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Soft local times and decoupling of random interlacements

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Abstract. In this paper we establish a decoupling feature of the random interlacement process $\mathcal{I}^u \subset \mathbb{Z}^d$ at level u, $d \geq 3$. Roughly speaking, we show that observations of $\mathcal{I}^u$ restricted to two disjoint subsets A_1 and A_2 of $\mathbb{Z}^d$ are approximately independent, once we add a sprinkling to the process $\mathcal{I}^u$ by slightly increasing the parameter u. Our results differ from previous ones in that we allow the mutual distance between the sets A_1 and A_2 to be much smaller than their diameters. We then provide an important application of this decoupling for which such flexibility is crucial. More precisely, we prove that, above a certain critical threshold u_{**}, the probability of having long paths that avoid $\mathcal{I}^u$ is exponentially small, with logarithmic corrections for $d = 3$.

To obtain the above decoupling, we first develop a general method for comparing the trace left by two Markov chains on the same state space. This method is based on what we call the soft local time of a chain. In another crucial step towards our main result, we also prove that any discrete set can be "smoothened" into a slightly enlarged discrete set, for which its equilibrium measure behaves in a regular way. Both these auxiliary results are interesting in themselves and are presented independently of the rest of the paper.

Keywords. Random interlacements, stochastic domination, soft local time, connectivity decay, smoothening of discrete sets.

1. Introduction and results

This work is mainly concerned with the decoupling of the random interlacements model introduced by A.-S. Sznitman [23]. In other words, we show that the restrictions of the interlacement set $\mathcal{I}^u$ to two disjoint subsets A_1 and A_2 of $\mathbb{Z}^d$ are approximately independent in a certain sense. To this end, we first develop a general method, based on what we call *soft local times*, to obtain an approximate stochastic domination between the ranges of two general Markov chains on the same state space.

To apply this coupling method to the model of random interlacements, we first need to modify the sets A_1 and A_2 through a procedure we call *smoothening*. This consists in en-

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closing a discrete set $A \subset \mathbb{Z}^d$ in a slightly enlarged set A', whose equilibrium distribution behaves "regularly", resembling what happens for a large discrete ball.

Finally, as an application of our decoupling result, we obtain upper bounds for the connectivity function of the vacant set $\mathcal{V}^u = \mathbb{Z}^d \setminus \mathcal{I}^u$, for intensities u above a critical threshold u_{**}. These bounds are considerably sharp, showing a behavior very similar to that of their corresponding lower bounds.

We believe that these four results are interesting in their own right. Therefore, we structured the article so that they can be read independently of each other. Below we give a more detailed description of each of these results.

1.1. Decoupling of random interlacements

The primary interest of this work lies in the study of the random interlacements process, recently introduced by A.-S. Sznitman [23]. The construction of random interlacements was originally motivated by the analysis of the trace left by simple random walks on large graphs, for instance a large discrete torus or a thick discrete cylinder. Intuitively speaking, this model describes the texture in the bulk left by these trajectories, when the random walk is let to run up to specific time scales.

Recently, great effort has been devoted to studying this model [18], [19], [31], [24], [25], [14], [3] as well as to establishing rigorously the relation between random interlacements and the trace left by random walks on large graphs [20], [34], [32], [4]. Recent works have also shown a connection between random interlacements, the Gaussian free field [28], [27] and cover times of random walks [2].

Roughly speaking, the model of random interlacements can be described as a Poissonian cloud of doubly infinite random walk trajectories on $\mathbb{Z}^d$, $d \geq 3$. The density of this cloud is governed by an intensity parameter $u > 0$ so that, as u increases, more and more trajectories enter the picture. We denote by $\mathcal{I}^u$ the so called *interlacement set*, given by the union of the ranges of these random walk trajectories. Regarding $\mathcal{I}^u$ as a random subset of $\mathbb{Z}^d$, its law $\mathcal{Q}^u$ can be characterized as the only distribution in $\{0, 1\}^{\mathbb{Z}^d}$ such that

$$\mathcal{Q}^u[K \cap \mathcal{I}^u = \emptyset] = \exp\{-u \operatorname{cap}(K)\} \quad \text{for every finite } K \subset \mathbb{Z}^d, \tag{1.1}$$

where $\operatorname{cap}(K)$ stands for the capacity of the set K defined in (2.6) (see [23, Proposition 1.5] for the characterization (1.1)).

The main difficulty in understanding the properties of $\mathcal{I}^u$ is related to its long range dependence. Note for instance that

$$\operatorname{Cov}(\mathbb{1}_{x\in\mathcal{I}^u}, \mathbb{1}_{y\in\mathcal{I}^u}) \sim \frac{c_d u}{\|x-y\|^{d-2}} \quad \text{as } \|x-y\| \to \infty \tag{1.2}$$

(see [23, (1.68)]). Such a slow decay of correlations imposes several obstacles to the analysis of random interlacements, especially in low dimensions. Various methods have been developed in order to circumvent this dependence, some of which we briefly summarize below.

Let us explain what is the type of statement we are after. Consider two subsets A_1 and A_2 of $\mathbb{Z}^d$ with diameters at most r and within distance at least $s \geq 1$ from each other. Suppose also that we are given two functions $f_1 : \{0,1\}^{A_1} \to [0,1]$ and $f_2 : \{0,1\}^{A_2} \to [0,1]$ that depend only on the configuration of the random interlacements inside the sets A_1 and A_2 respectively. In [23, (2.15)] it was established that

$$\mathrm{Cov}(f_1, f_2) \leq c_d u \frac{\mathrm{cap}(A_1)\,\mathrm{cap}(A_2)}{s^{d-2}} \leq c'_d u \left(\frac{r^2}{s}\right)^{d-2} \tag{1.3}$$

(see also [1, Lemma 2.1]). Although the above inequality retains the slow polynomial decay observed in (1.2), it has been useful in various situations, for instance in [23, Theorem 4.3] and [1, Theorem 0.1].

A first improvement on (1.3) appeared already in the pioneer work [23], where the author considers what he calls "sprinkling" of the law $\mathcal{I}^u$ (see Section 3). In the sprinkling procedure, "independent paths are thrown in, so as to dominate long range dependence" of $\mathcal{I}^u$.

Given two functions f_1 and f_2 as above, which are non-increasing in $\mathcal{I}^u$, the technique of [23, Section 3] allows one to conclude that, roughly speaking,

$$\mathcal{Q}^u[f_1 f_2] \leq \mathcal{Q}^{u(1+\delta)}[f_1]\mathcal{Q}^{u(1+\delta)}[f_2] + c_{d,\alpha}(r/s)^{\alpha}, \tag{1.4}$$

where α is arbitrary and the sprinkling parameter δ goes to zero as a polynomial of r/s. Note that the above represents a big improvement over (1.3): in exchange for restricting ourselves to non-increasing functions and introducing a sprinkling term, we obtain a much faster decay in the error term. Since its introduction, the sprinkling technique has been useful for several problems on random interlacements [21], [26], [32].

The most recent result on decoupling bounds for interlacements can be found in [26] and stands out for several reasons. First, it generalizes the ideas behind [19] and [31] to random interlacements on quite general classes of graphs (besides $\mathbb{Z}^d$), as long as they satisfy certain heat kernel estimates. Secondly, the tools developed in [26] work to show both existence and absence of percolation through a unified framework and give novel results even in the particular case of $\mathbb{Z}^d$; see also the beautiful applications in [15] and [6].

On the other hand, the results in [26] were designed having a renormalization scheme in mind. Thus, their use is restricted to bounding the so-called "cascading events", which behave in a certain hierarchical way (see the details in [26, Section 3]).

Although the results in (1.3), (1.4) and [26] complement each other, they suffer from the same drawback, as they implicitly assume that

$$\text{the distance between } A_1 \text{ and } A_2 \text{ is at least of the same order as their diameters.} \tag{1.5}$$

This can be a major obstruction in some applications, such as the one we present in Section 3 on the decay of connectivity.

Let us now state the main theorem of the present paper, which can be regarded as an improvement on (1.4). Later we will describe precisely how it differs quantitatively from previously known results.

Below, γ_0 and γ_1 are positive constants depending only on the dimension d.

Theorem 1.1. *Let A_1, A_2 be two non-intersecting subsets of $\mathbb{Z}^d$, at least one of them being finite. Let s be the distance between A_1 and A_2, and r be the minimum of their diameters. Then, for all $u > 0$ and $\varepsilon \in (0, 1)$, we have:*

(i) *for any increasing functions $f_1 : \{0, 1\}^{A_1} \to [0, 1]$ and $f_2 : \{0, 1\}^{A_2} \to [0, 1]$,*

$$\mathcal{Q}^u[f_1 f_2] \le \mathcal{Q}^{(1+\varepsilon)u}[f_1]\mathcal{Q}^{(1+\varepsilon)u}[f_2] + \gamma_0 (r + s)^d \exp(-\gamma_1 \varepsilon^2 u s^{d-2}); \tag{1.6}$$

(ii) *for any decreasing functions $f_1 : \{0, 1\}^{A_1} \to [0, 1]$ and $f_2 : \{0, 1\}^{A_2} \to [0, 1]$,*

$$\mathcal{Q}^u[f_1 f_2] \le \mathcal{Q}^{(1-\varepsilon)u}[f_1]\mathcal{Q}^{(1-\varepsilon)u}[f_2] + \gamma_0 (r + s)^d \exp(-\gamma_1 \varepsilon^2 u s^{d-2}). \tag{1.7}$$

We of course assume that the above functions f_1 and f_2 are measurable (recall that one of the sets A_1 or A_2 may be infinite).

The above theorem is a direct consequence of the slightly more general Theorem 2.1. Note that the opposite inequalities to (1.6) and (1.7) follow without error terms (and with $\varepsilon = 0$) from the FKG inequality, which was proved for random interlacements in [29, Theorem 3.1].

Let us now stress what are the main improvements offered by the above bounds over previously known results. First, there is no requirement that s should be larger than r as in (1.5) (and again, one of the sets may even be infinite). Moreover, these error bounds feature an explicit and fast decay on s, even as $\varepsilon = \varepsilon(s, r)$ goes (not too rapidly) to zero. In Remark 3.3 we include some observations on how close to optimal one can expect (1.6) and (1.7) to be.

1.2. Connectivity decay

As an application of Theorem 1.1, we establish a result on the decay of connectivity of the *vacant set* $\mathcal{V}^u = \mathbb{Z}^d \setminus \mathcal{I}^u$. More precisely, for u large enough (see Theorem 3.1 for details), for $d \ge 4$,

$$\mathcal{Q}^u[0 \stackrel{\mathcal{V}^u}{\longleftrightarrow} x] \le \gamma_2 \exp\{-\gamma_3 \|x\|\} \quad \text{for every } x \in \mathbb{Z}^d, \tag{1.8}$$

where γ_2 and γ_3 depend only on d. If $d = 3$ and u is large enough, then for any $b > 1$ there exist $\gamma_4 = \gamma_4(u, b)$ and $\gamma_5 = \gamma_5(u, b)$ such that

$$\mathcal{Q}^u[0 \stackrel{\mathcal{V}^u}{\longleftrightarrow} x] \le \gamma_4 \exp\left\{-\gamma_5 \frac{\|x\|}{\log^{3b} \|x\|}\right\} \quad \text{for every } x \in \mathbb{Z}^3 \tag{1.9}$$

(see Theorem 3.1 and Remark 3.2 for more details).

Let us stress that the above bounds greatly improve on the previously known results of [19, Theorem 0.1]. There, the authors establish similar bounds but with $\|x\|$ replaced by $\|x\|^\rho$ for some unknown exponent $\rho \in (0, 1)$. Our bounds on the other hand are considerably sharp, as they closely resemble the corresponding lower bounds (see Remark 3.2 for details).

Note that the exponential decay in (1.8) is also observed in independent percolation models (see for instance [8, Theorem (5.4), p. 88] and [11]). However, due to the strong dependence present in $\mathcal{V}^u$, its validity was at first not obvious to the authors. For one reason, it is known that the logarithmic factor in (1.9) cannot be dropped (see Remark 3.2 below). Similar types of non-exponential decay in dependent percolation models can be found for instance in [23, (1.65) and (2.21)] and [31, Remark 3.7 2)].

Finally we would like to stress that our proof of (1.8)–(1.9) is general enough in the sense that it could be adapted to other dependent percolation models, as long as they satisfy a suitable decoupling inequality. See the discussion in Remark 3.4.

1.3. Soft local times

In Section 4 we develop a technique to prove approximate stochastic domination of the trace left by a Markov chain on a metric space. This is an important ingredient in proving Theorem 1.1 and we also expect it to be useful in future applications. To illustrate this technique, consider an irreducible Markov chain $(Z_i)_{i\geq 1}$ on a finite state space Σ having π as its unique stationary measure.

A typical model to keep in mind is a random walk on a torus that jumps from z to a uniformly chosen point in the ball centered at z with radius k. By transitivity, the uniform distribution π is clearly invariant. Intuitively speaking, if we let this Markov chain run for a long time t, we expect the law of the covered set $\{Z_1, \dots, Z_t\}$ to be "reasonably close" to that of a collection $\{W_1, \dots, W_t\}$ of i.i.d. points in Σ distributed according to π. This is made precise in the following result, which is a consequence of Corollary 4.4.

Proposition 1.2. *Let $(Z_i)_{i\geq 1}$ be a Markov chain on a finite set Σ, with transition probabilities $p(z, z')$, initial distribution π_0, and stationary measure π. Then we can find a coupling $\mathbb{Q}$ between (Z_i) and an i.i.d. collection (W_i) (with law π) such that for any $\lambda > 0$ and $t \geq 0$,*

$$\mathbb{Q}[\{Z_1, \dots, Z_t\} \subset \{W_1, \dots, W_R\}] \geq \mathbb{Q}\Big[\xi_0\pi_0(z) + \sum_{j=1}^{t-1} \xi_j p(Z_j, z) \leq \lambda\pi(z) \text{ for all } z \in \Sigma\Big], \tag{1.10}$$

where ξ_i are i.i.d. $\mathrm{Exp}(1)$ *random variables, independent of R, a* $\mathrm{Poisson}(\lambda)$*-distributed random variable.*

Observe that the above statement may have interesting consequences in bounding the hitting time of a given subset of Σ (see (2.4) for a precise definition).

We call the sum $\sum_j \xi_j p(Z_j, z)$ the *soft local time* of the chain Z_j. To justify this terminology, observe that instead of counting the number of visits to a fixed site (which corresponds to the usual notion of local time), we are summing up the *chances* of visiting such site, multiplied by i.i.d. mean-one positive factors. See also Theorem 4.6.

In Remark 4.5 we describe the main advantages of Proposition 1.2 over previous domination techniques and how it allows us to drop the assumption (1.5).

Later in Section 4, we establish general estimates on the expectation, variance and exponential moments of the soft local time $\sum_j \xi_j p(Z_j, z)$. These are based on regularity assumptions on the transition probabilities $p(\cdot, \cdot)$ and are valuable when estimating the right-hand side of (1.10) by means of exponential Chebyshev inequalities (Theorems 4.6, 4.8 and 4.9).

Now, we comment on the main method employed to prove results such as Proposition 1.2 above. One can better visualize the picture in a continuous space, so we use another example to illustrate the method. Assume that we are given a sequence of (not necessarily independent or Markovian) random variables $S_1, S_2, \dots$ taking values in the interval $[0, 1]$, and let T be a finite stopping time. As in (1.10), we attempt to dominate this process by a sequence $U_1, \dots, U_N$, where (U_k) are i.i.d. Uniform$[0, 1]$ random variables, and N is a Poisson random variable independent of (U_k). More precisely, we want to construct a coupling between the two sequences in such a way that

$$\{S_1, \dots, S_T\} \subseteq \{U_1, \dots, U_N\} \tag{1.11}$$

with probability close to one. We assume that the law of S_k conditioned on $S_1, \dots, S_{k-1}$ is absolutely continuous with respect to the Lebesgue measure on $[0, 1]$ (see (4.6)).

Our method for obtaining such a coupling is illustrated in Figure 1. Consider a Poisson point process in $[0, 1] \times \mathbb{R}_+$ with rate 1. Then one can obtain a realization of the U-sequence by simply retaining the first coordinate of the points lying below a given threshold (the dashed line in Figure 1) corresponding to the parameter of the Poisson random variable N.

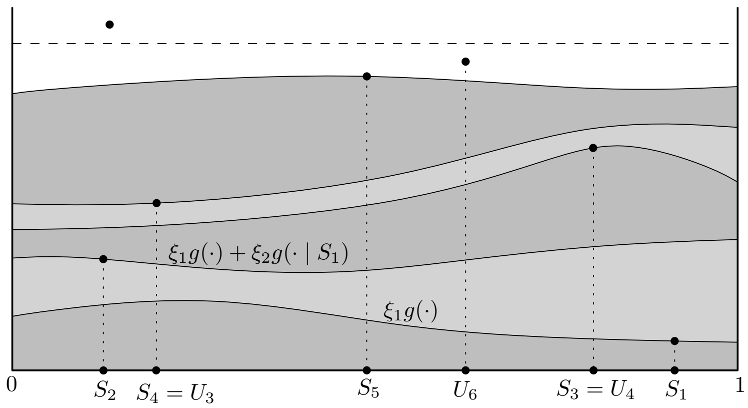

Fig. 1. Soft local times: the construction of the process S (here, $T = 5$, $N = 6$, $U_k = S_k$ for $k = 1, 2, 5$); it is important to observe that the points of the two processes need not necessarily appear in the same order with respect to the vertical axis (see Remark 4.5).

Now, in order to obtain a realization of the S-sequence using the same Poisson point process, one proceeds as follows:

- first, take the density $g(\cdot)$ of S_1 and multiply it by the unique positive number ξ_1 so that there is exactly one point of the Poisson process lying on the graph of $\xi_1 g$ and no point strictly below it;

- then consider the conditional density $g(\cdot \mid S_1)$ of S_2 given S_1 and find the smallest constant ξ_2 so that exactly two points lie underneath $\xi_2 g(\cdot \mid S_1) + \xi_1 g(\cdot)$;
- continue with $g(\cdot \mid S_1, S_2)$, and so on, up to time T, as shown in Figure 1.

In Proposition 4.3, we show that the collection of points obtained through the above procedure has the same law as $(S_1, S_2, \dots)$ and is independent of the random variables ξ_i, which are i.i.d. with law Exp(1). We call the sum $\xi_1 g(\cdot) + \xi_2 g(\cdot \mid S_1) + \cdots$ the *soft local time* of the process S (which coincides with the sum on the right-hand side of (1.10) in the Markovian case). Clearly, if the soft local time (the gray area in the picture) is below the dashed line, then the domination in (1.11) holds. To obtain the probability of a successful coupling, one has to estimate the probability that the soft local time lies below the dashed line. In several cases, this reduces to a large deviations estimate.

After developing a general version of this technique in Section 4, we adapt this theory to random interlacements in Section 5. More precisely, we present an alternative construction of the interlacement set $\mathcal{I}^u$ restricted to some $A \subset \mathbb{Z}^d$. In this construction, we split each trajectory composing $\mathcal{I}^u$ into a collection of excursions in and out of A. This induces a Markov chain on the space of excursions, and the technique of soft local times helps us control the range of such soup.

After completing this article, we learned that a technique similar to the soft local times was introduced in the special case $\Sigma = (0, 1) \subset \mathbb{R}$ in order to study local minima of the Brownian motion in [33, Claim 1.5].

We believe that the method of soft local times can be useful in other contexts besides random interlacements. For example, when considering a random walk trajectory on a finite graph (such as a torus or a discrete cylinder), one can naturally be interested in the degree of independence in the pictures left by the walker on disjoint subsets of the graph. The approach followed in this paper is likely to be successful in this situation as well. We also believe this technique could give alternative proofs or generalize results on the coupling of systems of independently moving particles (see [13, Proposition 5.1] for an example of such a statement).

1.4. Smoothening of discrete sets

As mentioned before, in order to estimate the probability of having a successful coupling using the soft local times technique, we need some regularity conditions on the transition densities of the Markov chain. When applying this to the excursions composing the random interlacements, this translates into a condition on the regularity of the entrance distributions on the sets A_1 and A_2, which may not hold in general (picture for instance a set with sharp points).

To overcome this difficulty, we develop a technique to enlarge the original discrete sets A_1 and A_2 into slightly bigger discrete sets with "sufficiently smooth" boundaries, so that their entrance probabilities satisfy the required regularity conditions.

The exact result we are referring to is given in Proposition 6.1, but we provide here a small preview of its statement. There exist positive constants c, c', c'', s_0 (depending only on dimension) such that for any $s \geq s_0$ and any finite set $A \subset \mathbb{Z}^d$, there exists a set $A^{(s)}$

with $A \subseteq A^{(s)} \subseteq B(A, s)$ and

$$P_x[X_H = y] \le c P_x[X_H = y'] \tag{1.12}$$

for all $y, y' \in \partial A^{(s)}$ with $\|y - y'\| \le c'' s$, and all x such that $\|x - y\| \ge c' s$. Here X is the simple random walk and H is the hitting time of the set $A^{(s)}$. That is, the entrance measure to the set $A^{(s)}$ is "comparable" in close sites of the boundary, as long as the starting point of the random walk is sufficiently far away.

It is important to observe that for example a large (discrete) ball has the above property, while a large box does not, since its entrance probabilities at the faces are typically much smaller than those at the corners (to see this, observe that using arguments similar e.g. to the proof of [12, Theorem 1.4] one can show that the harmonic measure at a corner of the box is at least $O(n^{-\gamma})$ for some $\gamma < 1$, while for "generic" sites on the faces it is $O(n^{-1})$.

