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Singular localization of $\mathfrak{g}$-modules and applications to representation theory

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Abstract. We prove a singular version of Beilinson–Bernstein localization for a complex semisimple Lie algebra following ideas from the positive characteristic case settled by [BMR06]. We apply this theory to translation functors, singular blocks in the Bernstein–Gelfand–Gelfand category O and Whittaker modules.

Keywords. Lie algebra, Beilinson–Bernstein localization, category O

1. Introduction

1.1.

Let $\mathfrak{g}$ be a semisimple complex Lie algebra with enveloping algebra U and center $\mathrm{Z} \subset \mathrm{U}$. Let $\mathfrak{h} \subset \mathfrak{g}$ be a Cartan subalgebra and $\mathcal{B}$ be the flag manifold of $\mathfrak{g}$. Let $\lambda \in \mathfrak{h}^*$ be regular and dominant and $I_\lambda \subset \mathrm{Z}$ be the corresponding maximal ideal determined by the Harish Chandra homomorphism. Set $\mathrm{U}^\lambda := \mathrm{U}/(I_\lambda)$. Let $\mathcal{D}^\lambda_{\mathcal{B}}$ be the sheaf of λ-twisted differential operators on $\mathcal{B}$. The celebrated localization theorem of Beilinson and Bernstein [BB81] states that the global section functor gives an equivalence $\mathrm{Mod}(\mathcal{D}^\lambda_{\mathcal{B}}) \cong \mathrm{Mod}(\mathrm{U}^\lambda)$. For applications and more information, see [HTT08].

A localization theory for singular λ was much later found in positive characteristic by Bezrukavnikov, Mirković and Rumynin [BMR06]. Let us sketch their basic construction (which makes sense in all characteristics):

Let G be a semisimple algebraic group such that $\mathit{Lie}\, G = \mathfrak{g}$. Instead of $\mathcal{B}$ consider a parabolic flag manifold $\mathcal{P} = G/P$, where $P \subseteq G$ is a parabolic subgroup whose parabolic roots coincide with the singular roots of λ. Replace the sheaf $\mathcal{D}^\lambda_{\mathcal{B}}$ by a sheaf $\mathcal{D}^\lambda_{\mathcal{P}} := \pi_*(\mathcal{D}_{G/R})^L$ modulo a certain ideal defined by λ. Here L is the Levi factor, R is the unipotent radical of P and $\pi : G/R \to \mathcal{P}$ is the projection. The L-invariants are taken with respect to the right L-action on G/R. The sheaf $\pi_*(\mathcal{D}_{G/R})^L$ is locally isomorphic to $\mathcal{D}_{\mathcal{P}} \otimes \mathrm{U}(\mathfrak{l})$, where $\mathfrak{l} = \mathit{Lie}\, L$. When $P = B$ we have $\mathcal{D}^\lambda_{\mathcal{P}} = \mathcal{D}^\lambda_{\mathcal{B}}$ and when $P = G$ we

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arrive at a tautological solution: $\mathcal{D}^{\lambda}_{\mathcal{P}} = \mathrm{U}^{\lambda} \otimes$ "sheaf of differential operators on a point" $= \mathrm{U}^{\lambda}$.

We use this construction to prove a singular localization theorem in characteristic zero, Theorem 5.1. This is probably well known to the experts but it is not in the literature. Our proof is similar to the original proof of [BB81], though parabolicity leads to some new complications. For instance, [BB81] introduced the method of tensoring a $\mathcal{D}_{\mathcal{B}}$-module with a trivial bundle and then to filter this bundle with G-equivariant line bundles as subquotients. In the parabolic setting the subquotients will necessarily be vector bundles—which are harder to control—since irreducible representations of P are generally not one-dimensional.

In Theorem 4.10 we show that the global sections $\Gamma(\mathcal{D}^{\lambda}_{\mathcal{P}})$ equal U^{λ} by passing to the associated graded level, i.e. to the level of a parabolic Springer resolution. That this works follows from the usual Springer resolution (Lemma 3.2).

Our localization theorem gives an equivalence at the level of abelian categories just like [BB81] does. This is different from positive characteristic where the localization theorem only holds at the level of derived categories.

1.2.

Our principal motivation comes from quantum groups. We do not wish to get into details here, but let us at least mention that we will need a singular localization theory for quantum groups in order to establish quantum analogs of fundamental constructions from [BMR08, BMR06, BMS13] that relate modular representation theory to (commutative) algebraic geometry. By our previous work [BK08], we know that the derived representation categories of quantum groups at roots of unity are equivalent to derived categories of coherent sheaves on Springer fibers in $T^*\mathcal{B}$.

To extend this to the level of abelian categories we must transport the tautological t-structure on the representation-theoretical derived category to a t-structure on the coherent sheaf side. It so happens that to describe this so called *exotic* t-structure (see also [Bez06]), a family of singular localizations is needed (even for a regular block).

We showed in [BK06] that a localization theory for quantum groups can be neatly formulated in terms of equivariant sheaves. The "space" G/B does not admit a quantization. However, one can quantize the function algebras $\mathcal{O}(G)$ and $\mathcal{O}(B)$ and thus the category of B-equivariant (= $\mathcal{O}(B)$-coequivariant) $\mathcal{O}(G)$-modules. This is just the category of quasicoherent sheaves on G/B. Therefore, to prepare for the quantum case we have taken thorough care to write down our results in an equivariant categorical language and at the same time to explain what is going on geometrically while this is still possible.

1.3.

The theory of singular localization of $\mathfrak{g}$-modules clarifies many aspects of representation theory and will have many applications in its own right. Here we discuss a few of them.

It is a basic principle in representation theory that understanding of representations at singular central characters enhances the understanding also at regular central characters. This is illustrated by our $\mathcal{D}$-module interpretation of translation functors (Section 6). Using regular localization only, such a theory was developed by Beilinson and Ginzburg [BG99]. Singular localization simplifies their picture for the plain reason that wall-crossing functors between regular blocks factor through a singular block. We shall also need these results in our work on quantum groups.

The localization theorem implies that a (perhaps singular) block O_λ in category O corresponds to certain bi-equivariant $\mathcal{D}$-modules on G (Section 7). From this we directly retrieve Bernstein and Gelfand's [BeGe81] classical result that O_λ is equivalent to a category of Harish-Chandra bimodules (Corollary 7.4).

Singular localization also leads to the useful observation that one should study Harish-Chandra $\mathfrak{g}$-$\mathfrak{l}$-bimodules, where $\mathfrak{l}$ is the Levi factor of $\mathfrak{p} = \mathit{Lie}\ P$, rather than $\mathfrak{g}$-$\mathfrak{g}$-bimodules (as well as the only proof we know that such bimodules are equivalent to O_λ). For instance, Theorem 8.1 gives this way a very short proof for Miličić and Soergel's [MS97] equivalence between O_λ and a block in the category of Whittaker modules, and Corollary 8.6 gives one for its parabolic generalization due to Webster [W09]. These Whittaker categories have encountered recent interest because they are equivalent to modules over finite W-algebras (e.g. [W09]). It is probably well worth the effort to further investigate the relationship between singular localization and finite W-algebras, in particular in the affine case.

We also retrieve and generalize some other known equivalences between representation categories (see e.g. [Soe86]).

1.4.

An interesting task will be to develop a theory for "holonomic" $\mathcal{D}^\lambda_\mathcal{P}$-modules. Those which are "smooth along the Bruhat stratification of $\mathcal{P}$" and have "regular singularities" will correspond to O_λ. One should then establish a "Riemann–Hilbert correspondence" between holonomic $\mathcal{D}^\lambda_\mathcal{P}$-modules with regular singularities and a suitable category of constructible sheaves on $\mathcal{P}$. Ideally the latter category would be accessible to the machinery of Hodge theory. This would further strengthen the interplay between representation theory and algebraic topology. Because of the simple local description of $\mathcal{D}^\lambda_\mathcal{P}$ we believe that all this can be done and is a good starting point for generalizing $\mathcal{D}$-module theory. We shall return to this topic later on.

Another topic we would like to approach via singular localization is the singular-parabolic Koszul duality for O of [BGS96].

2. Preliminaries

Here we fix notation and collect mostly well known results that we shall need.

2.1. Notation

We work over $\mathbb{C}$. Unless stated otherwise, $\otimes = \otimes_\mathbb{C}$. Let X be an algebraic variety, $\mathcal{O}_X$ the sheaf of regular functions on X and $\mathcal{O}(X)$ its global sections. $\mathrm{Mod}(\mathcal{O}_X)$ denotes the category of quasi-coherent sheaves on X, and $\Gamma := \Gamma_X : \mathrm{Mod}(\mathcal{O}_X) \to \mathrm{Mod}(\mathcal{O}(X))$ is the global section functor. If Y is another variety, π_X^Y will denote the obvious projection $X \to Y$ if there is such.

For $\mathcal{A}$ a sheaf of algebras on X such that $\mathcal{O}_X \subseteq \mathcal{A}$ (e.g., an algebra if $X = pt$) we abbreviate "an $\mathcal{A}$-module" for a sheaf of $\mathcal{A}$-modules that is quasi-coherent over $\mathcal{O}_X$. We denote by $\mathrm{Mod}(\mathcal{A})$ the category of $\mathcal{A}$-modules. More generally, we will encounter categories such as $\mathrm{Mod}(\mathcal{A}$, *additional data*) that consists of $\mathcal{A}$-modules with some *additional data*. We will then denote by $\mathrm{mod}(\mathcal{A}$, *additional data*) its full subcategory of noetherian objects.

Throughout this paper, G will denote a semisimple complex linear algebraic group. We have assumed semisimplicity to simplify notation; all our results can be straightforwardly extended to the case of G reductive. We mention this fact in those proofs that reduce to (reductive) Levi subgroups of G.

2.2. *Root data*

Let $B \subset G$ be a Borel subgroup of our semisimple group G and let $T \subset B$ be a maximal torus. Let $\mathfrak{h} \subset \mathfrak{b} \subset \mathfrak{g}$ be their respective Lie algebras. For any parabolic subgroup P of G containing B, denote by $R = R_P$ its unipotent radical, by $L := L_P$ its Levi subgroup and by $\mathfrak{p} = \mathit{Lie}\, P$, $\mathfrak{r} = \mathfrak{r}_P = \mathit{Lie}\, R$ and $\mathfrak{l} = \mathfrak{l}_P = \mathit{Lie}\, L$ their Lie algebras. We denote by $\mathcal{B} := G/B$ the flag manifold and by $\mathcal{P} := G/P$ the parabolic flag manifold corresponding to P.

Let Λ be the lattice of integral weights and let Λ_r be the root lattice. Let Λ_+ and Λ_{r+} be the positive weights and the positive linear combinations of the simple roots, respectively.

Let $\mathcal{W}$ be the Weyl group of $\mathfrak{g}$. Let Δ be the simple roots and let $\Delta_P := \{\alpha \in \Delta : \mathfrak{g}^{-\alpha} \subset \mathfrak{p}\}$ be the subset of P-parabolic roots. Let $\mathcal{W}_P$ be the subgroup of $\mathcal{W}$ generated by simple reflections s_α for $\alpha \in \Delta_P$. Note that $\mathfrak{h}$ is a Cartan subalgebra of the reductive Lie algebra $\mathfrak{l}_P$. Denote by $S(\mathfrak{h})^{\mathcal{W}_P}$ the $\mathcal{W}_P$-invariants in $S(\mathfrak{h})$ with respect to the $\bullet$-action (here $w \bullet \lambda := w(\lambda + \rho) - \rho$ for $\lambda \in \mathfrak{h}^*$ and $w \in \mathcal{W}$; ρ is the half sum of the positive roots).

