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Edge-reinforced random walk, vertex-reinforced jump process and the supersymmetric hyperbolic sigma model

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Abstract. Edge-reinforced random walk (ERRW), introduced by Coppersmith and Diaconis in 1986 [8], is a random process which takes values in the vertex set of a graph G and is more likely to cross edges it has visited before. We show that it can be represented in terms of a vertex-reinforced jump process (VRJP) with independent gamma conductances; the VRJP was conceived by Werner and first studied by Davis and Volkov [10, 11], and is a continuous-time process favouring sites with more local time. We calculate, for any finite graph G, the limiting measure of the centred occupation time measure of VRJP, and interpret it as a supersymmetric hyperbolic sigma model in quantum field theory, introduced by Zirnbauer in 1991 [35].

This enables us to deduce that VRJP and ERRW are positive recurrent on any graph of bounded degree for large reinforcement, and that the VRJP is transient on $\mathbb{Z}^d$, $d \geq 3$, for small reinforcement, using results of Disertori and Spencer [15] and Disertori, Spencer and Zirnbauer [16].

Keywords. Self-interacting random walk, reinforcement, random walk in random environment, sigma models, supersymmetry, de Finetti theorem

1. Introduction

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $G = (V, E, \sim)$ be a non-oriented connected locally finite graph without self-loops (i.e. edges connecting a vertex to itself). Let $(a_e)_{e\in E}$ be a sequence of positive initial weights associated to each edge $e \in E$.

Let $(X_n)_{n\in\mathbb{N}}$ be a random process that takes values in V, and let $\mathcal{F}_n = \sigma(X_0, \dots, X_n)$ be the filtration of its past. For any $e \in E$ and $n \in \mathbb{N} \cup \{\infty\}$, let

$$Z_n(e) = a_e + \sum_{k=1}^{n} \mathbb{1}_{\{\{X_{k-1}, X_k\}=e\}} \tag{1.1}$$

be the number of crosses of e up to time n plus the initial weight a_e.

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Then $(X_n)_{n\in\mathbb{N}}$ is called an *Edge-Reinforced Random Walk* (ERRW) with starting point $i_0 \in V$ and weights $(a_e)_{e\in E}$ if $X_0 = i_0$ and, for all $n \in \mathbb{N}$,

$$\mathbb{P}(X_{n+1} = j \mid \mathcal{F}_n) = \mathbb{1}_{\{j\sim X_n\}} \frac{Z_n(\{X_n, j\})}{\sum_{k\sim X_n} Z_n(\{X_n, k\})}. \tag{1.2}$$

The Edge-Reinforced Random Walk was introduced in 1986 by Diaconis [8]; on finite graphs it is a mixture of reversible Markov chains, and the mixing measure can be determined explicitly (the so-called Coppersmith–Diaconis measure, or "magic formula" [12], see also [17, 27]), which has applications in Bayesian statistics [14, 2, 3].

On infinite graphs, the research has focused so far on recurrence/transience criteria. In their seminal work Diaconis and Coppersmith [8] conjectured that the ERRW could be recurrent in any dimension.

On acyclic or directed graphs, the walk can be seen as a random walk in an *independent* random environment [25], and a recurrence/transience phase transition was first observed by Pemantle on trees [25, 18, 5]. In the case of infinite graphs with cycles, recurrence criteria and asymptotic estimates were obtained by Merkl and Rolles on graphs of the form $\mathbb{Z} \times G$, G a finite graph, and on a certain two-dimensional graph [22, 23, 24, 28], but recurrence on $\mathbb{Z}^2$ was still unresolved.

Also, this original ERRW model [8] has triggered a number of similar models of self-organization and learning behaviour; see for instance Davis [9], Limic and Tarrès [20, 21], Pemantle [26], Sabot [29, 30], Tarrès [32, 33] and Tóth [34], with different perspectives on the topic.

Our first result relates the ERRW to the Vertex-Reinforced Jump Process (VRJP), conceived by Werner and studied by Davis and Volkov [10, 11], Collevecchio [6, 7] and Basdevant and Singh [4].

We define a VRJP with conductances $(W_e)_{e\in E}$ to be a continuous-time process $(Y_t)_{t\geq 0}$ on V, starting at time 0 at some vertex $i_0 \in V$ and such that if Y is at a vertex $i \in V$ at time t, then, conditionally on $(Y_s, s \leq t)$, the process jumps to a neighbour j of i at rate $W_{\{i,j\}} L_j(t)$, where

$$L_j(t) := 1 + \int_0^t \mathbb{1}_{\{Y_s = j\}}\, ds.$$

The main results of the paper are the following. In Section 2, Theorem 1, we represent the ERRW in terms of a VRJP with independent gamma conductances. Section 3 is dedicated to showing, in Theorem 2, that the VRJP is a mixture of time-changed Markov jump processes, with a computation of the mixing law. In Section 6, we interpret that mixing law with the supersymmetric hyperbolic sigma model introduced by Zirnbauer [35] and Disertori, Spencer and Zirnbauer [16] and related to the Anderson model.

We prove positive recurrence of VRJP and ERRW in any dimension for large reinforcement in Corollaries 1 and 2, using a localization result of Disertori and Spencer [15], and transience of VRJP in dimension $d \geq 3$ at small reinforcement in Corollary 4 using a delocalization result of Disertori, Spencer and Zirnbauer [16]. Shortly after this paper appeared electronically, Angel, Crawford and Kozma [1] proposed another proof of recurrence of ERRW and VRJP under similar assumptions, without making the link with statistical physics (and using, for the VRJP, the representation as a mixture of time-changed Markov jump processes proved in this paper).

2. From ERRW to VRJP

It is convenient here to consider a time changed version of $(Y_s)_{s\geq 0}$: consider the positive continuous additive functional of $(Y_s)_{s\geq 0}$,

$$A(s) = \int_0^s \frac{1}{L_{Y_u}(u)}\, du = \sum_{x\in V} \log(L_x(s)),$$

and the time changed process

$$X_t = Y_{A^{-1}(t)}.$$

Let $(T_i(t))_{i\in V}$ be the local time of the process $(X_t)_{t\geq 0}$,

$$T_x(t) = \int_0^t \mathbb{1}_{\{X_u = x\}}\, du.$$

Lemma 1. *The inverse functional* A^{-1} *is given by*

$$A^{-1}(t) = \int_0^t e^{T_{X_u}(u)}\, du = \sum_{i\in V}(e^{T_i(t)} - 1).$$

The law of the process X_t *is described by the following: conditioned on the past at time* t, *if the process* X_t *is at the position* i, *then it jumps to a neighbour* j *of* i *at rate*

$$W_{i,j} e^{T_i(t)+T_j(t)}.$$

Proof. First note that

$$T_x(A(s)) = \log(L_x(s)), \tag{2.1}$$

since

$$(T_x(A(s)))' = A'(s)\mathbb{1}_{\{X_{A(s)}=x\}} = \frac{1}{L_{Y_s}(s)}\mathbb{1}_{\{Y_s=x\}}.$$

Hence,

$$(A^{-1}(t))' = \frac{1}{A'(A^{-1}(t))} = L_{X_t}(A^{-1}(t)) = e^{T_{X_t}(t)},$$

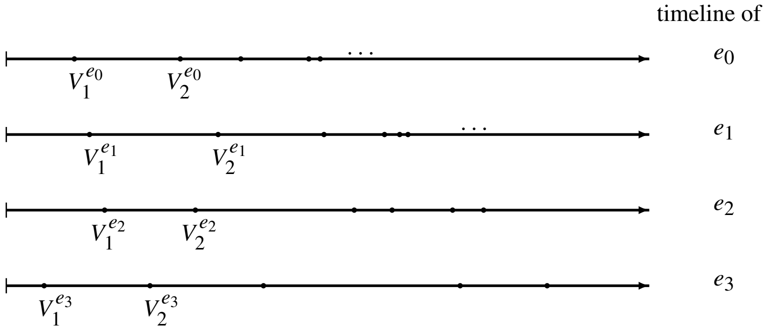
which yields the expression for A^{-1}. It remains to prove the last assertion:

$$\begin{aligned}\mathbb{P}(X_{t+dt} = j \mid \mathcal{F}_t) &= \mathbb{P}(Y_{A^{-1}(t+dt)} = j \mid \mathcal{F}_t)\\ &= W_{X_t,j}(A^{-1})'(t) L_j(A^{-1}(t))dt\\ &= W_{i,j} e^{T_{X_t}(t)} e^{T_j(t)} dt.\end{aligned}$$

□

In order to relate ERRW to VRJP, let us first define the following process $(\tilde{X}_t)_{t\in\mathbb{R}_+}$, initially introduced by Rubin, Davis and Sellke [9, 31], which we call here a *continuous-time ERRW with weights* $(a_e)_{e\in E}$ and starting at $\tilde{X}_0 := i_0$ at time 0.

- Define, on each edge $e \in E$, independent point processes (*alarm times*) as follows. Let $(\tau_k^e)_{e\in E,\, k\in\mathbb{Z}_+}$ be independent exponential random variables with parameter 1 and define
$$V_k^e = \sum_{l=0}^{k-1} \frac{1}{a_e + l}\tau_l^e, \qquad \forall k \in \mathbb{N}.$$
- Each edge $e \in E$ has its own clock, denoted by $\tilde{T}_e(t)$, which only runs when the process $(\tilde{X}_t)_{t\ge 0}$ is adjacent to e. This means that if $e = \{i, j\}$, then $\tilde{T}_{\{i,j\}}(t) = \tilde{T}_i(t) + \tilde{T}_j(t)$, where $\tilde{T}_i(t)$ is the local time of the process $\tilde{X}$ at vertex i and time t.
- When the clock of an edge $e \in E$ rings, i.e. when $\tilde{T}_e(t) = V_k^e$ for some $k > 0$, then $\tilde{X}_t$ crosses it instantaneously (of course, this can happen only when $\tilde{X}$ is adjacent to e).



Let τ_n be the n-th jump time of $(\tilde{X}_t)_{t\ge 0}$, with the convention that $\tau_0 := 0$.

Lemma 2 (Davis [9], Sellke [31]). *Let $(X_n)_{n\in\mathbb{N}}$ (resp. $(\tilde{X}_t)_{t\ge 0}$) be an ERRW (resp. continuous-time ERRW) with weights $(a_e)_{e\in E}$, starting at some vertex $i_0 \in V$. Then $(\tilde{X}_{\tau_n})_{n\ge 0}$ and $(X_n)_{n\ge 0}$ have the same distribution.*

Proof. The argument is based on the memoryless property of exponentials, and on the observation that if A and B are two independent random variables of parameters a and b, then $\mathbb{P}[A < B] = a/(a + b)$. □

On each timeline the alarm times follow a so-called *Yule process*, which, by a result of Kendall [19], can be described after an exponential change of time by a Poisson point process with constant (but random Gamma distributed) intensity. This observation applies to any discrete time random walk with linear reinforcement on its similarly defined continuous time version, and was initially made by Tarrès for the vertex-reinforced random walk [33]. Using that description and Lemma 1, we can deduce the following Theorem 1 linking up ERRW and VRJP.