1.5. Plan of the paper

The paper is organized in the following way. In Section 2 we formally define the model of random interlacements, and state our main decoupling result. In Section 3, we formally state the connectivity decay appearing in (1.7) and (1.6) (Theorem 3.1). In Section 4 we present a general version of the method of soft local times. Then, in Section 5 this method is used to introduce an alternative construction of random interlacements, which is better suited for decoupling configurations on disjoint sets. In the same section we reduce the proof of our main Theorem 2.1 to a large deviations estimate for the soft local time of excursions. In Section 6, we estimate the probability of these large deviation events and conclude the proof of Theorem 2.1 under a set of additional assumptions on the entrance measures of $A_{1,2}$. While this set of assumptions may not be satisfied for arbitrary $A_{1,2}$, we show in Section 8 that this is not really an issue, as one can always slightly enlarge the sets of interest (with the procedure referred to above as smoothening) so that the modified sets satisfy the necessary regularity assumptions. Before going to (quite technical) Section 8, in Section 7 we prove the result on the decay of connectivity for the vacant set, corresponding to (1.8) and (1.9).

2. Random interlacements: formal definitions and main result

In this paper, we use the following convention concerning constants: $c_1, c_2, \ldots$ as well as $\gamma_1, \gamma_2, \ldots$ denote strictly positive constants depending only on dimension d. Dependence of constants on additional parameters appears in the notation. For example, c_α denotes a constant depending only on d and α. Also c-constants are "local" (used only in a small neighborhood of the place of the first appearance) while γ-constants are "nonlocal" (they appear in propositions and "important" formulas).

Let us now introduce some notation and describe the model of random interlacements. In addition, we recall some useful facts concerning the model.

For $a \in \mathbb{R}$, we write $\lfloor a \rfloor$ for the largest integer smaller than or equal to a and recall that

$$\lfloor ta + (1-t)b \rfloor \in [\min\{a,b\}, \max\{a,b\}] \quad \text{for all } a, b \in \mathbb{Z} \text{ and } t \in [0,1]. \tag{2.1}$$

We say that two points $x, y \in \mathbb{Z}^d$ are *neighbors* if they are at Euclidean distance (denoted by $\|\cdot\|$) exactly 1 (we then write $x \leftrightarrow y$). This induces a graph structure and a notion of connectedness in $\mathbb{Z}^d$.

If $K \subset \mathbb{Z}^d$, we denote by K^c its complement and by $B(K,r)$ the r-neighborhood of K with respect to the Euclidean distance, i.e. the union of the balls $B(x,r)$ for $x \in K$. The diameter of K (denoted by $\mathrm{diam}(K)$) is the supremum of $\|x-y\|_\infty$ with $x, y \in K$, where $\|\cdot\|_\infty$ is the maximum norm. The *internal boundary* of K is $\partial K = \{x \in K;\ x \leftrightarrow y$ for some $y \in K^c\}$.

In this article the term *path* always denotes finite, nearest neighbor paths, i.e. some $\mathcal{T} : \{0, \dots, n\} \to \mathbb{Z}^d$ such that $\mathcal{T}(l) \leftrightarrow \mathcal{T}(l+1)$ for $l = 0, \dots, n-1$. In this case we say that the length of $\mathcal{T}$ is n.

Let us denote by W_+ and W the spaces of infinite, respectively doubly infinite, transient trajectories:

$$\begin{aligned} W_+ &= \Big\{ w : \mathbb{Z}_+ \to \mathbb{Z}^d;\ w(l) \leftrightarrow w(l+1) \text{ for each } l \geq 0 \text{ and } \|w(l)\| \xrightarrow[l\to\infty]{} \infty \Big\}, \\ W &= \Big\{ w : \mathbb{Z} \to \mathbb{Z}^d;\ w(l) \leftrightarrow w(l+1) \text{ for each } l \in \mathbb{Z} \text{ and } \|w(l)\| \xrightarrow[|l|\to\infty]{} \infty \Big\}. \end{aligned} \tag{2.2}$$

We endow these spaces with the σ-algebras $\mathcal{W}_+$ and $\mathcal{W}$ generated by the coordinate maps $\{X_n\}_{n \in \mathbb{Z}_+}$ and $\{X_n\}_{n \in \mathbb{Z}}$.

Let us also introduce the *entrance time* of a finite set $K \subset \mathbb{Z}^d$,

$$H_K(w) = \inf\{k;\ X_k(w) \in K\} \quad \text{for } w \in W_{(+)}, \tag{2.3}$$

and for $w \in W_+$, we define the *hitting time* of K as

$$\widetilde{H}_K(w) = \inf\{k \geq 1;\ X_k(w) \in K\}. \tag{2.4}$$

Let $\theta_k : W \to W$ stand for the time shift given by $\theta(w)(\cdot) = w(\cdot + k)$ (where k could also be a random time).

For $x \in \mathbb{Z}^d$ (recall that $d \geq 3$), we can define the law P_x of a simple random walk starting at x on the space $(W_+, \mathcal{W}_+)$. If ρ is a measure on $\mathbb{Z}^d$, we write $P_\rho = \sum_{x \in \mathbb{Z}^d} \rho(x) P_x$.

Let us introduce, for a finite $K \subset \mathbb{Z}^d$, the *equilibrium measure*

$$e_K(x) = \mathbb{1}_{x \in K} P_x[\widetilde{H}_K = \infty] \quad \text{for } x \in \mathbb{Z}^d, \tag{2.5}$$

the *capacity* of K

$$\mathrm{cap}(K) = e_K(\mathbb{Z}^d) \tag{2.6}$$

and the *normalized equilibrium measure*

$$\bar{e}_K(x) = e_K(x)/\mathrm{cap}(K) \quad \text{for } x \in \mathbb{Z}^d. \tag{2.7}$$

We mention the following bound on the capacity of a ball of radius $r \geq 1$:

$$\text{cap}(B(0,r)) \asymp r^{d-2} \tag{2.8}$$

(see [10, Proposition 6.5.2]; here and below we write $f(r) \asymp g(r)$ when $c_0 g(r) \leq f(r) \leq c_1 g(r)$ for strictly positive constants c_0, c_1 depending only on the dimension).

Let W^* stand for the space of doubly infinite trajectories in W modulo time shift,

$$W^* = W/\sim, \quad \text{where} \quad w \sim w' \text{ if } w(\cdot) = w'(k + \cdot) \text{ for some } k \in \mathbb{Z}, \tag{2.9}$$

endowed with the σ-algebra

$$\mathcal{W}^* = \{A \subset W^*;\ (\pi^*)^{-1}(A) \in \mathcal{W}\}, \tag{2.10}$$

which is the largest σ-algebra making the canonical projection $\pi^* : W \to W^*$ measurable. For a finite set $K \subset \mathbb{Z}^d$, we denote by W_K the set of trajectories in W which meet the set K, and define $W_K^* = \pi^*(W_K)$.

Now we are able to describe the intensity measure of the Poisson point process which governs the random interlacements.

For a finite set $K \subset \mathbb{Z}^d$, we consider the measure Q_K in $(W, \mathcal{W})$ supported in W_K such that given $A, B \in \mathcal{W}_+$ and $x \in K$,

$$Q_K[(X_{-n})_{n\geq 0} \in A, X_0 = x, (X_n)_{n \geq 0} \in B] = P_x[A \mid \tilde{H}_K = \infty] P_x[B] e_K(x). \tag{2.11}$$

Theorem 1.1 of [23] establishes the existence of a unique σ-finite measure ν in W^* such that

$$\mathbb{1}_{W_K^*} \cdot \nu = \pi^* \circ Q_K \qquad \text{for any finite set } K \subset \mathbb{Z}^d. \tag{2.12}$$

The above equation is the main tool to perform calculations on random interlacements.

We then introduce the spaces of point measures on $W^* \times \mathbb{R}_+$ and $W_+ \times \mathbb{R}_+$,

$$\Omega = \left\{ \omega = \sum_{i \geq 1} \delta_{(w_i^*, u_i)};\ \begin{array}{l} w_i^* \in W^*, u_i \in \mathbb{R}_+ \text{ and } \omega(W_K^* \times [0,u]) < \infty \\ \text{for every finite } K \subset \mathbb{Z}^d \text{ and } u \geq 0 \end{array} \right\} \tag{2.13}$$

and endowed with the σ-algebra $\mathcal{A}$ generated by the evaluation maps $\omega \mapsto \omega(D)$ for $D \in \mathcal{W}^* \otimes \mathcal{B}(\mathbb{R}_+)$. Here $\mathcal{B}(\cdot)$ denotes the Borel σ-algebra.

We let $\mathbb{P}$ be the law of a Poisson point process on Ω with intensity measure $\nu \otimes du$, where du denotes the Lebesgue measure on $\mathbb{R}_+$. Given $\omega = \sum_i \delta_{(w_i^*, u_i)} \in \Omega$, we define the *interlacement* and the *vacant set* at level u respectively as the random subsets of $\mathbb{Z}^d$:

$$\mathcal{I}^u(\omega) = \bigcup_{i : u_i \leq u} \text{Range}(w_i^*), \tag{2.14}$$

$$\mathcal{V}^u(\omega) = \mathbb{Z}^d \setminus \mathcal{I}^u(\omega). \tag{2.15}$$

In [23, (0.13)], Sznitman introduced the critical value

$$u_* = \inf\{u \geq 0;\ \mathbb{P}[\mathcal{V}^u \text{ contains an infinite connected component}] = 0\}, \tag{2.16}$$

where the vacant set undergoes a phase transition in connectivity. It is known that $0 < u_* < \infty$ for all $d \geq 3$ [23, Theorem 3.5], [18, Theorem 3.4]. Moreover, it is also proved that the infinite connected component of the vacant set (if any) must be unique [30, Theorem 1.1].

It is important to mention also that, as shown in [23],

$$\text{the law of the random set } \mathcal{I}^u \text{ is invariant and ergodic with respect to translations of the lattice } \mathbb{Z}^d. \tag{2.17}$$

2.1. Decoupling: the main result

We now state our main result on random interlacements. It provides us with a way to decouple the intersection of the interlacement set $\mathcal{I}^u$ with two disjoint subsets A_1 and A_2 of $\mathbb{Z}^d$. Namely, we couple the original interlacement process $\mathcal{I}^u$ with two *independent* interlacements processes $\mathcal{I}_1^u$ and $\mathcal{I}_2^u$ in such a way that $\mathcal{I}^u$ restricted on A_k is "close" to $\mathcal{I}_k^u$, for $k = 1, 2$, with probability rapidly going to 1 as the distance between the sets increases. This is formulated precisely in

Theorem 2.1. *Let A_1, A_2 be two non-intersecting subsets of $\mathbb{Z}^d$, at least one of them being finite. Set $s = \mathrm{d}(A_1, A_2)$ and $r = \min\{\mathrm{diam}(A_1), \mathrm{diam}(A_2)\}$. Then there are positive constants γ_0 and γ_1 (depending only on the dimension d) such that for all $u > 0$ and $\varepsilon \in (0, 1)$ there exists a coupling $\mathbb{Q}$ between $\mathcal{I}^u$ and two independent random interlacement processes, $(\mathcal{I}_1^u)_{u\geq 0}$ and $(\mathcal{I}_2^u)_{u\geq 0}$, such that*

$$\begin{aligned} &\mathbb{Q}[\mathcal{I}_k^{u(1-\varepsilon)} \cap A_k \subseteq \mathcal{I}^u \cap A_k \subseteq \mathcal{I}_k^{u(1+\varepsilon)},\ k = 1, 2] \\ &\qquad \geq 1 - \gamma_0 (r+s)^d \exp(-\gamma_1 \varepsilon^2 u s^{d-2}). \end{aligned} \tag{2.18}$$

It is straightforward to see that the above theorem implies the inequality on the covariance of increasing (or decreasing) functions depending only on A_1 and A_2, stated previously in Theorem 1.1. Also, we mention that the factor $(r+s)^d$ before the exponential in (2.18) can usually be reduced (see Remark 6.4).

3. Discussion, open problems, and an application of decoupling

We start this section with the following application of our main result. We are interested in the probability $\mathbb{P}[0 \stackrel{\mathcal{V}^u}{\longleftrightarrow} x]$ that two far away points are connected through the vacant set. In the subcritical case, $u > u_*$, this probability clearly converges to zero as $\|x\|$ goes to infinity. In what follows, we will be interested in the rate in which this convergence takes place.

In [23, Proposition 3.1], it was proven that $\mathbb{P}[0 \stackrel{\mathcal{V}^u}{\longleftrightarrow} x]$ decays at least as a polynomial in $\|x\|$ if u is chosen large enough. Then in [19] this was considerably improved, by showing that for u large enough, there exist c, c' and $\delta > 0$ (possibly depending on u) such that

$$\mathbb{P}[0 \stackrel{\mathcal{V}^u}{\longleftrightarrow} x] \leq c \exp\{-c' \|x\|^\delta\} \quad \text{for every } x \in \mathbb{Z}^d. \tag{3.1}$$

To be more precise, the above statement was established for all intensities u above the threshold

$$u_{**}(d) = \inf\Big\{u > 0;\ \text{for some } \alpha > 0,\ \lim_{L\to\infty} L^\alpha \mathbb{P}\big[[-L,L]^d \overset{\mathcal{V}^u}{\longleftrightarrow} \partial[-2L,2L]^d\big] = 0\Big\}. \tag{3.2}$$

The above critical value is known to satisfy $u_* \le u_{**} < \infty$ [22, Lemma 1.4] and a relevant question is whether u_* and u_{**} actually coincide.

In [26], an important class of decoupling inequalities was introduced, implying in particular that (3.2) can be written as

$$u_{**} = \inf\Big\{u > 0;\ \lim_{L\to\infty} \mathbb{P}\big[[-L,L]^d \overset{\mathcal{V}^u}{\longleftrightarrow} \partial[-2L,2L]^d\big] = 0\Big\}, \tag{3.3}$$

potentially enhancing the validity of (3.1). The above result could perhaps be seen as a step towards proving $u_* = u_{**}$.

Here, we further weaken the definition of u_{**} but, more importantly, we improve on the bound (3.1) for values of u above u_{**}. The improved result we present gives the correct exponents in the decay of the connectivity function, although for $d = 3$ they could be off by logarithmic corrections (see Remark 3.2 below).

Theorem 3.1. *For $d \ge 4$, given $u > u_{**}(d)$, there exist positive constants $\gamma_2 = \gamma_2(d,u)$ and $\gamma_3 = \gamma_3(d,u)$ such that*

$$\mathbb{P}[0 \overset{\mathcal{V}^u}{\longleftrightarrow} x] \le \gamma_2 \exp\{-\gamma_3 \|x\|\} \quad \textit{for every } x \in \mathbb{Z}^d. \tag{3.4}$$

If $d = 3$ and $u > u_{**}(3)$, then for any $b > 1$ there exist $\gamma_4 = \gamma_4(u,b)$ and $\gamma_5 = \gamma_5(u,b)$ such that

$$\mathbb{P}[0 \overset{\mathcal{V}^u}{\longleftrightarrow} x] \le \gamma_4 \exp\Big\{-\gamma_5 \frac{\|x\|}{\log^{3b}\|x\|}\Big\} \quad \textit{for every } x \in \mathbb{Z}^d. \tag{3.5}$$

Moreover, (3.2) can be written as

$$u_{**} = \inf\Big\{u > 0;\ \liminf_{L\to\infty} \mathbb{P}\big[[0,L]^d \overset{\mathcal{V}^u}{\longleftrightarrow} \partial[-L,2L]^d\big] < \frac{7}{2d\cdot 21^d}\Big\}. \tag{3.6}$$

Remark 3.2. The probability that a straight segment of length n is vacant is exponentially small in n when $d \ge 4$, while for $d = 3$, this probability is at least $c\exp(-c'n/\log n)$, which corresponds to the capacity of a line segment (this follows e.g. from [9, Proposition 2.4.5]). So, (3.4) is sharp (up to constants), but the situation with (3.5) is less clear, since in (3.5) the power of the logarithm in the denominator is at least 3. We believe, however, that (3.5) can be improved (by decreasing the power of the logarithm).

Remark 3.3. There is a general question about how sharp the result in (2.18) is (also in (1.6) and (1.7)). One could for instance ask whether the probability in (2.18) can be exactly 1, thus achieving equality in (1.6)–(1.7) (so that we would have a "perfect domination"). Interestingly enough, Theorem 3.1 sheds some light on this question, at least in dimension $d = 3$. Indeed, in the proof of Theorem 3.1 we use (1.7) with $\varepsilon \simeq \log^{-b} s$

to obtain the subexponential decay of (3.5); however, if the error term could be dropped altogether, or even if s could be replaced by $s^{1+\delta}$ (for some $\delta > 0$) in that term, then (compare with the proof for $d \geq 4$) one would obtain the exponential decay for $d = 3$ as well, which contradicts the previous remark. This is an indication that, in general, s^{d-2} in the exponent in the error term could be sharp, at least if ε is small enough. Also, one cannot hope to achieve perfect domination if $\varepsilon \ll s^{-(d-2)}$ simply due to (1.2).

It is less clear how small the parameter ε can be made (say, in the situation when s does not exceed r). Obviously, (2.18) stops working when $\varepsilon = O(s^{-(d-2)/2})$, but we are unsure about how much our main result can be improved in this direction. Also, it is interesting to observe that, unlike the bound (1.3), our estimates become *better* as the parameter u increases.

Remark 3.4. As mentioned in Section 1.1, one can obtain exponential decay as in (3.4) for any percolation model with suitable monotonicity and decoupling properties. Namely, let $\tilde{\mathcal{Q}}^u$ be a family of measures on $\{0, 1\}^{\mathbb{Z}^d}$, $d \geq 2$, indexed by a parameter $u \in [0, \infty)$. We assume that this family is monotone in the sense that $\tilde{\mathcal{Q}}^{u'}$ dominates $\tilde{\mathcal{Q}}^u$ if $u' < u$ (as happens for the vacant set in the random interlacement model). Also, assume that there are positive constants b, c, M, δ such that for any increasing events A_1, A_2 that depend on disjoint boxes of size r within distance at least s from each other, we have, for all $u > 0$ and $\varepsilon \in (0, 1)$,

$$\tilde{\mathcal{Q}}^u[A_1 A_2] \leq \tilde{\mathcal{Q}}^{(1-\varepsilon)u}[A_1]\tilde{\mathcal{Q}}^{(1-\varepsilon)u}[A_2] + c(r+s)^M \exp(-\gamma_1 \varepsilon^b u s^{1+\delta}).$$

Then for all $u > u^{**}$ (where u^{**} is defined as in (3.6) with obvious notational changes) we would obtain exponential decay as in (3.4) (again, with obvious notational changes). The proof would go through practically unaltered.

4. Soft local times and simulations with Poisson processes

In this section we prove a result about simulating sequences of random variables using Poisson processes. Besides being interesting in itself, this result will be a major ingredient in order to couple various random interlacements during the proof of Theorem 2.1.

Let Σ be a locally compact and Polish metric space. Suppose also that we are given a measure space $(\Sigma, \mathcal{B}, \mu)$ where $\mathcal{B}$ is the Borel σ-algebra on Σ and μ is a Radon measure, i.e., every compact set has finite μ-measure.

The above setup is standard for the construction of a Poisson point process on Σ. For this, we also consider the space of Radon point measures on $\Sigma \times \mathbb{R}_+$,

$$L = \left\{ \eta = \sum_{\lambda \in \Lambda} \delta_{(z_\lambda, v_\lambda)};\ z_\lambda \in \Sigma, v_\lambda \in \mathbb{R}_+ \text{ and } \eta(K) < \infty \text{ for all compact } K \right\}, \tag{4.1}$$

endowed with the σ-algebra $\mathcal{D}$ generated by the evaluation maps $\eta \mapsto \eta(S)$, $S \in \mathcal{B} \otimes \mathcal{B}(\mathbb{R})$.

Note that the index set Λ in the above sum has to be countable. However, we do not use $\mathbb{Z}_+$ for this indexing, because (z_λ, v_λ) will be ordered later and only then will we endow them with an ordered indexing set.

One can now canonically construct a Poisson point process η on the space $(L, \mathcal{D}, \mathbb{Q})$ with intensity given by $\mu \otimes dv$, where dv is the Lebesgue measure on $\mathbb{R}_+$. For more details on this construction, see for instance [16, Proposition 3.6, p. 130].

The proposition below provides us with a way to simulate a random element of Σ using the Poisson point process η. Although this result is very simple and intuitive, we provide here its proof for the sake of completeness and the reader's convenience.

Proposition 4.1. *Let $g : \Sigma \to \mathbb{R}_+$ be a measurable function with $\int g(z)\,\mu(dz) = 1$. For $\eta = \sum_{\lambda \in \Lambda} \delta_{(z_\lambda, v_\lambda)} \in L$, define*

$$\xi = \inf\{t \geq 0;\ \text{there exists } \lambda \in \Lambda \text{ such that } tg(z_\lambda) \geq v_\lambda\} \tag{4.2}$$

(see Figure 2). Then under the law $\mathbb{Q}$ of the Poisson point process η,

(i) *there exists a.s. a unique $\hat{\lambda} \in \Lambda$ such that $\xi g(z_{\hat{\lambda}}) = v_{\hat{\lambda}}$,*
(ii) *$(z_{\hat{\lambda}}, \xi)$ is distributed as $g(z)\mu(dz) \otimes \mathrm{Exp}(1)$,*
(iii) *$\eta' := \sum_{\lambda \neq \hat{\lambda}} \delta_{(z_\lambda, v_\lambda - \xi g(z_\lambda))}$ has the same law as η and is independent of $(\xi, \hat{\lambda})$.*

As mentioned in the introduction, a statement similar to the above proposition has already been established in the special case of $\Sigma = (0, 1) \subset \mathbb{R}$ in [33, Claim 1.5].