Let $\mathrm{Z}(\mathfrak{l})$ be the center of $\mathrm{U}(\mathfrak{l})$ and set $\mathrm{Z} := \mathrm{Z}(\mathfrak{g})$. We have the Harish-Chandra isomorphism $S(\mathfrak{h})^{\mathcal{W}_P} \cong \mathrm{Z}(\mathfrak{l})$ (thus $S(\mathfrak{h})^{\mathcal{W}} \cong \mathrm{Z}$).

Set $\Delta_\lambda := \{\alpha \in \Delta : \lambda(H_\alpha) = -1\}$ for $\lambda \in \mathfrak{h}^*$, where $H_\alpha \in \mathfrak{h}$ is the coroot corresponding to α. Let $\chi_{\mathfrak{l},\lambda} : \mathrm{Z}(\mathfrak{l}) \to \mathbb{C}$ be the character such that $I_{\mathfrak{l},\lambda} := \mathrm{Ker}\,\chi_{\mathfrak{l},\lambda}$ annihilates the Verma module M_λ (for $\mathrm{U}(\mathfrak{l})$) with highest weight λ. Thus, $\chi_{\mathfrak{l},\lambda} = \chi_{\mathfrak{l},\mu} \Leftrightarrow \mu \in \mathcal{W}_P \bullet \lambda$. We have $\lambda = \chi_{\mathfrak{h},\lambda}$ and we write $\chi_\lambda := \chi_{\mathfrak{g},\lambda}$ and $I_\lambda := \mathrm{Ker}\,\chi_\lambda$.

Let $\mathrm{U} := \mathrm{U}(\mathfrak{g})$ be the enveloping algebra of $\mathfrak{g}$ and $\widetilde{\mathrm{U}} := \mathrm{U} \otimes_{\mathrm{Z}} S(\mathfrak{h})$ the extended enveloping algebra; thus $\widetilde{\mathrm{U}}$ has a natural $\mathcal{W}$-action such that the invariant ring $\widetilde{\mathrm{U}}^{\mathcal{W}}$ is canonically isomorphic to U. Let $\mathrm{U}^\lambda := \mathrm{U}/(I_\lambda)$. We say that:

- $\lambda \in \mathfrak{h}^*$ is *P-dominant* if $\lambda(H_\alpha) \notin \{-2, -3, -4, \ldots\}$ for $\alpha \in \Delta_P$; and λ is *dominant* if it is G-dominant.
- λ is *P-regular* if $\Delta_\lambda \subseteq \Delta_P$, and *regular* if it is B-regular, that is, if $w \bullet \lambda = \lambda \Rightarrow w = e$, for $w \in \mathcal{W}$.

- λ is a *P-character* if it extends to a character of P; thus λ is a P-character iff λ is integral and $\lambda|_{\Delta_P} = 0$.

Suppose now that $\lambda \in \mathfrak{h}^*$ is integral and P-dominant. Then there is an irreducible finite-dimensional P-representation $V_P(\lambda)$ with highest weight λ. Note that $V_L(\lambda) := V_P(\lambda)$ is an irreducible representation for L. Of course, $\dim V_P(\lambda) = 1 \Leftrightarrow \lambda$ is a P-character.

The following is well-known:

Lemma 2.1. *Let $\lambda \in \mathfrak{h}^*$. Then λ is dominant iff $\chi_{\lambda+\mu} \neq \chi_\lambda$ for all $\mu \in \Lambda_{r+} \setminus \{0\}$.*

We also have

Lemma 2.2. *Let $\lambda \in \mathfrak{h}^*$ be P-regular and dominant. Let μ be a P-character and let V be the finite-dimensional irreducible representation of $\mathfrak{g}$ with extremal weight μ. Then for any weight ψ of V, $\psi \neq \mu$, we have $\chi_{\lambda+\mu} \neq \chi_{\lambda+\psi}$.*

Proof. This is well known for $P = B$. We reduce to that case as follows: Let $\mathfrak{g}'$ be the semisimple Lie subalgebra of $\mathfrak{g}$ generated by $X_{\pm\alpha}$, $\alpha \in \Delta \setminus \Delta_P$. Let $\mathfrak{h}' := \mathfrak{g}' \cap \mathfrak{h}$ be the Cartan subalgebra of $\mathfrak{g}'$. The inclusion $\mathfrak{h}' \hookrightarrow \mathfrak{h}$ gives the projection $p : \mathfrak{h}^* \to \mathfrak{h}'^*$. Consider the restriction $V|_{\mathfrak{g}'}$ and let V' denote the irreducible $\mathfrak{g}'$-module with highest weight $p(\mu)$; V' is a direct summand in $V|_{\mathfrak{g}'}$. Let $\Lambda(V)$ denote the set of weights of V. Then $p(\Lambda(V)) = \Lambda'(V|_{\mathfrak{g}'})$, the weights of $V|_{\mathfrak{g}'}$. By the assumption that μ is a P-character, it follows that $p(\Lambda(V))$ is contained in the convex hull $\overline{\Lambda'(V')}$ of $\Lambda'(V')$. Since $p(\lambda)$ is regular and dominant, it is well known that $p(\lambda+\mu) \notin \mathcal{W}'(p(\lambda) + \Lambda(V'))$. But then $p(\lambda+\mu) \notin \mathcal{W}'(p(\lambda) + \overline{\Lambda(V')})$. Now $\mathcal{W}' = p(\mathcal{W})$, so $\lambda + \mu \notin \mathcal{W}(\lambda + \Lambda(V))$. □

2.3. Equivariant $\mathcal{O}$-modules and induction

See [Jan83] for details on this material.

Let K be a linear algebraic group and J a closed algebraic subgroup. For X an algebraic variety equipped with a right (or left) action of K we denote by $\mathrm{Mod}(\mathcal{O}_X, K)$ the category of K-equivariant sheaves of (quasi-coherent) $\mathcal{O}_X$-modules. For M in $\mathrm{Mod}(\mathcal{O}_X, K)$ there is the sheaf $(\pi^{X/K}_{X*} M)^K$ on X/K of K-invariant local sections in the direct image $\pi^{X/K}_{X*} M$. If the K-action is free and the quotient is nice we have the equivalence

$$[\pi^{X/K}_{X*}(\ )]^K : \mathrm{Mod}(\mathcal{O}_X, K) \leftrightarrows \mathrm{Mod}(\mathcal{O}_{X/K}) : \pi^{X/K*}_X.$$

We denote by $\Gamma_{(K,J)}$ the global section functor on $\mathrm{Mod}(\mathcal{O}_K, J)$ that corresponds to $\Gamma_{K/J}$ under the equivalence $\mathrm{Mod}(\mathcal{O}_K, J) \cong \mathrm{Mod}(\mathcal{O}_{K/J})$. Then $\Gamma_{(K,J)}(M) = M^J$ for $M \in \mathrm{Mod}(\mathcal{O}_K, J)$.

Let $Rep(K)$ denote the category of algebraic representations of K. We have $\mathcal{O}(K) \in Rep(K)$ via $(gf)(x) := f(g^{-1}x)$ for $g, x \in K$ and $f \in \mathcal{O}(K)$. We shall also consider the left J-action on $\mathcal{O}(K)$ given by $(kf)(x) := f(xk)$ for $k \in J$, $x \in K$ and $f \in \mathcal{O}(K)$. These actions commute.

For $V \in Rep(J)$ we consider the diagonal left J-action on $\widetilde{V} := \mathcal{O}(K) \otimes V$. The left K-action on $\mathcal{O}(K)$ defines a left K-action on $\widetilde{V}$ that commutes with the J-action, and the multiplication map $\mathcal{O}(K) \otimes \widetilde{V} \to \widetilde{V}$ is K- and J-linear. Thus $\widetilde{V}$ belongs to the category $\mathrm{Mod}(K, \mathcal{O}(K), J)$ of K-J-bi-equivariant $\mathcal{O}(K)$-modules. This gives the functor

$$p^* : Rep(J) \to \mathrm{Mod}(K, \mathcal{O}(K), J), \qquad V \mapsto \widetilde{V}$$

(induced bundle of a representation, p symbolizes projection from K to pt/J).

Let $Ind_J^K\, V := \widetilde{V}^J \in Rep(K)$.

We have the factorization $Ind_J^K = (\,)^J \circ p^*$. One can show that $R(\,)^J \circ p^* \cong RInd_J^K$ where $R(\,)^J$ and $RInd_J^K$ are computed in suitable derived categories. An important formula is the *tensor identity*

$$RInd_J^K(V \otimes W) \cong RInd_J^K(V) \otimes W \quad \text{for } V \in Rep(J),\, W \in Rep(K). \tag{2.1}$$

(In particular $RInd_J^K(W) \cong W \otimes RInd_J^K(\mathbb{C})$ for $W \in Rep(K)$ and $\mathbb{C}$ the trivial representations.)

3. Parabolic Springer resolutions

In order to treat sheaves of extended differential operators on parabolic flag varieties in the next section, we will here gather information about their associated graded objects. This is encoded in the geometry of the parabolic Grothendieck–Springer resolution.

3.1. Parabolic flag varieties

The parabolic flag variety $\mathcal{P}$ has a natural left G-action. There is a bijection between representations of P and G-equivariant vector bundles on $\mathcal{P}$; a representation V of P corresponds to the induced bundle $G \times_P V$ on $\mathcal{P}$. We denote by $\mathcal{O}(V) := \mathcal{O}_{\mathcal{P}}(V)$ the corresponding locally free sheaf on $\mathcal{P}$, which hence has a left G-equivariant structure.

Let $\lambda \in \mathfrak{h}^*$ be a P-character and write $\mathcal{O}(\lambda) := \mathcal{O}(V_P(\lambda))$ for the line-bundle corresponding to the one-dimensional P-representation $V_P(\lambda)$. We have $Pic(\mathcal{P}) = Pic_G(\mathcal{P}) \cong$ group of P-characters (but note that not all vector bundles on $\mathcal{P}$ are G-equivariant). The ample line bundles $\mathcal{O}(-\mu)$ are given by P-characters μ such that $\mu(H_\alpha) > 0$ for all $\alpha \in \Delta \setminus \Delta_P$.

Next we define the parabolic Grothendieck resolution:

Definition 3.1. $\widetilde{\mathfrak{g}}_{\mathcal{P}} := \{(P', x) : P' \in \mathcal{P},\ x \in \mathfrak{g}^*,\ x|_{\mathfrak{r}_{P'}} = 0\}$.

Note that $\widetilde{\mathfrak{g}}_{\mathcal{P}} = G \times_P (\mathfrak{g}/\mathfrak{r}_P)^*$. Recall that $L = L_P$ is the Levi factor of P, $U = U_P$ its unipotent radical and $\mathfrak{l} = \mathfrak{l}_P$, $\mathfrak{r} = \mathfrak{r}_P$ their Lie algebras. We have a commutative square

$$\begin{array}{ccc} \widetilde{\mathfrak{g}}_{\mathcal{P}} & \longrightarrow & \mathfrak{l}^*/L = \mathfrak{h}^*/\mathcal{W}_P \\ \downarrow & & \downarrow \\ \mathfrak{g}^* & \longrightarrow & \mathfrak{h}^*/\mathcal{W} \end{array} \tag{3.1}$$

where the top map sends (P', x) to $x|\mathfrak{l}_{P'}/L_{P'} \in \mathfrak{l}^*_{P'}/L_{P'} \cong \mathfrak{l}^*/L$. Note that the last isomorphism is canonical. (We can call $\mathfrak{l}^*/L$ the universal coadjoint quotient of the Levi Lie subalgebra.) This induces a map

$$\pi_{\mathcal{P}} : \widetilde{\mathfrak{g}}_{\mathcal{P}} \to \mathfrak{g}^* \times_{\mathfrak{h}^*/\mathcal{W}} \mathfrak{h}^*/\mathcal{W}_P. \tag{3.2}$$

Lemma 3.2. $R\pi_{\mathcal{P}*}\mathcal{O}_{\widetilde{\mathfrak{g}}_{\mathcal{P}}} = \mathcal{O}_{\mathfrak{g}^*\times_{\mathfrak{h}^*/\mathcal{W}}\mathfrak{h}^*/\mathcal{W}_P}$.