Theorem 1. *Let $(\tilde{X}_t)_{t\ge 0}$ be a continuous-time ERRW with weights $(a_e)_{e\in E}$. Then there exists a sequence of independent random variables $W_e \sim \mathrm{Gamma}(a_e, 1)$, $e \in E$, such that, conditionally on $(W_e)_{e\in E}$, $(\tilde{X}_t)_{t\ge 0}$ has the same law as the time modification $(X_t)_{t\ge 0}$ of the VRJP with weights $(W_e)_{e\in E}$.*

In particular, the ERRW $(X_n)_{n\ge 0}$ is equal in law to the discrete time process associated to a VRJP with random independent conductances $W_e \sim \mathrm{Gamma}(a_e, 1)$.

Proof. For any $e \in E$, define the simple birth process $\{N_t^e, t \geq 0\}$ with initial population size a_e by

$$N_t^e := a_e + \sup\{k \in \mathbb{N} : V_k^e \leq t\}.$$

This process is sometimes called the *Yule process*; by a result of D. Kendall [19], there exists $W_e := \lim N_t^e e^{-t}$, with distribution Gamma$(a_e, 1)$, such that, conditionally on W_e, $\{N_{f_{W_e}(t)}^e, t \geq 0\}$ is a Poisson point process with unit parameter, where

$$f_W(t) := \log(1 + t/W).$$

Let us now condition on $(W_e)_{e\in E}$: N^e increases between times t and $t + dt$ with probability $W_e e^t \, dt = (f_{W_e}^{-1})'(t)\, dt$. A similar characterization of the timelines is also used in [33, Lemma 4.7]. If $\tilde{X}$ is at vertex x at time t, it jumps to a neighbour y of x at rate $W_{x,y} e^{T_x(t)+T_y(t)}$. □

3. The mixing measure of VRJP

Next we study VRJP. Given fixed weights $(W_e)_{e\in E}$, we denote by $(Y_t)_{t\geq 0}$ the VRJP and by $(X_t)_{t\geq 0}$ its time modification defined in the previous section, starting at site $X_0 := i_0$ at time 0; and $(T_i(t))_{i\in V}$ denotes its local time.

It is clear from the definition that the joint process $\Theta_t = (X_t, (T_i(t))_{i\in V})$ is a time continuous Markov process on the state space $V \times \mathbb{R}_+^V$ with generator $\tilde{L}$ defined on C^∞ bounded functions by

$$\tilde{L}(f)(i, T) = \left(\frac{\partial}{\partial T_i} f\right)(i, T) + L(T)(f(\cdot, T))(i), \qquad \forall (i, T) \in V \times \mathbb{R}_+^V,$$

where $L(T)$ is the generator of the jump process on V at frozen T defined for $g \in \mathbb{R}^V$:

$$L(T)(g)(i) = \sum_{j\in V} W_{i,j} e^{T_i+T_j}(g(j) - g(i)), \qquad \forall i \in V.$$

We denote by $\mathbb{P}_{i_0,T}$ the law of the Markov process with generator $\tilde{L}$ starting from the initial state (i_0, T).

Note that the law of $(X_t, T(t) - T)$ under $\mathbb{P}_{i_0,T}$ is equal to the law of the process starting from $(i_0, 0)$ with conductances

$$W_{i,j}^T = W_{i,j} e^{T_i+T_j}.$$

For simplicity, we let $\mathbb{P}_i := \mathbb{P}_{i,0}$.

We show, in Proposition 1, that for finite graphs the centred occupation times converge a.s., and we calculate the limiting measure in Theorem 2(i). In Theorem 2(ii) we show that the VRJP $(Y_s)_{s\geq 0}$ (as well as $(X_t)_{t\geq 0}$) is a mixture of time-changed Markov jump processes.

This limiting measure can be interpreted as a supersymmetric hyperbolic sigma model. We are grateful to a few specialists of field theory for their advice: Denis Perrot who mentioned that the limit measure of VRJP could be related to the sigma model, and Krzysztof Gawędzki who pointed out reference [16], which actually mentions a possible link of their model with ERRW, suggested by Kozma, Heydenreich and Sznitman (cf. [16, Section 1.5]).

Note that when G is a tree, if the edges are for instance oriented towards the root, and we let $V_e = e^{U_{\overline{e}}-U_{\underline{e}}}$, then the random variables (V_e) are independent and are distributed according to an inverse Gaussian law. This was proved in previous works on VRJP [10, 11, 6, 7, 4].

Theorems 1 and 2 enable us to retrieve, in Section 5, the limiting measure of ERRWs, computed by Coppersmith and Diaconis in [8] (see also [17]), by integration over the random gamma conductances $(W_e)_{e\in E}$. This explains its renormalization constant, which has remained mysterious so far.

Proposition 1. *Suppose that G is finite and set $N = |V|$. For all $i \in V$, the following limits exist $\mathbb{P}_{i_0}$-a.s.:*

$$U_i = \lim_{t\to\infty} (T_i(t) - t/N).$$

Theorem 2. *Suppose that G is finite and set $N = |V|$.*

(i) *Under $\mathbb{P}_{i_0}$, $(U_i)_{i\in V}$ has the following density distribution on $\mathcal{H}_0=\{(u_i) : \sum u_i=0\}$:*

$$\frac{N}{(2\pi)^{(N-1)/2}} e^{u_{i_0}} e^{-H(W,u)} \sqrt{D(W,u)}, \tag{3.1}$$

where

$$H(W,u) = 2 \sum_{\{i,j\}\in E} W_{i,j} \sinh^2\left(\frac{1}{2}(u_i - u_j)\right)$$

and $D(W,u)$ is any diagonal minor of the $N \times N$ matrix $M(W,u)$ with entries

$$m_{i,j} = \begin{cases} -W_{i,j} e^{u_i+u_j} & \text{if } i \neq j, \\ \sum_{k\sim i} W_{i,k} e^{u_i+u_k} & \text{if } i = j. \end{cases}$$

(ii) *Let C, resp. D, be positive continuous additive functionals of X, resp. Y, given by*

$$C(t) = \sum_{i\in V} (e^{2T_i(t)} - 1), \qquad D(s) = \sum_{i\in V} (L_i^2(s) - 1),$$

and let

$$Z_t = X_{C^{-1}(t)} \ (= Y_{D^{-1}(t)}).$$

Then, conditionally on $(U_i)_{i\in V}$, Z_t is a Markov jump process starting from i_0, with jump rate from i to j equal to

$$\tfrac{1}{2} W_{i,j} e^{U_j - U_i}.$$

In particular, the discrete time process associated with $(Y_s)_{s\geq 0}$ is a mixture of reversible Markov chains with conductances $W_{i,j} e^{U_i+U_j}$.

N.B.: 1) The density distribution in (3.1) is with respect to the Lebesgue measure on $\mathcal{H}_0$, which is $\prod_{i\in V\setminus\{j_0\}} du_i$ for any choice of j_0 in V. We simply write du for any of the $\prod_{i\in V\setminus\{j_0\}} du_i$.

2) The diagonal minors of the matrix $M(W,u)$ are all equal since the sum of the entries in any line or column of the matrix is null. By the matrix-tree theorem, if we let $\mathcal{T}$ be the set of spanning trees of $(V, E, \sim)$, then $D(W,u) = \sum_{T\in\mathcal{T}} \prod_{\{i,j\}\in\mathcal{T}} W_{\{i,j\}} e^{u_i+u_j}$.

Remark 1. Usually a result like (ii) makes use of de Finetti's theorem; here, we provide a direct proof exploiting the explicit form of the density. In Section 5, we apply Theorems 1 and 2 to give a new proof of the Diaconis–Coppersmith formula including its de Finetti part.

Remark 2. The fact that (3.1) is a density is not at all obvious. Our argument is probabilistic: (3.1) is the law of the random variables (U_i). This can also be explained directly as a consequence of supersymmetry [16, (5.1)]. The fact that the measure (3.1) normalizes at 1 is a fundamental property, which plays a crucial role in the proofs of the localization and delocalization results of Disertori, Spencer and Zirnbauer [15, 16].

Remark 3. (ii) implies that the VRJP (Y_s) is a mixture of Markov jump processes. More precisely, let $(U_i)_{i\in V}$ be a random variable distributed according to (3.1) and, conditionally on U, let Z be the Markov jump process with jump rates from i to j given by $\frac{1}{2}W_{i,j}e^{U_j-U_i}$. Then the time changed process $(Z_{B^{-1}(s)})_{s\ge 0}$ with

$$B(t) = \sum_{i\in V} \sqrt{1 + l_i^Z(t)} - 1,$$

where $(l_i^Z(t))$ is the local time of Z at time t, has the law of the VRJP (Y_s) with conductances W.

4. Proofs of Proposition 1 and Theorem 2

4.1. *Proof of Proposition 1*

By a slight abuse of notation, we also use the notation $L(T)$ for the $N\times N$ matrix of that operator in the canonical basis (which is equal to $-M(W,T)$ of Theorem 2). Let $\mathbf{1}$ be the $N\times N$ matrix with entries all equal to 1, i.e. $\mathbf{1}_{i,j} = 1$ for all $i, j \in V$, and let I be the identity matrix.

Let us define, for all $T\in\mathbb{R}^V$,

$$Q(T) := -\int_0^\infty \left(e^{uL(T)} - \mathbf{1}/N\right) du, \tag{4.1}$$

which exists since $e^{uL(T)}$ converges towards $\mathbf{1}/N$ at exponential rate.

Then $Q(T)$ is a solution of the Poisson equation for the Markov chain $L(T)$, namely

$$L(T)Q(T) = Q(T)L(T) = I - \mathbf{1}/N.$$

Observe that $L(T)$ is symmetric, and thus $Q(T)$ is as well.

For all $T \in \mathbb{R}^V$ and $i, j \in V$, let $E_i^T(\tau_j)$ denote the expectation of the first hitting time of site j for the continuous-time process with generator $L(T)$. Then

$$Q(T)_{i,j} = \frac{1}{N} E_i^T(\tau_j) + Q(T)_{j,j}$$

by the strong Markov property applied to (4.1). As a consequence, $Q(T)_{j,j}$ is non-positive for all j, using $\sum_{i \in V} Q(T)_{i,j} = 0$.

Let us fix $l \in V$. We want to study the asymptotics of $T_l(t) - t/N$ as $t \to \infty$:

$$\begin{aligned} T_l(t) - \frac{t}{N} &= \int_0^t \left(\mathbb{1}_{\{X_u = l\}} - \frac{1}{N} \right) du = \int_0^t (L(T(u)) Q(T(u)))_{X_u, l} \, du \\ &= \int_0^t \tilde{L}(Q(\cdot)_{\cdot,l})(X_u, T(u)) \, du - \int_0^t \frac{\partial}{\partial T_{X_u}} Q(T(u))_{X_u,l} \, du \\ &= Q(T(t))_{X_t,l} - Q(0)_{X_0,l} + M_l(t) - \int_0^t \frac{\partial}{\partial T_{X_u}} Q(T(u))_{X_u,l} \, du, \end{aligned} \tag{4.2}$$

where

$$M_l(t) := -Q(T(t))_{X_t,l} + Q(0)_{X_0,l} + \int_0^t \tilde{L}(Q(\cdot)_{\cdot,l})(X_u, T(u)) \, du$$

is a martingale for all l. Recall that $\tilde{L}$ is the generator of $(X_t, T(t))$.

The following lemma shows in particular the convergence of $Q(T(t))_{k,l}$ for all k, l as t goes to infinity. It is a purely deterministic statement, which does not depend on the trajectory of the process X_t (as long as it only performs finitely many jumps in a finite time interval), but only on the added local time in W^T.