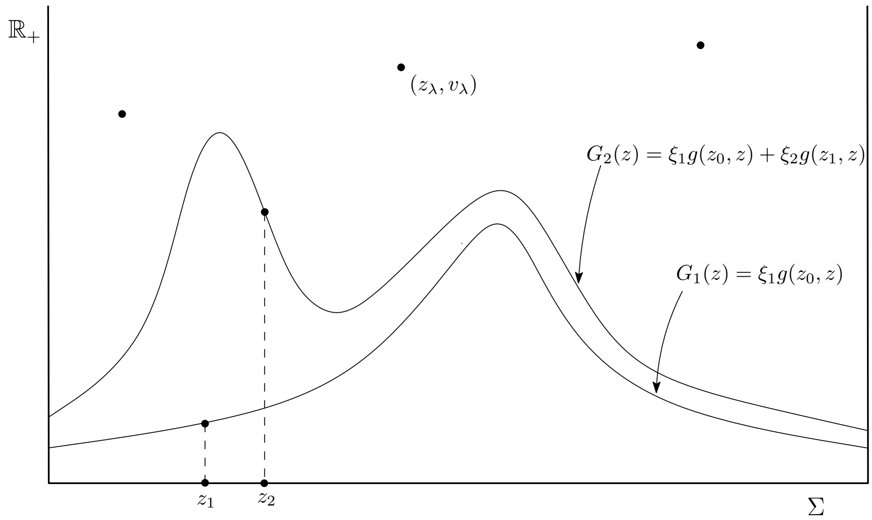

Fig. 2. An example illustrating the definition of ξ and $\hat{\lambda}$ in Proposition 4.1. More generally, ξ_1, z_1 and ξ_2, z_2 are as in (4.32).

Proof. Let us first define, for any measurable $A \subset \Sigma$, the random variable

$$\xi^A = \inf\{t \geq 0;\ \text{there exists } \lambda \in \Lambda \text{ such that } t\mathbb{1}_A g(z_\lambda) \geq v_\lambda\}. \tag{4.3}$$

Elementary properties of Poisson point processes (see for instance [16, (a) and (b), p. 130]) imply that

$$\begin{aligned} &\xi^A \text{ is exponentially distributed (with parameter } \textstyle\int_A g(z)\,\mu(dz)) \text{ and} \\ &\text{if } A \text{ and } B \text{ are disjoint, then } \xi^A \text{ and } \xi^B \text{ are independent.} \end{aligned} \tag{4.4}$$

Property (1) now follows from (4.4), using the fact that Σ is separable and that two independent exponential random variables are almost surely distinct. Observe also that

$$\mathbb{Q}[\xi \geq \alpha,\ z_{\hat{\lambda}} \in A] = \mathbb{Q}[\xi^{\Sigma \setminus A} > \xi^{A} \geq \alpha]. \tag{4.5}$$

Thus, using (4.4) we can prove property (2) using simple properties of the minimum of independent exponential random variables.

Finally, let us establish property (3). We first claim that, given ξ, $\eta'' := \sum_{\lambda \neq \hat{\lambda}} \delta_{(z_\lambda, v_\lambda)}$ is a Poisson point process, which is independent of $z_{\hat{\lambda}}$ and, conditioned on ξ, has intensity measure $\mathbb{1}_{\{v > \xi g(z)\}} \cdot \mu(dz) \otimes dv$.

This is a consequence of the strong Markov property for Poisson point processes and the fact that $\{(z, v) \in \Sigma \times \mathbb{R}_+;\ v \leq \xi g(z)\}$ is a stopping set [17, Theorem 4].

To finish the proof, we observe that, given ξ, η' is a mapping of η'' (in the sense of [16, Proposition 3.7, p. 134]). This mapping pulls back the measure $\mathbb{1}_{\{v > \xi g(z)\}} \cdot \mu(dz) \otimes dv$ to $\mu(dz) \otimes dv$. Noting that the latter distribution does not involve ξ, we conclude the proof of (3) and therefore of the lemma. □

Let us now use the same Poisson point process η to simulate not only a single random element of Σ, but a Markov chain $(Z_k)_{k \geq 1}$. For this, suppose that in some probability space $(L', \mathcal{D}', \mathcal{P})$ we are given a Markov chain $(Z_k)_{k \geq 1}$ on Σ with transition densities

$$\mathcal{P}[Z_{k+1} \in dz \mid Z_k] = g(Z_k, z)\mu(dz) \quad \text{ for } k \geq 1, \tag{4.6}$$

where $g(\cdot, \cdot)$ is $\mathcal{B}$-measurable in each of its coordinates and integrates to 1 with respect to μ in the second coordinate.

We moreover suppose that the starting distribution of the Markov chain is also absolutely continuous with respect to μ. In fact, in order to simplify the notation, we suppose that

$$Z_1 \text{ is distributed as } g(Z_0, z)\mu(dz). \tag{4.7}$$

Observe that the Markov chain starts at time one, so that there is no element Z_0 in the chain. In fact, (4.7) should be regarded as a notation for the distribution of Z_1, which is consistent with (4.6) for convenient indexing. This notation will be particularly useful in Theorem 4.8 below.

Remark 4.2. Observe that, in principle, Z_k could be any process adapted to a filtration and the arguments of this section would still work, as long as their conditional distribution are absolutely continuous with respect to μ. However, for simplicity we only deal with Markovian processes here, as the notation for general processes would be more complicated.

Using Proposition 4.1, we introduce

$$\begin{aligned} &\xi_1 := \inf\{t \geq 0;\ \text{there exists } \lambda \in \Lambda \text{ such that } t g(Z_0, z_\lambda) \geq v_\lambda\}, \\ &G_1(z) := \xi_1 g(Z_0, z) \text{ for } z \in \Sigma, \\ &(z_1, v_1) \text{ is the unique pair in } \{(z_\lambda, v_\lambda)\}_{\lambda \in \Lambda} \text{ with } \xi_1 G_1(z_1) = v_1 \end{aligned} \tag{4.8}$$

(see Figure 2).

It is clear from Proposition 4.1 that z_1 is distributed as Z_1 and that the point process $\sum_{(z_\lambda,v_\lambda)\neq(z_1,v_1)} \delta_{(z_\lambda, v_\lambda - G_1(z_\lambda))}$ is distributed as η. In fact we can continue this construction starting with η' to prove the following

Proposition 4.3. *We can proceed iteratively to define ξ_n, G_n and (z_n, v_n) as follows, for all $n \geq 1$:*

$$\xi_n := \inf\{t \geq 0;\ \exists (z_\lambda, v_\lambda) \notin \{(z_k, v_k)\}_{k=1}^{n-1} : G_{n-1}(z_\lambda) + t g(z_{n-1}, z_\lambda) \geq v_\lambda\}, \tag{4.9}$$

$$G_n(z) = G_{n-1}(z) + \xi_n g(z_{n-1}, z), \tag{4.10}$$

$$(z_n, v_n) \text{ is the unique pair } (z_\lambda, v_\lambda) \notin \{(z_k, v_k)\}_{k=1}^{n-1} \text{ with } G_n(z_\lambda) = v_\lambda, \tag{4.11}$$

$$(z_1, \dots, z_n) \overset{d}{\sim} (Z_1, \dots, Z_n) \text{ and they are independent of } \xi_1, \dots, \xi_n, \tag{4.12}$$

$$\sum_{(z_\lambda, v_\lambda)\notin\{(z_k,v_k)\}_{k=1}^{n}} \delta_{(z_\lambda, v_\lambda - G_n(z_\lambda))} \text{ is distributed as } \eta \text{ and independent of the above.} \tag{4.13}$$

See Figure 2 for an illustration of this iteration.

We call G_n the *soft local time* of the Markov chain, up to time n, with respect to the reference measure μ. We will justify the choice of this name in Theorem 4.6 below.

From the above construction we have the following

Corollary 4.4. *On the probability measure $\mathbb{Q}$ (where we defined the Poisson point process η) we can construct the Markov chain $(Z_k)_{k\geq 1}$, in such a way that for any measurable function $v : \Sigma \to \mathbb{R}_+$,*

$$\mathbb{Q}[\{Z_1, \dots, Z_T\} \subseteq \{z_\lambda;\ v_\lambda \leq v(z_\lambda)\}] \geq \mathbb{Q}[G_T(z) \leq v(z) \text{ for } \mu\text{-a.e. } z \in \Sigma] \tag{4.14}$$

for any finite stopping time $T \geq 1$.

Remark 4.5. Let us now comment on how the above corollary compares with other techniques for approximate domination present in the literature. One such method is called "Poissonization" and appears in various works, for instance [23], [22], [32]. Loosely speaking, the method of Poissonization attempts to compare the elements $Z_1, Z_2, \dots$ with $z_1, z_2, \dots$ one by one, so that one needs the transition densities $g(z, z')$ to be close to one (in $L^1(\mu)$) uniformly over z. Not having such a requirement is the main contribution of our technique, which will be useful later when working with random interlacements.

In order to estimate the right-hand side of (4.14), it is natural to resort to concentration inequalities or large deviations principles for the sum defining G_T. For this it is first necessary to obtain the expectation of the soft local time $G_T(z)$. The following proposition relates this with the expectation of the usual local time of the chain Z_k, and that is the main reason why we call G_k a soft local time.

We define the local time measure of the chain $(Z_k)_{k\geq 1}$ up to time n by

$$\mathcal{L}_n = \sum_{k \leq n} \delta_{Z_k}. \tag{4.15}$$

Observe that in some examples, the probability that $z \in \Sigma$ is visited by the Markov chain could be zero for every $z \in \Sigma$ (for instance if μ is the Lebesgue measure). Therefore, we need to use a test function in order to define what we call the *expected local time* of the chain. More precisely, we say that a measurable function $h : \Sigma \to \mathbb{R}_+$ is the *expected local time density* of $(Z_k)_{k \le n}$ with respect to μ if

$$E^{\mathcal{P}}(\mathcal{L}_n f) = \int_\Sigma f(z)h(z)\,\mu(dz) \quad \text{for every non-negative measurable } f. \tag{4.16}$$

Here n could also be replaced by a stopping time. An important special case occurs when Σ is countable and μ is the counting measure. In this case, the expected local time density $h(z)$ is given simply by the expectation of the local time $\mathcal{L}_n$ at z:

$$E^{\mathcal{P}}\Big(\sum_{k=1}^n f(Z_k)\Big) = \sum_{k=1}^n \sum_z f(z)\mathcal{P}[Z_k = z] = \sum_z f(z) E^{\mathcal{P}} \mathcal{L}_n(z). \tag{4.17}$$

For what follows, we suppose that the state space Σ contains a special element Δ which we refer to as the *cemetery*. We assume that $\mu(\{\Delta\}) = 1$ and $g(\Delta, \cdot) = 1_{\{\Delta\}}(\cdot)$, or in other words, that the cemetery is an absorbing state. We write T_Δ for the hitting time of Δ which is a *killing time* for the chain in the sense of [7, (2)]. We will also assume that test functions f as in (4.16) are zero at the cemetery.

The next result relates the expected local time density to the expectation of the soft local time.

Theorem 4.6. *Consider a state space $(\Sigma, \mathcal{B}, \mu)$ with a cemetery state Δ and a Markov chain $(Z_k)_{k\ge1}$ satisfying (4.7) and (4.6). Then*

$$E^{\mathbb{Q}}[G_{T_\Delta}(z)] \text{ is the expected local time density of } (Z_k)_{k \le T_\Delta} \text{ as in (4.16).} \tag{4.18}$$

The result is also true when T_Δ is replaced by a deterministic time.

Proof. Given some $n \ge 1$, let us calculate

$$\begin{aligned} E^{\mathcal{P}}\Big(\sum_{k=1}^n f(Z_k)\Big) &= E^{\mathcal{P}} f(Z_1) + E^{\mathcal{P}}\Big(\sum_{k=2}^n E^{\mathcal{P}}_{Z_{k-1}} f(Z_1)\Big) \\ &= E^{\mathcal{P}}\Big(\sum_{k=1}^n \int f(z) g(Z_{k-1}, z)\mu(dz)\Big) \\ &\overset{(4.12)}{=} E^{\mathbb{Q}} \int f(z) G_n(z)\,\mu(dz) = \int f(z) E^{\mathbb{Q}} G_n(z)\,\mu(dz), \end{aligned} \tag{4.19}$$

proving the validity of the proposition for the deterministic time n. We now let n go to infinity and the result follows from the monotone convergence theorem and the fact that f is zero at Δ. ◻

Let us remark that the above proof can be adapted to any *killing time*; on the other hand, one cannot put an arbitrary *stopping time* in place of T_Δ in Theorem 4.6.

Before stating the next result, let us discuss a bit further our convention on the starting distribution of the Markov chain. According to (4.7), Z_1 is distributed as $g(Z_0, z)\mu(dz)$, but this was seen as a mere notation for convenient indexing and Z_0 had no meaning whatsoever in that equation. However, it is clear that given any $z_0 \in \Sigma$, we could plug it in the first coordinate of $g(\cdot,\cdot)$ as in (4.6) to define the density of Z_1. Then the whole construction of ξ_k, G_k and (z_k, v_k) in Proposition 4.3 would depend on the specific choice of z_0. In the next proposition, we write $\mathbb{Q}_{z_0}$ for the measure $\mathbb{Q}$, where the construction of ξ_k, G_k and (z_k, v_k) (recall (4.9)) is obtained starting from the density $g(z_0, z)$. We also denote by $E^{\mathbb{Q}}_{z_0}$ the corresponding expectation.

Remark 4.7. Let us also observe that restricting the distribution of Z_1 to be $g(z_0, z)\mu(dz)$ for some $z_0 \in \Sigma$ does not represent any additional loss of generality, as z_0 could be an artificial state introduced in Σ, from which $g(z_0, z)$ is any desired density for Z_1.

The next two theorems are useful in estimating the second and exponential moments of the soft local times. This will be useful in the proofs of Lemma 6.2 and Theorem 2.1.

Besides calculating the expectation of G_k, it is useful to estimate its second moment.

Theorem 4.8. *For any $z, z_0 \in \Sigma$,*

$$E^{\mathbb{Q}}_{z_0}(G_{T_\Delta}(z))^2 \le 4E^{\mathbb{Q}}_{z_0}(G_{T_\Delta}(z)) \sup_{z_0'} E^{\mathbb{Q}}_{z_0'} G_{T_\Delta}(z). \tag{4.20}$$

The result is also true with T_Δ replaced by a deterministic time.

Proof. For $z \in \Sigma \setminus \Delta$ and $n \ge 1$, we write (recall that the expectation of $(\mathrm{Exp}(1))^2$ is 2)

$$\begin{aligned}
E^{\mathbb{Q}}_{z_0}(G_n(z))^2 &= E^{\mathbb{Q}}_{z_0}\Big(\sum_{k=1}^{n} \xi_k g(z_{k-1}, z)\Big)^2\\
&= E^{\mathbb{Q}}_{z_0}\Big(\sum_{k=1}^{n} \xi_k^2 g^2(z_{k-1}, z)\Big) + E^{\mathbb{Q}}_{z_0}\Big(2\sum_{k<k'\le n} \xi_k\xi_{k'} g(z_{k-1}, z) g(z_{k'-1}, z)\Big)\\
&\le \sum_{k=1}^{n} E\xi_k^2 \sup_{z'} g(z', z) E^{\mathbb{Q}}_{z_0} g(z_{k-1}, z) + 2\sum_{k=1}^{n-1}\sum_{k'=k+1}^{n} E^{\mathbb{Q}}_{z_0}\big(g(z_{k-1}, z) g(z_{k'-1}, z)\big)\\
&\le 2\sup_{z'} g(z', z) E^{\mathbb{Q}}_{z_0} G_n(z) + 2\sum_{k=1}^{n-1}\sum_{k'=k+1}^{n} E^{\mathbb{Q}}_{z_0}\big(g(z_{k-1}, z) E^{\mathbb{Q}}_{z_0}(g(z_{k'-1}, z) \mid z_{k-1})\big)\\
&\le 2\sup_{z'} E^{\mathbb{Q}}_{z'} G_n(z) E^{\mathbb{Q}}_{z_0} G_n(z) + 2\sum_{k=1}^{n-1} E^{\mathbb{Q}}_{z_0}\Big(g(z_{k-1}, z) E^{\mathbb{Q}}_{z_{k-1}}\Big(\sum_{m=1}^{n-k} g(z_{m-1}, z)\Big)\Big)\\
&\le 2\sup_{z'} E^{\mathbb{Q}}_{z'} G_n(z) E^{\mathbb{Q}}_{z_0} G_n(z) + 2\sup_{z'} E^{\mathbb{Q}}_{z'}\Big(\sum_{m=1}^{n-k} g(z_{m-1}, z)\Big) E^{\mathbb{Q}}_{z_0}\Big(\sum_{k=1}^{n-1} g(z_{k-1}, z)\Big)\\
&\le 4E^{\mathbb{Q}}_{z_0}(G_n(z)) \sup_{z_0'} E^{\mathbb{Q}}_{z_0'} G_n(z),
\end{aligned}$$

proving the result for the deterministic time n. Then we simply let n go to infinity and use the monotone convergence theorem. □

The next result provides an estimate on the exponential moments of G_{T_Δ}, which is clearly an important ingredient in bounding the right-hand side of (4.14). The next theorem imposes some regularity condition on the transition densities $g(\cdot,\cdot)$ (which will be encoded in ℓ and α below) to help obtain such fast decaying bounds. Intuitively speaking, the regularity condition says that if there is a big accumulation of densities g at some point $\hat{z}$, then there should be a big accumulation of densities in a large set Γ.

Theorem 4.9. *Given $\hat{z} \in \Sigma$ and measurable $\Gamma \subset \Sigma$, let*

$$\begin{aligned} \alpha &= \inf\left\{ \frac{g(z,z')}{g(z,\hat{z})};\ z \in \Sigma,\ z' \in \Gamma,\ g(z,\hat{z}) > 0 \right\}, \\ N(\Gamma) &= \#\{k \le T_\Delta;\ z_k \in \Gamma\}, \qquad \ell \ge \sup_{z' \in \Sigma} g(z',\hat{z}). \end{aligned} \tag{4.21}$$

Then, for any $v \ge 2$,

$$\begin{aligned} &\mathbb{Q}[G_{T_\Delta}(\hat{z}) \ge v\ell] \\ &\quad \le \mathbb{Q}[G_{T_\Delta}(\hat{z}) \ge \ell]\Big(\exp\big\{-\big(\tfrac{1}{2}v - 1\big)\big\} + \sup_{z'} \mathbb{Q}_{z'}\big[\eta\big(\Gamma \times \big[0, \tfrac{1}{2}v\ell\alpha\big]\big) \le N(\Gamma)\big]\Big), \end{aligned}$$

(*recall the definition of η in* (4.1) *and observe that $\eta(\Gamma \times [0, \frac{1}{2}v\ell\alpha])$ is a random variable with distribution* $\mathrm{Poisson}\big(\frac{1}{2}v\ell\alpha\mu(\Gamma)\big)$).

Before proving the above theorem, let us give an idea of what each term in the above bound represents. In order for $G_{T_\Delta}(\hat{z})$ to get past $v\ell$, it must first overcome ℓ, which explains the first term in the above bound. Then the two terms inside the parenthesis above correspond respectively to the overshooting probability and a large deviations term. We can expect the second term to decay fast as v grows, since $N(\Gamma)$ becomes much smaller than the expected value of $\eta(\Gamma \times [0, \frac{1}{2}v\ell\alpha])$.

Proof of Theorem 4.9. Define the stopping time (with respect to the filtration $\mathcal{F}_n = \sigma(z_k, \xi_k, k \le n)$)

$$T_\ell = \inf\{k \ge 1;\ G_k(\hat{z}) \ge \ell\}. \tag{4.22}$$

Now, for any $v \ge 2$, we can bound $\mathbb{Q}[G_{T_\Delta}(\hat{z}) \ge v\ell]$ by

$$\mathbb{Q}\big[T_\ell < \infty,\ G_{T_\ell}(\hat{z}) \ge \tfrac{1}{2}v\ell\big] + \mathbb{Q}\big[T_\ell < \infty,\ G_{T_\ell}(\hat{z}) < \tfrac{1}{2}v\ell,\ G_{T_\Delta}(\hat{z}) - G_{T_\ell}(\hat{z}) > \tfrac{1}{2}v\ell\big] \tag{4.23}$$

(observe that $\mathbb{Q}[G_{T_\Delta}(\hat{z}) \ge \ell] = \mathbb{Q}[T_\ell < \infty]$). We start by estimating the first term in the above sum, which equals (using the memoryless property of the exponential distribution)

$$\begin{aligned} &\sum_{n \ge 1} E^{\mathbb{Q}}\Big(G_{n-1}(\hat{z}) < \ell,\ \mathbb{Q}\big[\xi_n g(z_{n-1},\hat{z}) > \tfrac{1}{2}v\ell - G_{n-1}(\hat{z})\ \big|\ z_{n-1}, G_{n-1}\big]\Big) \\ &\quad \le \sum_{n \ge 1} E^{\mathbb{Q}}\Big(G_{n-1}(\hat{z}) < \ell,\ \mathbb{Q}[\xi_1 g(z_{n-1},\hat{z}) > \ell - G_{n-1}]\, \mathbb{Q}\big[\xi_1 g(z_{n-1},\hat{z}) > \big(\tfrac{1}{2}v - 1\big)\ell\big]\Big) \\ &\quad \le \mathbb{Q}[T_\ell < \infty] \sup_{z' \in \Sigma} \mathbb{Q}\big[\xi_1 g(z',\hat{z}) > \big(\tfrac{1}{2}v - 1\big)\ell\big] \le \mathbb{Q}[T_\ell < \infty] \exp\big\{-\big(\tfrac{1}{2}v - 1\big)\big\}. \end{aligned} \tag{4.24}$$

We now turn to the bound on the second term in (4.23), which is

$$E^{\mathbb{Q}}\big(T_\ell < \infty,\ G_{T_\ell}(\hat z) < \tfrac{1}{2}v\ell, \mathbb{Q}\big[G_{T_\Delta}(\hat z) - G_{T_\ell}(\hat z) > \tfrac{1}{2}v\ell \bigm| G_1, \dots, G_{T_\ell}\big]\big) \\ \le \mathbb{Q}[T_\ell < \infty] \sup_{z'} \mathbb{Q}_{z'}\big[G_{T_\Delta}(\hat z) > \tfrac{1}{2}v\ell\big]. \tag{4.25}$$

Now since for any $z' \in \Sigma$,

$$G_{T_\Delta}(z') = \sum_{k=1}^{T_\Delta} \xi_k g(z_{k-1}, z') \ge \sum_{k=1}^{T_\Delta} \alpha \xi_k g(z_{k-1}, \hat z) \mathbb{1}_\Gamma(z') = \alpha G_{T_\Delta}(\hat z) \mathbb{1}_\Gamma(z'), \tag{4.26}$$

we deduce that for all z',

$$\mathbb{Q}_{z'}\big[G_{T_\Delta}(\hat z) \ge \tfrac{1}{2}v\ell\big] \le \mathbb{Q}_{z'}\big[G_{T_\Delta}(z) \ge \tfrac{1}{2}v\ell\alpha \text{ for every } z \in \Gamma\big] \\ \le \mathbb{Q}_{z'}\big[\eta\big(\Gamma \times \big[0, \tfrac{1}{2}v\ell\alpha\big]\big) \le N(\Gamma)\big]. \tag{4.27}$$

Combining (4.23) with (4.24), (4.25) and the above we obtain the desired result. □

Unfortunately, the simulation of a single Markov chain will not suffice for our purposes in this work. As suggested by the definition of random interlacements in terms of a collection of random walks (see (2.14)), we will need to apply the above scheme to construct a sequence of independent Markov chains on Σ and to this end, we will make use of the same Poisson point process η. This is done in Proposition 4.10 below, which requires some further definitions.