Proof. We shall reduce to the well known case of the ordinary Grothendieck resolution for $\mathcal{P} = \mathcal{B}$. It states that

$$R\pi_{\mathcal{B}*}\mathcal{O}_{\widetilde{\mathfrak{g}}_{\mathcal{B}}} = \mathcal{O}_{\mathfrak{g}^*\times_{\mathfrak{h}^*/\mathcal{W}}\mathfrak{h}^*}. \tag{3.3}$$

Translating this to the equivariant language yields

$$RInd_B^G(S(\mathfrak{g}/\mathfrak{n})) = S(\mathfrak{g}) \otimes_{S(\mathfrak{h})^{\mathcal{W}}} S(\mathfrak{h}) \tag{3.4}$$

where $\mathfrak{n} := [\mathfrak{b}, \mathfrak{b}]$. To see this, observe first that, since $\mathfrak{g}^* \times_{\mathfrak{h}^*/\mathcal{W}} \mathfrak{h}^*$ is affine, the equality (3.3) is, after taking global sections, equivalent to the equality

$$R\Gamma(\mathcal{O}_{\widetilde{\mathfrak{g}}_{\mathcal{B}}}) = \mathcal{O}(\mathfrak{g}^* \times_{\mathfrak{h}^*/\mathcal{W}} \mathfrak{h}^*) = S(\mathfrak{g}) \otimes_{S(\mathfrak{h})^{\mathcal{W}}} S(\mathfrak{h})$$

of G-modules. Moreover, since the bundle projection $p : \widetilde{\mathfrak{g}}_{\mathcal{B}} \to \mathcal{B}$ with fiber $(\mathfrak{g}/\mathfrak{n})^*$ is affine, p_* is exact and hence $R\Gamma(\mathcal{O}_{\widetilde{\mathfrak{g}}_{\mathcal{B}}}) = R\Gamma(p_*(\mathcal{O}_{\widetilde{\mathfrak{g}}_{\mathcal{B}}}))$. Under the identification $\mathrm{Mod}(\mathcal{O}_{\mathcal{B}}) = \mathrm{Mod}(\mathcal{O}_G, B)$, $p_*(\mathcal{O}_{\widetilde{\mathfrak{g}}_{\mathcal{B}}})$ corresponds to $S(\mathfrak{g}/\mathfrak{n})\otimes\mathcal{O}(G)$, so its derived global sections are given by $RInd_B^G(S(\mathfrak{g}/\mathfrak{n}))$ as stated. This proves (3.4).

By a similar argument the statement of the lemma is equivalent to proving that

$$RInd_P^G(S(\mathfrak{g}/\mathfrak{r})) = S(\mathfrak{g}) \otimes_{S(\mathfrak{h})^{\mathcal{W}}} S(\mathfrak{h})^{\mathcal{W}_P}. \tag{3.5}$$

For any $M \in \mathrm{Mod}(B)$ we have an equality of P-modules

$$RInd_B^P(M) = RInd_{L\cap B}^L(M) \tag{3.6}$$

where the R-module structure on the RHS is defined by $(xf)(g) := g^{-1}xg \cdot f(g)$ for $f \in Mor(L, M)^{L\cap B} \cong Ind_{L\cap B}^L(M)$, $x \in U$, $g \in L$. Together with the given L-action this makes the RHS a P-module. In particular we have

$$RInd_B^P(S(\mathfrak{g}/\mathfrak{n})) = RInd_{L\cap B}^L(S(\mathfrak{g}/\mathfrak{n})). \tag{3.7}$$

We have a decomposition $\mathfrak{g} = \bar{\mathfrak{r}}_P \oplus \mathfrak{l} \oplus \mathfrak{r}$, where $\bar{\mathfrak{r}}_P$ is the image of $\mathfrak{r}$ under the Chevalley involution of $\mathfrak{g}$; thus $\mathfrak{g}/\mathfrak{n} = \mathfrak{l}/(\mathfrak{l}\cap\mathfrak{n}) \oplus \bar{\mathfrak{r}}_P$. Hence

$$\begin{aligned} RInd_{L\cap B}^L(S(\mathfrak{g}/\mathfrak{n})) &= RInd_{L\cap B}^L(S(\mathfrak{l}/\mathfrak{l}\cap\mathfrak{n}) \otimes S(\bar{\mathfrak{r}}_P)) \\ &= RInd_{L\cap B}^L(S(\mathfrak{l}/\mathfrak{l}\cap\mathfrak{n})) \otimes S(\bar{\mathfrak{r}}_P) = S(\mathfrak{g}/\mathfrak{r}) \otimes_{S(\mathfrak{h})^{\mathcal{W}_P}} S(\mathfrak{h}) \end{aligned} \tag{3.8}$$

where the last equality is given by (3.4) applied with G replaced by L, and the second equality is the tensor identity which applies since $S(\bar{\mathfrak{r}}_P)$ is an L-module. Since $RInd_B^G = RInd_P^G \circ RInd_B^P$ we deduce from (3.4), (3.7) and (3.8) that

$$S(\mathfrak{g}) \otimes_{S(\mathfrak{h})^{\mathcal{W}}} S(\mathfrak{h}) = RInd_P^G(S(\mathfrak{g}/\mathfrak{r}) \otimes_{S(\mathfrak{h})^{\mathcal{W}_P}} S(\mathfrak{h})) = RInd_P^G(S(\mathfrak{g}/\mathfrak{r})) \otimes_{S(\mathfrak{h})^{\mathcal{W}_P}} S(\mathfrak{h}).$$

Since $S(\mathfrak{h})$ is faithfully flat over $S(\mathfrak{h})^{\mathcal{W}_P}$, this implies (3.5). □

Let $P \subset Q$ be two parabolic subgroups. The projection $\pi_{\mathcal{P}}^{\mathcal{Q}} : \mathcal{P} \to \mathcal{Q}$ induces a map $\widetilde{\pi}_{\mathcal{P}}^{\mathcal{Q}} : \widetilde{\mathfrak{g}}_{\mathcal{P}} \to \widetilde{\mathfrak{g}}_{\mathcal{Q}}$ that fits into the following commutative square:

$$\begin{array}{ccc} \widetilde{\mathfrak{g}}_{\mathcal{P}} & \longrightarrow & \mathfrak{l}^*/L = \mathfrak{h}^*/\mathcal{W}_P \\ \downarrow{\scriptstyle \widetilde{\pi}_{\mathcal{P}}^{\mathcal{Q}}} & & \downarrow \\ \widetilde{\mathfrak{g}}_{\mathcal{Q}} & \longrightarrow & \mathfrak{l}_Q^*/L_Q = \mathfrak{h}^*/\mathcal{W}_Q \end{array} \tag{3.9}$$

With similar arguments to the proof of Lemma 3.2 one can prove

Lemma 3.3. $R\widetilde{\pi}_{\mathcal{P}*}^{\mathcal{Q}} \mathcal{O}_{\widetilde{\mathfrak{g}}_{\mathcal{P}}} = \mathcal{O}_{\widetilde{\mathfrak{g}}_{\mathcal{Q}} \times_{\mathfrak{h}^*/\mathcal{W}_Q} \mathfrak{h}^*/\mathcal{W}_P}$.

We observe that $\widetilde{\mathfrak{g}}_{\mathcal{P}}$ is an L-torsor over $T^*\mathcal{P}$. We set

Definition 3.4. $\widetilde{\mathfrak{g}}_{\mathcal{P}}^{\lambda} = \widetilde{\mathfrak{g}}_{\mathcal{P}} \times_{\mathfrak{h}^*/\mathcal{W}_P} \lambda$ for $\lambda \in \mathfrak{h}^*$.

We would like to view $\widetilde{\mathfrak{g}}_{\mathcal{P}}^{\lambda}$ as the classical Hamiltonian of $T^*(G/R)$ with respect to the (right) L-action. We have a moment map $\mu : T^*(G/R) \to \mathfrak{l}^*$. Recall that we can take the Hamiltonian reduction with respect to any subset of $\mathfrak{l}^*$ stable under the coadjoint action. Let $\mathcal{N}_\lambda \subset \mathfrak{l}^*$ be the preimage of $\lambda/\mathcal{W}_P \in \mathfrak{h}^*/\mathcal{W}_P \cong \mathfrak{l}_P^*/L$ under the quotient map. Then

$$T^*(G/R)//_{\mathcal{N}_\lambda} L = \mu^{-1}(\mathcal{N}_\lambda)/L = \widetilde{\mathfrak{g}}_{\mathcal{P}}^{\lambda}. \tag{3.10}$$

Note that we could also reduce with respect to $\lambda \in (\mathfrak{l}^*)^L$, in which case we would get twisted cotangent bundles.

4. Extended differential operators on $\mathcal{P}$

In this section we construct the sheaf of extended differential operators on a parabolic flag manifold and describe its global sections.

4.1. Torsors

Let X be an algebraic variety equipped with a free right action of a linear algebraic group K and let $p : X \to X/K$ be the projection. We assume that X, locally in the Zariski topology, is of the form $Y \times K$ for some variety Y, and p is the first projection. Such an X is called a *K-torsor*. We get induced right K-actions on the sheaf $\mathcal{D}_X$ of regular differential operators on X and on the direct image sheaf $p_*(\mathcal{D}_X)$. Denote by $\widetilde{\mathcal{D}}_{X/K} := p_*(\mathcal{D}_X)^K$ the sheaf on X/K of K-invariant local sections of $p_*(\mathcal{D}_X)$.

Let $\mathfrak{k} := \mathit{Lie}\, K$. The infinitesimal K-action gives algebra homomorphisms $\hat{\epsilon} : U(\mathfrak{k}) \to \mathcal{D}_X$ and $\tilde{\epsilon} : U(\mathfrak{k}) \to p_*\mathcal{D}_X$, which are injective since the K-action is free. It follows from the definition of differentiating a group action that $[\tilde{\epsilon}(U(\mathfrak{k})), \widetilde{\mathcal{D}}_{X/K}] = 0$.

Notice that $\tilde{\epsilon}(\mathrm{U}(\mathfrak{k})) \nsubseteq \widetilde{\mathcal{D}}_{X/K}$, unless K is abelian, but $\tilde{\epsilon}(\mathrm{Z}(\mathfrak{k})) \subseteq \widetilde{\mathcal{D}}_{X/K}$. We denote by $\epsilon : \mathrm{Z}(\mathfrak{k}) \to \widetilde{\mathcal{D}}_{X/K}$ the restriction of $\tilde{\epsilon}$ to $\mathrm{Z}(\mathfrak{k})$. By the discussion above it is a central embedding.

Now, since p is locally trivial, we can give a local description of $\widetilde{\mathcal{D}}_{X/K}$. Let $Y \times K$ be a Zariski open subset of X over which p is trivial. Then $\mathcal{D}_X|_{Y\times K} = \mathcal{D}_Y \otimes \mathcal{D}_K$ and $\widetilde{\mathcal{D}}_{X/K}|_Y = \mathcal{D}_Y \otimes \mathrm{U}(\mathfrak{k})$, where $\mathrm{U}(\mathfrak{k})$ is identified with the algebra $\mathcal{D}_K^K$ of right K-invariant differential operators on K.

Note that $\tilde{\epsilon}(\mathrm{U}(\mathfrak{k}))|_{Y\times K} = 1 \otimes {}^K\mathcal{D}_K$ is the algebra of left K-invariant differential operators on $Y \times K$, with respect to the natural left K-action on $Y \times K$, that are constant along Y. Since $\mathrm{Z}({}^K\mathcal{D}_K) = \mathrm{Z}(\mathcal{D}_K^K)$ we infer that ϵ is locally given by the embedding

$$\mathrm{Z}(\mathfrak{k}) \hookrightarrow \mathrm{U}(\mathfrak{k}) \cong 1 \otimes \mathrm{U}(\mathfrak{k}) \hookrightarrow \mathcal{D}_Y \otimes \mathrm{U}(\mathfrak{k}).$$

This implies that $\epsilon(\mathrm{Z}(\mathfrak{k})) = \mathrm{Z}(\widetilde{\mathcal{D}}_{X/K})$.

