa_n}[\underline{\psi}_n](x)$ for all $x \in \mathbb{R}$. Consequently, by the monotonicity of Q and $\underline{\psi}(x) \leq Q[\underline{\psi}](x+\bar{c})$ (see Lemma 3.2) we obtain

$$\underline{\psi}_n(x) \leq Q[\underline{\psi}_n](x+\bar{c}) \leq Q \circ A_{\kappa_n}[\underline{\psi}_n](x) = G_n[\underline{\psi}_n](x).$$

Similarly, $\bar{\psi}_n \geq G_n[\bar{\psi}_n]$. It follows that

$$\underline{\psi}_n \leq G_n^k[\underline{\psi}_n] \leq G_n^k[\bar{\psi}_n] \leq \bar{\psi}_n, \qquad \forall k \in \mathbb{N}. \tag{3.2}$$

For any $k \geq 1$, we have

$$G_n^k[\underline{\psi}_n] = G_n \circ G_n^{k-1}[\underline{\psi}_n] \in G_n[\mathcal{C}_\beta]. \tag{3.3}$$

Since G_n is order preserving and $\underline{\psi}_n(x)$ is nondecreasing in x, we know that $G_n^k[\underline{\psi}_n](x)$ is nondecreasing in both k and x. Recall that G_n is compact due to assumption (A4). It follows that $G_n^k[\underline{\psi}_n]$ converges in $\mathcal{C}_\beta$. Denote the limit by ϕ_n. By (3.2), we also get $\underline{\psi}_n \leq \phi_n \leq \bar{\psi}_n$. Moreover, $\phi_n(x)$ is nondecreasing due to Proposition 7.1(2). Obviously,

$$\phi_n = \lim_{k\to\infty} G_n^{k+1}[\underline{\psi}_n] = G_n\Big[\lim_{k\to\infty} G_n^k[\underline{\psi}_n]\Big] = G_n[\phi_n].$$

This completes the proof. □

The following lemma reveals a relation between the wave speeds of monostable traveling waves in sub-monostable systems and the numbers defined in (2.6) and (2.8).

Lemma 3.4. *Let $c_-^*(\alpha, \beta)$ and $c_+^*(0, \alpha)$ be defined as in* (2.6) *and* (2.8). *Assume that Q satisfies* (A3). *Then the following statements are valid:*

(1) *If $\psi(x + ct)$ is a monotone traveling wave connecting α to β of the discrete semiflow $\{Q^n\}_{n\geq 1}$, then $c \geq c_-^*(\alpha, \beta)$.*
(2) *If $\psi(x + ct)$ is a monotone traveling wave connecting 0 to α of the discrete semiflow $\{Q^n\}_{n\geq 1}$, then $c \leq -c_+^*(0, \alpha)$.*

Proof. We only prove (1) since the proof of (2) is similar. In view of Lemma 3.1, there exists a neighborhood $\mathcal{N}$ of β in $[\alpha, \beta]_{\mathcal{X}}$ such that

$$v_\alpha^- < \gamma, \quad \forall \gamma \in \mathcal{N},$$

where v_α^- is defined in (2.4). Since $\psi(-\infty) = \alpha$ and $\psi(\infty) = \beta$, there must exist a translation of ψ, still denoted by ψ, such that $\phi_\alpha^- \leq \psi$. Assume for contradiction that $c < c_-^*(\alpha, \beta)$. Choose $q/p \in (c, c_-^*(\alpha, \beta))$ with $p, q \in \mathbb{Z}$. It follows from (2.6) that

$$\begin{aligned}\beta &= \lim_{n\to\infty} Q^{pn}[\phi_\alpha^-]\left(-\frac{q}{p}\cdot pn\right) \leq \lim_{n\to\infty} Q^{pn}[\psi](-qn) \\ &= \lim_{n\to\infty} \psi(-qn + cpn) = \psi(-\infty) = \alpha,\end{aligned}$$

a contradiction. Thus, $c \geq c_-^*(\alpha, \beta)$. □

Now we are ready to prove the main result of this subsection.

Theorem 3.1. *Assume that Q satisfies* (A1)–(A6). *Then there exists $c \in \mathbb{R}$ such that the discrete semiflow $\{Q^n\}_{n\geq 1}$ admits a nondecreasing traveling wave with speed c and connecting 0 to β.*

Proof. The proof is in three steps. Firstly, we construct $\phi_+, \phi_- \in \mathcal{C}_\beta$ and $c_+ \leq c_- \in \mathbb{R}$ such that

$$Q[\phi_+](x) = \phi_+(x + c_+) \quad \text{and} \quad Q[\phi_-](x) = \phi_-(x + c_-)$$

with

$$\phi_-(0) \in (0, \delta e_0]_{\mathcal{X}} \quad \text{and} \quad \phi_+(0) \in [\beta - \delta e_\beta, \beta)_{\mathcal{X}}.$$

Indeed, let ϕ_n be as in Lemma 3.3. Since $0 \ll \bar{\psi}(-1) = \delta e_0 \ll \underline{\psi}(1) = \beta - \delta e_\beta \ll \beta$ and $\underline{\psi}_n \leq \phi_n \leq \bar{\psi}_n$, we have

$$\begin{aligned}\bar{\psi}(-1) &= \bar{\psi}_n(-1 - (n + \bar{c})) \geq \phi_n(-1 - (n + \bar{c})), \\ \underline{\psi}(1) &= \underline{\psi}_n(1 + (n + \bar{c})) \leq \phi_n(1 + (n + \bar{c})).\end{aligned}$$

Now we define

$$a_n := \sup_{x\in\mathbb{R}}\{\phi_n(x) \in [0, \delta e_0]_{\mathcal{X}}\}, \qquad b_n := \inf_{x\in\mathbb{R}}\{\phi_n(x) \in [\beta - \delta e_\beta, \beta]_{\mathcal{X}}\}.$$

Then

$$-1 - (n + \bar{c}) \leq a_n \leq b_n \leq 1 + (n + \bar{c}), \quad \phi_n(a_n) \leq \delta e_0 \leq \beta - \delta e_\beta \leq \phi_n(b_n).$$

Define $\phi_{-,n}(x) := \phi_n(x + a_n)$ and $\phi_{+,n}(x) := \phi_n(x + b_n)$. Then

$$\phi_{-,n} = \phi_n(\cdot + a_n) = G_n[\phi_n](\cdot + a_n) = Q[\phi_n(\kappa_n \cdot)](\cdot + a_n) = Q[\phi_n(\kappa_n(\cdot + a_n))] \in Q[\mathcal{C}_\beta].$$

Similarly, $\phi_{+,n} = Q[\phi_n(\kappa_n(\cdot + b_n))] \in Q[\mathcal{C}_\beta]$. Thus, there exists a subsequence (still indexed by n), two nondecreasing functions $\phi_-, \phi_+ \in \mathcal{C}_\beta$ and $\xi_-, \xi_+ \in [-1, 1]$ with $\xi_- \leq \xi_+$ such that

$$\lim_{n\to\infty} \frac{a_n}{n} = \xi_-, \quad \lim_{n\to\infty} \frac{b_n}{n} = \xi_+, \quad \lim_{n\to\infty} \phi_{-,n} = \phi_-, \quad \lim_{n\to\infty} \phi_{+,n} = \phi_+.$$

Obviously, $\phi_-(0) = \lim_{n\to\infty} \phi_n(a_n)$ and $\phi_+(0) = \lim_{n\to\infty} \phi_n(b_n)$. By the definitions of a_n and b_n, we immediately have $\phi_-(0) \neq 0$ and $\phi_+(0) \neq \beta$, and hence $0 < \phi_-(0) \leq \bar{\psi}(-1) = \delta e_0$ and $\beta - \delta e_\beta = \underline{\psi}(1) \leq \phi_+(0) < \beta$. Define $c_- := -\bar{c}\xi_-$ and $c_+ := -\bar{c}\xi_+$. Obviously, $c_- \geq c_+$ because $\xi_- \leq \xi_+$.

We only prove $Q[\phi_-](x) = \phi_-(x + c_-)$ because the proof of the other identity is similar. Note that the following limit is uniform for x in any bounded subset $M \subset \mathbb{R}$:

$$\lim_{n\to\infty} \kappa_n(x + a_n) - a_n = \lim_{n\to\infty} \left(x + \bar{c} \cdot \frac{x + a_n}{n} \right) = x - c_-.$$

Hence for any $x \in \mathbb{R}$,

$$\begin{aligned} \phi_-(x + c_-) &= \lim_{n\to\infty} \phi_{-,n}(x + c_-) = \lim_{n\to\infty} \phi_n(x + c_- + a_n) = \lim_{n\to\infty} G_n[\phi_n](x + c_- + a_n) \\ &= \lim_{n\to\infty} Q[\phi_n(\kappa_n \cdot)](x + c_- + a_n) = \lim_{n\to\infty} Q[\phi_n(\kappa_n(\cdot + a_n))](x + c_-) \\ &= \lim_{n\to\infty} Q[\phi_{-,n}(\kappa_n(\cdot + a_n) - a_n)](x + c_-) = Q[\phi_-](x), \end{aligned} \tag{3.4}$$

where the last equality is obtained from Proposition 7.2(2) and the continuity of Q.

Secondly, we prove that $\phi_\pm(x)$ obtained in the first step have the following properties:

(i) $\phi_-(-\infty) = 0$ and $\phi_+(\infty) = \beta$;
(ii) $\phi_-(\infty)$ and $\phi_+(-\infty)$ are ordered.

Indeed, let $x_n \to \infty$ be an increasing sequence in $\mathbb{R}$. Note that $\phi_-(x_n) = Q[\phi_-(\cdot - c_- + x_n)](0) \in Q[\mathcal{C}_\beta](0)$, which is precompact in $\mathcal{X}_\beta$. Therefore, there exists a subsequence $\{n_l\}$ and $v \in \mathcal{X}_\beta$ such that $\lim_{l\to\infty} \phi_-(x_{n_l}) = v$, which, together with the fact that ϕ_- is nondecreasing and Proposition 7.2(1), implies that $\phi_-(\infty) := \lim_{x\to\infty} \phi_-(x) = v$. Moreover, from (3.4) we see that $\phi_+(\infty) \in \mathcal{X}_\beta$ is a fixed point of Q. Similar results hold for $\phi_-(-\infty)$ and $\phi_+(\pm\infty)$. Recall that $\phi_-(-\infty) \leq \phi_-(0) \leq \delta e_0$ and $\phi_+(\infty) \geq \phi_+(0) \geq \beta - \delta e_\beta$, which, together with the choice of δ, implies that $\phi_-(-\infty) = 0$ and $\phi_+(\infty) = \beta$. Further, since any two real numbers are ordered, there exist sequences

$$\{n\}_{n\geq 0} \supset \{n_{1m}\}_{m\geq 1} \supset \{n_{2m}\}_{m\geq 2} \supset \cdots \supset \{n_{km}\}_{m\geq 1} \supset \cdots$$

such that for each $k \geq 1$,

$$k + a_{n_{km}} \leq -k + b_{n_{km}}, \ \forall m \geq 1, \quad \text{or} \quad k + a_{n_{km}} \geq -k + b_{n_{km}}, \ \forall m \geq 1.$$

Define $\Gamma_1 := \{k \in \mathbb{N} : k + a_{n_{km}} \le -k + b_{n_{km}}, \forall m \ge 1\}$ and $\Gamma_2 := \mathbb{N} \setminus \Gamma_1$. Then either Γ_1 or Γ_2 has infinitely many elements. If Γ_1 does, then

$$\phi_{-,n_{km}}(k) = \phi_{n_{km}}(k + a_{n_{km}}) \le \phi_{n_{km}}(-k + b_{n_{km}}) = \phi_{+,n_{km}}(-k), \quad \forall k \in \Gamma_1, m \in \mathbb{N}.$$

This implies that $\phi_-(k) \le \phi_+(-k)$ for all $k \in \Gamma_1$, and hence $\phi_-(\infty) \le \phi_+(-\infty)$. If Γ_2 has infinitely many elements, then $\phi_-(\infty) \ge \phi_+(-\infty)$ by a similar argument. Thus, $\phi_-(\infty)$ and $\phi_+(-\infty)$ must be ordered in $\mathcal{X}_\beta$.

Finally, we prove that either ϕ_- or ϕ_+ connects 0 to β. Indeed, we have shown in the second step that $\phi_-(\infty)$ and $\phi_+(-\infty)$ are ordered. It then follows from the bistability assumption (A5) that there are only three possibilities:

(i) $\beta = \phi_-(\infty) \ge \phi_+(-\infty)$;
(ii) $\phi_-(\infty) \ge \phi_+(-\infty) = 0$;
(iii) $\phi_-(\infty) = \alpha = \phi_+(-\infty)$ for some $\alpha \in E \setminus \{0, \beta\}$.

We further claim that the possibility (iii) cannot happen: Otherwise, Lemma 3.4 implies that $c_+ \ge c_-^*(\alpha, \beta)$ and $c_- \le -c_+^*(0, \alpha)$. Since $c_- \ge c_+$, it then follows that

$$0 \ge c_+ + (-c_-) \ge c_-^*(\alpha, \beta) + c_+^*(0, \alpha),$$

which contradicts (A6). Thus, either (i) or (ii) holds, completing the proof. □

3.2. *Discrete-time semiflows in a discrete habitat*

In this case, both $\mathcal{T}$ and $\mathcal{H}$ are discrete: $\mathcal{T} = \mathbb{Z}^+$ and $\mathcal{H} = \mathbb{Z}$. Without confusion, we consider the semiflow $\{Q^n\}_{n \ge 0}$ in a metric space $\mathcal{E} \subset \mathcal{C}$. Since the habitat is discrete, we cannot use the definition of traveling waves with an unknown speed as in Definition 3.1. This is because the wave profile $\psi(x)$ may not be well defined for all $x \in \mathbb{R}$. So we modify the definition of traveling waves in a discrete habitat.

Definition 3.2. $\psi(x + cn)$ with $\psi \in \mathcal{B}$ is said to be a *traveling wave with speed* $c \in \mathbb{R}$ of the discrete semiflow $\{Q^n\}_{n \ge 0}$ if there exists a countable set $\Gamma \subset \mathbb{R}$ such that $Q[\psi(\cdot + x)](i) = \psi(i + x + c)$ for all $i \in \mathbb{Z}$ and $x \in \mathbb{R} \setminus \Gamma$.

By Definition 3.2 and Proposition 7.3, there exists $x_0 \in \mathbb{R}$ such that $Q^n[\psi(\cdot + x_0)](i) = \psi(i + x_0 + cn)$ for all $i \in \mathbb{Z}$ and $n \ge 0$. Define $\phi(x) := \psi(x + x_0)$ for all $x \in \mathbb{R}$. Then, with a little abuse of notation, $Q^n[\phi](i) = \phi(i + cn)$ for all $i \in \mathbb{Z}$ and $n \ge 0$. Therefore, Definition 3.2 is a generalized version of the classical one, which is an analogue of Definition 3.1.

Let $\beta \gg 0$ be a fixed point of Q. Define $\tilde{Q} : \mathcal{B}_\beta \to \mathcal{B}_\beta$ by

$$\tilde{Q}[\phi](x) = Q[\phi(\cdot + x)](0), \qquad \forall x \in \mathbb{R}.$$

Then we see from [25, Lemma 2.1] that $\tilde{Q}$ satisfies (A1)–(A3) and (A5) with $Q = \tilde{Q}$ and $\mathcal{C}_\beta = \mathcal{B}_\beta$ if Q itself satisfies (A1)–(A3) and (A5). Further, if Q satisfies (A4), then the set $\tilde{Q}[\mathcal{B}_\beta](x) \subset \mathcal{X}_\beta$ is precompact for any $x \in \mathbb{R}$.

For $\tilde{Q} : \mathcal{B}_\beta \to \mathcal{B}_\beta$, we have similar results to Lemmas 3.2 and 3.3.

Lemma 3.5. *Assume that Q satisfies* (A1)–(A3) *and* (A5). *Then there exists a positive rational number $\bar{c}$ such that for any $c \geq \bar{c}$, we have*

$$\tilde{Q}[\underline{\psi}](x) \geq \underline{\psi}(x-c) \quad \text{and} \quad \tilde{Q}[\bar{\psi}](x) \leq \bar{\psi}(x+c) \quad \text{for any } x \in \mathbb{R}.$$

Lemma 3.6. *Assume that Q satisfies* (A1)–(A5). *Then for each $n \in \mathbb{N}$, $\tilde{G}_n := \tilde{Q} \circ A_{\kappa_n}$ has a fixed point $\tilde{\phi}_n$ in $\mathcal{B}_\beta$ such that $\tilde{\phi}_n$ is nondecreasing and $\underline{\psi}_n \leq \tilde{\phi}_n \leq \bar{\psi}_n$.*

Proof. By the same arguments as in the proof of Lemma 3.3, we can obtain a similar inequality to (3.2):

$$\underline{\psi}_n \leq \tilde{G}_n^k[\underline{\psi}_n] \leq \tilde{G}_n^k[\bar{\psi}_n] \leq \bar{\psi}_n, \qquad \forall k \in \mathbb{N}.$$

Define $w_{n,1} := \underline{\psi}_n$ and $w_{n,k+1} := \tilde{G}_n[w_{n,k}]$, $k \geq 1$. Then

$$w_{n,k+1}(x) = \tilde{Q} \circ A_{\kappa_n}[w_{n,k}](x) = \tilde{Q}[w_{n,k}(\kappa_n \cdot)](x) = Q[w_{n,k}(\kappa_n(\cdot + x))](0). \tag{3.5}$$

Note that $Q[\mathcal{C}_\beta]$ is compact and $w_{n,k}$ is nondecreasing in k. Hence, for any fixed $x \in \mathbb{R}$, $w_{n,k}(x)$ converges in $\mathcal{X}_\beta$. Denote the limit by $\tilde{\phi}_n(x)$. Then $\tilde{\phi}_n(x)$ is nondecreasing in $x \in \mathbb{R}$ and $\underline{\psi}_n \leq \tilde{\phi}_n \leq \bar{\psi}_n$. Letting $k \to \infty$ in (3.5), we arrive at $\tilde{\phi}_n(x) = Q[\tilde{\phi}_n(\kappa_n(\cdot + x))](0)$. Consequently,

$$\tilde{\phi}_n = \tilde{Q}[\tilde{\phi}_n(\kappa_n \cdot)] = \tilde{Q} \circ A_{\kappa_n}[\tilde{\phi}_n] = \tilde{G}_n[\tilde{\phi}_n].$$

This completes the proof. □

Due to the lack of compactness for $\tilde{Q}$, we will use the properties of monotone functions established in the Appendix to show the convergence of a sequence in $\tilde{Q}[\mathcal{B}_\beta]$.

