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# A presentation by generators and relations of Nichols algebras of diagonal type and convex orders on root systems

Received September 12, 2010 and in revised form August 13, 2012

**Abstract.** We obtain a presentation by generators and relations of any Nichols algebra of diagonal type with finite root system. We prove that the defining ideal is finitely generated. The proof is based on Kharchenko's theory of PBW bases of Lyndon words. We prove that the lexicographic order on Lyndon words is convex for PBW generators and so the PBW basis is orthogonal with respect to the canonical non-degenerate form associated to the Nichols algebra.

**Keywords.** Nichols algebras, quantized enveloping algebras, pointed Hopf algebras

## 1. Introduction

The interest in pointed Hopf algebras has grown since the appearance of quantized enveloping algebras [Dr, Ji]. The finite-dimensional analogues, called small quantum groups, were introduced and described by Lusztig [L1, L2].

The lifting method of Andruskiewitsch and Schneider is the leading method for the classification of finite-dimensional pointed Hopf algebras. The method depends on answers to some questions, including the following one:

**Question 1.1** ([And, Question 5.9]). Given a braided vector space of diagonal type, determine if the associated Nichols algebra is finite-dimensional, and in that case compute its dimension. Give a nice presentation by generators and relations.

The first part of this question has been answered by Heckenberger [H2], who gives a list of all diagonal braidings whose associated Nichols algebra has a finite root system, but neither an explicit formula for the dimension nor a finite set of defining relations are given. Some of them are Lusztig's examples, which are associated with the so-called Cartan braidings and for which the dimension and a presentation by generators and relations are known. Standard braidings were introduced in [AA] and they constitute a family which properly includes the family of Cartan braidings. Nichols algebras with standard braidings have been presented by generators and relations in [Ang1], where also an explicit

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*Mathematics Subject Classification (2010):* Primary 16T05; Secondary 17B37, 17B22

formula for the dimension has been given. Another result about presentation of Nichols algebras is given in [Y] for quantized enveloping algebras associated with semisimple Lie superalgebras, and for quantized enveloping algebras of Lie algebras [K1]. Some other preliminary considerations on the relations of a Nichols algebra of diagonal type appear in [He], and in [H3] for the rank-two case.

Using the lifting method Andruskiewitsch and Schneider [AS3] have classified finite-dimensional pointed Hopf algebras whose group of group-like elements is abelian of order not divisible by some small primes; all the possible such braidings are of finite Cartan type. They confirmed the following conjecture for  $H_0 = k\Gamma$ ,  $\Gamma$  an abelian group as above:

**Conjecture 1.2** ([AS1, Conj. 1.4]). *Let  $H$  be a finite-dimensional pointed Hopf algebra over  $k$ . Then  $H$  is generated by group-like and skew-primitive elements.*

This result was proved within the proof of the main theorem in [AS3] using a presentation by generators and relations. The conjecture was recently proved in a more general context [AnGa], when the braiding is of standard type. The proof also uses a presentation by generators and relations.

Because of the braidings of Cartan type we see that there exists a close relation between pointed Hopf algebras and the classical Lie theory. In this direction the notions of the Weyl groupoid and the root system [H1, HS, HY] associated to a Nichols algebra  $\mathfrak{B}(V)$  of diagonal type have shown to be good extensions of Weyl groups and root systems associated to semisimple Lie algebras. The root system is obtained as the set of degrees of the generators of any PBW basis, and controls coideal subalgebras between other structures associated to  $\mathfrak{B}(V)$  [HS].

In the classical case, convex orders over the root system were described in order to characterize the quantized enveloping algebras  $U_q(\mathfrak{g})$  for  $\mathfrak{g}$  semisimple [KhT, Le, R2], and to obtain Lusztig isomorphisms in the affine case [Be]. This kind of orders was first introduced in [Z]. A characterization of convex orders is therefore necessary, and it has been given for finite [P] and affine [I] root systems.

It seems natural to consider analogues of convex orders for Nichols algebras of diagonal type, and this is part of our work. As a consequence we obtain our main result, Theorem 4.9: we obtain a presentation by generators and relations for any Nichols algebra of diagonal type whose root system is finite. We obtain two kinds of relations that are enough to present  $\mathfrak{B}(V)$ : powers of root vectors (generators of a PBW basis), and some generalizations of quantum Serre relations which express the braided bracket of two root vectors as a linear combination of other root vectors in an explicit way (Lemmata 4.7, 4.5).

Theorem 4.9 follows by consideration of PBW bases as in [K1]. Such PBW bases consist of homogeneous polynomials associated to Lyndon letters (called hyperletters) and inherit the lexicographical order. Another important element is a characterization of convex orders for generalized root systems. Such orders are related to reduced expressions of elements of the Weyl groupoid. These reduced expressions also characterize right coideal subalgebras of Nichols algebras, so we can relate convex orders and coideal subalgebras. In particular, the following result holds by Theorem 4.9:

**Theorem 1.3.** *Let  $V$  be a braided vector space of diagonal type whose associated root system is finite, and let  $I(V)$  be the ideal of  $T(V)$  such that  $\mathfrak{B}(V) = T(V)/I(V)$ . Then  $I(V)$  is finitely generated.*

Theorem 4.9 extends the presentation by generators and relations of Nichols algebras of standard type (Remark 5.4), and gives a new proof for braidings of Cartan type. In particular we obtain the classical presentation of the quantized enveloping algebras  $U_q(\mathfrak{g})$  and Lusztig's small quantum groups  $u_q(\mathfrak{g})$ , with a different proof.

This result was used recently to obtain a minimal presentation of Nichols algebras of diagonal type, and this presentation was crucial to confirming Conjecture 1.2 when the group of group-like elements is abelian [Ang2].

The plan of this article is the following. In Section 2 we recall the definition of a Nichols algebra. We also consider results from [K1, R2] concerning PBW bases for Nichols algebras of diagonal type.

In Section 3 we deal with root systems and coideal subalgebras of Nichols algebras of diagonal type. In Subsection 3.1 we recall the notion of Weyl groupoid and root system, and give some properties of these objects. In Subsection 3.2 we characterize convex orders on finite root systems, generalizing the results in [P]. In Subsection 3.3 we recall some results from [HS] involving coideal subalgebras of Nichols algebras of diagonal type with finite root systems, and use these results to characterize PBW bases of hyperletters. In particular we show that the lexicographical order on the hyperletters is convex.

In Section 4 we obtain the desired presentation by generators and relations. First we prove that Kharchenko's PBW basis is orthogonal for the canonical non-degenerate bilinear form of Proposition 2.1 when the braiding matrix is symmetric. Power root vector relations hold in  $\mathfrak{B}(V)$  by Lemma 4.7, and generalized quantum Serre relations hold by Lemma 4.5. These two sets of relations are enough to give the presentation. The proof follows for symmetric braidings from the orthogonality of the PBW, and can be extended to the non-symmetric case by considering twistings. We show in Section 5 how the main theorem allows us to obtain explicit presentations of some specific Nichols algebras.

**Notation.**  $\mathbb{N}$  denotes the set of positive integers, and  $\mathbb{N}_0$  the set of non-negative integers.

We fix an algebraically closed field  $k$  of characteristic 0; all vector spaces, Hopf algebras and tensor products are considered over  $k$ .

For each  $N > 0$ ,  $\mathbb{G}_N$  denotes the group of  $N$ -th roots of 1 in  $k$ .

Given  $n \in \mathbb{N}$ , we define the following polynomials in  $q$ :

$$\binom{n}{j}_q = \frac{(n)_q!}{(k)_q!(n-k)_q!}, \quad \text{where } (n)_q! = \prod_{j=1}^n (k)_q \quad \text{and} \quad (k)_q = \sum_{j=0}^{k-1} q^j.$$

## 2. Preliminaries

We recall some definitions and results that we shall need in the subsequent sections. They are related to characterizations of Nichols algebras of diagonal type and PBW bases of such algebras.

Recall that a *braided vector space* is a pair  $(V, c)$ , where  $V$  is a vector space and  $c \in \text{Aut}(V \otimes V)$  is a solution of the *braid equation*:

$$(c \otimes \text{id})(\text{id} \otimes c)(c \otimes \text{id}) = (\text{id} \otimes c)(c \otimes \text{id})(\text{id} \otimes c).$$

A braided vector space  $(V, c)$  is of *diagonal type* if there exists a basis  $x_1, \dots, x_\theta$  of  $V$  and scalars  $q_{ij} \in k^\times$  such that

$$c(x_i \otimes x_j) = q_{ij}x_j \otimes x_i, \quad 1 \leq i, j \leq \theta. \tag{2.1}$$

Following [K1] we describe an appropriate PBW-basis of a braided graded Hopf algebra  $\mathfrak{B} = \bigoplus_{n \in \mathbb{N}} \mathfrak{B}^n$  such that  $\mathfrak{B}^1 \cong V$ , where  $(V, c)$  is of diagonal type. In particular we obtain PBW bases for Nichols algebras  $\mathfrak{B}(V)$  of diagonal type. This construction is based on the notion of Lyndon words. Each Lyndon word has a canonical decomposition as a product of a pair of smaller Lyndon words, called the Shirshov decomposition. Using that decomposition and the braided bracket, we define inductively a set of hyperwords, which are the elements of a PBW basis for braided graded Hopf algebras of diagonal type. We also recall some properties of this PBW basis.

2.1. Braided vector spaces of diagonal type and Nichols algebras

Given a braided vector space  $(V, c)$ , one can canonically extend the braiding to  $c : T(V) \otimes T(V) \rightarrow T(V) \otimes T(V)$  (see (2.3) for the diagonal case). For each pair  $x, y \in T(V)$  we define the *braided commutator* as follows:

$$[x, y]_c := \text{multiplication} \circ (\text{id} - c)(x \otimes y). \tag{2.2}$$

Fix a braided vector space  $(V, c)$  of diagonal type and an ordered basis  $X = \{x_1, \dots, x_\theta\}$  of  $V$  as in (2.1). Let  $\mathbb{X}$  be the corresponding vocabulary (the set of words in letters of  $X$ ) and consider the lexicographical order on  $\mathbb{X}$ . We will identify the vector space  $k\mathbb{X}$  with  $T(V)$ . We shall consider two different gradings of the algebra  $T(V)$ . First, we consider the usual  $\mathbb{N}_0$ -grading  $T(V) = \bigoplus_{n \geq 0} T^n(V)$ . If we denote by  $\ell$  the length of a word in  $\mathbb{X}$ , then  $T^n(V) = \bigoplus_{x \in \mathbb{X}, \ell(x)=n} kx$ .

Second, let  $\alpha_1, \dots, \alpha_\theta$  be the canonical basis of  $\mathbb{Z}^\theta$ . Then  $T(V)$  is  $\mathbb{Z}^\theta$ -graded, where the degree is given by  $\deg x_i = \alpha_i, 1 \leq i \leq \theta$ . Consider the bilinear form  $\chi : \mathbb{Z}^\theta \times \mathbb{Z}^\theta \rightarrow k^\times$  given by  $\chi(\alpha_i, \alpha_j) = q_{ij}, 1 \leq i, j \leq \theta$ . Then

$$c(u \otimes v) = q_{u,v}v \otimes u, \quad u, v \in \mathbb{X}, \tag{2.3}$$

where  $q_{u,v} = \chi(\deg u, \deg v) \in k^\times$ . The braided commutator satisfies a “braided” derivation equation which gives rise to a “braided” Jacobi identity, namely

$$[[u, v]_c, w]_c = [u, [v, w]_c]_c - \chi(\alpha, \beta)v[u, w]_c + \chi(\beta, \gamma)[u, w]_c v, \tag{2.4}$$

$$[u, vw]_c = [u, v]_c w + \chi(\alpha, \beta)v[u, w]_c, \tag{2.5}$$

$$[uv, w]_c = \chi(\beta, \gamma)[u, w]_c v + u[v, w]_c, \tag{2.6}$$

for any homogeneous  $u, v, w \in T(V)$ , of degrees  $\alpha, \beta, \gamma \in \mathbb{N}^\theta$ , respectively.

We denote by  ${}^H_H\mathcal{YD}$  the category of Yetter–Drinfeld modules over  $H$ , where  $H$  is a Hopf algebra with bijective antipode. Any  $V \in {}^H_H\mathcal{YD}$  becomes a braided vector space [Mo, Section 10.6]. If  $H = k\Gamma$ , where  $\Gamma$  is a finite abelian group, then any  $V \in {}^H_H\mathcal{YD}$  is a braided vector space of diagonal type: if  $V_g = \{v \in V \mid \delta(v) = g \otimes v\}$ ,  $V^\chi = \{v \in V \mid g \cdot v = \chi(g)v \text{ for all } g \in \Gamma\}$  and  $V_g^\chi = V^\chi \cap V_g$ , then  $V = \bigoplus_{g \in \Gamma, \chi \in \widehat{\Gamma}} V_g^\chi$ . In this setting the braiding is given by

$$c(x \otimes y) = \chi(g)y \otimes x, \quad x \in V_g, g \in \Gamma, y \in V^\chi, \chi \in \widehat{\Gamma}.$$

Conversely, any braided vector space of diagonal type can be realized as a Yetter–Drinfeld module over the group algebra of many abelian groups. For example let  $(V, c)$  be a braided vector space of diagonal type. Let  $\Gamma$  be the free abelian group of rank  $\theta$  with basis  $g_1, \dots, g_\theta$ , and define the characters  $\chi_1, \dots, \chi_\theta$  of  $\Gamma$  by

$$\chi_j(g_i) = q_{ij}, \quad 1 \leq i, j \leq \theta.$$

We can consider  $V$  as a Yetter–Drinfeld module over  $k\Gamma$  for which  $x_i \in V_{g_i}^{\chi_i}$ .

Given  $V \in {}^H_H\mathcal{YD}$ , the tensor algebra  $T(V)$  admits a unique structure of graded braided Hopf algebra in  ${}^H_H\mathcal{YD}$  such that the elements of  $V$  are primitive. As in [AS2], we define the Nichols algebra  $\mathfrak{B}(V)$  associated to  $V$  as the quotient of  $T(V)$  by the maximal element  $I(V)$  of the family  $\mathfrak{S}$  of homogeneous two-sided ideals  $I \subseteq \bigoplus_{n \geq 2} T(V)$  such that  $I$  is a Yetter–Drinfeld submodule of  $T(V)$  and a Hopf ideal:  $\Delta(I) \subset I \otimes T(V) + T(V) \otimes I$ .

The following proposition characterizes the Nichols algebra associated to  $V$  in a very interesting way.