Lemma 3. *For all $k, l \in V$, $Q(T(t))_{k,l}$ converges as $t \to \infty$, and*

$$\int_0^\infty \left| \frac{\partial}{\partial T_{X_u}} Q(T(u))_{X_u,l} \right| du < \infty.$$

Proof. For all $i, k, l \in V$, let us compute $\frac{\partial}{\partial T_i} Q(T)_{k,l}$: by differentiation of the Poisson equation,

$$\frac{\partial}{\partial T_i} Q(T)_{k,l} = -\left(Q(T) \left(\frac{\partial}{\partial T_i} L \right) Q(T) \right)_{k,l}.$$

Now, for any real function f on V,

$$\frac{\partial}{\partial T_i} Lf(k) = \begin{cases} \sum_{j \sim i} W_{i,j}^T (f(j) - f(i)) & \text{if } k = i, \\ W_{i,k}^T (f(i) - f(k)) & \text{if } k \sim i, \ k \neq i, \\ 0 & \text{otherwise.} \end{cases}$$

Hence

$$\frac{\partial}{\partial T_i} Lf(k) = \sum_{j \sim i} W_{i,j}^T (f(j) - f(i)) (\mathbb{1}_{\{i=k\}} - \mathbb{1}_{\{j=k\}}),$$

and therefore

$$\frac{\partial}{\partial T_i} Q(T)_{k,l} = \sum_{j\sim i} W^T_{i,j}(Q(T)_{k,i} - Q(T)_{k,j})(Q(T)_{i,l} - Q(T)_{j,l})$$
$$= \sum_{j\sim i} W^T_{i,j} Q(T)_{k,\nabla_{i,j}} Q(T)_{\nabla_{i,j},l} = \sum_{j\sim i} W^T_{i,j} Q(T)_{\nabla_{i,j},k} Q(T)_{\nabla_{i,j},l}, \tag{4.3}$$

where we use the notation $f(\nabla_{i,j}) := f(j) - f(i)$ in the second equality, and the fact that $Q(T)$ is symmetric in the third one.

In particular, for all $l \in V$ and $t \geq 0$,

$$\frac{d}{dt} Q(T(t))_{l,l} = \frac{\partial}{\partial T_{X_t}} Q(T(t))_{l,l} = \sum_{j\sim X_t} W_{X_t,j}\big(Q(T(t))_{\nabla_{X_t,j},l}\big)^2. \tag{4.4}$$

Now recall that $Q(T(t))_{l,l}$ is non-positive for all $t \geq 0$; therefore it must converge, and

$$\int_0^\infty \sum_{j\sim X_t} W_{X_t,j}\big(Q(T(t))_{\nabla_{X_t,j},l}\big)^2\, dt = (Q(T(\infty)) - Q(0))_{l,l} < \infty.$$

The convergence of $Q(T(t))_{k,l}$ now follows from the Cauchy–Schwarz inequality, using (4.3): for all $t \geq s$,

$$|(Q(T(t)) - Q(T(s)))_{k,l}| = \int_s^t \sum_{j\sim X_u} W^T_{X_u,j} Q(T(u))_{\nabla_{X_u,j},k} Q(T(u))_{\nabla_{X_u,j},l}\, du$$
$$\leq \sqrt{(Q(T(t)) - Q(T(s)))_{k,k}}\sqrt{(Q(T(t)) - Q(T(s)))_{l,l}};$$

thus $Q(T(t))_{k,l}$ is a Cauchy sequence, which converges as t goes to infinity. Now, using again the Cauchy–Schwarz inequality, we get

$$\int_0^\infty \left|\frac{\partial}{\partial T_{X_u}} Q(T(u))_{X_u,l}\right| du = \int_0^\infty \left|\sum_{j\sim X_u} W^T_{X_u,j} Q(T(u))_{\nabla_{X_u,j},X_u} Q(T(u))_{\nabla_{X_u,j},l}\right| du$$
$$\leq \sqrt{\sum_{k\in V}(Q(T(\infty)) - Q(T(0)))_{k,k}}\sqrt{(Q(T(\infty)) - Q(T(0)))_{l,l}},$$

which enables us to conclude the proof. □

Next, we show that $(M_l(t))_{t\geq 0}$ converges, which will complete the proof of Proposition 1: indeed, this implies that the size of the jumps in that martingale goes to 0 a.s., and therefore, by (4.2), $Q(T(t))_{X_t,l}$ must converge as well, again by (4.2).

Let us compute the quadratic variation of the martingale $(M_l(t))_{t\geq 0}$ at time t:

$$\frac{\partial}{\partial t}\langle M, M\rangle_t = \left(\frac{d}{d\varepsilon}\mathbb{E}\big((M_l(t+\varepsilon) - M_l(t))^2 \mid \mathcal{F}_t\big)\right)_{\varepsilon=0}$$
$$= \left(\frac{d}{d\varepsilon}\mathbb{E}\big((Q(T(t+\varepsilon))_{X_{t+\varepsilon},l} - Q(T(t))_{X_t,l})^2 \mid \mathcal{F}_t\big)\right)_{\varepsilon=0}$$
$$= R(T(t))_{X_t,l}$$

where, for all $(i, l, T) \in V \times V \times \mathbb{R}^V$, we let

$$R(T)_{i,l} := \tilde{L}(Q^2(\cdot)_{\cdot,l})(i, T) - 2Q(T)_{i,l}\tilde{L}(Q(\cdot)_{\cdot,l})(i, T);$$

here $Q^2(T)$ denotes the matrix with entries $(Q(T)_{i,j})^2$, rather than $Q(T)$ composed with itself. But

$$\tilde{L}(Q^2(\cdot)_{\cdot,l})(i, T) = 2(Q(T))_{i,l}\left(\frac{\partial}{\partial T_i}Q(T)\right)_{i,l} + (L(T)Q^2(T)_{\cdot,l}(i))_{i,l}$$
$$Q(T)_{i,l}\tilde{L}(Q(\cdot)_{\cdot,l})(i, T) = (Q(T))_{i,l}\left(\frac{\partial}{\partial T_i}Q(T)\right)_{i,l} + Q(T)_{i,l}(L(T)Q(T)_{\cdot,l}(i))_{i,l},$$

so that

$$\begin{aligned} R(T)_{i,l} &= L(T)(Q^2(T)_{\cdot,l})_{i,l} - 2Q(T)_{i,l}(L(T)Q(T)_{\cdot,l})_{i,l} \\ &= \sum_{j\sim i} W^T_{i,j}\big((Q(T)_{j,l})^2 - (Q(T)_{i,l})^2\big) - 2Q(T)_{i,l}\sum_{j\sim i} W^T_{i,j}(Q(T)_{j,l} - Q(T)_{i,l}) \\ &= \sum_{j\sim i} W^T_{i,j}\big(Q(T)_{\nabla_{i,j},l}\big)^2 = \frac{\partial}{\partial T_i}Q(T)_{l,l}, \end{aligned}$$

where we have used (4.3) in the last equality. Thus

$$\langle M_l, M_l\rangle_\infty = \int_0^\infty \frac{d}{du}Q(T(u))_{l,l}\,du = Q(T(\infty))_{l,l} - Q(0)_{l,l} \le -Q(0)_{l,l} < \infty.$$

Therefore $(M_l(t))_{t\ge 0}$ is a martingale bounded in L^2, which converges a.s.

Remark 4. Once we know that $T_i(t) - t/N$ converges, then $T_i(\infty) = \infty$ for all $i \in V$, hence $Q(T(\infty))_{l,l} = 0$, and the last inequality is in fact an equality, i.e. $\langle M_l, M_l\rangle_\infty = -Q(0)_{l,l}$.

4.2. *Proof of Theorem 2(i)*

For $i_0 \in V$, $T \in \mathbb{R}^V$, $\lambda \in \mathcal{H}_0$, we consider

$$\Psi(i_0, T, \lambda) = \int_{\mathcal{H}_0} e^{u_{i_0}} e^{i\langle\lambda,u\rangle}\phi(W^T, u)\,du, \tag{4.5}$$

where

$$\phi(W^T, u) = e^{-H(W^T,u)}\sqrt{D(W^T, u)}, \tag{4.6}$$

and $W^T_{i,j} = W_{i,j}e^{T_i+T_j}$. We will prove that

$$\frac{N}{(2\pi)^{(N-1)/2}}\Psi(i_0, T, \lambda) = \mathbb{E}_{i_0,T}(e^{i\langle\lambda,U\rangle})$$

for all $i_0 \in V$, $T \in \mathbb{R}^V$.

Lemma 4. *The function* Ψ *is a solution of the Feynman–Kac equation*

$$i\lambda_{i_0}\Psi(i_0,T,\lambda)+(\tilde{L}\Psi)(i_0,T,\lambda)=0.$$

Proof. Let $\overline{T}_i=T_i-\frac{1}{N}\sum_{j\in V}T_j$. With the change of variables $\tilde{u}_i=u_i+\overline{T}_i$, we obtain

$$\Psi(i_0,T,\lambda)=\int_{\mathcal{H}_0}e^{\tilde{u}_{i_0}-\overline{T}_{i_0}}e^{i\langle\lambda,\tilde{u}-\overline{T}\rangle}\phi(W^T,\tilde{u}-\overline{T})\,d\tilde{u}. \tag{4.7}$$

Note that $H(W^T,\tilde{u}-\overline{T})=H(W^T,\tilde{u}-T)$ since $H(W^T,u)$ only depends on the differences u_i-u_j. We observe that the entries of the matrix $M(W^T,u)$ only contain terms of the form $W_{i,j}e^{u_i+T_i+u_j+T_j}$, hence

$$\sqrt{D(W^T,\tilde{u}-\overline{T})}=e^{(N-1)/N\sum_j T_j}\sqrt{D(W,\tilde{u})}.$$

Finally, $\langle\lambda,\overline{T}\rangle=\langle\lambda,T\rangle$ since $\lambda\in\mathcal{H}_0$. This implies that

$$\Psi(i_0,T,\lambda)=\int_{\mathcal{H}_0}e^{\sum_j T_j}e^{\tilde{u}_{i_0}-T_{i_0}}e^{i\langle\lambda,\tilde{u}-T\rangle}e^{-H(W^T,\tilde{u}-T)}\sqrt{D(W,\tilde{u})}\,d\tilde{u}. \tag{4.8}$$

We have

$$\begin{aligned}\frac{\partial}{\partial T_{i_0}}H(W^T,\tilde{u}-T)&=\frac{\partial}{\partial T_{i_0}}\Big(2\sum_{\{i,j\}\in E}W_{i,j}e^{T_i+T_j}\sinh^2\big(\tfrac{1}{2}(\tilde{u}_i-\tilde{u}_j-T_i+T_j)\big)\Big)\\&=2\sum_{j\sim i_0}W_{i_0,j}e^{T_{i_0}+T_j}\big(\sinh^2\big(\tfrac{1}{2}(\tilde{u}_{i_0}-\tilde{u}_j-T_{i_0}+T_j)\big)-\tfrac{1}{2}\sinh(\tilde{u}_{i_0}-\tilde{u}_j-T_{i_0}+T_j)\big)\\&=\sum_{j\sim i_0}W_{i_0,j}e^{T_{i_0}+T_j}(e^{-\tilde{u}_{i_0}+\tilde{u}_j+T_{i_0}-T_j}-1)=e^{-(\tilde{u}_{i_0}-T_{i_0})}L(T)(e^{\tilde{u}-T})(i_0).\end{aligned}$$

Hence,

$$\begin{aligned}&-\frac{\partial}{\partial T_{i_0}}\Psi(i_0,T,\lambda)\\&\quad=\int_{\mathcal{H}_0}\big(i\lambda_{i_0}e^{\tilde{u}_{i_0}-T_{i_0}}+L(T)(e^{\tilde{u}-T})(i_0)\big)e^{\sum_j T_j}e^{i\langle\lambda,\tilde{u}-T\rangle}e^{-H(W^T,\tilde{u}-T)}\sqrt{D(W,\tilde{u})}\,d\tilde{u}\\&\quad=i\lambda_{i_0}\Psi(i_0,T,\lambda)+(L(T)\Psi)(i_0,T,\lambda).\end{aligned}$$

This gives the conclusion. □

Since Ψ is a solution of the Feynman–Kac equation we deduce that for all $t>0$, $i_0\in V$, $\lambda\in\mathcal{H}_0$, $T\in\mathbb{R}^V$,

$$\Psi(i_0,T,\lambda)=\mathbb{E}_{i_0,T}(e^{i\langle\lambda,\overline{T}(t)\rangle}\Psi(X_t,T(t),\lambda)),$$

where we recall that $\overline{T}_i(t)=T_i(t)-t/N$. Let us now prove that $\Psi(X_t,T(t),\lambda)$ is dominated and that $\mathbb{P}_{i_0}$-a.s.,

$$\lim_{t\to\infty}\Psi(X_t,T(t),\lambda)=(2\pi)^{(N-1)/2}/N. \tag{4.9}$$

We will need several times the computation of the following Gaussian integral.