Theorem 3.2. *Assume that $\mathcal{X} = C(M, \mathbb{R}^d)$ and Q satisfies* (A1)–(A6). *Then there exists $c \in \mathbb{R}$ such that the semiflow $\{Q^n\}_{n \geq 1}$ on $\mathcal{C}_\beta$ admits a nondecreasing traveling wave $\psi(x+cn)$ with speed c and connecting 0 to β. Further, ψ is either left or right continuous.*

Proof. As in the proof of Theorem 3.1, we define

$$\tilde{a}_n := \sup_{x \in \mathbb{R}}\{\tilde{\phi}_n(x) \in [0, \delta e_0]_{\mathcal{X}}\}, \qquad \tilde{b}_n := \inf_{x \in \mathbb{R}}\{\tilde{\phi}_n(x) \in [\beta - \delta e_\beta, \beta]_{\mathcal{X}}\}.$$

Then $-1 - (n + \bar{c}) \leq \tilde{a}_n \leq \tilde{b}_n \leq 1 + (n + \bar{c})$. Note that for any $x \in \mathbb{R}$, we have

$$\tilde{\phi}_n(x) = \tilde{G}_n[\tilde{\phi}_n](x) = \tilde{Q}[\tilde{\phi}_n(\kappa_n \cdot)](x) = Q[\tilde{\phi}_n(\kappa_n(\cdot + x))](0) \in Q[\mathcal{C}_\beta](0).$$

Since $Q[\mathcal{C}_\beta](0)$ is precompact in $\mathcal{X}_\beta$, for any $x \in \mathbb{R}$ the limits $\tilde{\phi}_n(x^-) := \lim_{y \uparrow x} \tilde{\phi}_n(y)$ and $\tilde{\phi}_n(x^+) := \lim_{y \downarrow x} \tilde{\phi}_n(y)$ both exist. Hence, by the definitions of $\tilde{a}_n$ and $\tilde{b}_n$,

$$\tilde{\phi}_n(\tilde{a}_n^-) \leq \delta e_0 \leq \beta - \delta e_\beta \leq \tilde{\phi}_n(\tilde{b}_n^+),$$

but

$$\tilde{\phi}_n(\tilde{a}_n^+) \notin [0, \delta e_0]]_{\mathcal{X}} \quad \text{and} \quad \tilde{\phi}_n(\tilde{b}_n^-) \notin [[\beta - \delta e_\beta, \beta]_{\mathcal{X}}.$$

Define $\tilde{\phi}_{-,n}(x) := \tilde{\phi}_n(x + \tilde{a}_n)$ and $\tilde{\phi}_{+,n}(x) := \tilde{\phi}_n(x + \tilde{b}_n)$. Then

$$\tilde{\phi}_{-,n}(0^-) \leq \delta e_0 \leq \beta - \delta e_\beta \leq \tilde{\phi}_{+,n}(0^+),$$

but

$$\tilde{\phi}_{-,n}(0^+) \notin [0, \delta e_0]]_{\mathcal{X}} \quad \text{and} \quad \tilde{\phi}_{+,n}(0^-) \notin [[\beta - \delta e_\beta, \beta]_{\mathcal{X}}.$$

Since $\tilde{\phi}_n = \tilde{G}_n[\tilde{\phi}_n]$, we have

$$\tilde{\phi}_{-,n}(x) = \tilde{G}_n[\tilde{\phi}_n](x + \tilde{a}_n) = Q[\tilde{\phi}_n(\kappa_n(\cdot + \tilde{a}_n + x))](0) \in Q[\mathcal{C}_\beta](0).$$

Similarly, $\tilde{\phi}_{+,n}(x) = Q[\tilde{\phi}_n(\kappa_n(\cdot + \tilde{b}_n + x))](0) \in Q[\mathcal{C}_\beta](0)$. Let $\mathbb{Q}$ be the set of all rational numbers, and $\{x_l\}_{l\geq 1} \subset \mathbb{Q}$ be an increasing sequence converging to x. Using $\tilde{\phi}_n = \tilde{G}_n[\tilde{\phi}_n]$ again, we see that for any $i \in \mathbb{Z}$ and $l \geq 1$,

$$\tilde{\phi}_n(\kappa_n(i + \tilde{a}_n + x_l)) = Q[\tilde{\phi}_n(\kappa_n(\cdot + \kappa_n(i + \tilde{a}_n + x_l)))](0) \in Q[\mathcal{C}_\beta](0).$$

Similarly, $\tilde{\phi}_n(\kappa_n(i + \tilde{b}_n + x_l)) \in Q[\mathcal{C}_\beta](0)$. Note that $Q[\mathcal{C}_\beta](0)$ is precompact in $\mathcal{X}_\beta$ and $\mathbb{Q}$ is countable. Hence there exists a subsequence (still indexed by $\{n\}$) and $\xi_- \leq \xi_+ \in \mathbb{R}$ such that $\lim_{n\to\infty} \tilde{a}_n/n = \xi_-$, $\lim_{n\to\infty} \tilde{b}_n/n = \xi_+$ and for any $x \in \mathbb{Q}$, $i \in \mathbb{Z}$ and $l \geq 1$, the sequences $\tilde{\phi}_{\pm,n}(x)$, $\tilde{\phi}_n(\kappa_n(i + \tilde{a}_n + x_l))$ and $\tilde{\phi}_n(\kappa_n(i + \tilde{b}_n + x_l))$ converge in $\mathcal{X}_\beta$. Hence, the limits

$$\begin{aligned}
\lim_{l\to\infty}\lim_{n\to\infty} \tilde{\phi}_{-,n}(x_l) &= \lim_{l\to\infty}\lim_{n\to\infty} Q[\tilde{\phi}_n(\kappa_n(\cdot + \tilde{a}_n + x_l))](0),\\
\lim_{l\to\infty}\lim_{n\to\infty} \tilde{\phi}_{+,n}(x_l) &= \lim_{l\to\infty}\lim_{n\to\infty} Q[\tilde{\phi}_n(\kappa_n(\cdot + \tilde{b}_n + x_l))](0)
\end{aligned}$$

both exist. This means the limits

$$\lim_{y\in\mathbb{Q},\, y\uparrow x}\lim_{n\to\infty} \tilde{\phi}_{\pm,n}(y) \quad \text{and} \quad \lim_{y\in\mathbb{Q},\, y\downarrow x}\lim_{n\to\infty} \tilde{\phi}_{\pm,n}(y)$$

exist for all $x \in \mathbb{R}$. Define

$$\hat{\phi}_-(x) := \begin{cases} \lim_{n\to\infty} \tilde{\phi}_{-,n}(x), & x \in \mathbb{Q},\\ \lim_{y\in\mathbb{Q},\, y\uparrow x} \lim_{n\to\infty} \tilde{\phi}_{-,n}(x), & x \in \mathbb{R}\setminus\mathbb{Q}, \end{cases}$$

$$\hat{\phi}_+(x) := \begin{cases} \lim_{n\to\infty} \tilde{\phi}_{+,n}(x), & x \in \mathbb{Q},\\ \lim_{y\in\mathbb{Q},\, y\downarrow x} \lim_{n\to\infty} \tilde{\phi}_{+,n}(x), & x \in \mathbb{R}\setminus\mathbb{Q}. \end{cases}$$

Clearly, $\hat{\phi}_\pm$ are nondecreasing functions in $\mathcal{B}_\beta$ and for any $x \in \mathbb{R}\setminus\mathbb{Q}$, $\hat{\phi}_\pm(x^\pm)$ all exist. Hence, we see from Theorem 7.1 that there exists a countable subset Γ_1 of $\mathbb{R}$ such that $\tilde{\phi}_{\pm,n}(x)$ converges to $\hat{\phi}_\pm(x)$ for all $x \in \mathbb{R}\setminus\Gamma_1$. Define

$$\tilde{\phi}_-(x) := \lim_{y\in\mathbb{Q},\, y\uparrow x}\lim_{n\to\infty} \tilde{\phi}_{-,n}(y), \qquad \tilde{\phi}_+(x) := \lim_{y\in\mathbb{Q},\, y\downarrow x}\lim_{n\to\infty} \tilde{\phi}_{+,n}(y), \qquad \forall x \in \mathbb{R}.$$

Thus, $\tilde{\phi}_-(x)$ is left continuous and $\tilde{\phi}_+(x)$ is right continuous. Note that $\tilde{\phi}_\pm(x) = \hat{\phi}_\pm(x)$ for all $x \in \mathbb{R}\setminus\mathbb{Q}$. It then follows that $\tilde{\phi}_{\pm,n}(x)$ converges to $\tilde{\phi}_\pm(x)$ for $x \in \mathbb{R}\setminus\Gamma_2$, where $\Gamma_2 := \mathbb{Q}\cup\Gamma_1$ is also countable.

Let $y_k \in \mathbb{R} \setminus \Gamma_2$ be an increasing sequence converging to 0 and $z_k \in \mathbb{R} \setminus \Gamma_2$ be an increasing sequence converging to 1. Note that

$$\tilde{\phi}_-(0) = \lim_{k\to\infty} \tilde{\phi}_-(y_k) = \lim_{k\to\infty}\lim_{n\to\infty} \tilde{\phi}_{-,n}(y_k) \le \delta e_0,$$
$$\tilde{\phi}_-(1) = \lim_{k\to\infty} \tilde{\phi}_-(z_k) = \lim_{k\to\infty}\lim_{n\to\infty} \tilde{\phi}_{-,n}(z_k) \notin [0, \delta e_0]]_{\mathcal{X}}.$$

Similarly, $\tilde{\phi}_+(0) \ge \beta - \delta e_\beta$ but $\tilde{\phi}_+(-1) \notin [[\beta - \delta e_\beta, \beta]_{\mathcal{X}}$. Define $c_- := -\bar{c}\xi_-$ and $c_+ := -\bar{c}\xi_+$. Obviously, $c_- \ge c_+$ since $\xi_- \le \xi_+$.

Now we want to prove $Q[\tilde{\phi}_-(\cdot + x)](0) = \tilde{\phi}_-(x + c_-)$ for all $x \in \mathbb{R} \setminus \Gamma_2$. Note that

$$\lim_{n\to\infty} \kappa_n(x + \tilde{a}_n) - \tilde{a}_n = \lim_{n\to\infty}\left(x + \bar{c}\cdot\frac{x + \tilde{a}_n}{n}\right) = x - c_-.$$

It follows that

$$\begin{aligned}\tilde{\phi}_-(x+c_-) &= \lim_{n\to\infty} \tilde{\phi}_{-,n}(x+c_-) = \lim_{n\to\infty} \tilde{\phi}_n(x+c_-+\tilde{a}_n) = \lim_{n\to\infty} \tilde{G}_n[\tilde{\phi}_n](x+c_-+\tilde{a}_n)\\ &= \lim_{n\to\infty} \tilde{Q}[\tilde{\phi}_n(\kappa_n\cdot)](x+c_-+\tilde{a}_n) = \lim_{n\to\infty} \tilde{Q}[\tilde{\phi}_n(\kappa_n(\cdot+a_n))](x+c_-)\\ &= \lim_{n\to\infty} \tilde{Q}[\tilde{\phi}_{-,n}(\kappa_n(\cdot+\tilde{a}_n)-\tilde{a}_n)](x+c_-)\\ &= \lim_{n\to\infty} Q[\tilde{\phi}_{-,n}(\kappa_n(\cdot+x+c_-+\tilde{a}_n)-\tilde{a}_n)](0).\end{aligned}$$

In view of Proposition 7.5, we obtain $\tilde{\phi}_-(x + c_-) = Q[\tilde{\phi}_-(\cdot + x)](0)$ for all $x \in \mathbb{R} \setminus \Gamma_2$. A similar result holds for $\tilde{\phi}_+$.

Now, the same argument as in the proof of Theorem 3.1 completes the proof. □

3.3. *Time-periodic semiflows*

Let $\omega \in \mathcal{T}$, where $\mathcal{T} = \mathbb{R}^+$ or $\mathbb{Z}^+$. Recall that a family $\{Q_t\}_{t\in\mathcal{T}}$ of mappings is said to be an *ω-time periodic semiflow* on a metric space $\mathcal{E} \subset \mathcal{C}$ provided that:

(i) $Q_0[\phi] = \phi$ for all $\phi \in \mathcal{E}$.
(ii) $Q_t \circ Q_\omega[\phi] = Q_{t+\omega}[\phi]$ for all $t \ge 0$ and $\phi \in \mathcal{E}$.
(iii) $Q_t[\phi]$ is jointly continuous in (t, ϕ) on $[0, \infty) \times \mathcal{E}$.

The mapping Q_ω is called the *Poincaré map* associated with this periodic semiflow.

Definition 3.3. (i) In the case where $\mathcal{H} = \mathbb{R}$, $U(t, x + ct)$ is said to be an *ω-time periodic traveling wave with speed c* of the semiflow $\{Q_t\}_{t\in\mathcal{T}}$ if $Q_t[U(0,\cdot)](x) = U(t, x + ct)$ and $U(t, x) = U(t + \omega, x)$ for all $t \in \mathcal{T}$ and $x \in \mathbb{R}$.

(ii) In the case where $\mathcal{H} = \mathbb{Z}$, $U(t, x + ct)$ is said to be an *ω-time periodic traveling wave with speed c* of $\{Q_t\}_{t\in\mathcal{T}}$ if there exists a countable subset $\Gamma \subset \mathbb{R}$ such that $Q_t[U(0, \cdot+x)](0) = U(t, x+ct)$ for all $t \in \mathcal{T}$ and $x \in \mathbb{R}$, and $U(t, x) = U(t+\omega, x)$ for all $t \in \mathcal{T}$ and $x \in \mathbb{R} \setminus \Gamma$.

Theorem 3.3. *Let $\beta(t)$ be a strongly positive ω-time periodic orbit of $\{Q_t\}_{t\in\mathcal{T}}$ restricted to $\mathcal{X}$. Assume that $Q := Q_\omega$ satisfies hypotheses* (A1)–(A6) *with $\beta = \beta(0)$. Then $\{Q_t\}_{t\in\mathcal{T}}$ admits a traveling wave $U(t, x + ct)$ with $U(t, -\infty) = 0$ and $U(t, \infty) = \beta(t)$ uniformly for $t \in \mathcal{T}$. Furthermore, $U(t, x)$ is nondecreasing in $x \in \mathbb{R}$.*

Proof. Assume that $\mathcal{H} = \mathbb{R}$. Since Q_ω satisfies (A1)–(A6), there exist $c \in \mathbb{R}$ and a nondecreasing function $\phi \in \mathcal{C}$ connecting 0 to $\beta(0)$ such that $Q_\omega[\phi](x) = \phi(x + c\omega)$. Clearly, $T_{c\omega} Q_\omega[\phi] = \phi$. Define $U(t, x) := T_{ct} Q_t[\phi](x)$. Then $U(t, x + ct) = Q_t[\phi](x) = Q_t[U(0, \cdot)](x)$, and

$$U(t + \omega, x) = T_{ct+c\omega} Q_{t+\omega}[\phi](x) = T_{ct} Q_t T_{c\omega} Q_\omega[\phi](x) = T_{ct} Q_t[\phi](x) = U(t, x).$$

Note that $Q_t[\beta(0)] = \beta(t)$ and ϕ is nondecreasing and connects 0 to $\beta(0)$. It follows that $U(t, -\infty) = 0$ and $U(t, \infty) = \beta(t)$.

Assume now $\mathcal{H} = \mathbb{Z}$. Since Q_ω satisfies (A1)–(A6), there exists $c \in \mathbb{R}$, a countable subset $\Gamma \subset \mathbb{R}$ and a nondecreasing function $\phi \in \mathcal{B}$ connecting 0 to $\beta(0)$ such that $\tilde{Q}_\omega[\phi](x) = \phi(x + c\omega)$ for all $x \in \mathbb{R} \setminus \Gamma$. Clearly, $T_{c\omega} \tilde{Q}_\omega[\phi](x) = \phi(x)$ for all $x \in \mathbb{R} \setminus \Gamma$. Define $U(t, x) := T_{ct} \tilde{Q}_t[\phi](x)$. Thus,

$$U(t, x + ct) = \tilde{Q}_t[\phi](x) = \tilde{Q}_t[U(0, \cdot)](x) = Q_t[U(0, \cdot + x)](0), \quad \forall x \in \mathbb{R},$$

and

$$U(t + \omega, x) = T_{ct+c\omega} \tilde{Q}_{t+\omega}[\phi](x) = T_{ct} \tilde{Q}_t T_{c\omega} \tilde{Q}_\omega[\phi](x) = T_{ct} \tilde{Q}_t[\phi](x) = U(t, x)$$

for all $x \in \mathbb{R} \setminus \Gamma$. Note that $Q_t[\beta(0)] = \beta(t)$ and ϕ is nondecreasing and connects 0 to $\beta(0)$. Hence $U(t, -\infty) = 0$ and $U(t, \infty) = \beta(t)$. □

3.4. Continuous-time semiflows in a continuous habitat

In this subsection, we consider continuous-time semiflows in the continuous habitat $\mathcal{H} = \mathbb{R}$. Recall that a family $\{Q_t\}_{t\geq 0}$ of mappings $Q_t : \mathcal{E} \to \mathcal{E}$ is said to be a *semiflow* on a metric space $\mathcal{E} \subset \mathcal{C}$ provided that

(1) $Q_0[\phi] = \phi$ for all $\phi \in \mathcal{E}$.
(2) $Q_t \circ Q_s[\phi] = Q_{t+s}[\phi]$ for all $t, s \geq 0$ and $\phi \in \mathcal{E}$.
(3) $Q_t[\phi]$ is jointly continuous in (t, ϕ) on $[0, \infty) \times \mathcal{E}$.

Before moving to the study of traveling waves of $\{Q_t\}_{t\geq 0}$, we first investigate the spatially homogeneous system, that is, the system restricted to $\mathcal{X}$. Let $\beta \gg 0$ be an equilibrium in $\mathcal{X}$. For each $t > 0$, we use Σ_t to denote the set of all fixed points of Q_t restricted to $\mathcal{X}_\beta$. Clearly, the equilibrium set of the semiflow is $\Sigma := \bigcap_{t>0} \Sigma_t$, which is a subset of Σ_t for any $t > 0$. The subsequent result indicates that the instability of intermediate equilibria of the semiflow implies the unordering property of all intermediate fixed points of each time-t map.

Proposition 3.1. *For any given $t > 0$, if the map Q_t satisfies the bistability assumption* (A5$'$) *with $E = \Sigma$, then Q_t satisfies* (A5) *with $E = \Sigma_t$.*

Proof. Let $t_0 > 0$ be given. We first show that any two points $u \in \Sigma \setminus \{0, \beta\}$ and $v \in \Sigma_{t_0} \setminus \{0, \beta\}$ are unordered. Assume for contradiction that u and v are ordered, say $u < v$. Then the eventual strong monotonicity implies that $u \ll v$. Since u is strongly unstable from above, there exist a unit vector $e \in \mathrm{Int}(\mathcal{X}^+)$ and $\delta > 0$ such that $Q_{t_0}[u+\delta e] \gg u+\delta e$ with $u + \delta e \in [[u, v]]_{\mathcal{X}}$. From [34, Theorem 1.2.1], $(Q_{t_0})^n[u + \delta e]$ is eventually strongly increasing and converges to some $\alpha \in \Sigma$. Note that $\alpha \in [[u, v]_{\mathcal{X}}$ is strongly unstable from below. Hence, by the same arguments as in the proof of Proposition 2.1, we obtain a contradiction.