Denote by $\mathrm{Mod}(\mathcal{D}_X, K)$ the category of weakly equivariant $(\mathcal{D}_X, K)$-modules. In order to simplify the description of this category we assume henceforth that X is quasi-affine. Its object M is then a left $\mathcal{D}_X$-module equipped with an algebraic right action $\rho := \{\rho_U\}$, where $\rho_U : K \to \mathrm{Aut}_{\mathbb{C}_U}(M(U))^{\mathrm{op}}$ are homomorphisms compatible with the restriction maps in M, for each Zariski open K-invariant subset of X. We require that $\mathcal{D}_X \otimes M \to M$ is K-linear (over K-invariant open sets) with respect to the diagonal K-action on a tensor. (For a general X, ρ must be replaced by a given isomorphism $pr^*M \cong act^*M$ satisfying a cocycle condition, where pr and $act : X \times K \to X$ are the projection and the action map, respectively.)

Denote by $\mathrm{Mod}(\mathcal{D}_X, K, \mathfrak{k})$ the category of strongly equivariant $(\mathcal{D}_K, K)$-modules. Its object (M, ρ) is a weakly equivariant $(\mathcal{D}_X, K)$-module such that $d\rho(x)m = \hat{\epsilon}(x)m$ for $x \in \mathfrak{k}$ and $m \in M$.

For $M \in \mathrm{Mod}(\mathcal{D}_X, K)$ we consider the sheaf $(p_*M)^K$ of K-invariant local sections in p_*M; it has a natural $\widetilde{\mathcal{D}}_{X/K}$-module structure. Thus we get a functor p_* whose right adjoint is p^* (the pullback in the category of $\mathcal{O}$-modules with its natural equivariant structure). The following is standard (see [BB93]):

Lemma 4.1. *The functors*

(i) $p_*(\,)^K : \mathrm{Mod}(\mathcal{D}_X, K) \leftrightarrows \mathrm{Mod}(\widetilde{\mathcal{D}}_{X/K}) : p^*$ *and*
(ii) $p_*(\,)^K : \mathrm{Mod}(\mathcal{D}_X, K, \mathfrak{k}) \leftrightarrows \mathrm{Mod}(\mathcal{D}_{X/K}) : p^*$

are mutually inverse equivalences of categories.

4.2. Definition of extended differential operators

On G/R we shall always consider the right L-action $(\overline{g}, h) \mapsto \overline{gh}$ for $g \in G$ and $h \in L$. Thus, G/R is an L-torsor. We set:

Definition 4.2. $\widetilde{\mathcal{D}}_{\mathcal{P}} := \pi^{\mathcal{P}}_{G/R*}(\mathcal{D}_{G/R})^L$.

By the results of the previous section, locally on $\mathcal{P}$ we have $\widetilde{\mathcal{D}}_{\mathcal{P}} \cong \mathcal{D}_{\mathcal{P}} \otimes \mathrm{U}(\mathfrak{l})$, and we have the central algebra embedding $\epsilon : \mathrm{Z}(\mathfrak{l}) \to \widetilde{\mathcal{D}}_{\mathcal{P}}$.

For $\lambda \in \mathfrak{h}^*$ we define:

Definition 4.3. $\mathcal{D}^\lambda_{\mathcal{P}} := \widetilde{\mathcal{D}}_{\mathcal{P}} \otimes_{\epsilon(\mathrm{Z}(\mathfrak{l}))} \mathbb{C}_\lambda$.

4.3. Equivariant description

For any $\mathrm{Z}(\mathfrak{l})$-algebra S and $\lambda \in \mathfrak{h}^*$ let $\mathrm{Mod}^{\widehat{\lambda}}(S)$ be the category of left S-modules which are locally annihilated by some power of $I_{\mathfrak{l},\lambda}$.

We shall give equivariant descriptions on G and on G/R of the category $\mathrm{Mod}(\widetilde{\mathcal{D}}_{\mathcal{P}})$ and its subcategories $\mathrm{Mod}(\mathcal{D}^{\lambda}_{\mathcal{P}})$ and $\mathrm{Mod}^{\widehat{\lambda}}(\widetilde{\mathcal{D}}_{\mathcal{P}})$. It is best to work on G. We start with G/R as an intermediate step.

By Lemma 4.1 we have mutually inverse equivalences

$$\pi^{\mathcal{P}}_{G/R*}(\,)^L : \mathrm{Mod}(\mathcal{D}_{G/R}, L) \leftrightarrows \mathrm{Mod}(\widetilde{\mathcal{D}}_{\mathcal{P}}) : \pi^{\mathcal{P}*}_{G/R}. \tag{4.1}$$

Differentiating the right L-action on G/R gives an algebra embedding $\mathrm{U}(\mathfrak{l}) \hookrightarrow \mathcal{D}_{G/R}$. This allows us to consider $\mathrm{Z}(\mathfrak{l}) \subseteq \mathrm{U}(\mathfrak{l})$ as a subalgebra of $\mathcal{D}_{G/R}$. Transporting conditions from the right hand side to the left hand side of (4.1) we see that $\mathrm{Mod}(\mathcal{D}^{\lambda}_{\mathcal{P}})$ is equivalent to the full subcategory $\mathrm{Mod}(\mathcal{D}_{G/R}, L, \lambda)$ of $\mathrm{Mod}(\mathcal{D}_{G/R}, L)$ whose objects M satisfy $I_{\mathfrak{l},\lambda} \cdot M^L = 0$. Similarly, $\mathrm{Mod}^{\widehat{\lambda}}(\widetilde{\mathcal{D}}_{\mathcal{P}})$ is equivalent to the full subcategory $\mathrm{Mod}(\mathcal{D}_{G/R}, L, \widehat{\lambda})$ of $\mathrm{Mod}(\mathcal{D}_{G/R}, L)$ whose objects M are such that $I_{\mathfrak{l},\lambda}$ is locally nilpotent on M^L.

Now we pass to G. Let us introduce some notation. We have left and right actions μ_l and μ_r of G on $\mathcal{O}(G)$ defined by $\mu_l(g)f(h) := f(g^{-1}h)$ and $\mu_r(g)f(h) := f(hg^{-1})$ for $f \in \mathcal{O}(G)$ and $g, h \in G$. Differentiating μ_l, resp., μ_r, gives an injective algebra homomorphism $\epsilon_l : \mathrm{U} \to \mathcal{D}_G$, resp., an anti-homomorphism $\epsilon_r : \mathrm{U} \to \mathcal{D}_G$. We see that $\epsilon_l(\mathrm{U}) = \mathcal{D}^G_G$ consists of *right* invariant differential operators on G, and $\epsilon_r(\mathrm{U}) = {}^G\mathcal{D}_G$ consists of *left* invariant differential operators on G; moreover $\mathrm{Z} = \epsilon_l(\mathrm{U}) \cap \epsilon_r(\mathrm{U})$ and $\epsilon_l|_{\mathrm{Z}} = \epsilon_r|_{\mathrm{Z}}$.

The actions μ_l and μ_r induce left and right actions of G on $\mathcal{D}_G$ that we denote by the same symbols.

Let $\mathrm{Mod}(\mathcal{D}_G, P, \mathfrak{r})$ be the category whose objects are (M, ρ) where

(1) M is a left $\mathcal{D}_G$-module.
(2) ρ is a right algebraic P-action on M such that $\mathcal{D}_G \otimes M \to M$ is P-linear, with respect to the right P-action $\mu_r|_P$ on $\mathcal{D}_G$ and the diagonal P-action on the tensor product.
(3) $d\rho|_{\mathfrak{r}} = \epsilon_r|_{\mathfrak{r}}$ on M.

In particular, by (3) the action $\epsilon_r|_{\mathfrak{r}}$ is integrable, i.e. this $\mathfrak{r}$-action is locally nilpotent. By (4.1) and Lemma 4.1(ii) (applied to $X = G$ and $K = R$) we have an equivalence

$$\pi^{\mathcal{P}}_{G*}(\,)^P : \mathrm{Mod}(\mathcal{D}_G, P, \mathfrak{r}) \leftrightarrows \mathrm{Mod}(\widetilde{\mathcal{D}}_{\mathcal{P}}) : \pi^{\mathcal{P}*}_G. \tag{4.2}$$

Note that the functor on the left hand side (that corresponds to) the global section functor is the functor of taking P-invariants.

Let $\widetilde{M}_P := \mathrm{U}\,/\,\mathrm{U}\cdot\mathfrak{r}$ be a sort of "P-universal" Verma module for U, and equip it with the P-action that is induced from the right adjoint action of P on U. Note that the object $\mathcal{O}_G \otimes \epsilon_r(\widetilde{M}_P) \in \mathrm{Mod}(\mathcal{D}_G, P, \mathfrak{r})$ represents global sections and therefore corresponds to $\widetilde{\mathcal{D}}_{\mathcal{P}} \in \mathrm{Mod}(\widetilde{\mathcal{D}}_{\mathcal{P}})$.

Our next task is to describe the (full) subcategories $\mathrm{Mod}(\mathcal{D}_G, P, \mathfrak{r}, \lambda)$ and $\mathrm{Mod}(\mathcal{D}_G, P, \mathfrak{r}, \widehat{\lambda})$ of $\mathrm{Mod}(\mathcal{D}_G, P, \mathfrak{r})$ corresponding to the subcategories $\mathrm{Mod}(\mathcal{D}^{\lambda}_{\mathcal{P}})$ and $\mathrm{Mod}^{\widehat{\lambda}}(\widetilde{\mathcal{D}}_{\mathcal{P}})$ of $\mathrm{Mod}(\widetilde{\mathcal{D}}_{\mathcal{P}})$, respectively.

Let us consider the smash product $\mathcal{D}_G * \mathrm{U}(\mathfrak{l})$ with respect to the adjoint action of $\mathfrak{l}$ on $\mathfrak{g}$. Thus, $\mathcal{D}_G * \mathrm{U}(\mathfrak{l}) = \mathcal{D}_G \otimes \mathrm{U}(\mathfrak{l})$ as a vector space and its (associative) multiplication is defined by

$$y \otimes x \cdot y' \otimes x' := y[\epsilon_r(x), y'] \otimes x' + yy' \otimes xx', \quad x \in \mathfrak{l},\ x' \in \mathrm{U}(\mathfrak{l}),\ y, y' \in \mathcal{D}_G.$$

Observe that a $(\mathcal{D}_G, L)$-module is the same thing as a $\mathcal{D}_G * \mathrm{U}(\mathfrak{l})$-module on which the action of $1 \otimes \mathfrak{l}$ is integrable (i.e. it is the differential of the given L-action). We have an algebra isomorphism

$$\mathcal{D}_G \otimes \mathrm{U}(\mathfrak{l}) \xrightarrow{\sim} \mathcal{D}_G * \mathrm{U}(\mathfrak{l}), \quad y \otimes 1 \mapsto y \otimes 1,\ 1 \otimes x \mapsto 1 \otimes x - \epsilon_r(x) \otimes 1,\ y \in \mathcal{D}_G,\ x \in \mathfrak{l}.$$

This restricts to the algebra homomorphism

$$\alpha_{\mathfrak{l}} : \mathrm{U}(\mathfrak{l}) \to \mathcal{D}_G * \mathrm{U}(\mathfrak{l}), \quad 1 \otimes x \mapsto 1 \otimes x - \epsilon_r(x) \otimes 1,\ x \in \mathfrak{l}. \tag{4.3}$$

Note that the algebra *anti*-isomorphism $^* : \mathrm{U}(\mathfrak{l}) \xrightarrow{\sim} \mathrm{U}(\mathfrak{l})$, $x \mapsto -x$ for $x \in \mathfrak{l}$, restricts to an isomorphism $^* : \mathrm{Z}(\mathfrak{l}) \xrightarrow{\sim} \mathrm{Z}(\mathfrak{l})$.

