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Generating series and asymptotics of classical spin networks

Abstract. We study classical spin networks with group SU_2. In the first part, using Gaussian integrals, we compute their generating series in the case where the edges are equipped with holonomies; this generalizes Westbury's formula. In the second part, we use an integral formula for the square of the spin network and perform stationary phase approximation under some non-degeneracy hypothesis. This gives a precise asymptotic behavior when the labels are rescaled by a constant going to infinity.

Keywords. Spin networks, generating series, asymptotical behavior, saddle point method, coherent states

1. Introduction

A classical spin network is a pair (Γ, c) where Γ is a trivalent graph equipped with a cyclic ordering of the edges around each vertex and c is a map from the edges to the natural numbers called *coloring* and satisfying some simple conditions. According to Penrose [P71], one may associate to such a pair a rational number $\langle \Gamma, c \rangle$ obtained by contracting a tensor with values in some representations of SU_2.

When Γ is a theta graph (⬡) or a tetrahedron (△), these quantities were introduced by Racah and Wigner in 1940–1950 for the study of atomic spectra [W59]. Later, Ponzano and Regge used them as a discrete model for gravity. Immediately, physicists were interested in the study of the asymptotic behavior of $\langle \Gamma, kc \rangle$ when k goes to infinity. This behavior corresponds to the classical limit of quantum mechanics and is expected to be related to Euclidean geometric quantities.

In the nineties physicists used spin networks in spin foam models for quantum gravity [B98], and the study of the asymptotical behavior was extended from 3j (theta) and 6j (tetrahedron) to more complicated networks like 9j (complete bipartite 3,3 graph), 10j, 15j (skeleta of the 4- and 5 simplices) (see [BS03, FL03]). Mathematicians got interested

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in spin networks through the work of Kirillov and Reshetikhin [KR89] and Kauffman [K94] who introduced and studied their "quantum versions" which are a key ingredient in the construction of quantum invariants of knots and 3-manifolds. In this paper, we will not deal with these "quantum versions" but restrict to the classical case.

The first rigorous proof for the asymptotical behavior of the 6j-symbols was obtained (in the so-called Euclidean case) by Roberts [R99] and then recovered and extended using different techniques [AHHJLY12, C10, GV13, LY09, WT05]. For general graphs, Garoufalidis and Van der Veen [GV13] proved that the generating series of the sequence $k \mapsto \langle \Gamma, kc \rangle$ is a G-function, implying that the sequence $\langle \Gamma, kc \rangle$ is of Nilsson type and thus that the asymptotic behavior does exist. Abdesselam [A12] obtained estimations on the growth of spin-network evaluations, in particular for generalized drum graphs. A general strategy based on WKB approximation for studying these asymptotical behaviors was proposed in [AHHJLY12].

Another interesting approach to the study of spin networks was proposed by Westbury [W98] who computed the generating series of spin networks as follows (we refer to Section 2 for notation). Let $R_X = \mathbb{C}[[X_\alpha, \alpha \in A]]$ be the ring of formal series in variables associated to the angles of Γ. For any coloring c, we denote by c_e, c_α and X^c respectively the color of the edge e, that of the angle α and the monomial $\prod_{\alpha \in A} X_\alpha^{c_\alpha}$. Then we define the following formal series:

$$Z(\Gamma) = \sum_c \langle\!\langle \Gamma, c \rangle\!\rangle X^c, \quad \langle\!\langle \Gamma, c \rangle\!\rangle = \langle \Gamma, c \rangle \frac{\prod_{e \in E} c_e!}{\prod_{\alpha \in A} c_\alpha!}.$$

Given $\delta \subset \Gamma$, a subgraph which is a disjoint union of cycles, we denote by c_δ the coloring which associates to an angle 1 or 0 according to whether or not δ contains the two edges forming the angle. Let then $P_\Gamma = \sum_{\delta \subset \Gamma} X^{c_\delta}$.

Theorem 1 (Westbury, [W98]). *If Γ is a planar graph then $Z(\Gamma) = P_\Gamma^{-2}$.*

This was generalized to all trivalent graphs by Garoufalidis and Van der Veen [GV13]. Below we describe our results, which fall into two independent parts.

1.1. Generating series with holonomies

In gauge theory it is natural to consider spin networks whose edges are decorated by a *holonomy* ψ in $SL_2(\mathbb{C})$; we shall denote them by $\langle \Gamma, c, \psi \rangle$ (see Definition 2.4) and their renormalization by $\langle\!\langle \Gamma, c, \psi \rangle\!\rangle$ (defined as above). In the present paper we generalize Westbury's theorem to the case of a general graph Γ equipped with any holonomy ψ and we give a closed formula for the generating series $Z(\Gamma, \psi) = \sum_c \langle\!\langle \Gamma, c, \psi \rangle\!\rangle X^c$.

Fix a numbering of the vertices of Γ compatible with the planar structure (see Figure 2), and let Γ' be the graph obtained from Γ by blowing up vertices, i.e. replacing $\bigcup | \bigcup \to \bigcup \bigcup$. The vertices of Γ' are in 1-to-1 correspondence with the set H of half-edges of Γ, and the holonomy may be seen as a map $\psi : H \to SL_2(\mathbb{C})$. Then we set $F_h = \mathbb{R}^2$ for every half-edge, and for any pair of half-edges g, h we define $b_{g,h} : F_g \times F_h \to \mathbb{C}$ by $b_{g,h}(z_g, w_g, z_h, w_h) = i(z_g w_h - z_h w_g)$.

We define the following R_X-valued quadratic form on $\bigoplus_{h \in H} F_h$:

$$\mathcal{Q}(x) = 2 \sum_{\alpha: g \to h} b_{g,h}(\psi_g^{-1} x_g, \psi_h^{-1} x_h) X_\alpha + 2 \sum_{e: g \to h} (-1)^{\mathrm{wind}(e)+1/2} b_{g,h}(x_g, x_h).$$

The expression $\alpha : g \to h$ means that α is an angle between the half-edges g and h such that the opposite vertex of g is lower than the opposite vertex of h. In the same way, $e : g \to h$ means that the edge e contains the half-edges g and h in such a way that the vertex contained in g is lower than the vertex contained in h. Moreover, $\mathrm{wind}(e)$ is the winding number of e oriented from h to g (it is always an odd multiple of $1/2$). Using Gaussian integrals, we prove the following result:

Theorem 1.1. *For a planar graph Γ with holonomy $\psi : H \to \mathrm{SL}_2(\mathbb{C})$, we have $Z(\Gamma, \psi)$ $= \det(\mathcal{Q})^{-1/2}$.*

In this expression, the determinant is computed in the canonical basis. We remark that strictly speaking, the formula makes sense only for holonomies in $\mathrm{SL}_2(\mathbb{R})$ because of the indeterminacy in the square root. By analytic continuation, the formula holds in general. The non-planar case can be easily treated as a corollary of Theorem 1.1 (see Subsection 3.2). As a corollary we observe that $Z(\Gamma, \psi)^{-2}$ is a polynomial with integer coefficients in the entries of ψ. In Theorem 3.2, we provide a combinatorial interpretation of our formula which extends Westbury's formula in the case when ψ takes values in the diagonal subgroup of $\mathrm{SL}_2(\mathbb{C})$.

1.2. Asymptotics from integral formulas

In his book [W59], Wigner showed that the square of a 6j-symbol may be computed by a simple integral formula over four copies of SU_2. Barrett [BS03] observed that this formula may be generalized to any graph. In Section 4, after recalling the integral formula, we compute the generating series of squares of spin networks:

Theorem 1.2. *Let (Γ, c, ψ) be a spin network equipped with a holonomy with values in $\mathrm{SL}_2(\mathbb{C})$. Define*

$$[\Gamma, c, \psi] = \frac{\langle \Gamma, c, \psi \rangle^2}{\prod_{v:(e_1, e_2, e_3)} \langle \Theta, c_{e_1}, c_{e_2}, c_{e_3} \rangle}$$

where by $v : (e_1, e_2, e_3)$ we indicate the edges touching v, and $\langle \Theta, c_{e_1}, c_{e_2}, c_{e_3} \rangle$ is the value of the Θ-graph colored by $c_{e_1}, c_{e_2}, c_{e_3}$. Considering the generating series $W(\Gamma, \psi)$ $= \sum_c [\Gamma, c, \psi] Y^{c_e}$, the following holds in $R_Y = \mathbb{C}[Y_e, e \in E]$:

$$W(\Gamma, \psi) = \int_{G^V} \frac{dg}{\prod_{e=(h_1,h_2)=(v,w)} \det(\psi_{h_2} g_w \psi_{h_2}^{-1} - \psi_{h_1} g_v \psi_{h_1}^{-1} Y_e)}.$$

Using Kirillov's trace formula, the integral formulas may be transformed in order to apply the stationary phase approximation. This method was applied in [BS03, FL03] to compute the asymptotical behavior of some spin networks but it faces some technical difficulties because of the existence of so-called "degenerate configurations". In Section 5 we propose a different transformation which allows us to treat uniformly all configurations

corresponding to critical points. We precisely describe the critical points of the integrand and compute the associated Hessian. Under suitable genericity hypotheses on Γ and c described in Section 5.2, we compute the dominating terms in the asymptotical development of $[\Gamma, kc, 1]$ for general Γ.

Fix a trivalent graph Γ and a coloring $c : E \to \mathbb{N}$. Let I be the set of maps P from oriented edges of Γ to S^2 which satisfy the following relations:

- Denoting by $-e$ the edge e with opposite orientation, we have $P_{-e} = -P_e$.
- For every vertex v with outgoing edges e_1, e_2, e_3 we have $\sum_i c_{e_i} P_{e_i} = 0$.

If Γ is planar, an element $P \in I$ may be considered, as a geometric realization in $\mathbb{R}^3$, to be a possibly non-convex polyhedron with triangular faces of the graph Γ^* dual to Γ.

Given any graph Γ, for each oriented edge $e : v \to w$ of Γ and for each $P \in I$ we define $\alpha^P(e) \in [0, 2\pi[$ as the oriented dihedral angle at $c(e)P_e$ formed by the two triangles in $\mathbb{R}^3$ whose edges are the vectors $c(e)P_e, c(e_2)P_{e_2}, c(e_3)P_{e_3}$ and $c(e)P_e, c(e_4)P_{e_4}, c(e_5)P_{e_5}$, where e, e_2, e_3 share the vertex v, and e, e_3, e_4 share the vertex w. Given a pair (P, Q) of non-isometric elements of I, we define its *phase function* τ in Subsection 5.2 as a map from E to $S^1/\{\pm 1\}$ by $\tau_e = \pm \exp\left(i \frac{\alpha^Q(e) - \alpha^P(e)}{2}\right)$.

Given $P \in I$ we define $r_P(\xi) = \sum_e c_e \|P_e \times \xi\|^2$ for $\xi \in \mathbb{R}^3$ and $q_P(\xi_v) = \sum_{e:(v,w)} c_e \|P_e \times (\xi_v - \xi_w)\|^2$ for $(\xi_v) \in \bigoplus_{v \in V} \mathbb{R}^3$. In these formulas, $\times$ is the cross-product in $\mathbb{R}^3$.

Finally, given a pair (P, Q) of non-isometric elements of I, we set, for (ξ_v) in $\bigoplus_{v \in V} \mathbb{R}^3$,

$$q_{P,Q}^\kappa(\xi_v) = \sum_e c_e \left(\frac{\kappa^2 \tau_e^2 + 1}{\kappa^2 \tau_e^2 - 1} \|Q_e \times (\xi_v - \xi_w)\|^2 + 2i\langle Q_e, \xi_v \times \xi_w \rangle \right).$$

Then, given a quadratic form q on $\mathbb{R}^n$, we denote by $\det'(q)$ the determinant of the restriction of q to the orthogonal of the kernel of q, that is, the product of all non-zero eigenvalues of the matrix of q. We also set $\det'(q_{P,Q}) = \lim_{\kappa \to 1} (\kappa - 1)^{-3} \det'(q_{P,Q}^\kappa)$.

The following gives a general answer to the question of understanding the asymptotical behavior of classical spin network evaluations. We recall the following equalities:

$$[\Gamma, c] = \frac{\langle \Gamma, c \rangle^2}{\prod_{v:(e_1,e_2,e_3)} \langle \Theta, c_{e_1}, c_{e_2}, c_{e_3} \rangle}$$

where $\langle \Gamma, c \rangle$ is the standard evaluation of spin networks and $\langle \Theta, c_{e_1}, c_{e_2}, c_{e_3} \rangle$ is the standard evaluation of the theta graph with given colors.

Theorem 1.3. *Let (Γ, c) be a colored graph satisfying the conditions of Subsection 5.2. Then denoting by N the opposite of the Euler characteristic of Γ, one has*

$$[\Gamma, kc] = \frac{(2N)^{3/2}}{(\pi k^3)^{N-1}} \left(\sum_{P \in I} \frac{\det(r_P)^{1/2}}{\det'(q_P)^{1/2}} \right.$$
$$\left. + \sum_{(P,Q) \in I^2,\, P \neq Q} \mathrm{Re}\left(\frac{i^N \det(r_P)^{1/2} e^{i \sum_e (kc_e + 1)\theta_e}}{\det'(q_{P,Q})^{1/2} \prod_e \sin(\theta_e)} \right) + O(k^{-1}) \right),$$

while $[\Gamma, kc]$ decays exponentially fast if $I = \emptyset$.

The above formula was numerically tested in the case of the tetrahedron and of course its results are in line with the known asymptotics [AHHJLY12], [R99], [GV13], [BS03], [C10]. The non-degeneracy conditions of Subsection 5.2 are equivalent to the non-vanishing of all determinants in the formula of Theorem 1.3.

- The quantity $\det(r_P)$ is zero only if the configuration P is planar, which can occur only for very special values of c.
- The non-vanishing of $\det'(q_P)$ is equivalent to the infinitesimal rigidity of the configuration P. In particular, it does not hold if the set I is not discrete: this happens for instance for the regular cube, or more generally the cube whose edges are colored by the lengths of Bricard's flexible octahedron.
- We do not have a geometric interpretation of the determinant $\det'(q_{P,Q})$ but in our numerical experiments on the spin networks formed by the 1-skeleton of a tetrahedron this determinant was non-zero. In any case, the conditions define a Zariski open set of configurations.

We believe that describing when the non-degeneracy conditions hold is a very difficult task as it contains the problem of flexibility of polyhedra, a notoriously hard problem. Still, we expect that for planar graphs whose colors correspond to the lengths of a generic convex configuration of the dual graph, there is a simple geometric condition ensuring that the non-degeneracy conditions hold—but this question is not addressed in this article.

1.2.1. About our proof and comparison with other approaches. The proof of Theorem 1.3 is based on a new integral formula (equation (7)) for the evaluation $[\Gamma, kc]$ obtained by applying a key lemma (Lemma 5.1) to another formula (equation (5)) which was first proved by J. W. Barrett [BS03] in the case of spin networks without holonomies on the edges and is proven here in the general case. One of the nice features of this formula is that the colors c_e of Γ intervene only as exponents of a product in the integrand, so when replacing them by kc_e, the integral is very well suited for stationary phase approximation. Then, our analysis takes care of all the technical points in applying that method, namely: identifying the critical points (Proposition 5.3), making the necessary assumptions for these points to be isolated and non-degenerate (Subsection 5.2), computing the Hessian of the integrand at the critical points (see equation (8)), dealing with the inevitable degeneracies due to the action of a group of symmetries (Subsection 5.4.1), computing the contribution of each critical point (Subsection 5.4.2) and finally summing up all the contributions of the critical points (Subsection 5.4.3).

Different other approaches to the study of asymptotics of classical spin networks have been proposed in the literature. Among the most recent ones, we mention those of Aquilanti et al. [AHHJLY12] who construct the WKB approximation of the wave functions associated to two "halves" of the spin network (called the A- and B-part in this paper); these functions concentrate on two lagrangian submanifolds which are cut out not by a set of commuting operators (as in the standard integrable systems) but obtained by a construction generalizing the symplectic reduction. Then a careful application of the WKB method on these functions yields the asymptotical behavior of their scalar product, hence of the evaluation of the initial classical spin network. Independently L. Charles

[C10] applied the geometric quantization techniques to the moduli spaces of polygons and retrieved Wigner's formula for the asymptotical behavior of ⬡. Garoufalidis and Van der Veen [GV13] used a completely different approach and gave a new proof of the formula based on the recursion formulas satisfied by the different evaluations of ⬡.