Next we show that $\Sigma_{t_0} \setminus \Sigma$ is unordered. By the first step it suffices to prove that for any two ordered elements $u < v$ in $\Sigma_{t_0} \setminus \Sigma$, $[u, v]_{\mathcal{X}} \cap \Sigma \neq \emptyset$. Indeed, by eventual strong monotonicity, we have $u \ll v$. Thus, we can choose a sequence $\{u_n\}$ on the segment connecting u and v such that $u \ll u_n \ll u_{n+1} \ll v$ for all $n \geq 1$. By [34, Theorem 1.3.7], $\omega(u) \leq \omega(u_n) \leq \omega(u_{n+1}) \leq \omega(v)$ for all $n \geq 1$. Clearly, $\omega(u) = \{Q_t u : t \in [0, t_0]\}$ and $\omega(v) = \{Q_t v : t \in [0, t_0]\}$, and hence $u \leq \omega(u_n) \leq v$ for all $n \geq 1$. Note that $\bigcup_{n\geq 1} \omega(u_n)$ is contained in the compact set $\overline{Q_{t_0}[\mathcal{X}_\beta]}$. In the compact metric space consisting of all nonempty compact subsets of $\overline{Q_{t_0}[\mathcal{X}_\beta]}$ with Hausdorff distance d_H, the sequence $\{\omega(u_n) : n \geq 1\}$ has a convergent subsequence. Without loss of generality, we assume that for some nonempty compact set $\varpi \subset \overline{Q_{t_0}[\mathcal{X}_\beta]}$, $\lim_{n\to\infty} d_H(\omega(u_n), \varpi) = 0$. Since each $\omega(u_n)$ is invariant for the semiflow $\{Q_t\}_{t\geq 0}$, so is the compact set ϖ, that is, $Q_t\varpi = \varpi$ for all $t \geq 0$. For any given $x, y \in \omega$, there exist sequences of points $x_n, y_n \in \varpi(u_n)$ such that $x_n \to x$ and $y_n \to y$ as $n \to \infty$. Since $\omega(u_n) \leq \omega(u_{n+1})$, we have $x_n \leq y_{n+1}$ and $y_n \leq x_{n+1}$ for all $n \geq 1$. Letting $n \to \infty$, we then have $x \leq y$ and $y \leq x$, and hence $x = y$. This implies that ϖ is a singleton, that is, $\varpi = \{\alpha\}$. By the invariance of ϖ for the semiflow, α is an equilibrium. Since $u \leq \omega(u_n) \leq v$ for all $n \geq 1$, it follows that $\alpha \in [u, v]_X$. □

For a continuous-time semiflow $\{Q_t\}_{t\geq 0}$, we need the following definition of traveling waves.

Definition 3.4. $\psi(x + ct)$ with $\psi \in \mathcal{C}$ is said to be a *traveling wave with speed* $c \in \mathbb{R}$ of the continuous-time semiflow $\{Q_t\}_{t\geq 0}$ if $Q_t[\psi](x) = \psi(x + ct)$ for all $x \in \mathbb{R}$ and $t \geq 0$. We say that ψ *connects* 0 *to* β if $\psi(-\infty) = 0$ and $\psi(\infty) = \beta$.

Theorem 3.4. *Assume that for each $t > 0$, the map Q_t satisfies assumptions* (A1) *and* (A3)–(A5) *with $E = \Sigma_t$, and the time-one map Q_1 satisfies* (A6) *with $E = \Sigma$. Then there exists $c \in \mathbb{R}$ such that $\{Q_t\}_{t\geq 0}$ admits a nondecreasing traveling wave with speed c and connecting* 0 *to* β .

Proof. Let e_0, e_β and δ be chosen as in (2.7), (2.3) and (3.1), respectively. We proceed in three steps.

Firstly, we show that there exists $s_k \downarrow 0$ such that each discrete semiflow $\{Q^n_{s_k}\}_{n\geq 0}$ admits nondecreasing traveling waves $\psi_{\pm,s_k}(x + c_{\pm,s_k} n)$ with $c_{-,s_k} \geq c_{+,s_k}$ and

$$0 < \psi_{-,s_k}(0) \leq \delta e_0 \quad \text{and} \quad \beta - \delta e_\beta \leq \psi_{+,s_k}(0) < \beta,$$

but

$$\psi_{-,s_k}(0) \notin [0, \delta e_0]]_{\mathcal{X}} \quad \text{and} \quad \psi_{+,s_k}(0) \notin [[\beta - \delta e_\beta, \beta]_{\mathcal{X}}.$$

Indeed, since for each $s > 0$ the map Q_s satisfies (A1)–(A5), from the first two steps of the proof of Theorem 3.1 we see that for the discrete semiflow $\{(Q_s)^n\}_{n\geq 0}$, there exist two nondecreasing traveling waves $\phi_{\pm,s}(x + c_{\pm,s}t)$ with the following properties:

(1) $\phi_{-,s}$ connects 0 to some $\alpha_{-,s} \in E_s \setminus \{0\}$ and $\phi_{+,s}$ connects some $\alpha_{+,s} \in E_s \setminus \{\beta\}$ to β;
(2) $\alpha_{-,s}$ and $\alpha_{+,s}$ are ordered and $c_{-,s} \geq c_{+,s}$.

By a similar argument to that in [47, Theorem 1.3.7], both $\alpha_{\pm,s}$ have subsequences $\alpha_{\pm,s_k}$ which tend to two equilibria of the semiflow as $s_k \to 0$, say α_- and α_+. Since $\alpha_{-,s}$ and $\alpha_{+,s}$ are ordered, it follows from Proposition 3.1 that there are only three possibilities for the relation of α_- and α_+:

$$\text{(i) } \beta = \alpha_- \geq \alpha_+; \text{ (ii) } \alpha_- \geq \alpha_+ = 0; \text{ (iii) } \alpha_- = \alpha_+ \in E \setminus \{0, \beta\}.$$

If $\alpha_- = \beta$, then for sufficiently large k we can define

$$a_{s_k} := \sup\{x \in \mathbb{R} : \phi_{-,s_k}(x) \in [0, \delta e_0]_{\mathcal{X}}\},$$
$$b_{s_k} := \inf\{x \in \mathbb{R} : \phi_{-,s_k}(x) \in [\beta - \delta e_\beta, \beta]_{\mathcal{X}}\}.$$

Hence, $\psi_{-,s}(x) := \psi_s(x + a_s)$ and $\psi_{+,s}(x) := \psi_s(x + b_s)$ are the required traveling waves. If $\alpha_+ = 0$, then for sufficiently large k we can define

$$a_{s_k} := \sup\{x \in \mathbb{R} : \phi_{+,s_k}(x) \in [0, \delta e_0]_{\mathcal{X}}\},$$
$$b_{s_k} := \inf\{x \in \mathbb{R} : \phi_{+,s_k}(x) \in [\beta - \delta e_\beta, \beta]_{\mathcal{X}}\}.$$

Hence, $\psi_{-,s_k}(x) := \phi_{+,s_k}(x + a_{s_k})$ and $\psi_{+,s_k}(x) := \phi_{+,s_k}(x + b_{s_k})$ are the required traveling waves. If $\alpha_- = \alpha_+ \in E \setminus \{0, \beta\}$, then by Lemma 3.1 we have $\alpha_- = \alpha_+ \in E \setminus \{[0, \delta e_0]_{\mathcal{X}} \cup [\beta - \delta e_\beta, \beta]_{\mathcal{X}}\}$. Consequently, for sufficiently large k we can define

$$a_{s_k} := \sup\{x \in \mathbb{R} : \phi_{-,s_k}(x) \in [0, \delta e_0]_{\mathcal{X}}\},$$
$$b_{s_k} := \inf\{x \in \mathbb{R} : \phi_{-,s_k}(x) \in [\beta - \delta e_\beta, \beta]_{\mathcal{X}}\}.$$

Hence, $\psi_{-,s_k}(x) := \phi_{-,s_k}(x + a_{s_k})$ and $\psi_{+,s_k}(x) := \phi_{-,s_k}(x + b_{s_k})$ are the required traveling waves.

Secondly, we show that there exists a subsequence, still denoted by s_k, such that $\psi_{\pm,s_k} \to \psi_\pm$ in $\mathcal{C}_\beta$ and $(1/s_k)c_{\pm,s_k} \to c_\pm \in \mathbb{R}$. Indeed, for each $s_k > 0$, there exists an integer $m_k > 0$ such that $m_k s_k > 2$. Then

$$\psi_{-,s_k} = T_{m_k c_{s_k}} \circ Q_{m_k s_k}[\psi_{-,s_k}] = Q_2 \circ Q_{m_k s_k - 2} \circ T_{m_k c_{s_k}}[\psi_{-,s_k}] \in Q_1 \circ Q_1[\mathcal{C}_\beta]. \tag{3.6}$$

Clearly, the compactness of Q_1 implies that $Q_1 \circ Q_1[\mathcal{C}_\beta]$ is precompact in $\mathcal{C}_\beta$. Thus, there exists a subsequence, still denoted by s_k, and nonincreasing functions $\psi_-, \psi_+ \in \mathcal{C}_\beta$ with $0 < \psi_-(0) \leq \delta e_0$ and $\beta - \delta e_\beta \leq \psi_+(0) < \beta$ such that $\psi_{-,s_k} \to \psi_-$ and $\psi_{+,s_k} \to \psi_+$ in $\mathcal{C}_\beta$. Also we claim that $\psi_{\pm,s_k}(\pm\infty)$ all exist. Indeed, from (3.6) there exists $\phi_{s_k} \in \mathcal{C}_\beta$ such that $Q_1 \circ Q_1[\phi_{s_k}] \to \psi_-$. Note that $\{Q_1[\phi_{s_k}]\}_{k\geq 1}$ also has a convergent subsequence with limit $\phi \in \mathcal{C}_\beta$. Hence, by the uniqueness of limit we have $Q_1[\phi] = \psi_-$. Note that

$\psi_-(k) = Q_1[\phi](k) = Q_1[\phi(\cdot + k)](0)$ and $\{Q_1[\phi(\cdot + k)]\}_{k\geq 1}$ has a convergent subsequence. Hence, $\psi_-(\pm\infty)$ exist because ψ_- is nonincreasing. Similarly, $\psi_+(\pm\infty)$ exist. Also,

$$\psi_-(-\infty) \leq \psi_-(0) \leq \delta e_0 \quad \text{and} \quad \psi_+(\infty) \geq \psi_+(0) \geq \beta - \delta e_\beta, \tag{3.7}$$

but

$$\psi_-(0) \notin [0, \delta e_0]]_{\mathcal{X}} \quad \text{and} \quad \psi_+(0) \notin [[\beta - \delta e_\beta, \beta]_{\mathcal{X}}.$$

Consequently, by the monotonicity of $\psi_\pm$,

$$\psi_-(x) \notin [0, \delta e_0]]_{\mathcal{X}},\ \forall x > 0, \quad \text{and} \quad \psi_+(x) \notin [[\beta - \delta e_\beta, \beta]_{\mathcal{X}},\ \forall x < 0. \tag{3.8}$$

Since ψ_- and ψ_+ are the limits of sequences of monotone functions with different translations, we can employ the same arguments as in the second step of the proof of Theorem 3.1 to show that $\psi_-(\infty)$ and $\psi_+(-\infty)$ are ordered.

To prove that $(1/s_k)c_{\pm,s_k}$ have convergent subsequences, we only need to prove that $(1/s_k)c_{-,s_k}$ is bounded above and $(1/s_k)c_{+,s_k}$ is bounded below because $c_{-,s_k} \geq c_{+,s_k}$. Assume for contradiction that some subsequence, still say $(1/s_k)c_{-,s_k}$, tends to ∞. Note that for each $s > 0$ there exists $n_s \in \mathbb{Z}^+$ such that the integer part of $1/s$, denoted by $\langle 1/s\rangle$, equals n_s and $1/(n_s + 1) < s \leq 1/n_s$. Hence, $s\langle 1/s\rangle \to 1$ as $s \to 0$. Then

$$\lim_{k\to\infty} \left\langle \frac{1}{s_k}\right\rangle c_{-,s_k} = \lim_{k\to\infty} \frac{1}{s_k} c_{-,s_k} \cdot s_k \left\langle \frac{1}{s_k}\right\rangle = \lim_{k\to\infty} \frac{1}{s_k} c_{-,s_k} = \infty.$$

Thus, using the first observation in (3.8), we have

$$\begin{aligned} Q_1[\delta e_0] &\geq Q_1[\psi_-(-\infty)] = Q_1[\psi_-(-\infty)](0) = \lim_{x\to-\infty} Q_1[\psi_-(\cdot + x)](0) \\ &= \lim_{x\to-\infty} Q_1[\psi_-](x) = \lim_{x\to-\infty} \lim_{k\to\infty} (Q_{s_k})^{\langle 1/s_k\rangle}[\psi_{-,s_k}](x) \\ &= \lim_{x\to-\infty} \lim_{k\to\infty} \psi_{-,s_k}\left(x + \left\langle \frac{1}{s_k}\right\rangle c_{-,s_k}\right) \geq \lim_{x\to-\infty} \lim_{y\to\infty} \lim_{k\to\infty} \psi_{-,s_k}(y) \\ &= \lim_{y\to\infty} \psi_-(y) = \psi_-(\infty) \notin [0, \delta e_0]]_{\mathcal{X}}, \end{aligned} \tag{3.9}$$

which contradicts $Q_1[\delta e_0] \ll \delta e_0$. Similarly, if $(1/s_k)c_{+,s_k} \to -\infty$, then the second observation in (3.8) implies that

$$\begin{aligned} Q_1[\beta - \delta e_\beta] &\leq Q_1[\psi_+(\infty)] = Q_1[\psi_+(\infty)](0) = \lim_{x\to\infty} Q_1[\psi_+(\cdot + x)](0) \\ &= \lim_{x\to\infty} Q_1[\psi_+](x) = \lim_{x\to\infty} \lim_{k\to\infty} (Q_{s_k})^{\langle 1/s_k\rangle}[\psi_{+,s_k}](x) \\ &= \lim_{x\to\infty} \lim_{k\to\infty} \psi_{+,s_k}\left(x + \left\langle \frac{1}{s_k}\right\rangle c_{+,s_k}\right) \leq \lim_{x\to\infty} \lim_{y\to-\infty} \lim_{k\to\infty} \psi_{+,s_k}(y) \\ &= \lim_{y\to-\infty} \psi_+(y) = \psi_+(-\infty) \notin [[\beta - \delta e_\beta, \beta]_{\mathcal{X}}, \end{aligned} \tag{3.10}$$

which contradicts $Q_1[\beta - \delta e_\beta] \gg \beta - \delta e_\beta$. Consequently, $(1/s_k)c_{\pm,s_k}$ are bounded.

Finally, we show that either $\psi_-(x + c_- t)$ or $\psi_+(x + c_+ t)$ established in the second step is a traveling wave connecting 0 to β. Indeed, for any $t > 0$, there exist $m_k \in \mathbb{Z}$ and $r_k \in [0, s_k)$ such that $t = m_k s_k - r_k$. Clearly, $r_k \to 0$ as $k \to \infty$. Then

$$\begin{aligned} Q_t[\psi_\pm] &= \lim_{k\to\infty} Q_{t+r_k}[\psi_{\pm,s_k}] = \lim_{k\to\infty} Q_{m_k s_k}[\psi_{\pm,s_k}] = \lim_{k\to\infty} \psi_{\pm,s_k}(\cdot + m_k c_{\pm,s_k}) \\ &= \lim_{k\to\infty} \psi_{\pm,s_k}\left(\cdot + (t + r_k)\frac{1}{s_k} c_{\pm,s_k}\right) = \psi_\pm(\cdot + c_\pm t), \end{aligned}$$

where the last equality follows from Proposition 7.2(2). From the equality $Q_t[\psi_\pm] = \psi_\pm(\cdot + ct)$ for all $t \geq 0$, we see that $\psi(\pm\infty)$ are equilibria. Recall that $\psi_-(-\infty) \leq \delta e_0 \leq \psi(\infty)$ and $\psi_+(\infty) \geq \beta - \delta e_\beta \geq \psi_+(-\infty)$. Then $\psi_-(-\infty) = 0$, $\psi_+(\infty) = \beta$, and there are only three possibilities for $\psi_-(\infty)$ and $\psi_+(-\infty)$:

(i) $\beta = \psi_-(\infty) > \psi_+(-\infty)$;
(ii) $\psi_-(\infty) > \psi_+(-\infty) = 0$;
(iii) $\psi_-(\infty) = \alpha = \psi_+(-\infty)$ for some $\alpha \in \Sigma \setminus \{0, \beta\}$.

Since the time-one map Q_1 satisfies (A6) with $E = \Sigma$, we can employ the same arguments as in the proof of Lemma 3.4 to exclude (iii). Thus, either (i) or (ii) holds, completing the proof. □

3.5. Continuous-time semiflows in a discrete habitat

In this case, $\mathcal{T} = \mathbb{R}^+$ and $\mathcal{H} = \mathbb{Z}$. Let $\beta \gg 0$ be an equilibrium of the semiflow $\{Q_t\}_{t\geq 0}$. We start with the definition of traveling waves for this case.

Definition 3.5. $\psi(i + ct)$ with $\psi \in \mathcal{B}_\beta$ is said to be a *traveling wave with speed* $c \in \mathbb{R}$ of the continuous-time semiflow $\{Q_t\}_{t\geq 0}$ if $Q_t[\psi](i) = \psi(i + ct)$ for all $i \in \mathbb{Z}$ and $t \geq 0$. Clearly, ψ is continuous if $c \neq 0$.

For each $t > 0$, define $\tilde{Q}_t : \mathcal{B}_\beta \to \mathcal{B}_\beta$ by $\tilde{Q}_s[\phi](x) = Q_s[\phi(\cdot + x)](0)$. Then it is easy to see the following result holds.

Lemma 3.7. $\{\tilde{Q}_t\}_{t\geq 0}$ *has the following properties:*

(i) $\tilde{Q}_0[\phi] = \phi$ *for all* $\phi \in \mathcal{B}$.
(ii) $\tilde{Q}_t \circ \tilde{Q}_s[\phi] = \tilde{Q}_{t+s}[\phi]$ *for all* $t, s \geq 0$ *and* $\phi \in \mathcal{B}$.
(iii) *For fixed* $x \in \mathbb{R}$, *if* $t_n \to t$ *and* $\phi_n(i + x) \to \phi(i + x)$ *in* $\mathcal{X}$ *for any* $i \in \mathbb{Z}$, *then* $\tilde{Q}_{t_n}[\phi_n](x) \to \tilde{Q}_t[\phi](x)$ *in* $\mathcal{X}$.