**Proposition 2.1** ([L2, Prop. 1.2.3], [AS2, Prop. 2.10]). *Assume that  $(q_{ij})_{1 \leq i, j \leq \theta}$  is symmetric. For each family of scalars  $a_1, \dots, a_\theta \in k^\times$ , there exists a unique bilinear form  $(\cdot | \cdot) : T(V) \times T(V) \rightarrow k$  such that  $(1|1) = 1$ , and*

$$(x|yy') = (x_{(1)}|y)(x_{(2)}|y') \quad \text{for all } x, y, y' \in T(V), \tag{2.7}$$

$$(xx'|y) = (x|y_{(1)})(x'|y_{(2)}) \quad \text{for all } x, x', y \in T(V), \tag{2.8}$$

$$(x_i|x_j) = \delta_{ij}a_i \quad \text{for all } i, j. \tag{2.9}$$

*This form is symmetric and satisfies*

$$(x|y) = 0 \quad \text{for all } x \in T(V)_g, y \in T(V)_h, g, h \in \Gamma, g \neq h. \tag{2.10}$$

*The radical of this form,  $\{x \in T(V) : (x|y) = 0, \forall y \in T(V)\}$ , is  $I(V)$ , so  $(\cdot | \cdot)$  induces a non-degenerate bilinear form on  $\mathfrak{B}(V)$  denoted by the same symbol.  $\square$*

Consequently, if  $(V, c)$  is of diagonal type, then the ideal  $I(V)$  is  $\mathbb{Z}^\theta$ -homogeneous and  $\mathfrak{B}(V)$  is  $\mathbb{Z}^\theta$ -graded (see [AS2, Prop. 2.10]).

### 2.2. Lyndon words and PBW basis of braided graded Hopf algebras generated in degree zero and one

A word  $u \in \mathbb{X}$ ,  $u \neq 1$ , is Lyndon if  $u$  is smaller than any of its proper ends; that is, for any decomposition  $u = vw$ ,  $v, w \in \mathbb{X} - \{1\}$ , we have  $u < w$ . We denote by  $L$  the set of Lyndon words (see [Lo, Chapter 5]).

Note that  $X \subset L$ , and any Lyndon word begins with its smallest letter. They also satisfy the following properties:

1. Let  $u \in \mathbb{X} - X$ . Then  $u$  is Lyndon if and only if for any decomposition  $u = vw$  with  $v, w \in \mathbb{X} - \{1\}$  we have  $vw = u < wv$ .
2. If  $v, w \in L$  and  $v < w$ , then  $vw \in L$ .
3. Let  $u \in \mathbb{X} - X$ . Then  $u \in L$  if and only if there exist  $v, w \in L$  with  $v < w$  such that  $u = vw$ .

**Definition 2.2.** Let  $u \in L - X$ . The *Shirshov decomposition* of  $u$  is the decomposition  $u = vw$  with  $v, w \in L$  such that  $w$  is the smallest among proper non-empty ends of  $u$  (see [Lo]). Following [He], we denote  $\text{Sh}(u) = (v, w) \in L \times L$ . Then  $w$  is the longest among the ends of  $u$  that are Lyndon words.

Given  $u, v, w \in L$  such that  $u = vw, u \neq 1$ , we have  $\text{Sh}(u) = (v, w)$  if and only if either  $v \in X$ , or else  $\text{Sh}(v) = (v_1, v_2)$  satisfies  $w \leq v_2$ .

The Lyndon Theorem says that any word  $u \in \mathbb{X}$  admits a unique decomposition  $u = l_1 \cdots l_r$  as a product of non-increasing Lyndon words:  $l_i \in L, l_r \leq \cdots \leq l_1$  (see [Lo, Thm. 5.1.5]). This is called the *Lyndon decomposition* of  $u \in \mathbb{X}$ , and the  $l_i$  are the *Lyndon letters* of  $u$ .

We recall the endomorphism  $[-]_c$  (see [K1]), defined inductively on  $k\mathbb{X}$  using Shirshov and Lyndon decomposition:

$$[u]_c := \begin{cases} u & \text{if } u = 1 \text{ or } u \in X, \\ [[v]_c, [w]_c]_c & \text{if } u \in L, \ell(u) > 1 \text{ and } \text{Sh}(u) = (v, w), \\ [u_1]_c \cdots [u_r]_c, & \text{if } u \in \mathbb{X} - L \text{ with Lyndon decomposition } u = u_1 \cdots u_r. \end{cases}$$

**Definition 2.3** ([K1]). The *hyperletter* corresponding to  $l \in L$  is  $[l]_c$ . A *hyperword* is a word in hyperletters, and a *monotone hyperword* is a hyperword  $[u_1]_c^{k_1} \cdots [u_m]_c^{k_m}$  such that  $u_1 > \cdots > u_m$ .

Note that for any  $u \in L$ ,  $[u]_c$  is a homogeneous polynomial with coefficients in the subring  $\mathbb{Z}[q_{ij}]$  and  $[u]_c \in u + \mathbb{Z}[q_{ij}]\mathbb{X}_{>u}^{\ell(u)}$ .

The hyperletters inherit the order from the Lyndon words; this induces in turn the lexicographical ordering in the hyperwords. We now describe the braided commutator of hyperwords.

**Theorem 2.4** ([R2, Thm. 10]). *Let  $m, n \in L$  with  $m < n$ . Then  $[[m]_c, [n]_c]_c$  is a  $\mathbb{Z}[q_{ij}]$ -linear combination of monotone hyperwords  $[l_1]_c \cdots [l_r]_c, l_i \in L$ , such that  $n > l_i \geq mn$ . Moreover,  $[mn]_c$  appears in the expansion with a non-zero coefficient, and for any hyperword of this decomposition,  $\deg(l_1 \cdots l_r) = \deg(mn)$ .  $\square$*

The coproduct of  $T(V)$  can also be described in the basis of hyperwords.

**Lemma 2.5** ([R2]). *Let  $u \in \mathbb{X}$ , and let  $u = u_1 \cdots u_r v^m$ ,  $v, u_i \in L$ ,  $v < u_r \leq \cdots \leq u_1$ , be the Lyndon decomposition of  $u$ . Then*

$$\Delta([u]_c) = 1 \otimes [u]_c + \sum_{i=0}^m \binom{m}{i}_{q_{v,v}} [u_1]_c \cdots [u_r]_c [v]_c^i \otimes [v]_c^{m-i} + \sum_{\substack{l_1 \geq \cdots \geq l_p > v, l_i \in L \\ 0 \leq j \leq m}} x_{l_1, \dots, l_p}^{(j)} \otimes [l_1]_c \cdots [l_p]_c [v]_c^j,$$

where each  $x_{l_1, \dots, l_p}^{(j)}$  is  $\mathbb{Z}^\theta$ -homogeneous,  $\deg(x_{l_1, \dots, l_p}^{(j)} l_1 \cdots l_p v^j) = \deg(u)$ . □

We then have the following result from [R2].

**Lemma 2.6.** *For each  $l \in L$  denote by  $W_l$  the subspace of  $T(V)$  generated by*

$$[l_1]_c \cdots [l_k]_c, \quad k \in \mathbb{N}_0, l_i \in L, l_1 \geq \cdots \geq l_k \geq l. \tag{2.11}$$

Then  $W_l$  is a right coideal subalgebra of  $T(V)$ .

*Proof.* This follows from Theorem 2.4 and Lemma 2.5. □

As in [U] and [K1], we consider another order in  $\mathbb{X}$ . Given  $u, v \in \mathbb{X}$ , we say that  $u \succ v$  if either  $\ell(u) < \ell(v)$ , or  $\ell(u) = \ell(v)$  and  $u > v$  for the lexicographical order. We call  $\succ$  the *deg-lex order*, which is a total order. The empty word 1 is the maximal element for  $\succ$ , and this order is invariant under right and left multiplication.

Let  $I$  be a proper ideal of  $T(V)$ , and set  $R = T(V)/I$ . Let  $\pi : T(V) \rightarrow R$  be the canonical projection. Define

$$G_I := \{u \in \mathbb{X} : u \notin k\mathbb{X}_{\succ u} + I\}.$$

This set satisfies:

- (a) If  $u \in G_I$  and  $u = vw$ , then  $v, w \in G_I$ .
- (b) Any  $u \in G_I$  factorizes uniquely as a non-increasing product of Lyndon words in  $G_I$ .

**Proposition 2.7** ([K1, R2]). *The set  $\pi(G_I)$  is a basis of  $R$ .* □

In what follows, we assume that  $I$  is a Hopf ideal. Consider now

$$S_I := G_I \cap L. \tag{2.12}$$

We then define the *height* function  $h_I : S_I \rightarrow \{2, 3, \dots\} \cup \{\infty\}$  by

$$h_I(u) := \min\{t \in \mathbb{N} : u^t \in k\mathbb{X}_{\succ u^t} + I\}. \tag{2.13}$$

One can find a PBW-basis consisting of hyperwords of the quotient  $R$  of  $T(V)$  using the set  $S_I$  and the height previously defined.

**Theorem 2.8** ([K1]). *The following set is a PBW-basis of  $R = T(V)/I$ :*

$$\{[u_1]_c^{n_1} \cdots [u_k]_c^{n_k} : k \in \mathbb{N}_0, u_1 > \cdots > u_k \in S_I, 0 \leq n_i < h_I(u_i)\}. \tag{2.14}$$

Proofs are in [K1], where the next consequences are also considered.

**Proposition 2.9.** For any  $v \in S_I$  such that  $h_I(v) < \infty$ ,  $q_{v,v}$  is a root of unity whose order coincides with  $h_I(v)$ .  $\square$

**Corollary 2.10.** A word  $u$  does not belong to  $G_I$  if and only if the associated hyperletter  $[u]_c$  is a linear combination, modulo  $I$ , of hyperwords  $[w]_c$ ,  $w \succ u$ , with all hyperletters in  $S_I$ . Moreover, if  $h_I(v) =: h < \infty$ , then  $[v]^h$  is a linear combination of hyperwords  $[w]_c$ ,  $w \succ v^h$ .  $\square$

### 3. Root systems and coideal subalgebras

In this section we recall the definition of Weyl groupoid and the associated generalized root system given in [CH1] and [HY]. We also recall some properties of these objects that we shall use in the subsequent sections, and the relation to Nichols algebras of diagonal type. Next, we describe convex orders for subsets of the root systems as a generalization of Papi's results [P] for Weyl groups. We then consider a family of coideal subalgebras of a Nichols algebra of diagonal type with finite root system in order to prove that the ordering on the Lyndon words of a PBW basis as in Section 2.2 is convex. The proof of the convexity uses the characterization of coideal subalgebras given in [HS].

#### 3.1. Weyl groupoid and root systems

The notation used here is the same as in [CH1].

Fix a non-empty set  $\mathcal{X}$  and a non-empty finite set  $I$ , and let  $\{\alpha_i\}_{i \in I}$  be the canonical basis of  $\mathbb{Z}^I$ . For each  $i \in I$  consider a map  $r_i : \mathcal{X} \rightarrow \mathcal{X}$ , and for each  $X \in \mathcal{X}$  a generalized Cartan matrix  $A^X = (a_{ij}^X)_{i,j \in I}$  in the sense of [Ka].

**Definition 3.1** ([HY, CH1]). The quadruple  $\mathcal{C} := \mathcal{C}(I, \mathcal{X}, (r_i)_{i \in I}, (A^X)_{X \in \mathcal{C}})$  is a *Cartan scheme* if

- $r_i^2 = \text{id}$  for all  $i \in I$ ,
- $a_{ij}^X = a_{ij}^{r_i(X)}$  for all  $X \in \mathcal{X}$  and  $i, j \in I$ .

For each  $i \in I$  and  $X \in \mathcal{X}$  denote by  $s_i^X$  the automorphism of  $\mathbb{Z}^I$  given by

$$s_i^X(\alpha_j) = \alpha_j - a_{ij}^X \alpha_i, \quad j \in I.$$

The *Weyl groupoid* of  $\mathcal{C}$  is the groupoid  $\mathcal{W}(\mathcal{C})$  whose set of objects is  $\mathcal{X}$  and whose morphisms are generated by  $s_i^X$ , where we consider  $s_i^X \in \text{Hom}(X, r_i(X))$ ,  $i \in I$ ,  $X \in \mathcal{X}$ .

In general we shall denote  $\mathcal{W}(\mathcal{C})$  simply by  $\mathcal{W}$ , and for any  $X \in \mathcal{X}$ , we set

$$\text{Hom}(\mathcal{W}, X) := \bigcup_{Y \in \mathcal{X}} \text{Hom}(Y, X), \quad (3.1)$$

$$\Delta^{X \text{ re}} := \{w(\alpha_i) : i \in I, w \in \text{Hom}(\mathcal{W}, X)\}. \quad (3.2)$$



$\Delta^{X \text{ re}}$  is the set of *real roots* of  $X$ . Each  $w \in \text{Hom}(\mathcal{W}, X_1)$  can be written as a product  $s_{i_1}^{X_1} \cdots s_{i_m}^{X_m}$ , where  $X_j = r_{i_{j-1}} \cdots r_{i_1}(X_1)$ ,  $i \geq 2$ . We denote  $w = \text{id}_{X_1} s_{i_1} \cdots s_{i_m}$ : this means that  $w \in \text{Hom}(\mathcal{W}, X_1)$ , because the elements  $X_j \in \mathcal{X}$  are uniquely determined. The *length* of  $w$  is defined by

$$\ell(w) = \min\{n \in \mathbb{N}_0 : \exists i_1, \dots, i_n \in I \text{ such that } w = \text{id}_X s_{i_1} \cdots s_{i_n}\}.$$

In what follows we will assume that the groupoid  $\mathcal{W}$  is connected:

$$\text{Hom}(Y, X) \neq \emptyset, \quad \forall X, Y \in \mathcal{X}.$$

**Definition 3.2** ([HY, CHI]). Fix a Cartan scheme  $\mathcal{C}$ , and for each  $X \in \mathcal{X}$  a set  $\Delta^X \subset \mathbb{Z}^I$ . Then  $\mathcal{R} := \mathcal{R}(\mathcal{C}, (\Delta^X)_{X \in \mathcal{X}})$  is a *root system of type  $\mathcal{C}$*  if

- (1)  $\Delta^X = (\Delta^X \cap \mathbb{N}_0^I) \cup -(\Delta^X \cap \mathbb{N}_0^I)$  for all  $X \in \mathcal{X}$ ,
- (2)  $\Delta^X \cap \mathbb{Z}\alpha_i = \{\pm\alpha_i\}$  for all  $i \in I$  and  $X \in \mathcal{X}$ ,
- (3)  $s_i^X(\Delta^X) = \Delta^{r_i(X)}$  for all  $i \in I$  and  $X \in \mathcal{X}$ ,
- (4) if  $m_{ij}^X := |\Delta^X \cap (\mathbb{N}_0\alpha_i + \mathbb{N}_0\alpha_j)|$ , then  $(r_i r_j)^{m_{ij}^X}(X) = X$  for all  $i \neq j \in I$  and  $X \in \mathcal{X}$ .