Lemma 5.

$$\int_{\mathcal{H}_0} e^{-\frac{1}{2}\sum_{\{i,j\}\in V} W_{i,j}(u_i-u_j)^2} du = \frac{(2\pi)^{(N-1)/2}}{N\sqrt{D(W,0)}}.$$

Proof. Indeed, change variables to $t_i = u_i - u_{i_0}$. The Jacobian is

$$\det(\mathrm{Id}_{N-1} + \mathbf{1}_{N-1}) = N. \tag{4.10}$$

where $\mathbf{1}_{N-1}$ is the matrix with all entries 1, and the integral becomes (with $t_{i_0} = 0$)

$$\int_{\mathbb{R}^{V\setminus\{i_0\}}} e^{-\frac{1}{2}\sum_{\{i,j\}\in V} W_{i,j}(t_i-t_j)^2} \left(\frac{\prod_{i\neq i_0} dt_i}{N}\right) = \frac{(2\pi)^{(N-1)/2}}{N\sqrt{D(W,0)}}.$$ □

By the matrix-tree theorem, denoting by $\mathcal{T}$ the set of spanning trees of G, and using again the notation ϕ of (4.6), we have

$$\begin{aligned} e^{u_{i_0}}\phi(W^T,u) &= e^{u_{i_0}} e^{-H(W^T,u)} \sqrt{\sum_{\Lambda\in\mathcal{T}} \prod_{\{i,j\}\in\Lambda} W^T_{i,j} e^{u_i+u_j}} \\ &\leq e^{N\max_{i\in V}|u_i|} e^{-\frac{1}{2}\sum_{\{i,j\}\in V} W^T_{i,j}(u_i-u_j)^2} \sqrt{D(W^T,0)} \\ &\leq \Big(\sum_{i\in V} e^{Nu_i} + e^{-Nu_i}\Big) e^{-\frac{1}{2}\sum_{\{i,j\}\in V} W^T_{i,j}(u_i-u_j)^2} \sqrt{D(W^T,0)}. \end{aligned} \tag{4.11}$$

This is a Gaussian integrand: for any real a and $j_0 \in V$,

$$\begin{aligned} &\int_{\mathcal{H}_0} e^{au_{j_0}} e^{-\frac{1}{2}\sum_{\{i,j\}\in V} W^T_{i,j}(u_i-u_j)^2} \sqrt{D(W^T,0)}\, du \\ &\quad = e^{-\frac{1}{2}a^2 Q(T)_{j_0,j_0}} \int_{\mathcal{H}_0} e^{-\frac{1}{2}\sum_{\{i,j\}\in V} W^T_{i,j}((u_i-aQ(T)_{j_0,i})-(u_j-aQ(T)_{j_0,j}))^2} \sqrt{D(W^T,0)}\, du \end{aligned}$$

where $Q(T)$ is defined at the beginning of Section 4.1. Changing variables to $\tilde{u}_i = u_i - aQ(T)_{j_0,i}$ and using Lemma 5 gives

$$\int_{\mathcal{H}_0} e^{au_{j_0}} e^{-\frac{1}{2}\sum_{\{i,j\}\in V} W^T_{i,j}(u_i-u_j)^2} \sqrt{D(W^T,0)}\, du \leq e^{-\frac{1}{2}a^2 Q(T)_{j_0,j_0}} (2\pi)^{(N-1)/2}/N.$$

Therefore for all $i_0 \in V$ and $(T_i) \in \mathbb{R}^V$,

$$|\Psi(i_0,T,\lambda)| \leq 2\sum_{i\in V} \frac{(2\pi)^{(N-1)/2}}{N} e^{-\frac{1}{2}N^2 Q(T)_{i,i}}.$$

By (4.4), $Q(T(t))_{i,i}$ increases in t, hence

$$|\Psi(X_t,T(t),\lambda)| \leq 2\sum_{i\in V} \frac{(2\pi)^{(N-1)/2}}{N} e^{-\frac{1}{2}N^2 Q(0)_{i,i}}$$

for all $t \geq 0$. Let us now prove (4.9). We have

$$\Psi(X_t, T(t), \lambda) = \int e^{i\langle\lambda,u\rangle} e^{u_{X_t}} e^{-2\sum_{\{i,j\}\in E} W_{i,j}^{T(t)} \sinh^2(\frac{1}{2}(u_i-u_j))} \sqrt{D(W^{T(t)}, u)}\, du$$

$$= \int e^{i\langle\lambda,u\rangle} e^{u_{X_t}} e^{-2\sum_{\{i,j\}\in E} e^{2t/N} W_{i,j}^{\overline{T}(t)} \sinh^2(\frac{1}{2}(u_i-u_j))} \sqrt{D(W^{\overline{T}(t)}, u) e^{(N-1)t/N}}\, du.$$

Changing variables to $\tilde{u}_i = e^{t/N} u_i$, we deduce that $\Psi(X_t, T(t), \lambda)$ equals

$$\int e^{i\langle\lambda, e^{-t/N}\tilde{u}\rangle} e^{e^{-t/N}\tilde{u}_{X_t}} e^{-2\sum_{\{i,j\}\in E} W_{i,j}^{\overline{T}(t)} e^{2t/N} \sinh^2(\frac{1}{2}e^{-t/N}(\tilde{u}_i-\tilde{u}_j))} \sqrt{D(W^{\overline{T}(t)}, e^{-t/N}\tilde{u})}\, d\tilde{u}.$$

Since $\lim_{t\to\infty} \overline{T}_i(t) = U_i$, the integrand converges pointwise to the Gaussian integrand

$$e^{-\frac{1}{2}\sum_{\{i,j\}\in V} W_{i,j}^{U}(\tilde{u}_i-\tilde{u}_j)^2} \sqrt{D(W^U, 0)}.$$

By Lemma 5, its integral on $\mathcal{H}_0$ is $(2\pi)^{(N-1)/2}/N$.

Consider $\overline{U}_i = \sup_{t\geq 0} \overline{T}_i(t)$ and $\underline{U}_i = \inf_{t\geq 0} \overline{T}_i(t)$. Proceeding as in (4.11) we can dominate the integrand for all t by

$$e^{Ne^{-t/N} \max_{i\in V} |\tilde{u}_i|} e^{-\frac{1}{2}\sum_{\{i,j\}\in V} W_{i,j}^{\overline{T}(t)}(\tilde{u}_i-\tilde{u}_j)^2} \sqrt{D(W^{\overline{T}(t)}, 0)}$$

$$\leq \Big(\sum_{i\in V} e^{N\tilde{u}_i} + e^{-N\tilde{u}_i}\Big) e^{-\frac{1}{2}\sum_{\{i,j\}\in V} W_{i,j}^{\underline{U}}(\tilde{u}_i-\tilde{u}_j)^2} \sqrt{D(W^{\overline{U}}, 0)},$$

which is integrable, yielding (4.9) by dominated convergence.

4.3. *Proof of Theorem 2(ii)*

The same change of variables as in (4.8), applied to $T_i = \log \lambda_i$, implies that, for any $j_0 \in V$ and $(\lambda_i)_{i\in V}$ positive reals,

$$N \frac{\prod_{i\in V} \lambda_i}{(2\pi)^{(N-1)/2}} e^{u_{j_0} - \log(\lambda_{j_0})} e^{-\frac{1}{2}\sum_{\{i,j\}\in E} W_{i,j}\lambda_i\lambda_j (e^{\frac{1}{2}(u_j-u_i)}\sqrt{\lambda_i/\lambda_j} - e^{\frac{1}{2}(u_j-u_i)}\sqrt{\lambda_j/\lambda_i})^2} \sqrt{D(W, u)}$$

is the density of a probability measure, called ν^{λ, j_0} (we use the fact that (3.1) defines a probability measure). Note that this density can be rewritten as

$$N \frac{\prod_{i\in V} \lambda_i}{(2\pi)^{(N-1)/2}} e^{u_{j_0} - \log(\lambda_{j_0})} e^{-\frac{1}{2}\sum_i \sum_{j\sim i} W_{i,j}(\lambda_i^2 e^{u_j-u_i} - \lambda_i\lambda_j)} \sqrt{D(W, u)}.$$

Let (U_i) be a random variable distributed according to (3.1), and, conditionally on U, let (Z_t) be the Markov jump process starting at i_0, and with jump rates from i to j equal to $\frac{1}{2}W_{i,j}e^{U_j-U_i}$. Let $(\mathcal{F}_t^Z)_{t\geq 0}$ be the filtration generated by Z, and let E_i^U be the law of the process Z starting at i, conditionally on U.