We combine the ideas in the proofs of Theorems 3.2 and 3.4 to prove the following result for continuous-time semiflows in a discrete habitat.

Theorem 3.5. *Let* $\mathcal{X} = C(M, \mathbb{R}^d)$. *Assume that for each* $t > 0$, *the map* Q_t *satisfies* (A1) *and* (A3)–(A5) *with* $E = \Sigma_t$, *and the time-one map* Q_1 *satisfies* (A6) *with* $E = \Sigma$. *Then there exists* $c \in \mathbb{R}$ *such that* $\{Q_t\}_{t\geq 0}$ *admits a nondecreasing traveling wave with speed* c *and connecting* 0 *to* β.

Proof. Let δ, e_0, e_β be chosen as in (3.1), (2.7) and (2.3). We proceed in three steps.

Firstly, since for any $s > 0$ the map Q_s satisfies (A1)–(A5), the proofs of Theorems 3.2 and 3.4 show that there exists $s_k \downarrow 0$ such that $\{(Q_{s_k})^n\}_{n\geq 0}$ admits two nondecreasing traveling waves $\tilde{\psi}_{\pm,s_k}(x + c_{\pm,s_k}n)$ with $c_{-,s_k} \geq c_{+,s_k}$, that is, there exists a countable subset Θ_k such that

$$\tilde{Q}_{s_k}[\tilde{\psi}_{\pm,s_k}](x) = \tilde{\psi}_{\pm,s_k}(x + c_{\pm,s_k}), \qquad \forall x \in \mathbb{R} \setminus \Theta_k.$$

Furthermore, $\tilde{\psi}_{-,s_k}$ is left continuous and $\tilde{\psi}_{+,s_k}$ is right continuous, with

$$0 < \tilde{\psi}_{-,s_k}(0) \leq \delta e_0 \quad \text{and} \quad \beta - \delta e_\beta \leq \tilde{\psi}_{+,s_k}(0) < \beta,$$

but

$$\tilde{\psi}_{-,s_k}(0) \notin [0, \delta e_0]]_{\mathcal{X}} \quad \text{and} \quad \tilde{\psi}_{+,s_k}(0) \notin [[\beta - \delta e_\beta, \beta]_{\mathcal{X}}.$$

Secondly, we show that for the above sequence s_k, there exist a countable set $\tilde{\Gamma} \subset \mathbb{R}$ and a subsequence, still denoted by s_k, such that $(1/s_k)c_{\pm,s_k} \to c_\pm \in \mathbb{R}$ and $\tilde{\psi}_{\pm,s_k}(x)$ converges in $\mathcal{X}$ for all $x \in \mathbb{R} \setminus \tilde{\Gamma}$. Indeed, let $\Theta = \bigcup_{k=1}^{\infty} \Theta_k$. Hence, Θ is countable and

$$\tilde{Q}_{s_k}[\tilde{\psi}_{\pm,s_k}](x) = \tilde{\psi}_{\pm,s_k}(x + c_{\pm,s_k}), \qquad \forall k \geq 1,\ x \in \mathbb{R} \setminus \Theta.$$

From Proposition 7.4, there exists another countably dense set $\Gamma \subset \mathbb{R}$ such that $\Gamma \cap \Theta = \emptyset$. By the same arguments as in the proof of Theorem 3.2, the limits

$$\tilde{\psi}_-(x) := \lim_{y\in\Gamma, y\uparrow x} \lim_{k\to\infty} \tilde{\psi}_{-,s_k}(y), \qquad \tilde{\psi}_+(x) := \lim_{y\in\Gamma, y\downarrow x} \lim_{k\to\infty} \tilde{\psi}_{+,s_k}(y), \qquad \forall x \in \mathbb{R},$$

are well defined and all $\tilde{\psi}_\pm(\pm\infty)$ exist. Furthermore, $\tilde{\psi}_-(\infty)$ and $\tilde{\psi}_+(-\infty)$ are ordered in $\mathcal{X}$, and

$$\tilde{\psi}_-(-\infty) \leq \psi_-(0) \leq \delta e_0 \quad \text{and} \quad \tilde{\psi}_+(\infty) \geq \tilde{\psi}_+(0) \geq \beta - \delta e_\beta,$$

but

$$\tilde{\psi}_-(0) \notin [0, \delta e_0]]_{\mathcal{X}} \quad \text{and} \quad \tilde{\psi}_+(0) \notin [[\beta - \delta e_\beta, \beta]_{\mathcal{X}}.$$

Further, $\tilde{\psi}_\pm(x^\pm)$ exist for all $x \in \mathbb{R} \setminus \Gamma$. Hence, Theorem 7.1 yields a countable subset $\tilde{\Gamma}$ of $\mathbb{R}$ such that

$$\tilde{\psi}_{\pm,s_k}(x) \to \tilde{\psi}_\pm(x), \qquad \forall x \in \mathbb{R} \setminus \tilde{\Gamma}. \tag{3.11}$$

By similar arguments to the second step of the proof of Theorem 3.4, $(1/s_k)c_{\pm,s_k}$ are bounded.

Finally, we prove that either $\tilde{\psi}_-(x+c_-t)$ or $\tilde{\psi}_+(x+c_+t)$ is a nondecreasing traveling wave connecting 0 to β. Indeed, from (3.11) and Proposition 7.3, there exists a countable subset Γ_1 of $\mathbb{R}$ such that

$$\tilde{\psi}_{\pm,s_k}(i + x) \to \tilde{\psi}_\pm(i + x), \qquad \forall i \in \mathbb{Z},\ x \in \mathbb{R} \setminus \Gamma_1.$$

Hence, for any $x \in \mathbb{R} \setminus \Gamma_1$ and $t > 0$,

$$\begin{aligned}\tilde{Q}_t[\tilde{\psi}_-](x) &= Q_t[\tilde{\psi}_-(\cdot + x)](0) = \lim_{k\to\infty} Q_{t+r_k}[\tilde{\psi}_{-,s_k}(\cdot + x)](0)\\ &= \lim_{k\to\infty} Q_{m_k s_k}[\tilde{\psi}_{-,s_k}(\cdot + x)](0) = \lim_{k\to\infty} \tilde{Q}_{m_k s_k}[\tilde{\psi}_{-,s_k}](x)\\ &= \lim_{k\to\infty} (\tilde{Q}_{s_k})^{m_k}[\tilde{\psi}_{-,s_k}](x) = \lim_{k\to\infty} \tilde{\psi}_{-,s_k}(x + m_k c_{-,s_k})\\ &= \lim_{k\to\infty} \tilde{\psi}_{-,s_k}\left(x + (t + r_k)\frac{1}{s_k}c_{-,s_k}\right). \end{aligned} \tag{3.12}$$

If $c_- = 0$, we can choose x_0 such that

$$Q_t[\tilde{\psi}_-(x_0 + \cdot)](i) = \lim_{k\to\infty} \tilde{\psi}_{-,s_k}\left(x + (t + r_k)\frac{1}{s_k}c_{-,s_k}\right) = \tilde{\psi}_-(x_0 + i), \quad \forall i \in \mathbb{Z}.$$

If $c_- \neq 0$, there exists a countable subset Γ_2 of $\mathbb{R}$ such that

$$\tilde{Q}_t[\tilde{\psi}_-](x) = \lim_{k\to\infty} \tilde{\psi}_{-,s_k}\left(x+(t+r_k)\frac{1}{s_k}c_{-,s_k}\right) = \tilde{\psi}_-(x+c_-t), \quad \forall x \notin \Gamma_1,\ x+c_-t \in \Gamma_2.$$

Without loss of generality, we assume that $c_- > 0$. For any $y \in \mathbb{R}$, we can choose $x_0 \in \mathbb{R}$ and $t_0 \geq 0$ such that $x_0 + c_-t_0 = y$ and $\tilde{\psi}_-(x)$ is continuous at $x = x_0 + i$ for all $i \in \mathbb{Z}$. Now one can find $x_{\pm,k} \in \mathbb{R} \setminus \Gamma_1$ and $t_{\pm,k} \to t_0$ with $y_{\pm,k} := x_\pm + c_-t_{\pm,k} \in \mathbb{R} \setminus \Gamma_2$ such that $y_{-,k} \uparrow y$ and $y_{+,k} \downarrow y$. Note that

$$\begin{aligned}\tilde{\psi}_-(y^-) &:= \lim_{k\to\infty} \tilde{\psi}_-(y_{-,k}) = \lim_{k\to\infty} Q_{t_{-,k}}[\tilde{\psi}_-(\cdot + x_{-,k})](0) = Q_{t_0}[\tilde{\psi}_-(\cdot + x_0)](0),\\ \tilde{\psi}_-(y^+) &:= \lim_{k\to\infty} \tilde{\psi}_-(y_{+,k}) = \lim_{k\to\infty} Q_{t_{+,k}}[\tilde{\psi}_-(\cdot + x_{+,k})](0) = Q_{t_0}[\tilde{\psi}_-(\cdot + x_0)](0).\end{aligned}$$

Thus, $\tilde{\psi}_-(x)$ is continuous in $x \in \mathbb{R}$. Hence, again by Proposition 7.5 and (3.12), we have $\tilde{Q}_t[\tilde{\psi}_-](x) = \tilde{\psi}_-(x+c_-t)$ for all $x \in \mathbb{R}$ and $t \geq 0$. Therefore, $\tilde{\psi}_-(x+c_-t)$ is a traveling wave connecting 0 to some $\alpha_- \in \Sigma \setminus \{0\}$. Similarly, we can construct the traveling wave $\tilde{\psi}_+(x+c_+t)$ connecting some $\alpha_+ \in \Sigma \setminus \{\beta\}$ to β. Moreover, α_- and α_+ are ordered. The rest of the proof is essentially the same as for Theorem 3.4. □

4. Semiflows in a periodic habitat

A typical example of evolution systems in a periodic habitat is

$$u_t = (d(x)u_x)_x + f(u), \qquad t > 0,\ x \in \mathbb{R}, \tag{4.1}$$

where $d(x)$ is a positive periodic function of $x \in \mathbb{R}$. Under the assumption that f has exactly three ordered zeros $0 < a < 1$ with $f'(0) < 0$, $f'(a) > 0$ and $f'(1) < 0$, Xin [43] employed perturbation methods to obtain the existence of a spatially periodic traveling wave $V(x + ct, x)$ with $V(-\infty, \cdot) = 0$ and $V(\infty, \cdot) = 1$ provided that $d(x)$ is sufficiently close to a positive constant in a certain sense (see also [42]). For a general positive periodic function $d(x)$, the existence of such a traveling wave remains open. We will revisit this problem in Subsection 6.3. Below we extend the previous results in homogeneous habitats to periodic ones.

A map $Q : \mathcal{E} \to \mathcal{E} \subset \mathcal{C}$ is said to be *spatially periodic* with a positive period $r \in \mathcal{H}$ if $Q \circ T_r = T_r \circ Q$, where T_r is the r-translation operator. Similarly, a semiflow $\{Q_t\}_{t\in\mathcal{T}}$ on $\mathcal{E} \subset \mathcal{C}$ is said to be spatially periodic with a positive period $r \in \mathcal{H}$ if $Q_t \circ T_r = T_r \circ Q_t$ for all $t \in \mathcal{T}$, where $\mathcal{H} = \mathbb{Z}$ or $\mathbb{R}$ and $\mathcal{T} = \mathbb{Z}^+$ or $\mathbb{R}^+$.

Definition 4.1. (i) An r-periodic function $\beta(x)$ is said to be an *r-periodic steady state* of the map Q [semiflow $\{Q_t\}_{t\in\mathcal{T}}$] if $Q[\beta] = \beta$ [$Q_t[\beta] = \beta$ for all $t \in \mathcal{T}$].

(ii) $V(x + ct, x)$ is said to be a *spatially r-periodic traveling wave with speed c* of the semiflow $\{Q_t\}_{t\in\mathcal{T}}$ if $Q_t[V(\cdot,\cdot)](x) = V(x + ct, x)$ and $V(\cdot, x)$ is r-periodic in x. Moreover, we say that $V(\xi, x)$ *connects* 0 *to* $\beta(x)$ if $\lim_{\xi\to-\infty} \|V(\xi, x)\|_{\mathcal{X}} = 0$ and $\lim_{\xi\to\infty} \|V(\xi, x) - \beta(x)\|_{\mathcal{X}} = 0$ uniformly for $x \in \mathcal{H}$.

Motivated by [26, Section 5], we can regard a spatially periodic semiflow on $\mathcal{E} \subset \mathcal{C}$ as a spatially homogeneous semiflow on another phase space. For any positive $h \in \mathcal{H}$, define $[0, h]_{\mathcal{H}} := \{l \in \mathcal{H} : 0 \le l \le h\}$. We use $\mathcal{Y}$ to denote $C([0, r]_{\mathcal{H}}, \mathcal{X})$ and $\mathcal{S}$ to denote the set of all bounded functions from $r\mathbb{Z}$ to $\mathcal{Y}$. Clearly, $\mathcal{Y}$ can be regarded as a subspace of $\mathcal{S}$. Let $\mathcal{Y}^+ = C([0, r]_{\mathcal{H}}, \mathcal{X}^+)$ and $\mathcal{S}^+$ be the set of all bounded functions from $r\mathbb{Z}$ to $\mathcal{Y}^+$. We equip $\mathcal{Y}$ with the norm $\|u\|_{\mathcal{Y}} = \max\{\|u(x)\|_{\mathcal{X}} : x \in [0, r]_{\mathcal{H}}\}$, and $\mathcal{S}$ with the compact-open topology. Thus, $\mathcal{Y}$ is a Banach lattice with the norm $\|\cdot\|_{\mathcal{Y}}$ and the cone $\mathcal{Y}^+$.

Let

$$\mathcal{K} := \{f \in \mathcal{S} : f(ri)(r) = f(r(i + 1))(0),\ \forall i \in \mathbb{Z}\}.$$

It is easy to see that

$$\mathcal{K} \cap \mathcal{Y} = \{f \in \mathcal{S} : f(ri) \equiv f(rj) \text{ and } f(ri)(0) = f(ri)(r),\ \forall i, j \in \mathbb{Z}\}.$$

For any $\phi \in \mathcal{C}$, define $\tilde{\phi} \in \mathcal{S}$ by

$$\tilde{\phi}(ri)(y) = \phi(ri + y), \quad \forall i \in \mathbb{Z},\ y \in [0, r]_{\mathcal{H}}.$$

Then we have the following observation.

Lemma 4.1. *For any $f \in \mathcal{K}$, there exists a unique $\phi_f \in \mathcal{C}$ such that $\tilde{\phi_f} = f$. Further, if $f \in \mathcal{K} \cap \mathcal{Y}$, then ϕ_f is r-periodic.*

Proof. For any $x \in \mathcal{H}$, we can find $i \in \mathbb{Z}$ and $y \in [0, r]_{\mathcal{H}}$ such that $x = ri + y$. It is easy to see that this decomposition of x is unique when $x \in \mathcal{H} \setminus r\mathbb{Z}$, and is in two possible ways when $x \in r\mathbb{Z}$. More precisely, each $x \in r\mathbb{Z}$ can be decomposed into either $x = r(i + 1) + 0$ or $x = ri + r$ for some $i \in \mathbb{Z}$. Note that $f(r(i + 1))(0) = f(ri)(r)$. Thus $\phi_f(x) = \phi_f(ri + y) := f(ri)(y)$ is a well defined function in $\mathcal{C}$. Clearly, $\tilde{\phi_f} = f$. If $f(ri) = u$ for all $i \in \mathbb{Z}$, then $\phi_f(ri) = u$ for all $i \in \mathbb{Z}$, which implies that ϕ_f is r-periodic. □

If we define $F : \mathcal{C} \to \mathcal{K}$ by $F(\phi) = \tilde{\phi}$, then F is a homeomorphism between $\mathcal{C}$ and $\mathcal{K}$. Let $\beta(x)$ be a strongly positive r-periodic steady state of $\{Q_t\}_{t\ge 0}$. With a little abuse of notation, we use $\mathcal{C}_\beta$ to denote the set $\{\phi \in \mathcal{C} : 0 \le \phi \le \beta\}$. Now we can define a semiflow $\{P_t\}_{t\in\mathcal{T}}$ on $\mathcal{K}_{\tilde{\beta}} := \{f \in \mathcal{K} :\ 0 \le f \le \tilde{\beta}\}$ by

$$P_t[f] = F \circ Q_t[\phi_f], \qquad \forall f \in \mathcal{K}_{\tilde{\beta}},\ t \in \mathcal{T}. \tag{4.2}$$

Clearly, $P_t \circ F = F \circ Q_t$ for all $t \in \mathcal{T}$, which implies that the semiflows $\{Q_t\}_{t\in\mathcal{T}}$ and $\{P_t\}_{t\in\mathcal{T}}$ are topologically conjugate. Moreover, $\{P_t\}_{t\in\mathcal{T}}$ is spatially homogeneous and $\tilde{\beta}$ is its equilibrium. Thus, $\{Q_t\}_{t\in\mathcal{T}}$ on $\mathcal{C}_\beta$ has a spatially r-periodic traveling wave if $\{P_t\}_{t\in\mathcal{T}}$ on $\mathcal{K}_{\tilde{\beta}}$ has a traveling wave. Before stating the main result, we first introduce the bistability assumption. Let $\beta(x) \gg 0$ be an r-periodic steady state of $\{Q_t\}_{t\in\mathcal{T}}$. Assume that 0 is a trivial steady state. Define

$$\Pi_\beta := \{\phi \in \mathcal{C} : \phi(x) = \phi(x + r),\ 0 \le \phi(x) \le \beta(x),\ \forall x \in \mathcal{H}\}.$$

As in Definition 2.1, we can define the strong stability of periodic steady states for a map Q in the space of periodic functions.

Definition 4.2. A steady state $\alpha \in \Pi_\beta$ is said to be *strongly stable from below* for the map $Q : \Pi_\beta \to \Pi_\beta$ if there exist $\delta_\alpha^+ > 0$ and a strongly positive element $e_\alpha^+ \in \Pi_\beta$ such that

$$Q[\alpha - \eta e_\alpha^+] \gg \alpha - \eta e_\alpha^+, \qquad \forall \eta \in (0, \delta_\alpha^+]. \tag{4.3}$$

The strong instability from below is defined by reversing the inequality (4.3). Similarly, we can define strong stability (and instability) from above.

We need the following bistability assumption on the spatially r-periodic map Q.

(A5$''$) (*Bistability*) 0 and $\beta \gg 0$ are strongly stable r-periodic steady states from above and below, respectively, for $Q : \Pi_\beta \to \Pi_\beta$, and the set of all intermediate r-periodic steady states is totally unordered in Π_β.