We call  $\Delta_+^X := \Delta^X \cap \mathbb{N}_0^I$  the set of *positive roots*, and  $\Delta_-^X := -\Delta_+^X$  the set of *negative roots*.

By (3) we have  $w(\Delta^X) = \Delta^Y$  for any  $w \in \text{Hom}(Y, X)$ .

We say that  $\mathcal{R}$  is *finite* if  $\Delta^X$  is finite for some  $X \in \mathcal{X}$ . By [CHI, Lemma 2.11], this is equivalent to the fact that the sets  $\Delta^X$  are finite for all  $X \in \mathcal{X}$ , and  $\mathcal{W}$  is finite.

The following result plays a fundamental role in the next subsection.

**Theorem 3.3** ([CH2, Thm. 2.10]). Let  $\alpha \in \Delta_+^X \setminus \{\alpha_i : i = 1, \dots, \theta\}$ . There exist  $\beta, \gamma \in \Delta_+^X$  such that  $\alpha = \beta + \gamma$ . □

Now we recall some results involving real roots and the length of elements in  $\mathcal{W}$ .

**Lemma 3.4** ([HY, Cor. 3]). Let  $m \in \mathbb{N}$ ,  $X, Y \in \mathcal{X}$  and  $i_1, \dots, i_m, j \in I$ . Set  $w = \text{id}_X s_{i_1} \cdots s_{i_m} \in \text{Hom}(Y, X)$ , and assume that  $\ell(w) = m$ . Then:

- $\ell(ws_j) = m + 1$  if and only if  $w(\alpha_j) \in \Delta_+^X$ ,
- $\ell(ws_j) = m - 1$  if and only if  $w(\alpha_j) \in \Delta_-^X$ . □

**Proposition 3.5** ([CHI, Prop. 2.12]). For each  $w = \text{id}_X s_{i_1} \cdots s_{i_m}$  such that  $\ell(w) = m$ , the roots  $\beta_j = s_{i_1} \cdots s_{i_{j-1}}(\alpha_{i_j}) \in \Delta^X$  are positive and pairwise different. If  $w$  is an element of maximal length and  $\mathcal{R}$  is finite, then  $\{\beta_j\} = \Delta_+^X$ . Hence all the roots are real, i.e., for each  $\alpha \in \Delta_+^X$  there exist  $i_1, \dots, i_k, j \in I$  such that  $\alpha = s_{i_k} \cdots s_{i_1}(x_j)$ . □

As in [HS], for  $X \in \mathcal{X}$ ,  $m \in \mathbb{N}$  and  $(i_1, \dots, i_m) \in I^m$  consider the sets

$$\Lambda^X(i_1, \dots, i_m) := \{\beta_k := \text{id}_X s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k}) : 1 \leq k \leq m\} \subset \Delta^X, \tag{3.3}$$

$$\Lambda_+^X(i_1, \dots, i_m) := \{\beta \in \Delta_+^X : |\{k \in \{1, \dots, m\} : \beta = \pm\beta_k\}| \text{ is odd}\}. \tag{3.4}$$

By [HS, Prop. 1.9], given other elements  $j_1, \dots, j_n \in I$ , we have

$$\Lambda_+^X(i_1, \dots, i_m) = \Lambda_+^X(j_1, \dots, j_n) \Leftrightarrow \text{id}_X s_{i_1} \cdots s_{i_m} = \text{id}_X s_{j_1} \cdots s_{j_n},$$

and moreover

$$|\Lambda_+^X(i_1, \dots, i_m)| = \ell(\text{id}_X s_{i_1} \cdots s_{i_m}). \quad (3.5)$$

In this way, if  $w = \text{id}_X s_{i_1} \cdots s_{i_m}$  is any expression of  $w \in \text{Hom}(\mathcal{W}, X)$ , we can define  $\Lambda_+^X(w) := \Lambda_+^X(i_1, \dots, i_m)$ .

### 3.2. Convex orders on root systems

Now we characterize convex orders on subsets of root systems of finite Weyl groupoids, extending the results given in [P] for Weyl groups.

**Definition 3.6.** Consider a root system  $\Delta_+^X$  with a fixed total order  $<$ . We say that the order is

- *convex* if for any  $\alpha, \beta \in \Delta^+$  such that  $\alpha < \beta$  and  $\alpha + \beta \in \Delta^+$  we have

$$\alpha < \alpha + \beta < \beta;$$

- *subconvex* if for any  $\alpha, \beta \in \Delta^+$  such that  $\alpha < \beta$  and  $\alpha + \beta \in \Delta^+$  we have

$$\alpha < \alpha + \beta;$$

- *strongly convex* if for each ordered subset  $\alpha_1 \leq \dots \leq \alpha_k$  of  $\Delta^+$  with  $\alpha := \sum \alpha_i \in \Delta^+$  we have  $\alpha_1 < \alpha < \alpha_k$ .

**Definition 3.7.** Let  $L = \{\beta_1, \dots, \beta_m\}$  be an ordered subset of  $\Delta_+^X$ . We say that  $L$  is *associated to*  $w \in \text{Hom}(\mathcal{W}, X)$  if there exists a reduced expression  $w = \text{id}_X s_{i_1} \cdots s_{i_m}$  such that

$$\beta_j = s_{i_1} \cdots s_{i_{j-1}}(\alpha_{i_j}), \quad \forall 1 \leq j \leq m.$$

Compare this with [P]. For any  $w \in \text{Hom}(Y, X)$  define

$$R_w := \{\alpha \in \Delta_+^X : w^{-1}(\alpha) \in \Delta_-^Y\}.$$

Now we generalize some results about Weyl groups to the context of Weyl groupoids. First we consider the analogue of a result in [Bo].

**Proposition 3.8.** *For any ordered set  $L$  associated to  $w$ , we have  $L = R_w$ . Consequently,  $|R_w| = \ell(w)$  and two ordered sets associated to the same  $w$  differ at most by the ordering.*

*Proof.* Note that for any  $\beta_j = s_{i_1} \cdots s_{i_{j-1}}(\alpha_{i_j})$ ,

$$w^{-1}(\beta_j) = -s_{i_m} \cdots s_{i_{j+1}}(\alpha_j) s_{i_m} \cdots s_{i_{j+1}} s_{i_j}$$

is a reduced expression because it is contained in a reduced expression, so we have  $w^{-1}(\beta_j) \in \Delta_-^Y$  by Lemma 3.4. Therefore  $L \subseteq R_w$ .

Conversely, let  $\alpha \in R_w$ . As  $w^{-1}(\alpha) \in \Delta_-^Y$  and  $s_{i_1} \cdots s_{i_m}(w^{-1}(\alpha)) = \alpha \in \Delta_+^X$ , consider the greatest  $j$  such that  $s_{i_j} \cdots s_{i_m} w^{-1}(\alpha)$  is positive. Then  $s_{i_{j+1}} \cdots s_{i_m} w^{-1}(\alpha)$  is negative, so  $s_{i_j} \cdots s_{i_m} w^{-1}(\alpha) = \alpha_{i_j}$ , and hence  $\alpha_{i_j} = s_{i_j} \cdots s_{i_m} w^{-1}(\alpha)$ ; that is,  $\alpha = s_{i_1} \cdots s_{i_{j-1}}(\alpha_j) \in L$ .  $\square$

Second, we relate our sets  $R_w$  to the ones in [HS] (see (3.4)). Although the sets are equal, our definition is more convenient to prove statements about convexity.

**Lemma 3.9.** *For each  $w \in \text{Hom}(\mathcal{W}, X)$ ,  $R_w = \Lambda_+^X(w)$ .*

*Proof.* Fix a reduced expression  $w = \text{id}_X s_{i_1} \cdots s_{i_m}$ , so  $\beta_j = s_{i_1} \cdots s_{i_{j-1}}(\alpha_{i_j})$  is a positive root, and  $\alpha \in \Delta_+^X$  is equal to  $\pm\beta_j$  if and only if  $\alpha = \beta_j$ . Therefore  $\Lambda_+^X(w) = L$ .  $\square$

Now we extend another result from [P]. Note that condition (a) in our result is weaker than the one in [P], but the proof is very similar. This weaker condition will simplify some proofs in what follows.

**Theorem 3.10.** *Let  $L$  be an ordered subset of  $\Delta_+^X$ . There exists  $w \in \text{Hom}(\mathcal{W}, X)$  such that  $L$  is associated to  $w$  if and only if the following conditions are satisfied:*

- (a) *For each pair  $\lambda < \mu \in L$  such that  $\lambda + \mu \in \Delta_+^X$ , we have  $\lambda + \mu \in L$  and  $\lambda < \lambda + \mu$ .*
- (b) *If  $\lambda + \mu \in L$  and  $\lambda, \mu \in \Delta_+^X$ , then at least one of them belongs to  $L$  and precedes  $\lambda + \mu$ .*

*Proof.* Assume that  $L$  is associated to  $w = \text{id}_X s_{i_1} \cdots s_{i_m}$  for some  $w \in \text{Hom}(Y, X)$ . If  $\lambda = s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k})$  and  $\mu = s_{i_1} \cdots s_{i_{j-1}}(\alpha_{i_j})$  are such that  $1 \leq k < j \leq m$  and  $\lambda + \mu \in \Delta_+^X$ , we have  $\lambda + \mu \in L = R_w$ , because

$$w^{-1}(\lambda + \mu) = w^{-1}(\lambda) + w^{-1}(\mu) \in \Delta_-^Y.$$

Suppose that  $\lambda + \mu < \lambda$ . Then  $\lambda + \mu = s_{i_1} \cdots s_{i_{l-1}}(\alpha_{i_l})$  for some  $1 \leq l < k$ , so  $s_{i_l} \cdots s_{i_1}(\lambda + \mu) = -\alpha_l \in \Delta_-^{r_{i_l} \cdots r_{i_1}(X)}$ . But as  $l < k < j$ , we have

$$s_{i_l} \cdots s_{i_1}(\lambda), s_{i_l} \cdots s_{i_1}(\mu) \in \Delta_-^{r_{i_l} \cdots r_{i_1}(X)},$$

which is a contradiction. Therefore  $\lambda < \lambda + \mu$ , and  $L$  satisfies (a).

For (b), suppose that  $\lambda + \mu \in L$ , but  $\lambda, \mu \notin L$ ; then  $w^{-1}(\lambda), w^{-1}(\mu) \in \Delta_+^Y$ , so  $w^{-1}(\lambda + \mu)$  is positive, which contradicts the fact that  $\lambda + \mu \in R_w$ . If both  $\lambda, \mu \in L$ , a similar proof to (a) gives that one of them precedes  $\lambda + \mu$ . Now, suppose that  $\lambda \in L, \mu \notin L$  and  $\lambda + \mu < \lambda$ . If  $l < k$  is such that  $\lambda + \mu = s_{i_1} \cdots s_{i_{l-1}}(\alpha_{i_l})$ , we have  $s_{i_l} \cdots s_{i_1}(\lambda) \in \Delta^+$  and

$$s_{i_l} \cdots s_{i_1}(\lambda) + s_{i_l} \cdots s_{i_1}(\mu) = s_{i_l} \cdots s_{i_1}(\lambda + \mu) = -\alpha_l \in \Delta_-^{r_{i_l} \cdots r_{i_1}(X)},$$

so  $s_{i_l} \cdots s_{i_1}(\mu) \in \Delta_-^{r_{i_l} \cdots r_{i_1}(X)}$ , and then  $\mu \in R_{\text{id}_X s_{i_1} \cdots s_{i_l}} \subset R_{\text{id}_X s_{i_1} \cdots s_{i_m}} = L$ , a contradiction.

Conversely, we will prove that an ordered set  $L$  satisfying (a) and (b) is associated to some  $w$  by induction on  $m := |L|$ . If  $m = 1$ , let  $\alpha \in L$ . If  $\alpha$  is not simple, then by

Theorem 3.3,  $\alpha = \beta + \gamma$  for some positive roots  $\beta, \gamma$ , and by condition (b) one of them belongs to  $L$ , so  $m \geq 2$ , which is a contradiction. Therefore  $L = \{\alpha_j\} = R_{s_j}$  for some  $1 \leq j \leq \theta$ .

Now assume  $m > 1$  and let  $\beta_1 < \dots < \beta_m$  be the elements of  $L$ . Notice that  $L' = \{\beta_1, \dots, \beta_{m-1}\}$  satisfies conditions (a) and (b), so by the inductive hypothesis there exists a reduced expression  $v = \text{id}_X s_{i_1} \dots s_{i_{m-1}}$  such that

$$\beta_1 = \alpha_{i_1}, \quad \beta_j = s_{i_1} \dots s_{i_{j-1}}(\alpha_{i_j}), \quad j = 2, \dots, m - 1.$$

Let  $Z = r_{i_{m-1}} \dots r_{i_1}(X)$ . Then  $v^{-1}(\beta_m) \in \Delta_+^Z$  because  $\beta_m \notin L' = R_v$ . Suppose that  $v^{-1}(\beta_m)$  is not simple. Then there exist  $\alpha, \beta \in \Delta_+^Z$  such that  $\alpha + \beta = v^{-1}(\beta_m)$ , i.e.  $\beta_m = \alpha' + \beta'$ , where  $\alpha' = v(\alpha), \beta' = v(\beta) \in \Delta^X$ . Therefore  $\alpha' \in \Delta_+^X$  or  $\beta' \in \Delta_+^X$ . On the other hand, if both are positive then one of them is  $\beta_k$  for some  $k < m$ ; assume  $\alpha' = \beta_k$ ; then  $\alpha = v^{-1}(\beta_k) \in \Delta_-^Z$ , a contradiction. Consequently, we can consider  $\alpha' \in \Delta_+^X$  and  $\beta' \in \Delta_-^X$ . For this case,  $\alpha' \notin R_v = L'$  and  $-\beta' \in R_v = L' \subset L$ . As  $\alpha' = \beta_m + (-\beta')$ , hypothesis (a) implies that  $\alpha' \in L$ , so  $\alpha' = \beta_m \in L - L'$ , a contradiction. Therefore,  $v^{-1}(\beta_m) = \alpha_{i_m}$  for some  $i_m \in I, w = vs_{i_m} \in \text{Hom}(r_{i_m}(Z), X)$  is a reduced expression by Lemma 3.4, and  $L = R_w$ .  $\square$

**Theorem 3.11.** *Given an order on  $\Delta_+^X$ , the following statements are equivalent:*

- (1) *the order is associated with a reduced expression of the longest element,*
- (2) *the order is strongly convex,*
- (3) *the order is convex.*

*Proof.* (1) $\Rightarrow$ (2). Let  $\omega = \text{id}_X s_{i_1} \dots s_{i_m}$  be an element of maximal length in  $\text{Hom}(\mathcal{W}, X)$ . By Proposition 3.5,  $m = |\Delta_+^X|$  and the elements

$$\beta_k := s_{i_1} \dots s_{i_{k-1}}(\alpha_{i_k}), \quad k = 1, \dots, m,$$

are all different, so  $\{\beta_k\} = \Delta_+^X$ . This induces an order on  $\Delta_+^X$ :

$$\beta_1 < \dots < \beta_m.$$

To prove that this order is strongly convex, consider  $\beta, \beta_{k_1}, \dots, \beta_{k_l} \in \Delta_+^X$  such that  $k_1 < \dots < k_l$  and  $\beta = \beta_{k_1} + \dots + \beta_{k_l}$ . Suppose that  $\beta = \beta_k$  with  $k < k_1$ . Then  $u = \text{id}_X s_{i_1} \dots s_{i_k}$  satisfies  $\ell(u) = k, \beta \in R_u$  but  $\beta_{k_j} \notin R_u$  for all  $j = 1, \dots, l$ , which is a contradiction because  $u(\beta) \in \Delta_-^{r_{i_k} \dots r_{i_1} X}$  should be the sum of the positive roots  $u(\beta_j)$ . We obtain a similar contradiction if we assume  $k > k_l$ . Therefore  $k_1 < k < k_l$ .