Proposition 4.4. (i) *Let $M \in \mathrm{Mod}(\widetilde{\mathcal{D}}_{\mathcal{P}})$ and $z \in \mathrm{Z}(\mathfrak{l})$. Then $\epsilon_l(z) \in \mathrm{Z}(\mathfrak{l}) = \mathrm{Z}(\widetilde{\mathcal{D}}_{\mathcal{P}})$ defines a morphism $\epsilon_l(z) : M \to M$. By functoriality we get a morphism $\pi_G^{\mathcal{P}*}(\epsilon_l(z)) : \pi_G^{\mathcal{P}*}(M) \to \pi_G^{\mathcal{P}*}(M)$. We have $\pi_G^{\mathcal{P}*}(\epsilon_l(z)) = \alpha_{\mathfrak{l}}(z^*)|_{\pi_G^{\mathcal{P}*}(M)}$.*

(ii) *Let $M \in \mathrm{Mod}(\mathcal{D}_G, P, \mathfrak{r})$. Then $M \in \mathrm{Mod}(\mathcal{D}_G, P, \mathfrak{r}, \lambda)$ iff*

(4) $(\alpha_{\mathfrak{l}}(z^*) - \chi_{\mathfrak{l},\lambda}(z))m = 0$ *for $m \in M$ and $z \in \mathcal{Z}(\mathfrak{l})$.*

(iii) *Let $M \in \mathrm{Mod}(\mathcal{D}_G, P, \mathfrak{r})$. Then $M \in \mathrm{Mod}(\mathcal{D}_G, P, \mathfrak{r}, \widehat{\lambda})$ iff*

$(\widehat{4})$ $\alpha_{\mathfrak{l}}(z) - \chi_{\mathfrak{l},\lambda}(z)$ *is locally nilpotent on M for $z \in \mathcal{Z}(\mathfrak{l})$.*

Proof. (i) We have $\pi_G^{\mathcal{P}*}(M) = \mathcal{O}_G \otimes_{\pi_G^{\mathcal{P}-1}(\mathcal{O}_{\mathcal{P}})} \pi_G^{\mathcal{P}-1}(M)$. Let $f \in \mathcal{O}_G$ and $m \in \pi_G^{\mathcal{P}-1}(M)$. Then for $x \in \mathfrak{l}$ we have $d\rho(x)m = 0$, and consequently

$$\alpha_{\mathfrak{l}}(-x)(f \otimes m) = (\epsilon_r(x) - d\rho(x))(f \otimes m) = f \otimes \epsilon_r(x)m.$$

Since $\alpha_{\mathfrak{l}}$ is an algebra homomorphism, for $z \in \mathrm{Z}(\mathfrak{l})$ we get

$$\alpha_{\mathfrak{l}}(z^*)(f \otimes m) = f \otimes \epsilon_r(z)m = \pi_G^{\mathcal{P}*}(\epsilon_l(z))(f \otimes m).$$

This proves (i); (ii) follows from (i); and (iii) is similar to (ii) and left to the reader. □

Let $M_{P,\lambda} := \mathrm{U} / \mathrm{U} \cdot (\mathfrak{r} + \mathrm{Ker}\, \chi_{\mathfrak{l},\lambda})$ be a left U-module equipped with the right P-action that is induced from the adjoint action of P on U. Note that the object $\mathcal{O}_G \otimes \epsilon_r(M_{P,\lambda})$ of $\mathrm{Mod}(\mathcal{D}_G, P, \mathfrak{r}, \lambda)$ represents global sections (= taking P-invariants) and therefore corresponds to $\mathcal{D}^{\lambda}_{\mathcal{P}} \in \mathrm{Mod}(\mathcal{D}^{\lambda}_{\mathcal{P}})$.

Remark 4.5. Note that when $\mathfrak{l} = \mathfrak{h}$ condition (4) becomes the traditional condition of [BB93]: $\epsilon_r(x)m - d\rho(x)m = \lambda(x)m$ for $x \in \mathfrak{h}$ and $m \in M$.

Remark 4.6. Assume that $M \in \mathrm{Mod}(\mathcal{D}_G, P, \mathfrak{r})$. Then condition (4) holds for M iff

(4′) $(\epsilon_r(z) - \chi_{\mathfrak{l},\lambda}(z))m = 0$ for $m \in M^L$ and $z \in \mathcal{Z}(\mathfrak{l})$.

(Because (4′) is obviously equivalent to $(\pi^{\mathcal{P}}_{G*}M)^L \in \mathrm{Mod}(\mathcal{D}^{\lambda}_{\mathcal{P}})$.)

If we consider M^L as a sheaf on G/L, its global sections equal $\Gamma_G(M)^L$, where $\Gamma_G(M)$ is the $\mathcal{O}(G)$-module corresponding to the $\mathcal{O}_G$-module M. Since L is reductive, G/L is affine [Mat60], and therefore we may replace M^L by $\Gamma_G(M)^L$ in (4′).

However, condition (4) is better to work with than (4′), particularly while considering modules with an additional equivariance condition from the left side (see Section 7).

Example 4.7. Let us consider the simplest case when $P = G$. Then $\mathfrak{r} = 0$ and we write $\mathrm{Mod}(\mathcal{D}_G, G, \lambda) := \mathrm{Mod}(\mathcal{D}_G, G, \mathfrak{r}_G, \lambda)$ for simplicity.

The equivalence $\mathrm{Mod}(\mathbb{C}) \cong \mathrm{Mod}(\mathcal{O}_G, G)$, $V \mapsto \mathcal{O}_G \otimes V$, induces for any $\lambda \in \mathfrak{h}^*$ the equivalence $\mathrm{Mod}(\mathrm{U}^{\lambda}) \cong \mathrm{Mod}(\mathcal{D}_G, G, \lambda)$ given by $V \mapsto \mathcal{O}_G \otimes V$ where $(\mathcal{O}_G \otimes V)^G = V$ is a left module for $\epsilon_l(\mathrm{U})^{\lambda}$ (and similarly with χ_λ replaced by $\widehat{\chi_\lambda}$).

Example 4.8. Let $P = B$. Let $\lambda \in \mathfrak{h}^*$ and let M_λ be the Verma module for $\epsilon_r(\mathrm{U})$ with highest weight λ. Let $\mu \in \mathfrak{h}^*$ be integral. Consider the algebraic B-action ρ on M_λ which after differentiation satisfies

$$d\rho(x)m = (x - \lambda(x) + \mu(x))m, \quad m \in M_\lambda,\ x \in \mathfrak{b}.$$

Denote by $M_{\lambda,\mu}$ the Verma module M_λ equipped with this B-action. Then

$$\mathcal{O}_G \otimes M_{\lambda,\mu} \in \mathrm{Mod}(\mathcal{D}_G, B, \mathfrak{n}, \lambda - \mu).$$

For $\mu = 0$ we have mentioned that the functor $\mathrm{Hom}_{\mathrm{Mod}(\mathcal{D}_G,B,\mathfrak{n},\lambda)}(\mathcal{O}_G \otimes M_{\lambda,0},\ )$ is naturally equivalent to the global section functor on $\mathrm{Mod}(\mathcal{D}_G, B, \mathfrak{n}, \lambda)$, so that we have $\mathcal{O}_G \otimes M_{\lambda,0} \cong \pi^{G*}_{\mathcal{B}}\mathcal{D}^{\lambda}_{\mathcal{B}}$. This implies

$$\mathrm{End}_{\mathrm{Mod}(\mathcal{D}_G,B,\mathfrak{n},\lambda)}(\mathcal{O}_G \otimes M_\lambda) = \Gamma(\mathcal{D}^{\lambda}_{\mathcal{B}}) = \mathrm{U}^{\lambda}. \tag{4.4}$$

To get an idea of a general $\mathcal{O}_G \otimes M_{\lambda,\mu}$ assume for instance that $\mu \geq 0$. Then there is an injective map

$$f : \mathcal{O}_G \otimes M_{\lambda,\mu} \to \mathcal{O}_G \otimes M_{\lambda-\mu,0}. \tag{4.5}$$

By the Peter–Weyl theorem $\mathcal{O}_G \cong \bigoplus_{\phi\in\Lambda_+} V^*_G(\phi) \otimes V_G(\phi)$ as a G-bimodule. Let $v_\phi \in V_G(\phi)$ be a highest weight vector. Let 1_λ and $1_{\lambda-\mu}$ be highest weight vectors in $M_{\lambda,\mu}$ and $M_{\lambda-\mu,0}$, respectively. We can define f by $f(1 \otimes 1_\lambda) := (v \otimes v_\mu) \otimes 1_{\lambda-\mu}$ where $v \in V^*_G(\mu)$ is any non-zero vector. Then f is injective since both sides of (4.5) are free over the integral domain $\mathcal{O}_G \otimes \epsilon_r(\mathrm{U}(\mathfrak{n}_-))$. Note that f is not an isomorphism (and the two objects of (4.5) must be non-isomorphic) unless $\mu = 0$.

4.4. Global sections

The left G-action on G/R, $(g, \overline{g'}) \mapsto \overline{gg'}$, commutes with the right L-action and therefore induces a homomorphism $\mathrm{U} \to \widetilde{\mathcal{D}}_{\mathcal{P}}$. There is also the map $\epsilon : S(\mathfrak{h})^{\mathcal{W}_P} = \mathrm{Z}(\mathfrak{l}) \to \widetilde{\mathcal{D}}_{\mathcal{P}}$. These maps agree on $S(\mathfrak{h})^{\mathcal{W}}$ and hence induce a map

$$\widetilde{\mathrm{U}}^{\mathcal{W}_P} = \mathrm{U} \otimes_{\mathrm{Z}} S(\mathfrak{h})^{\mathcal{W}_P} \to \widetilde{\mathcal{D}}_{\mathcal{P}}.$$

This induces a homomorphism $\mathrm{U}^{\lambda} = \widetilde{\mathrm{U}}^{\mathcal{W}_P}/(I_{\mathfrak{l},\lambda}) \to \mathcal{D}^{\lambda}_{\mathcal{P}}$.

Consider the sheaf of algebras $\mathcal{O}_{\mathcal{P}} \otimes \mathrm{U}$ on $\mathcal{P}$ with multiplication determined by those in $\mathcal{O}_{\mathcal{P}}$ and in U and by the requirement that $[A, f] = \epsilon(A)(f)$ for $A \in \mathfrak{g}$ and $f \in \mathcal{O}_{\mathcal{P}}$. Then we have a surjective algebra homomorphism $\eta : \mathcal{O}_{\mathcal{P}} \otimes \mathrm{U} \to \widetilde{\mathcal{D}}_{\mathcal{P}}$. Its kernel is the ideal generated by $\xi \in \mathcal{O}_{\mathcal{P}} \otimes \mathfrak{r}$, $\xi(x) \in \mathfrak{p}_x$ for $x \in \mathcal{P}$ and $\mathfrak{p}_x \subseteq \mathfrak{g}$ the corresponding parabolic subalgebra.

Hence, to define a $\widetilde{\mathcal{D}}_{\mathcal{P}}$-module structure on an $\mathcal{O}_{\mathcal{P}}$-module M is the same thing as defining a U-module structure on M such that $\mathrm{Ker}\,\eta$ vanishes on M and $A(fm) = f(Am) + \epsilon(A)(f)m$ for $A \in \mathfrak{g}$, $f \in \mathcal{O}_{\mathcal{P}}$ and $m \in M$.