We note that a sufficient condition for the unordering property of all intermediate r-periodic steady states is: $Q : \Pi_\beta \to \Pi_\beta$ is eventually strongly monotone and all intermediate fixed points are strongly unstable from both above and below.

Theorem 4.1. *Let $\mathcal{X} = C(M, \mathbb{R}^d)$. Assume that for any $t > 0$, the map Q_t satisfies* (A2)–(A4) *and the bistability assumption* (A5$''$). *Further, assume that the map $P_1 := FQ_1F^{-1}$ satisfies assumption* (A6) *with $\mathcal{C}$ and β replaced by $\mathcal{K}$ and $\tilde{\beta}$, respectively. Then the spatially r-periodic semiflow $\{Q_t\}_{t\in\mathcal{T}}$ has an r-periodic traveling wave $V(x, x + ct)$. Moreover, $V(x, \xi)$ is nondecreasing in ξ and connects 0 to $\beta(x)$.*

Proof. Fix $t \ge 0$ and define P_t as in (4.2). Then it is easy to see that P_t satisfies (A1)–(A5) with $\mathcal{C}_\beta$ replaced by $\mathcal{K}_{\tilde{\beta}}$. From Theorems 3.2 and 3.5, $\{P_t\}_{t\in\mathcal{T}}$ admits a traveling wave $U(x + ct)$ with U connecting 0 to $\tilde{\beta}$. By the definitions of traveling waves in a discrete habitat (see Definitions 3.2 and 3.5), we can find $x_0 \in \mathbb{R}$ such that $g := U(\cdot + x_0) \in \mathcal{K}_{\tilde{\beta}}$ and $P_t[g](ri) = U(ri + ct + x_0)$ for all $i \in \mathbb{Z}$. By Lemma 4.1, we can find $\psi, h_t \in \mathcal{C}$ such that $\tilde{\psi} = g$ and $\tilde{h}_t = U(\cdot + ct + x_0)$, and hence $P_t[\tilde{\psi}] = \tilde{h}_t$. By the topological conjugacy of Q_t and P_t, we have $Q_t[\psi] = h_t$. Note that $\tilde{\psi} = g = U(\cdot + x_0) = \tilde{h}_0$. Lemma 4.1 yields $\psi = h_0$. If $c = 0$, then $Q_t[\psi] = h_t \equiv h_0 = \psi$, which implies that ψ is a traveling wave with speed zero. If $c \ne 0$, then we define $V(\xi, x) := h_{(\xi - x)/c}(x)$.

Consequently,

$$V(x+ct,x)=h_t(x)=Q_t[\psi](x)=Q_t[h_0](x)=Q_t[V(\cdot,\cdot)](x), \quad \forall x\in\mathcal{H}, t\geq 0.$$

This completes the proof. □

Remark 4.1. In applications of Theorem 4.1, to verify assumption (A6) for the map P_1 it suffices to show that for any intermediate periodic steady state α, the sum of the rightward spreading speed of $Q_1 : [0,\alpha]_{\mathcal{C}} \to [0,\alpha]_{\mathcal{C}}$ and the leftward spreading speed of $Q_1 : [\alpha,\beta]_{\mathcal{C}} \to [\alpha,\beta]_{\mathcal{C}}$ is positive since P_1 and Q_1 are topologically conjugate.

To finish this section, we note that the bistability structure can be obtained for equation (4.1) under appropriate conditions so that the existence result in [43, 42] is improved (see the details in Subsection 6.3). Further, Theorem 4.1 with $\mathcal{H}=\mathbb{Z}$ and $\mathcal{X}=\mathbb{R}$ can be used to rediscover the existence result of [14] for a one-dimensional lattice equation under the bistability assumption.

5. Semiflows with weak compactness

In assumption (A4) of Section 2, we assume that $Q:\mathcal{C}_\beta\to\mathcal{C}_\beta$ is compact with respect to the compact-open topology. In this section, we establish the existence of bistable waves under some weaker compactness assumptions.

Fix $\tau>0$. It is well known that the time-t solution map of time-delayed reaction-diffusion equations such as

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}+f(u(t,x),u(t-\tau,x)) \tag{5.1}$$

is compact in the compact-open topology if and only if $t>\tau$, where the phase space $\mathcal{C}$ is chosen as $C(\mathbb{R},C([-\tau,0],\mathbb{R}))$. The first purpose of this section is to show that our results are still valid for this kind of evolution equations by introducing an alternative assumption (A4′).

To state it, we need some notation for time-delayed evolution systems. Let $\tau\in\mathcal{T}$, let $\mathcal{F}$ be a Banach lattice with the positive cone $\mathcal{F}^+$ having nonempty interior, $\beta\in\mathrm{Int}(\mathcal{F}^+)$, and $\mathcal{X}_\beta=C([-\tau,0],\mathcal{F}_\beta)$. Any $\phi\in\mathcal{C}_\beta$ can be regarded as an element in $C([-\tau,0]\times\mathcal{H},\mathcal{F}^+)$. For any subset B of $[-\tau,0]\times\mathcal{H}$, we define $\phi|_B$ as the restriction of ϕ to B.

(A4′) (*Compactness*) There exists $s\in(0,\tau]$ such that:

- (i) $Q[\phi](\theta,x)=\phi(\theta+s,x)$ whenever $\theta+s\leq 0$.
- (ii) For any $\epsilon\in(0,s)$, the set $Q[\mathcal{C}_\beta]|_{[-s+\epsilon,0]\times\mathcal{H}}$ is precompact.
- (iii) For any subset $\mathcal{J}\subset\mathcal{C}_\beta$ with $\mathcal{J}(0,\cdot)\subset C(\mathcal{H},\mathcal{Y}_\beta)$ being precompact, the set $Q[\mathcal{J}]|_{[-s,0]\times\mathcal{H}}$ is precompact.

This assumption was motivated by [25, Assumption (A6′)]. Let us use equation (5.1) to explain (A4′). For any $t>\tau$, one can directly verify that the solution map Q_t satisfies (A4) by rewriting (5.1) in integral form (see, e.g., [41]); and for any $t\in(0,\tau]$, one can show that Q_t satisfies (A4′)(i) & (ii) by the same arguments. For (A4′)(iii), we provide a proof below.

Let $T(0) = I$, and for any $t \in (0, \tau]$, let $T(t)$ be the time-t map of the heat equation $u_t = \Delta u$. Then (5.1) can be written as

$$u(t, x; \phi) = T(t)\phi(x) + \int_0^t T(t-s) f(u(s, u(s-\tau)))(x)\, ds,$$

and hence $Q_t[\phi](\theta, x) = u(t+\theta, x)$. Note that for any $\phi \in \mathcal{C}_\beta$, $T(t)\phi \to \phi$ in the compact-open topology as $t \to 0$. It then follows from the triangular inequality and the absolute continuity of integrals that for any compact subset $\mathcal{H}_1 \subset \mathbb{R}$, the set $Q_t[\mathcal{J}]|_{[-t,0]\times\mathcal{H}_1}$ is equicontinuous, and hence $Q[\mathcal{J}]|_{[-t,0]\times\mathbb{R}}$ is precompact in $\mathcal{C}_\beta$.

Lemma 5.1. *Let A_ξ, $\xi \ge 1$, be defined as in Section* 3 *and let $\beta \in \mathrm{Int}(\mathcal{F}^+)$. Assume that the map $Q : \mathcal{C}_\beta \to \mathcal{C}_\beta$ satisfies* (A4′). *Then there exists an integer m_0 such that $\bigcup_{\xi\in[1,1+\delta]}(Q \circ A_\xi)^{m_0}[\mathcal{C}_\beta] \subset \mathcal{C}_\beta$ is precompact when $\mathcal{H} = \mathbb{R}$, and $\bigcup_{\xi\in[1,2]}(\tilde{Q} \circ A_\xi)^{m_0}[\mathcal{B}_\beta](x) \subset \mathcal{X}_\beta$ is precompact for any $x \in \mathbb{R}$ when $\mathcal{H} = \mathbb{Z}$.*

Proof. We only handle the case $\mathcal{H} = \mathbb{R}$ since the proof for $\mathcal{H} = \mathbb{Z}$ is essentially similar. Let s and τ be as in (A4′). There exists $m_0 \in \mathbb{N}$ such that $s \in \left(\frac{1}{m_0+1}\tau, \frac{1}{m_0}\tau\right]$. By assumption (A4′)(i), for any $\xi \ge 1$ and $\phi_0 \in \mathcal{C}_\beta$,

$$\phi_1^\xi(\theta, x) := Q \circ A_\xi[\phi_0](\theta, x) = \begin{cases} \phi_0(\theta+s, \xi x), & \theta + s \le 0, \\ Q[\phi_0(\xi\cdot)](\theta, x), & \theta + s > 0 \end{cases}$$

This implies that for any $\xi \ge 1$ and $\epsilon < s - \frac{1}{m_0+1}\tau$,

$$\bigcup_{\xi\in[1,2]} Q \circ A_\xi[\mathcal{C}_\beta]|_{[-s+\epsilon,0]\times\mathbb{R}} \subset Q[\mathcal{C}_\beta]|_{[-s+\epsilon,0]\times\mathbb{R}}.$$

Since $Q[\mathcal{C}_\beta]|_{[-s+\epsilon,0]\times\mathbb{R}}$ is precompact, as assumed in (A4′)(ii), it then follows that $\bigcup_{\xi\in[1,2]} Q \circ A_\xi[\mathcal{C}_\beta](0,\cdot) \subset C(\mathbb{R}, \mathcal{Y}_\beta)$ is precompact. By (A4′)(iii) and similar arguments, we have

$$\begin{aligned} \phi_2^\xi(\theta, x) := Q \circ A_\xi[\phi_1^\xi](\theta, x) &= \begin{cases} \phi_1^\xi(\theta+s, \xi x), & \theta + s \le 0, \\ Q[\phi_1^\xi(\xi\cdot)](\theta, x), & \theta + s > 0, \end{cases} \\ &= \begin{cases} \phi_0(\theta+2s, \xi^2 x), & \theta + 2s \le 0, \\ Q[\phi_0(\xi\cdot)](\theta+s, \xi x), & 0 < \theta + 2s \le s, \\ Q[\phi_1^\xi(\xi\cdot)](\theta, x), & \theta + s > 0, \end{cases} \end{aligned}$$

This implies that $\bigcup_{\xi\in[1,2]}(Q\circ A_\xi)^2[\mathcal{C}_\beta]|_{[-2s+\epsilon,0]\times\mathbb{R}}$ is precompact. Consequently, the set $\bigcup_{\xi\in[1,2]}(Q \circ A_\xi)^2[\mathcal{C}_\beta](0,\cdot) \subset C(\mathbb{R}, \mathcal{Y}_\beta)$ is compact. By induction, we have

$$\begin{aligned} \phi_{m_0+1}^\xi(\theta, x) := Q \circ A[\phi_{m_0}^\xi](\theta, x) &= \begin{cases} \phi_{m_0}^\xi(\theta+s, \xi x), & \theta + s \le 0, \\ Q[\phi_{m_0}^\xi(\xi\cdot)](\theta, x), & \theta + s > 0, \end{cases} \\ &= \cdots \end{aligned}$$

$$= \begin{cases} Q[\phi_0(\xi\cdot)](\theta + (m_0+1)s, \xi^{m_0}x), & 0 < \theta + (m_0+1)s \le s, \\ Q[\phi_1^{\xi}(\xi\cdot)](\theta + m_0 s, \xi^{m_0-1}x), & 0 < \theta + m_0 s \le s, \\ \cdots \\ Q[\phi_{m_0-1}^{\xi}(\xi\cdot)](\theta + s, \xi x), & 0 < \theta + s \le s, \\ Q[\phi_{m_0}^{\xi}](\theta, x), & \theta + s > 0. \end{cases}$$

This implies that $\bigcup_{\xi\in[1,2]}(Q \circ A_\xi)^{m_0}[\mathcal{C}_\beta]$ is precompact in $\mathcal{C}_\beta$. □

Theorem 5.1. *All results in Theorems 3.1-3.5 and 4.1 are valid if we replace* (A4) *with* (A4′).

Proof. Following the proof of these theorems, we only need to modify the parts where we use the compactness assumption (A4). At these parts, we apply Lemma 5.1. □

Note that the solution maps of the nonlocal dispersal equation

$$u_t = J * u - u + f(u)$$

satisfy neither (A4) nor (A4′). The second purpose of this section is to modify our theory so that it applies to such integro-differential systems in a homogeneous habitat.

Let $\mathcal{M}$ denote the set of all nondecreasing functions from $\mathbb{R}$ to $\mathcal{X}$, and $\beta \in \mathcal{X}^+$. We equip $\mathcal{M}$ with the compact-open topology. Assume that Q maps $\mathcal{M}_\beta$ to $\mathcal{M}_\beta$. Let E denote the set of fixed points of Q restricted to $\mathcal{X}_\beta$. Suppose that 0 and β are in E. We impose the following assumptions on Q:

(B1) (*Translation invariance*) $T_y \circ Q[\phi] = Q \circ T_y[\phi]$ for all $\phi \in \mathcal{M}_\beta$ and $y \in \mathbb{R}$.

(B2) (*Continuity*) $Q : \mathcal{M}_\beta \to \mathcal{M}_\beta$ is continuous in the sense that if $\phi_n \to \phi$ in $\mathcal{M}_\beta$, then $Q[\phi_n](x) \to Q[\phi](x)$ in $\mathcal{X}_\beta$ for almost all $x \in \mathbb{R}$.

(B3) (*Monotonicity*) Q is order preserving in the sense that $Q[\phi] \ge Q[\psi]$ whenever $\phi \ge \psi$ in $\mathcal{M}_\beta$.

(B4) (*Weak compactness*) For any fixed $x \in \mathbb{R}$, the set $Q[\mathcal{M}_\beta](x)$ is precompact in $\mathcal{X}_\beta$.

(B5) (*Bistability*) The fixed points 0 and β are strongly stable from above and below, respectively, for the map $Q : \mathcal{X}_\beta \to \mathcal{X}_\beta$, and the set $E \setminus \{0, \beta\} \subset \mathcal{X}_\beta$ is totally unordered.

(B6) (*Counter-propagation*) For each $\alpha \in E \setminus \{0, \beta\}$, $c_-^*(\alpha, \beta) + c_+^*(0, \alpha) > 0$.

Comparing assumptions (A1)–(A6) and (B1)–(B6), one can find that the assumptions of translation invariance, monotonicity, bistability and counter-propagation are the same. The difference lies in the assumptions of continuity and compactness. Clearly, the compactness assumption (B4) is much weaker than (A4).

Theorem 5.2. *Let $\mathcal{X} = C(M, \mathbb{R}^d)$ and assume $Q : \mathcal{M}_\beta \to \mathcal{M}_\beta$ satisfies* (B1)–(B6). *Then there exist $c \in \mathbb{R}$ and $\psi \in \mathcal{M}_\beta$ connecting* 0 *to* β *such that $Q[\psi](x) = \psi(x + c)$ for all $x \in \mathbb{R}$.*

Proof. We combine the proofs of Theorems 3.1 and 3.2. More precisely, one can repeat the proof of Theorem 3.1 except for the parts where the compactness assumption (A4) is used. For these parts, one needs to use the idea in Theorem 3.2, where $\tilde{Q}$ has the same compactness property as Q. □

In the rest of this section, we say $\{Q_t\}_{t\geq 0}$ is a semiflow on $\mathcal{M}_\beta$ provided that $Q_0 = I$; $Q_t \circ Q_s = Q_{t+s}$ for all $t, s > 0$; and $Q_{t_n}[\phi_n](x) \to Q_t[\phi](x)$ in $\mathcal{X}_\beta$ for almost all $x \in \mathbb{R}$ whenever $t_n \to t$ and $\phi_n \to \phi$ in $\mathcal{M}_\beta$.

Theorem 5.3. *Let $\mathcal{X} = C(M, \mathbb{R}^d)$. Assume that $\{Q_t\}_{t\geq 0}$ is a semiflow on $\mathcal{M}_\beta$, and for any $t > 0$, the map Q_t satisfies* (B1) *and* (B3)–(B6). *Then there exist $c \in \mathbb{R}$ and $\psi \in \mathcal{M}_\beta$ connecting* 0 *to β such that $Q_t[\psi](x) = \psi(x + ct)$ for all $x \in \mathbb{R}$.*

Proof. As in the proof of Theorem 5.2, we need to combine the proofs of Theorems 3.4 and 3.5. □

Similarly, we can define ω-time periodic semiflows on $\mathcal{M}_\beta$ and obtain the following result.

Theorem 5.4. *Let $\mathcal{X} = C(M, \mathbb{R}^d)$. Assume that $\{Q_t\}_{t\geq 0}$ is an ω-time periodic semiflow on $\mathcal{M}_\beta$. Let $\beta(t)$ be a strongly positive periodic solution of $\{Q_t\}_{t\geq 0}$ restricted to $\mathcal{X}_\beta$. Further, assume that the Poincaré map Q_ω satisfies* (B1) *and* (B3)–(B6) *with $\beta = \beta(0)$. Then there exist $c \in \mathbb{R}$ and $\phi(t, x)$ with $\phi(t, -\infty) = 0$ and $\phi(t, \infty) = \beta(t)$ such that $Q_t[\psi](x) = \psi(t, x + ct)$ for all $x \in \mathbb{R}$. Moreover, $\phi(t, \cdot) \in \mathcal{M}_\beta$ and $\phi(t, \cdot)$ is ω-periodic in $t \geq 0$.*

6. Applications

In this section, we apply the abstract results to four kinds of monotone evolution systems: a time-periodic reaction-diffusion system, a parabolic system in a cylinder, a parabolic equation with variable diffusion, and a nonlocal and time-delayed reaction-diffusion equation.

6.1. A time-periodic reaction-diffusion system

Consider the time-periodic reaction-diffusion system

$$\frac{\partial u}{\partial t} = A\Delta u + f(t, u), \qquad x \in \mathbb{R}, \tag{6.1}$$

where $u = (u_1, \ldots, u_n)^T$, $A = \mathrm{diag}\{d_1, \ldots, d_n\}$ with each $d_i > 0$ and $f = (f_1, \ldots, f_n)^T$ is ω-periodic in $t \geq 0$ (i.e., $f(t, \cdot) = f(t + \omega, \cdot)$). The existence of periodic bistable traveling waves of (6.1) with $n = 1$ was proved in [1]. Here we generalize this result to $n \geq 1$.