(2) $\Rightarrow$ (3) is clear.

(3) $\Rightarrow$ (1). Assume that a given order on  $\Delta_+^X$  is convex; then it trivially satisfies condition (a) of Theorem 3.10 because we consider  $L = \Delta_+^X$ . Therefore it also satisfies condition (b) by convexity, so the order is associated to some  $w$ . As  $\ell(w) = |\Delta_+^X|$  by Proposition 3.8, it is the element of maximal length.  $\square$

### 3.3. Coideal subalgebras and convex orders for PBW bases

Now we recall a description of coideal subalgebras of Nichols algebras with finite root system given in [HS]. We will use this result to prove that the lexicographical order on the PBW generators of Kharchenko’s basis is convex. First, we recall some results about the Weyl groupoid attached to a braided vector space of diagonal type. Given a braided vector space  $(V, c)$  of diagonal type, fix a basis  $\{x_1, \dots, x_\theta\}$  and scalars  $q_{ij} \in k^\times$  as in (2.1), and the bilinear form as in (2.3). As in [H1], let  $\Delta_+^V$  be the set of degrees of a PBW basis of  $\mathfrak{B}(V)$ , counted with their multiplicities. This set does not depend on the PBW basis, as remarked in [H1] and proved in [AA].

In what follows, we fix a braided vector space  $(V, c)$  of diagonal type and assume that the root system  $\Delta_+^V$  is finite. In that case we can attach to it a Cartan scheme  $\mathcal{C}$ , a Weyl groupoid  $\mathcal{W}$  and the corresponding root system  $\mathcal{R}$  (see [HS, Thms. 6.2, 6.9] and the references therein); here  $\mathcal{W}$  coincides with the Weyl groupoid defined in [H1] for a braided vector space of diagonal type. This Weyl groupoid can be built as follows (see [AA]). Let  $\mathcal{X}$  be the set of ordered bases of  $\mathbb{Z}^\theta$ , and for each  $F = \{f_1, \dots, f_\theta\} \in \mathcal{X}$ , set  $\tilde{q}_{ij} = \chi(f_i, f_j)$ . For each  $1 \leq i \neq j \leq \theta$ , define

$$m_{ij}(F) := \min\{n \in \mathbb{N}_0 : (n + 1)\tilde{q}_{ii}(1 - \tilde{q}_{ii}^n \tilde{q}_{ij} \tilde{q}_{ji}) = 0\}, \tag{3.6}$$

and let  $s_{i,F} \in \text{Aut}(\mathbb{Z}^\theta)$  be such that  $s_{i,F}(f_j) = f_j + m_{ij}(F)f_i$ . Here  $m_{ii} = -2$ .

Note that  $\mathcal{G} = \text{Aut}(\mathbb{Z}^\theta) \times \mathcal{X}$  is a groupoid whose set of objects is  $\mathcal{X}$  and whose morphisms are

$$x \xrightarrow{(g,x)} g(x).$$

The Weyl groupoid  $W(\chi)$  of  $\chi$  is the least subgroupoid of  $\mathcal{G}$  such that

- $(\text{id}, E) \in W(\chi)$ ,
- if  $(\text{id}, F) \in W(\chi)$  and  $s_{i,F}$  is defined, then  $(s_{i,F}, F) \in W(\chi)$ .

Recall that the (right) *Duflo order* on  $\text{Hom}(\mathcal{W}, X)$  is defined as follows: if  $x \in \text{Hom}(Y, X)$  and  $y \in \text{Hom}(Z, Y)$ , then  $x \leq_D xy$  iff  $\ell(xy) = \ell(x) + \ell(y)$  (see [HS, Def. 1.11]). By [HS, Thm. 1.13], given  $v, w \in \text{Hom}(\mathcal{W}, X)$  we have  $v \leq_D w$  if and only if  $\Lambda_+^X(v) \subset \Lambda_+^X(w)$ .

**Remark 3.12.** Let  $w_1 \leq_D \dots \leq_D w_k$  be a maximal chain in  $\text{Hom}(\mathcal{W}, X)$ . Then there exists a reduced expression  $\text{id}_X s_{i_1} \dots s_{i_k}$  for some  $i_1, \dots, i_k \in I$  such that  $w_j = \text{id}_X s_{i_1} \dots s_{i_j}$  for each  $1 \leq j \leq k$ .

In particular, a chain  $w_1 \leq_D \dots \leq_D w_k$  has maximal length iff it is associated to a reduced expression of the longest element in  $\text{Hom}(\mathcal{W}, X)$ , and so  $k = |\Delta_+^X|$ .

We now recall some results from [HS] about the classification of coideal subalgebras of  $\mathfrak{B}(V)$ . As there, we denote by  $\mathcal{K}(V)$  the set of all  $\mathbb{N}_0^\theta$ -graded left coideal subalgebras of  $\mathfrak{B}(V)$ . We rewrite these results in the context of diagonal braidings (in [HS] the authors work in a more general context).

First results about the classification of coideal subalgebras were obtained in [K3, KL, Po] for the quantized enveloping algebras  $U_q(\mathfrak{g})$  of type  $A_n, B_n$  and  $G_2$ , respectively,

where it was proved that coideal subalgebras admit a PBW basis and these subalgebras were classified.

Given  $n = (n_1, \dots, n_\theta) \in \mathbb{N}_0^\theta$ , we set  $X^n = X_1^{n_1} \cdots X_\theta^{n_\theta}$  in  $k[[x_1, \dots, x_\theta]]$ . We also set

$$q_h(t) := \frac{t^h - 1}{t - 1} \in k[[t]], \quad h \in \mathbb{N}; \quad q_\infty(t) := \frac{1}{1 - t} = \sum_{s=0}^\infty t^s \in k[[t]].$$

For each  $\mathbb{N}_0^\theta$ -graded  $k$ -vector space  $W = \bigoplus_{\alpha \in \mathbb{N}_0^\theta} W_\alpha$ , we denote its Hilbert series by

$$\mathcal{H}_W := \sum_{\alpha \in \mathbb{N}_0^\theta} (\dim W_\alpha) X^\alpha \in k[[x_1, \dots, x_\theta]].$$

For any  $\alpha \in \mathbb{N}_0^\theta$ , we set  $q_\alpha = \chi(\alpha, \alpha)$ , where  $\chi$  is the bicharacter over  $\mathbb{Z}^\theta$  as in (2.3), and  $N_\alpha = \text{ord}(q_\alpha)$ , where  $N_\alpha = \infty$  if  $q_\alpha$  is not a root of unity.

**Theorem 3.13** ([HS]). *For each  $w \in \text{Hom}(\mathcal{W}, V)$  there exists a unique left coideal subalgebra  $F(w) \in \mathcal{K}(V)$  with Hilbert series*

$$\mathcal{H}_{F(w)} = \prod_{\beta \in \Lambda_+^V(w)} q_{N_\beta}(X^\beta). \tag{3.7}$$

Moreover, the correspondence  $w \mapsto F(w)$  gives an order preserving and order reflecting bijection between  $\text{Hom}(\mathcal{W}, V)$  and  $\mathcal{K}(V)$ , where we consider the Duflo order over  $\text{Hom}(\mathcal{W}, V)$  and the inclusion order over  $\mathcal{K}(V)$ ; i.e.

$$w_1 \leq_D w_2 \Leftrightarrow F(w_1) \subset F(w_2).$$

*Proof.* Note that in [HS] the authors classify the right coideal subalgebras, but  $E$  is a right coideal subalgebra if and only if  $\mathcal{S}(E)$  is a left coideal subalgebra, where  $\mathcal{S}$  denotes the antipode of  $\mathfrak{B}(V)$ . Moreover, if they are  $\mathbb{N}^\theta$ -graded, then  $\mathcal{H}_E = \mathcal{H}_{\mathcal{S}(E)}$ , because  $\mathcal{S}$  is  $\mathbb{N}^\theta$ -graded, and the order given by inclusion on the family of left coideal subalgebras corresponds with the one on the family of right coideal subalgebras because  $\mathcal{S}$  is bijective. In this context we define  $F(w) = \mathcal{S}(E^V(w))$ , where  $E^V(w)$  is as in [HS, Thm. 6.12].

By [HS, Lemma 6.11], we have an isomorphism of  $\mathbb{N}_0^\theta$ -graded spaces

$$F(w) \cong \bigotimes_{\beta \in \Lambda_+^V(w)} \mathfrak{B}(V_\beta),$$

where  $V_\beta$  corresponds to  $N_\beta$  of [HS, Def. 6.5]. In this way  $V_\beta$  is a 1-dimensional braided vector space of diagonal type generated by a non-zero vector  $v_\beta$ , such that  $c(v_\beta \otimes v_\beta) = q_\beta v_\beta \otimes v_\beta$ . Therefore,  $\mathcal{H}_{\mathfrak{B}(V_\beta)} = q_{N_\beta}(X^\beta)$ , and (3.7) follows.

The uniqueness of a coideal subalgebra with a given Hilbert series follows from [HS, Lemma 6.4]. The map  $\text{Hom}(\mathcal{W}, V) \rightarrow \mathcal{K}(V)$  is bijective and preserves the order in both directions by [HS, Thms. 6.12, 6.15] (note that we can apply these theorems because we assume that  $V$  has diagonal braiding and  $\Delta_+^V$  is finite).  $\square$

Consider the PBW basis of Lyndon words given in Theorem 2.8 for the fixed basis  $\{x_1, \dots, x_\theta\}$  of  $V$ . We assume that  $\Delta_+^V$  is finite, so all the roots are real and have multiplicity one. In this way, we can label PBW generators by elements  $\beta \in \Delta_+^V$ : the generators are  $x_\beta = [l_\beta]_c$  for some Lyndon word  $l_\beta$  of degree  $\beta$ . This induces a total order on the roots: if  $l_{\beta_1} < \dots < l_{\beta_M}$  are ordered lexicographically, we consider  $\beta_1 < \dots < \beta_M$ , where  $M = |\Delta_+^V|$  and in particular  $l_{\beta_1} = x_1, l_{\beta_M} = x_\theta$ . Let  $B$  be the basis of  $\mathfrak{B}(V)$  consisting of hyperwords as above.

Let  $\pi : T(V) \rightarrow \mathfrak{B}(V) = T(V)/I(V)$  be the canonical projection. Recall the definition of the coideal subalgebras  $W_{l_\beta}$  in Lemma 2.6, and set

$$W_\beta := \pi(W_{l_\beta}), \quad \beta \in \Delta_+^V.$$

**Remark 3.14.**  $W_\beta$  is a left coideal subalgebra of  $\mathfrak{B}(V)$ , because  $\pi$  is a morphism of braided Hopf algebras and  $W_{l_\beta}$  is a left coideal subalgebra of  $T(V)$ . Also  $W_{\beta_j} \subseteq W_{\beta_i}$  if  $i < j$ , and

$$W_{\beta_1} = \mathfrak{B}(V), \quad W_{\beta_M} = k\langle x_\theta \rangle.$$

**Lemma 3.15.** *With the notation above,  $x_{\beta_i} \notin W_{\beta_j}$  if  $i < j$ . Hence,*

$$\mathfrak{B}(V) = W_{\beta_1} \supsetneq W_{\beta_2} \supsetneq \dots \supsetneq W_{\beta_M}.$$

*Proof.* Suppose that  $x_{\beta_i} \in W_{\beta_j}$  with  $i < j$ . Then  $x_{\beta_i} \in G_{I(V)}$  is a linear combination of hyperwords greater than or equal to  $x_{\beta_j}$  in  $\mathfrak{B}(V)$ , contrary to Corollary 2.10. Therefore  $x_{\beta_i} \notin W_{\beta_j}$ . The second statement follows from Remark 3.14.  $\square$

We now prove the main result of this section.

**Theorem 3.16.** *Keep the notation above. The order  $\beta_1 < \dots < \beta_M$  on  $\Delta_+^V$  is convex.*

*Proof.* Each  $W_{\beta_i}$  corresponds to one  $F(w_i)$ . As we have a chain as in the previous lemma, by Theorem 3.13 we have  $w_1 \geq_D \dots \geq_D w_M$ .

As the  $w_i$ 's are pairwise different, we have a chain of maximal length, and by Remark 3.12 there exists a reduced expression of the longest element  $\omega^V = \text{id}_V s_{i_M} \dots s_{i_1}$  such that  $w_k = \text{id}_V s_{i_M} \dots s_{i_k}$  for each  $1 \leq k \leq M$ .

We will prove by descending induction on  $j$  that  $\beta_j = s_{i_M} \dots s_{i_{j+1}}(\alpha_j)$ . This will conclude the proof because of Theorem 3.11. For the base step, notice that  $\mathcal{H}_{w_M} = \mathfrak{q}_{N_{\alpha_\theta}}(x_\theta)$  by Theorem 3.13, and by Remark 3.14 we have  $i_M = \theta$ .