We denote by $(l_i(t))_{i\in V}$ the vector of local times of the process Z at time t, and consider the positive continuous additive functional of Z given by

$$B(t) = \int_0^t \frac{1}{2}\frac{1}{\sqrt{1+l_{Z_u}(u)}}\,du = \sum_{i\in V}\left(\sqrt{1+l_i(t)}-1\right),$$

and the time changed process

$$\tilde{Y}_s = Z_{B^{-1}(s)}.$$

Let us first prove that the law of U conditioned on $\mathcal{F}_t^Z$ is

$$\mathcal{L}(U \mid \mathcal{F}_t^Z) = \nu^{\lambda(t),Z_t}, \tag{4.12}$$

where $\lambda_i(t) = \sqrt{1+l_i(t)}$. Indeed, let $t>0$; if $\tau_1,\dots,\tau_{K(t)}$ denote the jumping times of the Markov process Z_t up to time t, then for any positive test function ψ,

$$\begin{aligned}&E_{i_0}^U\left(\psi(\tau_1,\dots,\tau_{K(t)},Z_{\tau_1},\dots,Z_{\tau_{K(t)}})\right)\\ &=\sum_{k=0}^{\infty}\sum_{i_1,\dots,i_k}\left(\prod_{l=0}^{k-1}\frac{1}{2}W_{i_l,i_{l+1}}\right)\int_{[0,t]^k}\psi((t_j),(i_j))e^{U_{i_k}-U_{i_0}}e^{-\frac{1}{2}\sum_{l=0}^{k-1}(\sum_{j\sim i_l}W_{i_l,j}e^{U_j-U_{i_l}})(t_{l+1}-t_l)}\\ &\qquad dt_1\cdots dt_k\end{aligned}$$

with the convention $t_{k+1}=t$. Hence, for any test function G,

$$\mathbb{E}(G(U)\mid\mathcal{F}_t^Z) = \frac{\int_{\mathcal{H}_0}G(u)e^{u_{Z_t}}e^{-H(W,u)-\frac{1}{2}\sum_{i\in V}(\sum_{j\sim i}W_{i,j}e^{u_j-u_i})l_i(t)}\sqrt{D(W,u)}\,du}{\int_{\mathcal{H}_0}e^{u_{Z_t}}e^{-H(W,u)-\frac{1}{2}\sum_{i\in V}(\sum_{j\sim i}W_{i,j}e^{u_j-u_i})l_i(t)}\sqrt{D(W,u)}\,du}.$$

Using the fact that we can write $H(W,u)=\frac{1}{2}\sum_{i\in V}\sum_{j\sim i}W_{i,j}(e^{u_j-u_i}-1)$, and introducing suitable constants in the numerator and denominator we have

$$\begin{aligned}&\mathbb{E}(G(U)\mid\mathcal{F}_t^Z) =\\ &\frac{(2\pi)^{-(N-1)/2}\int_{\mathcal{H}_0}G(u)(\prod\lambda_i)e^{u_{Z_t}-\log\lambda_{Z_t}}e^{-\frac{1}{2}\sum_i\sum_{j\sim i}W_{i,j}(\lambda_i(t)^2e^{u_j-u_i}-\lambda_i(t)\lambda_j(t))}\sqrt{D(W,u)}\,du}{(2\pi)^{-(N-1)/2}\int_{\mathcal{H}_0}(\prod\lambda_i)e^{u_{Z_t}-\log\lambda_{Z_t}}e^{-\frac{1}{2}\sum_i\sum_{j\sim i}W_{i,j}(\lambda_i(t)^2e^{u_j-u_i}-\lambda_i(t)\lambda_j(t))}\sqrt{D(W,u)}\,du}\end{aligned}$$

(recall that $\lambda_i(t)=\sqrt{1+l_i(t)}$). The denominator is 1 since it is the integral of the density of $\nu^{\lambda(t),Z_t}$. This proves (4.12).

Subsequently, by (4.12), conditioned on $(\mathcal{F}_t^Z)$, if the process Z is at i at time t, then it jumps to a neighbour j of i with rate

$$\frac{1}{2}W_{i,j}\mathbb{E}^{\nu^{\lambda(t),i}}(e^{U_j-U_i}) = \frac{1}{2}W_{i,j}\frac{\lambda_j(t)}{\lambda_i(t)}.$$

In order to conclude, we now compute the corresponding rate for $\tilde{Y}$: by definition,

$$B'(t) = \frac{1}{2}\frac{1}{\sqrt{1+l_{Z_t}(t)}}.$$

Therefore, similarly to the proof of Lemma 1,

$$\begin{aligned}\mathbb{P}(\tilde{Y}_{s+ds} = j \mid \mathcal{F}_s^Z) &= \mathbb{P}(Z_{B^{-1}(s+ds)} = j \mid \mathcal{F}_s^Z) \\ &= \frac{1}{2} W_{Y_s,j} \frac{1}{B'(B^{-1}(s))} \frac{\lambda_j(B^{-1}(s))}{\lambda_{Y_s}(B^{-1}(s))} ds \\ &= W_{Y_s,j} \lambda_j(B^{-1}(s))\, ds.\end{aligned}$$

Let $(\tilde{l}_i(s))$ be the local time of the process $\tilde{Y}$. Then

$$(\tilde{l}_i(B(t)))' = B'(t) \mathbb{1}_{\{\tilde{Y}_{B(s)=i}\}} = \tfrac{1}{2}(1 + l_i(t))^{-1/2} \mathbb{1}_{\{Z_t=i\}}.$$

This implies

$$\tilde{l}_i(B(t)) = \sqrt{1 + l_i(t)} - 1 \tag{4.13}$$

and

$$\mathbb{P}(\tilde{Y}_{s+ds} = j \mid \mathcal{F}_s^Z) = W_{\tilde{Y}_s,j}(1 + \tilde{l}_j(s))\, ds.$$

This means that the annealed law of $\tilde{Y}$ is the law of a VRJP with conductances $(W_{i,j})$ (this is the content of Remark 3).

Therefore, the process defined, for all $t \geq 0$, by $\tilde{Y}_{A^{-1}(t)} = Z_{(A\circ B)^{-1}(t)}$ is equal in law to $(X_t)_{t\geq 0}$; let us denote by T its local time, and show that $T_i(t) - t/N$ converges to U_i as $t \to \infty$, which will complete the proof.

First note, using (2.1) and (4.13), that, for all $i \in V$,

$$T_i((A \circ B)(t)) = \log(\tilde{l}_i(B(t)) + 1) = \log(1 + l_i(t))/2.$$

On the other hand, conditionally on U, the Markov chain Z has invariant measure $(Ce^{2U_i})_{i\in V}$, $C := (\sum_{i\in V} e^{2U_i})^{-1}$, so that $l_i(t)/(Ce^{2U_i} t)$ converges to 1 as $t \to \infty$, for all $i \in V$.

Therefore, for all $i \in V$,

$$T_i(t) - T_{i_0}(t) = \frac{1}{2} \log\left(\frac{1 + l_i((A \circ B)^{-1}(t))}{1 + l_{i_0}((A \circ B)^{-1}(t))}\right),$$

which converges to $U_i - U_{i_0}$ as $t \to \infty$, enabling us to conclude the proof.

5. Back to Diaconis–Coppersmith formula

It follows from de Finetti's theorem for Markov chains [13, 27] that the law of the ERRW is a mixture of reversible Markov chains; its mixing measure was explicitly described by Coppersmith and Diaconis ([8], see also [17]).

Theorems 1 and 2 enable us to retrieve this so-called Coppersmith–Diaconis formula, including its de Finetti part: they imply that the ERRW $(X_n)_{n\in\mathbb{N}}$ follows the annealed law of a reversible Markov chain in a random conductance network $x_{i,j} = W_{i,j} e^{U_i+U_j}$ where

$W_e \sim \mathrm{Gamma}(a_e, 1)$, $e \in E$, are independent random variables and, conditioned on W, the random variables (U_i) are distributed according to the law (3.1).

Let us compute the law it induces on the random variables (x_e). The random variable (x_e) is only significant up to a scaling factor, hence we consider a 0-homogeneous bounded measurable test function ϕ; by Theorem 2,

$$\mathbb{E}(\phi((x_e))) = \frac{N}{(2\pi)^{(N-1)/2}} \int_{\mathbb{R}_+^E \times \mathcal{H}_0} \phi(x) \Big(\prod_{e\in E} \frac{1}{\Gamma(a_e)} W_e^{a_e} e^{-W_e}\Big) e^{u_{i_0}} \sqrt{D(W,u)} e^{-H(W,u)} \frac{dW}{W}\, du$$

where we write $\frac{dW}{W} = \prod_{e\in E} \frac{dW_e}{W_e}$. Changing coordinates to $\overline{u}_i = u_i - u_{i_0}$, the Jacobian being N (cf. (4.10)), we get

$$C(a) \int_{\mathbb{R}_+^E \times \mathbb{R}^{V\setminus\{i_0\}}} \phi(x) \Big(\prod_{e\in E} W_e^{a_e} e^{-W_e}\Big) e^{-\sum_{i\neq i_0} \overline{u}_i} \sqrt{D(W,\overline{u})} e^{-H(W,\overline{u})} \frac{dW}{W}\, d\overline{u}$$

with $d\overline{u} = \prod_{i\neq i_0} d\overline{u}_i$ and $C(a) = \frac{1}{(2\pi)^{(N-1)/2}} \prod_{e\in E} \frac{1}{\Gamma(a_e)}$. But

$$-\sum_{e\in E} W_e - H(W,\overline{u}) = -\frac{1}{2} \sum_{\{i,j\}\in E} W_{i,j} e^{\overline{u}_i+\overline{u}_j} (e^{-2\overline{u}_j} + e^{-2\overline{u}_i}).$$

The change of variables

$$\big((x_{i,j} = W_{i,j} e^{\overline{u}_i+\overline{u}_j})_{\{i,j\}\in E}, (v_i = e^{-2\overline{u}_i})_{i\in V\setminus\{i_0\}}\big)$$

with $v_{i_0} = 1$ implies

$$-\sum_{e\in E} W_e - H(W,\overline{u}) = -\frac{1}{2} \sum_{i\in V} v_i x_i,$$

where $x_i = \sum_{j\sim i} x_{i,j}$, and $\mathbb{E}(\phi((x_e)))$ is equal to the integral

$$C'(a) \int \phi(x) \Big(\prod_{e\in E} x_e^{a_e}\Big) \Big(\prod_{i\in V} v_i^{(a_i+1)/2}\Big) v_{i_0}^{-1/2} \sqrt{D(x)} e^{-\frac{1}{2}\sum_{i\in V} v_i x_i} \Big(\prod_{e\in E} \frac{dx_e}{x_e}\Big) \Big(\prod_{i\neq i_0} \frac{dv_i}{v_i}\Big)$$

with $a_i = \sum_{j\sim i} a_{i,j}$, $D(x)$ the determinant of any diagonal minor of the $N \times N$ matrix

$$m_{i,j} = \begin{cases} -x_{i,j} & \text{if } i \neq j, \\ \sum_{k\sim i} x_{i,k} & \text{if } i = j, \end{cases}$$

and

$$C'(a) = \frac{2^{-N+1}}{(2\pi)^{(N-1)/2}} \prod_{e\in E} \frac{1}{\Gamma(a_e)}.$$

Let e_0 be a fixed edge; we normalize the conductance to be 1 at e_0 by changing variables to

$$\left(\left(y_e = \frac{x_e}{x_{e_0}}\right)_{e\neq e_0}, (z_i = x_{e_0} v_i)_{i\in V}\right),$$

with $y_{e_0} = 1$. Now, observe that

$$\left(\prod_{e\in E} \frac{dx_e}{x_e}\right)\left(\prod_{i\neq i_0} \frac{dv_i}{v_i}\right) = \left(\prod_{e\in E,\, e\neq e_0} \frac{dy_e}{y_e}\right)\left(\prod_{i\in V} \frac{dz_i}{z_i}\right).$$

We deduce that $\mathbb{E}(\phi((x_e)))$ equals the integral

$$C(a)\int_{\mathbb{R}_+^V\times\mathbb{R}_+^{E\setminus\{e_0\}}} \phi(y)\left(\prod_{e\in E} y_e^{a_e}\right)\left(\prod_{i\in V} z_i^{a_i/2}\right) z_{i_0}^{-1/2}\sqrt{D(y)}e^{-\frac{1}{2}\sum_{i\in V} z_i y_i}\frac{dy}{y}\frac{dz}{z}$$

with $\frac{dy}{y} = \prod_{e\neq e_0}\frac{dy_e}{y_e}$ and $\frac{dz}{z} = \prod_{i\in V}\frac{dz_i}{z_i}$. Therefore, integrating over the variables z_i yields

$$\mathbb{E}(\phi((x_e))) = C''(a)\int_{\mathbb{R}_+^{E\setminus\{e_0\}}} \phi(y) y_{i_0}^{1/2}\left(\frac{\prod_{e\in E} y_e^{a_e}}{\prod_{i\in V} y_i^{(a_i+1)/2}}\right)\sqrt{D(y)}\frac{dy}{y},$$

where

$$C''(a) = \frac{2^{1-N-\sum_{e\in E} a_e}}{\pi^{(N-1)/2}}\frac{\Gamma(a_{i_0}/2)\prod_{i\neq i_0}\Gamma((a_i+1)/2)}{\prod_{e\in E}\Gamma(a_e)},$$

which is the Diaconis–Coppersmith formula: the extra term $(|E|-1)!$ in [17, 14] arises from the normalization of $(x_e)_{e\in E}$ on the simplex $\Delta = \{\sum x_e = 1\}$ (see [14, Section 2.2]).