Let $f \in C^1(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n)$. In order to apply Theorem 3.3 to system (6.1), we choose $\mathcal{C} := C(\mathbb{R}, \mathbb{R}^n)$, $\mathcal{X} := \mathbb{R}^n$, and $\mathcal{E} \subset \mathcal{C}$ to be the set of all bounded functions from $\mathbb{R}$ to $\mathbb{R}^n$. Using the solution maps $\{T(t)\}_{t\geq 0}$ of the heat equation $\frac{\partial u}{\partial t} = A\Delta u$, we write (6.1) in integral form:

$$u(t; \phi) = T(t)\phi + \int_0^t T(t-s) f(s, u(s; \phi))\, ds. \tag{6.2}$$

Define $Q_t[\phi] := u(t; \phi)$ for all $\phi \in \mathcal{E}$. Let 0 and $\beta \gg 0$ be two fixed points of the Poincaré map Q_ω in $\mathcal{X}$, and let E be the set of all spatially homogeneous fixed points of Q_ω in $\mathcal{X}_\beta$. We impose the following assumptions:

(C1) The Jacobian matrix $Df(t,u)$ $(D = D_u)$ is cooperative and irreducible for all $t \geq 0$ and $u \geq 0$. For any ω-periodic solution $p(t)$ of $u' = f(t,u)$ with $0 < p(0) < \beta$, every off-diagonal entry of $Df(t, p(t))$ is either strictly positive for all t or identically zero.

(C2) The spatially homogeneous system $u' = f(t,u)$ is of bistable type, that is, 0 and β are stable fixed points of Q_ω in the sense that $s\big(\frac{d}{du}Q_\omega[0]\big) < 0$ and $s\big(\frac{d}{du}Q_\omega[\beta]\big) < 0$, and any $\alpha \in E \setminus \{0,\beta\}$ is unstable in the sense that $s\big(\frac{d}{du}Q_\omega[\alpha]\big) > 0$, where $s(M)$ is the stability modulus of the matrix M defined by $s(M) = \max\{\mathrm{Re}\,\lambda : \lambda \text{ is an eigenvalue}\}$.

Theorem 6.1. *Assume that* (C1)–(C2) *hold, and let $\beta(t)$ be the periodic solution of $u' = f(t,u)$ with $\beta(0) = \beta$. Then there exists $c \in \mathbb{R}$ such that* (6.1) *admits a time-periodic traveling wave $U(t, x + ct)$ connecting* 0 *to $\beta(t)$.*

Proof. It is easy to see that the discrete semiflow $\{Q_\omega^n\}_{n\geq 1}$ on $\mathcal{C}_\beta$ satisfies (A1)–(A5) with $Q = Q_\omega$. Next we show that (A6) holds with $Q = Q_\omega$.

Note that for any $\alpha \in E \setminus \{0,\beta\}$, $\{Q_\omega^n\}_{n\geq 1} : [\alpha,\beta]_{\mathcal{C}} \to [\alpha,\beta]_{\mathcal{C}}$ has monostable dynamics, where α is unstable and β is stable. By the theory developed in [26], Q_ω admits leftward and rightward spreading speeds $c_-^*(\alpha,\beta)$ and $c_+^*(\alpha,\beta)$. Since Q_ω is reflectively invariant, we further have $c_-^*(\alpha,\beta) = c_+^*(\alpha,\beta) =: c^*(\alpha,\beta)$, which is called the spreading speed of this monostable subsystem. Note that $\{Q_\omega^n\}_{n\geq 1} : [0,\alpha]_{\mathcal{C}} \to [0,\alpha]_{\mathcal{C}}$ also has monostable dynamics, where 0 is stable and α is unstable. Similarly, this monostable subsystem also admits a spreading speed $c^*(0,\alpha)$.

Denote the periodic solution $Q_t[\alpha]$ by $\alpha(t)$, and define

$$\bar{f}(t,u) = f(t,\alpha(t) + u) - f(t,\alpha(t)). \tag{6.3}$$

Clearly, $\alpha(0) = \alpha$ and $\bar{f}(t,0) \equiv 0$. Then the semiflow Q_t has the same dynamics on $[\alpha,\beta]_{\mathcal{C}}$ as the semiflow $\bar{Q}_t$ on $\mathcal{C}_{\beta-\alpha}$ associated with

$$u_t = A\Delta u + \bar{f}(t,u). \tag{6.4}$$

In particular, these two semiflows share the spreading speed $c^*(\alpha,\beta)$.

To give a lower bound on $c^*(\alpha,\beta)$, we employ a similar idea to that in the proof of [40, Lemma 4.1]. Choose $\rho > 0$ such that the diagonal entries of $D\bar{f}(t,0) + \rho I$ are strictly positive. We define for each i the projection

$$\{\Pi_i[u]\}_j = \begin{cases} u_j & \text{if } \{D\bar{f}(t,0) + \rho I\}_{ij} > 0, \\ 0 & \text{if } \{D\bar{f}(t,0) + \rho I\}_{ij} = 0, \end{cases} \tag{6.5}$$

and

$$\sigma(t) = \min\left\{ \frac{\partial \bar{f}_i}{\partial u_j}(t,0) + \rho I_{ij} : \frac{\partial \bar{f}_i}{\partial u_j}(t,0) + \rho I_{ij} > 0 \right\}.$$

It is easy to see from assumption (C1) that Π_i is independent of $t \in [0,\omega]$, and $\sigma := \inf\{\sigma(t) : t \in [0,\omega]\}$ is positive. Further,

$$u \geq \Pi_i[u] \quad \text{and} \quad \bar{f}_i(t,u) \geq \bar{f}_i(t,\Pi_i[u]), \quad \forall t \geq 0,\ u \geq 0. \tag{6.6}$$

By the differentiability of $\bar{f} = (\bar{f}_1, \ldots, \bar{f}_n)$, for any $\epsilon \in (0, 1)$, there exists $\eta = \eta(\epsilon) \gg 0$ in $\mathbb{R}^n$ such that

$$\bar{f}_i(t, \Pi_i[u]) \geq \nabla \bar{f}_i(t, 0) \cdot \Pi_i[u] - \sigma\epsilon \|\Pi_i[u]\|, \quad \forall t \in [0, \omega],\ u \in [0, \eta]. \tag{6.7}$$

Note that

$$\nabla \bar{f}_i(t, 0) \cdot \Pi_i[u] = (D\bar{f}(t, 0)u)_i, \quad \forall u \in \mathbb{R}^n, \tag{6.8}$$

and

$$\begin{aligned} \|\Pi_i[u]\| &\leq \sum_j (\Pi_i[u])_j \leq \sum_j \frac{(D\bar{f}(t, 0) + \rho I)_{ij}}{\sigma} (\Pi_i[u])_j \\ &= \sum_j \frac{\nabla \bar{f}_i(t, 0) \cdot \Pi_i[u] + \rho(\Pi_i[u])_i}{\sigma} \\ &= \sigma^{-1}(D\bar{f}(t, 0)u + \rho u)_i, \quad \forall u \geq 0. \end{aligned} \tag{6.9}$$

Combining (6.6)–(6.9), we obtain

$$\bar{f}(t, u) \geq D\bar{f}(t, 0)u - \epsilon[D\bar{f}(t, 0)u + \rho u], \quad \forall t \in [0, \omega],\ u \in [0, \eta]. \tag{6.10}$$

For the above $\eta = \eta(\epsilon)$, there exists $\gamma = \gamma(\epsilon) \gg 0$ in $\mathbb{R}^n$ such that

$$\bar{Q}_t[\phi] \leq \bar{Q}_t[\gamma] \leq \eta, \quad \forall \phi \in \mathcal{C}_\gamma,\ t \in [0, \omega]. \tag{6.11}$$

Thus, the comparison theorem implies that

$$\bar{Q}_t[\phi] \geq M_t^\epsilon[\phi], \quad \forall \phi \in \mathcal{C}_\gamma,\ t \in [0, \omega], \tag{6.12}$$

where M_t^ϵ is the solution map of

$$u_t = A\Delta u + D\bar{f}(t, 0)u - \epsilon(D\bar{f}(t, 0) + \rho I)u. \tag{6.13}$$

Let $\rho_\epsilon(\mu)$ be the principal Floquet multiplier of the linear periodic cooperative and irreducible system

$$\frac{dv}{dt} = [\mu^2 A + D\bar{f}(t, 0)]v - \epsilon(D\bar{f}(t, 0) + \rho I)v. \tag{6.14}$$

Since $\lim_{\epsilon \to 0^+} \rho_\epsilon(0) = \rho_0(0) > 1$, we can fix an $\epsilon \in (0, 1)$ such that $\rho_\epsilon(0) > 1$. Let $v(t, w)$ be the solution of (6.14) satisfying $v(0, w) = w \in \mathbb{R}^n$. It is easy to see that $u(t, x) = e^{-\mu x} v(t, w)$ is a solution of the linear periodic system (6.13). Define $\Phi_\epsilon(\mu) := \ln \rho_\epsilon(\mu)/\mu$. From [25, Theorem 3.10] and (6.12), we have

$$c^*(\alpha, \beta) \geq \inf_{\mu > 0} \Phi_\epsilon(\mu). \tag{6.15}$$

Now we prove that $\Phi_\epsilon(\infty) = \infty$. Let $\lambda_\epsilon(\mu) = (1/\omega)\ln \rho_\epsilon(\mu)$. By Floquet theory, there exists a positive ω-periodic function $\xi(t) := (\xi_1(t), \dots, \xi_n(t))^T$ such that $v(t) := e^{\lambda_\epsilon(\mu)t}\xi(t)$ is a solution of (6.14). In particular,

$$\xi_1'(t) = (d_1\mu^2 - \lambda_\epsilon(\mu) - \epsilon\rho)\xi_1(t) + (1-\epsilon)\sum_{i=1}^n \frac{\partial}{\partial u_i}\bar{f}_1(t,0)\xi_i(t).$$

Dividing by $\xi_1(t)$ and integrating from 0 to ω gives

$$0 = (d_1\mu^2 - \lambda_\epsilon(\mu) - \epsilon\rho)\omega + (1-\epsilon)\int_0^\omega \sum_{i=1}^n \frac{\partial}{\partial u_i}\bar{f}_1(t,0)\xi_i(t)/\xi_1(t)\,dt, \qquad \forall \mu > 0.$$

Since the matrix $D\bar{f}(t,0)$ is cooperative and $\xi(t)$ is positive, we obtain

$$0 \geq (d_1\mu^2 - \lambda_\epsilon(\mu) - \epsilon\rho)\omega + (1-\epsilon)\int_0^\omega \frac{\partial}{\partial u_1}\bar{f}_1(t,0)\,dt.$$

This implies that

$$\Phi_\epsilon(\mu) = \frac{\omega\lambda_\epsilon(\mu)}{\mu} \geq d_1\mu\omega + \frac{1}{\mu}\left[-\epsilon\rho\omega + (1-\epsilon)\int_0^\omega \frac{\partial}{\partial u_1}\bar{f}_1(t,0)\,dt\right],$$

and hence $\Phi_\epsilon(\infty) = \infty$. By [25, Lemma 3.8], we have $\inf_{\mu>0}\Phi_\epsilon(\mu) > 0$. Therefore, $c^*(\alpha,\beta) > 0$. Similarly, we can prove that $c_*(0,\alpha) > 0$. Thus, assumption (A6) with $Q = Q_\omega$ holds. Consequently, Theorem 3.3 completes the proof. □

6.2. A reaction-diffusion-advection system in a cylinder

In this subsection, we consider the system

$$\begin{cases} \frac{\partial u}{\partial t} = A\frac{\partial^2 u}{\partial x^2} + B\Delta_y u + E(y)\frac{\partial u}{\partial x} + f(u), & x \in \mathbb{R},\ y \in \Omega \subset \mathbb{R}^{m-1},\ t > 0, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } (0,\infty)\times\mathbb{R}\times\partial\Omega, \end{cases} \tag{6.16}$$

where A, B are positive definite diagonal $n\times n$ matrices, E is a diagonal matrix of smooth functions of y, Ω is a bounded and convex open subset in $\mathbb{R}^{m-1}$ with smooth boundary $\partial\Omega$, $\Delta_y = \sum_{i=1}^{m-1}\partial^2/\partial y_i^2$, and ν is the outer unit normal vector to $\partial\Omega\times\mathbb{R}$.

The existence of bistable traveling waves for (6.16) with $n = 1$ was obtained in [11]. Here we extend this result to the case $n \geq 2$. Assume that $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ satisfies the following conditions:

(D1) The Jacobian matrix $Df(u)$ is cooperative and irreducible for all $u \geq 0$.
(D2) f is of bistable type in the sense that it has exactly three ordered zeros: $0 < a < \beta$ and $s(Df(0)) < 0$, $s(Df(a)) > 0$, $s(Df(\beta)) < 0$.

Theorem 6.2. *Assume that* (D1)–(D2) *hold. Then there exists* $c \in \mathbb{R}$ *such that system* (6.16) *admits a traveling wave connecting* 0 *to* β *with speed* c.

Proof. In order to employ Theorem 3.4, we choose $\mathcal{X} := C(\bar{\Omega}, \mathbb{R}^n)$ and $\mathcal{C} := C(\mathbb{R}, \mathcal{X})$ with the standard cones $\mathcal{X}^+$ and $\mathcal{C}^+$, respectively. Let $G(t, x, y, w)$ be the Green function of the linear equation

$$\begin{cases} \frac{\partial u}{\partial t} = A\frac{\partial^2 u}{\partial x^2} + B\Delta_y u + E(y)\frac{\partial u}{\partial x}, & x \in \mathbb{R},\ y \in \Omega,\ t > 0, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } (0,\infty) \times \mathbb{R} \times \partial\Omega. \end{cases} \tag{6.17}$$

Then the solution of (6.17) with initial value $u(0,\cdot) = \phi(\cdot) \in \mathcal{C}$ can be expressed as

$$u(t, x, y; \phi) = \int_{\mathbb{R}}\int_{\Omega} G(t, x - z, y, w)\phi(z, w)\,dw\,dz.$$

Define $T(t)\phi = u(t, \cdot; \phi)$ for $\phi \in \mathcal{C}_\beta$. Using the variation of constants formula, we write (6.16) subject to $u(0,\cdot) = \phi(\cdot) \in \mathcal{C}_\beta$ as an integral equation

$$u(t, x, y; \phi) = T(t)[\phi](x, y) + \int_0^t T(t - s) f(u(s, x, y))\,ds. \tag{6.18}$$

By linear operator theory, for any $\phi \in \mathcal{C}_\beta$, system (6.16) has a unique solution $u(t; \phi)$ with $u(0; \phi) = \phi$, which exists globally on $[0, \infty)$. Define $Q_t[\phi] := u(t, \phi)$. Then $\{Q_t\}_{t\geq 0}$ is a subhomogeneous semiflow on $\mathcal{C}_\beta$ (see [25, Section 5.3]). Also, assumption (D1) ensures that the semiflow $\{Q_t\}_{t\geq 0}$ restricted to $\mathcal{X}_\beta$ is strongly monotone (see [34]). Further, it is easy to see that Q_t, $t \geq 0$, satisfies assumptions (A1)–(A4). Since the domain Ω is convex, it follows from the result in [24] that any nonconstant steady state of the x-independent system

$$\begin{cases} \frac{\partial u}{\partial t} = B\Delta_y u + f(u), & y \in \Omega,\ t > 0, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } (0,\infty) \times \partial\Omega, \end{cases}$$

is linearly unstable. This then implies that Q_t satisfies (A5$'$). Now it remains to show that (A6) holds for Q_1.

For each x-independent steady state $\alpha = \alpha(y)$ in $[[0, \beta]]_{\mathcal{X}}$, system (6.16) has monostable dynamics on $[\alpha, \beta]_{\mathcal{C}}$. To better understand the dynamics of this subsystem, we make the transform $u = w + \alpha$ and define $g(w, y) := f(w + \alpha(y)) - f(\alpha(y))$. Then the dynamics is equivalent to that of the following system on $[0, \beta - \alpha]_{\mathcal{C}}$:

$$\begin{cases} \frac{\partial w}{\partial t} = A\frac{\partial^2 w}{\partial x^2} + B\Delta_y w + E(y)\frac{\partial w}{\partial x} + g(w, y), & x \in \mathbb{R},\ y \in \Omega,\ t > 0, \\ \frac{\partial w}{\partial \nu} = 0 & \text{on } (0,\infty) \times \mathbb{R} \times \partial\Omega. \end{cases} \tag{6.19}$$

System (6.19) has exactly two x-independent steady states $S_1 := 0$ and $S_2 := \beta - \alpha \gg 0$. By the theory developed in [26], (6.19) has a leftward spreading speed $\hat{c}_-^*$ in a strong sense. Let $c_-^*(\alpha, \beta)$ be defined as in (2.6) with $Q = Q_1$. Then $c_-^*(\alpha, \beta) \geq \hat{c}_-^*$.

To verify (A6) for Q_1, we first estimate the speed $\hat{c}_-^*$. Consider the linearized system of (6.19) at the equilibrium S_1:

$$\begin{cases} \frac{\partial w}{\partial t} = A\frac{\partial^2 w}{\partial x^2} + B\Delta_y w + E(y)\frac{\partial w}{\partial x} + \frac{\partial g(0,y)}{\partial w} w, & x \in \mathbb{R},\ y \in \Omega,\ t > 0, \\ \frac{\partial w}{\partial \nu} = 0 & \text{on } (0,\infty) \times \mathbb{R} \times \partial\Omega. \end{cases} \tag{6.20}$$

Suppose $w(t, x, y) := e^{\mu x} v(t, y)$ is a solution of (6.20). Then $v(t, y)$ satisfies the μ-parameterized linear parabolic equation

$$\begin{cases} \frac{\partial v}{\partial t} = B\Delta_y u + \left[\mu^2 A + \mu E(y) + \frac{\partial g(0,y)}{\partial w}\right] v, & y \in \Omega,\ t > 0, \\ \frac{\partial v}{\partial \nu} = 0 & \text{on } (0, \infty) \times \partial\Omega. \end{cases} \tag{6.21}$$

Let $\lambda^-(\mu)$ be the principal eigenvalue of (6.21). Then by the results in [25, Section 3], it follows that $\hat{c}_-^* \geq \inf_{\mu>0} \lambda^+(\mu)/\mu$, and $\lambda^-(\mu)$ is convex in μ. Moreover, it is easy to see that $\lim_{\mu\to\infty} \lambda^-(\mu)/\mu = \infty$ and $\lim_{\mu\to 0^+} \lambda^-(\mu)/\mu = \infty$, and hence $\lambda^-(\mu)/\mu$ attains its infimum at some $\mu_1 \in (0, \infty)$.