Assume now that  $k < M$  and  $\beta_j = s_{i_M} \dots s_{i_{j+1}}(\alpha_j)$  for  $j = k + 1, \dots, M$ . Set  $\gamma = s_{i_M} \dots s_{i_{k+1}}(\alpha_k)$ . By the inductive hypothesis we have

$$\mathcal{H}_{W_{\beta_{k+1}}} = \prod_{j=k+1}^M \mathfrak{q}_{N_{\beta_j}}(X^{\beta_j}), \quad \mathcal{H}_{W_{\beta_k}} = \mathfrak{q}_{N_\gamma}(X^\gamma) \left( \prod_{j=k+1}^M \mathfrak{q}_{N_{\beta_j}}(X^{\beta_j}) \right).$$

On the other hand,  $\{x_{\beta_M}^{n_M} \dots x_{\beta_k}^{n_k} : 0 \leq n_j < N_{\beta_j}\}$  is a linearly independent set in  $W_{\beta_k}$ , so

$$\mathcal{H}_{W_{\beta_k}} \geq \prod_{j=k}^M \mathfrak{q}_{N_{\beta_j}}(X^{\beta_j}),$$

where the inequality between the series means the inequality for all the corresponding coefficients. Looking at the coefficient of  $X^{\beta_k}$  we find that there exists an expression

$$\beta_k = n\gamma + \sum_{j=k+1}^M n_j \beta_j, \quad n \in \mathbb{N}, n_j \in \mathbb{N}_0.$$

Note that  $R_{w_k} = \Delta_+^V = \{\gamma, \beta_{k+1}, \dots, \beta_M\}$ , so if we apply  $w_k$  to the last equality, we deduce that  $w_l^{-1}(\beta_k) \in \Delta_m^{r_1 \dots r_M(V)}$ . Therefore  $\beta_k \in R_{w_k}$ , and as  $\beta_k \neq \beta_j$  for all  $j > k$ , we conclude  $\beta_k = \gamma$ .  $\square$

The next result is analogous to the one for the positive part of the quantized enveloping algebra  $U_q(\mathfrak{g})$  given in [Le], and gives an inductive way to obtain the words  $l_\beta$  for  $\beta \in \Delta_+^V$ .

**Corollary 3.17.** *For each  $\beta \in \Delta_+^V$ ,  $\beta \neq \alpha_1, \dots, \alpha_\theta$ , we have*

$$l_\beta = \max\{l_{\delta_1} l_{\delta_2} : \delta_1, \delta_2 \in \Delta_+^V, \delta_1 + \delta_2 = \beta, l_{\delta_1} < l_{\delta_2}\}. \tag{3.8}$$

*Proof.* Any factor of an element of  $G_{I(V)}$  is in  $G_{I(V)}$  (see Subsection 2.2). If  $l_\beta = uv$  is the Shirshov decomposition of  $l_\beta$ , then there exist  $\gamma_1, \gamma_2 \in \Delta_+^V$  such that  $u = l_{\gamma_1} < v = l_{\gamma_2}$  and  $\beta = \gamma_1 + \gamma_2$ .

On the other hand, let  $\delta_1, \delta_2 \in \Delta_+^V$  be such that  $\delta_1 + \delta_2 = \beta$  and  $l_{\delta_1} < l_{\delta_2}$ . By the previous theorem,  $l_{\delta_1} < l_\beta < l_{\delta_2}$ . If  $l_\beta$  does not begin with  $l_{\delta_1}$ , then  $l_{\delta_1}u < l_\beta$  for every word  $u$ , so in particular  $l_{\delta_1}l_{\delta_2} < l_\beta$ . If  $l_\beta$  begins with  $l_{\delta_1}$ , then  $l_\beta = l_{\delta_1}u$ , where  $u$  has degree  $\delta_2$ . Let  $u = l_p l_{p-1} \dots l_1$  be its Lyndon decomposition. Then each  $l_i$  is in  $G_{I(V)}$ , so  $u = l_{\beta_M}^{n_M} \dots l_{\beta_1}^{n_1}$  for some  $n_i \in \mathbb{N}_0$ . Let  $k = \max\{j : n_j \neq 0\}$ . As the order is strongly convex,  $x_{\beta_k} \geq x_{\delta_2}$ , i.e.  $l_{\beta_k} \geq l_{\delta_2}$ , so  $u \geq l_{\delta_2}$  and hence  $l_\beta = l_{\delta_1}u \geq l_{\delta_1}l_{\delta_2}$ . In any case,  $l_\beta = l_{\delta_1}u \geq l_{\delta_1}l_{\delta_2}$ .  $\square$

Another consequence is that the coideal subalgebras  $W_\beta$  (which are in particular left  $\mathfrak{B}(V)$ -comodules) behave as a kind of modules of highest weight.

**Theorem 3.18.** *The set  $B_k = \{x_{\beta_M}^{n_M} \dots x_{\beta_k}^{n_k} : 0 \leq n_j < N_{\beta_j}\}$  is a basis of  $W_{\beta_k}$ . Moreover, if  $W_{\beta_k} = \bigoplus_{\alpha \in \mathbb{N}_0^\theta} W_{\beta_k}(\alpha)$  denotes the decomposition into  $\mathbb{N}_0^\theta$ -homogeneous components, then  $\dim W_{\beta_k}(\beta_k) = 1$ .*

*Proof.* The first statement follows because  $B_k$  is included in  $W_{\beta_k}$ , it is linearly independent and the Hilbert series of the  $k$ -linear subspace spanned by  $B_k$  coincides with the Hilbert series of  $W_{\beta_k}$ .

For the second statement, if  $\sum_{i=1}^M n_i \beta_i = \beta_k$  for some  $n_i \in \mathbb{N}_0$ , then  $n_i = \delta_{i,k}$  or there exists  $i < k$  such that  $n_i > 0$ , by Theorem 3.11.  $\square$

The first consequence of the description of the coideal subalgebras  $W_\alpha$  in the previous theorem is a new expression for the coproduct of hyperwords which we will use in the next section. We set

$$C_k := \{x_{\beta_k}^{n_k} x_{\beta_{k-1}}^{n_{k-1}} \dots x_{\beta_1}^{n_1} : 0 \leq n_j < N_{\beta_j}\}, \tag{3.9}$$

$$D_k := \{x_{\beta_M}^{n_M} \dots x_{\beta_1}^{n_1} : 0 \leq n_j < N_{\beta_j}, \exists j \geq k \text{ such that } n_j \neq 0\}. \tag{3.10}$$



**Lemma 3.19.** *Let  $a \in B_k - \{1\}$  and  $b \in B_l$  with  $l \leq k$ . Then either  $ab = 0$  or  $ab$  is spanned by elements of  $B_l \cap D_k$ .*

*Proof.* If  $l = k$ , the conclusion follows directly. Assume that  $l < k$  and write  $b = b_1 b_2$  with  $b_1 \in B_k$  and  $b_2 \in C_{k-1} \cap B_l$  (possibly  $b_1 = 1$ ). Then  $ab_1 \in W_{\beta_k}$ , because  $W_{\beta_k}$  is a subalgebra, so it is spanned by  $B_k$ . Finally, note that if  $c \in B_k$ , then  $cb_2 \in B_l \cap D_k$ .  $\square$

We also set  $\text{ht}(u) := \sum n_i$  if  $u = x_{\beta_M}^{n_M} x_{\beta_{k-1}}^{n_{k-1}} \cdots x_{\beta_1}^{n_1}$ .

**Lemma 3.20.** *Let  $u = x_{\beta_k}^{n_k} \cdots x_{\beta_l}^{n_l} \in B_l - D_{k+1}$ ,  $l \leq k$ , be such that  $n_k, n_l \neq 0$ . Then*

$$\Delta(u) \in \left( \bigoplus_{v \in B, w \in D_k \cap B_l} \mathbb{k} v \otimes w \right) \oplus \left( \bigoplus_{v \in D_k, w \in B_l - D_k} \mathbb{k} v \otimes w \right).$$

*Proof.* We use induction on the height. If  $\text{ht}(u) = 1$ , then  $u = x_{\beta_i}$  for some  $i$ . Thus,  $\Delta(u) \in u \otimes 1 + 1 \otimes u + \mathfrak{B}(V) \otimes W_{\beta_i}$ , so the result follows.

Assume the conclusion holds for  $\text{ht}(w) < n$ , and  $u = x_{\beta_k}^{n_k} \cdots x_{\beta_l}^{n_l}$  is such that  $\text{ht}(u) = n$ . Write  $u = x_{\beta_k} w$ , so by the inductive hypothesis,

$$\Delta(u) \in \left( \bigoplus_{v \in B, w \in D_s \cap B_l} \mathbb{k} v \otimes w \right) \oplus \left( \bigoplus_{v \in D_s, w \in B_l - D_s} \mathbb{k} v \otimes w \right),$$

where  $s = k - 1$  if  $n_k = 1$ , or  $s = k$  if  $n_k > 1$ . We calculate  $\Delta(u) = \Delta(x_{\beta_k})\Delta(w)$ . Using the fact that the braiding is diagonal, and Lemma 3.19, we conclude that

$$(\Delta(x_{\beta_k}) - x_{\beta_k} \otimes 1)\Delta(w) \in \bigoplus_{v \in B, w \in D_k \cap B_l} \mathbb{k} v \otimes w.$$

Also, for any  $v \in B$  we have  $x_{\beta_k} v \in D_k$ , because if  $v \in B_k$  then  $x_{\beta_k} v \in W_{\beta_k}$  and if  $v \in B_i$  for  $i < k$  then we apply Lemma 3.19 again, and we conclude the proof.  $\square$

#### 4. Presentation by generators and relations of Nichols algebras of diagonal type

In this section we use the convex order of a PBW basis of hyperletters to prove that, when the diagonal braiding is symmetric, the PBW basis is orthogonal with respect to the bilinear form of Proposition 2.1. This gives a way to obtain relations which hold in Nichols algebras, even when the braiding is not symmetric. We then obtain a presentation by generators and relations for any Nichols algebra of diagonal type whose root system is finite, by considering a suitable set of relations.

##### 4.1. A general presentation

We continue with the setting fixed in Subsection 3.3. To begin with, we prove the orthogonality of the PBW basis with respect to the bilinear form in Proposition 2.1. This extends [Ang1, Prop. 5.1], and the proof is very similar; anyway we rewrite it in this general setting.

**Proposition 4.1.** *Consider a PBW basis of  $\mathfrak{B}(V)$  as above given by Lyndon words, and assume that the braiding matrix is symmetric. Then the PBW basis is orthogonal with respect to the bilinear form in Proposition 2.1.*

*Proof.* We prove by induction on  $k = \ell(u) + \ell(v)$  that  $(u|v) = 0$ , where  $u \neq v$  are ordered products of PBW generators. If  $k = 1$ , then either  $u = 1, v = x_j$ , or  $u = x_i, v = 1$ , for some  $i, j \in \{1, \dots, \theta\}$ , and  $(1|x_j) = (x_i|1) = 0$ .

Suppose the statement is valid when  $\ell(u) + \ell(v) < k$ , and let  $u \neq v$  be hyperwords such that  $\ell(u) + \ell(v) = k$ . If both are hyperletters, they have different degrees  $\alpha \neq \beta \in \mathbb{Z}^\theta$ , so  $u = x_\alpha, v = x_\beta$ , and  $(x_\alpha|x_\beta) = 0$ , since the homogeneous components are orthogonal for  $(\cdot|\cdot)$ .

Suppose that  $u = x_\alpha$  and  $v = x_{\beta_k}^{h_k} x_{\beta_{k-1}}^{h_{k-1}} \dots x_{\beta_1}^{h_1}$  for some  $1 \leq i \leq k \leq M$  (we consider  $h_k, h_l \neq 0$ ). If they have different  $\mathbb{Z}^\theta$ -degree, they are orthogonal. Assume that  $\alpha = \sum_{j=i}^k h_j \beta_j$ , so  $\beta_i < \alpha$  because the ordered root system is strongly convex by Theorem 3.16. Using Lemma 2.5 and (2.1), we have

$$(u|v) = (x_\alpha|w)(1|x_{\beta_i}) + (1|w)(x_\alpha|x_{\beta_i}) + \sum_{l_1 \geq \dots \geq l_p > l_\alpha, l_i \in L} (x_{l_1, \dots, l_p}|w)([l_1]_c \cdots [l_p]_c|x_{\beta_i})$$

where  $v = wx_{\beta_i}$ . Note that  $(1|x_{\beta_i}) = (1|w) = 0$ . Also,  $[l_1]_c \cdots [l_p]_c$  is a linear combination of greater hyperwords of the same degree and an element of  $I(V)$ , so by inductive hypothesis and the fact that  $I(V)$  is the radical of the bilinear form, we conclude  $([l_1]_c \cdots [l_p]_c|x_{\beta_i}) = 0$ . Therefore  $(u|v) = 0$ .

Finally, we consider

$$u = x_{\beta_k}^{h_k} \dots x_{\beta_i}^{h_i}, \quad 1 \leq i \leq k \leq M, \quad v = x_{\beta_q}^{f_q} \dots x_{\beta_p}^{f_p}, \quad 1 \leq p \leq q \leq M.$$

The bilinear form is symmetric, so we can assume  $i \leq p$ . By Lemma 2.5 and (2.7),

$$(u|v) = (w|1)(x_{\beta_i}|v) + \sum_{j=0}^{f_p} \binom{f_p}{j}_{q\beta_p} (w|x_{\beta_q}^{f_q} \dots x_{\beta_{p-1}}^{f_{p-1}} x_{\beta_p}^j)(x_{\beta_i}|x_{\beta_p}^{f_p-j}) + \sum_{\substack{l_1 \geq \dots \geq l_t > l_{\beta_p}, l_s \in L \\ 0 \leq j \leq f_p}} (w|x_{l_1, \dots, l_t}^{(j)})(x_{\beta_i}|[l_1]_c \cdots [l_t]_c x_{\beta_p}^j)$$

where  $u = wx_{\beta_i}$ . Note that  $(w|1) = 0$ , and  $[l_1]_c \cdots [l_t]_c x_{\beta_p}^j$  is a combination of hyperwords of the PBW basis greater than or equal to it and an element of  $I(V)$ . Using the induction hypothesis and the fact that  $I(V)$  is the radical of this bilinear form, we conclude that  $(x_{\beta_i}|[l_1]_c \cdots [l_t]_c x_{\beta_p}^j) = 0$ . As also  $x_{\beta_i}, x_{\beta_p}^{f_p-j}$  are different elements of the PBW basis for  $f_p - j \neq 1$ , we have

$$(u|v) = (f_p)_{q\beta_p} (w|x_{\beta_q}^{f_q} \dots x_{\beta_{p-1}}^{f_{p-1}} x_{\beta_p}^{f_p-1})(x_{\beta_i}|x_{\beta_p}). \tag{4.1}$$

Then it is zero if  $i < p$ , but also if  $i = p$ , because in that case  $w \neq x_{\beta_q}^{f_q} \dots x_{\beta_{p-1}}^{f_{p-1}} x_{\beta_p}^{f_p-1}$  and we use the induction hypothesis.  $\square$

**Corollary 4.2.** *If  $u = x_{\beta_M}^{n_M} \cdots x_{\beta_1}^{n_1}$ , where  $0 \leq n_j < N_{\beta_j}$ , then*

$$c_u := (u|u) = \prod_{j=1}^M (n_j)!_{q_{\beta_j}} c_{x_{\beta_j}}^{n_j} \neq 0. \tag{4.2}$$

*Proof.* We use induction on  $\text{ht}(w)$ . If  $\text{ht}(w) = 1$ ,  $w$  is a hyperletter. If we assume the conclusion holds for  $\text{ht}(w) < k$ , and  $\text{ht}(u) = k$ , we use the orthogonality of the PBW basis and a calculation as in (4.1) for  $v = u$  to deduce (4.2) from the inductive hypothesis.