1.3. Some questions

- The generating series $Z(\Gamma, \psi)^2$ has coefficients which are integral polynomials in the entries of ψ. Is this still true for $Z(\Gamma, \psi)$ as suggested by the abelian case?
- Is there a combinatorial interpretation of $Z(\Gamma, \psi)$ for general ψ?
- Find a direct relation between the series $Z(\Gamma, \psi)$ and $W(\Gamma, \psi)$.
- Find sufficient conditions on Γ and c which ensure that the non-degeneracy hypothesis holds, in particular in the case of convex polyhedra.
- Interpret geometrically the leading terms in the asymptotic formula of Theorem 1.3.

2. Equivalent descriptions of spin network evaluations

Let Γ be a trivalent graph, possibly with loops and multiple edges. We denote by E the set of edges, V the set of vertices and we divide each edge $e \in E$ into two subarcs called *half-edges*. We will then let H be the set of half-edges commonly described as pairs (e, v) where e is an edge and v is an end of e. We shall moreover orient a priori each half-edge $h = (e, v)$ so that it goes out of the vertex v. We assume that for each vertex the set of half-edges incoming to that vertex have a cyclic order. We define an *angle* to be a pair of half-edges touching the same vertex and denote by A the set of angles. In the whole article, we will suppose that Γ is connected and contains at least one vertex; we will denote by N the opposite of the Euler characteristic of Γ so that $\#V = 2N, \#E = 3N$ and $\#H = \#A = 6N$.

An *admissible coloring* is a map $c : E \to \mathbb{N}$ satisfying the triangle conditions:

$$\forall v : (i, j, k), \quad c_i + c_j + c_k \in 2\mathbb{N} \text{ and } c_i \le c_j + c_k. \tag{T}$$

For convenience, we wrote $v : (i, j, k)$ meaning that the edges i, j, k are incoming at v with that cyclic order. Associated to each coloring c there is an *internal coloring*, i.e. a map from A to $\mathbb{N}$ also denoted by c and defined as follows: if α is the angle between edges i and j around v then we set $c_\alpha = (c_i + c_j - c_k)/2$. We remark than one can recover the original coloring from the internal coloring; to avoid confusion, we will denote edges with Latin letters and angles with Greek ones.

Let (V, ω) be a complex symplectic vector space of rank 2 and $\text{SL}(V)$ be its symmetry group. A *discrete connection* is a map $\psi : H \to \text{SL}(V)$; we define the *gauge group* as the group of maps $\{g : V \sqcup E \to \text{SL}(V)\}$. An element g of the gauge group acts on a discrete connection ψ as follows: for each $h = (e, v) \in H, (g \cdot \psi)_h = g_e \psi_h g_v^{-1}$. Two discrete connections $\psi_1, \psi_2 : E \to \text{SL}(V)$ are said to be *gauge equivalent* if they are in the same orbit of the gauge group.

Definition 2.1 (Holonomy). • A holonomy on Γ is an equivalence class of discrete connections ψ on Γ.

• The *trivial holonomy* (denoted by 1) is the class of the constant discrete connection defined by $\psi_h = 1 \in \mathrm{SL}(V)$ for all half-edges $h \in H$.

• Let γ be an oriented path in Γ described as a sequence of half-edges $h_1, \ldots, h_n$. We define the holonomy $\psi(\gamma)$ of ψ along γ to be the product $\psi_{h_n}^{\varepsilon_n} \cdots \psi_{h_1}^{\varepsilon_1}$ where ε_i is 1 or -1 depending on whether the half-edge is oriented coherently with γ or not. If ψ_1 and ψ_2 are gauge equivalent and γ is closed then $\psi_1(\gamma)$ and $\psi_2(\gamma)$ are conjugate.

In the following section, we give two equivalent descriptions of the spin network evaluation which consists in associating to a triple (Γ, c, ψ) a complex number $\langle \Gamma, c, \psi \rangle$: the *abstract* and *computational* descriptions. The computational description is one of the many equivalent descriptions already known in the literature and is based on standard algebra. The abstract one makes use of super vector spaces to take care of all the annoying signs appearing when dealing with spin networks; through the abstract description one may simply define a spin network as a contraction of tensors without drawing any graph or inserting additional signs. Although the abstract definition is esthetically nicer, in the rest of the paper we will use the computational one. In any case, when the holonomy is trivial, both definitions coincide with the standard evaluation of spin networks defined by Penrose [P71].

2.1. Abstract description

Let (V, ω) be a complex symplectic vector space of rank 2. In this section we will consider that V is an odd superspace. We refer to Appendix for a basic review of supersymmetry adapted to our purposes.

The symplectic form is considered as a supersymmetric map $\omega : V \otimes V \to \mathbb{C}$ in the sense that $\omega \circ \varphi_{(12)} = \omega$. For any integer $n \in \mathbb{N}$, ω induces a map $\omega^{\otimes n} : V^{\otimes n} \otimes V^{\otimes n} \to \mathbb{C}$ defined by the formula

$$\omega^{\otimes n}(v_1 \otimes \cdots \otimes v_n, w_1 \otimes \cdots \otimes w_n) = \prod_{i=1}^{n} \omega(v_{n+1-i}, w_i).$$

Denote by V_n the subspace of $V^{\otimes n}$ consisting of anti-supersymmetric tensors (or, equivalently, symmetric in the standard sense); the form $\omega^{\otimes n}$ restricts to a supersymmetric form $\omega_n : V_n \otimes V_n \to \mathbb{C}$. The vector space V_n (of parity n) is the $(n+1)$-dimensional irreducible representation of the group $\mathrm{SL}(V)$; we will sometimes use the notation $\rho_n : \mathrm{SL}(V) \to \mathrm{End}(V_n)$. Denote by ω^{-1} the unique element of $V \otimes V$ such that the contraction of the two middle terms in $\omega \otimes \omega^{-1}$ is the identity of V.

Let a, b, c be three integers. It is well known that the set of $\mathrm{SL}(V)$-invariant elements in $V_a \otimes V_b \otimes V_c$ is 1-dimensional if a, b, c satisfy the triangle conditions (T) unless it is 0. One can find an explicit generator $\varepsilon_{a,b,c} \in V_a \otimes V_b \otimes V_c$ given by the symmetrization of the element

$$\omega_{a,b,c} = (\omega^{-1})^{\otimes(a+b-c)/2} \otimes (\omega^{-1})^{\otimes(b+c-a)/2} \otimes (\omega^{-1})^{\otimes(a+c-b)/2}$$

where the supersymmetric tensor product is reordered as in Figure 1.

Remark 2.2. Note that the sign of the permutation which reorders the $a + b + c$ factors in this tensor product is $+1$.
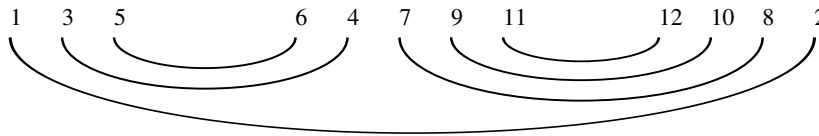

Fig. 1. Invariant tensor at vertices with $a = 3, b = 5, c = 4$.

Then we define $\varepsilon_{a,b,c} = (\Pi_a \otimes \Pi_b \otimes \Pi_c)\omega_{a,b,c}$ where Π_a is the anti-supersymmetrization (i.e. standard symmetrization) map projecting $V^{\otimes a}$ onto V_a.

One may check that this element is supersymmetric: if we permute cyclically a, b and c to the left the result is multiplied by $(-1)^a = (-1)^{a(b+c)}$, which is the sign of the cycle $V_a \otimes V_b \otimes V_c \to V_b \otimes V_c \otimes V_a$ according to the supersymmetric rule.

Definition 2.3 (Spin network). Let Γ be a trivalent graph and $c : E \to \mathbb{N}$ be an admissible coloring. Then $\langle \Gamma, c, 1 \rangle$ is the result of the supersymmetric contraction of

$$\left\langle \bigotimes_{e \in E} \omega_{c_e}, \bigotimes_{v \in V, \, v:(i,j,k)} \varepsilon_{c_i, c_j, c_k} \right\rangle.$$

By supersymmetric contraction, we mean that we reorder the tensors on the right hand side according to the sign rule so that factors corresponding to the same edge are consecutive, and then contract with the maps ω_n.

Given a discrete connection $\psi : H \to \mathrm{SL}(V)$, let us define Hol_ψ to be the endomorphism

$$\bigotimes_{v \in V, \, v:(i,j,k)} \rho_{c_i}(\psi_{i,v}) \otimes \rho_{c_j}(\psi_{j,v}) \otimes \rho_{c_k}(\psi_{k,v}) \in \mathrm{End}\left(\bigotimes_{v \in V, \, v:(i,j,k)} V_{c_i} \otimes V_{c_j} \otimes V_{c_k} \right)$$

Definition 2.4 (Spin network with holonomy). Let Γ be a trivalent graph, $\psi : H \to \mathrm{SL}(V)$ be a discrete connection and $c : E \to \mathbb{N}$ be an admissible coloring. Then $\langle \Gamma, c, \psi \rangle$ is the result of the following supersymmetric contraction:

$$\left\langle \bigotimes_{e \in E} \omega_{c_e}, \mathrm{Hol}_\psi \left(\bigotimes_{v \in V, \, v:(i,j,k)} \varepsilon_{c_i, c_j, c_k} \right) \right\rangle.$$

One can check directly that this definition does not depend on the gauge equivalence class of ψ.

2.2. Computational description

The computational description follows directly from the abstract one, by stipulating now that V is an even vector space and then taking care of the signs which are now no longer natural and need to be inserted ad hoc.

Set $V = \mathbb{C}^2$ and $\omega(v, w) = \det(v, w)$. Then $V^* = \mathbb{C}^2$ and we denote by z and w the corresponding coordinates. In this way, $\omega^{-1} \in V \otimes V$ corresponds to the linear polynomial on $V^* \times V^*$ which in coordinates (z_1, w_1, z_2, w_2) reads as $z_2 w_1 - z_1 w_2$.

The form $\omega_n : V_n \otimes V_n \to \mathbb{C}$ is a linear form on the space of homogeneous polynomials of bidegree (n, n) on $V^* \times V^*$. Here we identify homogeneous polynomials and symmetric tensors via

$$z^i w^{n-i} \mapsto \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} X_{\sigma_1} \otimes \cdots \otimes X_{\sigma_n}$$

where $X_k = z$ if $k \le i$ and w otherwise. One checks directly that in coordinates, ω_n is expressed by

$$\omega_n = \frac{1}{n!^2} \left(\frac{\partial}{\partial z_1} \frac{\partial}{\partial w_2} - \frac{\partial}{\partial z_2} \frac{\partial}{\partial w_1} \right)^n.$$

In the same way, the element $\varepsilon_{a,b,c} \in V_a \otimes V_b \otimes V_c$ corresponds to a polynomial on $V^* \times V^* \times V^*$ which in coordinates $(z_1, w_1, z_2, w_2, z_3, w_3)$ reads as

$$\varepsilon_{a,b,c} = (z_2 w_1 - z_1 w_2)^{\frac{a+b-c}{2}} (z_3 w_2 - z_2 w_3)^{\frac{b+c-a}{2}} (z_3 w_1 - z_1 w_3)^{\frac{a+c-b}{2}}.$$

In these expressions we consider the variables z_i, w_i and the derivations as even quantities, and these expressions coincide with those in the preceding section because of Remark 2.2.
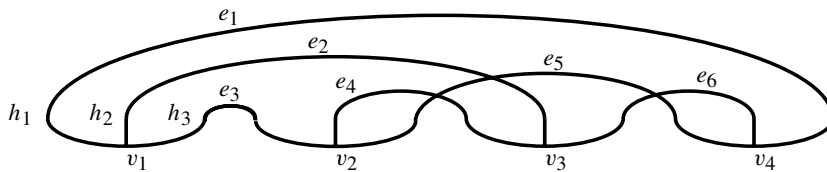

Fig. 2. Planar presentation of the tetrahedron.

Finally, let us compute the sign needed to perform the final contraction as explained in the preceding subsection. Suppose first that the trivalent graph is presented as in Figure 2; this presentation induces an ordering of the set of half-edges H. For any edge $e \in E$, let e_1 and e_2 be the respective left and right half-edges of e, and for any vertex v in V, let v_1, v_2, v_3 be the three half-edges incoming to v in increasing order. Writing the tensor product of the invariant elements $\varepsilon_{a,b,c}$ by ordering them from left to right as the vertices in the figure gives a big tensor which has to be reordered in such a way that half-edges are matched in pairs as indicated in the figure (so for instance h_1 is to be matched with h_2). Let X be the set of pairs of crossing edges; the sign of this permutation (see Remark 2.2) is $(-1)^{\sum_{\{e,e'\} \in X} c_e c_{e'}}$; since now we treat all the vector spaces as even we need to re-insert this sign in the contraction of tensors.

We then compute

$$\langle \Gamma, c, 1 \rangle = (-1)^{\sum_{\{e,e'\} \in X} c_e c_{e'}} \prod_e \frac{1}{c_e!^2} \left(\frac{\partial}{\partial z_{e_1}} \frac{\partial}{\partial w_{e_2}} - \frac{\partial}{\partial z_{e_2}} \frac{\partial}{\partial w_{e_1}} \right)^{c_e}$$

$$\cdot \prod_v (z_{v_2} w_{v_1} - z_{v_1} w_{v_2})^{\frac{c_{v_1}+c_{v_2}-c_{v_3}}{2}} (z_{v_3} w_{v_2} - z_{v_2} w_{v_3})^{\frac{c_{v_2}+c_{v_3}-c_{v_1}}{2}}$$

$$\cdot (z_{v_3} w_{v_1} - z_{v_1} w_{v_3})^{\frac{c_{v_1}+c_{v_3}-c_{v_2}}{2}}. \quad (1)$$

In order to use the machinery of Gaussian integration, it will be more comfortable to introduce $i = \sqrt{-1}$ in our formulas. The formula is unchanged if we replace ω_n and $\varepsilon_{a,b,c}$ by $\overline{\omega}_n$ and $\overline{\varepsilon}_{a,b,c}$ where we set

$$\overline{\omega}_n = \frac{i^n}{n!^2}\left(\frac{\partial}{\partial z_1}\frac{\partial}{\partial w_2} - \frac{\partial}{\partial z_2}\frac{\partial}{\partial w_1}\right)^n,$$

$$\overline{\varepsilon}_{a,b,c} = i^{\frac{a+b+c}{2}}(z_1w_2 - z_2w_1)^{\frac{a+b-c}{2}}(z_2w_3 - z_3w_2)^{\frac{b+c-a}{2}}(z_1w_3 - z_3w_1)^{\frac{a+c-b}{2}}.$$

The formula for $\langle \Gamma, c, \psi \rangle$ is obtained from (1) by replacing the variables (z_i, w_i) at the vertex v with $\psi_{i,v}^{-1}(z_i, w_i)$.

More generally, if Γ is such that all the vertices look like $\smile\,\cup$ but the edges may pass under some vertex then the sign correction can be shown to be equal to $(-1)^{\sum_e c_e(\mathrm{wind}(e)+1/2)+\sum_{\{e,e'\}\in X} c_e c_{e'}}$ where for each edge we compute its winding number by orienting it from its leftmost to its rightmost endpoint (and it is therefore an odd multiple of $1/2$). Indeed it is sufficient to check what happens when one edge e passing below a vertex v is pushed by an isotopy over the vertex: if v is not an endpoint of e, the global contribution of the three resulting crossings is 1 (because of the parity condition). Otherwise a kink is created along e whose contribution to the sign is $(-1)^{c_e^2} = (-1)^{c_e}$; deleting the kink one changes the sign exactly by the change in $(-1)^{c_e(\mathrm{wind}(e)+1/2)}$.