Let $\mu \in \mathfrak{h}^*$ be integral and P-dominant. Recall that $V_P(\mu)$ denotes the corresponding irreducible representation of P with highest weight μ, and $\mathcal{O}(V_P(\mu))$ the corresponding left G-equivariant locally free sheaf on $\mathcal{P}$.

Let $M \in \mathrm{Mod}(\widetilde{\mathcal{D}}_{\mathcal{P}})$. We shall show that the $\mathcal{O}_{\mathcal{P}}$-module $M \otimes_{\mathcal{O}_{\mathcal{P}}} \mathcal{O}(V_P(\mu))$ is naturally a $\widetilde{\mathcal{D}}_{\mathcal{P}}$-module. We proceed as follows:

The G-action on $\mathcal{O}(V_P(\mu))$ differentiates to a left $\mathfrak{g}$-action on it, which extends to a $\mathfrak{g}$-action on $M \otimes_{\mathcal{O}_{\mathcal{P}}} \mathcal{O}(V_P(\mu))$ by Leibniz's rule. Since $V_P(\mu)$ is an irreducible P-module we know that R acts trivially on it (recall $V_P(\mu) = V_L(\mu)$). Hence, $\mathfrak{r}$ acts trivially on $\mathcal{O}(V_P(\mu))$ and from this it now follows that the compatibilities for being a $\widetilde{\mathcal{D}}_{\mathcal{P}}$-module are satisfied by $M \otimes_{\mathcal{O}_{\mathcal{P}}} \mathcal{O}(V_P(\mu))$.

Assume that $M \in \mathrm{Mod}(\widetilde{\mathcal{D}}_{\mathcal{P}})$. In the equivariant language on G we see that M and $M \otimes_{\mathcal{O}_{\mathcal{P}}} \mathcal{O}(V_P(\mu))$ correspond to $\pi_G^{\mathcal{P}*}M$ and $M_{V_P(\mu)} := (\pi_G^{\mathcal{P}*}M) \otimes V_P(\mu) \in \mathrm{Mod}(\mathcal{D}_G, P, \mathfrak{r})$, respectively. Here, the $\mathcal{D}_G$-action on $M_{V_P(\mu)}$ is given by the action on the first factor, and the P-action is diagonal. Again, it is the fact that R acts trivially on $V_P(\mu)$ that shows that $M_{V_P(\mu)}$ is an object of $\mathrm{Mod}(\mathcal{D}_G, L, \mathfrak{r})$.

Lemma 4.9. *Let $\lambda \in \mathfrak{h}^*$, $M \in \mathrm{Mod}(\mathcal{D}^{\lambda}_{\mathcal{P}})$ and let $\mu \in \mathfrak{h}^*$ be integral and P-dominant. Then $M \otimes_{\mathcal{O}_{\mathcal{P}}} \mathcal{O}(V_P(\mu)) \in \bigoplus_{\nu \in \Lambda(V_P(\mu))} \mathrm{Mod}^{\widehat{\lambda+\nu}}(\widetilde{\mathcal{D}}_{\mathcal{P}})$, where $\Lambda(V_P(\mu))$ denotes the set of weights of $V_P(\mu)$.*

Proof. In equivariant translation we want to prove that

$$M_{V_P(\mu)} \in \bigoplus_{\nu \in \Lambda(V_P(\mu))} \mathrm{Mod}(\mathcal{D}_G, P, \mathfrak{r}, \widehat{\lambda + \nu}). \tag{4.6}$$

We use Proposition 4.4(i). We have an action $\widetilde{\alpha}_{\mathfrak{l}} : \mathrm{U}(\mathfrak{l}) \to \mathrm{End}(M_{V_P(\mu)})$. We see that this action is actually the tensor product of the $\widetilde{\alpha}_{\mathfrak{l}}$-action of $\mathrm{U}(\mathfrak{l})$ on $\pi_G^{\mathcal{P}*}M$ and the $\mathrm{U}(\mathfrak{l})$-action on $V_P(\mu)$, which is the differential of the given L-action. Now, since for $z \in \mathrm{Z}(\mathfrak{l})$, by assumption $\alpha_{\mathfrak{l}}(z) = \widetilde{\alpha}_{\mathfrak{l}}(z)$ acts by $\chi_{\mathfrak{l},\lambda}(z)$ on $\pi_G^{\mathcal{P}*}M$, it follows from [BeGe81] that (4.6) holds. ◻

Theorem 4.10. (i) $R\pi^{\mathcal{P}}_{\mathcal{B}*}\widetilde{\mathcal{D}}_{\mathcal{B}} = \widetilde{\mathcal{D}}_{\mathcal{P}} \otimes_{Z(\mathfrak{l})} S(\mathfrak{h})$,
(ii) $R\pi^{\mathcal{Q}}_{\mathcal{P}*}\widetilde{\mathcal{D}}_{\mathcal{P}} = \widetilde{\mathcal{D}}_{\mathcal{Q}} \otimes_{Z(\mathfrak{l}_Q)} S(\mathfrak{h})^{\mathcal{W}_P}$,
(iii) $R\Gamma(\widetilde{\mathcal{D}}_{\mathcal{P}}) = \widetilde{U}^{\mathcal{W}_P}$,
(iv) $R\Gamma(\mathcal{D}^{\lambda}_{\mathcal{P}}) = U^{\lambda}$.

Proof. By Lemmas 3.2 and 3.3 the associated graded maps (i) and (ii) are isomorphisms; hence (i) and (ii) are also isomorphisms. (iii) is a special case of (ii), and (iv) follows from (iii) because $R\Gamma$ commutes with $(\ ) \otimes_{Z(\mathfrak{l})} \mathbb{C}_{\lambda}$, since $\widetilde{\mathcal{D}}_{\mathcal{P}}$ is locally free over $Z(\mathfrak{l})$. □

The functor $\Gamma : \mathrm{Mod}(\mathcal{D}^{\lambda}_{\mathcal{P}}) \to \mathrm{Mod}(U^{\lambda})$ has a left adjoint $\mathcal{L} := \mathcal{D}^{\lambda}_{\mathcal{P}} \otimes_{U^{\lambda}} (\ )$, called the *localization functor*. Also $\Gamma : \mathrm{Mod}^{\widehat{\lambda}}(\widetilde{\mathcal{D}}_{\mathcal{P}}) \to \mathrm{Mod}^{\widehat{\lambda}}(U)$ has a left adjoint $\mathcal{L} := \varprojlim_n \mathcal{D}_{\mathcal{P}}/(I_{\lambda})^n \otimes_U (\ )$.

5. Singular localization

Here we prove the singular version of Beilinson–Bernstein localization.

Theorem 5.1. *Let λ be dominant and P-regular. Then $\Gamma : \mathrm{Mod}(\mathcal{D}^{\lambda}_{\mathcal{P}}) \to \mathrm{Mod}(U^{\lambda})$ is an equivalence of categories.*

Proof (essentially taken from [BB81]). Since Γ has a left adjoint $\mathcal{L}$ which is right exact and since $\Gamma \circ \mathcal{L}(U^{\lambda}) = \Gamma(\mathcal{D}^{\lambda}_{\mathcal{P}}) = U^{\lambda}$, the theorem will follow from the following two claims:

(a) Let λ be dominant. Then $\Gamma : \mathrm{Mod}(\mathcal{D}^{\lambda}_{\mathcal{P}}) \to \mathrm{Mod}(U^{\lambda})$ is exact.
(b) Let λ be dominant and P-regular and $M \in \mathrm{Mod}(\mathcal{D}^{\lambda}_{\mathcal{P}})$. Then $\Gamma(M) = 0$ implies that $M = 0$.

Let V be a finite-dimensional irreducible G-module and let

$$0 = V_{-1} \subset V_0 \subset \cdots \subset V_n = V$$

be a filtration of V by P-submodules such that $V_i/V_{i-1} \cong V_P(\mu_i)$ is an irreducible P-module.

We first choose V so that its highest weight μ_0 is a P-character. Thus $M \otimes_{\mathcal{O}} \mathcal{O}(V_0) = M(-\mu_0)$ and we get an embedding $M(-\mu_0) \hookrightarrow M \otimes_{\mathcal{O}} \mathcal{O}(V)$, which twists to the embedding $M \hookrightarrow M(\mu_0) \otimes_{\mathcal{O}} \mathcal{O}(V) \cong M(\mu_0)^{\dim V}$. Now, by Lemmas 2.1, 4.9 and Theorem 4.10(iii), this inclusion splits on derived global sections, so $R\Gamma(M)$ is a direct summand of $R\Gamma(M(\mu_0))^{\dim V}$. Now, for μ_0 large enough and if M is $\mathcal{O}$-coherent, we have $R^{>0}\Gamma(M(\mu_0)) = 0$ (since $\mathcal{O}(\mu_0)$ is very ample). Hence, $R^{>0}\Gamma(M) = 0$ in this case. A general M is the union of coherent submodules and by a standard limit argument it follows that $R^{>0}\Gamma(M) = 0$. This proves (a).

Now, for (b) we assume instead that the lowest weight μ_n of V is a P-character. Then we have a surjection $M^{\dim V} \cong M \otimes_{\mathcal{O}} \mathcal{O}(V) \to M(-\mu_n)$. Applying global sections and using Lemmas 2.2, 4.9 and Theorem 4.10(iv) we find that $\Gamma(M(-\mu_n))$ is a direct summand of $\Gamma(M)^{\dim V}$. For μ_n small enough we conclude that $\Gamma(M(-\mu_n)) \neq 0$. Hence, $\Gamma(M) \neq 0$. This proves (b). □

Assume that λ is P-regular. Then the projection $\mathfrak{h}^*/\mathcal{W}_P \to \mathfrak{h}^*/\mathcal{W}$ is unramified at λ, from which one deduces (see [BG99]) that restriction defines an equivalence of categories $\mathrm{Mod}^{\widehat{\lambda}}(\widetilde{\mathrm{U}}^{\mathcal{W}_P}) \xrightarrow{\sim} \mathrm{Mod}^{\widehat{\lambda}}(\mathrm{U})$.

Theorem 5.2. *Let λ be dominant and P-regular. Then $\Gamma : \mathrm{Mod}^{\widehat{\lambda}}(\widetilde{\mathcal{D}}_{\mathcal{P}}) \to \mathrm{Mod}^{\widehat{\lambda}}(\widetilde{\mathrm{U}}^{\mathcal{W}_P}) \cong \mathrm{Mod}^{\widehat{\lambda}}(\mathrm{U})$ is an equivalence of categories.*

Proof. This follows from Theorem 5.1 and a simple devissage. □

6. Translation functors

We geometrically describe translation functors on $\mathfrak{g}$-modules in the context of singular localization. For regular localization this was worked out in [BG99]. Singular localization clarifies the picture.

6.1. Translation functors

Let A be a $\mathrm{Z}(\mathfrak{l})$-algebra (or a sheaf of algebras). Let $\mathrm{Mod}^{\mathrm{Z}(\mathfrak{l})\text{-fin}}(A)$ be the category of (quasi-coherent) A-modules that are locally finite over $\mathrm{Z}(\mathfrak{l})$. Thus we have $\mathrm{Mod}^{\mathrm{Z}(\mathfrak{l})\text{-fin}}(A) = \bigoplus_{\mu\in\mathfrak{h}^*} \mathrm{Mod}^{\widehat{\lambda}}(A)$ and there are exact projections $pr_{\mathfrak{l},\widehat{\mu}} : \mathrm{Mod}^{\mathrm{Z}(\mathfrak{l})\text{-fin}}(A) \to \mathrm{Mod}^{\widehat{\mu}}(A)$.