The scalar is not zero because  $u \notin I(V)$  and the PBW basis generates a  $k$ -linear complement to  $I(V)$ , the radical of this bilinear form.  $\square$

**Remark 4.3.** Note that

$$(x_{\beta_i} x_{\beta_j} | u) = (x_{\beta_i} | u_{(1)})(x_{\beta_j} | u_{(2)}) = d_{i,j} c_{x_{\beta_i}} c_{x_{\beta_j}},$$

where  $d_{i,j}$  is the coefficient of  $x_{\beta_i} \otimes x_{\beta_j}$  in the expression of  $\Delta(u)$  as a linear combination of elements of the PBW basis on both sides of the tensor product.

We return to the general case where the braiding matrix is not necessarily symmetric. We obtain some relations and then prove a presentation of Nichols algebras by generators and relations. To obtain these relations is the key step to finding the presentation in Theorem 4.9. Note that  $B_i \cap C_j$  is the set of monotone hyperwords whose hyperletters are between  $x_{\beta_i}$  and  $x_{\beta_j}$  (see Theorem 3.18 and the definition of  $C_j$  in Subsection 3.3).

Let  $(W, d)$  be a braided vector space of diagonal type,  $\hat{x}_1, \dots, \hat{x}_\theta$  a basis of  $W$  and  $\hat{q}_{ij} \in k^\times$  such that  $d(\hat{x}_i \otimes \hat{x}_j) = \hat{q}_{ij} \hat{x}_j \otimes \hat{x}_i$ . Assume that  $\hat{q}_{ij} = \hat{q}_{ji}$  for all  $1 \leq i, j \leq \theta$ , and that  $(V, c)$  and  $(W, d)$  are twist equivalent:

$$q_{ij} q_{ji} = \hat{q}_{ij} \hat{q}_{ji}, \quad q_{ii} = \hat{q}_{ii}, \quad 1 \leq i \neq j \leq \theta.$$

We define  $\hat{x}_\beta = [l_\beta]_d$ , that is, the corresponding hyperletter to  $l_\beta$ , but where we change the braiding  $c$  to  $d$ . By Corollary 3.17 and the invariance of the root system under twist equivalence, the set of all  $\hat{x}_\beta, \beta \in \Delta_+^V = \Delta_+^W$ , is a set of generators of a PBW basis as in Kharchenko’s Theorem. If  $u = x_{\beta_M}^{n_M} \cdots x_{\beta_1}^{n_1}$ , then we denote  $\hat{u} = \hat{x}_{\beta_M}^{n_M} \cdots \hat{x}_{\beta_1}^{n_1}$ .

Let  $\sigma : \mathbb{Z}^\theta \times \mathbb{Z}^\theta \rightarrow k^\times$  be the bilinear form given by

$$\sigma(g_i, g_j) = \begin{cases} \hat{q}_{ij} q_{ij}^{-1}, & i \leq j, \\ 1, & i > j. \end{cases} \tag{4.3}$$

By [AS2, Prop. 3.9, Rem. 3.10] there exists a linear isomorphism  $\Psi : \mathfrak{B}(W) \rightarrow \mathfrak{B}(V)$  such that  $\Psi(\hat{x}_i) = x_i$  and for any  $x \in \mathfrak{B}(W)_\alpha, y \in \mathfrak{B}(W)_\beta, \alpha, \beta \in \mathbb{N}_0^\theta$ ,

$$\Psi(xy) = \sigma(\alpha, \beta) \Psi(x) \Psi(y), \tag{4.4}$$

$$\Psi([x, y]_d) = \sigma(\alpha, \beta) [\Psi(x), \Psi(y)]_d. \tag{4.5}$$

Define  $t_{\alpha_i} = 1$  for all  $1 \leq i \leq \theta$ , and inductively

$$t_\beta = \sigma(\beta_1, \beta_2) t_{\beta_1} t_{\beta_2}, \quad \text{Sh}(l_\beta) = (l_{\beta_1}, l_{\beta_2}).$$

Also for any  $u = x_{\beta_M}^{n_M} \cdots x_{\beta_1}^{n_1}$  define

$$f(u) := \prod_{1 \leq i < j \leq M} \sigma(\beta_j, \beta_i)^{n_i n_j} \prod_{1 \leq i \leq M} \sigma(\beta_i, \beta_i)^{\binom{n_i}{2}} t_{\beta_i}^{n_i}. \tag{4.6}$$

**Lemma 4.4.** For any  $u = x_{\beta_M}^{n_M} \cdots x_{\beta_1}^{n_1}$ ,  $\Psi(\hat{u}) = f(u)u$ .

*Proof.* We first prove by induction on  $\ell(l_\beta)$ ,  $\beta \in \Delta_+^V$ , that  $\Psi(\hat{x}_\beta) = t_\beta x_\beta$ . This follows by definition when  $\ell(l_\beta) = 1$ , i.e. when  $\beta = \alpha_i$  for some  $1 \leq i \leq \theta$ . Now assume it holds for  $\ell(l_\gamma) < k$ , and consider  $\beta \in \Delta_+^V$  such that  $\ell(l_\beta) = k$ . Let  $\text{Sh}(l_\beta) = (\beta_1, \beta_2)$ . Then

$$\begin{aligned} \Psi(\hat{x}_\beta) &= \Psi([\hat{x}_{\beta_1}, \hat{x}_{\beta_2}]_d) = \sigma(\beta_1, \beta_2)[\Psi(\hat{x}_{\beta_1}), \Psi(\hat{x}_{\beta_2})]_c \\ &= \sigma(\beta_1, \beta_2)t_{\beta_1}t_{\beta_2}[x_{\beta_1}, x_{\beta_2}]_c = t_\beta x_\beta, \end{aligned}$$

by (4.5) and the inductive hypothesis.

Now we prove that  $\Psi(\hat{u}) = f(u)u$  by induction on  $\text{ht}(u)$ . Note that if  $\text{ht}(u) = 1$ , this reduces to  $\Psi(\hat{x}_\beta) = t_\beta x_\beta$ . Assume now that it holds for  $\text{ht}(v) < N$ , and consider  $u = x_{\beta_M}^{n_M} \cdots x_{\beta_k}^{n_k}$  such that  $\text{ht}(u) = N$  and  $n_k > 0$ . Set  $v = x_{\beta_M}^{n_M} \cdots x_{\beta_k}^{n_k-1}$ . Then

$$\begin{aligned} \Psi(\hat{u}) &= \sigma\left((n_k - 1)\beta_k + \sum_{i=k+1}^M n_i \beta_i, \beta_k\right) \Psi(\hat{v}) \Psi(\hat{x}_{\beta_k}) \\ &= \left(\prod_{i=k+1}^M \sigma(\beta_i, \beta_k)^{n_i}\right) \sigma(\beta_k, \beta_k)^{n_k-1} f(v)v t_{\beta_k} x_{\beta_k} = f(u)u, \end{aligned}$$

by (4.4) and the inductive hypothesis. □

For  $1 \leq i < j \leq \theta$  and  $u = x_{\beta_M}^{n_M} \cdots x_{\beta_1}^{n_1}$ , we define

$$c_{i,j}^u := \frac{f(u)(\hat{x}_{\beta_i} \hat{x}_{\beta_j} | \hat{u})}{\sigma(\beta_i, \beta_j) t_{\beta_i} t_{\beta_j} c_{\hat{u}}}, \tag{4.7}$$

where  $(\cdot | \cdot)$  denotes the bilinear form associated to  $(W, d)$ , and  $c_{\hat{u}}$  is computed as in Corollary 4.2. Note that if  $(q_{ij})$  is symmetric and we consider  $q_{ij} = \hat{q}_{ij}$ , then  $\sigma(\alpha, \beta) = 1$  for all  $\alpha, \beta \in \mathbb{Z}^\theta$  and so  $f(u) = 1$  for any  $u$ . Consequently,  $c_{i,j}^u = (x_{\beta_i} x_{\beta_j} | u) c_u^{-1}$ .

We obtain a first set of relations for our presentation.

**Lemma 4.5.** Let  $1 \leq i < j \leq M$  be such that  $l_{\beta_i} l_{\beta_j} \neq l_{\beta_k}$  for all  $k$ , and  $\text{Sh}(l_{\beta_i} l_{\beta_j}) = (l_{\beta_i}, l_{\beta_j})$ , and  $c_{i,j}^u \in \mathfrak{k}$  as above. Then

$$[x_{\beta_i}, x_{\beta_j}]_c = \sum_{u \in B_i \cap C_j - \{x_{\beta_j}, x_{\beta_i}\}: \deg u = \beta_i + \beta_j} c_{i,j}^u u. \tag{4.8}$$

*Proof.* Assume that the braiding is symmetric. As  $l_{\beta_i} l_{\beta_j} \neq l_{\beta_k}$  for all  $k$ , and  $\text{Sh}(l_{\beta_i} l_{\beta_j}) = (l_{\beta_i}, l_{\beta_j})$ , it follows that  $[l_{\beta_i} l_{\beta_j}]_c = [x_{\beta_i}, x_{\beta_j}]_c = x_{\beta_i} x_{\beta_j} - \chi(\beta_i, \beta_j) x_{\beta_j} x_{\beta_i}$  is a linear combination of greater monotone hyperwords by Corollary 2.10.

As  $x_{\beta_i}x_{\beta_j} \in W_{\beta_i}$ , it is a linear combination of elements in  $B_i$  by Theorem 3.18. Also,  $\mathfrak{B}(V)$  is  $\mathbb{N}_0^\theta$ -graded, so this linear combination is over elements of  $B_i$  of degree  $\beta_i + \beta_j$ . Moreover, if  $c_{i,j}^u \neq 0$  for  $u = x_{\beta_k}^{n_k} \cdots x_{\beta_l}^{n_l}$ ,  $l \leq k$ , such that  $n_k, n_l \neq 0$ , then  $x_{\beta_i} \otimes x_{\beta_j}$  appears in the expression of  $\Delta(u)$  by Remark 4.3. Note that  $x_{\beta_i} \otimes x_{\beta_j} \notin D_k \otimes (B_l - D_k)$ , because  $i < j$ . By Lemma 3.20, we have  $x_{\beta_j} \in B_k$ , so  $j \geq k$ , and  $u \in C_j$ .

The explicit formula for the coefficients comes from Proposition 4.1.

If we want to compute  $c_{i,j}^{x_{\beta_i}x_{\beta_j}}$ , we have to calculate the coefficient of  $x_{\beta_i} \otimes x_{\beta_j}$  in  $\Delta(x_{\beta_j}x_{\beta_i})$ , because of Remark 4.3 and the formula  $c_{x_{\alpha_j}x_{\alpha_i}} = c_{x_{\alpha_i}}c_{x_{\alpha_j}}$ . This coefficient is  $\chi(\beta_j, \beta_i)$ , but as the braiding matrix is symmetric,  $\chi(\beta_j, \beta_i) = \chi(\beta_i, \beta_j)$ . This concludes the proof when the matrix braiding is symmetric.

When the braiding is not symmetric, we use the linear isomorphism  $\Psi$  considered previously to reduce the computation to the symmetric case. Then

$$\begin{aligned} 0 &= \Psi\left([\hat{x}_{\beta_i}, \hat{x}_{\beta_j}]_d - \sum (\hat{x}_{\beta_i}\hat{x}_{\beta_j}|\hat{u})c_{\hat{u}}^{-1}\hat{u}\right) \\ &= \sigma(\beta_i, \beta_j)t_{\beta_i}t_{\beta_j}[x_{\beta_i}, x_{\beta_j}]_c - \sum (\hat{x}_{\beta_i}\hat{x}_{\beta_j}|\hat{u})c_{\hat{u}}^{-1}f(u)u, \end{aligned}$$

by (4.5) and Lemma 4.4, so (4.8) holds in  $\mathfrak{B}(V)$ . □

**Corollary 4.6.** *Assume that  $i, j$  are as in Lemma 4.5, and  $\beta_i + \beta_j = \sum_{k=i}^j n_k \beta_k$  with  $n_k \in \mathbb{N}_0$  if and only if  $n_i = n_j = 1$  and  $n_k = 0$  for  $k \neq i, j$ . Then*

$$[x_{\beta_i}, x_{\beta_j}]_c = 0. \tag{4.9}$$

*Proof.* This follows from the previous proposition. □

Now we extend [Ang1, Cor. 5.2]. Recall that  $N_\beta = \text{ord}(q_\beta) = h(x_\beta)$ .