Example 2.5 (The unknot colored by n).

$$\langle \bigcirc, n, 1 \rangle = \frac{i^n}{n!^2}\left(\frac{\partial}{\partial z_1}\frac{\partial}{\partial w_2} - \frac{\partial}{\partial z_2}\frac{\partial}{\partial w_1}\right)^n (i^n(z_1w_2 - z_2w_1)^n)$$

$$= \frac{(-1)^n}{n!^2}\sum_{h=0}^n \binom{n}{h}\left(\frac{\partial}{\partial z_1}\frac{\partial}{\partial w_2}\right)^h \left(\frac{\partial}{\partial w_1}\frac{\partial}{\partial z_2}\right)^{n-h}\sum_{k=0}^n \binom{n}{k}(z_1w_2)^k(z_2w_1)^{n-k}$$

$$= \frac{(-1)^n}{n!^2}\sum_{h=0}^n \binom{n}{h}^2 \frac{\partial^h}{\partial z_1^h}\frac{\partial^{n-h}}{\partial w_1^{n-h}}\frac{\partial^{n-h}}{\partial z_2^{n-h}}\frac{\partial^h}{\partial w_2^h}(z_1^h w_1^{n-h} z_2^{n-h} w_2^h) = (-1)^n(n+1).$$

More generally, if ψ is a matrix whose trace is $\lambda + \lambda^{-1}$, a similar computation yields

$$\langle \bigcirc, n, \psi \rangle = (-1)^n \frac{\lambda^{n+1} - \lambda^{-(n+1)}}{\lambda - \lambda^{-1}}.$$

Example 2.6. As already stated, we defined $\langle \Gamma, c, 1 \rangle$ to be identical to the original evaluation defined by Penrose, so in particular

$$\left\langle \overset{c}{\underset{a}{\overset{b}{\bigcirc}}}, 1 \right\rangle = (-1)^{\frac{a+b+c}{2}} \frac{\left(\frac{a+b+c}{2}+1\right)!\left(\frac{a+b-c}{2}\right)!\left(\frac{a-b+c}{2}\right)!\left(\frac{-a+b+c}{2}\right)!}{a!b!c!}$$

while the integral normalization $\langle\!\langle \ominus \rangle\!\rangle$ defined in the introduction gives

$$\left\langle\!\!\left\langle \overset{c}{\underset{a}{\overset{b}{\bigcirc}}}, 1 \right\rangle\!\!\right\rangle = (-1)^{\frac{a+b+c}{2}} \frac{\left(\frac{a+b+c}{2}+1\right)!}{\left(\frac{a+b-c}{2}\right)!\left(\frac{a-b+c}{2}\right)!\left(\frac{-a+b+c}{2}\right)!} \in \mathbb{Z}.$$

2.3. Why supersymmetry?

The abstract definition of spin network evaluations might puzzle the reader, so we dedicate this subsection to explain where this definition came from and why both our definitions coincide with the following standard evaluation of spin networks used in the literature:

Definition 2.7 (Evaluation of spin networks, [P71]). Let Γ be a trivalent graph in $\mathbb{R}^3$ equipped with an admissible coloring c of its edges and a cyclic ordering of the edges around each vertex. One defines $\langle \Gamma, c \rangle$ by the following algorithm:

1. Cable each edge e of Γ by replacing an edge e colored by a by the linear combination of braids given by the Jones–Wenzl projectors JW_a:

$$\left| a \rightarrow \boxed{JW_a} \doteq \sum_{\sigma \in \mathfrak{S}_a} \frac{(-1)^{-a(a-1)+3T(\sigma)}}{a!} \boxed{JW_{\hat{\sigma}}} \right.$$

   where $\hat{\sigma}$ is the minimal positive braid inducing the permutation σ, and $T(\sigma)$ is the number of crossings it contains.
2. Around each vertex, connect the (yet free) endpoints of the resulting strands in the unique planar way without self-returns:
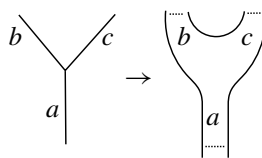
3. This way one associates to (Γ, c) a linear combination with coefficients $c_i \in \mathbb{Q}$ of links L_i. Define $\langle \Gamma, c \rangle := \sum_i c_i (-2)^{\#L_i}$, where $\#L_i$ is the number of components of L_i.

It is well known that the above definition is a special case (corresponding to $A = -1$) of the definition of the quantum spin network $\langle \Gamma, c \rangle \in \mathbb{Q}(A)$ associated to a framed, colored graph Γ in $\mathbb{R}^3$ via the representation theory of $U_q(\mathfrak{sl}_2)$, where $A^2 = q$; see for instance [K94]. Hence $\langle \Gamma, c \rangle$ can be computed by suitably composing morphisms of $U_q(\mathfrak{sl}_2)$ when $A = -1$.

This is basically what we do in our computational definition. The simple modules V_a of $U(\mathfrak{sl}_2)$ are homogeneous polynomials of a degree a in two variables z, w, and the key point is to observe that when $A = -1$ the operator associated to a crossing formed by two strands colored with V_1 is *minus* the flip. The computational description we gave in the preceding section is obtained by giving explicitly the invariant tensors in tensor products of V_a and writing them as polynomials, and remarking that the action of the R-matrix (when $A = -1$) introduces signs associated to the crossings of a diagram. Although this approach is more practical for computations, it is less intrinsic as it needs the choice of some embedding of the graph in $\mathbb{R}^3$ to identify the crossings and suitably take care of the signs.

This is why we searched for the abstract description which was obtained by taking seriously the hint of the signs appearing at crossings and so considering V_1 as an odd super vector space and its tensor powers V_a as having degrees $(-1)^a$. Using super vector spaces automatically sorts out a list of annoying sign problems: for instance the value associated to an unknot colored by n is $(-1)^n(n+1)$, which is exactly the supertrace of the identity if V_n has degree $(-1)^n$.

3. Generating series via Gaussian integrals

In Subsection 3.1 we will use Gaussian integration to prove Theorem 1.1 in the case of a planar graph Γ. This is a generalization of Westbury's result to the case when graphs are equipped with holonomies. In Subsection 3.3 we will provide a topological interpretation of the theorem when the holonomy is diagonal and recover Westbury's original result (always in the case of planar graphs). In Subsection 3.2 we will extend Theorem 1.1 to the case of non-planar graphs.

3.1. Computing the generating series of a spin network

Let Γ be a planar trivalent graph that we present by means of a planar diagram in $\mathbb{R}^2$ in which all the vertices have different horizontal coordinate and look like $\bigcup\bigcap$. Each edge has a left half-edge and a right one, $e : g \to h$; orienting the edges from left to right one can define the winding number wind(e) which is always an odd multiple of $1/2$. In particular if Γ can be presented as in Figure 2 and without crossings, all the winding numbers are $-1/2$.

Let F_h be a copy of $\mathbb{R}^2$ associated to each half-edge h; we will always consider the standard density on these spaces and then omit it in the notation, and let $F = \bigoplus_h F_h$. For two half-edges g, h we set $b_{g,h} : F_g \times F_h \to \mathbb{C}$ as being given in coordinates by the expression $i(z_g w_h - z_h w_g)$, and $b_{g,h}^{-1} : F_g^* \times F_h^* \to \mathbb{C}$ as $-i\left(\frac{\partial}{\partial z_g}\frac{\partial}{\partial w_h} - \frac{\partial}{\partial z_h}\frac{\partial}{\partial w_g}\right)$.

Define quadratic forms P on F^* and Q on F by

$$P = -2\sum_{e:g\to h}(-1)^{\text{wind}(e)+1/2}b_{g,h}^{-1} \quad \text{and} \quad Q = 2\sum_{\alpha:g\to h}X_\alpha b_{g,h}.$$

The notation $\alpha : g \to h$ and $e : g \to h$ means that α and e are composed of the half-edges g, h which appear in that order from left to right in Figure 2. For the moment, X_α should be interpreted as a real parameter.

Remark 3.1. The role of X_α is just that of a formal variable in the generating series used to keep track of the combinatorics of the colorings. So for this purpose, it is clear that one may also replace it by $Y_{l(\alpha)}Y_{r(\alpha)}$ where $l(\alpha), r(\alpha)$ are the half-edges at the left and right endpoint of α. In the following formulas we decided to use X_α because these variables always come in pairs, but we emphasize that it is just a matter of taste here.

Note that the quadratic form P^{-1} on F is expressed by

$$P^{-1} = 2\sum_{e:g\to h}(-1)^{\text{wind}(e)+1/2}b_{g,h}.$$

Consider P as a differential operator P^{op} on $C^\infty(F, \mathbb{C})$ and develop $\left(\exp\left(\frac{1}{2}P\right)^{\mathrm{op}} \cdot \exp\left(\frac{1}{2}Q\right)\right)\big|_0$ (see Appendix B). Collecting monomials in the variables X_α, by equation (1) one sees that the coefficient of $\prod_\alpha X_\alpha^{c_\alpha}$ is $\langle\!\langle \Gamma, c \rangle\!\rangle$ (defined in the Introduction). (In particular the winding number of an edge e shows up when contracting all the tensors associated to minima and maxima along e.) Therefore we have

$$Z(\Gamma, 1) = \left(\exp\left(\tfrac{1}{2}P\right)^{\mathrm{op}} \exp\left(\tfrac{1}{2}Q\right)\right)\big|_0.$$

Consider a deformation Q_ε of Q which is non-degenerate and has positive real part. For concreteness, we can pick $Q_0 = \sum_h (z_h^2 + w_h^2)$ and set $Q_\varepsilon = Q + \varepsilon Q_0$. We define $Z_\varepsilon = \left(\exp\left(\frac{1}{2}P\right)^{\mathrm{op}} \exp\left(\frac{1}{2}Q_\varepsilon\right)\right)\big|_0$ so that $Z(\Gamma, 1) = \lim_{\varepsilon \to 0} Z_\varepsilon$. Then if we replace Q by Q_ε^{-1} and $\frac{1}{2}P$ by $\exp(\frac{1}{2}P)$, formula (16) of Appendix B gives

$$Z_\varepsilon = (2\pi)^{-n/2} \det(Q_\varepsilon^{-1})^{1/2} \int_{F^*} \exp\left(\frac{1}{2}P(x) - \frac{1}{2}Q_\varepsilon^{-1}(x)\right) dx.$$

We now apply formula (15) to the integral, noting that the quadratic form $Q_\varepsilon^{-1} - P$ is still non-degenerate and has positive real part. Hence, we have

$$Z_\varepsilon^2 = \frac{\det(Q_\varepsilon^{-1})}{\det(Q_\varepsilon^{-1} - P)} = \det(Q_0 - Q_\varepsilon P)^{-1}.$$

Letting ε go to 0, we find that $Z(\Gamma, 1) = \det(Q_0 - QP)^{-1/2}$.

Suppose that ψ is represented by a discrete connection on Γ with values in $\mathrm{SL}_2(\mathbb{R})$. By formula (1), we know how to adapt the construction: for each angle α connecting two half-edges g and h, we need to replace $b_{g,h} : F_g \times F_h \to \mathbb{C}$ by $b_{g,h}(\psi_g^{-1}x_g, \psi_h^{-1}x_h)$. We denote by Q_ψ the resulting quadratic form. By the assumption that ψ lives in $\mathrm{SL}_2(\mathbb{R})$, P takes again only imaginary values. Hence, the argument above repeats exactly and we obtain $Z(\Gamma, \psi) = \det(Q_0 - Q_\psi P)^{-1/2}$. The general case, that is, for ψ taking values in $\mathrm{SL}_2(\mathbb{C})$, follows by analytic continuation.

One can simplify this formula by remarking that the matrix of P in the canonical basis satisfies $P^{-1} = -P$, and moreover $\det(P) = 1$. We obtain the formula of Theorem 1.1:

Theorem 1.1.
$$Z(\Gamma, \psi) = \det(P + Q_\psi)^{-1/2}.$$

3.2. The non-planar case

Let now Γ be a non-planar graph and let us fix a diagram of Γ as in Figure 2 containing crossings $x_1, \ldots, x_k$. For each coloring c of Γ, a crossing x_i between edges e_1 and e_2 induces a factor $(-1)^{c_{e_1} c_{e_2}}$, which we denote by $s(x_i, c)$. The previous results allow us to compute the "wrong" generating series for Γ where the signs coming from crossings are not taken into account, that is, we can compute $W(\Gamma, \psi) = \sum_c \langle\!\langle \Gamma, c, \psi \rangle\!\rangle X^c \prod_{x_i} s(x_i, c)$. To fix these signs, use the identity

$$(-1)^{ab} = \tfrac{1}{2}(1 + (-1)^a + (-1)^b - (-1)^{a+b}), \quad \forall a, b \in \mathbb{Z}.$$

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More explicitly, for each edge e let α_e, β_e be the leftmost angles formed by e and let $\mathrm{Op}_e : \mathbb{C}[[X]] \to \mathbb{C}[[X]]$ be the automorphism that changes the signs of X_{α_e} and X_{β_e}. Then, for a crossing x between edges e_1 and e_2, define $S_x : \mathbb{C}[[X]] \to \mathbb{C}[[X]]$ as $S_x = \frac{1}{2}(\mathrm{Id} + \mathrm{Op}_{e_1} + \mathrm{Op}_{e_2} - \mathrm{Op}_{e_1} \circ \mathrm{Op}_{e_2})$. Then we recover $Z(\Gamma, \psi) = \sum_c \langle\!\langle \Gamma, c, \psi \rangle\!\rangle X^c$ as

$$Z(\Gamma, \psi) = S_{x_1} \circ \cdots \circ S_{x_k}(W(\Gamma, \psi)).$$

In particular, when ψ is trivial this recovers Garoufalidis and Van der Veen's extension to non-planar graphs of Westbury's theorem [GV13].

3.3. A generalization of Westbury's theorem to the case of diagonal holonomies

Suppose that the holonomy ψ can be represented by a connection with values in the subgroup $\mathrm{D} \subset \mathrm{SL}_2(\mathbb{R})$ of diagonal matrices. We introduce a map $t : H \to \mathbb{R}^*$ such that for all h, one has $\psi_h = \begin{pmatrix} t_h & 0 \\ 0 & t_h^{-1} \end{pmatrix}$ in the basis (z_h, w_h). In this case, we can extend Westbury's Theorem 1 as follows.

Let $C(\Gamma)$ be the set of all oriented curves immersed in Γ which pass over an edge of Γ either 0, 1 or 2 times, in the latter case with opposite orientations. Given $\gamma \in C(\Gamma)$ we denote by $\mathrm{cr}(\gamma)$ the number of crossings modulo 2 of the corresponding immersion.

Let $X^\gamma = \prod_{\alpha \subset \gamma} X_\alpha$ (hence each angle may appear 0, 1 or 2 times) and $\mathrm{Tr}(\gamma) = \prod_{h \subset \gamma} t_h^{\varepsilon(\gamma, h)}$ ($\varepsilon(\gamma, h)$ being 1 if γ crosses h in the positive direction, -1 otherwise). The following result generalizes Westbury's result to the case of holonomies with values in diagonal matrices:

Theorem 3.2. *Let Γ be a planar graph equipped with an abelian holonomy ψ. Then*

$$Z(\Gamma, \psi) = \left(\sum_{\gamma \in C(\Gamma)} (-1)^{\mathrm{cr}(\gamma)} \mathrm{Tr}(\gamma) X^\gamma(c) \right)^{-1}.$$

Theorem 3.2 follows directly from Corollary 3.4 and Proposition 3.8; the details are given below.