Assume $\lambda, \mu \in \mathfrak{h}^*$ with $\lambda - \mu$ integral. Then there is the translation functor (see [BeGe81])

$$T^{\mu}_{\mathfrak{l},\lambda} : \mathrm{Mod}^{\widehat{\lambda}}(\mathrm{U}(\mathfrak{l})) \to \mathrm{Mod}^{\widehat{\mu}}(\mathrm{U}(\mathfrak{l})), \qquad M \mapsto pr_{\mathfrak{l},\widehat{\mu}}(M \otimes E),$$

where E is an irreducible finite-dimensional representation of $\mathfrak{l}$ with extremal weight $\mu - \lambda$. We set $pr_{\widehat{\mu}} := pr_{\mathfrak{g},\widehat{\mu}}$ and $T^{\mu}_{\lambda} := T^{\mu}_{\mathfrak{g},\lambda}$ in the case $\mathfrak{g} = \mathfrak{l}$.

Assume that $\mathfrak{l}$ is the Levi factor of the Lie algebra of a parabolic subgroup $Q \subseteq G$ and that $P \subseteq Q$ is another parabolic. We assume henceforth that $\lambda, \mu \in \mathfrak{h}^*$ are weights such that $\mu - \lambda$ is integral, $\mathcal{W}_P = \mathcal{W}_\lambda$ and $\mathcal{W}_Q = \mathcal{W}_\mu$. Then we have the algebra inclusion $S(\mathfrak{h})^{\mathcal{W}_Q} \cong \mathrm{Z}(\mathfrak{l}) \hookrightarrow S(\mathfrak{h})^{\mathcal{W}_P}$, which induces an algebra inclusion $A \hookrightarrow A \otimes_{\mathrm{Z}(\mathfrak{l})} S(\mathfrak{h})^{\mathcal{W}_P}$.

Lemma 6.1. *The inclusion $A \hookrightarrow A \otimes_{\mathrm{Z}(\mathfrak{l})} S(\mathfrak{h})^{\mathcal{W}_P}$ induces an equivalence of categories $Res : \mathrm{Mod}^{\widehat{\lambda}}(A \otimes_{\mathrm{Z}(\mathfrak{l})} S(\mathfrak{h})^{\mathcal{W}_P}) \to \mathrm{Mod}^{\widehat{\lambda}}(A)$.*

Proof. Since λ is P-regular, $\mathfrak{h}^*/\mathcal{W}_P \to \mathfrak{h}^*/\mathcal{W}_Q$ is unramified at λ. The argument of the proof in [BG99, Lemma 1.2], where the assertion is proved in the case $A = U$, works also for A. □

The functor $T^{\mu}_{\mathfrak{l},\lambda}$ can be extended to a functor

$$\widetilde{T}^{\mu}_{\mathfrak{l},\lambda} : \mathrm{Mod}^{\widehat{\lambda}}(\mathrm{U}(\mathfrak{l}) \otimes_{\mathrm{Z}(\mathfrak{l})} S(\mathfrak{h})^{\mathcal{W}_P}) \to \mathrm{Mod}^{\widehat{\mu}}(\mathrm{U}(\mathfrak{l}) \otimes_{\mathrm{Z}(\mathfrak{l})} S(\mathfrak{h})^{\mathcal{W}_P}). \tag{6.1}$$

We briefly describe its construction here (see [BG99, Proposition 1.4] for details). Let $V \in \mathrm{Mod}^{\widehat{\lambda}}(\mathrm{U}(\mathfrak{l}) \otimes_{\mathrm{Z}(\mathfrak{l})} S(\mathfrak{h})^{\mathcal{W}_P})$. Then $a \in S(\mathfrak{h})^{\mathcal{W}_P} \subset \mathrm{U}(\mathfrak{l}) \otimes_{\mathrm{Z}(\mathfrak{l})} S(\mathfrak{h})^{\mathcal{W}_P})$ acts on $T^{\mu}_{\mathfrak{l},\lambda}(Res(V))$ by

$$a * m := T^{\mu}_{\mathfrak{l},\lambda}(\pi_{\mu-\lambda}a)(m), \qquad m \in T^{\mu}_{\mathfrak{l},\lambda}(Res(V)),$$

where $\pi_{\mu-\lambda} : S(\mathfrak{h}) \xrightarrow{\sim} S(\mathfrak{h})$ is induced by the affine translation $x \mapsto x + \mu - \lambda$ on $\mathfrak{h}^*$. Combining this with the $U(\mathfrak{l})$-action on $T^{\mu}_{\mathfrak{l},\lambda}(Res(V))$ we get an action of $\mathrm{U}(\mathfrak{l}) \otimes_{\mathbb{C}} S(\mathfrak{h})^{\mathcal{W}_P}$ on $T^{\mu}_{\mathfrak{l},\lambda}(Res(V))$ which factors through a $\mathrm{U}(\mathfrak{l}) \otimes_{\mathrm{U}(\mathfrak{l})} S(\mathfrak{h})^{\mathcal{W}_P}$-action. We define $\widetilde{T}^{\mu}_{\mathfrak{l},\lambda}(V)$ to be $T^{\mu}_{\mathfrak{l},\lambda}(Res(V))$ equipped with this $\mathrm{U}(\mathfrak{l}) \otimes_{\mathrm{U}(\mathfrak{l})} S(\mathfrak{h})^{\mathcal{W}_P}$-action.

We shall now give a $\mathcal{D}$-module interpretation of these functors. We use the language of $\widetilde{\mathcal{D}}_{\mathcal{P}}$-modules; it is a simple task to pass to an equivariant description on G. Define a geometric translation functor

$$\mathbb{T}^{\mu}_{P,\lambda} : \mathrm{Mod}^{\widehat{\lambda}}(\widetilde{\mathcal{D}}_{\mathcal{P}}) \to \mathrm{Mod}^{\widehat{\mu}}(\widetilde{\mathcal{D}}_{\mathcal{P}}), \qquad M \mapsto pr_{\mathfrak{l},\widehat{\mu}}(M \otimes_{\mathcal{O}_{\mathcal{P}}} \mathcal{O}(E)),$$

for $M \in \mathrm{Mod}^{\widehat{\lambda}}(\widetilde{\mathcal{D}}_{\mathcal{P}})$, where E is an irreducible P-representation with highest weight in $\mathcal{W}_P(\mu - \lambda)$.

Note that if $\mu - \lambda$ is a P-character then $\mathcal{O}_{\mathcal{P}}(E) = \mathcal{O}_{\mathcal{P}}(\mu - \lambda)$ and in this case $\mathbb{T}^{\mu}_{P,\lambda} = (\,) \otimes_{\mathcal{O}_{\mathcal{P}}} \mathcal{O}(\mu - \lambda)$ is an equivalence with inverse given by $\mathbb{T}^{\lambda}_{P,\nu} = (\,) \otimes_{\mathcal{O}_{\mathcal{P}}} \mathcal{O}(\lambda - \mu)$. In particular, for $P = B$ we have $\mathbb{T}^{\mu}_{B,\lambda} = (\,) \otimes_{\mathcal{O}_{\mathcal{B}}} \mathcal{O}(\mu - \lambda)$ for any μ and λ.

Let us denote $\widetilde{\mathcal{D}}_{\mathcal{Q},\mathcal{P}} := \widetilde{\mathcal{D}}_{\mathcal{Q}} \otimes_{\mathrm{Z}(\mathfrak{l}_Q)} S(\mathfrak{h})^{\mathcal{W}_P}$. This gives the category $\mathrm{Mod}^{\widehat{\lambda}}(\widetilde{\mathcal{D}}_{\mathcal{Q},\mathcal{P}})$. By Theorem 4.10 we have $\pi^{\mathcal{Q}}_{\mathcal{P}*}\widetilde{\mathcal{D}}_{\mathcal{P}} \cong \widetilde{\mathcal{D}}_{\mathcal{Q},\mathcal{P}}$ and $\Gamma(\widetilde{\mathcal{D}}_{\mathcal{Q},\mathcal{P}}) \cong \Gamma(\widetilde{\mathcal{D}}_{\mathcal{P}}) \cong \widetilde{\mathrm{U}}^{\mathcal{W}_P}$.

By the same reasoning as in (6.1) the functor $\mathbb{T}^{\mu}_{Q,\lambda}$ extends to $\widetilde{\mathbb{T}}^{\mu}_{Q,\lambda} : \mathrm{Mod}^{\widehat{\lambda}}(\widetilde{\mathcal{D}}_{\mathcal{Q},\mathcal{P}}) \to \mathrm{Mod}^{\widehat{\mu}}(\widetilde{\mathcal{D}}_{\mathcal{Q},\mathcal{P}})$. We have

Lemma 6.2. *The diagram*

$$\begin{array}{ccc} \mathrm{Mod}^{\widehat{\lambda}}(\widetilde{\mathcal{D}}_{\mathcal{P}}) & \xrightarrow{\mathbb{T}^{\mu}_{P,\lambda}} & \mathrm{Mod}^{\widehat{\mu}}(\widetilde{\mathcal{D}}_{\mathcal{P}}) \\ \downarrow{\scriptstyle \pi^{\mathcal{Q}}_{\mathcal{P}*}} & & \downarrow{\scriptstyle \pi^{\mathcal{Q}}_{\mathcal{P}*}} \\ \mathrm{Mod}^{\widehat{\lambda}}(\widetilde{\mathcal{D}}_{\mathcal{Q},\mathcal{P}}) & \xrightarrow{\widetilde{\mathbb{T}}^{\mu}_{Q,\lambda}} & \mathrm{Mod}^{\widehat{\mu}}(\widetilde{\mathcal{D}}_{\mathcal{Q},\mathcal{P}}) \end{array}$$

of exact functors commutes up to natural equivalence.

In the case of $P = B$ and $Q = G$ this was proved in [BG99, Proposition 2.8].

Proof. Let V (resp., V') be an irreducible finite-dimensional representation for Q (resp., for P) whose highest weight belongs to $\mathcal{W}_Q(\mu - \lambda)$ (resp., $\mathcal{W}_P(\mu - \lambda)$). Let $M \in \mathrm{Mod}^{\widehat{\lambda}}(\widetilde{\mathcal{D}}_{\mathcal{P}})$. Then, since V is a Q-representation, we have $\mathcal{O}_{\mathcal{P}}(V) = \pi^{\mathcal{Q}*}_{\mathcal{P}}(\mathcal{O}_{\mathcal{Q}}(V))$, and therefore it follows from the projection formula that

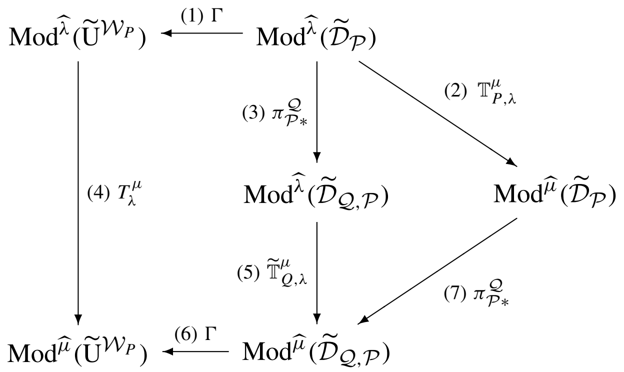
$$\pi^{\mathcal{Q}}_{\mathcal{P}*}(\mathcal{O}_{\mathcal{P}}(V) \otimes_{\mathcal{O}_{\mathcal{P}}} M) = \mathcal{O}_{\mathcal{Q}}(V) \otimes_{\mathcal{O}_{\mathcal{Q}}} \pi^{\mathcal{Q}}_{\mathcal{P}*}(M).$$

Thus we get

$$\begin{aligned} \widetilde{\mathbb{T}}^{\mu}_{Q,\lambda} \circ \pi^{\mathcal{Q}}_{\mathcal{P}*}(M) &= pr_{\mathfrak{l}_Q,\widehat{\mu}}(\mathcal{O}_{\mathcal{Q}}(V) \otimes_{\mathcal{O}_{\mathcal{Q}}} \pi^{\mathcal{Q}}_{\mathcal{P}*}(M)) \\ &= pr_{\mathfrak{l}_Q,\widehat{\mu}}(\pi^{\mathcal{Q}}_{\mathcal{P}*}(\mathcal{O}_{\mathcal{P}}(V) \otimes_{\mathcal{O}_{\mathcal{P}}} M)) = \pi^{\mathcal{Q}}_{\mathcal{P}*}(pr_{\mathfrak{l},\widehat{\mu}}(\mathcal{O}_{\mathcal{P}}(V) \otimes_{\mathcal{O}_{\mathcal{P}}} M)) \\ &\overset{(*)}{=} \pi^{\mathcal{Q}}_{\mathcal{P}*}(pr_{\mathfrak{l},\widehat{\mu}}(\mathcal{O}_{\mathcal{P}}(V') \otimes_{\mathcal{O}_{\mathcal{P}}} M)) = \pi^{\mathcal{Q}}_{\mathcal{P}*} \circ \widetilde{\mathbb{T}}^{\mu}_{P,\lambda}(M). \end{aligned}$$

The equality $(*)$ follows from Lemma 2.2 applied to the reductive Lie algebra $\mathfrak{l}_Q$ and its parabolic subalgebra $\mathfrak{l}_Q \cap \mathfrak{p}$ (compare with the proof of the localization theorem). □

Let us geometrically describe *translation to the wall*: We assume that λ and μ are dominant and chosen such that $\mu - \lambda$ is a P-character. . By Theorem 5.2 and Lemma 6.2 it follows that the diagram below commutes up to natural equivalence:



(6.2)

Note that (1) and (6) are equivalences by the choices of P and Q, and $(2) = (\ ) \otimes_{\mathcal{O}_{\mathcal{P}}} \mathcal{O}(\mu - \lambda)$ is an equivalence since $\mu - \lambda$ is a P-character.