6. The supersymmetric hyperbolic sigma model

We first relate VRJP to the supersymmetric hyperbolic sigma model studied by Disertori, Spencer and Zirnbauer [16, 15].

Let us start by a description of the measures defined in [16, 15]. Again let $G = (V, E, \sim)$ be a graph. Let $\beta_{i,j}$, $i, j \in V$, $i \sim j$, be some positive weights on the edges, and $\varepsilon = (\varepsilon_i)_{i\in V}$ be a vector of non-negative reals, $\varepsilon \neq 0$. Let $\mu_V^{\varepsilon,\beta}$ be a generalization of the measure studied in [15, (1.1)–(1.7)], namely

$$\begin{aligned} d\mu_V^{\varepsilon,\beta}(t) &:= \left(\prod_{j\in V}\frac{dt_j}{\sqrt{2\pi}}\right)e^{-\sum_{j\in V} t_j}e^{-F_V^\beta(\nabla t)}e^{-M_V^\varepsilon(t)}\sqrt{\det A_V^{\varepsilon,\beta}} \\ &= \left(\prod_{j\in V}\frac{dt_j}{\sqrt{2\pi}}\right)e^{-F_V^\beta(\nabla t)}e^{-M_V^\varepsilon(t)}\sqrt{\det D_V^{\varepsilon,\beta}} \end{aligned}$$

where $A_V^{\varepsilon,\beta} = A^{\varepsilon,\beta}$ and $D_V^{\varepsilon,\beta} = D^{\varepsilon,\beta}$ are defined, for all $i, j \in V$, by

$$A_{ij}^{\varepsilon,\beta} = e^{t_i} D_{ij}^{\varepsilon,\beta} e^{t_j} = \begin{cases} 0, & i \nsim j \text{ and } i \neq j, \\ -\beta_{ij} e^{t_i+t_j}, & i \sim j, \\ \sum_{l \sim i,\, l \in V} \beta_{il} e^{t_i+t_l} + \varepsilon_i e^{t_i}, & i = j, \end{cases}$$

$$F_V^{\beta}(\nabla t) := \sum_{\{i,j\} \in E} \beta_{ij} (\cosh(t_i - t_j) - 1)$$

$$M_V^{\varepsilon}(t) := \sum_{i \in V} \varepsilon_i (\cosh t_i - 1).$$

The fact that $\mu_V^{\varepsilon,\beta}$ is a probability measure can be seen as a consequence of supersymmetry (see [16, (5.1)]). This is also a consequence of Theorem 2(i), as we explain next.

The measure $\mu_V^{\varepsilon,\beta}$ is directly related to the measure (3.1) defined in Theorem 2 as follows. Let us add an extra point δ to V, $\tilde{V} = V \cup \{\delta\}$, and extra edges $\{i, \delta\}$ connecting any site $i \in V$ such that $\varepsilon_i > 0$ to δ, i.e.

$$\tilde{E}_V = E_V \cup \bigcup_{i \in V,\, \varepsilon_i > 0} \{i, \delta\}.$$

Consider the VRJP on this new graph with vertices $\tilde{V}$, starting at δ and with conductances $W_{i,j} = \beta_{i,j}$ if $i \sim j$ in V, and $W_{i,\delta} = \varepsilon_i$ if $\varepsilon_i > 0$.

Let us again write $(U_i)_{i \in \tilde{V}}$ for the limiting centred occupation times of VRJP on $\tilde{V}$ starting at δ, and consider the change of variables, from $\mathcal{H}_0$ into $\mathbb{R}^V$, which maps u_i to $t_i := u_i - u_\delta$ (the Jacobian is $|V| + 1$, cf. (4.10)). Then, by Theorem 2, for any test function ϕ, letting ι be the canonical injection $\mathbb{R}^V \to \mathbb{R}^{\tilde{V}}$, we have

$$\begin{aligned} \mathbb{E}_\delta^W(\phi(U - U_\delta)) &= \frac{|V| + 1}{(2\pi)^{|V|/2}} \int_{\mathcal{H}_0} \phi(u - u_\delta) e^{u_\delta} e^{-H(W,u)} \sqrt{D(W,u)}\, du \\ &= \frac{1}{(2\pi)^{|V|/2}} \int_{\mathbb{R}^V} \phi(t) e^{-\sum_{i \in V} t_i} e^{-H(W,\iota(t))} \sqrt{D(W,\iota(t))} \Big(\prod_{i \neq \delta} dt_i\Big) \\ &= \mu_V^{\varepsilon,\beta}(\phi(t)), \end{aligned}$$

which means that $U - U_\delta$ is distributed according to $\mu_V^{\varepsilon,\beta}$. Indeed, A_V^{ε} is the restriction to $V \times V$ of the matrix $M(W, \iota(t))$ (which is defined on $\tilde{V} \times \tilde{V}$) (so that $\det A_V^{\varepsilon} = D(W, \iota(t))$), and $F_V(\nabla t) + M_V^{\varepsilon}(t) = H(W, \iota(t))$.

We will be interested in the VRJP on finite subsets of $G = (V, E, \sim)$ starting at vertex i_0. For all $x, y \in G$, let $d(x, y)$ be the canonical distance between x and y on G, i.e. the minimal number of edges linking x to y. In order to directly apply results from [15], we consider the VRJP on G with an extra point δ uniquely connected to i_0 and with $W_{i_0,\delta} = \varepsilon_{i_0} = 1$ and $W_{i,j} = \beta_{i,j}$ if $i \sim j$ in G.

Clearly, the trace on G of the VRJP starting from δ has the law of the VRJP on G starting from i_0. When V contains i_0, the limiting occupation time $U_i - U_\delta$ of the VRJP on $\tilde{V} = V \cup \{\delta\}$ starting at δ is distributed according to $d\mu_V^{\delta_{i_0},\beta}$, where δ_{i_0} is the Dirac measure at i_0.

For all $\beta > 0$, set

$$I_\beta := \sqrt{\beta} \int_{-\infty}^{\infty} \frac{dt}{\sqrt{2\pi}} e^{-\beta(\cosh t - 1)},$$

which is strictly increasing in β. Let β_c^r be defined as the unique solution to the equation

$$I_{\beta_c^r} e^{\beta_c^r (r-2)} (r-1) = 1$$

for all $r > 2$, and $\beta_c^r := \infty$ if $r = 1, 2$.

Theorem 2 in [15] implies that the VRJP over any graph of degree bounded by r is recurrent if $\beta_e < \beta_c^r$ for all $e \in E$ (i.e. for large reinforcement). This fact is stated in [15] on $\mathbb{Z}^d$ and with fixed β_e, but it can readily be generalised. The reader will find in Proposition 2 below a self-contained proof of a close variant of estimate (6.1) below (see in particular (6.5), Lemma 6 and (6.10)).

Theorem 3 (Disertori and Spencer [15, Theorem 2]). *Let $G = (V, E, \sim)$ be a graph of degree bounded by $r \geq 2$. Then there exists a constant $C_0 := r/(r-1) > 0$ such that, for every finite connected subset $\Lambda \subseteq V$ containing i_0 and x, if $0 < \beta_e < \beta$ for all $e \in E$, for some $\beta > 0$, then*

$$\mu_\Lambda^{\eta\delta_{i_0},\beta}(e^{t_x/2}) \leq C_0 I_\eta [I_\beta e^{\beta(r-2)}(r-1)]^{d(i_0,x)}.$$

More precisely, if Γ_x is the set of non-intersecting paths from i_0 to x in Λ, then

$$\mu_\Lambda^{\eta\delta_{i_0},\beta}(e^{t_x/2}) \leq I_\eta \sum_{\gamma\in\Gamma_x} \prod_{e\sim\gamma} e^{\beta_e} \prod_{e\in\gamma} I_{\beta_e}. \tag{6.1}$$

Corollary 1. *Let G be a graph of degree bounded by $r \geq 2$, and assume $0 < \beta_e < \beta$ for all $e \in E$, for some $\beta < \beta_c^r$. Let (Y_n) be the discrete time process associated with the VRJP on G starting from i_0 with conductances $(\beta_e)_{e\in E}$. Then (Y_n) is a mixture of reversible positive recurrent Markov chains.*

Corollary 2. *The ERRW on a graph of degree bounded by $r \geq 2$ starting at i_0 with initial weights $a_e \in (0, a)$, $e \in E$, is a mixture of positive recurrent Markov chains for $a < a_c^r$, for some $a_c^r > 0$ sufficiently small.*

Proof of Corollary 1. We prove this for the VRJP on G with an extra point δ connected to i_0 only, and conductances $W_{x,y} = \beta_{x,y}$ and $W_{i_0,\delta} = 1$, which is clearly stronger. On a finite connected subset $\Lambda \subseteq V$ containing i_0, we know from Theorem 2 that $(Y_n)_{n\in\mathbb{N}}$, the discrete-time process associated with $(Y_s)_{s\geq 0}$, is a mixture of reversible Markov chains with conductances $c_{x,y} = \beta_{x,y} e^{t_x+t_y}$, where $(t_x)_{x\in\Lambda}$ has law $\mu_\Lambda^{\delta_{i_0},\beta}$.