3.4. Some general facts about dimers and determinants

The following are general well known facts which we shall apply to interpret topologically some of the determinants we will be dealing with. Let G^{or} be a graph whose vertices $v_1, \ldots, v_n$ are connected by oriented edges $e_{ij} : v_i \to v_j$ whose weights are the entries m_{ij} of an $n \times n$ matrix M (the diagonal terms then correspond to loops). The following is a standard well known fact:

Lemma 3.3. $\det(M) = (-1)^n \sum_c (-1)^{\#c} w(c)$ *where c runs over all the oriented curves embedded in G^{or} and passing through each vertex exactly once, $w(c)$ is the product of the weights m_{ij} of the oriented edges in c, and $\#c$ is the number of connected components of c.*

Let now G be the graph whose vertices $v_1, \ldots, v_n$ are connected by exactly one unoriented edge $e_{i,j}$ and suppose that M is a matrix such that $m_{ii} = 0$ for all i. Given a connected oriented curve c embedded in G connecting vertices $v_{i_1} \to v_{i_2} \to \cdots \to v_{i_k} \to v_{i_1}$ let $w(c) = m_{i_1, i_2} \cdots m_{i_k, i_1}$ and if c is a disjoint union of disjoint oriented curves $c_1, \ldots, c_k$, let $w(c) = \prod_i w(c_i)$. Also let a *dimer* be a disjoint union of edges, $d = e_{i_i, j_1} \sqcup \cdots \sqcup e_{i_k, j_k}$, and let $w(d) = m_{i_1, j_1} m_{j_1, i_1} \cdots m_{i_k, j_k} m_{j_1, i_k}$ and $\#d = k$. By a *configuration of curves and dimers* we will mean a disjoint union $c \cup d$ of dimers and oriented curves embedded in G such that each vertex is contained in exactly one component of $c \cup d$; its weight will be $w(c \cup d) = w(c)w(d)$; similarly a *configuration of dimers* will be a configuration of curves and dimers containing no curves. We shall denote by $\mathrm{Conf}(G)$ the set of configurations of curves and dimers on G, and by $\mathrm{DConf}(G)$ the set of dimer configurations. (Basically here we call dimers curves of length 2, recalling that by hypothesis, any two vertices share at most one edge.) Then the following holds:

Corollary 3.4. $\det(M) = (-1)^n \sum_{c \cup d \in \mathrm{Conf}(G)} (-1)^{\#c + \#d} w(c) w(d)$.

Finally if $M^t = -M$ then $\det(M) = \mathrm{Pfaff}(M)^2$ and $\mathrm{Pfaff}(M)$ can be interpreted as counting the dimer configurations in G (see [K63]):

Theorem 3.5. $\mathrm{Pfaff}(M) = \sum_{d \in \mathrm{DConf}(G)} \pm \sqrt{|w(d)|}$.

In the above theorem, the square root is due only to our definition of $w(d)$ while the choice of the signs is in general a delicate matter (see [K63]); in our specific cases it will be quite easy to determine it.

3.5. Proof of Theorem 3.2

Theorem 3.2 follows directly from Corollary 3.4 and Proposition 3.8; this subsection is dedicated to proving the latter. Let Γ be a planar spin network equipped with a holonomy with values in the diagonal matrices of $\mathrm{SL}_2(\mathbb{C})$ and let us use the notation introduced in Subsection 3.3. Let Γ' be the graph obtained from Γ by blowing up the vertices; each curve $\gamma \subset \Gamma$ can be lifted in a natural way to one in Γ' (which we will keep calling γ) and the connection ψ may be lifted to ψ on Γ' so that the holonomy on a curve and its lift coincide (see Figure 3).
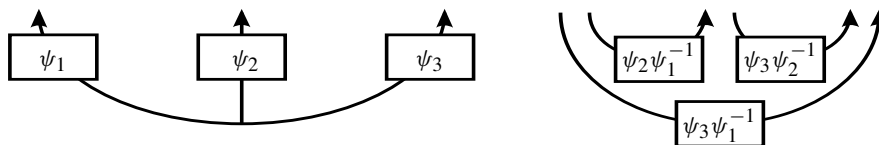

Fig. 3. Blowing up with holonomies.

For any half-edge h of Γ, the space F_h has a standard basis whose corresponding coordinates are z_h, w_h. We will say that a basis element has *type z or w*. Denote by W the matrix of $(1/i)(P + Q_\psi)$ in this basis. Observe that since ψ is diagonal, the coefficient $W_{i,j}$ does not vanish only if i, j correspond to adjacent vertices of Γ' with

distinct types. Recall that the set of half-edges is ordered and put first the type z basis, then the type w basis. Then W is block antidiagonal. Denote by W^1, W^2 the matrices indexed by H defined respectively by $W^1_{g,h} = W_{z_g,w_h}$ and $W^2_{g,h} = W_{w_g,z_h}$. We have $\det(W) = \det(W^1)\det(W^2)$.

Using the presentation of Γ' in $\mathbb{R}^2$ induced by that of Γ (Figure 2) we see that the edges corresponding to adjacent angles in Γ form an angle of 0 degrees (so a cusp) pointing upwards in Γ' (see Figure 3); we will call such edges *internal* and the other ones *external*.

Remark 3.6. If e is an edge of Γ', let l_e, r_e be the half-edges respectively at the left and right endpoint of e. If e is an external edge of Γ' oriented from left to right, then the entry of W_1 corresponding to e is $(-1)^{\text{wind}(e)}$, and that corresponding to e equipped with the opposite orientation is the opposite; indeed, both entries come from the matrix expressing $-iP$. Similarly, if α is an internal edge of Γ' oriented from left to right, then the entry of W_1 corresponding to α is $(t_{l_\alpha}^{-1} t_{r_\alpha}) X_\alpha$ and that corresponding to the opposite orientation is $-(t_{l_\alpha} t_{r_\alpha}^{-1}) X_\alpha$; both entries come from the matrix expressing $-iQ_\psi$.

Let us denote $\overline{t_h} = t_h^{-1}$. This extends to an automorphism of the ring of coefficients. We check that $W^2 = -\overline{W^1} = (W^1)^t$, hence $\det(W^2) = \overline{\det(W^1)} = \det(W^1)$ because there is an even number of half-edges, so, by Theorem 1.1, to prove Theorem 3.2 it is sufficient to interpret $\det(W^1)$ in terms of traces of curves. To this end, note that W^1 has 0 on the diagonal and we can apply Corollary 3.4 with $G = \Gamma'$ and $M = W^1$.

Now observe that if a vertex v of Γ' is contained in a curve c embedded in Γ' then either c contains an internal and an external edge containing v, or it contains two internal edges containing v; we will call the latter kind of vertices *cusps* of c; also a dimer covering an internal edge will be called *internal*.

Proposition 3.7. *Let $c \cup d$ be a configuration of curves and dimers containing each vertex of Γ' exactly once. Then:*

1. *There exists a unique set L of disjoint arcs in $\Gamma' \setminus c$ whose boundary vertices coincide with the set of cusps of c.*
2. *Let $d' = d \setminus L$. The set of internal dimers of d' can be joined by external edges of Γ' to form a unique embedded (possibly disconnected) curve $c' \subset \Gamma' \setminus (c \cup L)$.*

Proof. The first statement is proved by remarking that if e is an edge of Γ' whose endpoint is a cusp of a configuration, then its other endpoint can only be either another cusp (in which case set $l_i = e$), or contained in an internal dimer. By iterating this argument for the edge at the other endpoint of that internal dimer, one eventually constructs a path l_i which must end at another cusp of c. The second statement is proved by a similar inspection. □

There is a natural map $\pi : C(\Gamma) \to \text{Conf}(\Gamma')$: define $\pi(\gamma)$ as $c \cup d$ where c is the oriented curve in Γ' formed by the edges and angles of Γ contained in γ exactly once, and d is the dimer formed by all the angles contained twice in γ and all the edges not contained in γ.

Proposition 3.8. *The map π is a bijection, and letting $\pi(\gamma) = c \cup d$ we have*

$$(-1)^{\#c+\#d} w(c \cup d) = (-1)^{\mathrm{cr}(\gamma)} \mathrm{Tr}(\gamma) X^{\gamma}$$

where $\mathrm{cr}(\gamma)$ is the number of crossings of γ.

Proof. We construct the inverse map $s : \mathrm{Conf}(\Gamma') \to C(\Gamma)$ by associating to a configuration $c \cup d$ the unique oriented curve γ which passes once exactly over the edges contained in c (and with the same orientation as c), twice over all the edges contained in L (see Proposition 3.7 applied to $c \cup d$) and twice over all the edges contained in c' (in that case we consider the curve without crossings). This proves our first claim.

Set $\pi(\gamma) = c \cup d$. Then for any connected component δ of γ, we define $w(\delta) = \prod_{k=0}^{n} W^1_{h_k, h_{k+1}}$ where δ visits the half-edges $h_0, \dots, h_n, h_{n+1} = h_0$ in that order. We first claim that

$$(-1)^{\#c+\#d} w(c) w(d) = (-1)^{\#\gamma} w(\gamma).$$

We check this formula by looking first at the X_α terms, then the t_h terms and then the signs.

- The monomial in X_α occurring on the right hand side is X^γ, where each angle appears as many times as it is visited by γ. On the left hand side, each time an internal edge α is visited by c produces a monomial X_α, whereas each time a dimer occupies an internal edge X_α produces a monomial X_α^2. As internal edges occupied by a dimer are visited twice by γ, these monomials coincide.
- By Remark 3.6, if an internal edge is visited twice in opposite directions, the resulting monomial in t_h is equal to 1. Hence, monomials occur only for internal edges visited once. One sees that the sets of internal edges visited once by γ and once by $c \cup d$ coincide (with orientation), hence the resulting monomial is the same.
- By Remark 3.6, for any individual dimer d_0 one has $w(d_0) = -1$. Hence, $(-1)^{\#d} w(d) = 1$ (we drop here the X_α and t_h parts which have already been taken care of). In order to show $(-1)^{\#\gamma} w(\gamma) = (-1)^{\#c} w(c)$, we recall that to build γ from c, one needs to add the double arcs L and the double curve c'. For the latter, the weight is 1 because the number of curves is even and each curve goes through an even number of edges. Now, any added double arc contains exactly one more external edges than internal ones and thus produces a -1 factor. On the other hand, it either merges two components of c, or splits one component in two, and hence the formula is proven.

Let us show finally that $(-1)^{\#\gamma} w(\gamma) = (-1)^{\mathrm{cr}(\gamma)} \mathrm{Tr}(\gamma) X^\gamma$. We only have to check the sign, and one can suppose that γ is connected. Then one has to show that $\mathrm{sgn}\, w(\gamma) = (-1)^{\mathrm{cr}(\gamma)+1} = (-1)^{\mathrm{wind}(\gamma)}$ where $\mathrm{wind}(\gamma)$ is the winding number of γ (here we use the equality $\mathrm{wind}(\gamma) = \mathrm{cr}(\gamma) + 1 \bmod 2$).

The lift of γ to an oriented curve immersed in Γ' meets alternately an internal edge i and an external one e; by Remark 3.6, if both are covered in the same direction (e.g. from right to left) then the contribution to $w(\gamma)$ is $(-1)^{\mathrm{wind}(e)+1/2}$, and if not then the contribution is $(-1)^{\mathrm{wind}(e)-1/2}$. On the other hand, the tangent vector to γ rotates through either $\pm 2\pi(\mathrm{wind}(e) + 1/2)$ or $\pm 2\pi(\mathrm{wind}(e) - 1/2)$, depending on the same condition. Hence, $\mathrm{sgn}\, w(\gamma) = (-1)^{\mathrm{wind}(\gamma)}$, and thus the formula of the proposition is proved. $\qquad\square$

3.6. Recovering Westbury's theorem from Theorem 3.2

Theorem 1 can be recovered from Theorem 3.2 by remarking that if $t_h = 1$ for all $h \in H$ then the equalities $W^2 = -\overline{W^1}$ and $W^2 = (W^1)^t$ imply $(W^1)^t = -W^1$ and so $\det(W^1) = \text{Pfaff}(W^1)^2$. Hence we must show that

$$\text{Pfaff}(W^1) = \sum_{c \subset \Gamma} \prod_{\alpha \subset c} X_\alpha.$$

By Theorem 3.5 we have

$$\text{Pfaff}(W^1) = \sum_{d \in \text{DConf}(\Gamma')} \pm\sqrt{|w(d)|} = \sum_{d \in \text{DConf}(\Gamma')} \pm \prod_{\alpha \subset d} X_\alpha$$

and it is straightforward to check that for each $d \in \text{DConf}(\Gamma')$ the monomial $\sqrt{|w(d)|}$ equals $\pm w(c(d))$ for a unique (possibly empty or disconnected) curve $c(d)$ embedded in Γ; similarly for every $\gamma \subset \Gamma$ there exists exactly one $d \in \text{DConf}(\Gamma')$ such that $c(d) = \gamma$. So we have $\text{Pfaff}(W^1) = \sum_{\gamma \subset \Gamma} s(\gamma) \prod_{\alpha \subset \gamma} X_\alpha$ (with $s(\gamma) = \pm 1$) and it remains to show that $s(\gamma) = 1$ for all $\gamma \subset \Gamma$. Let us first remark that by Theorem 3.2 the coefficient of $w(\gamma_1 \sqcup \cdots \sqcup \gamma_k) = \prod_{\alpha \subset \gamma} X_\alpha$ in $\det(W^1)$ is 2^k. We now use the equality $\det(W^1) = \text{Pfaff}(W^1)^2$ and argue by induction on k. If $k = 1$ the coefficient of $w(\gamma)$ in $\det(W^1)$ is $2 = 2s(\gamma)$ and so $s(\gamma) = 1$. Now suppose $s(\gamma) = 1$ for all $\gamma \subset \Gamma$ such that $\#\gamma \le k$; then given $\gamma = \gamma_1 \sqcup \cdots \sqcup \gamma_{k+1}$, the coefficient of $w(\gamma)$ in $\det(W^1)$ is

$$2^{k+1} = 2s(\gamma) + 2 \sum_{\substack{I \subset \{1,\dots,k+1\} \\ 1 \le \#I \le (k+1)/2}} s\left(\bigsqcup_{i \in I} \gamma_i\right) s\left(\bigsqcup_{i \notin I} \gamma_i\right) = 2\left(s(\gamma) + \frac{1}{2}(2^{k+1} - 2)\right)$$

and so $s(\gamma) = 1$.

4. An integral formula for the square of a spin-network evaluation and its analysis

In this section we deduce an integral formula for the square of a spin-network evaluation. In the case of the tetrahedron, this formula was already known to Wigner [W59]. It was generalized by Barrett [BS03] to any graph. For reasons of clarity and normalizations, we provide a proof of it, then we derive a formula for the corresponding generating series. From now on we will no longer work in the supersymmetric category: since we are interested in squares of spin-network evaluations, sign matters are irrelevant.

4.1. Derivation of the integral formula

Let (V, ω, h) be a symplectic complex and Hermitian vector space of dimension 2. Both structures are supposed to be compatible in the sense that there is a Hermitian basis (e_1, e_2) of V such that $\omega(e_1, e_2) = 1$. We also suppose that the Hermitian product is antilinear on the left and denote by $SU(V)$ the symmetry group of the whole structure.

Naturally, $V^{\otimes n}$ has an Hermitian structure, as does its subspace V_n. Denote also by h the Hermitian product induced on the spaces V_n and their tensor products.

We would like to replace contractions with the forms ω_n by scalar products with some vectors. The compatibility of the Hermitian structure and the symplectic form immediately gives the formula $h(\omega^{-1}, v \otimes w) = \omega(w, v)$ or $h(\varpi, v \otimes w) = \omega(v, w)$ where we set $\varpi = -\omega^{-1}$. Let $\varpi_n = (\Pi_n \otimes \Pi_n) \bigotimes_{i=1}^{n} \varpi_{i,2n+1-i}$, where $\Pi_n : V^{\otimes n} \to V^{\otimes n}$ is the projector on symmetric tensors in the non-supersymmetric sense. After some computation, we obtain $h(\varpi_n, v \otimes w) = \omega_n(v, w)$ for any $v, w \in V_n$.

Then we can compute the spin-network evaluation using scalar product instead of contraction; more precisely, up to sign we have

$$\langle \Gamma, c \rangle = h\Big(\bigotimes_{e \in E} \varpi_{c_e}, \bigotimes_{v \in V, \, v:(i,j,k)} \varepsilon_{c_i, c_j, c_k} \Big). \tag{2}$$

Denote by G the group $\mathrm{SU}(V)$ and for any integer n, let $\rho_n : G \to \mathrm{U}(V_n)$ be the induced representation of G. We also write dg for the Haar measure on G satisfying $\int_G dg = 1$.