Suppose that in some probability space $(L, \mathcal{L}, \mathcal{P})$ we are given a collection $(Z_k^j)_{j,k\ge 1}$ of random elements of Σ such that

$$\text{for any given } j \ge 1, \text{ the sequence } (Z_1^j, Z_2^j, \dots) \text{ is a Markov chain on } \Sigma, \\ \text{characterized by } \mathcal{P}[Z_k^j \in dz \mid Z_{k-1}^j] = g(Z_{k-1}^j, z)\mu(dz) \text{ for } k = 1, 2, \dots, \tag{4.28}$$

$$\text{for distinct values of } j, \text{ the above Markov chains are independent.} \tag{4.29}$$

Recall that we interpret (4.28) for $k = 1$ as a notation for the starting distribution of the chain as we did in (4.7). However, we are allowed to impose different starting laws (for distinct values of j) by choosing the Z_0^j's. Although they have a possibly different starting distribution, they all evolve independently and under the same transition laws.

Suppose that for each $j \ge 1$,

$$\text{the hitting time of } \Delta \text{ (as below (4.17)) is } \mathcal{P}_{Z_0^j}\text{-a.s. finite,} \tag{4.30}$$

where $\mathcal{P}_z$ denotes the law of this Markov chain evolution starting from z.

In what follows, we are going to use a single Poisson point process η to simulate all the above Markov chains (Z_k^j) until they hit Δ. We do this by simply repeating the procedure of Proposition 4.3 following the lexicographic order $(j, k) \preccurlyeq (j', k')$ if either $j < j'$, or $j = j'$ and $k \le k'$. This construction results in the accumulation of the soft local times of all the chains, which is essential in proving our main theorem.

In the same spirit of the definition (4.9), we set $G_0^1 \equiv 0$ and define inductively, for $n = 1, 2, \ldots$,

$$\begin{aligned}
&\xi_n^1 := \inf\{t \geq 0;\ \exists (z_\lambda, v_\lambda) \notin \{(z_k^1, v_k^1)\}_{k=1}^{n-1} \colon G_{n-1}^1(z_\lambda) + t g(z_{n-1}^1, z_\lambda) \geq v_\lambda\},\\
&G_n^1(z) = G_{n-1}^1(z) + \xi_n^1 g(z_{n-1}^1, z),\\
&(z_n^1, v_n^1) = \text{the unique pair } (z_\lambda, v_\lambda) \notin \{(z_k^1, v_k^1)\}_{k=1}^{n-1} \text{ with } G_n^1(z_\lambda) = v_\lambda.
\end{aligned} \tag{4.31}$$

We write T_Δ^1 for the hitting time of Δ by the chain $(z_1^1, z_2^1, z_3^1, \ldots)$. Applying Proposition 4.3, we find that $(z_1^1, \ldots, z_{T_\Delta^1}^1)$ is distributed as $(Z_1^1, \ldots, Z_{T_\Delta}^1)$ under the law $\mathcal{P}$ and that

$$\eta' := \sum_{(z_\lambda, v_\lambda) \notin \{(z_n^1, v_n^1)\}_{n \leq T_\Delta^1}} \delta_{(z_\lambda, v_\lambda - G_{T_\Delta^1}^1(z_\lambda))}$$

is distributed as η and independent of the above.

Now that we are done simulating the first Markov chain up to time T_Δ^1 using η, let us continue the above procedure in order to obtain from η' the chain $(Z_k^2)_{k \geq 1}$ and so on. Supposing we have concluded the construction up to $m - 1$, let $G_0^m \equiv 0$ and define for $n = 1, \ldots, T_\Delta^m$ (T_Δ^m stands for the absorption time of the mth chain),

$$\begin{aligned}
&\xi_n^m := \inf\Big\{t \geq 0;\ \exists (z_\lambda, v_\lambda) \notin \{(z_k^j, v_k^j)\}_{(j,k) \preccurlyeq (m, n-1)} \colon\\
&\qquad\qquad \textstyle\sum_{j=1}^{m-1} G_{T_\Delta^j}^j(z_\lambda) + G_{n-1}^m(z_\lambda) + t g(z_{n-1}^m, z_\lambda) \geq v_\lambda\Big\},\\
&G_n^m(z) = G_{n-1}^m(z) + \xi_n^m g(z_{n-1}^m, z),\\
&(z_n^m, v_n^m) \notin \{(z_k^j, v_k^j)\}_{(j,k) \preccurlyeq (m, n-1)} \text{ with } \textstyle\sum_{j=1}^{m-1} G_{T_\Delta^j}^j(z_\lambda) + G_n^m(z_\lambda) = v_\lambda.
\end{aligned} \tag{4.32}$$

The following proposition summarizes the main properties of the above construction and its proof is a straightforward consequence of Proposition 4.3.

Proposition 4.10. *Suppose we are given starting densities $g(Z_0^j, \cdot)$ ($j \geq 1$) and transition densities $g(\cdot, \cdot)$ of a Markov chain as in (4.28). Then, defining ξ_k^j, G_k^j and z_k^j for $j = 1, 2, \ldots$ and $k = 1, \ldots, T_\Delta^j$ as in (4.32), one has:*

$$(\xi_k^j, j \geq 1, k \leq T_\Delta^j) \text{ are i.i.d. } \mathrm{Exp}(1)\text{-random variables,} \tag{4.33}$$

$$(z_k^j, j \geq 1, k \leq T_\Delta^j) \stackrel{d}{\sim} (Z_k^j, j \geq 1, k \leq T_\Delta^j) \text{ are independent of } \xi_k^j\text{'s.} \tag{4.34}$$

The most relevant conclusion of the proposition is (4.34), showing that our method indeed provides a way to simulate a sequence of independent Markov chains.

5. Construction of random interlacements from a soup of excursions

In this section we use Proposition 4.10 to construct random interlacements in an alternative way. The advantage of this new construction is that it is more "local" than the usual one, i.e., it does not reveal the interlacement configuration far away from the set of interest; this of course facilitates the decoupling of the configuration on different sets, and that is why we consider this construction to be the key idea of this paper. Note that the canonical construction of the random interlacements (presented in Section 2) does not have this property of "localization", since it is quite probable that many walkers would do *long* excursions away from the set of interest before eventually coming back.

Let us start with a simple decomposition of random interlacements that prepares the ground for the main construction of this section.

5.1. Decomposition of random interlacements

A crucial ingredient in proving our main result is a decomposition of the interlacement set $\mathcal{I}^u$ that we now describe. For the rest of this section, let K be a fixed finite subset of $\mathbb{Z}^d$.

Consider first the map $s_K : W_K^* \to W$ defined by

$$s_K(w^*) \text{ is the unique trajectory } w \in W \text{ with } \pi^*(w) = w^* \text{ and } H_K(w) = 0. \tag{5.1}$$

We also introduce, for $w \in W$, the one-sided trajectories $w^+ = (X_i(w))_{i \geq 0}$ and $w^- = (X_{-i}(w))_{i \geq 0}$ in W_+. These can be seen as the future and past of w.

Let us define the space of point measures

$$M = \left\{ \chi = \sum_{i \in I} \delta_{(w_i, u_i)};\ \begin{array}{l} I \subset \mathbb{N},\, w_i \in W_+,\ u_i \in \mathbb{R}_+ \text{ and} \\ \omega(W_+ \times [0, u]) < \infty \text{ for every } u \geq 0 \end{array} \right\}, \tag{5.2}$$

endowed with the σ-algebra $\mathcal{M}$ generated by the evaluation maps $\chi \mapsto \chi(D)$ for $D \in \mathcal{W}_+ \otimes \mathcal{B}(\mathbb{R}_+)$. And for $\chi = \sum_i \delta_{(w_i, u_i)}$ we extend the definition in (2.14) to M as follows:

$$\mathcal{I}^u(\chi) = \bigcup_{i:\, u_i < u} \text{Range}(w_i). \tag{5.3}$$

We can now introduce, for $\omega = \sum_i \delta_{(w_i^*, u_i)} \in \Omega$, maps $\chi_K^+, \chi_K^- : \Omega \to M$ by

$$\chi_K^+(\omega) = \sum_{i:\, w_i^* \in W_K^*} \delta_{(s_K(w_i^*)^+, u_i)} \quad \text{and} \quad \chi_K^-(\omega) = \sum_{i:\, w_i^* \in W_K^*} \delta_{(s_K(w_i^*)^-, u_i)} \quad \text{in } M. \tag{5.4}$$

We also define the analogous point processes $\chi_{K,u}^+$ and $\chi_{K,u}^-$ where the summations are taken only over $u_i \leq u$.

The main observation concerning these point processes is stated in the following proposition, which is a direct consequence of (2.11) and (2.12).

Proposition 5.1. *For any finite set $K \subset \mathbb{Z}^d$, the law of (χ_K^+, χ_K^-) under $\mathbb{P}$ is a Poisson point process on $(M \times M, \mathcal{M} \otimes \mathcal{M})$ with intensity measure characterized by*

$$\zeta_K(A\times[a,b]\times B\times[c,d]) = \Delta((a,b)\times(c,d)) \sum_{x\in K} e_K(x) P_x[A] P_x[B \mid \tilde{H}_K = \infty] \quad (5.5)$$

for $A, B \in \mathcal{W}_+$ and $a < b, c < d \in \mathbb{R}$. Here Δ is the Lebesgue measure on the diagonal in $\mathbb{R}^2$ divided by $\sqrt{2}$.

A way to rephrase the above proposition is to say that we can simulate the pair $(\chi_{K,u}^+, \chi_{K,u}^-)$ as follows:

- let Θ_u^K be a Poisson($u\,\mathrm{cap}(K)$)-distributed random variable,
- choose i.i.d. points $X_0^1, \dots, X_0^{\Theta_u^K}$ with law $\bar{e}_K$, and
- from each point X_0^j, start two trajectories, with laws given respectively by $P_{X_0^j}$ and $P_{X_0^j}[\,\cdot\,|\tilde{H}_K = \infty]$.

Given a finite set $K \subset \mathbb{Z}^d$, we are going to decompose the interlacement set $\mathcal{I}^u$ as the union of three sets $\mathcal{I}_{K,+}^u, \mathcal{I}_{K,-}^u$ and $\widehat{\mathcal{I}}_K^u$ given by

$$\mathcal{I}_{K,+}^u(\omega) = \mathcal{I}^u(\chi_K^+(\omega)), \quad \mathcal{I}_{K,-}^u(\omega) = \mathcal{I}^u(\chi_K^-(\omega)), \quad \widehat{\mathcal{I}}_K^u(\omega) = \mathcal{I}^u(\mathbb{1}\{W^* \setminus W_K^*\} \cdot \omega) \quad (5.6)$$

(recall the definitions (2.14) and (5.3)).

Roughly speaking, the sets $\mathcal{I}_{K,+}^u$ and $\mathcal{I}_{K,-}^u$ correspond respectively to the future and past of the trajectories of $\mathcal{I}^u$ that hit K, while $\widehat{\mathcal{I}}_K^u$ encompasses the trajectories not hitting K. This decomposition will be crucial for obtaining the decoupling in Theorem 2.1, and we now present its main properties.

Proposition 5.2. *For any finite $K \subset \mathbb{Z}^d$ and $u \geq 0$,*

$$\mathcal{I}^u = \mathcal{I}_{K,+}^u \cup \mathcal{I}_{K,-}^u \cup \widehat{\mathcal{I}}_K^u \quad \textit{for every } \omega \in \Omega, \quad (5.7)$$

$$\mathcal{I}^u \cap K = \mathcal{I}_{K,+}^u \quad \mathbb{P}\textit{-a.s.} \quad (5.8)$$

$$\widehat{\mathcal{I}}_{K,u} \textit{ is independent of } (\mathcal{I}_{K,+}^u, \mathcal{I}_{K,-}^u). \quad (5.9)$$

Proof. To prove (5.7), one should decompose the union giving $\mathcal{I}^u$ into W_K^* and $W^* \setminus W_K^*$, observing that for each $w^* \in W_K^*$, $\mathrm{Range}(w^*) = \mathrm{Range}(s_K(w^*)^+) \cup \mathrm{Range}(s_K(w^*)^-)$.

To see why the second statement is true, observe first that $\mathcal{I}^u \cap K \subset \mathcal{I}_{K,+}^u \cup \mathcal{I}_{K,-}^u$, since we have (5.7) and $\widehat{\mathcal{I}}_K^u$ is disjoint from K. Then, observe that $\mathcal{I}_{K,-}^u \cap K$ is $\mathbb{P}$-a.s. contained in $\mathcal{I}_{K,+}^u$, which follows from Proposition 5.1, since for every $x \in \mathrm{supp}(e_K)$, $\mathrm{Range}(w) \cap K = \{X_0(w)\}$, $P_x[\,\cdot \mid \tilde{H}_K = \infty]$-a.s.

Finally, to prove (5.9), we observe that these two sets are determined by the realization of the Poisson point process ω in the disjoint spaces of trajectories W^* and $W^* \setminus W_K^*$. □

We also observe that the random variable

$$\Theta_u^K = \chi_K^+(W_+ \times [0,u]) = \chi_K^-(W_+ \times [0,u]) \text{ is Poisson}(u\,\mathrm{cap}(K))\text{-distributed.} \quad (5.10)$$

5.2. Chopping into excursions

Fix a finite set $V \subset \mathbb{Z}^d$ and a set $C \subset \mathbb{Z}^d$ such that

$$\partial C \text{ is finite.} \tag{5.11}$$

The above condition is equivalent to C being either finite or having finite complement (see Figure 3 below). Suppose also that $C \cap V = \emptyset$. Although some of the definitions that follow will depend on both V and C, we will keep only the dependence on C explicit, since the set V will be kept unchanged throughout proofs.

We are interested at first in the trace left by $\mathcal{I}^u_{V,\pm}$ on the set C. The random walks composing $\mathcal{I}^u_{V,+}$ (see (5.6)) will perform various excursions between C and V until they finally escape to infinity. This decomposition of a random walk trajectory into excursions is crucial to our proofs and we now give the details of its definition. In fact, one can look at Figure 3 to have a feeling of what is going to happen.

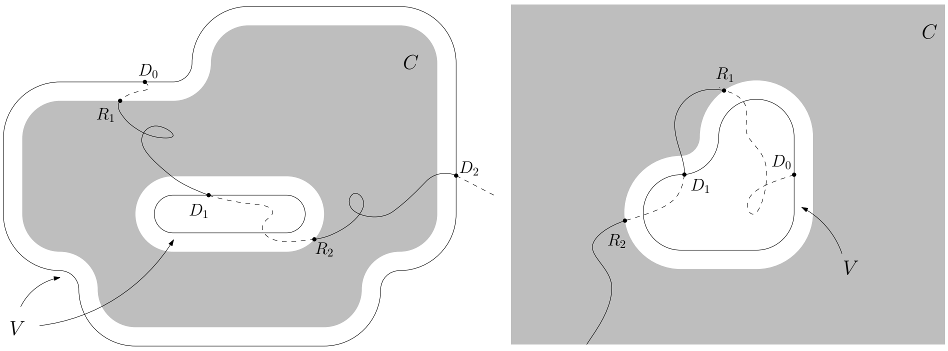

Fig. 3. Typical examples of sets C (gray) and V (closed curves). On the left C is finite, while on the right it has finite complement. The stopping times R_k and D_k are also pictured.

Given a trajectory $w_+ \in W_+$ (recall (2.2)), let us define its successive return and departure times between C and V:

$$\begin{aligned} D_0 &= 0, & R_1 &= H_C, \\ D_1 &= H_V \circ \theta_{R_1} + R_1, & R_2 &= H_C \circ \theta_{D_1} + D_1, \\ D_2 &= H_V \circ \theta_{R_2} + R_2 & &\text{and so on (see Figure 3).} \end{aligned}$$

Note that above we have omitted the dependence on w_+. Define

$$T^C = \inf\{k \geq 1;\ R_k = \infty\}, \tag{5.12}$$

which is equal to one plus the random number of excursions performed by w_+ until escaping to infinity. Since we have assumed that the set V is finite, and the random walk on $\mathbb{Z}^d$ ($d \geq 3$) is transient, T^C is finite P-almost surely.

The reason why we define T^C as *one plus* the number of excursions is to guarantee that it coincides with T_Δ as defined just after (4.17) in the construction that follows.

As mentioned before, we are interested in the intersection of $\mathcal{I}^u_{V,+}$ (recall (5.6)) with the set C. Writing $\chi^+_{V,u} = \sum_{j=1}^{\Theta^V_u} \delta_{w_j}$ (where the w's are ordered according to their corresponding u's), and abbreviating $T^C_j = T^C(w_j)$, we obtain

$$C\cap\mathcal{I}^u_{V,+} = C\cap \bigcup_{(w_+,u)\in\mathrm{supp}(\chi^+_{V,u})} (w_+) = C\cap\bigcup_{j=1}^{\Theta^V_u}\bigcup_{k=1}^{T^C_j-1}\{X_{R_k}(w_j),\dots,X_{D_k}(w_j)\}, \tag{5.13}$$

where it may occur that some of the $D_k(w_j)$'s above are infinite.

We are now going to employ the techniques of Section 4 to simulate the above collection of excursions using a Poisson point process. For this let Σ_C denote the following space of paths:

$$\begin{aligned}\Sigma_C = \{\Delta\}\cup &\left\{\begin{array}{c} w=(x_1,\dots,x_k) \text{ finite nearest neighbor path, starting} \\ \text{at } \partial C \text{ and ending at its first visit to } V\end{array}\right\} \\ \cup &\left\{\begin{array}{c} w=(x_1,x_2,\dots) \text{ infinite nearest neighbor path,} \\ \text{starting at } \partial C \text{ and never visiting } V\end{array}\right\},\end{aligned} \tag{5.14}$$

where Δ is a distinguished state that encodes the fact that a given trajectory has already diverged to infinity. Illustrations of finite and infinite paths in Σ_C can be found in Figure 3.

Consistently with the previous discussion, we use the shorthand $X^j_\cdot = X_\cdot(w_j)$; in other words, the superscript j means that we are dealing with the jth walk of the construction. The excursions induced by the random walks will be encoded as elements of Σ_C as follows

$$\begin{aligned} &Z^j_k = (X^j_{R_k},\dots,X^j_{D_k}) \in \Sigma_C \quad \text{for } k=1,\dots,T^C_j-1, \\ &Z^j_{T^C_j} = \Delta. \end{aligned} \tag{5.15}$$

The reason why we introduce the state Δ is to recover the description of Section 4, indicating that another trajectory is about to start.

In view of (5.13), in order to simulate $C\cap\mathcal{I}^u_{V,+}$, we only need to construct the excursions Z^j_k with the correct law. For this, we are going to use the construction of the previous section to simulate them from a Poisson point process. In (5.18) below, we will prove that for a fixed j, the sequence $Z^j_1, Z^j_2,\dots$ is a Markov chain, as required in (4.28) and (4.29).

Endow the space of paths Σ_C with the σ-algebra $\mathcal{S}$ generated by the canonical coordinates and with the measure μ_C given by

$$\mu_C(\mathcal{X}) = \sum_{x\in\partial C} P_x[(X_0,X_1,\dots,X_{H_V})\in\mathcal{X}] + \delta_\Delta(\mathcal{X}), \tag{5.16}$$

where $\mathcal{X}\in\mathcal{S}$. Note that μ_C is finite due to (5.11). We can therefore define a Poisson point process $\eta = \sum_i \delta_{(z_i,v_i)}$ on $\Sigma_C\times\mathbb{R}_+$ with intensity $\mu_C\otimes dv$ as in (4.1).