**Lemma 4.7.** *If  $\beta \in \Delta_+^V$  and  $N_\beta$  is finite, then*

$$x_\beta^{N_\beta} = 0 \quad \text{in } \mathfrak{B}(V). \tag{4.10}$$

*Proof.* Assume first that  $(q_{ij})$  is symmetric. Consider  $w = \tilde{w}x_\beta^m$ , where  $\beta \in \Delta^+$  and either  $\tilde{w}$  is a non-increasing product of hyperletters  $x_\gamma$ ,  $\gamma \in \Delta^+$ ,  $\gamma > \beta$ , or  $\tilde{w} = 1$ . If  $\beta > \alpha$ , then

$$\begin{aligned} (x_\alpha^{N_\alpha}|w) &= (x_\alpha^{N_\alpha-1}|1)(x_\alpha|w) + \sum_{i=0}^m \binom{m}{i}_{q_\beta} (x_\alpha^{N_\alpha-1}|\tilde{w}x_\beta^i)(x_\alpha|x_\beta^{m-i}) \\ &\quad + \sum_{l_1 \geq \dots \geq l_p > x_\beta, 0 \leq j \leq m} (x_\alpha^{N_\alpha-1}|x_{l_1, \dots, l_p}^{(j)})(x_\alpha|[l_1]_c \cdots [l_p]_c x_\beta^j) = 0, \end{aligned}$$

where we use the fact that  $(x_\alpha^{N_\alpha-1}|1) = (x_\alpha|x_\beta^{m-i}) = (x_\alpha|[l_1]_c \cdots [l_p]_c x_\beta^j) = 0$  by the orthogonality of the PBW basis.

If  $\beta \leq \alpha$ , then

$$(x_\alpha^{N_\alpha} | w) = (1 | \tilde{w} x_\beta^{m-1})(x_\alpha^{N_\alpha} | x_\beta) + \sum_{i=1}^{N_\alpha} \binom{N_\alpha}{i}_{q_\alpha} (x_\alpha^i | \tilde{w} x_\beta^{m-1})(x_\alpha^{N_\alpha-i} | x_\beta) + \sum_{l_1 \geq \dots \geq l_p > x_\alpha, 0 \leq j \leq N_\alpha} (x_{l_1, \dots, l_p}^{(j)} | \tilde{w} x_\beta^{m-1})([l_1]_c \cdots [l_p]_c x_\alpha^j | x_\beta)$$

where we use the fact that  $q_\alpha \in \mathbb{G}_{N_\alpha}$ , the orthogonality of the PBW basis and the fact that  $N\beta \notin \Delta^+$  if  $N > 1$  (so  $(x_\alpha^{N_\alpha} | x_\beta) = 0$ ).

Therefore  $(x_\alpha^{N_\alpha} | v) = 0$  for all  $v$  in the PBW basis. Also  $(I(V) | x_\alpha^{N_\alpha}) = 0$ , because  $I(V)$  is the radical of this bilinear form, so  $(T(V) | x_\alpha^{N_\alpha}) = 0$ , and hence  $x_\alpha^{N_\alpha} \in I(V)$ . That is, we have  $x_\alpha^{N_\alpha} = 0$  in  $\mathfrak{B}(V)$ .

For the general case, we recall that a diagonal braiding is twist equivalent to a braiding with a symmetric matrix (see [AS2, Theorem 4.5]). Also, there exists a linear isomorphism between the corresponding Nichols algebras. The corresponding  $x_\alpha$  are related by a non-zero scalar, because they are iterations of braided commutators between hyperwords. □

Before proving the main result of this section, we need another technical lemma.

**Lemma 4.8.** *Let  $\mathfrak{B}$  be a quotient of  $T(V)$  such that relations (4.8) hold. Then for any  $i < j$ ,  $x_{\beta_i}, x_{\beta_j}$  can be written as a linear combination of monotone hyperwords greater than  $x_{\beta_i}$  whose hyperletters are  $x_{\beta_k}$ ,  $i \leq k \leq j$ .*

*Proof.* This is similar to the proof of Theorem 2.4 (see [R2, Thm. 10]). For each  $n \geq 2$ , set

$$L_n := \{(x_{\beta_i}, x_{\beta_j}) : i < j, \ell(l_{\beta_i}) + \ell(l_{\beta_j}) = n\}.$$

We order  $L_k$  as follows:  $(x_{\beta_i}, x_{\beta_j}) < (x_{\beta_k}, x_{\beta_m})$  if either  $l_{\beta_i} l_{\beta_j} < l_{\beta_k} l_{\beta_m}$ , or  $l_{\beta_i} l_{\beta_j} = l_{\beta_k} l_{\beta_m}$  and  $l_{\beta_i} < l_{\beta_k}$ .

We prove the statement by induction on  $n = \ell(x_{\beta_i}) + \ell(x_{\beta_j})$ , and then by induction on the previous order on  $L_n$ . If  $n = 2$ , then  $\beta_i, \beta_j$  are simple, and  $[x_i, x_j]_c = x_{\alpha_i + \alpha_j}$  or  $[x_i, x_j]_c = 0$  in  $\mathfrak{B}$ .

Fix then a pair  $(x_{\beta_i}, x_{\beta_j}) \in L_n$  and assume that the statement holds for  $(x_{\beta_k}, x_{\beta_m}) \in L_n$ ,  $(x_{\beta_i}, x_{\beta_j}) > (x_{\beta_k}, x_{\beta_m})$ , and for  $(x_{\beta_k}, x_{\beta_m}) \in L_{n'}, n' < n$ . If  $\text{Sh}(l_{\beta_i} l_{\beta_j}) = (l_{\beta_i}, l_{\beta_j})$  then the assertion holds because

- if  $l_{\beta_i} l_{\beta_j} = l_{\beta_k}$  for some  $k$ , then necessarily (by the definition of the order)  $i < k < j$  and  $[x_{\beta_i}, x_{\beta_j}]_c = x_{\beta_k}$ ,
- otherwise it holds because we assume (4.8).

If  $\text{Sh}(l_{\beta_i} l_{\beta_j}) \neq (l_{\beta_i}, l_{\beta_j})$ , let  $\text{Sh}(l_{\beta_i}) = (l_{\beta_p}, l_{\beta_q})$ , so  $x_{\beta_i} = [x_{\beta_p}, x_{\beta_q}]_c$ . Therefore  $l_{\beta_q} < l_{\beta_j}$  (see Subsection 2.2). By (2.4),

$$[x_{\beta_i}, x_{\beta_j}]_c = [x_{\beta_p}, [x_{\beta_q}, x_{\beta_j}]_c]_c - \chi(\beta_p, \beta_q) x_{\beta_q} [x_{\beta_p}, x_{\beta_j}]_c + \chi(\beta_q, \beta_j) [x_{\beta_p}, x_{\beta_j}]_c x_{\beta_q}.$$

We apply the induction hypothesis and express  $[x_{\beta_q}, x_{\beta_j}]_c$  as a linear combination of monotone hyperwords whose hyperletters are between  $x_{\beta_q}$  and  $x_{\beta_j}$ . By (2.5) and the inductive hypothesis, we express  $[x_{\beta_p}, [x_{\beta_q}, x_{\beta_j}]_c]_c$  as a linear combination of monotone hyperwords whose letters are between  $x_{\beta_i}$  and  $x_{\beta_j}$ . The order in  $L_n$  is important here, because in such a linear combination a single hyperletter  $x_{\beta_k}$  can appear, which by hypothesis is between  $x_{\beta_q}$  and  $x_{\beta_j}$ , and so  $(l_{\beta_i}, l_{\beta_j}) > (l_{\beta_p}, l_{\beta_k})$ .

We also use the inductive hypothesis to express  $[x_{\beta_p}, x_{\beta_j}]_c$  as a linear combination of hyperwords whose hyperletters are between  $x_{\beta_p}$  and  $x_{\beta_j}$ . As in the previous step, we can reorder the hyperletters to find the desired expression by the inductive hypothesis.  $\square$

Now we are ready to prove the main result of this work.

**Theorem 4.9.** *Let  $(V, c)$  be a finite-dimensional braided vector space of diagonal type such that  $\Delta_+^V$  is finite. Let  $x_1, \dots, x_\theta$  be a basis of  $V$  such that  $c(x_i \otimes x_j) = q_{ij}x_j \otimes x_i$ , where  $(q_{ij}) \in (\mathbb{k}^\times)^{\theta \times \theta}$  is the braiding matrix, and let  $\{x_{\beta_k}\}_{\beta_k \in \Delta_+^V}$  be the associated set of hyperletters. Then  $\mathfrak{B}(V)$  is presented by the generators  $x_1, \dots, x_\theta$  and the relations*

$$x_\beta^{N_\beta} = 0, \quad \beta \in \Delta_+^V, \text{ord}(q_\beta) = N_\beta < \infty, \tag{4.11}$$

$$[x_{\beta_i}, x_{\beta_j}]_c = \sum_{u \in B_i \cap C_j - \{x_{\beta_j}, x_{\beta_i}\}: \text{deg } u = \beta_i + \beta_j} c_{i,j}^u u, \tag{4.12}$$

$$\text{Sh}(l_{\beta_i} l_{\beta_j}) = (l_{\beta_i}, l_{\beta_j}), \quad 1 \leq i < j \leq M, \quad l_{\beta_i} l_{\beta_j} \neq l_{\beta_k}, \forall k,$$

where the  $c_{i,j}^u$  are as in (4.7). Moreover,  $\{x_{\beta_M}^{n_M} \cdots x_{\beta_1}^{n_1} : 0 \leq n_j < N_{\beta_j}\}$  is a basis of  $\mathfrak{B}(V)$ .

*Proof.* The statement about the basis follows from Kharchenko’s theory of PBW bases (Subsection 2.2) and the definition of  $\Delta_+^V$  (see Subsection 3.1), where the hyperletters  $x_{\beta_k}$  are uniquely determined by Corollary 3.17.

Let  $\mathfrak{B} := T(V)/I$ , where  $I$  is the ideal of  $T(V)$  generated by (4.11), (4.12): by Lemmata 4.5 and 4.7,  $I \subseteq I(V)$ , so the projection  $\pi : T(V) \rightarrow \mathfrak{B}(V)$  induces canonically a projection  $\pi' : \mathfrak{B} \rightarrow \mathfrak{B}(V)$ . Let  $W$  be the subspace of  $\mathfrak{B}$  spanned by  $B$ , where  $B$  is the PBW basis of  $\mathfrak{B}(V)$ ; we have  $1 \in W$ . For each pair  $1 \leq i \leq j \leq M$ , we let  $W_{i,j}$  be the subspace of  $W$  spanned by  $B_i \cap C_j$ .

We assert that

$$x_{\beta_k} W_{i,j} \subset W_{\min\{i,k\}, \max\{j,k\}}. \tag{4.13}$$

We shall prove this by induction on  $k$ . When  $k = M$ , fix  $i \leq j$ . For each  $w \in B_i \cap C_j$ , we have either  $x_{\beta_M} w \in B_i \cap C_M = B_i$ , or  $x_{\beta_M} w = 0$  if  $j = M$ ,  $N_M < \infty$  and  $w$  begins with  $x_{\beta_M}^{N_M-1}$ , so  $x_{\beta_M} W_{i,j} \subset W_{i,M}$ .

Now assume that (4.13) holds for all  $l > k$  and all  $i \leq j$ . We argue by induction on  $j$ . If  $i \leq j \leq k$ , for each  $w \in B_i \cap C_j$  we have either  $x_{\beta_k} w \in B_i \cap C_k$  or  $x_{\beta_k} w = 0$  as in the initial step, so  $x_{\beta_k} W_{i,j} \subset W_{i,k}$ . Now assume  $j > k$ , and consider  $w \in B_i \cap C_j$ ; it is enough to prove that  $x_{\beta_k} w \in W_{\min\{i,k\}, j}$ . Moreover, we can assume  $w = x_{\beta_j} w'$  for some monotone hyperword  $w'$  in  $W_{i,j}$  (if  $w$  begins with another hyperletter  $x_{\beta_l}$ ,  $l < j$ , we con-

sider  $w \in W_{i,l} \subset W_{i,j}$ ). By Lemma 4.8, we can write  $x_{\beta_k} x_{\beta_j}$  as a linear combination of monotone hyperwords whose hyperletters belong to  $B_k \cap C_j$ . Therefore the result follows by the inductive hypothesis: any of these hyperwords either has no letters  $x_{\beta_k}$  and we use the first inductive hypothesis (it holds for all  $l > k$ ), or it ends with hyperletters  $x_{\beta_k}$  and we write  $x_{\beta_k} w'$  as a linear combination of hyperwords in  $B_{\min\{i,k\}} \cap C_j$  by the second inductive hypothesis.

In this way we find that  $W$  is a left ideal which contains 1, so  $W = \mathfrak{B}$ . But then the projection  $\pi'$  is an isomorphism, and  $\mathfrak{B} = \mathfrak{B}(V)$ .  $\square$

**Remark 4.10.** Recall that we have defined, for  $i, j \in \{1, \dots, \theta\}$ ,

$$m_{ij} := \max\{m : (\text{ad}_c x_i)^m x_j \neq 0\}$$

(see (3.6)), and so  $m\alpha_i + \alpha_j \in \Delta_+^V$  iff  $0 \leq m \leq m_{ij}$ . Moreover assume  $i < j$ . Then  $x_{m\alpha_i + \alpha_j} = (\text{ad}_c x_i)^m x_j$ , and a pair as in Corollary 4.6 is  $(x_i, x_i^{m_{ij}} x_j)$ , so the corollary implies the well-known quantum Serre relation in  $\mathfrak{B}(V)$ :  $(\text{ad}_c x_i)^{m_{ij}+1} x_j = 0$ . If  $i > j$ , then the pair changes to  $(x_j x_i^{m_{ij}}, x_i)$ , but then  $0 = [x_{m\alpha_i + \alpha_j}, x_i]_c = a(\text{ad}_c x_i)^{m_{ij}+1} x_j$  for some  $a \in k^\times$ . In any case we have  $(\text{ad}_c x_i)^{m_{ij}+1} x_j = 0$ .

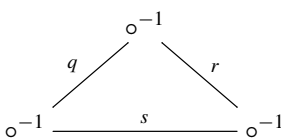
This shows that the set of relations (4.8), (4.10) is not minimal: if  $\text{ord}(q_{ii}) = m_{ij} + 1$ , then  $x_i^{m_{ij}+1}$  is one of the relations (4.10), and then  $(\text{ad}_c x_i)^{m_{ij}+1} x_j$  belongs to the ideal generated by  $x_i^{m_{ij}+1}$ .

### 5. Explicit presentations by generators and relations of some Nichols algebras of diagonal type

We shall apply the previous theory concerning a PBW basis (Corollary 3.17) and a presentation of the corresponding Nichols algebra (Theorem 4.9) in some concrete examples.