Consider the element

$$\mathcal{P} = \int_G \rho_n(g) \otimes \rho_n(g) \, dg \in \mathrm{End}(V_n \otimes V_n).$$

We check directly that $\mathcal{P} \circ \mathcal{P} = \mathcal{P}$ and $\mathcal{P}^* = \mathcal{P}$. Moreover, the image of $\mathcal{P}$ is the one-dimensional space $(V_n \otimes V_n)^G$. Hence, $\mathcal{P}$ is the orthogonal projector on $\mathbb{C}\varpi_n$. Using the formula $h(\varpi_n, v \otimes w) = \langle \omega_n, v \otimes w \rangle$ we have $h(\varpi_n, \varpi_n) = \langle \omega_n, \varpi_n \rangle$. But in the contraction of $\omega_n \otimes \varpi_n$, one can first contract the middle terms. One finds n times the evaluation of $\omega \otimes \varpi$, which is equal to $-\mathrm{Id}_{V^*}$. Thus, the middle contraction is $(-1)^n \mathrm{Id}_{V_n^*}$. The final contraction computes the super-trace. Hence, one finds $h(\varpi_n, \varpi_n) = n + 1$. Finally, we see that $\mathcal{P}(v) = \frac{1}{n+1} h(\varpi^n, v) \varpi^n$ for any $v \in V_n \otimes V_n$.

In the same spirit, for any triple (a, b, c) satisfying the triangular relations consider the element

$$\mathcal{E} = \int_G \rho_a(g) \otimes \rho_b(g) \otimes \rho_c(g) \, dg \in \mathrm{End}(V_a \otimes V_b \otimes V_c).$$

The same reasoning as above shows that for any $v \in V_a \otimes V_b \otimes V_c$ one has $\mathcal{E}(v) = \frac{h(\varepsilon_{a,b,c}, v)}{\langle a,b,c \rangle} v$ where we set $\langle a, b, c \rangle = h(\varepsilon_{a,b,c}, \varepsilon_{a,b,c})$.

Lemma 4.1.

$$\langle a, b, c \rangle = |\langle \Theta, (a, b, c) \rangle| = \frac{\left(\frac{a+b+c}{2} + 1\right)! \left(\frac{a+b-c}{2}\right)! \left(\frac{a-b+c}{2}\right)! \left(\frac{-a+b+c}{2}\right)!}{a! \, b! \, c!}.$$

Proof. A proof may be found for instance in [BL81] or deduced from the generating series given in [W98, Theorem] for the theta graph, taking care of the renormalization from $\langle\langle \Theta, (a, b, c) \rangle\rangle$ to $\langle \Theta, (a, b, c) \rangle$. $\square$

In equation (2), replace ε by $\mathcal{E}$ and ϖ by $\mathcal{P}$ and consider the scalar product $h(\bigotimes_{e \in E} \mathcal{P}_e, \bigotimes_{v \in V} \mathcal{E}_v)$; using the fact that all matrices involved are self-adjoint, one sees

that it equals $\mathrm{Tr}(\bigotimes_{e \in E} \mathcal{P}_e \circ \bigotimes_{v \in V} \mathcal{E}_v)$. For any edge e colored by c_e, let $\langle e \rangle = c_e + 1$ and $\mathcal{P}_e$ be the projector acting on $V_{c_e} \otimes V_{c_e}$. Similarly, for any vertex v whose incoming edges are i, j, k, we set $\langle v \rangle = \langle c_i, c_j, c_k \rangle$ and denote by $\mathcal{E}_v$ the projector acting on $V_{c_i} \otimes V_{c_j} \otimes V_{c_k}$. Let

$$[\Gamma, c] = \prod_e \langle e \rangle \, \mathrm{Tr}\left(\bigotimes_{e \in E} \mathcal{P}_e \circ \bigotimes_{v \in V} \mathcal{E}_v \right) = \prod_e \langle e \rangle h\left(\bigotimes_{e \in E} \mathcal{P}_e, \bigotimes_{v \in V} \mathcal{E}_v \right).$$

The first term in the scalar product defining $[\Gamma, c]$ is the orthogonal projector on $\bigotimes_e \varpi^{c_e}$, and the second term is the orthogonal projector on $\bigotimes_v \varepsilon_v$. Hence, we have

$$[\Gamma, c] = \frac{|\langle \Gamma, c \rangle|^2}{\prod_v \langle v \rangle}. \tag{3}$$

Here we use the fact that h induces an Hermitian form on $\mathrm{End}(W) = W \otimes W^*$ and that the scalar product of the projectors P_u and P_v on the unit vectors u and v is $|\langle u, v \rangle|^2$. On the other hand, we can interchange Hermitian product and integration, finding

$$[\Gamma, c] = \prod_e \langle e \rangle \int_{G^{V \sqcup E}} h\left(\bigotimes_e \rho_{c_e}(g_e) \otimes \rho_{c_e}(g_e), \bigotimes_{v:(i,j,k)} \rho_{c_i}(g_v) \otimes \rho_{c_j}(g_v) \otimes \rho_{c_k}(g_v) \right) dg$$

$$= \prod_e \langle e \rangle \int_{G^{V \sqcup E}} \prod_{h=(v,e)} \mathrm{Tr}_{c_e}(g_v g_e) \, dg = \int_{G^V} \prod_{e:(v,w)} \mathrm{Tr}_{c_e}(g_v g_w^{-1}) \, dg.$$

Note that the last equation holds even if c is a non-admissible coloring on Γ (in which case both sides are 0). In order to prove the last equality, we have used the following lemma:

Lemma 4.2. *For any integer c and all $A, B \in G$ one has*

$$\int_G \mathrm{Tr}_c(Ag) \, \mathrm{Tr}_c(Bg) \, dg = \frac{1}{c+1} \mathrm{Tr}_c(AB^{-1}).$$

Proof. We have

$$\int_G \mathrm{Tr}_c(Ag) \, \mathrm{Tr}_c(Bg) \, dg = \int_G \mathrm{Tr}_c(AB^{-1}g) \, \mathrm{Tr}_c(g) \, dg$$

$$= \mathrm{Tr}\, \rho_c \left(AB^{-1} \int_G \rho_c(g) \, \mathrm{Tr}_c(g) \, dg \right).$$

Writing $U = \int_G \rho_c(g) \, \mathrm{Tr}_c(g) \, dg$ one finds that U commutes with G, hence U is proportional to the identity. Moreover, $\mathrm{Tr}(U) = \int_G \mathrm{Tr}_c(g)^2 \, dg = 1$, hence $U = \frac{1}{c+1}\mathrm{Id}_{V_c}$, which proves the lemma. $\qquad \square$

In summary we have obtained the following formula, noting that $\langle \Gamma, c \rangle$ is real:

$$\langle \Gamma, c \rangle^2 = \prod_v \langle v \rangle \int_{G^V} \prod_{e:(v,w)} \mathrm{Tr}_{c_e}(g_v g_w^{-1}) \, dg = \prod_v \langle v \rangle [\Gamma, c].$$

Notice that to write this formula we need to orient the edges of Γ arbitrarily; this detail will be important in the next section.

Let $R = \mathbb{C}[[Y_e, e \in E]]$ be the ring of formal series in variables associated to the edges. For any coloring c, we denote by Y^c the monomial $\prod_{e \in E} Y_e^{c_e}$. For any A in G, the following identity holds:

$$\sum_{n \in \mathbb{N}} \mathrm{Tr}_n(A) Y^n = \frac{1}{\det(\mathrm{Id} - YA)}.$$

Consider then $W(\Gamma) = \sum_c [\Gamma, c] Y^c$ where the sum is taken also over non-admissible colorings by setting $[\Gamma, c] = 0$ in that case. We deduce that

$$W(\Gamma) = \sum_{c \text{ admissible}} \frac{\langle \Gamma, c \rangle^2 Y^c}{\prod_v \langle v \rangle} = \int_{G^V} \frac{dg}{\prod_{e:(v,w)} \det(\mathrm{Id} - Y_e g_v g_w^{-1})}. \tag{4}$$

Example 4.3. Consider the case of the graph $\ominus$. Then for any coloring c given by three colors a, b, c, one has by construction $\langle \Theta, a, b, c \rangle = \langle a, b, c \rangle$, and hence the generating series (4) is simply

$$\sum_{c \text{ admissible}} Y^c = \frac{1}{(1 - Y_1 Y_2)(1 - Y_2 Y_3)(1 - Y_1 Y_3)} = \int_G \frac{dg}{\prod_{i=1}^3 \det(\mathrm{Id} - Y_i g)}.$$

The coefficient of $Y_1^a Y_2^b Y_3^c$ in the above integral is $\int_G \mathrm{Tr}_a(g) \mathrm{Tr}_b(g) \mathrm{Tr}_c(g) \, dg$. Orthonormality of characters in $L^2(G, dg)$ and the Clebsch–Gordan rules imply that this integral is 1 if a, b, c are admissible and 0 if not.

We can generalize the formula to the holonomy case. Fix a holonomy $\psi : H \to G$. Then, at each vertex v, one has to replace the tensor $\varepsilon_{c_i, c_j, c_k}$ with $\mathrm{Hol}_\psi(\varepsilon_{c_i, c_j, c_k}) = \rho_{c_i}(\psi_{i,v}) \otimes \rho_{c_j}(\psi_{j,v}) \otimes \rho_{c_k}(\psi_{k,v}) \varepsilon_{c_i, c_j, c_k}$. Consequently, in the computation above, we need to replace the projector $\mathcal{E}$ with the projector on $\mathrm{Hol}_\psi(\varepsilon_{c_i, c_j, c_k})$, that is, $\mathrm{Hol}_\psi \mathcal{E} \mathrm{Hol}_\psi^{-1}$. Therefore we have

$$[\Gamma, c, \psi] = \prod_e \langle e \rangle \int_{G^V \sqcup E} \prod_{h = (v,e)} \mathrm{Tr}_{c_e}(g_e \psi_h g_v \psi_h^{-1}) \, dg$$

$$= \int_{G^V} \prod_{e:(v,w)} \mathrm{Tr}_{c_e}(\psi_{e,v} g_v \psi_{e,v}^{-1} \psi_{e,w} g_w^{-1} \psi_{e,w}^{-1}) \, dg. \tag{5}$$

Since as before the above equalities still hold if c is not admissible (all terms are 0), defining $W(\Gamma, \psi)$ as in (4), the following holds:

Theorem 4.4. *Let (Γ, ψ) be a trivalent graph equipped with a connection with values in* $\mathrm{SL}_2(\mathbb{C})$, *and let $R_Y = \mathbb{C}[[Y_e, e \in E]]$. Then the following equality holds in R_Y:*

$$W(\Gamma, \psi) = \int_{G^V} \frac{dg}{\prod_{e:(v,w)} \det(\mathrm{Id} - \psi_{e,v} g_v \psi_{e,v}^{-1} \psi_{e,w} g_w^{-1} \psi_{e,w}^{-1} Y_e)}.$$

4.2. Comparison between generating series

Let Γ be a trivalent graph. We denote by $\overline{R}_X$ be the subalgebra of R_X linearly generated by X^c for all admissible colorings c, and by $\overline{R}_Y$ the subalgebra of R_Y linearly generated by Y^c for all admissible colorings c. We define a morphism (antilinear on the left) $(\cdot, \cdot)$: $\overline{R}_X \times \overline{R}_X \to \overline{R}_Y$ in the following way: if c and c' are distinct then $(X^c, X^{c'}) = 0$; if not, set

$$(X^c, X^c) = \frac{Y^c}{\prod_{v:(e_1,e_2,e_3)} \langle\!\langle \Theta, c_{e_1}, c_{e_2}, c_{e_3} \rangle\!\rangle}.$$

Summing over all colorings we find

$$(Z(\Gamma, \psi), Z(\Gamma, \psi)) = W(\Gamma, \psi). \tag{6}$$

It would be interesting to interpret this formula at the level of generating series for which we should interpret the product $(\cdot, \cdot)$ as an Hermitian product over $\overline{R}_X$ given by some integral formula. We do not address this question here.

We end this section with some remarks on the orthogonality of spin networks as functions of the holonomy. Let $\mathcal{M}$ be the moduli space of holonomies on Γ with values in $G = \mathrm{SU}_2$. It is equal to the quotient $G^H / G^{V \sqcup E}$; the Haar measure on G^H produces a measure μ on $\mathcal{M}$.

The map from $\mathcal{M}$ to $\mathbb{C}$ sending ψ to $\langle \Gamma, c, \psi \rangle$ is a polynomial function and it is well-known that the family $\langle \Gamma, c, \cdot \rangle$ forms an orthogonal basis of $L^2(\mathcal{M}, \mu)$. Equation (3) relating $\langle \Gamma, c \rangle$ and $[\Gamma, c]$ generalizes directly to the case with holonomies. We can use it to compute the normalization of the basis:

$$\int_{\mathcal{M}} |\langle \Gamma, c, \psi \rangle|^2 \, d\mu(\psi) = \prod_v \langle v \rangle \prod_e \langle e \rangle \int_{\mathcal{M}} \prod_{h=(v,e)} \mathrm{Tr}_{c_e}(g_e \psi_h g_v \psi_h^{-1}) \, d\mu(h).$$

We have $\int_G \mathrm{Tr}_c(AgBg^{-1}) \, dg = \frac{1}{c+1} \mathrm{Tr}_c(A) \mathrm{Tr}_c(B)$ by an argument similar to Lemma 4.2. If we apply this formula to each term of the product, the term $\prod_e \langle e \rangle$ gets inverted. By the orthonormality of characters and the Clebsch–Gordan rule, we find $\int_{\mathcal{M}} |\langle \Gamma, c, \psi \rangle|^2 \, d\mu(\psi) = \prod_v \langle v \rangle \prod_e \langle e \rangle^{-1}$.

5. Asymptotics of spin-network evaluations

In this section, we compute the first order asymptotic behavior of $[\Gamma, kc, 1]$ (the square of the spin-network evaluation divided by $\prod_v \langle v \rangle$), for general Γ. In Subsection 5.1, we transform the integral formula to adapt it to stationary phase approximation and describe the critical set of the integrand. Then we discuss some sufficient non-degeneracy hypotheses (Subsection 5.2). The computation of the Hessian occupies Subsection 5.3, in Subsection 5.4 we apply the stationary phase method, and Subsection 5.4.3 ends the proof. Subsection 5.5 discusses an example of application of our formula to the case of the tetrahedron.

5.1. Description of the critical set

From now on, to simplify the notation, we will write $h(v, w) = \langle v, w \rangle$ for any $v, w \in V$. Denote by S^3 the unit sphere of V, and by dv the Haar measure on it (i.e. $\int_{S^3} 1 \, dv = 1$).

Lemma 5.1. *For any $g \in G$, $\mathrm{Tr}_n(g) = (n + 1) \int_{S^3} \langle v, \rho(g)v \rangle^n \, dv$.*

Proof. Recall that $\bigoplus_n V_n$ is the algebra of polynomials on V^* and that G acts on it by algebra morphisms. For any $v \in S^3$, we have $\langle v^n, w^n \rangle = \langle v, w \rangle^n$. Consider the endomorphism U of V_n given by $U = \int_{S^3} v^n \langle v^n, \cdot \rangle \, dv$. Since U clearly commutes with the action of G, it is proportional to Id_{V_n}. As $\mathrm{Tr}(v^n \langle v^n, \cdot \rangle) = 1$, we find $\mathrm{Tr}(U) = 1$, hence $U = \frac{1}{n+1} \mathrm{Id}_{V_n}$. Now, given any $g \in G$, we have

$$\mathrm{Tr}_n(g) = \mathrm{Tr}\,\rho_n(g) = (n + 1)\,\mathrm{Tr}\,\rho_n(g)U = (n + 1) \int_{S^3} \langle v^n, \rho(g)v^n \rangle \, dv$$

$$= (n + 1) \int_{S^3} \langle v, \rho(g)v \rangle^n \, dv. \qquad \square$$

We deduce the following formula:

$$[\Gamma, c] = \prod_e \langle e \rangle \int_{G^V} \int_{(S^3)^E} \prod_{e:v \to w} \langle g_v u_e, g_w u_e \rangle^{c_e} \, dg \, du. \tag{7}$$

The notation $e : v \to w$ means that e is an oriented edge joining v to w; and u_e is an element of S^3 corresponding to the edge e.