In order to apply Proposition 4.10, we first observe that for fixed $j\ge 1$, Z^j_k is a Markov chain, due to the Markovian character of the simple random walk. We then define

$$f^C_y(x) := P_y[X_{\tilde{H}_C} = x] \tag{5.17}$$

and apply the strong Markov property at D_{k-1} to obtain the Radon–Nikodym derivative

$$\frac{dP[Z_k^j \in \cdot \mid Z_{k-1}^j = z]}{d\mu_C}(z') = \begin{cases} 1 & \text{if } z = z' = \Delta, \\ & \text{or } z' = \Delta,\, D_{k-1} = \infty, \\ f^C_{X^j_{D_{k-1}}(z)}(X_0(z')) & \text{if } z, z' \neq \Delta,\, D_{k-1} < \infty, \\ P_{X^j_{D_{k-1}}(z)}[H_C = \infty] & \text{if } z' = \Delta \neq z,\, D_{k-1} < \infty, \\ 0 & \text{otherwise,} \end{cases} \tag{5.18}$$

for all $k \geq 2$.

The above not only shows that the sequence $Z_1^j, Z_2^j, \ldots$ is Markovian, but also that the transition density of the chain satisfies

$$g_C((x_0, \ldots, x_l), (y_0, \ldots, y_m)) = f_{x_l}(y_0) \tag{5.19}$$

(g is a density with respect to μ_C, as in (5.16)). We are now left with the starting distributions of the Markov chains Z_k^j.

Recall that we are attempting to construct the measure $\chi^+_{V,u}$, which is not independent of $\chi^-_{V,u}$. In fact, they are conditionally independent given $\{X_0(w)\}_{w \in \mathrm{supp}(\chi^-_{V,u})}$. Therefore, conditioning on $\{X_0^j\}_{j=1,\ldots,\Theta_u^V}$, the starting density of the jth chain (with respect to μ_C) satisfies

$$g_C^j(x_0, \ldots, x_l) = f_{X_0^j}(x_0). \tag{5.20}$$

Finally, we set $Z_0^j = w$ where w is any trajectory with $X_0(w) = X_0^j = X^j_{D_0}$, so that (5.18) is also satisfied for $k = 1$, in compliance with the notation in (4.28) (see also Remark 4.7).

We can now follow the construction of $\xi^C_{j,k}$ and $G^C_{j,k}$, for $j \geq 1$, $k = 1, \ldots, T_j^C$, as in (4.32). Then, using Proposition 4.10, we obtain a way to simulate the excursions Z_k^j as promised. In particular, we can show that

$$C \cap \mathcal{I}^u_{V,+} \text{ is distributed as } C \cap \bigcup_{j=1}^{\Theta_u^V} \bigcup_{k=1}^{T_j^C} \mathrm{Range}(z^C_{j,k}) \text{ under } \mathbb{Q}. \tag{5.21}$$

See Figure 4 for an illustration of the first two steps (for the first particle) of the construction of random interlacements on the set C.

We now prove a proposition that relates our main result, Theorem 2.1, to the above construction. To simplify the notation for the soft local time, we abbreviate the accumulated soft local time up to the Θ_u^Cth trajectory as

$$G_u^C = G^C_{1,T_1^C} + G^C_{2,T_2^C} + \cdots + G^C_{\Theta_u^C, T^C_{\Theta_u^C}}. \tag{5.22}$$

We can use Theorem 4.6 to obtain a short expression for $EG_v^C(z)$. For this, given $j \geq 1$, we let

$$\rho_j^C(x) = \sum_{k=1}^{T_j^C} \mathbb{1}_x(X^j_{R_k}) \tag{5.23}$$

count the number of times the jth trajectory starts an excursion through x.

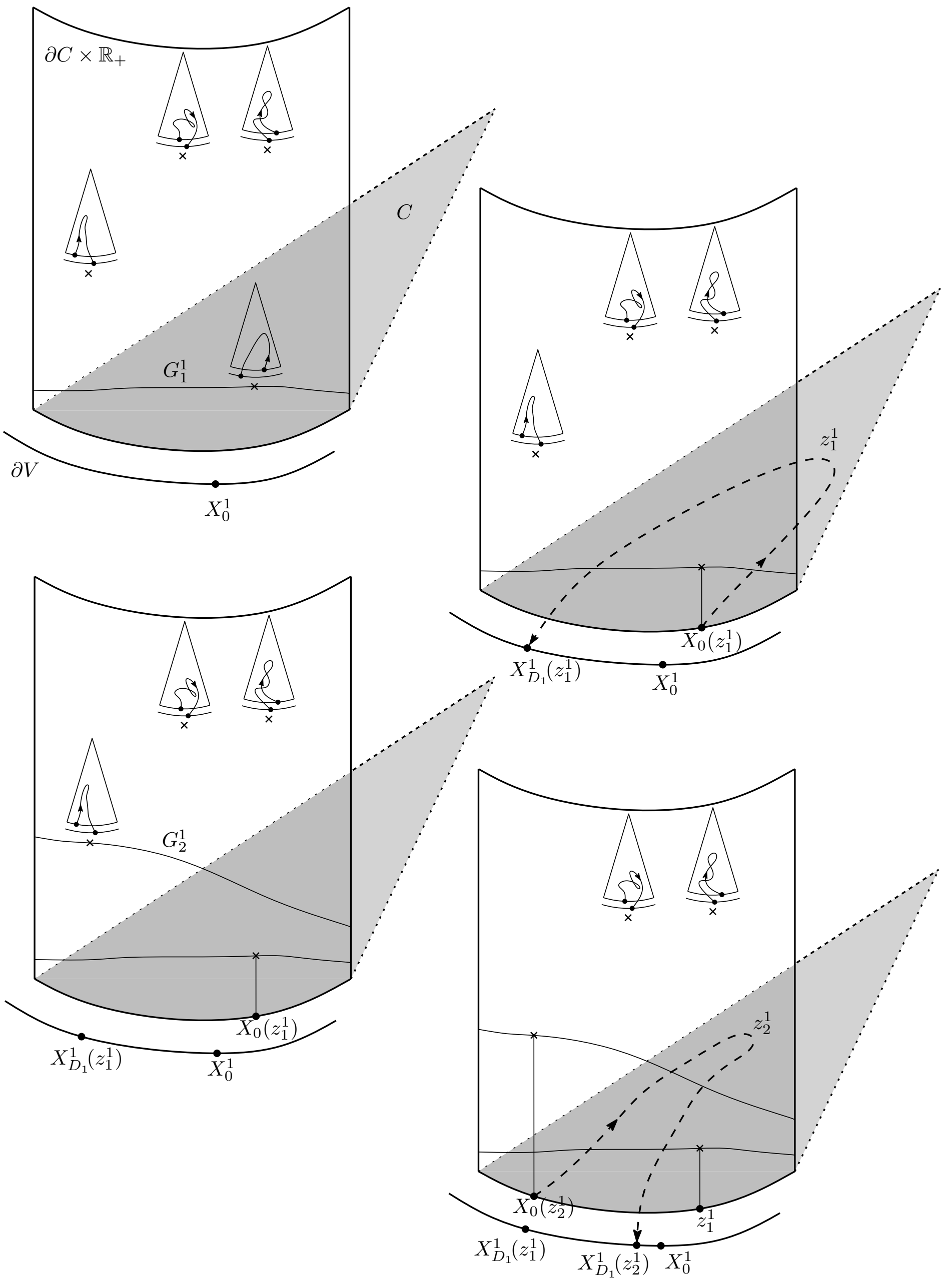

Fig. 4. The construction of random interlacements on the set C; the points of Σ_C are substituted by points in $\partial C \times \mathbb{R}_+$ with marks representing the corresponding trajectories, and the state Δ is not pictured.

Let us first recall, from (5.19), that G_v^C depends on $z = (x_0, x_1, \dots)$ solely through x_0. Thus, given $z, z' \in \Sigma_C$, we define $q(z, z') = \mathbb{1}\{X_0(z) = X_0(z')\}$ to obtain

$$E^{\mathbb{Q}} G^C_{1,T_1^C}(z) \overset{(5.19)}{=} \int q(z,z') E^{\mathbb{Q}} G^C_{1,T_1^C}(z')\, \mu_C(dz') \overset{\text{Theorem 4.6}}{=} E^{\mathcal{P}}\Big(\sum_{k=1}^{T_j^C} q(Z_k^j, z)\Big) = E^{\mathcal{P}} \rho_j^C(X_0(z)), \tag{5.24}$$

for every $z \in \Sigma_C$. Clearly, this implies that

$$E^{\mathbb{Q}} G_v^C(z) = E^{\mathbb{Q}} \Theta_v^C \times E^{\mathcal{P}} \rho_j^C(X_0(z)) = v\, \mathrm{cap}(V) E^{\mathcal{P}} \rho_j^C(X_0(z)). \tag{5.25}$$

Proposition 5.3. *Let A_1 and A_2 be two disjoint subsets of $\mathbb{Z}^d$ with A_2 having finite complement. Now suppose that*

$$V \subset \mathbb{Z}^d \text{ is such that any path from } A_1 \text{ to } A_2 \text{ crosses } V. \tag{5.26}$$

Then for every $u > 0$ and $\varepsilon \in (0,1)$ there exists a coupling $\mathbb{Q}$ between $\mathcal{I}^u$ and two independent random interlacements processes, $(\mathcal{I}_1^u)_{u \ge 0}$ and $(\mathcal{I}_2^u)_{u\ge 0}$, such that

$$\begin{aligned} &\mathbb{Q}[\mathcal{I}_k^{u(1-\varepsilon)} \cap A_k \subseteq \mathcal{I}^u \cap A_k \subseteq \mathcal{I}_k^{u(1+\varepsilon)},\ k = 1,2] \\ &\quad \ge 1 - \sum_{\substack{(v,C)=(u(1\pm\varepsilon),A_1),\\ (u(1\pm\varepsilon),A_2),(u,A_1\cup A_2)}} \mathbb{Q}\big[|G_v^C(z) - E^{\mathbb{Q}} G_v^C(z)| \ge \tfrac{1}{3}\varepsilon E^{\mathbb{Q}} G_v^C(z) \text{ for some } z \in \Sigma_C\big], \end{aligned} \tag{5.27}$$

where the soft local times above are determined in terms of V.

We note that the above proposition is an important ingredient for the proof of Theorem 2.1, since it relates the success probability of our decoupling to an estimate on the soft local times. In Section 6, we will bound the right-hand side of (5.27) using large deviations. One should not be worried that the set Σ_C may be uncountable (in case the excursions are infinite). Later we will deal with this, using the fact that the soft local time depends on z only through its starting point.

Proof of Proposition 5.3. We are going to follow the scheme in Section 5.1 in order to construct the triple $\mathcal{I}^u$, $(\mathcal{I}_1^u)_{u\ge0}$, $(\mathcal{I}_2^u)_{u\ge0}$, distributed as random interlacements on $\mathbb{Z}^d$ as stated in the proposition. However, we will need two independent copies of some of the ingredients appearing in that construction. More precisely,

let $\chi_{V,1}^- = \sum_i \delta_{(w_i^1, u_i^1)}$ and $\chi_{V,2}^- = \sum_i \delta_{(w_i^2,u_i^2)}$ be two independent random variables on M (i.e., Poisson point processes on the space of labeled trajectories) with the same law as χ_V^- in (5.4), (5.28)

let the counting processes $\Theta_u^{V,1} = \chi_{V,1}^-(W_+ \times [0,u])$ and $\Theta_u^{V,2} = \chi_{V,2}^-(W_+ \times [0,u])$ be as in (5.10), for $u \ge 0$, and finally (5.29)

define two independent processes $\widehat{\mathcal{I}}_{V,1}^u$ and $\widehat{\mathcal{I}}_{V,2}^u$ as in (5.6). (5.30)

The only missing ingredients in order to construct two independent random interlacement processes following the construction of Section 5.1 are the random walks composing χ_V^+ (see (5.4)). The construction will be based on Proposition 4.10, and that is where the coupling will take place.

Let us introduce the sets

$$\Sigma_{A_1\cup A_2},\ \Sigma_{A_1} \text{ and } \Sigma_{A_2} \text{ given by (5.14) with } V \text{ as in (5.26)}. \tag{5.31}$$

Note that we have replaced the set C by the three above choices, while keeping V fixed.

We also let $\mu_{A_1\cup A_2}$, μ_{A_1} and μ_{A_2} be the respective measures on these sets, given by (5.16). The first crucial observation for this proof is that

$$\Sigma_{A_1\cup A_2} \text{ is the disjoint union of } \Sigma_{A_1} \text{ and } \Sigma_{A_2} \text{ and } \mu_{A_1\cup A_2} = \mu_{A_1} + \mu_{A_2}. \tag{5.32}$$

Note that we are duplicating the cemetery on $\Sigma_{A_1\cup A_2}$ for the above to hold.

We define a Poisson point process η on $\Sigma_{A_1\cup A_2} \times \mathbb{R}_+$ with intensity $\mu_{A_1\cup A_2} \otimes dv$ as below (5.16). From (5.32) we conclude that

$$\begin{aligned}&\eta \text{ restricted to } \Sigma_{A_1} \text{ and } \Sigma_{A_2} \text{ are Poisson point processes with respective}\\ &\text{intensities } \mu_{A_1} \otimes dv \text{ and } \mu_{A_2} \otimes dv, \text{ which are independent of each other.}\\ &\text{Moreover, an excursion } z \in \Sigma_{A_k} \text{ cannot intersect } A_{k'} \text{ with } k' \neq k \text{ (see (5.26)).}\end{aligned} \tag{5.33}$$

We use $\chi_{V,1}^-$ and $\chi_{V,2}^-$ in order to define the starting points $\{X_0^{V,1,j}\}_{j=1,\dots,\Theta_u^{A_1}}$ and $\{X_0^{V,2,j}\}_{j=1,\dots,\Theta_u^{A_2}}$. Let us finally recall the definitions of T^C from (5.12), and of $G_{j,k}^C$ and $z_{j,k}^C$ from (4.32), where C can be replaced by either of the three sets $A_1 \cup A_2$, A_1, or A_2. It is important to observe that we use the starting points $X_0^{V,1,j}$ for the case $C = A_1$ and $X_0^{V,2,j}$ for both $C = A_2, A_1 \cup A_2$. We can finally introduce

$$\mathcal{J}_C^u = C \cap \bigcup_{j=1}^{\Theta_u^C} \bigcup_{k=1}^{T_j^C} \operatorname{Range}(z_{j,k}^C) \quad \text{with } C = A_1 \cup A_2, A_1 \text{ or } A_2 \tag{5.34}$$

(note that we use the same Poisson point process to define the three sets above) and

$$\begin{aligned}&\mathcal{I}^u = \mathcal{J}_{A_1\cup A_2}^u \cup \mathcal{I}^u(\chi_{V,2}^-) \cup \widehat{\mathcal{I}}_{V,2}^u,\\ &\mathcal{I}_1^u = \mathcal{J}_{A_1}^u \cup \mathcal{I}^u(\chi_{V,1}^-) \cup \widehat{\mathcal{I}}_{V,1}^u, \qquad \mathcal{I}_2^u = \mathcal{J}_{A_2}^u \cup \mathcal{I}^u(\chi_{V,2}^-) \cup \widehat{\mathcal{I}}_{V,2}^u.\end{aligned}$$

We independently modify the above sets on $(A_1 \cup A_2)^c$ to obtain the correct distributions, although this is immaterial for the statement of the proposition.

To conclude the proof of the proposition, let us observe that

- $(\mathcal{J}_C^u)_{u\geq 0}$ is distributed as $(C \cap \mathcal{I}_{V,+}^u)_{u\geq 0}$ for $C = A_1 \cup A_2$, A_1 or A_2 (see (5.21)), so that $((A_1 \cup A_2) \cap \mathcal{I}^u)_{u\geq 0}$, $(A_1 \cap \mathcal{I}_1^u)_{u\geq 0}$ and $(A_2 \cap \mathcal{I}_2^u)_{u\geq 0}$ have the right distributions as under the random interlacements;
- $\mathcal{J}_{A_1}^u$ and $\mathcal{J}_{A_2}^u$ are independent (see (5.28), (5.29) and (5.33)), which means $(A_1 \cap \mathcal{I}_1^u)_{u\geq 0}$ and $(A_2 \cap \mathcal{I}_2^u)_{u\geq 0}$ are also independent.

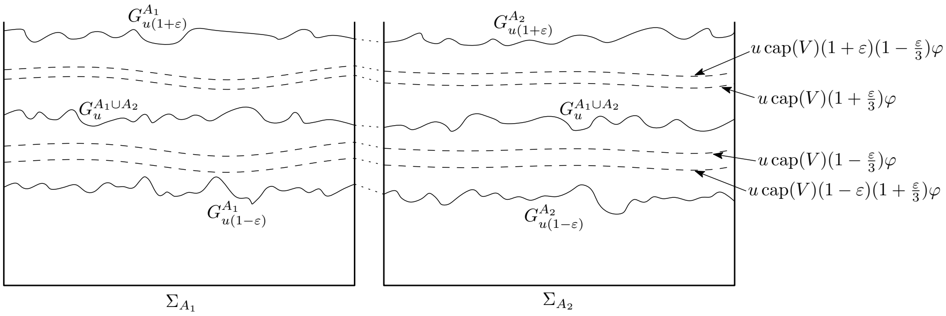

Fig. 5. Proof of Proposition 5.3; φ was defined in the last paragraph of the proof (observe that $1+\varepsilon/3 \le (1-\varepsilon/3)(1+\varepsilon)$ for $\varepsilon \in [0,1]$).

Hence, using the definition of $\mathcal{I}^u$, $\mathcal{I}^u_1$ and $\mathcal{I}^u_2$, we see that

$$\begin{aligned}
&\mathbb{Q}[\mathcal{I}_k^{u(1-\varepsilon)} \cap A_k \subseteq \mathcal{I}^u \cap A_k \subseteq \mathcal{I}_k^{u(1+\varepsilon)},\ k=1,2]\\
&\quad \ge \mathbb{Q}[\mathcal{J}_{A_k}^{u(1-\varepsilon)} \subseteq \mathcal{J}^u_{A_1\cup A_2} \cap A_k \subseteq \mathcal{J}_{A_k}^{u(1+\varepsilon)},\ k=1,2]\\
&\quad \ge \mathbb{Q}[G^{A_k}_{u(1-\varepsilon)}(z) \le G^{A_1\cup A_2}_u(z) \le G^{A_k}_{u(1+\varepsilon)}(z) \text{ for all } z\in\Sigma_{A_k} \text{ and } k=1,2].
\end{aligned} \tag{5.35}$$

Now, (5.26) implies that for $x \in \partial A_k$ we have $\varphi(x) := E^{\mathcal{P}}\rho_1^{A_k}(x) = E^{\mathcal{P}}\rho_1^{A_1\cup A_2}(x)$. The conclusion of (5.27) is now a simple consequence of the above display and the fact that the expectation of G^C_u is linear in u according to (5.25) (see Figure 5). □

6. Proof of Theorem 2.1

In this section we will prove our main result, modulo a set of additional assumptions that will be proved in the next section.

Recall that we use the notation $B(x,r) = \{y \in \mathbb{Z}^d;\ \|x-y\| \le r\}$ for discrete balls. Also, for $A \subset \mathbb{Z}^d$ we write $B(A,r) = \bigcup_{x\in A} B(x,r)$.

Suppose we are given sets A_1 and A_2 as in Theorem 2.1 and suppose without loss of generality that the diameter of A_1 is not greater than the diameter of A_2. It is clear that we can assume that $A_2 = \mathbb{Z}^d \setminus B(A_1,s)$, since the function f_2 can be seen as a function in $\{0,1\}^{\mathbb{Z}^d\setminus B(A_1,s)}$; so, from now on we work with this assumption.

The proof of the main theorem will require some estimates on the entrance distribution of a random walk on the sets A_1, A_2 and $A_1 \cup A_2$, which are closely related to the regularity conditions mentioned before Theorem 4.9. However, the problem is that, in general, these estimates need not be satisfied for an *arbitrary* finite set A_1 and $A_2 = \mathbb{Z}^d \setminus B(A_1,s)$. So, in order to fix this problem, we will replace A_1 and A_2 by slightly larger sets $A_1^{(s)}$ and $A_2^{(s)}$, using Proposition 6.1 below. Roughly speaking, these "fattened" sets will have the following properties (below, C stands for any of the three sets $A_1^{(s)}$, $A_2^{(s)}$, or $A_1^{(s)} \cup A_2^{(s)}$):

- the probability that the simple random walk enters C through some point y is at most $O(s^{-(d-1)})$ for starting points at distance at least of order s from C;
- this probability should be at least of order $s^{-(d-1)}$ for "many" starting points which are at distance of order s from y;
- the probabilities of entering C through two near points y and y' in ∂C can be different by at most a (fixed) constant factor (this should be valid as soon as the random walk starts far from $\{y, y'\}$);
- finally, we also need some additional geometric properties of ∂C.

A typical example of a set having these properties is a discrete ball of radius s; in fact, we will prove that any set with "sufficiently smooth boundary" will do. More rigorously, the fact that we need is formulated in the following way (one may find it helpful to look at Figure 6):

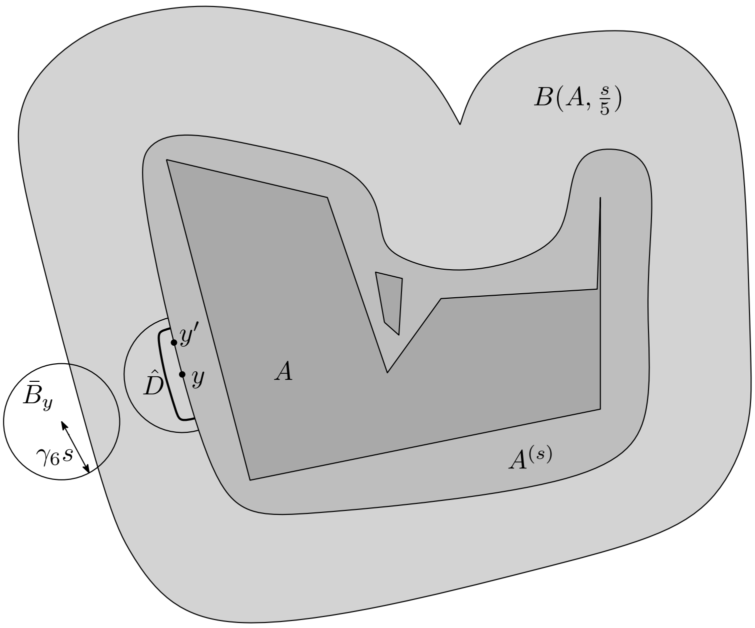

Fig. 6. The sets in Proposition 6.1.