Now Theorem 3 implies that $\mu_\Lambda^{\delta_{i_0},\beta}((c_e/c_{\delta,i_0})^{1/4})$ decreases exponentially with the distance from e to i_0: indeed, by the Cauchy–Schwarz inequality,

$$\begin{aligned}
\mu_\Lambda^{\delta_{i_0},\beta}((c_{x,y}/c_{\delta,i_0})^{1/4}) &\leq \beta^{1/4}[\mu_\Lambda^{\delta_{i_0},\beta}(e^{t_x/2})\mu_\Lambda^{\delta_{i_0},\beta}(e^{(t_y-t_{i_0})/2})]^{1/2} \\
&\leq \beta^{1/4} C[\mu_\Lambda^{\delta_{i_0},\beta}(e^{t_x/2})\mu_\Lambda^{\delta_{i_0},\beta}(e^{\frac{1}{2}(\cosh(t_{i_0})-1)} e^{t_y/2})]^{1/2} \\
&\leq 2\beta^{1/4} C[\mu_\Lambda^{\delta_{i_0},\beta}(e^{t_x/2})\mu_\Lambda^{\delta_{i_0}/2,\beta}(e^{t_y/2})]^{1/2}
\end{aligned}$$

for some $C > 0$ such that $|z| \le 4\log C + \cosh(z) - 1$. This implies that there exist constants $c_1, c_2 > 0$ such that $\mu_\Lambda^{\delta_{i_0},\beta}((c_{x,y}/c_{\delta,i_0}) > e^{-c_1|x|}) \le e^{-c_2|x|}$. Following [23, proof of Lemma 5.1] this implies that (Y_n) is a mixture of positive recurrent Markov chains. □

Proof of Corollary 2. For any connected finite set Λ containing i_0, by Theorems 1 and 2, the ERRW on Λ starting at i_0 and with initial weights a_e, $e \in E$, is a mixture of reversible Markov chains with conductances $c_{x,y} = \beta_{x,y}e^{t_x+t_y}$, where β_e are Gamma$(a_e, 1)$ independent random variables for $e \in E$; let $\mathbb{E}$ be the expectation with respect to the variables β_e, $e \in E$. As above add an extra vertex δ and edge $\{i_0, \delta\}$, and assume $\beta_{i_0,\delta} = 1$. As in Corollary 1, there exist constants $C, C', C'' > 0$ such that, for all $\varepsilon \le 1/4$,

$$\begin{aligned}
\mathbb{E}(\mu_\Lambda^{\delta_{i_0},\beta}((c_{x,y}/c_{\delta,i_0})^\varepsilon)) &\le C[\mathbb{E}((\beta_{x,y})^{2\varepsilon}\mu_\Lambda^{\delta_{i_0},\beta}(e^{2\varepsilon t_x}))]^{1/2}[\mathbb{E}((\beta_{x,y})^{2\varepsilon}\mu_\Lambda^{\delta_{i_0}/2,\beta}(e^{2\varepsilon t_y}))]^{1/2} \\
&\le C[\mathbb{E}((\beta_{x,y})^{2\varepsilon}(\mu_\Lambda^{\delta_{i_0},\beta}(e^{t_x/2}))^{4\varepsilon})]^{1/2}[\mathbb{E}((\beta_{x,y})^{2\varepsilon}(\mu_\Lambda^{\delta_{i_0}/2,\beta}(e^{t_y/2}))^{4\varepsilon})]^{1/2} \\
&\le C'\mathbb{E}\Big[\sum_{\gamma\in\Gamma_x}\prod_{e\sim\gamma,\, e\ne\{x,y\}} e^{4\varepsilon\beta_e}\prod_{e\in\gamma,\, e\ne\{x,y\}} I_{\beta_e}^{4\varepsilon}\Big]^{1/2} \\
&\quad\times \mathbb{E}\Big[\sum_{\gamma\in\Gamma_y}\prod_{e\sim\gamma,\, e\ne\{x,y\}} e^{4\varepsilon\beta_e}\prod_{e\in\gamma,\, e\ne\{x,y\}} I_{\beta_e}^{4\varepsilon}\Big]^{1/2} \\
&\le C''[(r-1)g^{r-2}h]^{d(i_0,x)}
\end{aligned}$$

where $g = \sup_{e\in E}\mathbb{E}(e^{4\varepsilon\beta_e})$ and $h = \sup_{e\in E}\mathbb{E}(I_{\beta_e}^{4\varepsilon})$. We use Jensen's inequality in the second inequality, and (6.1) in the third inequality. Now $I_\beta \le (\log\beta^{-1})\sqrt{\beta}$ for $\beta < 0.15$ (see [15, (1.22)]), and $I_\beta < 1$ for all $\beta > 0$, so that $h \to 0$ when $a = \sup_{e\in E} a_e \to 0$. Hence, if $\varepsilon < 1/4$ and a is sufficiently small, then $(r-1)g^{r-2}h < 1$. The rest of the proof is similar to the proof of Corollary 1. □

We give in Proposition 2 another estimate of $\mu_\Lambda^{\eta\delta_{i_0},\beta}(e^{t_x/2})$ (better for large conductances than (6.1)), which enables us to deduce in Corollary 3 positive recurrence for any mixture of VRJPs where the conductances β_e, $e \in E$, are independent random variables such that $\sup\mathbb{E}(\beta_e^\varepsilon)$ is sufficiently small. Again, $\mathbb{E}$ denotes the expectation with respect to the environment of conductances. The same Corollary 3 implies for the ERRW that $\mathbb{E}\mu_\Lambda^{\eta\delta_{i_0},\beta}((c_e/c_{\delta,i_0})^{1/4})$ decreases exponentially with the distance from e to i_0.

Given $\varepsilon > 0$ and independent positive random variables β_e, $e \in E$, let

$$\hat{I}_\varepsilon = \sup_{e\in E} E(I_{\beta_e}^\varepsilon), \qquad \hat{J}_\varepsilon = \sup_{e\in E}\mathbb{E}\big((\max(\beta_e, 1)e^{\min(\beta_e,1)})^\varepsilon\big).$$

Proposition 2. *Let $G = (V, E, \sim)$ be a graph of degree bounded by $r \ge 2$. For every finite connected subset $\Lambda \subseteq V$ containing i_0 and x, if Γ_x is the set of non-intersecting paths from i_0 to x in Λ, then*

$$\mu_\Lambda^{\eta\delta_{i_0},\beta}(e^{t_x/2}) \le I_\eta \sum_{\gamma\in\Gamma_x}\Big(\prod_{e\sim\gamma}\sqrt{\max(\beta_e, 1)}\, e^{\min(\beta_e,1)}\Big)\Big(\prod_{e\in\gamma} I_{\beta_e}\Big).$$

Corollary 3. *Let $G = (V, E, \sim)$ be a graph of degree bounded by $r \geq 2$, and assume that the conductances β_e, $e \in E$, are independent random variables. Denote by $\mathbb{E}$ the expectation with respect to the random variables $(\beta_e)_{e\in E}$. Then there exists a constant $C > 0$ such that, for all $\varepsilon \leq 1/4$, all $x, y \in V$ with $x \sim y$, and every finite connected subset $\Lambda \subseteq V$ containing i_0,*

$$\mathbb{E}\big(\mu_\Lambda^{\eta\delta_{i_0},\beta}((c_{x,y}/c_{\delta,i_0})^\varepsilon)\big) \leq C[(r-1)\hat{I}_{4\varepsilon}(\hat{J}_{4\varepsilon})^{r-2}]^{d(i_0,x)}.$$

In particular, if for some $\varepsilon \leq 1/4$, $\mathbb{E}(\beta_e^\varepsilon)$ is sufficiently small, then the VRJP with random conductances $(\beta_e)_{e\in E}$ is a mixture of positive recurrent Markov chains.

Corollary 3 follows from Proposition 2, similarly to the proof of Corollary 2.

Proof of Proposition 2. The strategy is to follow the proof of [15, Theorem 2], and to truncate the random variables β_e at suitable positions. For convenience we provide a self-contained proof but the only new input compared to [15, Theorem 2] lies in the truncating argument (6.6)–(6.8) below. Let us define, for any $\Lambda \subseteq \mathbb{Z}^d$ and $\varepsilon = (\varepsilon_i)_{i\in\Lambda} \in \mathbb{R}_+^\Lambda$

$$d\nu_\Lambda^{\varepsilon,\beta}(t) := \bigg(\prod_{i\in\Lambda}\frac{dt_i}{\sqrt{2\pi}}\bigg)e^{-F_\Lambda^\beta(\nabla t)}e^{-M_\Lambda^\varepsilon(t)},$$

which is not a probability measure in general.

We now fix a finite connected subset $\Lambda \subseteq \mathbb{Z}^d$ containing i_0 and x. Let Γ_x be the set of non-intersecting paths in Λ from i_0 to x. For notational purposes, any element γ in Γ_x is defined here as the set of non-oriented edges in the path. We let Λ_γ and Λ_γ^c be respectively the set of vertices in the path and its complement. We say that an edge e is *adjacent* to the path γ if e is not in γ and has one adjacent vertex in γ, i.e. if $e = \{i, j\}$ with $i \in \Lambda_\gamma$, $j \notin \Lambda_\gamma$; we write $e \sim \gamma$.

We first proceed similarly to [15, Lemma 2, (3.1)–(3.4)]. For a subset $\Lambda \subseteq \mathbb{Z}^d$ we denote by E_Λ the set of edges with both extremities in Λ. Let $\mathcal{T}_\Lambda$ be the set of spanning trees of Λ.

By the matrix-tree theorem,

$$\det(A_\Lambda^{\eta\delta_{i_0},\beta}) = \eta e^{t_{i_0}} \sum_{T\in\mathcal{T}_\Lambda}\prod_{\{i,j\}\in T}\beta_{\{i,j\}}e^{t_i+t_j}.$$

In a spanning tree T there is a unique path between i_0 and $x \in \Lambda$. Decomposing this sum depending on this path we deduce

$$\det(A_\Lambda^{\eta\delta_{i_0},\beta}) = \eta e^{t_{i_0}} \sum_{\gamma\in\Gamma_x}\bigg(\prod_{\{i,j\}\in\gamma}\beta_{\{i,j\}}e^{t_i+t_j}\bigg)\sum_{T'\in\mathcal{T}_\Lambda^\gamma}\prod_{\{i,j\}\in T'}\beta_{\{i,j\}}e^{t_i+t_j}$$

where $\mathcal{T}_\Lambda^\gamma$ is the set of subsets $T' \subseteq E_\Lambda \setminus \gamma$ such that $\gamma \cup T'$ is a spanning tree. By the matrix-tree theorem, we have

$$\sum_{T'\in\mathcal{T}_\Lambda^\gamma}\prod_{\{i,j\}\in T'}\beta_{\{i,j\}}e^{t_i+t_j} = \det(A_{\Lambda_\gamma^c}^{\varepsilon,\beta}) \tag{6.2}$$

where $(\varepsilon_i)_{i\in\Lambda_\gamma^c}$ is the vector defined by

$$\varepsilon_i := \sum_{k\in\Lambda_\gamma, k\sim i} \beta_{\{i,k\}} e^{t_k}, \qquad \forall i \in \Lambda_\gamma^c.$$

It follows that

$$\det D_\Lambda^{\eta\delta_{i_0},\beta} = \eta e^{-t_x} \sum_{\gamma\in\Gamma_x} \Big(\prod_{e\in\gamma} \beta_e\Big) \det D_{\Lambda_\gamma^c}^{\varepsilon,\beta}. \tag{6.3}$$

Let us define, similarly to [15, (2.12), (2.14)], for $t_\gamma = t_{|\Lambda_\gamma}$ the restriction of t to the vertices on the path γ,

$$\begin{aligned} Z_{\Lambda_\gamma^c}^{\gamma,\beta}(t_\gamma) &:= \nu_{\Lambda_\gamma^c}^{\eta\delta_{i_0},\beta}\Big(\sqrt{\det D_{\Lambda_\gamma^c}^{\varepsilon,\beta}}\, e^{-F_{\partial\gamma}^{\beta}(\nabla t)}\Big) \\ F_{\partial\gamma}^{\beta}(\nabla t) &:= \sum_{k\in\Lambda_\gamma,\, j\in\Lambda_\gamma^c,\, k\sim j} \beta_{kj}(\cosh(t_j - t_k) - 1). \end{aligned} \tag{6.4}$$