In order to analyze this integral, we restrict the integration domain to a subset X of $G^V \times (S^3)^E$ such that the integrand does not vanish and symmetries behave nicely; we will see that this restriction does not affect the asymptotic analysis. More precisely, let $\pi : S^3 \to S^2$ be the Hopf fibration. The sphere S^2 is either identified to the projective space $\mathbb{P}(V)$ or to the unit sphere of the Lie algebra $\mathcal{G}$ of G (equipped with the scalar product $|\xi|^2 = -\frac{1}{2} \mathrm{Tr}(\xi^2)$). Set

$$X = \{(g_v, u_e) \in G^V \times (S^3)^E \mid \forall e : v \to w, \langle g_v u_e, g_w u_e \rangle \neq 0$$
$$\text{and the family } (\pi(u_e))_{e \in E} \text{ has rank at least 2 in } \mathcal{G}\}.$$

Let $F : X \to \mathbb{C}/2i\pi\mathbb{Z}$ be the map

$$F(g, u) = \sum_{e:v \to w} c_e \ln \langle g_v u_e, g_w u_e \rangle$$

and write $P_e = \pi(u_e) \in \mathcal{G}$ if the orientation of e coincides with the chosen one and $P_e = -\pi(u_e)$ if not. We have the following proposition:

Proposition 5.2. *The critical points of F are the elements $(g, u) \in X$ satisfying:*

1. *For all edges $e : v \to w$, $g_v u_e = \tau_e g_w u_e$ for some $\tau_e \in S^1$.*
2. *For all vertices v with outgoing edges e_1, e_2, e_3, one has $c_{e_1} P_{e_1} + c_{e_2} P_{e_2} + c_{e_3} P_{e_3} = 0$.*

Proof. Differentiating F with respect to the variable u_e in the direction v_e, we get

$$DF(v_e) = \frac{c_e}{\langle g_v u_e, g_w u_e \rangle}(\langle g_v v_e, g_w u_e \rangle + \langle g_v u_e, g_w v_e \rangle).$$

Denote by $u_e^\perp$ the element of S^3 satisfying $\langle u_e, u_e^\perp \rangle = 0$ and $\det(u_e, u_e^\perp) = 1$. Taking $v_e = u_e^\perp$ and $v_e = iu_e^\perp$, we find that $g_v^{-1} g_w$ is diagonal in the basis $(u_e, u_e^\perp)$. This is equivalent to the first item of the proposition.

Consider an element ξ in $\mathcal{G}$ and replace g_v by $g_v e^{t\xi}$ for some $t > 0$. Then compute the derivative of F at $t = 0$. Three terms contribute to the sum, corresponding to the edges incoming to v. Suppose that they are oriented from v to v_1, v_2, v_3 by edges denoted by e_1, e_2, e_3 respectively. For simplicity, we write $g = g_v$, $g_i = g_{v_i}$, $u_i = u_{e_i}$ and $c_i = c_{e_i}$.

We have $F'(0) = \sum_{i=1}^3 c_i \frac{\langle g\xi u_i, g_i u_i \rangle}{\langle gu_i, g_i u_i \rangle} = \sum_{i=1}^3 c_i \langle \xi u_i, u_i \rangle$. The last equality comes from the first item. Let Π_i be the element of $\mathrm{End}(V)$ acting by i on u_i and $-i$ on $u_i^\perp$. Then, from $\mathrm{Tr}(\xi \Pi_i) = 2i \langle u_i, \xi u_i \rangle$ we see that $\sum_i c_i \Pi_i$ is 0. Using the standard identification between anti-Hermitian operators and $\mathcal{G}$ which sends Π_i to P_i, we obtain the second item of the proposition. $\square$

Suppose that the coloring c satisfies the strict triangle inequalities (T); then at any vertex, the triple $P_{e_1}, P_{e_2}, P_{e_3}$ has rank 2. In particular, the corresponding pair (g_v, u_e) is in X. We will make this assumption in the following.

The integral presents some obvious symmetries: the integrand depends on u_e through its image $\pi(u_e) \in S^2$. Hence, we can replace the integral over $(S^3)^E$ by an integral over $(S^2)^E$ integrating over the fiber $(S^1)^E$ of the Hopf fibration. Moreover replacing (g_v, u_e) by (gg_v, u_e) does not change the integral, nor does replacing (g_v, u_e) by $(g_v g^{-1}, gu_e)$. Hence, the integral can be performed over $Y = X/(S^1)^E \times G \times G$ where (α_e, g, h) acts on (g_v, u_e) by $(gg_v h^{-1}, h(\alpha_e u_e))$. Notice that the stabilizer of the action of $(S^1)^E \times G \times G$ on X is $\{\pm 1\}$, hence the quotient Y is a smooth manifold of dimension $12N - 6$. Let $\tilde{F} : Y \to \mathbb{C}/2i\pi\mathbb{Z}$ be the induced map.

We denote by $I \subset \mathcal{G}^E/G$ the isometry classes of tuples $(P_e)_{e\in E}$ satisfying the equation (2) of Proposition 5.2. They encode the critical points of $\tilde{F}$ in the following way:

Proposition 5.3. *The set C of critical points of $\tilde{F}$ are in bijection with the set of triples $(P, Q, (g_v))$ in $I \times I \times (G^V/\{\pm 1\})$ where for all half-edges (e, v) we have $g_v P_e = Q_e$. Moreover, the map $C \to I \times I$ which forgets the third term is surjective and its fibers have cardinality 2^{2N-1}.*

We remark that if $N > 1$, then I is never reduced to one element, because if P belongs to I, then $-P$ is an element of I distinct from P.

Proof. Let (g, u) be a critical point of F. Then the family $P_e = \pi(u_e)$ belongs to I. For any half-edge (e, v), set $Q_e = \pi(g_v u_e) = \pi(g_w u_e)$. For any vertex v with incoming edges e_1, e_2, e_3, the relation $\sum_i c_{e_i} P_{e_i} = 0$ implies $\sum_i c_{e_i} Q_{e_i} = 0$, hence (Q_e) also belongs to I. Conversely, given two families $(P_e), (Q_e)$ in I, we choose vectors $u_e, s_e \in S^3$

satisfying $\pi(u_e) = P_e$ and $\pi(s_e) = Q_e$. Then for every vertex v surrounded by e_1, e_2, e_3, we need to find $g_v \in G$ such that for all i, $g_v u_{e_i}$ is proportional to s_{e_i}. As we required that the coefficients c_{e_i} satisfy the strict triangular inequalities, the pair P_{e_1}, P_{e_2} is linearly independent in $\mathcal{G}$. As Q_{e_1}, Q_{e_2} have the same lengths and the same angle by hypothesis, there is a unique rotation of $\mathcal{G}$ which sends P_{e_i} to Q_{e_i} for all i. This rotation lifts in two possible ways to G, giving our choices for g_v. Note that the action of $\{\pm 1\} \in G \times G$ divides by 2 the number of critical points. $\qquad\square$

5.2. Non-degeneracy hypotheses

We would like to understand which conditions ensure that the critical points of $\tilde{F}$ are isolated in Y. At first, we concentrate on the family (P_e) of vectors associated to critical points.

Let $\mathcal{E}$ be the set of maps from E to $\mathcal{G}$. We consider it as a Euclidean space where the scalar product is induced from the scalar product $|\xi|^2 = -\frac{1}{2}\,\mathrm{Tr}(\xi^2)$ on $\mathcal{G}$. This space will then be identified alternatively as the chain or cochain complex of Γ with coefficients in $\mathcal{G}$. As edges of E are oriented, there is a boundary operator $\partial : \mathcal{E} \to C_0(\Gamma, \mathcal{G})$ and we set $\mathcal{H} = \ker \partial$. We see that any critical point in X produces an element ζ of $\mathcal{H}$ by setting $\zeta_e = c_e P_e$. Conversely, any element ζ of $\mathcal{H}$ with $|\zeta_e| = c_e$ produces an element of I (see Proposition 5.3).

A tangent vector to ζ is an element $\zeta' \in \mathcal{H}$ such that $\langle \zeta_e, \zeta'_e \rangle = 0$ for all e (because $|\zeta_e| = c_e$). We interpret this formula as the scalar product of $(\zeta_e \delta_e)$ and ζ'. On the other hand, a tangent vector correspond to a global symmetry if and only if there exists $\xi \in \mathcal{G}$ such that $\zeta'_e = \xi \times \zeta_e$ for all e. We count that the dimension of $\mathcal{H}$ is $3(N + 1)$: the number of conditions imposed by ζ is $3N$ (one for each edge), whereas the symmetry gives three dimensions (because the family ζ_e has rank at least 2). Hence, an element of $\mathcal{G}^E/G$ associated to P is isolated if the family $(P_e \delta_e)$ projected orthogonally on $\mathcal{H}$ is linearly independent. With this interpretation in mind, we make the following assumption:

(H1) Let (c_e) be a coloring of Γ such that all triangle inequalities in (T) are strict. We suppose that all elements $\zeta \in \mathcal{H}$ such that $|\zeta_e| = c_e$ satisfy the non-degeneracy condition that the family $(\Pi_{\mathcal{H}}(\zeta_e \delta_e))_{e \in E}$ is linearly independent, where $\Pi_{\mathcal{H}} : \mathcal{E} \to \mathcal{H}$ is the orthogonal projector on $\mathcal{H}$. This implies that I is a finite set.

Remark 5.4. The preceding hypothesis is equivalent to saying that for any (ξ_v) in $C_0(\Gamma, \mathcal{G})$, if $P_e \times (\xi_w - \xi_v) = 0$ for all edges $e : v \to w$ then all the ξ_v are equal. Let us show this:

The image of $d : C_0(\Gamma, \mathcal{G}) \to \mathcal{E}$ is orthogonal to the space $\mathcal{H}$. Let $\mathcal{U}$ be the subspace $\mathcal{U} = \bigoplus_e \mathbb{R} P_e \subset \mathcal{E}$. The non-degeneracy assumption is that the orthogonal projection of $\mathcal{U}$ on $\mathcal{H}$ is injective. Hence, by standard linear algebra, the orthogonal projection of $\mathcal{U}^\perp$ on $\mathcal{H}^\perp$ is surjective. This implies that if some vector ζ_e in $\mathcal{H}^\perp$ satisfies $P_e \times \zeta_e = 0$ for all e then $\zeta_e = 0$.

For any critical point x, one can define the *phase function* $\tau^x : E \to S^1$ in the following way. Set $x = (g_v, u_e)$. Then, by the first item of Proposition 5.2, $g_v u_e$ and $g_w u_e$ are

proportional unit vectors, moreover the phase factor $\tau_e^x = \langle g_v u_e, g_w u_e \rangle$ depends only on the class of the critical point x in Y.

Remark 5.5. The proof of Proposition 5.3 gives a way to compute up to sign the phase factor τ_e associated to an oriented edge e and two configurations $P, Q \in I$. Indeed, a configuration P allows us to define an *angle at* $e : v \to w$ denoted by $\alpha_P(e_1)$ as the interior dihedral angle formed by the triangles adjacent to $c_e P_e$, namely those formed by $c_e P_e, c_{e_2} P_{e_2}, c_{e_3} P_{e_3}$ and $c_e P_e, c_{e_4} P_{e_4}, c_{e_5} P_{e_5}$ with $e_2 \cap e = e_3 \cap e = \{v\}$ and $e_4 \cap e = e_5 \cap e = \{w\}$. Then applying the ideas of the proof in the case when $g_v = \mathrm{Id}$ (which can be achieved by acting via $1^E \times \{g_v^{-1}\} \times \mathrm{Id}$ on the critical points of F), one sees that $\tau_e = \pm \exp\left(i \frac{\alpha_Q(e) - \alpha_P(e)}{2}\right)$.

We remark that the generalized phase of a critical point indexed by $(P, Q, g_v) \in I \times I \times G^V$ depends on g_v only up to a sign and is necessarily equal to ± 1 if $P = Q$. We assume that this is the only case when the phase function takes the value ± 1:

(H2) For any distinct $(P, Q) \in I$, the phase function of the associated critical points does not take the value ± 1.

Let P, Q be distinct elements of I, and τ be the associated phase function. Write $\tau_e = e^{i\theta_e}$. Then θ_e is well-defined modulo π. Consider the following quadratic form on $\mathcal{G}^V$:

$$q = \sum_e 2c_e \left(-i \cot(\theta_e) \| Q_e \times (\xi_v - \xi_w) \|^2 + i \langle Q_e, \xi_v \times \xi_w \rangle \right).$$

We assume

(H3) For any distinct $(P, Q) \in I$, the quadratic form q has corank 6.

Let us develop an example which is the main motivation of this section:

Example 5.6. Suppose that Γ is a planar graph and $I = \{P, -P\}$. Then there is a unique (up to isometry) polyhedron $\Delta \subset \mathcal{G}$ whose 1-skeleton is dual to Γ (hence whose faces are triangles) and such that for any oriented edge e of Γ the dual edge in Δ is vectorially equal to $c_e P_e$; in particular, it has length c_e.

The non-degeneracy condition (H1) is equivalent to an infinitesimal rigidity condition on Δ. For instance, if Δ is convex, this condition is automatically satisfied by Cauchy's theorem (see for instance [AZ10, Chapter 13]). The generalized phase function has the following nice interpretation. Let (g_v, u_e) be a critical point associated to the pair $(P, -P)$. This means that $\pi(u_e) = P_e$ and $\pi(g_v u_e) = -P_e$. Let v be a vertex of Γ, dual to a face F_v of Δ. Suppose that v has three outgoing edges e_1, e_2, e_3. Then g_v lifts the unique rotation mapping P_{e_i} to $-P_{e_i}$, that is, the rotation through π in the plane supporting F_v. Now, given an edge e, one has $\tau_e = \langle g_v u_e, g_w u_e \rangle = \pm e^{i\theta}$ where 2θ is the angle of the rotation $g_v^{-1} g_w$ around P_e. Hence, θ is the angle modulo π between the faces F_v and F_w. In particular, if Δ is a non-degenerate polyhedron, its corresponding graph satisfies (H2). We do not know which geometric condition on Δ would imply that (H3) is valid.

5.3. Second variation formula

Given a point $(g_v, u_e) \in X$, we can build a a coordinate system around it using the variables $(\xi_v) \in \mathcal{G}^V$, $(\lambda_e) \in \mathbb{C}^E$ and $\alpha_e \in (S^1)^E$. The parametrization is given by $(e^{\xi_v} g_v, \alpha_e u_e(\lambda_e))$ where for any $u \in S^3$ and $\lambda \in \mathbb{C}$ we set

$$u(\lambda) = \frac{1}{\sqrt{1 + |\lambda|^2}}(u + \lambda u^\perp) = (1 - |\lambda|^2/2)u + \lambda u^\perp + o(|\lambda|^2).$$

Let $s_v^e = g_v u_e$ and notice that $g_v u_e(\lambda_e) = s_v^e(\lambda_e)$. Let $i x_v^e = \langle s_v^e, \xi_v s_v^e \rangle$, $\mu_v^e = \langle s_v^{e\perp}, \xi_v s_v^e \rangle$, $\tau_e = \langle s_v^e, s_w^e \rangle$. In what follows, the values of g_v, u_e, s_v^e are to be considered as fixed and ξ_v, λ_e (and consequently those of x_v^e, μ_v^e) as variable. Notice that $x_v^e \in \mathbb{R}$, that τ_e was already defined in Proposition 5.2, and that $F : X \to \mathbb{R}$ does not depend on the coordinates $\{\alpha_e\}$, so we set them to 1 in the following:

Proposition 5.7. *The Taylor expansion of F in a neighborhood of a point $(g_v, u_e) \in X$ is, up to the second order,*

$$F(e^{\xi_v} g_v, u_e(\lambda_e)) = \sum_e c_e \ln \tau_e + \sum_e c_e i(x_w^e - x_v^e) + \sum_e c_e \frac{\overline{\tau_e}}{\tau_e}(\overline{\mu_v^e} + \overline{\lambda_e})(\mu_w^e + \lambda_e)$$

$$- \sum_e c_e(|\lambda_e|^2 + |\mu_v^e|^2/2 + |\mu_w^e|^2/2 + \overline{\lambda_e}\mu_v^e + \lambda_e\overline{\mu_w^e})$$

$$+ o\left(\sum_v |\xi_v|^2 + \sum_e |\lambda_e|^2\right).$$

In particular, at critical points, $\sum_e i c_e(x_v^e - x_w^e) = 0$ and $\overline{\tau_e}/\tau_e = \tau_e^{-2}$.