Proposition 6.1. *There exist positive constants* $\gamma_6 \in (0, \frac{1}{10})$, γ_7, $\gamma_8 < \gamma_6/2$, γ_9, γ_{10}, γ_{11} $\in (0, 1)$, s_0 *(depending only on dimension) such that, for any* $s \geq s_0$ *and any set* $A \subset \mathbb{Z}^d$ *such that* $\mathbb{Z}^d \setminus B(A, s)$ *is non-empty, there is a set* $A^{(s)}$ *with the following properties:*

$$A \subseteq A^{(s)} \subseteq B(A, s/5); \tag{6.1}$$

for any $y \in \partial A^{(s)}$,

$$\sup_{\substack{x \in \mathbb{Z}^d: \\ d(x,y) \geq \gamma_6 s/2}} P_x[X_{H_{A^{(s)}}} = y] \leq \gamma_7 s^{-(d-1)} \tag{6.2}$$

and there exists a ball $\bar{B}_y$ *of radius* $\gamma_6 s$ *such that* $\mathrm{d}(\bar{B}_y, y) \in [\gamma_6 s, 2\gamma_6 s]$ *and*

$$\inf_{x \in \bar{B}_y} P_x[X_{H_{A^{(s)}}} = y,\ H_{A^{(s)}} < H_{\mathbb{Z}^d \setminus B(y, 4\gamma_6 s)}] \geq \gamma_7^{-1} s^{-(d-1)}. \tag{6.3}$$

Moreover, for any $y \in \partial A^{(s)}$,

$$|\{z \in \partial A^{(s)};\ \|y - z\| \leq \gamma_8 s\}| \geq \gamma_9 s^{d-1}, \tag{6.4}$$

and if $y' \in \partial A^{(s)}$ *is such that* $\|y - y'\| \leq \gamma_8 s$*, then there exists a set* $\hat{D}$ *(depending on* y, y'*) that separates* $\{y, y'\}$ *from* $\partial B(y, \gamma_6 s)$ *(i.e., any nearest neighbor path starting at* $\partial B(y, \gamma_6 s)$ *that enters* $A^{(s)}$ *at* $\{y, y'\}$*, must pass through* $\hat{D}$*) such that*

$$\sup_{\substack{x \in \hat{D}: \\ P_x[X_{H_{A^{(s)}}} = y'] > 0}} \frac{P_x[X_{H_{A^{(s)}}} = y]}{P_x[X_{H_{A^{(s)}}} = y',\ H_{A^{(s)}} < H_{\mathbb{Z}^d \setminus B(y', 5\gamma_6 s)}]} \leq \gamma_{10}. \tag{6.5}$$

The proof of this proposition is postponed to Section 8. We are now going to use the above result to prove Theorem 2.1.

Recall that we define $A_2 = \mathbb{Z}^d \setminus B(A_1, s)$. The idea is to use Proposition 5.3 for $A_1^{(s)}$ and $A_2^{(s)}$ provided by Proposition 6.1, and V defined as

$$V = \{y \in \mathbb{Z}^d;\ \mathrm{d}(y, A_1^{(s)} \cup A_2^{(s)}) \geq \gamma_6 s\}. \tag{6.6}$$

Let $y, y' \in \partial A_1^{(s)} \cup \partial A_2^{(s)}$ be such that $\|y - y'\| \leq \gamma_8 s$ (in fact, in this case both y and y' must be in the same set, either $\partial A_1^{(s)}$ or $\partial A_2^{(s)}$). Let $\hat{D}$ be the corresponding separating set, as in (6.5) of Proposition 6.1. Now, consider an arbitrary site $x \in V$, and write, for $C = A_1^{(s)}, A_2^{(s)}, A_1^{(s)} \cup A_2^{(s)}$,

$$P_x[X_{H_C} = y] = \sum_{z \in \hat{D}} P_x[X_{H_{C \cup \hat{D}}} = z] P_z[X_{H_C} = y] \quad \text{and similarly with } y', \tag{6.7}$$

where we have used the strong Markov property at $H_{C \cup \hat{D}}$ and dropped vanishing terms. So, by construction, we have

$$\sup_{\substack{x \in V: \\ P_x[X_{H_C} = y'] > 0}} \frac{P_x[X_{H_C} = y]}{P_x[X_{H_C} = y']} \leq \gamma_{10} \quad \text{for } C = A_1^{(s)}, A_2^{(s)}, A_1^{(s)} \cup A_2^{(s)}, \quad \text{when } \|y - y'\| \leq \gamma_8 s. \tag{6.8}$$

With the above, we can now start estimating the soft local times appearing in (5.27).

In the rest of this section, C stands for one of the sets $A_1^{(s)}, A_2^{(s)}, A_1^{(s)} \cup A_2^{(s)}$; we will obtain the same estimates for all of them. Recalling the definition of T_ℓ in (4.22), we consider $x \in \partial C$ and fix any $z \in \Sigma_C$ such that $x = X_0(z)$; then we denote by

$$F_j^C(x) = G_{j, T_j^C}^C(z) \tag{6.9}$$

the contribution of the jth particle to the soft local time in trajectories starting at x, in the construction of the corresponding interlacement set for C, so that $G_u^C(z) = \sum_{j=1}^{\Theta_u^C} F_j^C(x)$.

We also introduce

$$\pi^C(x) = E[F_1^C(x)], \text{ which also equals } E^{\mathcal{P}} \rho_1^C(x) \text{ due to (5.24);} \tag{6.10}$$

recall the definition of ρ_j^C from (5.23).

Lemma 6.2. *For C being either $A_1^{(s)}$, $A_2^{(s)}$ or $A_1^{(s)} \cup A_2^{(s)}$, and V as in (6.6), for all $x \in \partial C$ we have*

(i) $\gamma_{12} s^{-1} \operatorname{cap}(V)^{-1} \le \pi^C(x) \le \gamma_{13} s^{-1} \operatorname{cap}(V)^{-1}$;

(ii) $E(F_1^C(x))^2 \le \gamma_{14} s^{-d} \operatorname{cap}(V)^{-1}$.

Proof. Instead of estimating the expected soft local time directly, we rather work with the "real" local time $\rho_1^C(x)$, with the assistance of Theorem 4.6.

Consider the discrete sphere $\tilde{V}$ of radius $3(r+s)$ centered at any fixed point of A_1. Given a trajectory $w^* \in W^*$, the number of excursions $\rho_1^C(x)$ between V and C entering at x is the same for both $s_V(w^*)$ and $s_{\tilde{V}}(w^*)$. Thus, their expected values are the same and can be written respectively as $u \operatorname{cap}(V)\pi^C(x)$ and $u \operatorname{cap}(\tilde{V})\tilde{\pi}^C(x)$, where $\tilde{\pi}^C(x)$ is the expected number of such (V, x)-crossings under $P_{\tilde{e}_{\tilde{V}}}$. So,

$$\pi^C(x) = \operatorname{cap}(V)^{-1} \operatorname{cap}(\tilde{V}) \tilde{\pi}^C(x).$$

We know that $\operatorname{cap}(\tilde{V}) \asymp (r+s)^{d-2}$ (see (2.8)), so in order to prove (i), it will be enough to obtain

$$\tilde{\pi}^C(x) \asymp s^{-1}(r+s)^{-(d-2)}. \tag{6.11}$$

For $x' \in \mathbb{Z}^d \setminus C$ such that $\mathrm{d}(x', x) \ge \gamma_6 s$, we use the Markov property at H_C to obtain

$$E_{x'}\rho_1^C(x) \le P_{x'}[X_{H_C} = x] + \sup_{y \in V} P_y[H_{B(x,\gamma_6 s/2)} < \infty] \sup_{z:\, \mathrm{d}(z,x) \ge \gamma_6 s/2} E_z \rho_1^C(x). \tag{6.12}$$

Then taking the supremum in x' and using (6.2), we get

$$\sup_{x':\, \mathrm{d}(x',x) \ge \gamma_6 s/2} E_{x'}\rho_1^C(x) \le \frac{\sup_{x':\, \mathrm{d}(x',x) \ge \gamma_6 s/2} P_{x'}[X_{H_C} = x]}{1 - \sup_{y \in V} P_y[H_{B(x,\gamma_6 s/2)} < \infty]} \le c_2 s^{-(d-1)}. \tag{6.13}$$

So, by [10, Proposition 6.4.2],

$$\begin{aligned} E_{\tilde{e}_{\tilde{V}}}\rho_1^C(x) &\le \sup_{x' \in \tilde{V}} P_{x'}[H_{B(x,\gamma_6 s/2)} < \infty] \sup_{x':\, \mathrm{d}(x',x) \ge \gamma_6 s/2} E_{x'}\rho_1^C(x) \\ &\le c_3 \left(\frac{s}{s+r}\right)^{-(d-2)} s^{-(d-1)} = c_3 s^{-1}(r+s)^{-(d-2)}. \end{aligned}$$

We are now left with the lower bound

$$E_{\bar{e}_{\tilde{V}}}\rho_1^C(x) \geq \inf_{x'\in\partial\tilde{V}} P_{x'}[H_{\bar{B}_x} < \infty] \inf_{x''\in\partial\bar{B}_x} P_{x''}[X_{H_C} = x] \overset{(6.3)}{\geq} c_4\left(\frac{s}{s+r}\right)^{-(d-2)} s^{-(d-1)},$$

proving (6.11) and consequently (i).

Part (ii) then immediately follows from (6.13) and Theorem 4.8 (see also Remark 4.7). □

Next, we need the following large deviation bound for $F_1^C(x)$:

Lemma 6.3. *For $C = A_1^{(s)}, A_2^{(s)}, A_1^{(s)} \cup A_2^{(s)}$ and V as in (6.6), for all $x \in \partial C$ we have*

$$P[F_1^C(x) > v\gamma_7 s^{-(d-1)}] \leq \gamma_{15} s^{d-2}\,\mathrm{cap}(V)^{-1}\exp(-\gamma_{16}v) \quad \textit{for any } v \geq 2 \tag{6.14}$$

(*also, without loss of generality we suppose that* $\gamma_{16} \leq 1$).

Proof. The idea is to apply Theorem 4.9 for $F_1^C(x)$ and with $\Gamma_x = \{z \in \Sigma_C;\ \|x - X_0(z)\| \leq \gamma_8 s\}$; observe that $\mu_C(\Gamma_x) \geq \gamma_9 s^{d-1}$ by (6.4). With the notation of Theorem 4.9, we set

$$\ell = \gamma_7 s^{-(d-1)} \quad \text{and observe that} \quad \alpha \geq 1/\gamma_{10},$$

by (6.2) and (6.8).

Chebyshev's inequality together with Lemma 6.2(i) then implies that

$$P[T_\ell < \infty] = P[F_1^C(x) \geq \gamma_7 s^{-(d-1)}] \leq \frac{\pi^C(x)}{\gamma_7 s^{-(d-1)}} \leq \gamma_7^{-1}\gamma_{13} s^{d-2}\,\mathrm{cap}(V)^{-1}. \tag{6.15}$$

Now, denoting by $N(\Gamma_x)$ the number of crossings between V and C that enter Γ_x, and by η_x the number of points of the Poisson process (from the construction in Section 5) in $\Gamma_x \times [0, \frac{\gamma_7}{2\gamma_{10}} v s^{-(d-1)}]$, we write

$$\mathbb{Q}_{z'}[\eta_x \leq N(\Gamma_x)] \leq \mathbb{Q}_{z'}\left[\eta_x \leq \frac{\gamma_7\gamma_9}{4\gamma_{10}} v\right] + \mathbb{Q}_{z'}\left[N(\Gamma_x) \geq \frac{\gamma_7\gamma_9}{4\gamma_{10}} v\right].$$

To see that both terms on the right-hand side of the above display are exponentially small in v, we observe that

- η_x has Poisson distribution with parameter at least $\frac{\gamma_7\gamma_9}{2\gamma_{10}} v$, and
- starting from any $y \in V$, with uniformly positive probability the random walk does not enter Γ_x (recall that $\gamma_8 < \gamma_6/2$, which implies that $P_y[H_{\Gamma_x} < \infty] < c_5 < 1$ uniformly in $y \in V$). Therefore N_x is dominated by a Geometric(c_5) random variable having exponential tail as well.

Together with (6.15) and Theorem 4.9, this finishes the proof of Lemma 6.3. □

Now, we are able to finish the proof of our main result.

Proof of Theorem 2.1. For $C = A_1, A_2, A_1 \cup A_2$ and $x \in \partial C$, let $\psi_C^x(\lambda) = Ee^{\lambda F_1^C(x)}$ be the moment generating function of $F_1^C(x)$. It is elementary that $e^t - 1 \leq t + t^2$ for all $t \in [0,1]$. Using this observation, for $0 \le \lambda \le \frac{1}{2}\gamma_7^{-1}\gamma_{16}s^{d-1}$ (where γ_{16} is from Lemma 6.3) we write

$$\begin{aligned}
\psi_C^x(\lambda) - 1 &= E(e^{\lambda F_1^C(x)} - 1)\mathbb{1}_{\lambda F_1^C(x)\le 1} + E(e^{\lambda F_1^C(x)} - 1)\mathbb{1}_{\lambda F_1^C(x)>1} \\
&\le E\big(\lambda F_1^C(x) + \lambda^2 (F_1^C(x))^2\big) + Ee^{\lambda F_1^C(x)}\mathbb{1}_{F_1^C(x)>\lambda^{-1}} \\
&\le \lambda\pi^C(x) + \gamma_{14}\lambda^2 s^{-d}\operatorname{cap}(V)^{-1} + \lambda\int_{\lambda^{-1}}^{\infty} e^{\lambda y} P[F_1^C(x) > y]\,dy \\
&\le \lambda\pi^C(x) + \gamma_{14}\lambda^2 s^{-d}\operatorname{cap}(V)^{-1} + \lambda\gamma_{15}s^{d-2}\operatorname{cap}(V)^{-1}\int_{\lambda^{-1}}^{\infty}\exp\Big(-\frac{\gamma_{16}}{2\gamma_7}s^{d-1}y\Big)\,dy \\
&\le \lambda\pi^C(x) + \gamma_{14}\lambda^2 s^{-d}\operatorname{cap}(V)^{-1} + c_6 s^{-1}\operatorname{cap}(V)^{-1}\lambda\exp(-c_7\lambda^{-1}s^{d-1}) \\
&\le \lambda\pi^C(x) + c_8\lambda^2 s^{-d}\operatorname{cap}(V)^{-1}, \qquad (6.16)
\end{aligned}$$

where we have used Lemmas 6.2(ii) and 6.3. Analogously, since $e^{-t} - 1 \le -t + t^2$ for *all* $t > 0$, we obtain, for $\lambda \ge 0$,

$$\psi_C^x(-\lambda) - 1 \le -\lambda\pi^C(x) + c_9\lambda^2 s^{-d}\operatorname{cap}(V)^{-1} \qquad (6.17)$$

(in this case we do not need the large deviation bound of Lemma 6.3).

Observe that if $(Y_k, k \ge 1)$ are i.i.d. random variables with common moment generating function ψ, and N is an independent Poisson random variable with parameter θ, then $E\exp(\lambda\sum_{k=1}^N Y_k) = \exp(\theta(\psi(\lambda) - 1))$. So, using (6.16) and Lemma 6.2(ii), we write, for any $\delta > 0$, $z \in \Sigma$ and $x = X_0(z)$,

$$\begin{aligned}
\mathbb{Q}[G_{\hat u}^C \ge (1+\delta)\hat u\operatorname{cap}(V)\pi^C(x)] &= \mathbb{Q}\Big[\sum_{k=1}^{\Theta_{\hat u}^C} F_k^C(x) \ge (1+\delta)\hat u\operatorname{cap}(V)\pi^C(x)\Big] \\
&\le \frac{E\exp(\lambda\sum_{k=1}^{\Theta_{\hat u}^C} F_k^C(x))}{\exp(\lambda(1+\delta)\hat u\operatorname{cap}(V)\pi^C(x))} \\
&= \exp\big(-\lambda(1+\delta)\hat u\operatorname{cap}(V)\pi^C(x) + \hat u\operatorname{cap}(V)(\psi(\lambda) - 1)\big) \\
&\le \exp\big(-(\lambda\delta\hat u\operatorname{cap}(V)\pi^C(x) - c_8\lambda^2\hat u s^{-d})\big) \le \exp\big(-(c_{10}\lambda\delta\hat u s^{-1} - c_8\lambda^2\hat u s^{-d})\big),
\end{aligned}$$

and, analogously, with (6.17) instead of (6.16) one can obtain

$$\mathbb{Q}[G_{\hat u}^C \le (1-\delta)\hat u\operatorname{cap}(V)\pi^C(x)] \le \exp\big(-(c_{12}\lambda\delta\hat u s^{-1} - c_{13}\lambda^2\hat u s^{-d})\big).$$

Choosing $\lambda = c_{14}\delta s^{d-1}$ with small enough c_{14} depending on $c_8, c_{10}, c_{12}, c_{13}$ (and such that $c_{14} \le (\delta^{-1}\gamma_{16}/2) \wedge \frac{\gamma_{16}}{3\gamma_7}$), we thus obtain, using also the union bound (clearly, the cardinality of ∂C is at most $O((r+s)^d)$),

$$\begin{aligned}
\mathbb{Q}\big[(1-\delta)\hat u\operatorname{cap}(V)\pi^C(x) \le G_{\hat u}^C \le (1+\delta)\hat u\operatorname{cap}(V)\pi^C(x) \text{ for all } x \in \partial C\big]& \\
\ge 1 - c_{15}(r+s)^d\exp(-c_{16}\delta^2\hat u s^{d-2}).& \qquad (6.18)
\end{aligned}$$

Using (6.18) with $\delta = \varepsilon/3$ and $u, (1-\varepsilon)u, (1+\varepsilon)u$ in place of $\hat u$ together with Proposition 5.3, we conclude the proof of Theorem 2.1. □

Remark 6.4. As mentioned in the introduction, the factor $(r+s)^d$ before the exponential in (2.18) can usually be reduced. Observe that this factor (times a constant) appears in the proof as an upper bound for the cardinality of $\partial(A_1^{(s)} \cup A_2^{(s)})$. In the typical situation when s is smaller than r and the sets have a sufficiently regular boundary (e.g., boxes or balls), one can replace $(r+s)^d$ by r^{d-1}.

7. Connectivity decay

Proof of Theorem 3.1. We start by introducing the renormalization scheme on which the proof will be based. Fix $b \in (1, 2]$; clearly, one can consider only this range of the parameter b in proving (3.5), and any particular value of $b \in (1, 2]$ (in fact, any $b \in (0, \infty)$) will work for (3.4). Given $L_1 \geq 100$, we define

$$L_{k+1} = 2\left(1 + \frac{1}{(k+5)^b}\right)L_k \quad \text{for } k \geq 1. \tag{7.1}$$

Note that L_k grows roughly as 2^k and it need not be an integer in general. Before moving further, let us first establish some important properties of the rate of growth of this sequence. First, it is obvious that

$$2L_k = L_{k+1} - \frac{2}{(k+5)^b}L_k \leq L_{k+1} - \frac{2^k L_1}{(k+5)^b} \leq \lfloor L_{k+1} \rfloor - \frac{2^{k-1}L_1}{(k+5)^b} \tag{7.2}$$

for all $k \geq 1$ (here we have used $\frac{50 \cdot 2^{k+1}}{(k+5)^2} > 1$ for every $k \geq 1$). Moreover, it is clear that

$$\begin{aligned} \log L_k &= \log L_1 + (k-1)\log 2 + \sum_{j=1}^{k-1} \log\left(1 + \frac{1}{(j+5)^b}\right) \\ &\leq \log L_1 + (k-1)\log 2 + \sum_{j=1}^{k-1} \frac{1}{(j+5)^b}, \end{aligned}$$

so

$$L_1 2^{k-1} \leq L_k \leq e^{\zeta(b)} L_1 2^{k-1}. \tag{7.3}$$

We use the above scale sequence to define boxes entering our renormalization scheme. For $x \in \mathbb{Z}^d$ and $k \geq 1$, let

$$C_x^k = [0, L_k)^d \cap \mathbb{Z}^d + x \quad \text{and} \quad D_x^k = [-L_k, 2L_k) \cap \mathbb{Z}^d + x. \tag{7.4}$$

(Observe that the L_k's above need not be integers in general.)

Given $u > u^{**}$, $k \geq 1$ and a point $x \in \mathbb{Z}^d$, we will be interested in the probability of the event

$$A_x^k(u) = \{C_x^k \overset{\mathcal{V}^u}{\longleftrightarrow} \mathbb{Z}^d \setminus D_x^k\}, \tag{7.5}$$

pictured in Figure 7. Our main objective is to bound the probabilities

$$p_k(u) = \sup_{x\in\mathbb{Z}^d} \mathbb{P}[A_x^k(u)] \overset{(2.17)}{=} \mathbb{P}[A_0^k(u)]. \tag{7.6}$$

In order to employ a renormalization scheme, we will need to relate the events A^k for different scales, as done in the following observation. Given $k \geq 1$,

there exist two collections of points $\{x_i^k\}_{i=1}^{3^d}$ and $\{y_j^k\}_{j=1}^{2d\cdot 7^{d-1}}$ such that
(i) $C_0^{k+1} = \bigcup_{i=1}^{3^d} C_{x_i^k}^k$, (7.7)
(ii) $\bigcup_{j=1}^{2d\cdot 7^{d-1}} C_{y_j^k}^k$ is disjoint from D_0^{k+1} and contains $\partial(\mathbb{Z}^d \setminus D_0^{k+1})$

(see Figure 7). The above statement is a consequence of (7.2) and the fact that for all $k \geq 1$ we have $2\big(1+\frac{1}{(k+5)^b}\big) < 3$ and $6\big(1+\frac{1}{(k+5)^b}\big) < 7$. It implies that

$$A_0^{k+1} \subset \bigcup_{\substack{i\leq 3^d \\ j\leq 2d\cdot 7^{d-1}}} A_{x_i^k}^k \cap A_{y_j^k}^k \tag{7.8}$$

(see Figure 7).