Now

$$\begin{aligned} \mu_\Lambda^{\eta\delta_{i_0},\beta}(e^{t_x/2}) &= \nu_\Lambda^{\eta\delta_{i_0},\beta}\Big(\sqrt{\det D_\Lambda^{\eta\delta_{i_0},\beta} e^{t_x}}\Big) = \sqrt{\eta}\, \nu_\Lambda^{\eta\delta_{i_0},\beta}\Big(\sqrt{\sum_{\gamma\in\Gamma_x}\prod_{e\in\gamma}\beta_e \det D_{\Lambda_\gamma^c}^{\varepsilon,\beta}}\Big) \\ &\le \sqrt{\eta} \sum_{\gamma\in\Gamma_x}\Big(\prod_{e\in\gamma}\sqrt{\beta_e}\Big)\nu_{\Lambda_\gamma}^{\eta\delta_{i_0},\beta}(Z_{\Lambda_\gamma^c}^{\gamma,\beta}(t_\gamma)), \end{aligned} \tag{6.5}$$

using (6.3) in the second equality and, in the inequality, the fact that for all $\gamma \in \Gamma_x$,

$$d\nu_\Lambda^{\eta\delta_{i_0},\beta}(t) = d\nu_{\Lambda_\gamma}^{\eta\delta_{i_0},\beta}(t)\, d\nu_{\Lambda_\gamma^c}^{\eta\delta_{i_0},\beta}(t) e^{-F_{\partial\gamma}(\nabla t)}.$$

The new argument compared to Theorem 3 which allows us to handle the case of random parameters β is the following truncation. Given $\gamma \in \Gamma_x$, let $(\tilde\beta_e)$ be the set of random variables defined by

$$\tilde\beta_e = \begin{cases} \min(\beta_e, 1) & \text{if } e \sim \gamma, \\ \beta_e & \text{otherwise.} \end{cases} \tag{6.6}$$

First note that, trivially,

$$e^{-F_{\partial\gamma}^{\beta}(\nabla t)} \le e^{-F_{\partial\gamma}^{\tilde\beta}(\nabla t)}. \tag{6.7}$$

On the other hand, identity (6.2) implies that

$$\det(D_{\Lambda_\gamma^c}^{\varepsilon,\beta}) \le \det(D_{\Lambda_\gamma^c}^{\tilde\varepsilon,\tilde\beta}) \prod_{e\sim\gamma} \max(\beta_e, 1), \tag{6.8}$$

where $(\tilde\varepsilon_i)_{i\in\Lambda_\gamma^c}$ is the vector defined by

$$\tilde\varepsilon_i := \sum_{k\in\Lambda_\gamma,\, i\sim k} \tilde\beta_{\{i,k\}} e^{t_k}, \qquad \forall i \in \Lambda_\gamma^c.$$

(In the last argument we have used the fact that $\beta_{i,j} = \tilde{\beta}_{i,j}\max(1, \beta_{i,j})$ for any $\{i, j\}$ adjacent to γ.) Therefore

$$Z^{\gamma,\beta}_{\Lambda^c_\gamma}(t_\gamma) \le Z^{\gamma,\tilde{\beta}}_{\Lambda^c_\gamma}(t_\gamma)\prod_{e\sim\gamma}\sqrt{\max(\beta_e, 1)} \tag{6.9}$$

with $Z^{\gamma,\tilde{\beta}}_{\Lambda^c_\gamma}(t_\gamma)$ defined as in (6.4) with ε, β replaced by $\tilde{\varepsilon}, \tilde{\beta}$. Hence we can replace β by $\tilde{\beta}$ at the cost of the term $\prod_{e\sim\gamma}\sqrt{\max(\beta_e, 1)}$.

The following lemma, which adapts [15, Lemma 3], provides an upper bound of $Z^{\gamma,\tilde{\beta}}_{\Lambda^c_\gamma}(t_\gamma)$.

Lemma 6. *For any configuration of* $t_\gamma = t_{|\Lambda_\gamma}$, $Z^{\gamma,\tilde{\beta}}_{\Lambda^c_\gamma}(t_\gamma) \le e^{\sum_{e\sim\gamma}\tilde{\beta}_e}$.

Proof. We have

$$Z^{\gamma,\tilde{\beta}}_{\Lambda^c_\gamma}(t_\gamma) = \int\left(\prod_{j\in\Lambda^c_\gamma}\frac{dt_j}{\sqrt{2\pi}}\right)e^{-F^{\tilde{\beta}}_{\Lambda^c_\gamma}(\nabla t)-F^{\tilde{\beta}}_{\partial\gamma}(\nabla t)}\sqrt{\det(D^{\tilde{\varepsilon},\tilde{\beta}}_{\Lambda^c_\gamma})}.$$

Let $t^* = \max\{t_k : k \in \Lambda_\gamma\}$. We translate the variables, $t_j \to t_j + t^*$ for $j \in \Lambda^c_\gamma$; then in the previous integral the term $F^{\tilde{\beta}}_{\Lambda^c_\gamma}(\nabla t)$ does not change, the term $F^{\tilde{\beta}}_{\partial\gamma}(\nabla t)$ becomes

$$\sum_{k\in\Lambda_\gamma,\, j\in\Lambda^c_\gamma,\, k\sim j}\tilde{\beta}_{kj}(\cosh(t_j + t^* - t_k) - 1),$$

and the term $\det(D^{\tilde{\varepsilon},\tilde{\beta}}_{\Lambda^c_\gamma})$ is replaced by $\det(D^{e^{-t^*}\tilde{\varepsilon},\tilde{\beta}}_{\Lambda^c_\gamma})$. Since $t^* - t_k \ge 0$, we have

$$\cosh(t_j + t^* - t_k) - 1 \ge e^{t_k - t^*}(\cosh(t_j) - 1) + (e^{t_k - t^*} - 1).$$

This implies that

$$\sum_{k\in\Lambda_\gamma,\, j\in\Lambda^c_\gamma,\, k\sim j}\tilde{\beta}_{kj}(\cosh(t_j+t^*-t_k)-1) \ge M^{e^{-t^*}\tilde{\varepsilon}}_{\Lambda^c_\gamma}(t) + \sum_{k\in\Lambda_\gamma,\, j\in\Lambda^c_\gamma,\, k\sim j}\tilde{\beta}_{k,j}(e^{t_k-t^*}-1),$$

and

$$Z^{\gamma,\tilde{\beta}}_{\Lambda^c_\gamma}(t_\gamma) \le e^{\sum_{k\in\Lambda_\gamma,\, j\in\Lambda^c_\gamma,\, k\sim j}\tilde{\beta}_{k,j}(1-e^{t_k-t^*})}\mu^{e^{-t^*}\tilde{\varepsilon},\tilde{\beta}}_{\Lambda^c_\gamma}(1) \le e^{\sum_{e\sim\gamma}\tilde{\beta}_e},$$

since $\mu^{e^{-t^*}\tilde{\varepsilon},\tilde{\beta}}_{\Lambda^c_\gamma}$ is a probability measure. □

Combining (6.5), (6.9), Lemma 6, and integration over the variables $(\nabla t_e)_{e\in\gamma}$, we obtain

$$\mu^{\eta\delta_{i_0},\beta}_{\Lambda}(e^{t_x/2}) \le I_\eta\sum_{\gamma\in\Gamma_x}\left(\prod_{e\sim\gamma}\sqrt{\max(\beta_e, 1)}e^{\min(\beta_e,1)}\right)\left(\prod_{e\in\gamma}I_{\beta_e}\right). \tag{6.10}$$

□

Fix $d \geq 3$. Theorem 1 of [16] (see also the remark above its statement) implies transience of VRJP with constant conductance $\beta_e = \beta > 0$, $e \in E$, sufficiently large on $\mathbb{Z}^d$, $d \geq 3$; the result is stated for constant pinning, but its proof does not require that assumption, as we checked through careful reading.

Let $\Lambda_n = \{i \in \mathbb{Z}^d : \|i\|_\infty \leq n\}$ be the ball centred at 0 with radius n and $\partial\Lambda_n = \{i \in \mathbb{Z}^d : \|i\|_\infty = n\}$ its boundary.

Theorem 4 (Disertori, Spencer and Zirnbauer [16, Theorem 1]). *For any $m > 0$, there exists $\tilde{\beta}_c(m)$ such that, for any $\beta > \tilde{\beta}_c(m)$, and all $n \in \mathbb{N}$ and $x, y \in \Lambda_n$,*

$$\mu_{\Lambda_n}^{\delta_0,\beta}(\cosh^m(t_x - t_y)) \leq 2. \tag{6.11}$$

Corollary 4. *For any $d \geq 3$, there exists $\beta_c(d)$ such that, for all $\beta > \beta_c(d)$, the VRJP on $\mathbb{Z}^d$ with constant conductance β is transient.*

Proof. Let E_n be the set of edges contained in Λ_n. We consider the VRJP on $\mathbb{Z}^d$ with constant conductances $W_{i,j} = \beta$ and denote by $\mathbb{P}_0^\beta(\cdot)$ its law starting from 0. We denote by P_0^c the law of the Markov chain with conductances $c_{i,j} = \beta e^{t_i+t_j}$ starting from 0, where (t_i) is distributed according to $\mu_{\Lambda_n}^{\delta_0,\beta}$. Let $H_{\partial\Lambda_n}$ be the first hitting time of the boundary $\partial\Lambda_n$ and $\tilde{H}_0$ be the first return time to the point δ. Let $R(0, \partial\Lambda_n)$ (resp. $R(0, \partial\Lambda_n, c)$) be the effective resistance between 0 and $\partial\Lambda_n$ for conductances 1 (resp. $c_{i,j}$). Classically

$$c_0 R(0, \partial\Lambda_n, c) = \frac{1}{P_0^c(H_{\partial\Lambda_n} < \tilde{H}_0)}$$

with $c_0 = \sum_{j\sim 0} c_{0,j}$. By Theorem 2 and Jensen's inequality,

$$\frac{1}{\mathbb{P}_0^\beta(H_{\partial\Lambda_n} < \tilde{H}_0)} \leq \mu_{\Lambda_n}^{\delta_0,\beta}(P_0^c(H_{\partial\Lambda_n} < \tilde{H}_0)^{-1}) \leq \mu_{\Lambda_n}^{\delta_0,\beta}(c_0 R(0, \partial\Lambda_n, c)). \tag{6.12}$$

Let us now show that for all $\beta > \tilde{\beta}_c(2)$,

$$\mu_{\Lambda_n}^{\delta_0,\beta}[c_0 R(0, \partial\Lambda_n, c)] \leq 16 d R(0, \partial\Lambda_n). \tag{6.13}$$

This will enable us to conclude the proof: since $\limsup R(0, \partial\Lambda_n) < \infty$, (6.12) and (6.13) imply that $\mathbb{P}_0^\beta(\tilde{H}_0 = \infty) > 0$.

Indeed, let θ be the unit flow from 0 to $\partial\Lambda_n$ which minimizes the L^2 norm. Then

$$R(0, \partial\Lambda_n, c) \leq \sum_{\{i,j\}\in E_n} \frac{1}{c_{i,j}} \theta^2(i, j), \qquad R(0, \partial\Lambda_n) = \sum_{\{i,j\}\in E_n} \theta^2(i, j).$$

Now, for all $\beta > \tilde{\beta}_c(2)$, using identity (6.11), we obtain

$$\mu_{\Lambda_n}^{\delta_0,\beta}(c_0/c_{i,j}) \leq \sum_{l\sim 0} \mu_{\Lambda_n}^{\delta_0,\beta}(e^{2(t_0-t_i)})^{1/2} \mu_{\Lambda_n}^{\delta_0,\beta}(e^{2(t_l-t_j)})^{1/2} \leq 16d.$$ □

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