#### 5.1. Examples when $\dim V = 3$

We consider the Weyl equivalence classes 9, 10, 11 in [H2, Table 2]. We fix the following notation: let  $q, r, s \in k^\times$  be such that  $qrs = 1$ . Let  $M, N, P \in \mathbb{N}$  be the orders of these scalars, if they are finite. Such a Weyl equivalence class includes the following generalized Dynkin diagrams:

- $\circ^q \xrightarrow{q^{-1}} \circ^{-1} \xrightarrow{r^{-1}} \circ^r$
- $\circ^q \xrightarrow{q^{-1}} \circ^{-1} \xrightarrow{s^{-1}} \circ^s$
- $\circ^r \xrightarrow{r^{-1}} \circ^{-1} \xrightarrow{s^{-1}} \circ^s$
- 



Notice that 10, 11 are particular cases of 9 when  $q = r, q = r = s \in \mathbb{G}_3$ , respectively. Also the second and the third diagrams are analogous to the first one, so it is enough to obtain the presentation for the first and the last braidings.

If  $i < j$ , then  $l_{\alpha_i+\alpha_j} = x_i x_j$ , so  $x_{\alpha_i+\alpha_j} = [x_i, x_j]_c = (\text{ad}_c x_i)x_j$ . Also,

$$l_{\alpha_1+\alpha_2+\alpha_3} = \begin{cases} x_1 x_2 x_3 & \text{if } (\text{ad}_c x_1)x_3 = 0 \text{ and } x_{\alpha_1+\alpha_2+\alpha_3} = [x_1, x_{\alpha_2+\alpha_3}]_c, \\ x_1 x_3 x_2 & \text{if } (\text{ad}_c x_1)x_3 \neq 0 \text{ and } x_{\alpha_1+\alpha_2+\alpha_3} = [x_{\alpha_1+\alpha_3}, x_2]_c. \end{cases}$$

When  $(\text{ad}_c x_1)x_3 = 0$ , we also have

$$l_{\alpha_1+2\alpha_2+\alpha_3} = x_1 x_2 x_3 x_2 \quad \text{and} \quad x_{\alpha_1+2\alpha_2+\alpha_3} = [x_{\alpha_1+\alpha_2+\alpha_3}, x_2]_c.$$

**Proposition 5.1.** *Let  $(V, c)$  be a braided vector space such that  $\dim V = 3$ , and the corresponding generalized Dynkin diagram is*

$$\circ^q \xrightarrow{q^{-1}} \circ^{-1} \xrightarrow{r^{-1}} \circ^r .$$

Then  $\mathfrak{B}(V)$  is presented by generators  $x_1, x_2, x_3$  and the relations

$$x_1^M = x_2^2 = x_3^N = x_{\alpha_1+2\alpha_2+\alpha_3}^P = 0, \tag{5.1}$$

$$(\text{ad}_c x_1)^2 x_2 = (\text{ad}_c x_3)^2 x_2 = (\text{ad}_c x_1)x_3 = 0, \tag{5.2}$$

$$[x_{\alpha_1+\alpha_2}, x_{\alpha_1+\alpha_2+\alpha_3}]_c = [x_{\alpha_1+\alpha_2+\alpha_3}, x_{\alpha_2+\alpha_3}]_c = 0. \tag{5.3}$$

Moreover,  $\mathfrak{B}(V)$  has the following PBW basis:

$$\{x_3^{n_3} x_{\alpha_2+\alpha_3}^{n_{23}} x_2^{n_2} x_{\alpha_1+2\alpha_2+\alpha_3}^{n_{1232}} x_{\alpha_1+\alpha_2+\alpha_3}^{n_{123}} x_{\alpha_1+\alpha_2}^{n_{12}} x_1^{n_1} : 0 \leq n_1 < M, 0 \leq n_2 < N, 0 \leq n_{1232} < P, n_{12}, n_{123}, n_2, n_{23} \in \{0, 1\}\}.$$

If  $M, N, P < \infty$ , then  $\dim \mathfrak{B}(V) = 16MNP$ .

*Proof.* For this case,

$$\Delta_+^V = \{\alpha_3, \alpha_2 + \alpha_3, \alpha_2, \alpha_1 + 2\alpha_2 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3, \alpha_1 + \alpha_2, \alpha_1\}.$$

Therefore we easily obtain  $l_\beta, \beta \in \Delta_+^V$ , from Corollary 3.17.

By Remark 4.10, we consider the relations

$$(\text{ad}_c x_1)^2 x_2 = (\text{ad}_c x_3)^2 x_2 = (\text{ad}_c x_1)x_3 = 0,$$

because  $(\text{ad}_c x_2)^2 x_1, (\text{ad}_c x_2)^2 x_3 = 0$  follow from  $x_2^2 = 0$ .

We have the following decompositions:

$$\text{Sh}(l_{\alpha_1+\alpha_2} l_{\alpha_1+\alpha_2+\alpha_3}) = (l_{\alpha_1+\alpha_2}, l_{\alpha_1+\alpha_2+\alpha_3}),$$

$$\text{Sh}(l_{\alpha_1+\alpha_2+\alpha_3} l_{\alpha_2+\alpha_3}) = (l_{\alpha_1+\alpha_2+\alpha_3}, l_{\alpha_2+\alpha_3}).$$

Relation (5.3) then follows from Corollary 4.6.

Also  $\text{Sh}(l_{\alpha_1}l_{\alpha_1+\alpha_2+\alpha_3}) = (l_{\alpha_1}, l_{\alpha_1+\alpha_2+\alpha_3})$ , so

$$[x_1, x_{\alpha_1+\alpha_2+\alpha_3}]_c = 0.$$

Note that  $x_{\alpha_1+\alpha_2+\alpha_3} = [x_{\alpha_1+\alpha_2}, x_3]_c$  from  $(\text{ad}_c x_1)x_3 = 0$  and the identity (2.4). Therefore, the displayed relation is redundant because of (2.4) and  $x_1^2 = 0$ . The same holds for the relation  $[x_{\alpha_1+\alpha_2+\alpha_3}, x_3]_c = 0$ , coming from the decomposition  $\text{Sh}(l_{\alpha_1+\alpha_2+\alpha_3}l_{\alpha_3}) = (l_{\alpha_1+\alpha_2+\alpha_3}, l_{\alpha_3})$ .

Also  $\text{Sh}(l_{\alpha_1}l_{\alpha_1+2\alpha_2+\alpha_3}) = (l_{\alpha_1}, l_{\alpha_1+2\alpha_2+\alpha_3})$ , so by Lemma 4.5 there exists  $a \in k$  such that

$$[x_1, x_{\alpha_1+2\alpha_2+\alpha_3}]_c = ax_{\alpha_1+\alpha_2+\alpha_3}x_{\alpha_1+\alpha_2}.$$

This relation is also redundant:

$$\begin{aligned} [x_1, x_{\alpha_1+2\alpha_2+\alpha_3}]_c &= [[x_1, x_{\alpha_1+\alpha_2+\alpha_3}]_c, x_2]_c + q_{11}q_{12}q_{13}x_{\alpha_1+\alpha_2+\alpha_3}x_{\alpha_1+\alpha_2} \\ &\quad - q_{12}q_{22}q_{32}x_{\alpha_1+\alpha_2}x_{\alpha_1+\alpha_2+\alpha_3} \\ &= q_{11}q_{12}q_{13}(1-s)x_{\alpha_1+\alpha_2+\alpha_3}x_{\alpha_1+\alpha_2}, \end{aligned}$$

where we use (2.4) and the previous relations.

We finally have

$$\begin{aligned} \text{Sh}(l_{\alpha_1+\alpha_2}l_{\alpha_1+2\alpha_2+\alpha_3}) &= (l_{\alpha_1+\alpha_2}, l_{\alpha_1+2\alpha_2+\alpha_3}), \\ \text{Sh}(l_{\alpha_1+\alpha_2+\alpha_3}l_{\alpha_1+2\alpha_2+\alpha_3}) &= (l_{\alpha_1+\alpha_2+\alpha_3}, l_{\alpha_1+2\alpha_2+\alpha_3}), \end{aligned}$$

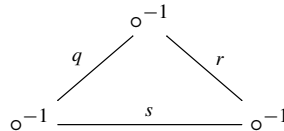
which yields the following relations:

$$[x_{\alpha_1+\alpha_2}, x_{\alpha_1+2\alpha_2+\alpha_3}]_c = [x_{\alpha_1+\alpha_2+\alpha_3}, x_{\alpha_1+2\alpha_2+\alpha_3}]_c = 0.$$

These relations also follow from the previous ones using (2.4).

We can prove in the same way that  $x_{\alpha_1+\alpha_2}^2, x_{\alpha_2+\alpha_3}^2, x_{\alpha_1+\alpha_2+\alpha_3}^2 = 0$  are redundant. The proposition then follows by Theorem 4.9, where we omit some redundant relations.  $\square$

**Proposition 5.2.** *Let  $(V, c)$  be a braided vector space such that  $\dim V = 3$ , and the corresponding generalized Dynkin diagram is*



Then  $\mathfrak{B}(V)$  is presented by generators  $x_1, x_2, x_3$  and the relations

$$x_1^2 = x_2^2 = x_3^2 = x_{\alpha_1+\alpha_2+\alpha_3}^2 = 0, \tag{5.4}$$

$$x_{\alpha_1+\alpha_2}^M = x_{\alpha_2+\alpha_3}^N = x_{\alpha_1+\alpha_3}^P = 0, \tag{5.5}$$

$$[x_{\alpha_i+\alpha_j}, x_{\alpha_i+\alpha_k}]_c = 0, \quad \{i, j, k\} = \{1, 2, 3\}, \tag{5.6}$$

$$[x_1, x_{\alpha_2+\alpha_3}]_c = \frac{1-s}{q_{23}(1-r)}x_{\alpha_1+\alpha_2+\alpha_3} + q_{12}(1-s)x_2x_{\alpha_1+\alpha_3}. \tag{5.7}$$

Moreover,  $\mathfrak{B}(V)$  has a PBW basis

$$\{x_3^{n_3} x_{\alpha_2+\alpha_3}^{n_{23}} x_2^{n_2} x_{\alpha_1+\alpha_3}^{n_{13}} x_{\alpha_1+\alpha_2+\alpha_3}^{n_{123}} x_{\alpha_1+\alpha_2}^{n_{12}} x_1^{n_1} : 0 \leq n_{12} < M, 0 \leq n_{23} < N, 0 \leq n_{13} < P, n_1, n_{123}, n_2, n_3 \in \{0, 1\}\}.$$

If  $M, N, P < \infty$ , then  $\dim \mathfrak{B}(V) = 16MNP$ .

*Proof.* Again we obtain  $l_\beta, \beta \in \Delta_+^V$ , easily from Corollary 3.17, because

$$\Delta_+^V = \{\alpha_3, \alpha_2 + \alpha_3, \alpha_2, \alpha_1 + \alpha_3, \alpha_1 + \alpha_2 + \alpha_3, \alpha_1 + \alpha_2, \alpha_1\}.$$

By Remark 4.10, all the quantum Serre relations  $(\text{ad}_c x_i)^2 x_j = 0, i \neq j$ , follow from  $x_i^2 = 0, i = 1, 2, 3$ .

We have the decompositions

$$\begin{aligned} \text{Sh}(l_{\alpha_1+\alpha_2} l_{\alpha_1+\alpha_3}) &= (l_{\alpha_1+\alpha_2}, l_{\alpha_1+\alpha_3}), \\ \text{Sh}(l_{\alpha_1+\alpha_2} l_{\alpha_2+\alpha_3}) &= (l_{\alpha_1+\alpha_2}, l_{\alpha_2+\alpha_3}), \\ \text{Sh}(l_{\alpha_1+\alpha_3} l_{\alpha_2+\alpha_3}) &= (l_{\alpha_1+\alpha_3}, l_{\alpha_2+\alpha_3}), \end{aligned}$$

which yield relations (5.6) by Corollary 4.6.

The decomposition  $\text{Sh}(l_{\alpha_1} l_{\alpha_2+\alpha_3}) = (l_{\alpha_1}, l_{\alpha_2+\alpha_3})$  tells us that  $[x_1, x_{\alpha_2+\alpha_3}]_c$  is a linear combination of  $x_{\alpha_1+\alpha_2+\alpha_3}$  and  $x_2 x_{\alpha_1+\alpha_3}$  by Lemma 4.5, and we calculate the corresponding coefficients using Lemma 4.4.

Also  $\text{Sh}(l_{\alpha_1} l_{\alpha_1+\alpha_2+\alpha_3}) = (l_{\alpha_1}, l_{\alpha_1+\alpha_2+\alpha_3})$ , so

$$[x_1, x_{\alpha_1+\alpha_2+\alpha_3}]_c = 0.$$

This relation is again redundant because of (2.4),  $x_1^2 = 0$  and the first relation in (5.6). The same holds for the relation  $[x_{\alpha_1+\alpha_2+\alpha_3}, x_2]_c = 0$ , coming from the decomposition  $\text{Sh}(l_{\alpha_1+\alpha_2+\alpha_3} l_{\alpha_2}) = (l_{\alpha_1+\alpha_2+\alpha_3}, l_{\alpha_2})$ .

Also  $\text{Sh}(l_{\alpha_1+\alpha_2} l_{\alpha_1+\alpha_2+\alpha_3}) = (l_{\alpha_1+\alpha_2}, l_{\alpha_1+\alpha_2+\alpha_3})$ , so

$$[x_{\alpha_1+\alpha_2}, x_{\alpha_1+\alpha_2+\alpha_3}]_c = 0.$$

This relation is also redundant by the previous relations and (2.4). In the same way,  $[x_{\alpha_1+\alpha_2+\alpha_3}, x_{\alpha_1+\alpha_3}]_c = [x_{\alpha_1+\alpha_2+\alpha_3}, x_{\alpha_2+\alpha_3}]_c = 0$  are redundant. The proposition follows by Theorem 4.9.  $\square$

**Remark 5.3.** We can prove that if  $(V, c)$  is a braided vector space as in Proposition 5.1 or Proposition 5.2, and  $R = \bigoplus_{n \geq 0} R_n$  is a finite-dimensional graded braided Hopf algebra such that  $R_0 = k1$  and  $R_1 \cong V$  as braided vector spaces, then  $R$  is generated by  $R_1$  as an algebra. The proof is exactly as in [AnGa, Thm. 2.7], using the corresponding presentation by generators and relations.

**Remark 5.4.** When the braiding is of standard type, we obtain the presentation by generators and relations given in [Ang1, Section 5]. In fact, Corollary 3.17 gives the set of Lyndon words obtained in [Ang1, Section 4B]. Then we obtain a set of relations as in Theorem 4.9, where the set of relations (4.12) includes the ones of [Ang1, Theorems 5.14, 5.19, 5.22, 5.25] which are not root vector powers. Then we can reduce this set of relations because of (2.4) as in this paper, in order to obtain a minimal set of relations.

*Acknowledgments.* I want to thank my advisor Nicolás Andruskiewitsch for his excellent guidance, his suggestions and his reading of this work.

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