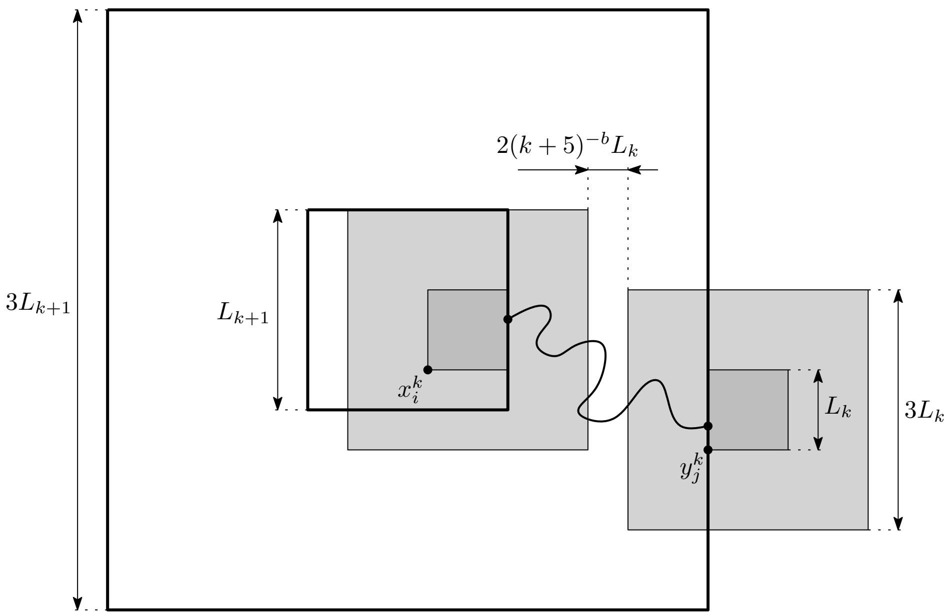

Fig. 7. An illustration of the event in (7.5) and the inclusion in (7.8).

It is also important to observe from (7.7) that for any $i \le 3^d$ and $j \le 2d \cdot 7^{d-1}$,

$$d\big(D^k_{x^k_i}, D^k_{y^k_j}\big) \ge \lfloor L_{k+1} \rfloor - 2L_k \overset{(7.2)}{\ge} \frac{2^{k-1}L_1}{(k+5)^b}. \tag{7.9}$$

Abbreviate $\hat{u} = (u^{**} + u)/2$; since $u > u^{**}$, we have $u > \hat{u} > u^{**}$. Then, choose a sufficiently small $\varepsilon > 0$ in such a way that

$$\prod_{k=1}^{\infty}\Big(1 - \frac{\varepsilon}{k^b}\Big) > \frac{\hat{u}}{u},$$

and define

$$u_k = \frac{\hat{u}}{\prod_{j=1}^{k-1}(1 - \varepsilon k^{-b})};$$

by construction, $u_k < u$ for all k. Abbreviate $\kappa_d = 2d \cdot 21^d/7$ and set

$$\varrho_d = \liminf_{L\to\infty} \mathbb{P}\big[[0, L]^d \overset{\mathcal{V}^{\hat{u}}}{\longleftrightarrow} \partial[-L, 2L]^d\big]; \tag{7.10}$$

then (recall (3.6)) $0 \le \varrho_d < \kappa_d^{-1}$. The above event is that there exists a connecting path between these two sets through $\mathcal{V}^{\hat{u}}$.

Now, we obtain a recursive relation for $p_k(u_k)$ (recall (7.6)).

We use (1.7) with $r = 3\sqrt{d}\, L_k$, $s \ge 2^{k-1}L_1(k+5)^{-b}$ (recall (7.9)), u_{k+1} in place of u and εk^{-b} in place of ε (observe that $u_k = (1 - \varepsilon k^{-b})u_{k+1}$), and use also (7.3) and (7.8) to obtain

$$p_{k+1}(u_{k+1}) \le \kappa_d p_k^2(u_k) + c_{17} 2^{kd} L_1^d \exp\bigg(-c_{18} k^{-2b}\Big(\frac{L_1 2^k}{(k+5)^b}\Big)^{d-2}\bigg), \tag{7.11}$$

where $c_{18} = c_{18}(u, b, \varepsilon)$.

Now, let us first consider the case $d \ge 4$ (as mentioned above, for this case any particular value of $b \in (1, 2]$ will do the job, so in the calculations below one can assume for definiteness that e.g. $b = 2$). Let $h_1 > 0$ be such that $\varrho_d < e^{-h_1} < \kappa_d^{-1}$. Choose a sufficiently large $L_1 \ge 100$ in such a way that $p_1(\hat{u}) < e^{-h_1}$ and

$$c_{17} 2^{kd} L_1^d \exp\bigg(-c_{18} k^{-2b}\Big(\frac{L_1 2^k}{(k+5)^b}\Big)^{d-2} + h_1 + 2^{k+1}\bigg) < 1 - \kappa_d e^{-h_1} \tag{7.12}$$

for all $k \ge 1$ (here we have used $d \ge 4$). Then we can find $h_2 \in (0, 1)$ small enough that

$$p_1(\hat{u}) \le \exp(-h_1 - 2h_2). \tag{7.13}$$

We then prove by induction that

$$p_k(u_k) \le \exp(-h_1 - h_2 2^k). \tag{7.14}$$

Indeed, the base for the induction is provided by (7.13); then, by (7.11),

$$p_{k+1}(u_{k+1}) \le \kappa_d \exp(-2h_1 - h_2 2^{k+1}) + c_{17} 2^{kd} L_1^d \exp\biggl(-c_{18} k^{-2b} \biggl(\frac{L_1 2^k}{(k+5)^b}\biggr)^{d-2}\biggr),$$

so, by (7.12) (recall that $h_2 < 1$)

$$\frac{p_{k+1}(u_{k+1})}{\exp(-h_1 - h_2 2^{k+1})} \le \kappa_d e^{-h_1} + c_{17} 2^{kd} L_1^d \exp\biggl(-c_{18} k^{-2b} \biggl(\frac{L_1 2^k}{(k+5)^b}\biggr)^{d-2} + h_1 + h_2 2^{k+1}\biggr),$$

which is smaller than one, thus proving (7.14).

Observe that for all x,

$$\mathbb{P}[0 \overset{\mathcal{V}^u}{\longleftrightarrow} x] \le \mathbb{P}\bigl[[-L_k, L_k]^d \overset{\mathcal{V}^u}{\longleftrightarrow} \partial[-2L_k, 2L_k]^d\bigr] \tag{7.15}$$

with $k = \max\{m;\ \frac{3}{2}L_m < \|x\|\}$; also, $L_k = O(2^k)$ by (7.3). Since $u_k < u$ for all k, (7.14) implies that $p_k(u) \le \exp(-h_1 - h_2 2^k)$, and we obtain (3.4) from (7.15).

Let us now deal with the case $d = 3$. Again, let $h_1' > 0$ with $\varrho_3 < e^{-h_1'} < \kappa_3^{-1}$. Choose a sufficiently large $L_1 \ge 100$ in such a way that $p_1(\hat{u}) < e^{-h_1'}$ and

$$c_{17} 2^{3k} L_1^3 \exp(-c_{18}(k+5)^{-3b} L_1 2^{k-1} + h_1) < 1 - \kappa_3 e^{-h_1'} \tag{7.16}$$

for all $k \ge 1$. Then, we can find $h_2' \in (0, \frac{1}{4}c_{18})$ small enough that

$$p_1(\hat{u}) \le \exp(-h_1' - 2h_2'). \tag{7.17}$$

Now, in three dimensions we are going to prove by induction that

$$p_k(u_k) \le \exp(-h_1' - h_2'(k+5)^{-3b} 2^k). \tag{7.18}$$

Indeed, by (7.11) we have

$$p_{k+1}(u_{k+1}) \le \kappa_3 \exp(-2h_1' - h_2'(k+5)^{-3b} 2^{k+1}) + c_{17} 2^{3k} L_1^3 \exp(-c_{18}(k+5)^{-3b} L_1 2^k),$$

so, by (7.16) (recall that $h_2 < \frac{1}{4}c_{18}$),

$$\begin{aligned}\frac{p_{k+1}(u_{k+1})}{\exp(-h_1' - h_2'(k+6)^{-3b} 2^{k+1})} &\le \kappa_3 e^{-h_1'} \exp\bigl(-h_2'((k+5)^{-3b} - (k+6)^{-3b})\bigr) \\ &\quad + c_{17} 2^{3k} L_1^3 \exp\biggl(-c_{18} \frac{L_1 2^k}{(k+5)^{3b}} + h_1' + h_2' \frac{2^{k+1}}{(k+6)^{3b}}\biggr) \\ &\le \kappa_3 e^{-h_1'} + c_{17} 2^{3k} L_1^3 \exp(-c_{18}(k+5)^{-3b} L_1 2^{k-1} + h_1') \\ &< 1,\end{aligned}$$

thus proving (7.18). Again, since $u_k < u$ for all k, (7.14) implies that $p_k(u) \le \exp(-h_1' - h_2'(k+5)^{-3b} 2^k)$ for all k, and then we obtain (3.5) with the help of (7.15) analogously to the case $d \ge 4$. This concludes the proof of Theorem 3.1. ◻

8. Smoothing of discrete sets: proof of Proposition 6.1

In this section we show that any set can be enclosed in a slightly larger set with "smooth enough" boundaries, and this larger set has the desired properties (in particular, the entrance probabilities behave in a good way), as described in Proposition 6.1.

To facilitate reading, throughout this section we will adopt the following convention for denoting points and subsets of $\mathbb{R}^d$ which are not (generally) in $\mathbb{Z}^d$: they will be respectively denoted by $\mathsf{x}, \mathsf{y}, \mathsf{z}$ and $\mathsf{A}, \mathsf{B}, \mathsf{D}$, using the sans serif font. The usual fonts are reserved for points and subsets of $\mathbb{Z}^d$. Also, we use the following (a bit loose but convenient) notation: if a set $\mathsf{A} \subset \mathbb{R}^d$ is defined, then we denote by $A \subset \mathbb{Z}^d$ its discretization: $A = \mathsf{A} \cap \mathbb{Z}^d$; conversely, if $A \subset \mathbb{Z}^d$ is a discrete set, then A just equals A, but is regarded as a subset of $\mathbb{R}^d$.

Similarly to the notation in the discrete case, let us write $\mathsf{B}(\mathsf{x}, s) = \{\mathsf{y} \in \mathbb{R}^d; \|\mathsf{x} - \mathsf{y}\| \le s\}$ for the ball with radius s; recall that $\| \cdot \|$ stands for the Euclidean norm. We abbreviate $\mathsf{B}(s) = \mathsf{B}(0, s)$. It will be convenient to define, for $\mathsf{A} \subseteq \mathbb{R}^d$, the ball $\mathsf{B}(\mathsf{A}, s) = \bigcup_{\mathsf{x} \in \mathsf{A}} \mathsf{B}(\mathsf{x}, s)$.

Definition 8.1. Let $\mathsf{D} \subset \mathbb{R}^d$ be an open set (not necessarily connected) with smooth boundary $\partial \mathsf{D}$. We say that D is *s-regular* if for any $\mathsf{x} \in \partial \mathsf{D}$ there exist two balls $\mathsf{B}^{\mathsf{x}}_{\text{in}} \subset \bar{\mathsf{D}}$ and $\mathsf{B}^{\mathsf{x}}_{\text{out}} \subset \mathbb{R}^d \setminus \mathsf{D}$ of radius s such that $\partial \mathsf{D} \cap \mathsf{B}^{\mathsf{x}}_{\text{out}} = \partial \mathsf{D} \cap \mathsf{B}^{\mathsf{x}}_{\text{in}} = \{\mathsf{x}\}$. Informally speaking, the definition means that one can touch the boundary of D by spheres of radius s from the inside and outside. We also adopt the convention that $\mathbb{R}^d$ is s-regular for any $s > 0$.

Observe that if D is an s-regular set, then for each $\mathsf{x} \in \partial \mathsf{D}$ the balls $\mathsf{B}^{\mathsf{x}}_{\text{in}}$ and $\mathsf{B}^{\mathsf{x}}_{\text{out}}$ are unique. Let us denote by x^{in} and x^{out} their respective centers, which lie on the line normal to $\partial \mathsf{D}$ at x. Also, it is important to keep in mind that if D is s-regular then it is also s'-regular for all $s' \le s$.

First, we will show that any set $\mathsf{A} \subset \mathbb{R}^d$ can be thickened into a smooth and regular $\mathsf{A}^{(s)}$ which is "close" to A (see Figure 6). This is made precise in the following

Lemma 8.2. *There exists a constant* $\gamma_{17} \in (0, 1/5)$ *such that, for any set* $\mathsf{A} \subset \mathbb{R}^d$ *and* $s > 0$*, there exists a set* $\mathsf{A}^{(s)} \subset \mathbb{R}^d$ *with smooth boundary such that:*

(1) $\mathsf{A} \subseteq \mathsf{A}^{(s)} \subseteq \mathsf{B}(\mathsf{A}, s/5)$*;*
(2) $\mathsf{A}^{(s)}$ *is* $\gamma_{17} s$*-regular in the sense of Definition* 8.1.

Proof. Assume that $\mathbb{R}^d \setminus \mathsf{B}(\mathsf{A}, s/5)$ is non-empty, otherwise the claim is straightforward. Since we suppose that $\mathsf{A} \subset \mathbb{R}^d$ is arbitrary, we can suppose that $s = 5$ (so that $s/5 = 1$) by scaling A if necessary.

Let us first tile the space $\mathbb{R}^d$ with compact cubes K_m of side length $1/(8\sqrt{d})$. More precisely, for $m = (m_1, \dots, m_d) \in \mathbb{Z}^d$, let

$$\mathsf{K}_m = \tfrac{1}{8\sqrt{d}}[m_1, m_1 + 1] \times \cdots \times [m_d, m_d + 1]. \tag{8.1}$$

With the above definition, $\operatorname{diam}(\mathsf{K}_{m_1} \cup \mathsf{K}_{m_2}) \le \frac{1}{4}$ if K_{m_1} and K_{m_2} have at least one common point.

We first consider the set

$$\hat{\mathsf{A}} = \bigcup \mathsf{K}_m,$$

where the union is taken over all cubes that either intersect A or have at least one common point with another cube that intersects A.

Define now the function $\hat{f}$ to be the convolution of $\mathbb{1}_{\hat{\mathsf{A}}}(\cdot)$ with a smooth test function $\psi \geq 0$, with $\int \psi \, d\mathsf{x} = 1$ and supported on $\mathsf{B}(1/8)$. Clearly, for any $\alpha \in (0, 1)$,

$$\mathsf{A} \subseteq \{\mathsf{x};\ \hat{f}(\mathsf{x}) > \alpha\}, \tag{8.2}$$

so it remains to show that, for some α, the set $\{\mathsf{x};\ \hat{f}(\mathsf{x}) > \alpha\}$ is γ_{17}-regular for some small enough $\gamma_{17} < 1/5$ independent of A.

To understand how the above construction depends on the choice of A, let us scale and recenter the function $\hat{f}$. More precisely, let $\varphi_{\mathsf{A},m} : \mathsf{B}(0, 1) \to \mathbb{R}_+$ be the function that associates a point $\mathsf{x} \in \mathsf{B}(0, 1)$ to $\hat{f}(\mathsf{x} - m)$. It is important to observe that

$$\begin{aligned}&\text{as we vary } \mathsf{A} \subset \mathbb{Z}^d \text{ and } m \in \mathbb{Z}^d, \text{ the functions}\\ &\varphi_{\mathsf{A},m} \text{ range over a finite collection of smooth functions,}\end{aligned} \tag{8.3}$$

since $\varphi_{\mathsf{A},m}$ is determined by the finitely many possible configurations of boxes $\mathsf{K}_{m'}$ that intersect K_m (whether they appear or not in the union defining $\hat{\mathsf{A}}$).

From Sard's theorem and the implicit function theorem one can deduce that for some $\alpha \in (0, 1)$ (in fact, for generic values of $\alpha \in (0, 1)$) the boundary $\{\mathsf{x};\ \hat{f}(\mathsf{x}) = \alpha\}$ is smooth. Therefore, using (8.3) we can choose $\alpha_o \in (0, 1)$ such that $\{\mathsf{x};\ \hat{f}(\mathsf{x}) = \alpha_o\}$ is smooth, independently of the choice of A. We now let $\mathsf{A}' = \{\mathsf{x};\ \hat{f}(\mathsf{x}) > \alpha_o\}$. From (8.2), we conclude that $\mathsf{A} \subset \mathsf{A}'$ and from the definition of $\hat{f}$, we see that $\mathsf{A}' \subseteq \mathsf{B}(\mathsf{A}, 1)$. To finish the proof, we should show that A' is γ_{17}-regular (with some small enough constant γ_{17} independent of A).

Since $\partial \mathsf{A}'$ is smooth, we can show that for every $\mathsf{x} \in \partial \mathsf{A}'$, there exist B_{in} and B_{out} as in Definition 8.1. Observe that the existence of such balls with radius smaller than or equal to $1/4$ only depends on the values of $\hat{f}$ in $\mathsf{B}(\mathsf{x}, 1)$. So that the independence of γ_{17} from the choice of A follows from (8.3). □

At this point, we can collect the first ingredient for Proposition 6.1: we take $A^{(s)}$ to be the discretization of the set $\mathsf{A}^{(s)}$ provided by Lemma 8.2.

Now, we prove several geometric properties of regular sets and their discretizations.

Lemma 8.3. *Abbreviate* $\gamma_{18} = 1/200$ *and* $\gamma_{19} = (1 + \sqrt{799})/200 < 1/6$. *Then for any* s*-regular set* A *and for any* $\mathsf{v}_1, \mathsf{v}_2 \in \partial \mathsf{A}$ *such that* $\|\mathsf{v}_1 - \mathsf{v}_2\| \leq \gamma_{18} s$*, we have*

$$\|\mathsf{v}_1^{\text{out}} - \mathsf{v}_2^{\text{out}}\| \leq \gamma_{19} s \tag{8.4}$$

(*by symmetry, the same holds for* $\mathsf{v}_1^{\text{in}}, \mathsf{v}_2^{\text{in}}$).