Similarly, system (6.16) has monostable dynamics on $[0, \alpha]_{\mathcal{C}}$. To better understand this dynamics of this subsystem, we make the transform $w = \alpha - u$ and define $h(w, y) := f(\alpha(y)) - f(\alpha(y) - w)$. Then the dynamics is equivalent to that of the following system on $[0, \alpha]_{\mathcal{C}}$:

$$\begin{cases} \frac{\partial w}{\partial t} = A\frac{\partial^2 w}{\partial x^2} + B\Delta_y w + E(y)\frac{\partial w}{\partial x} + h(w, y), & x \in \mathbb{R},\ y \in \Omega,\ t > 0, \\ \frac{\partial w}{\partial \nu} = 0 & \text{on } (0, \infty) \times \mathbb{R} \times \partial\Omega. \end{cases}$$

By the same arguments, this system has a rightward spreading speed $\hat{c}_+^*$ and $c_+^*(0, \alpha) \geq \hat{c}_+^*$. Also, by a similar procedure, we define $\lambda^+(\mu)$ as the principal eigenvalue of the problem

$$\begin{cases} \frac{\partial v}{\partial t} = B\Delta_y u + \left[\mu^2 A - \mu E(y) + \frac{\partial g(0,y)}{\partial w}\right] v, & y \in \Omega,\ t > 0, \\ \frac{\partial v}{\partial \nu} = 0 & \text{on } (0, \infty) \times \partial\Omega. \end{cases} \tag{6.22}$$

Then $\hat{c}_+^* \geq \inf_{\mu>0} \lambda^+(\mu)/\mu$, and $\lambda^+(\mu)/\mu$ attains its infimum at some $\mu_2 \in (0, \infty)$. Clearly, $\lambda^+(\mu) = \lambda^-(-\mu)$ because $\frac{\partial g(0,y)}{\partial w} = \frac{\partial h(0,y)}{\partial w} = f'(\alpha(y))$.

From assumption (D2), we see that S_1 [S_2] is a linearly unstable [stable] steady state of the x-independent system

$$\begin{cases} \frac{\partial w}{\partial t} = B\Delta_y w + g(w, y), & y \in \Omega,\ t > 0, \\ \frac{\partial w}{\partial \nu} = 0 & \text{on } (0, \infty) \times \partial\Omega. \end{cases} \tag{6.23}$$

More precisely, letting λ_0 be the principal eigenvalue of the linearized equation at S_1, we have $\lambda_0 > 0$. Obviously, equations (6.21) and (6.22) with $\mu = 0$ are the same, and hence $\lambda^+(0) = \lambda_0 = \lambda^-(0) > 0$.

With the information above, now we can show that Q_1 satisfies (A6). Let $\theta = \frac{\mu_2}{\mu_1+\mu_2}$ and $\lambda^+(\mu) = \lambda^-(-\mu)$. Note that $\theta\mu_1 + (1-\theta)(-\mu_2) = 0$. Hence,

$$\begin{aligned} c_-^*(\alpha, \beta) + c_+^*(0, \alpha) &\geq \frac{\lambda^-(\mu_1)}{\mu_1} + \frac{\lambda^+(\mu_2)}{\mu_2} = \frac{\mu_1 + \mu_2}{\mu_1\mu_2}[\theta\lambda^-(\mu_1) + (1-\theta)\lambda^-(-\mu_2)] \\ &\geq \frac{\mu_1 + \mu_2}{\mu_1\mu_2}\lambda^-(\theta\mu_1 + (1-\theta)(-\mu_2)) \\ &= \frac{\mu_1 + \mu_2}{\mu_1\mu_2}\lambda^-(0) = \frac{\mu_1 + \mu_2}{\mu_1\mu_2}\lambda_0 > 0. \end{aligned} \tag{6.24}$$

Consequently, Theorem 3.4 completes the proof. □

6.3. A parabolic equation with periodic diffusion

In this subsection, we study the existence of spatially periodic traveling waves of the parabolic equation

$$u_t = (d(x)u_x)_x + f(u), \quad t > 0,\ x \in \mathbb{R}, \tag{6.25}$$

where $f(u) = u(1-u)(u-a)$, $a \in (0, 1)$, and $d(x)$ is a positive, C^1, and r-periodic function on $\mathbb{R}$ for some real $r > 0$.

For any $\phi \in C(\mathbb{R}, [0, 1])$, equation (6.25) admits a unique solution $u(t; \phi)$ with $u(0; \phi) = \phi$. Define $Q_t : C(\mathbb{R}, [0, 1]) \to C(\mathbb{R}, [0, 1])$ by $Q_t[\phi] = u(t; \phi)$. Then $\{Q_t\}_{t\geq 0}$ is a continuous, compact and monotone semiflow on $C(\mathbb{R}, [0, 1])$ equipped with the compact-open topology. Let $C_{\text{per}}(\mathbb{R}, [0, 1])$ be the set of all continuous and r-periodic functions from $\mathbb{R}$ to $[0, 1]$. Then the semiflow $\{Q_t\}_{t\geq 0}$ restricted to $C_{\text{per}}(\mathbb{R}, [0, 1])$ is strongly monotone. Choosing $\mathcal{H} = \mathbb{R}$ and $\mathcal{X} = \mathbb{R}$ in Theorem 4.1, one can easily verify that $\{Q_t\}_{t\geq 0}$ satisfies assumptions (A2)–(A4). If (6.25) admits the bistability structure, then Proposition 3.1 implies (A5$''$) and a similar argument to that in the previous section shows that (A6) also holds. Thus, we focus on finding sufficient conditions on $d(x)$ under which (6.25) admits the bistability structure.

Let $\bar{u}$ be an r-periodic steady state of (6.25). As in [10], we define $\lambda_1(\bar{u}, d)$ as the largest number such that there exists a function $\phi > 0$ which satisfies

$$\begin{cases} (d\phi_x)_x + f'(\bar{u})\phi = \lambda_1(\bar{u}, d)\phi, & x \in \mathbb{R}, \\ \phi \text{ is } r\text{-periodic and } \|\phi\|_\infty = 1. \end{cases} \tag{6.26}$$

We call $\lambda_1(\bar{u}, d)$ the *principal eigenvalue* of $\bar{u}$, and ϕ the corresponding eigenfunction. We say $\bar{u}$ is *linearly unstable* if $\lambda_1(\bar{u}, d) > 0$, and *linearly stable* if $\lambda_1(\bar{u}, d) < 0$. Define

$$W := \{\psi \in C^1(\mathbb{R}, \mathbb{R}) : \psi(x) = \psi(x + r),\ \forall x \in \mathbb{R}\}$$

and equip it with the topology induced by the maximum norm. We say $\psi \in W$ has *property* (P) if every possible nonconstant r-periodic steady state of (6.25) with $d = \psi$ is linearly unstable, that is, (6.25) with $d = \psi$ admits no nonconstant r-periodic steady state $\bar{u}$ such that $\lambda_1(\bar{u}, \psi) \leq 0$. Define

$$Y := \{\psi \in W : \psi(x) > 0 \text{ and } \psi \text{ has property (P)}\}.$$

Lemma 6.1. *Any positive constant function is in Y.*

Proof. Let $d(x) \equiv \bar{d}$ be given. If (6.25) has no nonconstant r-periodic steady state, we are done. Let $\bar{u}$ be a nonconstant r-periodic steady state of (6.25). We need to prove $\lambda_1(\bar{u}, \bar{d}) > 0$. Assume for contradiction that $\lambda_1(\bar{u}, \bar{d}) \leq 0$. Let ϕ be the positive eigenfunction associated with $\lambda_1(\bar{u}, \bar{d})$. Define $M := \max_{0\leq x\leq r}\{|\bar{u}_x|/\phi\}$ and $\psi(x, t) := e^{-\gamma t}(|\bar{u}_x|^2/\phi - M^2\phi)$. It is easy to see that $\psi(t, x) \leq 0$ for all x and t. Let $\xi := |\bar{u}_x|^2/\phi$ and $\eta := M^2\phi$. Then

$$\begin{aligned} \xi_x &= (|\bar{u}_x|^2\phi^{-1})_x = 2\bar{u}_x\bar{u}_{xx}\phi^{-1} - |\bar{u}_x|^2\phi^{-2}\phi_x, \\ \xi_{xx} &= 2\phi^{-3}[\bar{u}_{xx}\phi - \bar{u}_x\phi_x]^2 + \phi^{-3}[2\bar{u}_x\bar{u}_{xxx}\phi^2 - |\bar{u}_x|^2\phi\phi_{xx}]. \end{aligned}$$

Note that

$$0 = [\bar{d}\bar{u}_{xx} + f(\bar{u})]_x = \bar{d}\bar{u}_{xxx} + f'(\bar{u})\bar{u}_x.$$

It follows that

$$\begin{aligned}
&e^{\gamma t}\big(\psi_t - \bar{d}\psi_{xx} + [\gamma - f'(\bar{u})]\psi\big)\\
&\quad = -[\bar{d}\xi_{xx} + f'(\bar{u})\xi] + \lambda_1(\bar{u},\bar{d})\eta\\
&\quad = -2\bar{d}\phi^{-3}[\bar{u}_{xx}\phi - \bar{u}_x\phi_x]^2 - \bar{d}\phi^{-3}[2\bar{u}_x\bar{u}_{xxx}\phi^2 - |\bar{u}_x|^2\phi\phi_{xx}] - f'(\bar{u})|\bar{u}_x|^2\phi^{-1}\\
&\qquad + \lambda_1(\bar{u},\bar{d})\eta\\
&\quad \le \bar{d}|\bar{u}_x|^2\phi^{-2}\phi_{xx} + f'(\bar{u})\phi^{-1}|\bar{u}_x|^2 - 2f'(\bar{u})|\bar{u}_x|^2\phi^{-1} - 2\bar{d}\phi^{-1}\bar{u}_x\bar{u}_{xxx} + \lambda_1(\bar{u},\bar{d})\eta\\
&\quad = \lambda_1(\bar{u},\bar{d})\xi + \lambda_1(\bar{u},\bar{d})\eta - 2\bar{u}_x\phi^{-1}[f'(\bar{u})\bar{u}_x + \bar{d}\bar{u}_{xxx}] = \lambda_1(\bar{u},\bar{d})[\xi + \eta].
\end{aligned}$$

Hence, $\psi_t - \bar{d}\psi_{xx} + [\gamma - f'(\bar{u})]\psi \le 0$ because $\lambda_1(\bar{u},\bar{d}) \le 0$.

Since $\bar{u}$ is not a constant and $\psi(t,x)$ is r-periodic in $x \in \mathbb{R}$, we can choose x_0 such that $\psi(x_0,t) = \psi(x_0+r,t) = \min_{x\in\mathbb{R}}\psi(x,t) < 0$, and hence $\psi_x|_{x=x_0} = \psi_x|_{x=x_0+r} = 0$. Thus, $\psi(t,x)$ with $x \in [x_0, x_0+r]$ satisfies the equation

$$\begin{cases}\psi_t - \bar{d}\psi_{xx} + [\gamma - f'(\bar{u})]\psi \le 0, & x \in (x_0, x_0+r),\\ \psi_x|_{x=x_0} = \psi_x|_{x=x_0+r} = 0,\end{cases} \tag{6.27}$$

and $\psi(t,x)$ attains its maximum 0 at (x^*,t) with $x^* \in (x_0, x_0+r)$. By the strong maximum principle, we see that $\psi(t,x) \equiv 0$, which implies that $\bar{u}_x/\phi$ is a constant. Since $\bar{u}_x/\phi$ is r-periodic, it follows that $\bar{u}_x \equiv 0$, and hence $\bar{u}$ is a constant, a contradiction. □

Remark 6.1. By the proof above, the conclusion of Lemma 6.1 is valid for any $f \in C^1$.

Lemma 6.2. *Y is nonempty and open in W.*

Proof. Clearly, Lemma 6.1 implies that $Y \ne \emptyset$. Let $d^* \in Y$. We need to show that d^* is an interior point of Y. Assume for contradiction that there is a sequence of points $d_n \in W \setminus Y$ such that $d_n \to d^*$ in W as $n \to \infty$. Then (6.25) with $d = d_n$ admits a nonconstant r-periodic steady state u_n with the principal eigenvalue $\lambda_1(u_n, d_n) \le 0$. Using the transformation $v_n = d_n(u_n)_x$, we see that (u_n, v_n) is a periodic solution of the ordinary differential system

$$\begin{cases}(u_n)_x = v_n/d_n,\\ (v_n)_x = -f(u_n).\end{cases} \tag{6.28}$$

By elementary phase plane arguments, it then follows that

$$0 \le \inf_{x\in\mathbb{R}} u_n(x) \le a \le \sup_{x\in\mathbb{R}} u_n(x) \le 1, \qquad \forall n \ge 1,\ x \in \mathbb{R}. \tag{6.29}$$

Thus, the sequence of functions $((u_n)_x, (v_n)_x)$ is uniformly bounded and equicontinuous, and hence (u_n, v_n) has a uniformly convergent subsequence, still denoted by (u_n, v_n).

Let (u^*, v^*) be the limiting function of (u_n, v_n). Then u^* is an r-periodic steady state of (6.25) with $d = d^*$. It is easy to see from (6.29) that u^* is not the constant function 0 or 1.

Let ϕ_n be the positive eigenfunction associated with $\lambda_1(u_n, d_n)$. Then

$$(d_n(\phi_n)_x)_x + f'(u_n(x))\phi_n = \lambda_1(u_n, d_n)\phi_n. \tag{6.30}$$

Dividing (6.30) by ϕ_n and integrating from 0 to r, we obtain

$$\int_0^r \frac{d_n[(\phi_n)_x]^2}{\phi_n^2}\, dx + \int_0^r f'(u_n(x))\, dx = \lambda_1(u_n, d_n)r \le 0. \tag{6.31}$$

Since $f \in C^1$ and $f'(a) > 0$, we see that u^* cannot be the constant a: otherwise, the uniform convergence of u_n to a implies that $f'(u_n(x)) > 0$ for all $x \in [0, r]$ and sufficiently large n, which contradicts (6.31). Thus, u^* is a nonconstant r-periodic function. Since $d^* \in Y$, we have $\lambda_1(u^*, d^*) > 0$.

Note that $u_n \to u^*$ in $C(\mathbb{R}, \mathbb{R})$ and $d_n \to d^*$ in W. By the variational characterization of the principal eigenvalue $\lambda_1(u_n, d_n)$ (see [10, (5.2)]), it then follows that $0 \ge \lambda_1(u_n, d_n) \to \lambda_1(u^*, d^*) > 0$, a contradiction. □

The following counter-example shows that the parabolic equation (6.25) admits no bistability structure for the general periodic function $d(x)$.

Lemma 6.3. *Let either $f(u) = u(1-u^2)$, or $f(u) = u(1-u)(u-1/2)$. Then there exists a positive function $d \in W$ such that (6.25) admits a pair of linearly stable, nonconstant, and r-periodic steady states.*

Proof. We only consider the case where $f(u) = u(1-u^2)$ since the other one can be obtained under appropriate scalings. Our proof is based on the main result in [21, Theorem 3]. Without loss of generality, we assume that $r = 4$. In what follows, we use some notation of [21].

Let $l \in (0, 1)$ be fixed and c^0 be the step function on $[-1, 1]$ defined by

$$c^0(x) = \begin{cases} 1, & x \in [-1, -l] \cup (l, 1], \\ 0, & x \in (-l, l]. \end{cases} \tag{6.32}$$

Define $D := \{(x, y) : x = \pm l,\ y \in [0, 1]\} \cup$ graph of c^0. By [21, Theorem 3], for any positive even function $c \in C^1([-1, 1], \mathbb{R}^+)$ which is sufficiently close to c^0 (in the sense that the distance between D and the graph of c is small enough), the Neumann boundary problem

$$\begin{cases} u_t = (cu_x)_x + u(1-u^2), & x \in (-1, 1), \\ u_x(t, \pm 1) = 0, \end{cases} \tag{6.33}$$

admits an odd increasing steady state u_c which is linearly stable. That is, there exist $\lambda_1 < 0$ and $\phi > 0$ such that

$$\begin{cases} (c\phi_x)_x + f'(u_c)\phi = \lambda_1\phi, & x \in (-1, 1), \\ \phi_x(\pm 1) = 0. \end{cases} \tag{6.34}$$

In particular, we can choose c such that $c_x(-1) = c_x(1) = 0$. Since c is even and f is odd, we see that $v_c(x) := u_c(-x)$ is also a steady state, and λ_1 is the corresponding eigenvalue with the positive eigenfunction $\phi(-x)$.

Now we can construct a linearly stable 4-periodic steady state of (6.25). Define two 4-periodic functions by

$$\tilde{d}(x) = \begin{cases} c(x), & x \in [-1, 1], \\ c(2-x), & x \in (1, 3), \end{cases} \qquad w_1(x) = \begin{cases} u_c(x), & x \in [-1, 1], \\ u_c(2-x), & x \in (1, 3). \end{cases}$$

Then $w_1(x)$ is a 4-periodic steady state of (6.25) with $d = \tilde{d}$. Let the positive 4-periodic function $\rho(x)$ be defined by

$$\rho(x) = \begin{cases} \phi(x), & x \in [-1, 1], \\ \phi(2-x), & x \in (1, 3). \end{cases}$$

Then λ_1 and ρ solve the eigenvalue problem

$$\begin{cases} (\tilde{d}\rho_x)_x + f'(w_1)\rho = \lambda_1 \rho, & x \in \mathbb{R}, \\ \rho \text{ is } r\text{-periodic}. \end{cases}$$

This implies that w_1 is a linearly stable periodic steady state of (6.25) with $d = \tilde{d}$. Similarly, so is $w_2(x) := w_1(x + 2)$. □

As a consequence of Theorem 4.1, together with Lemmas 6.1 and 6.2, we have the following result on the existence of bistable traveling waves for (6.25).

Theorem 6.3. *Let $\bar{d}$ be a given positive constant. Then there exists $\delta_0 > 0$ such that for any $d \in W$ with $\|d - \bar{d}\|_W < \delta_0$, (6.25) admits a spatially periodic traveling wave solution $u(t, x) := V(x + ct, x)$ with some speed $c \in \mathbb{R}$ and connecting 0 to 1. Moreover, $V(\xi, x)$ is nondecreasing in ξ.*

We remark that Theorem 6.3 is a C^0-perturbation result in W, and hence it improves the existence result in [42, Theorem 3.1], where H^s-perturbation is used for some $s > 2$.

6.4. A nonlocal and time-delayed reaction-diffusion equation

Fix $\tau > 0$. Choose $\mathcal{X} := C([-\tau, 0], \mathbb{R})$, $\mathcal{Y} := C(\mathbb{R}, \mathbb{R})$ and $\mathcal{C} := C([-\tau, 0], \mathcal{Y})$. We equip $\mathcal{X}$ with the maximum norm, and $\mathcal{Y}$ and $\mathcal{C}$ with norms similar to (2.1). Define $\mathcal{Y}_+ := C(\mathbb{R}, \mathbb{R}_+)$. Let d be the metric in $\mathcal{C}(\mathcal{Y})$ induced by the norm. We are interested in bistable traveling waves of the nonlocal and time-delayed reaction-diffusion equation

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} = \frac{\partial^2 u(t,x)}{\partial x^2} + f(u_t)(x), & t > 0,\ x \in \mathbb{R}, \\ u_0 = \phi \in \mathcal{C}, \end{cases} \tag{6.35}$$

where $f : \mathcal{C} \to \mathcal{Y}$ is Lipschitz continuous and for each $t \geq 0$, $u_t \in \mathcal{C}$ is defined by

$$u_t(\theta, x) := u(t + \theta, x), \quad \forall \theta \in [-\tau, 0],\ x \in \mathbb{R}.$$

If the functional f takes the form $f(\phi)(x) = F(\phi(0,x), \phi(-\tau,x))$, then (6.35) becomes a local and time-delayed reaction-diffusion equation

$$\frac{\partial u(t,x)}{\partial t} = \frac{\partial^2 u(t,x)}{\partial x^2} + F(u(t,x), u(t-\tau,x)). \tag{6.36}$$

The bistable traveling waves of (6.36) were studied in [31]. If $f(\phi)(x) = -d\phi(0,x) + \int_{\mathbb{R}} b(\phi(-\tau,y))k(x-y)\,dy$, then (6.35) becomes a nonlocal and time-delayed reaction-diffusion equation

$$\frac{\partial u(t,x)}{\partial t} = \frac{\partial^2 u(t,x)}{\partial x^2} - du(t,x) + \int_{\mathbb{R}} b(u(t-\tau,y))k(x-y)\,dy. \tag{6.37}$$

The existence, uniqueness and stability of bistable waves of (6.37) were established in [29].