Proof. Notice that in the basis $(s_v^e, (s_v^e)^\perp)$ we have

$$\xi_v = \begin{pmatrix} i x_v^e & -\overline{\mu_v^e} \\ \mu_v^e & -i x_v^e \end{pmatrix};$$

moreover since $\langle s_v^e, s_w^e \rangle = \tau_e$, it follows that $\langle (s_v^e)^\perp, (s_w^e)^\perp \rangle = \overline{\tau_e} (= \tau_e^{-1}$ at critical points by Proposition 5.2). To compute the first terms of $F(e^{\xi_v} g_v, u_e(\lambda_e)) = \sum_{e: v \to w} c_e \ln \langle e^{\xi_v} s_v^e(\lambda_e), e^{\xi_w} s_w^e(\lambda_e) \rangle$, we first compute

$$e^{\xi_v} s_v^e(\lambda_e) = \begin{pmatrix} 1 + i x_v^e - |\xi_v|^2/2 & -\overline{\mu_v^e} \\ \mu_v^e & 1 - i x_v^e - |\xi_v|^2/2 \end{pmatrix} \begin{pmatrix} 1 - |\lambda_e|^2/2 \\ \lambda_e \end{pmatrix} + \text{h.o.t.}$$

Then we get

$$\langle e^{\xi_v} s_v^e(\lambda_e), e^{\xi_w} s_w^e(\lambda_e) \rangle$$
$$= \tau_e(1 - i x_v^e + i x_w^e + x_v^e x_w^e - |\lambda_e|^2 - |\xi_v|^2/2 - |\xi_w|^2/2 - \mu_v^e \overline{\lambda_e} - \overline{\mu_w^e}\lambda_e)$$
$$+ \overline{\tau_e}(\overline{\mu_v^e} + \overline{\lambda_e})(\mu_w^e + \lambda_e) + \text{h.o.t.}$$

Taking the logarithm, expanding it up to order 2 terms, and recalling that $|\xi_v|^2 = -\frac{1}{2}\mathrm{Tr}(\xi_v^2) = (x_v^e)^2 + |\mu_v^e|^2$ we get the formula of the proposition. The last statement is a consequence of Proposition 5.2. $\square$

To obtain a nicer formula, in the case of a critical point, we introduce the variables $z_v^e = \mu_v^e$ and $z_w^e = \tau_e^{-2}\mu_w^e$. After some computation, by Proposition 5.7, one finds that at critical points we have $\mathrm{Hess}(F)_x(\xi, \lambda) = -q(\xi, \lambda)$ where

$$
\begin{aligned}
q(\xi, \lambda) &= \sum_e c_e \big(2(1 - \tau_e^{-2})|\lambda_e|^2 + 2\lambda_e \tau_e^{-2}(\overline{z_w^e} - \overline{z_v^e}) + 2\overline{\lambda_e}(z_v^e - z_w^e) \\
&\qquad + |z_v - z_w|^2 + z_v\overline{z_w} - z_w\overline{z_v} \big) \\
&= \sum_e c_e \Bigg(2(1 - \tau_e^{-2}) \bigg| \lambda_e - \frac{z_w^e - z_v^e}{1 - \tau_e^{-2}} \bigg|^2 \\
&\qquad + \frac{\tau_e + \tau_e^{-1}}{\tau_e - \tau_e^{-1}} |z_w^e - z_v^e|^2 + z_v^e\overline{z_w^e} - z_w^e\overline{z_v^e} \Bigg) \quad \text{if } \tau_e^2 \neq 1 \; \forall e \in E. \quad (8)
\end{aligned}
$$

5.4. Applying the stationary phase method

Let us now perform the stationary phase approximation when replacing the coloring c by kc and letting k go to infinity in $\int_Y \exp(k\tilde{F})\, d\mu$. In this formula, the measure μ is obtained from the Haar measure on $(S^3)^{2N} \times (S^3)^{3N}$ by integration over the action of $(S^1)^E \times G \times G/\{\pm 1\}$, equipped with its Haar density. In this subsection as well as in the following one, we rely on the notation and machinery recalled in Appendix B.

By assumption, any critical point x is isolated in Y. Provided that $\mathrm{Hess}(\tilde{F})_x$ is non-degenerate, we can apply the stationary phase expansion theorem [H83, Theorem 7.7.5] to $\tilde{F}$, which is a smooth function with non-positive real part, and find that the local contribution of x to $[\Gamma, kc]$ when $k \to \infty$ is

$$
I(x) = \prod_e \langle e \rangle e^{k\tilde{F}(x)} \left(\frac{(2\pi)^{12N-6}}{\det(-k\,\mathrm{Hess}(\tilde{F})_x, \mu)} \right)^{1/2} = \prod_e \langle e \rangle e^{k\tilde{F}(x)} I(kq). \quad (9)
$$

The proof of Theorem 1.3 then consists in computing, for any critical point, the Gaussian integral $I(q)$ where q is the opposite of the Hessian computed in the preceding subsection (if there are no critical points then the stationary phase expansion theorem yields an exponential decay of the integral). We observe that in equation (8) this quadratic form is expressed in terms of $\{\lambda_e, z_v^e, \overline{z_v^e}\}$ and that its restriction to $\{z_v^e = z_w^e = 0\}$ is diagonal. Thus we would like to perform a partial integration on the λ_e terms which form local coordinates for the $(S^2)^E$-part of Y. There are two different cases to handle. In the first case one has $\tau_e^2 = 1$ for all e in E. The quadratic form q is then degenerate with respect to the λ coordinates and we deform it with a parameter κ. In the second case, one has $\tau_e^2 \neq 1$ for all e, thanks to (H2). Fix a critical point described by a pair $(P, Q) \in I \times I$ (as in Proposition 5.3); to avoid cumbersome notation, in what follows we will suppress the indices P, Q except in the definition of the quadratic forms used in the statement of Theorem 1.3.

5.4.1. Deformation and degeneracies. The kernel of q contains the image of $\mathcal{G} \times \mathcal{G}$. Let ξ and η be the two variables acting on (g_v, u_e) by $(e^\xi g_v e^{-\eta}, e^\eta u_u)$. For an edge $e : v \to w$ set $p_e = \langle u_e^\perp, \eta u_e \rangle$ and $q_e = \langle s_v^{e\perp}, \xi s_v^e \rangle$. The infinitesimal action of ξ and η in the tangent space at (g_v, u_e) is $(\xi - g_v \eta g_v^{-1}, \eta)$. One computes this action in terms of p_e and q_e as $\lambda_e = p_e$, $z_v^e = \mu_v^e = \langle s_v^{e\perp}, (\xi - g_v \eta g_v^{-1}) s_v^e \rangle = q_e - p_e$ and $z_w^e = q_e - \tau_e^{-2} p_e$.

Consider the deformation q_κ of q obtained by replacing τ_e by $\tau_{e\kappa} = \kappa \tau_e$. One recovers the original one by letting $\kappa \in (1, \infty)$ go to 1. After some computations one may check that $q_\kappa|_{\mathcal{G} \times \mathcal{G}} = 4(\kappa - 1) \sum_e c_e |p_e|^2 + o(\kappa - 1)$. Hence, denoting by r_P the quadratic form on $\mathcal{G}$ given by $r_P(\eta) = \sum_e c_e |p_e|^2$ we can remove the indeterminacy in η as explained in Appendix B. We have

$$I(q) = \lim_{\kappa \to 1} \frac{I(q_\kappa)}{I(q_\kappa|_{\mathcal{G} \times \mathcal{G}})} = \lim_{\kappa \to 1} \frac{(4\kappa - 4)^{3/2} I(q_\kappa)}{I(r_P)}.$$

5.4.2. Partial integration. We now remark that this deformation also allows us to integrate by part over the coordinates (λ_e). More precisely, let σ_κ be the restriction of q_κ to the λ-coordinates. Setting $\sigma_\kappa = \sum_e 2c_e(1 - \tau_{e\kappa}^{-2})|\lambda_e|^2$, one obtains $I(q_\kappa) = I(\sigma_\kappa) I(q^\kappa)$ where

$$q^\kappa = \sum_e c_e \left(\frac{\tau_{e\kappa} + \tau_{e\kappa}^{-1}}{\tau_{e\kappa} - \tau_{e\kappa}^{-1}} |z_v^e - z_w^e|^2 + z_v^e \overline{z_w^e} - \overline{z_v^e} z_w^e \right).$$

We can reformulate the preceding quadratic form by introducing only operations on $\mathcal{G}$. For instance, one has $|z_v^e|^2 = |Q_e \times \xi_v|^2$. We also have $z_v^e \overline{z_w^e} - z_w^e \overline{z_v^e} = -i \operatorname{Tr}(\Pi \xi_v \xi_w) = 2i \langle Q_e, \xi_v \times \xi_w \rangle$, where Π is the matrix whose eigenvalues in the basis $(s_v^e, s_v^{e\perp})$ are i and $-i$ and $2\xi_v \times \xi_w = [\xi_v, \xi_w]$. This may be proved by direct computation in the basis $(s_v^e, s_v^{e\perp})$. Hence, we can write

$$q_{P,Q}^\kappa = \sum_e c_e \left(\frac{\tau_{e\kappa} + \tau_{e\kappa}^{-1}}{\tau_{e\kappa} - \tau_{e\kappa}^{-1}} |Q_e \times (\xi_v - \xi_w)|^2 + 2i \langle Q_e, \xi_v \times \xi_w \rangle \right). \tag{10}$$

First case: $\tau_e^2 = 1$ for all e in E. Write $q^\kappa = \frac{\kappa^2 + 1}{\kappa^2 - 1} q_Q + q'$ where we set

$$q' = \sum_e 2i c_e \langle Q_e, \xi_v \times \xi_w \rangle \quad \text{and} \quad q_Q = \sum_e c_e |Q_e \times (\xi_v - \xi_w)|^2.$$

By Remark 5.4, the kernel of q_Q is the subspace of $\mathcal{G}$ given by the equations $\xi_v = \xi$ for all v. When κ goes to 1, the quadratic form q' becomes negligible. Hence, $I(q^\kappa) \simeq (\kappa - 1)^{3N - 3/2} I(q_Q)$, and using $I(|\lambda|^2, \mu_H^{\mathbb{P}^1}) = I(|\lambda|^2, \mu_{euc}^{\mathbb{P}^1})/\pi = 2$ one computes

$$I = \lim_{\kappa \to 1} \frac{(4\kappa - 4)^{3/2}}{I(r_P)} \frac{2^{3N}}{(4\kappa - 4)^{3N} \prod_e c_e} \frac{I(q_Q)}{(\kappa - 1)^{3/2 - 3N}} = \frac{2^{3 - 3N} I(q_Q)}{I(r_P) \prod_e c_e}.$$

Second case: $\tau_e^2 \neq 1$ for all e in E. Setting $\kappa = 1$ in q^κ, we get the quadratic form q'':

$$q_{P,Q}'' = \sum_e c_e \left(-i \cot(\theta_e) |\Pi_{Q_e^\perp}(\xi_v - \xi_w)|^2 + 2i \langle Q_e, \xi_v \times \xi_w \rangle \right). \tag{11}$$

By hypothesis (H3) this form has corank 6, hence the following formula makes sense:

$$I = \lim_{\kappa \to 1} \frac{(4\kappa - 4)^{3/2}}{I(r_P)} I(\sigma_1) I(q^\kappa) = \lim_{\kappa \to 1} \frac{(4\kappa - 4)^{3/2}}{I(r_P)} \frac{2^{3N}}{\prod_e 2c_e(1 - \tau_e^{-2})} I(q^\kappa)$$

$$= \lim_{\kappa \to 1} \frac{2^3(\kappa - 1)^{3/2} I(q^\kappa)}{I(r_P) \prod_e c_e(1 - \tau_e^{-2})} = \lim_{\kappa \to 1} \frac{2^{3-3N} i^N \prod_e \tau_e (\kappa - 1)^{3/2} I(q^\kappa)}{\prod_e c_e \sin(\theta_e) I(r_P)}. \qquad (12)$$

5.4.3. Collecting the critical points. In this section, we apply the previous computations to all critical points and collect the results. There are two cases: the critical point corresponds to a pair (P, P) or to a pair (P, Q) with $P \neq Q$.

First case: Fix a $3N$-tuple (u_e) representing P_e. All critical points associated to the pair (P, P) have the form $((-1)^{\varepsilon_v}, u_e)$ where $\varepsilon_v \in \{0, 1\}$. In the preceding subsection, we found that the contribution of a single critical point $(1, u_e)$ to (9) is $2^{3-3N} I(q_P)/I(r_P)$, remarking that the term $\prod_e c_e$ cancels and using the fact that $\tilde{F}(1, u_e) = 0$. The other pairs $((-1)^{\varepsilon_v}, u_e)$ differ only by the value of $e^{k\tilde{F}} = \prod_{e:v \to w}(-1)^{(\varepsilon_v + \varepsilon_w)kc_e}$. Adding the contributions and dividing by 2 (see Proposition 5.3), we get

$$\frac{1}{2}\Big(\sum_{\varepsilon_v} \prod_{v:(e_1, e_2, e_3)} (-1)^{\varepsilon_v k(c_{e_1} + c_{e_2} + c_{e_3})}\Big) \frac{2^{3-3N} I(q_P)}{I(r_P)} = \frac{2^{2-N} I(q_P)}{I(r_P)}.$$

We note that the contribution of $(-P, -P)$ is the same because $q_{-P} = q_P$ and $r_{-P} = r_P$. We will multiply the result by 2 when summing over $P \in I/\{\pm 1\}$.

Second case: When $P \neq Q$, we have found in (12) that the contribution of a critical point (g_v, u_e) is

$$\lim_{\kappa \to 1} \frac{2^{3-3N} i^N (\kappa - 1)^{3/2} I(q^\kappa_{P, Q})}{I(r_P) \prod_e \sin(\theta_e)} e^{i \sum_e (kc_e + 1)\theta_e}.$$

As before, taking into account all critical points associated to the same pair (P, Q) amounts to multiplying the result by 2^{2N-1}.

   When replacing (P, Q) by $(-P, -Q)$, we also have $r_{-P} = r_P$. Given a critical point (g_v, u_e) corresponding to (P, Q), one checks directly that the point $(g_v, u_e^\perp)$ corresponds to $(-P, -Q)$ and that the phase function gets inverted. Letting $q^\kappa_{P, Q}$ be defined as in equation (10), one sees that $q^\kappa_{-P, -Q} = -q^\kappa_{P, Q}$ and since $q^\kappa_{P, Q}$ is purely imaginary when $\kappa \to 1$, we have $\lim_{\kappa \to 1}(\kappa - 1)^{3/2} I(-q^\kappa_{P, Q}) = \lim_{\kappa \to 1}(\kappa - 1)^{3/2} \overline{I(q^\kappa_{P, Q})}$. Collecting the contributions of (P, Q) and $(-P, -Q)$, we get the following formula:

$$\frac{2^{3-N} \operatorname{Re}\big(i^N \lim_{\kappa \to 1}(\kappa - 1)^{3/2} I(q^\kappa_{P, Q}) e^{i \sum_e (kc_e + 1)\theta_e}\big)}{I(r_P) \prod_e \sin(\theta_e)}.$$

   It remains to compute explicitly the Gaussian integrals involved in the preceding computation. Considering on $\mathcal{G} \times \mathcal{G}$ the usual Haar measure, one has to divide the result by 2 because of the isotropy subgroup $\{\pm 1\} \in G \times G$. Using Appendix B, we can replace

all occurrences of I by a usual det in the Euclidean basis of $\mathcal{G}$ and its tensor powers. Denoting by $\det'$ the product of the non-zero eigenvalues of a matrix, we have

$$I(r_P) = (2/\pi)^{1/2} \det(r_P)^{-1/2},$$
$$I(q_P) = (2N)^{3/2} (2/\pi)^{N-1/2} \det'(q_P)^{-1/2},$$
$$I(q_{P,Q}^\kappa, \mu) = (2N)^{3/2} (2/\pi)^{N-1/2} \det'(q_{P,Q}^\kappa)^{-1/2},$$

where the factors $(2N)^{3/2}$ are due to the fact that the Haar density on the kernels of q_P and of $q_{P,Q}^\kappa$ is $\mu_{Haar} = \mu_{euc}/(2N)^{3/2}$ because each vector of the kernel is represented in $\mathcal{G}^{2N}$ by $2N$ copies of the same vector of $\mathcal{G}$.