Proof. Consider the plane $\mathcal{L}$ generated by the points v_1, $\mathsf{v}_1^{\text{out}}$, and $\mathsf{v}_2^{\text{out}} + (\mathsf{v}_1 - \mathsf{v}_2)$ (see Figure 8; note that, as indicated in the picture, v_2 need not lie on this plane). Let x be the point that lies in the intersection of $\partial \mathsf{B}_{\text{in}}^{\mathsf{v}_1} \cap (\partial \mathsf{B}_{\text{out}}^{\mathsf{v}_2} + \mathsf{v}_1 - \mathsf{v}_2)$ with $\mathcal{L}$ and is different from v_1, and let y be the middle point on the arc of the circle $(\partial \mathsf{B}_{\text{out}}^{\mathsf{v}_2} + \mathsf{v}_1 - \mathsf{v}_2) \cap \mathcal{L}$

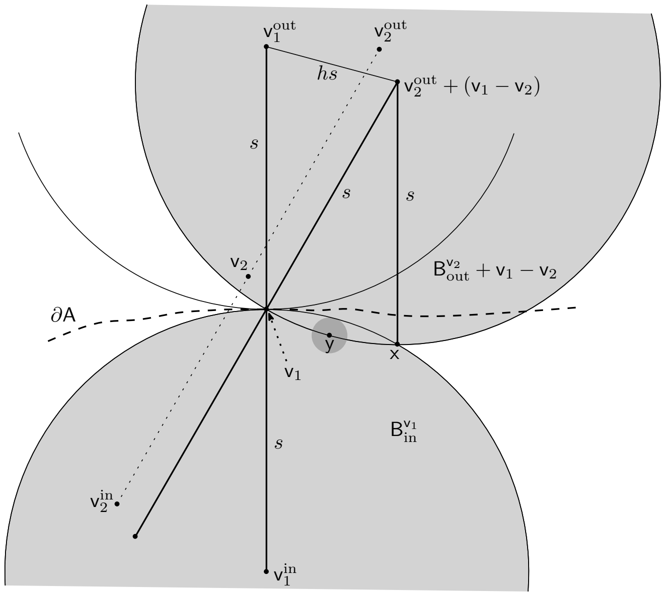

Fig. 8. The plane $\mathcal{L}$ in the proof of Lemma 8.3. The radius of the small gray circle centered in y is as; also, in this picture the segment between v_2^{in} and v_2^{out} (containing also v_2) does not intersection with the plane $\mathcal{L}$. That is why v_2 appears not to intersect ∂A.

between v_1 and x (of course, we mean the arc that lies inside $B_{in}^{v_1}$). Abbreviate also $h = s^{-1}\|v_2^{out} + (v_1 - v_2) - v_1^{out}\|$ and $a = s^{-1}\, d(y, \partial B_{in}^{v_1})$; with some elementary geometry, we obtain

$$h = 2\sqrt{a - a^2/4}.$$

But, we must necessarily have

$$d(y, \partial B_{in}^{v_1}) \leq \|v_1 - v_2\|,$$

because otherwise the point $y - v_1 + v_2 \in B_{out}^{v_2}$ would also belong to the interior of $B_{in}^{v_1}$, a contradiction. So, we have

$$h \leq 2\sqrt{\gamma_{18} - \gamma_{18}^2/4} = \sqrt{799}/200,$$

which means that

$$\|v_1^{out} - v_2^{out}\| < (\sqrt{799}/200 + \gamma_{18})s = \gamma_{19}s. \qquad \square$$

The next lemma is a consequence of the obvious observation that the boundary of discretized s-regular sets looks locally flat for large s:

Lemma 8.4. *There exist (large enough) s_0, h_0 with the following properties. Assume that* A *is s-regular for some $s \geq s_0$ and $x, y \notin A$ are such that $\|x - y\| \leq 3\sqrt{d}$. Then there exists a path between x and y of length at most h_0 that does not intersect A.*

Proof. This result is fairly obvious, so we give only a sketch of the proof (certainly, not the most "economic" one). First, without restricting generality, one can assume that $\max(\mathsf{d}(x, \mathsf{A}), \mathsf{d}(y, \mathsf{A})) < 3\sqrt{d}$ (otherwise, the ball of radius $3\sqrt{d}$ centered at one of the points does not intersect A and contains the other point; then, use the fact that this discrete ball is a connected graph). Let $\mathsf{z} \in \partial\mathsf{A}$ be a point on the boundary closest to x, let z be the point in A closest to z, and consider the cube

$$G = \{z' \in \mathbb{Z}^d;\ \|z' - z\|_\infty \le \lceil 7\sqrt{d}\rceil\}$$

(where $\|\cdot\|_\infty$ is the maximum norm). Assume without loss of generality that the projection of the normal vector to $\partial\mathsf{A}$ at z on the first coordinate vector is at least $1/\sqrt{d}$. Then the claim of the lemma follows once we prove that for all large enough s,

$$\text{the set } G \setminus \mathsf{A} \text{ is connected.} \tag{8.5}$$

Indeed, for s large enough, $\{v, v + e_1\}$ is not fully inside $\mathbb{R}^d \setminus (\mathsf{B}^{\mathsf{z}}_{\text{in}} \cup \mathsf{B}^{\mathsf{z}}_{\text{out}})$ for any v in G. This implies that $G \setminus \mathsf{A}$ is given by $G \cap \mathsf{B}^{z}_{\text{out}}$ together with some extra points in the neighborhood of this set, implying (8.5) and concluding the proof of the lemma. □

Observe that Lemma 8.4 implies that for any $x \in \partial A$ and $y \notin A$ such that $\|x-y\| \le 2\sqrt{d}$, we have

$$P_y[X_{H_A} = x] \ge (2d)^{-h_0}. \tag{8.6}$$

Next, we need an elementary result about escape probabilities from spheres:

Lemma 8.5. *There exist positive constants $s_1, c_{19}, c_{20}, c_{21}, c_{22}$ (depending only on the dimension) such that for all $\mathsf{y} \in \mathbb{R}^d$, all $s \ge s_1$ and every $x \in B(\mathsf{y}, 2s) \setminus B(\mathsf{y}, s)$, we have*

$$c_{19}\frac{\|x-\mathsf{y}\| - s}{s} \le P_x[H_{B(\mathsf{y},s)} > H_{\mathbb{Z}^d\setminus B(\mathsf{y},2s)}] \le c_{20}\frac{\|x-\mathsf{y}\| - s + 1}{s}, \tag{8.7}$$

and for all $x \in B(\mathsf{y}, 3s) \setminus B(\mathsf{y}, s)$,

$$c_{21}\frac{3s - \|x-\mathsf{y}\|}{s} \le P_x[H_{B(\mathsf{y},s)} < H_{\mathbb{Z}^d\setminus B(\mathsf{y},3s)}] \le c_{22}\frac{3s - \|x-\mathsf{y}\| + 1}{s}. \tag{8.8}$$

Proof. By a direct and elementary calculation for large enough s (not depending on y), the process $\|X_{n\wedge H_{B(s)}} - \mathsf{y}\|^{-(d-1)}$ is a supermartingale, and $\|X_{n\wedge H_{B(s)}} - \mathsf{y}\|^{-(d-5/2)}$ is a submartingale (see e.g. [5, proof of Lemma 1]). From the Optional Stopping Theorem,

$$\begin{aligned}\frac{s^{-(d-1)} - \|x-\mathsf{y}\|^{-(d-1)}}{s^{-(d-1)} - (2s+1)^{-(d-1)}} &\le P_x[H_{B(s)} > H_{\mathbb{Z}^d\setminus B(2s)}] \\ &\le \frac{(s-1)^{-d+5/2} - \|x-\mathsf{y}\|^{-d+5/2}}{(s-1)^{-d+5/2} - (2s)^{-d+5/2}},\end{aligned} \tag{8.9}$$

where the above balls are centered at y. Then (8.7) follows from (8.9) with the observation that $0 < \|x-\mathsf{y}\|/s \le 2$ and some elementary calculus. The proof of (8.8) is completely analogous.

In fact, with some more effort, one can show that $s_1 = \sqrt{d}/2$ (observe that $B(\mathsf{y}, \sqrt{d}/2)$ is non-empty for all $\mathsf{y} \in \mathbb{R}^d$), but we do not need this stronger fact in the present paper. □

We now need estimates on the entrance measure of a set in $\mathbb{Z}^d$ which has been obtained from the discretization of a regular set $\mathsf{D} \subset \mathbb{R}^d$. For this, we will need the following definitions. Let $D = \mathsf{D} \cap \mathbb{Z}^d$ and fix $x \in \partial D$, write x for the closest point to x in $\partial\mathsf{D}$ (it can be chosen arbitrarily in case of ties) and note that $\|x - \mathsf{x}\| \leq 1$. We define x^{in} and x^{out} to be the closest points to x^{in} and x^{out} in $\mathbb{Z}^d$ (again chosen arbitrarily in case of ties). Observe that $\|x^{\text{out}} - \mathsf{x}^{\text{out}}\|$ is at most $\sqrt{d}/2$ (and the same holds for x^{in} and x^{in}).

Lemma 8.6. (i) *Suppose that* A *is an* s*-regular set for some* $s \geq s_0 + \sqrt{d}$ *and let* $y \in \partial A$, $x \in \mathbb{Z}^d \setminus A$ *be such that* $\|x - y\| \geq 2s$. *Then* $P_x[X_{H_A} = y] \leq c_{24} s^{-(d-1)}$.
(ii) *Assume that* A *is* s*-regular with* $s \geq s_0 + \sqrt{d}$ *and* $y \in \partial A$. *Then for every* x *in* $B(y^{\text{out}}, s/2)$*, we have* $P_x[X_{H_A} = y, H_A < H_{\mathbb{Z}^d \setminus B(y^{\text{out}}, s+\sqrt{d})}] \geq c_{25} s^{-(d-1)}$.

Proof. Given A and $y \in \partial A$ as above, recall that y stands for the closest point to y in $\partial\mathsf{A}$ (chosen arbitrarily in case of ties). By Definition 8.1, the ball $\mathsf{B}^{\mathsf{y}}_{\text{in}} \subset \mathbb{R}^d$ of radius s lies fully inside A. Moreover, since y^{in} is at distance at most $\sqrt{d}/2$ from y^{in}, we conclude that

$$B^y_{\text{in}} := B(y^{\text{in}}, s - \sqrt{d}/2) \subseteq \mathsf{B}^{\mathsf{y}}_{\text{in}} \quad \text{and} \quad B^y_{\text{out}} := B(y^{\text{out}}, s - \sqrt{d}/2) \subseteq \mathsf{B}^{\mathsf{y}}_{\text{out}} \qquad (8.10)$$

(see Figure 9).

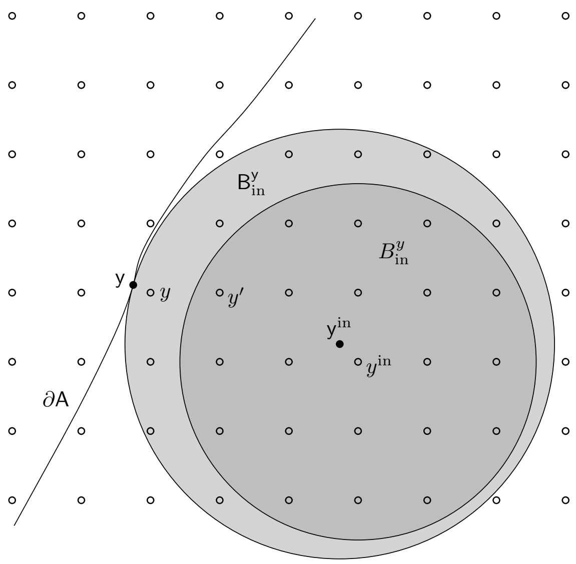

Fig. 9. Proof of Lemma 8.6.

Let $y' \in \partial B^y_{\text{in}}$ be the point closest to y (it could happen that y' is y itself). By construction, $\mathrm{d}(y, B^y_{\text{in}}) \le \sqrt{d}$, therefore $\|y - y'\| \le \frac{3}{2}\sqrt{d}$, and so by Lemma 8.4,

$$P_x[X_{H_B} = y'] \ge c_{26} P_x[X_{H_A} = y]. \tag{8.11}$$

Employing [10, Proposition 6.5.4], we obtain

$$P_x[X_{H_B} = y'] \le c_{24} s^{-(d-1)}, \tag{8.12}$$

which together with (8.11) proves (i).

A discretization argument analogous to the above gives (ii) for all x in $B(y^{\text{out}}, s/2 - \sqrt{d}/4)$ as a direct consequence of [10, Lemma 6.3.7]; then, using Lemma 8.4, we obtain the desired statement for all $x \in B(y^{\text{out}}, s/2)$. □

Next, aiming at the proof of (6.5), we prove the following result:

Proposition 8.7. *There exist constants $s_0, \gamma_8, \gamma_{10} > 0$ such that if $s \ge s_0$ and $\mathsf{A} \subset \mathbb{R}^d$ is $\gamma_{17}s$-regular and if $y_1, y_2 \in \partial A$ are such that $\|y_1 - y_2\| \le \gamma_8 s$, then there exists a set $\hat{D}$ (depending on y_1, y_2) that separates $\{y_1, y_2\}$ from $\partial B(y_1, 2\gamma_{17}s)$ (i.e., any nearest neighbor path starting at $\partial B(y_1, 2\gamma_{17}s)$ that enters A at $\{y_1, y_2\}$, must pass through $\hat{D}$) such that*

$$\sup_{\substack{x \in \hat{D};\\ P_x[X_{H_A} = y_1] > 0}} \frac{P_x[X_{H_A} = y_2]}{P_x[X_{H_A} = y_1,\ H_A < H_{\mathbb{Z}^d \setminus B(y_1, \frac{5}{2}\gamma_{17}s)}]} \le \gamma_{10}. \tag{8.13}$$

Let us mention that the constants γ_8, γ_{10} here are exactly those that we need in Proposition 6.1.

Proof of Proposition 8.7. Define

$$s_2 = \max\{\gamma_{17}^{-1} s_0, 36(\gamma_{17}\gamma_{18})^{-1}(s_1 + \sqrt{d})\} \ge 18(\gamma_{17}\gamma_{18})^{-1}. \tag{8.14}$$

Also, define $\gamma_8 = \frac{1}{3}\gamma_{17}\gamma_{18}$. Given y_1 and y_2 in ∂A such that $\|y_1 - y_2\| < \gamma_8 s$, set

$$D = \{z \in \mathbb{Z}^d \setminus A;\ \mathrm{d}(z, \mathsf{A}) \le \tfrac{1}{2}\gamma_8 s \text{ and } \mathrm{d}(z, y_k) \le 2\gamma_8 s,\ k = 1, 2\}, \tag{8.15}$$

$$\hat{D} = \{z \in D;\ \text{there exists } v \in \mathbb{Z}^d \setminus (A \cup D) \text{ such that } z \text{ and } v \text{ are neighbors}\} \tag{8.16}$$

(see Figure 10). Intuitively speaking, $\hat{D}$ is the part of the boundary of D not adjacent to A.

We now claim that

$$\text{all sites of } \hat{D} \text{ are at distance at least } \tfrac{1}{2}\gamma_8 s - 1 \text{ from } \{y_1, y_2\}. \tag{8.17}$$

To see this, observe first that for $z \in \hat{D}$, the point $v \leftrightarrow z$ as in (8.16) is not in D. Therefore, either $\mathrm{d}(v, \{y_1, y_2\}) > 2\gamma_8 s$ or $\mathrm{d}(v, \mathsf{A}) \ge \frac{1}{2}\gamma_8 s$; in both cases (8.17) holds.

In fact, to prove (8.13), it is enough to prove that for all $z \in \hat{D}$,

$$P_x[X_{H_A} = y_1,\ H_A < H_{\mathbb{Z}^d \setminus B(y_1, \frac{5}{2}\gamma_{17}s)}] \ge c_{29}\mathrm{d}(z, A)/s^d, \tag{8.18}$$

$$P_z[X_{H_A} = y_2] \le c_{30}\mathrm{d}(z, A)/s^d. \tag{8.19}$$

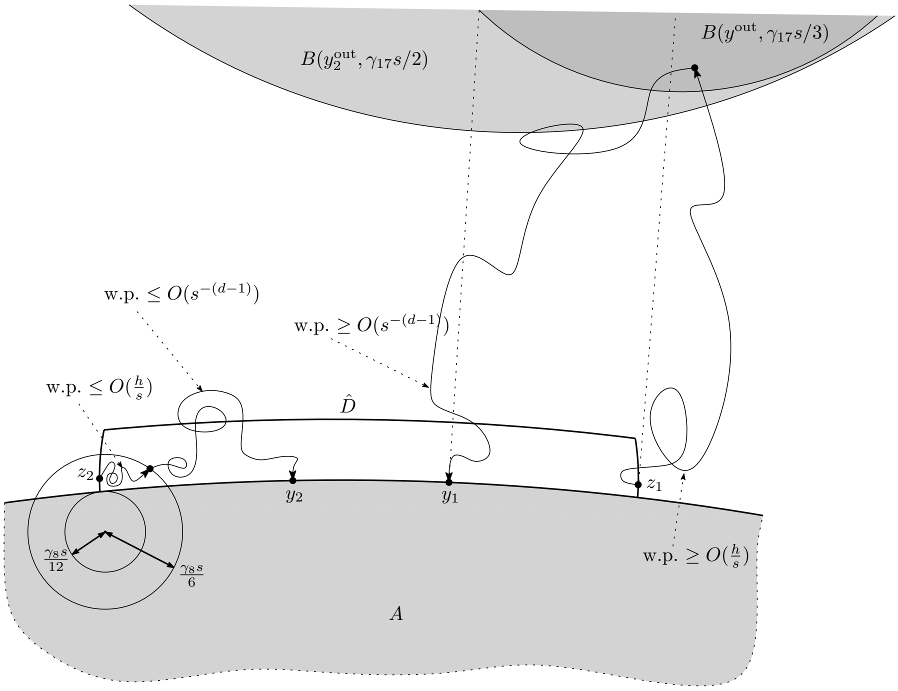

Fig. 10. Proof of Proposition 8.7: lower bound for $P_{z_1}[X_{H_A} = y_1, \dots]$ and upper bound for $P_{z_2}[X_{H_A} = y_2]$; we have $h \simeq \mathrm{d}(z_{1,2}, A)$, and "w.p." stands for "with probability".

The idea behind these two bounds is depicted in Figure 10, which we now turn into a rigorous proof. To obtain (8.18), we proceed in the following way. Consider some $\mathsf{y} \in \partial A$ such that $\mathrm{d}(z, A) \geq \|z - \mathsf{y}\|$, and observe that (8.8) implies that

$$P_z[H_{B(\mathsf{y}^{\text{out}}, \gamma_{17}s/3)} < H_A] \geq c_{31}\, \mathrm{d}(z, A)/s. \tag{8.20}$$

Let $\mathsf{y}_k \in \partial A$ be the closest boundary point to y_k; clearly, $\|\mathsf{y}_k - y_k\| \leq 1$. Then $\|y_1 - z\| \leq 2\gamma_8 s$ and $\|z - \mathsf{y}\| \leq \frac{1}{2}\gamma_8 s$, and thus, by (8.14),

$$\|\mathsf{y}_1 - \mathsf{y}\| \leq \tfrac{5}{2}\gamma_8 s + 1 < 3\gamma_8 s = \gamma_{17}\gamma_{18} s.$$

So, by Lemma 8.3 we have $\|\mathsf{y}^{\text{out}} - \mathsf{y}_1^{\text{out}}\| \leq \gamma_{17}\gamma_{19} s < \frac{1}{6}\gamma_{17} s$, which implies that $\mathsf{B}(\mathsf{y}^{\text{out}}, \frac{1}{3}\gamma_{17} s) \subset \mathsf{B}(\mathsf{y}_1^{\text{out}}, \frac{1}{2}\gamma_{17} s)$. Observing that $B(y_1^{\text{out}}, \gamma_{17}s + \sqrt{d}) \subset B(y_1, \frac{5}{2}\gamma_{17} s)$, applying Lemma 8.6(ii) and using (8.20), we obtain (8.18).

To prove (8.19), we proceed in the following way. Recall that if a set is r-regular then it is r'-regular for all $r' \leq r$; so, if $\mathrm{d}(z, A) \geq \frac{1}{3}\gamma_8 s$ then Lemma 8.6(i) already implies (8.19). Assume now that $z \in \hat{D}$ is such that $\mathrm{d}(z, A) < \frac{1}{3}\gamma_8 s$ and let $\mathsf{y} \in \partial \mathsf{A}$ be such that $\mathrm{d}(z, A) \geq \|z - \mathsf{y}\|$. Let us show that then $\|\mathsf{y} - y_2\| \geq \frac{1}{2}\gamma_8 s$. Indeed, by construction of $\hat{D}$ there exists $v \notin A \cup G$ such that $\|z - v\| = 1$ and either $\mathrm{d}(v, A) \geq \frac{1}{2}\gamma_8 s$

or $\min(\|v - y_1\|, \|v - y_2\|) > 2\gamma_8 s$. The first possibility is ruled out since then we would have $\mathsf{d}(z, A) > \frac{1}{2}\gamma_8 s - 1$, which contradicts $\mathsf{d}(z, A) < \frac{1}{3}\gamma_8 s$ because of (8.14). The second alternative implies that $\|v - y_2\| > \gamma_8 s$, so $\|z - y_2\| > \gamma_8 s - 1$. This means that

$$\|\mathsf{y} - y_2\| \geq \gamma_8 s - 1 - \tfrac{1}{3}\gamma_8 s \geq \tfrac{1}{2}\gamma_8 s$$

again because of (8.14).

Let $\hat{\mathsf{v}}$ be the center of the ball with radius $\frac{1}{12}\gamma_8 s$ that touches $\partial\mathsf{A}$ at y from the inside; by (8.14) we have

$$\inf_{v' \in \mathsf{B}(\hat{\mathsf{v}}, \frac{1}{6}\gamma_8 s)} \|v' - y_2\| \geq \tfrac{1}{2}\gamma_8 s - \tfrac{1}{4}\gamma_8 s - 1 \geq \tfrac{1}{12}\gamma_8 s + 1.$$

Then, one can write

$$P_z[X_{H_A} = y_2] \leq P_z[H_{\mathbb{Z}^d \setminus B(\hat{\mathsf{v}}, \frac{1}{6}\gamma_8 s)} < H_{B(\hat{\mathsf{v}}, \frac{1}{12}\gamma_8 s)}] \sup_{z' : \|z' - y_2\| \geq \frac{1}{12}\gamma_8 s} P_{z'}[X_{H_A} = y_2],$$

and use Lemma 8.5 to find that the first term on the right-hand side is at most $c_{32} s^{-1} \mathsf{d}(z, A)$. By Lemma 8.6(i), the second term is bounded above by $c_{33} s^{-(d-1)}$. This concludes the proof of (8.19) and hence of Proposition 8.7. □

We now collect the ingredients necessary for the proof of Proposition 6.1:

- as already mentioned just before Lemma 8.3, the sets $A_{1,2}^{(s)}$ are the discretizations of the sets $\mathsf{A}_{1,2}^{(s)}$ provided by Lemma 8.2;
- we take the same s_2 provided by (8.14) and define $\gamma_6 = \frac{1}{2}\gamma_{17}$;
- existence of γ_7 suitable for (6.2) and (6.3) follows from Lemma 8.6;
- the claim (6.5) follows from Proposition 8.7, with the right constants γ_8, γ_{10}, as already mentioned.

So, the only unattended item in Proposition 6.1 is (6.4). But it is straightforward to obtain (6.4) from a projection argument: Let $\mathsf{y} \in \partial\mathsf{A}_k^{(s)}$ be a closest point to $y \in \partial A_k^{(s)}$ and assume without lost of generality that the projection of the normal vector at y to the first coordinate is at least $1/\sqrt{d}$. Then the intersection of the projections of $\mathsf{B}_{\text{in}}^{\mathsf{y}} \cap \mathsf{B}(\mathsf{y}, \gamma_8 s - 1)$ and $\mathsf{B}_{\text{out}}^{\mathsf{y}} \cap \mathsf{B}(\mathsf{y}, \gamma_8 s - 1)$ along the first coordinate axis contains a ($(d-1)$-dimensional) ball of radius $O(s)$, and this proves (6.4) (since on the preimage of each integer point which lies within this intersection there should be at least one point of $\partial A_k^{(s)} \cap B(y, \gamma_8 s)$). This concludes the proof of Proposition 6.1. □

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