Note that $\mathbb{R}$ can be regarded as a subspace of $\mathcal{X}$, and the latter can also be regarded as a subspace of $\mathcal{C}$. Define $\bar{f} : \mathcal{X} \to \mathbb{R}$ by $\bar{f}(\varphi) = f(\varphi)$ and $\hat{f} : \mathbb{R} \to \mathbb{R}$ by $\hat{f}(\xi) = f(\xi)$. In order to obtain the existence of bistable waves for system (6.35), we impose the following assumptions on the functional f:

(E1) $0 < \alpha < \beta$ are three equilibria and there are no other equilibria between 0 and β.

(E2) The functional $f : \mathcal{C}_\beta \to \mathcal{Y}$ is quasi-monotone in the sense that

$$\lim_{h\to 0^+} \frac{1}{h} d\big([\phi(0) - \psi(0)] + h[f(\phi) - f(\psi)]; \mathcal{Y}_+\big) = 0 \quad \text{whenever } \phi \geq \psi \text{ in } \mathcal{C}_\beta.$$

(E3) The equilibria 0 and β are stable, and α is unstable in the sense that $\hat{f}'(0) < 0$, $\hat{f}'(\alpha) > 0$ and $\hat{f}'(\beta) < 0$.

(E4) For each $\varphi \in \mathcal{X}_\beta$, the derivative $\bar{L}(\varphi) := D\bar{f}(\varphi)$ of $\bar{f}$ can be represented as

$$\bar{L}(\varphi)\chi = a(\varphi)\chi(0) + \int_{-\tau}^{0} \chi(\theta)\,d_\theta\eta(\varphi) =: a(\varphi)\chi(0) + L_1(\varphi)\chi,$$

where $\eta(\varphi)$ is a positive Borel measure on $[-\tau, 0]$ and $\eta(\varphi)([-\tau, -\tau+\epsilon]) > 0$ for all small $\epsilon > 0$.

(E5) For any small $\epsilon > 0$, there exists $\delta \in (0, \beta)$ and a linear operator $L_\epsilon : \mathcal{C}_\beta \to \mathcal{Y}$ such that $L_\epsilon\phi \to Df(\alpha)\phi$ for all $\phi \in \mathcal{C}_\beta$ as $\epsilon \to 0$ and

$$f(\alpha+\phi) \geq L_\epsilon(\phi) \quad \text{and} \quad f(\alpha-\phi) \leq -L_\epsilon(\phi), \qquad \forall \phi \in \mathcal{C}_\delta.$$

Using the solution maps $\{T(t)\}_{t\geq 0}$ generated by the heat equation $\frac{\partial u(t,x)}{\partial t} = \frac{\partial^2 u(t,x)}{\partial x^2}$, we write system (6.35) in integral form

$$\begin{cases} u(t,\cdot) = T(t)\phi(0,\cdot) + \int_0^t T(t-r)f(u_r(\cdot,\cdot))\,dr, & t > 0, \\ u(\theta,\cdot) = \phi(\theta,\cdot), & \theta \in [-\tau, 0]. \end{cases} \tag{6.38}$$

Note that traveling waves of system (6.38) are those of system (6.35). It then remains to show that (6.38) admits a bistable traveling wave.

Theorem 6.4. *Under assumptions* (E1)–(E5)*, system* (6.35) *admits a nondecreasing traveling wave* $\phi(x+ct)$ *with* $\phi(-\infty)=0$ *and* $\phi(\infty)=\beta$.

Proof. From assumptions (E1)–(E2), we see that system (6.38) generates a monotone semiflow $\{Q_t\}_{t\geq 0}$ on $\mathcal{C}_\beta$ with

$$Q_t[\phi](\theta,x)=u_t(\theta,x;\phi),\qquad \forall(\theta,x)\in[-\tau,0]\times\mathbb{R},$$

where $u(t,x;\phi)$ is the unique solution of system (6.38) satisfying $u_0(\cdot,\cdot;\phi)=\phi\in\mathcal{C}_\beta$. By similar arguments to those in Section 5, the map Q_t satisfies (A4) if $t>\tau$ and (A4$'$) if $t\in(0,\tau]$.

Let $\bar{Q}_t$ be the restriction of Q_t to $\mathcal{X}_\beta$. Denote the derivative $D\bar{Q}_t[\hat{0}]$ of $\bar{Q}_t$ by $\bar{M}_{0,t}$. Then $\bar{M}_{0,t}$ is the solution map of the functional equation

$$\frac{du}{dt}=\bar{L}(0)u_t=a(0)u(t)+L_1(0)u_t. \tag{6.39}$$

By assumptions (E2) and (E4), system (6.39) admits a principal eigenvalue s_0 with an associated eigenfunction $v_0:=e^{s_0\theta}$ (see [34, Theorem 5.5.1]). More precisely, $\bar{M}_{0,t}[v_0]=e^{s_0t}v_0$. Furthermore, [34, Corollary 5.5.2] implies that $s_0<0$ since $\hat{f}'(0)<0$. Therefore, there exists $\delta_0(t)>0$ such that

$$\begin{aligned}\bar{Q}_t[\delta v_0]&=\bar{Q}_t[0]+D\bar{Q}_t[0][\delta v_0]+o(\delta^2)=\delta\bar{M}_{0,t}[v_0]+o(\delta^2)\\&=\delta e^{s_0t}v_0+o(\delta^2)=\delta v_0+\delta[e^{s_0t}-1]v_0+o(\delta^2)\ll\delta v_0,\qquad\forall\delta\in(0,\delta_0(t)].\end{aligned}$$

Similarly, there exist $\delta_\alpha(t)$, v_α and $\delta_\beta(t)$, v_β such that

$$\bar{Q}_t[\beta-\delta v_\beta]\gg\beta-\delta v_\beta,\qquad\forall\delta\in(0,\delta_\beta(t)],$$

and

$$\bar{Q}_t[\alpha+\delta v_\alpha]\gg\alpha+\delta v_\alpha,\qquad\bar{Q}_t[\alpha-\delta v_\alpha]\ll\alpha-\delta v_\alpha,\qquad\forall\delta\in(0,\delta_\alpha(t)].$$

It remains to show that (A6) is also true. Indeed, from [25, Theorem 2.17] the solution semiflows $\{Q_t\}_{t\geq 0}$ restricted to $[0,\alpha]_{\mathcal{C}}$ and $[\alpha,\beta]_{\mathcal{C}}$ admit spreading speeds $c^*(0,\alpha)$ and $c^*(\alpha,\beta)$, respectively. Let M_ϵ^t be the solution maps of the linear system

$$\begin{cases}u(t,\cdot)=T(t)\phi(0,\cdot)+\int_0^t T(t-r)L_\epsilon(u_r(\cdot,\cdot))\,dr, & t>0,\\ u(\theta,\cdot)=\phi(\theta,\cdot), & \theta\in[-\tau,0].\end{cases}$$

Then assumption (E5) guarantees that $Q_t[\phi]\geq M_t^\epsilon[\phi]$ when $\phi\in\mathcal{C}_\delta$, where $\delta=\delta(\epsilon)$ is defined in (E5). Therefore, we see from [25, Theorem 3.10] that $c^*(0,\alpha)\geq\bar{c}$ and $c^*(\alpha,\beta)\geq\bar{c}$, where $\bar{c}$ is a positive number determined by the linearized system of (6.35) at $u\equiv\alpha$, and hence (A6) holds. Consequently, Theorem 5.1 completes the proof. $\square$

Remark 6.2. At this moment we are unable to present a general result on the uniqueness and global attractivity of bistable waves under the current abstract setting. However, one may use the convergence theorem for monotone semiflows (see [47, Theorem 2.2.4]) and similar arguments to the proofs of [47, Theorem 10.2.1] and [45, Theorem 3.1] to obtain the global attractivity (and hence uniqueness) of bistable waves for the four examples in this section.

7. Appendix

In this appendix, we present certain properties of Banach lattices and countable subsets of $\mathbb{R}$, and some convergence results for sequences of monotone functions, including an abstract variant of Helly's theorem.

Proposition 7.1. *Every Banach lattice $\mathcal{X}$ has the following properties:*

(1) *For any $u, v \in \mathcal{X}$ with $v \in \mathcal{X}^+$, if $-v \leq u \leq v$, then $\|u\|_{\mathcal{X}} \leq \|v\|_{\mathcal{X}}$.*
(2) *If $u_k \to u$ and $v_k \to v$ in $\mathcal{X}$ with $u_k \geq v_k$, then $u \geq v$.*

Proposition 7.2. *The space $\mathcal{C}$ has the following properties:*

(1) *Let ϕ be a monotone function in $\mathcal{C}$. If $x_k \in \mathcal{H}$ nondecreasingly tends to $x \in \mathcal{H} \cup \{\infty\}$ and $\lim_{k\to\infty} \phi(x_k) = u \in \mathcal{X}$, then $\lim_{y\uparrow x} \phi(y) = u$. A similar result holds if x_k nonincreasingly tends to $x \in \mathcal{H} \cup \{-\infty\}$.*
(2) *Assume that $h, h_k : \mathcal{H} \to \mathcal{H}$ are continuous and $\phi_k \to \phi$ in $\mathcal{C}$. If $h_k(x) \to h(x)$ uniformly for x in any bounded subset of $\mathcal{H}$, then $\phi_k \circ h_k \to \phi \circ h$ in $\mathcal{C}$.*

We omit the easy proofs of Propositions 7.1 and 7.2.

Proposition 7.3. *Assume that D is a countable subset of $\mathbb{R}$. Then for any $c \in \mathbb{R}$, there exists another countable subset A of $\mathbb{R}$ such that $(\mathbb{R} \setminus A) + cm \subset \mathbb{R} \setminus D$ for all $m \in \mathbb{Z}^+$.*

Proof. It suffices to show the set $A := \{x \in \mathbb{R} : \text{there exists } m \text{ such that } x + cm \in D\}$ is countable. Indeed, we have $A = \bigcup_{m=1}^{\infty}(D - cm)$. This implies A is countable. □

Proposition 7.4. *For any countable subset Γ_1 of $\mathbb{R}$, there exists another countable set Γ_2 that is dense in $\mathbb{R}$ and $\Gamma_1 \cap \Gamma_2 = \emptyset$.*

Proof. Since Γ_1 is countable and $\bigcup_{\alpha\in\mathbb{R}}(\alpha + \mathbb{Q}) = \mathbb{R}$, there must exist a sequence α_n such that $\Gamma_1 \subset \bigcup_{n=1}^{\infty}(\alpha_n + \mathbb{Q})$. Note that $\bigcup_{n=1}^{\infty}(\alpha_n + \mathbb{Q})$ is countable. Hence, there exists $\alpha \in \mathbb{R}$ such that $\alpha \notin \bigcup_{n=1}^{\infty}(\alpha_n + \mathbb{Q})$. This means that $\alpha - \alpha_n \notin \mathbb{Q}$ for all $n \geq 1$, and hence $(\alpha + \mathbb{Q}) \cap (\alpha_n + \mathbb{Q}) = \emptyset$ for all $n \geq 1$. Define $\Gamma_2 := \alpha + \mathbb{Q}$. We then see that Γ_2 is countable and dense in $\mathbb{R}$, and $\Gamma_1 \cap \Gamma_2 = \emptyset$. □

Proposition 7.5. *Assume that $f, f_n : \mathbb{R} \to \mathcal{X}$ are nondecreasing and the set D is dense in $\mathbb{R}$. If $s_n \to 0$, $f(s)$ is continuous on D and $f_n(s) \to f(s)$ for every $s \in D$, then $f_n(s + s_n) \to f(s)$ for every $s \in D$.*

Proof. Let $s \in D$ be fixed. For any $\delta > 0$, since $D - s$ is dense in $\mathbb{R}$, we can choose $\delta_+ \in (D - s) \cap (0, \delta)$ and $\delta_- \in (D - s) \cap (-\delta, 0)$. Clearly, $s + \delta_+, s + \delta_- \in D$. Thus, there exists an integer N_δ such that $s + s_n \in (s + \delta_-, s + \delta_+)$ for all $n \geq N_\delta$. Since

$$f_n(s+\delta_-) - f_n(s+\delta_+) \leq f_n(s+s_n) - f_n(s) \leq f_n(s+\delta_+) - f_n(s+\delta_-), \quad \forall n \geq N_\delta,$$

we have

$$\|f_n(s + s_n) - f_n(s)\|_{\mathcal{X}} \leq \|f_n(s + \delta_+) - f_n(s + \delta_-)\|_{\mathcal{X}}, \quad \forall n \geq N_\delta.$$

It then follows that

$$\begin{aligned}\|f_n(s+s_n)-f(s)\|_{\mathcal{X}} &\le \|f_n(s+s_n)-f_n(s)\|_{\mathcal{X}}+\|f_n(s)-f(s)\|_{\mathcal{X}}\\ &\le \|f_n(s+\delta_+)-f_n(s+\delta_-)\|_{\mathcal{X}}+\|f_n(s)-f(s)\|_{\mathcal{X}}\\ &\le \|f(s+\delta_+)-f_n(s+\delta_+)\|_{\mathcal{X}}+\|f(s+\delta_+)-f(s+\delta_-)\|_{\mathcal{X}}\\ &\quad+\|f_n(s+\delta_-)-f(s+\delta_-)\|_{\mathcal{X}}+\|f_n(s)-f(s)\|_{\mathcal{X}}\end{aligned}$$

for all $n \ge N_\delta$. Now the pointwise convergence of f_n in D and the continuity of f on D complete the proof. ◻

To end this section, we prove a convergence theorem for sequences of monotone functions from $\mathbb{R}$ to the special Banach lattice $C(M, \mathbb{R}^d)$ defined in Section 2, which is a variant of Helly's theorem [17, p. 165] for sequences of monotone functions from $\mathbb{R}$ to $\mathbb{R}$.

Theorem 7.1. *Let D be a dense subset of $\mathbb{R}$ and $f_n, n \ge 1$, be a sequence of nondecreasing functions from $\mathbb{R}$ to the Banach lattice $\mathcal{X} := C(M, \mathbb{R}^d)$. Assume that:*

(i) *For any $s \in D$, $f_n(s)$ is convergent in $\mathcal{X}$.*
(ii) *There exists a countable set $D_1 \subset \mathbb{R}$ such that for any $s \in \mathbb{R} \setminus D_1$, the limits $\lim_{m\to\infty} \lim_{n\to\infty} f_n(s_{\pm,m})$ exist in $\mathcal{X}$, where $s_{-,m} \uparrow s$ and $s_{+,m} \downarrow s$ with $s_{\pm,m} \in D$.*

Then $f_n(s)$ is convergent in $\mathcal{X}$ for almost all $s \in \mathbb{R}$.

Proof. Due to assumption (ii), we can define $f : \mathbb{R} \to \mathcal{X}$ by

$$f(s) := \begin{cases} \lim_{x\uparrow s} \lim_{x\in D,\, n\to\infty} f_n(x), & s \in \mathbb{R}\setminus D_1,\\ \text{any value}, & s \in D_1. \end{cases} \tag{7.1}$$

We first show that the set of discontinuity points of f is at most countable. Define

$$A := \{s \in \mathbb{R}\setminus D_1 : f(s^-), f(s^+) \text{ both exist}\}, \qquad B := \{s \in A : f(s^-) < f(s^+)\}.$$

For any $s \in B$, there are $x_1 \in M$ and $1 \le i \le d$ such that $(f(s^-)(x_1))_i < (f(s^+)(x_1))_i$. Recall that M is compact, so there is a countable dense subset M_1. It then follows that there must be $x_2 \in M_1$ such that $(f(s^-)(x_2))_i < (f(s^+)(x_2))_i$. Therefore,

$$B = \bigcup_{i=1}^{m} \bigcup_{x\in M_1} \{s \in A : (f(s^-)(x))_i < (f(s^+)(x))_i\}.$$

Since for each fixed i and x, $(f(s)(x))_i$ is a nondecreasing function from $\mathbb{R}\setminus D_1$ to $\mathbb{R}$, the set $\{s \in A : (f(s^-)(x))_i < (f(s^+)(x))_i\}$ is at most countable, and hence so is B.

Now we can prove the conclusion. Assume that $s \in \mathbb{R}$ is a continuity point of f. For any $\delta > 0$, choose $\delta_- \in D \cap (s-\delta, s)$ and $\delta_+ \in D \cap (s, s+\delta)$. Then we have

$$f_n(\delta_-) - f_n(\delta_+) \le f_n(s) - f_n(\delta_-) \le f_n(\delta_+) - f_n(\delta_-), \qquad \forall n \ge 1,$$

which, together with Proposition 7.1(2), implies that

$$\|f_n(s) - f_n(\delta_-)\|_{\mathcal{X}} \le \|f_n(\delta_+) - f_n(\delta_-)\|_{\mathcal{X}}, \qquad \forall n \ge 1. \tag{7.2}$$

On the other hand, by (7.2) and the triangular inequality we have

$$\begin{aligned}\|f_n(s)-f(s)\|_{\mathcal{X}} &\le \|f_n(s)-f_n(\delta_-)\|_{\mathcal{X}}+\|f_n(\delta_-)-f(s)\|_{\mathcal{X}}\\&\le \|f_n(\delta_+)-f_n(\delta_-)\|_{\mathcal{X}}+\|f_n(\delta_-)-f(s)\|_{\mathcal{X}}\\&\le \|f_n(\delta_+)-f(\delta_+)\|_{\mathcal{X}}+\|f(\delta_+)-f(\delta_-)\|_{\mathcal{X}}\\&\quad+\|f(\delta_-)-f_n(\delta_-)\|_{\mathcal{X}}+\|f_n(\delta_-)-f(\delta_-)\|_{\mathcal{X}}\\&\quad+\|f(\delta_-)-f(s)\|_{\mathcal{X}},\qquad \forall n\ge 1.\end{aligned}$$

Now the pointwise convergence of f_n in D and the continuity of f at s complete the proof. ◻

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