5.5. An explicit example: the tetrahedron

This subsection is dedicated to providing some examples of applications of Theorem 1.3 and comparisons with previously known results.

First, let us remark that the first two hypotheses in Subsection 5.2 become geometrically meaningful when Γ is a planar graph. Indeed, in this case, let Γ^* be its dual graph and observe that each $P \in I$ provides a geometric realization in $\mathcal{G} = \mathbb{R}^3$ of Γ^* by a (possibly non-convex) polyhedron Δ whose edges are vectorially equal to $c_e P_e, e \in E$ (where we identify the edges of Γ and Γ^* in the natural way). As explained at the end of Subsection 5.2, if Γ is planar and all $P \in I$ correspond to convex polyhedra, then hypotheses (H1) and (H2) are automatically satisfied.

To provide an explicit example, let now $\Gamma = \bigotimes$ and c be an admissible coloring on it; in this case $N = 2$, and the graph Γ^* is still a tetrahedron. Given six edge lengths $c_1, \ldots, c_6$, under suitable conditions concerning the determinant of the associated Cayley–Menger matrix (see for instance [GV13, Proposition 9.2]) there exist exactly two (up to positive isometry) Euclidean tetrahedra whose edge lengths are c_i, and they are the mirror images of each other. In our language this translates to the fact that there are exactly two configurations: $I = \{P, -P\}$; by the above general discussion we already know that hypotheses (H1) and (H2) of Subsection 5.2 are satisfied. Moreover it was conjectured by Ponzano and Regge and then proved by Roberts [R99], but also more recently through new techniques by Garoufalidis and Van der Veen [GV13], by Aquilanti et al. [AHHJLY12] (generalizing the approach of Littlejohn and Yu [LY09]) and by L. Charles [C10] that the following holds for $k \to \infty$:

$$\left\langle \bigotimes, kc \right\rangle^U = \frac{\sqrt{2}}{\sqrt{3\pi k^3 V}} \cos\left(\frac{\pi}{4} + \sum_e \frac{(kc_e + 1)\theta_e}{2}\right)(1 + O(1/k)) \qquad (13)$$

where V is the common volume of the above Euclidean tetrahedra and θ_e is the exterior dihedral angle at e. The above formula was based on another normalization which is sometimes used in the literature (see [GV13]), known as the *unitary normalization*, which relates to our symbol $[\Gamma, c]$ as follows:

$$\langle \Gamma, c \rangle^U := \frac{\langle \Gamma, c \rangle}{\prod_v \sqrt{\langle v \rangle}}, \qquad [\Gamma, c] = (\langle \Gamma, c \rangle^U)^2.$$

So taking the square of (13), after few manipulations we get

$$[\Gamma, kc] = \frac{1 + \cos(\pi/2 + \sum_e (kc_e + 1)\theta_e)}{3\pi k^3 V}(1 + O(1/k)). \tag{14}$$

Now in the main formula of Theorem 1.3 there are four summands (corresponding to $P, -P$ in the first sum and to $(P, -P), (-P, P)$ in the second sum). By the definition of the quadratic forms q_P and r_P it is easy to verify that the first two summands give the same contribution (this is a general fact). Moreover numerical experiments have shown that the quadratic form $q_{P,-P}$ has corank 6 (so that hypothesis (H3) is satisfied) and that the following equalities hold true:

$$8\sqrt{\frac{\det(r_P)}{\det'(q_P)}} = \frac{1}{6V},$$

$$8\frac{\mathrm{Re}\left(-\sqrt{\frac{\det(r_P)}{\det'(q_{P,-P})}} \exp(i \sum_e (kc_e + 1)\theta_e)\right)}{\prod_e \sin(\theta_e)} = \frac{\cos(\pi/2 + \sum_e (kc_e + 1)\theta_e)}{6V}.$$

Inserting this in (14) we recover our initial formula in the case of a tetrahedron.

For the interested reader, we make here some remarks concerning the computational aspects of our formula. In the case of the tetrahedron, we have $I = \{P, -P\}$, and so as already stated our formula is made up of four terms. To find the configurations is an easy task for a computer as there are $2N$ linear equations (one per vertex) on the vectors $P_e, e \in E$, and $3N$ equations imposing that the lenghts of P_e are 1. Moreover, up to rotations one may fix $P_1 = (1, 0, 0)$ and P_2 to have zero z-coordinate and, say, negative y-coordinate; this completely fixes the indeterminacy due to the action of $G \times G$. In the case of the tetrahedron the equations can be easily solved and have two solutions, which are obtained from each other by reflection in the xy-plane.

The matrices r_P and q_P have respectively size 3×3 (this is always the case) and $(3 \cdot 2N) \times (3 \cdot 2N) = 12 \times 12$, and are computed in terms of the solutions found in the preceding step. To compute q_P' (which we recall is the product of all the non-zero eigenvalues of the matrix expressing q_P) one may compute minus the first non-zero coefficient (which is that of degree 3 if the hypotheses are satisfied) of the characteristic polynomial of the matrix which represents the form. A similar (but more tedious) computation yields $\lim_{k \to 1}(k - 1)^3 \det'(q_{P,Q})$: for generic k minus the degree three coefficient of the characteristic polynomial is non-zero (if the hypotheses are satisfied), and the above limit is non-zero. In any case the computations we carried out were exact but with fixed numerical values of c_e, and matched the known asymptotical behavior (14).

Appendix A. Supersymmetric rules

We refer to [DM99] for a detailed discussion. We collect here some definitions and warnings important in the article. A super vector space V is a direct sum of two finite-dimensional complex vector spaces V_0 and V_1 called respectively the *even* and *odd parts*.

We will always suppose that our spaces are *homogeneous*, that is, one of the two components vanishes, in particular any element $x \in V$ has a degree $|x|$ which is 0 or 1 depending on the parity of V. Super vector spaces form a category where morphisms are linear maps respecting the decomposition.

The tensor product $V \otimes W$ is defined as the vector space whose even part is $V_0 \otimes W_0 \oplus V_1 \otimes W_1$ and odd part is $V_1 \otimes W_0 \oplus V_0 \otimes W_1$. There is an isomorphism $c_{V,W} : V \otimes W \to W \otimes V$ sending $x \otimes y$ to $(-1)^{|x\|y|} y \otimes x$. The even vector space $\mathbb{C}$ is neutral for tensor product.

This allows one to define the unordered tensor product of homogeneous super spaces in the following way: Given a finite family $(V_i)_{i \in I}$ of vector spaces of parity p_i, we set

$$\bigotimes_{i \in I} V_i = \varinjlim(\varphi_{(\sigma')^{-1} \circ \sigma} : V_\sigma \to V_{\sigma'}).$$

The projective system is defined as follows. Let n be the cardinality of I, and for any bijection $\sigma : \{1, \ldots, n\} \to I$, set $V_\sigma = \bigotimes_{i=1}^n V_i$. Then define an isomorphism $\varphi_{(\sigma')^{-1} \circ \sigma} : V_\sigma \to V_{\sigma'}$ by the formula

$$\varphi_\tau(v_1 \otimes \cdots \otimes v_n) = (-1)^s v_{\tau_1} \otimes \cdots \otimes v_{\tau_n} \quad \text{where} \quad s = \sum_{i < i', \tau_i > \tau_{i'}} p_i\, p_{i'}.$$

There is an internal functor $\underline{\mathrm{Hom}}$ satisfying the following adjunction formula:

$$\mathrm{Hom}(U, \underline{\mathrm{Hom}}(V, W)) = \mathrm{Hom}(U \otimes V, W).$$

The space $\underline{\mathrm{Hom}}(U, V)$ is the space of all linear maps where a map is considered even if it respects the parity and odd if it reverses it. In particular, if we set $U^* = \underline{\mathrm{Hom}}(U, \mathbb{C})$ we have an isomorphism $\underline{\mathrm{Hom}}(U, V) = V \otimes U^*$ and an evaluation map $\mathrm{ev}_U : U^* \otimes U \to \mathbb{C}$.

With these identifications, bilinear forms are elements of $\mathrm{Hom}(U \otimes V, \mathbb{C}) = \mathrm{Hom}(U, V^*) = V^* \otimes U^*$. The inversion of the terms should be noticed.

There is another tricky issue: the natural isomorphism $\theta_U : U \to (U^*)^*$ is not the identity but sends x to $(-1)^{|x|}$. With that convention, we have the identity $\mathrm{ev}_{U^*} = \mathrm{ev}_U \circ c_{U,U^*} \circ (\theta_U^{-1} \otimes 1)$. Informally, this formula is explained by the transposition of the terms in the Gelfand transform: $\theta(x)(\lambda) = (-1)^{|x||\lambda|} \lambda(x)$.

Appendix B. Gaussian integrals and densities

In this subsection let F be a real vector space of dimension n, F^* be its dual, Q a quadratic form on F, and $S(F)$ the symmetric algebra of F. If Q is non-degenerate, one can identify F and F^* and thus transport it to a quadratic form denoted Q^{-1} on F^*. Note that if F is equipped with a basis and F^* with the dual, the matrices expressing Q and Q^{-1} in these bases are inverse to each other.

Definition B.1 (Densities). Let $\mathcal{B}(F)$ be the set of bases of F. A *density* μ on F is a map $\mu : \mathcal{B}(F) \to \mathbb{R}$ such that $\mu(A \cdot b) = |\det(A)| \mu(b)$ for each $A \in \mathrm{GL}(F)$. The set of densities is a real 1-dimensional vector space denoted $|\Lambda|(F)$. A *density* on a manifold M is a continuous section of the bundle $|\Lambda|(TM)$.

Fixing $b \in \mathcal{B}(F)$ defines an isomorphism $F \simeq \mathbb{R}^n$ and thus F can be equipped with a density which we denote μ_{euc} which satisfies $\mu_{\text{euc}}(b) = 1$. If F is equipped with a density μ and $Q : F \to \mathbb{C}$ is a quadratic form, then one defines $\det(Q, \mu) \in \mathbb{C}$ as $\det(Q(b))/\mu(b)^2$ where $Q(b)$ is the matrix expressing Q in the basis b of F (this clearly does not depend on b).

Lemma B.2 (Gaussian integrals). *Given a density μ on F and a quadratic form Q such that* $\text{Re}(Q) > 0$*, we have*

$$I(Q) = \int_F \exp\left(-\frac{Q}{2}\right) d\mu = \frac{(2\pi)^{n/2}}{\sqrt{\det(Q, \mu)}} \tag{15}$$

where the square root is the analytical extension of the positive one on the set of real and positive quadratic forms.

An element P of $S(V^*)$ may be interpreted either as a polynomial function on V or as a differential operator on $C^\infty(V^*; \mathbb{C})$. To distinguish the two cases we shall denote by P^{op} the element P interpreted as a differential operator. The following is a generalization of the Gaussian integration formula (15):

Proposition B.3 (Fourier transforms of Gaussian functions).

$$\int_V P(x) \exp\left(-\frac{1}{2}Q(x)\right) d\mu = \frac{(2\pi)^{n/2}}{\sqrt{\det(Q, \mu)}} \left(P^{\text{op}} \exp\left(\frac{1}{2}Q^{-1}\right)\right)\Big|_0. \tag{16}$$

If $K \subset F$ is a k-dimensional subspace, one has $|\Lambda(F)| = |\Lambda(K)| \otimes_{\mathbb{R}} |\Lambda(F/K)|$. Indeed, given bases b_K and $b_{F/K}$ for K and F/K, one can construct easily a basis of b of F using the inclusion of K and choosing an arbitrary complement to K (the choice of the complement affects b only up to elements of $\text{SL}(F)$). Thus given densities μ and μ_F on F and K respectively, the *quotient* density $\mu_{F/K}$ on F/K is defined so that $\mu = \mu_K \otimes \mu_{F/K}$.

Suppose that the quadratic form Q is degenerate. Then denoting by K its kernel, we can apply the formula (15) to the reduced quadratic form $\overline{Q}$ on F/K. To do so, we need to fix a density μ_K on K. Then we can set $I(Q, \mu_K) = \int_{F/K} \exp(-\frac{1}{2}\overline{Q}(x)) d\mu_{F/K}(x)$.

This new integral can be computed without considering the quotient, by the following perturbative argument. Let Q' be a quadratic form on F with positive real part and non-degenerate on K. Then, for ε positive and small enough, the quadratic form $Q + \varepsilon Q'$ is non-degenerate on K and on F. Moreover

$$I(Q, \mu_K) = \lim_{\varepsilon \to 0} \frac{I(Q + \varepsilon Q')}{I(Q'|_K)}.$$

As a consequence, if $\alpha \in \mathbb{R}$ is positive, then $I(\alpha Q, \mu_k) = \alpha^{(k-n)/2} I(Q, \mu_k)$. Suppose now that $F = F_1 \oplus F_2$, Q is non-degenerate on F_1 and both F and F_1 are equipped with densities μ and μ_1 (consequently, F_2 inherits a density μ_2). Let Q_1 be the restriction of Q to F_1 and let $A : F_2 \to F_1$ be defined by $Q(x, y) = Q_1(x, Ay)$ for $y \in F_2$ and $x \in F_1$. Then setting $Q'(y) = Q(y, y) - Q(Ay, Ay)$ we have

$$I(Q) = I(Q_1) I(Q'). \tag{17}$$

Given a manifold M equipped with a density μ one defines $\int_M f\, d\mu \in \mathbb{C}$ for $f \in C^\infty(M)$ in the natural way. In particular, a compact Lie group G can be equipped with a G-invariant density μ_H defined by $\int_G 1\, d\mu_H = 1$. If G acts freely on a manifold M equipped with a density μ preserved by G, one defines a density $\mu_{M/G}$ on the quotient in a natural way.

Example B.4 (The Euclidean density on $\mathcal{G}$). On $\mathcal{G}$ equipped with the scalar product $|\xi|^2 = -\frac{1}{2}\operatorname{Tr}(\xi^2)$, let μ_{euc} be the density whose value on an orthonormal basis is 1. Then μ_{euc} coincides with the density induced by the identification of $\mathcal{G}$ with the tangent space to $S^3 \subset \mathbb{C}^2$ at a point, and the Haar density is $\mu_H = \mu_{\mathrm{euc}}/(2\pi^2)$.

Example B.5 (The Hopf fibration). Let $S^3 = \mathrm{SU}(2)$ be the unit sphere in $\mathbb{C}^2$ and let $\mathbb{P}^1$ be the quotient by the diagonal action of S^1. Let $\mu_{\mathrm{Haar}}^{\mathrm{SU}(2)}$, $\mu_{\mathrm{Haar}}^{S^1}$ and $\mu_{\mathrm{Haar}}^{\mathbb{P}^1}$ be the Haar densities on $\mathrm{SU}(2)$ and the quotient density on $\mathbb{P}^1$ respectively. Let also $\mu_{\mathrm{euc}}^{\mathrm{SU}(2)}$ and $\mu_{\mathrm{euc}}^{\mathbb{P}^1}$ be the Euclidean densities. Then $\mu_{\mathrm{Haar}}^{\mathrm{SU}(2)} = (2\pi^2)^{-1}\mu_{\mathrm{euc}}^{\mathrm{SU}(2)}$ and $\mu_{\mathrm{Haar}}^{\mathbb{P}^1} = \pi^{-1}\mu_{\mathrm{euc}}^{\mathbb{P}^1}$.

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