We see that (3) is an equivalence of categories because both the source and the target categories are D-affine, since λ is P- and Q-regular, and $\Gamma \circ \pi^{Q}_{\mathcal{P}*} = \Gamma$. On the other hand, the functor (7) is not faithful, because μ is not P-regular, and (5) is not faithful either. We recall that these functors are all exact.

Translation out of the wall can now be described by considering adjoint functors to the functors involved in (6.2). For example the adjoint (left and right) of $\mathrm{T}^{\mu}_{\mathfrak{l},\lambda}$ is $T^{\lambda}_{\mathfrak{l},\mu}$, and the left adjoint of (7) is $\pi^{Q*}_{\mathcal{P}}$. To describe the (in general non-exact) adjoints of the extended translation functors we refer to [BG99, Proposition 1.5 and surrounding discussion]. We do not write down the details here.

7. Category O and Harish-Chandra (bi-)modules

Singular localization allows us to interpret blocks of category O as bi-equivariant $\mathcal{D}_G$-modules which in turn are equivalent to categories of Harish-Chandra (bi-)modules. As we mentioned in the introduction, the novelty here is that we are led to consider $\mathfrak{g}$-$\mathfrak{l}$-bimodules, which we believe is a better notion. *Parabolic* (and singular) blocks of O are discussed in Section 8.2.

The material here is related to Section 6 because translation functors restrict to functors between blocks in O.

7.1. Category O *and generalized twisted Harish-Chandra modules*

See [Hum08] for generalities on category O and [Dix77] for generalities on Harish-Chandra modules.

We are interested in the Bernstein–Gelfand–Gefand category O of finitely generated left U-modules which are locally finite over $\mathrm{U}(\mathfrak{n})$ and semisimple over $\mathfrak{h}$. For $\lambda \in \mathfrak{h}^*$ we let $\mathrm{O}_\lambda, \mathrm{O}_{\widehat{\lambda}} \subset \mathrm{O}$ be the subcategories of modules with central character, respectively, generalized central character, χ_λ.

Generalized twisted Harish-Chandra modules. Let $K \subset G$ be a subgroup and let $\mathfrak{k} := \mathit{Lie}\, K$ be its Lie algebra. A *weak Harish-Chandra* (K, U)-*module* (or simply a (K, U)-module) is a left U-module M equipped with an algebraic left action of K such that the action map $\mathrm{U} \otimes M \to M$ is K-equivariant with respect to the adjoint action of K on U. A *Harish-Chandra* (K, U)-*module* (or simply a $(\mathfrak{k}, K, \mathrm{U})$-module) is a weak Harish-Chandra module such that the differential of the K-action coincides with the action of $\mathfrak{k} \subset \mathrm{U}$.

Similarly, there are (K, U^λ)-modules and $(\mathfrak{k}, K, \mathrm{U}^\lambda)$-modules, for $\lambda \in \mathfrak{h}^*$.

Let $\mu \in K^*$. A μ-*twisted Harish-Chandra module* is a (K, U)-module M on which the action of $\mathfrak{k} \subset \mathrm{U}$ minus the differential of the K-action is equal to μ.

We shall now give certain generalizations of twisted Harish-Chandra modules in the case when $K = P$. Consider the smash-product algebra $\mathrm{U} * \mathrm{U}(\mathfrak{l})$ with respect to the adjoint action of $\mathfrak{l}$ on U. Observe that an (L, U)-module is the same thing as a $\mathrm{U} * \mathrm{U}(\mathfrak{l})$-module on which $1 \otimes \mathfrak{l}$ acts semisimply, and $1 \otimes H_\alpha$ has integral eigenvalues for each simple coroot H_α. The algebra *anti-homomorphism* $\mathrm{U}(\mathfrak{l}) \to \mathrm{U} * \mathrm{U}(\mathfrak{l})$ defined by $x \mapsto x \otimes 1 - 1 \otimes x$ for $x \in \mathfrak{l}$ restricts to a *homomorphism*

$$\overline{\alpha}_{\mathfrak{l}} : \mathrm{Z}(\mathfrak{l}) \to \mathrm{Z}(U(\mathfrak{g}) * \mathrm{U}(\mathfrak{l})). \tag{7.1}$$

(Compare with the map $\alpha_{\mathfrak{l}}(z^*)$ from (4.3).) We define $\mathrm{Mod}(\widehat{\lambda}, \mathfrak{r}, P, \mathrm{U}^{\lambda'})$ to be the category of $(P, \mathrm{U}^{\lambda'})$-modules M such that, if ρ denotes the P-action on M, then $d\rho|_{\mathfrak{r}}$ coincides with the action of $\mathfrak{r} \subset \mathrm{U}^{\lambda'}$ on M and for $z \in \mathrm{Z}(\mathfrak{l})$, $\overline{\alpha}_{\mathfrak{l}}(z) - \chi_{\mathfrak{l},\lambda}(z)$ acts locally nilpotently on M.

Similarly, one defines categories $\mathrm{Mod}^{\widehat{\lambda'}}(\widehat{\lambda}, \mathfrak{r}, P, \mathrm{U})$ and $\mathrm{Mod}(\lambda, \mathfrak{r}, P, \mathrm{U}^{\lambda'})$, etc.

We see that if $\lambda, \lambda' \in \mathfrak{h}^*$ and $\lambda - \lambda'$ is integral then

$$\mathrm{O}_\lambda = \mathrm{mod}(\lambda', \mathfrak{n}, B, \mathrm{U}^\lambda) \quad \text{and} \quad \mathrm{O}_{\widehat{\lambda}} = \mathrm{mod}^{\widehat{\lambda}}(\lambda', \mathfrak{n}, B, \mathrm{U})$$

are (non-generalized) categories of twisted Harish-Chandra modules. For $P \neq B$ we like to think of $\mathrm{mod}(\widehat{\lambda}, \mathfrak{r}, P, \mathrm{U}^{\lambda'})$ and $\mathrm{mod}(\lambda, \mathfrak{r}, P, \mathrm{U}^{\lambda'})$ as "non-standard parabolic blocks in O", although, in reality, they are not even subcategories of O, since the $\mathfrak{b}$-action is not locally finite.

7.2. *Harish-Chandra modules to bimodules*

The categories of the previous section can be described in terms of Harish-Chandra bimodules [BeGe81]. Let $\widetilde{\mathcal{H}}(\mathfrak{l})$ be the category of U-$\mathrm{U}(\mathfrak{l})$-bimodules on which the adjoint action of $\mathfrak{l}$ is integrable and the left action of $\mathfrak{r}$ is locally nilpotent. Write $\widetilde{\mathcal{H}} := \widetilde{\mathcal{H}}(\mathfrak{g})$ and replacing $\mathfrak{g}$ by $\mathfrak{l}$ write $\widetilde{\mathcal{H}}(\mathfrak{l}, \mathfrak{l})$ for the category of $\mathrm{U}(\mathfrak{l})$-$\mathrm{U}(\mathfrak{l})$-bimodules on which the adjoint $\mathfrak{l}$-action is integrable.

Let $\mathcal{H}(\mathfrak{l}) \subset \widetilde{\mathcal{H}}(\mathfrak{l})$ be the subcategory of noetherian objects. Note that for $M \in \widetilde{\mathcal{H}}(\mathfrak{l})$ we have $M \in \mathcal{H}(\mathfrak{l}) \Leftrightarrow M$ is f.g. as a U-U($\mathfrak{l}$)-bimodule $\Leftrightarrow M$ is f.g. as a left U-module (and in case $\mathfrak{l} = \mathfrak{g}$ this holds if and only if M is f.g. as a right U-module). Set

$$\begin{aligned}
{}_{\text{Z-fin}}\mathcal{H}(\mathfrak{l}) &:= \{M \in \mathcal{H}(\mathfrak{l}) : \mathrm{Z} \text{ acts locally finitely on } M \text{ from the left}\},\\
\mathcal{H}(\mathfrak{l})_{\mathrm{Z}(\mathfrak{l})\text{-fin}} &:= \{M \in \mathcal{H}(\mathfrak{l}) : \mathrm{Z}(\mathfrak{l}) \text{ acts locally finitely on } M \text{ from the right}\},\\
{}_{\text{Z-fin}}\mathcal{H}(\mathfrak{l})_{\mathrm{Z}(\mathfrak{l})\text{-fin}} &:= {}_{\text{Z-fin}}\mathcal{H}(\mathfrak{l}) \cap \mathcal{H}(\mathfrak{l})_{\mathrm{Z}(\mathfrak{l})\text{-fin}}.
\end{aligned}$$

Observe that

$$ {}_{\text{Z-fin}}\mathcal{H} = \mathcal{H}_{\text{Z-fin}} = {}_{\text{Z-fin}}\mathcal{H}_{\text{Z-fin}}. \tag{7.2}$$

We set ${}_{\lambda'}\mathcal{H}(\mathfrak{l}) := \{M \in \mathcal{H}(\mathfrak{l}) : I_{\lambda'}M = 0\}$, $\mathcal{H}(\mathfrak{l})_{\lambda} := \{M \in \mathcal{H}(\mathfrak{l}) : MI_{\mathfrak{l},\lambda} = 0\}$ and ${}_{\widehat{\mu}}\mathcal{H}(\mathfrak{l}) := \{M \in \mathcal{H}(\mathfrak{l}) : I_{\lambda'}$ acts locally nilpotently on $M\}$, etc. Similarly, we define ${}_{\lambda'}\mathcal{H}(\mathfrak{l})_{\widehat{\lambda}} := {}_{\lambda'}\mathcal{H}(\mathfrak{l}) \cap \mathcal{H}(\mathfrak{l})_{\widehat{\lambda}}$, $\widetilde{\mathcal{H}}(\mathfrak{l})_{\lambda}$, etc.

Lemma 7.1. $\mathrm{Mod}(\lambda, \mathfrak{r}, P, \mathrm{U}^{\lambda'}) \cong {}_{\lambda'}\mathcal{H}(\mathfrak{l})_{\lambda}$ *and* $\mathrm{Mod}(\widehat{\lambda}, \mathfrak{r}, P, \mathrm{U}^{\lambda'}) \cong {}_{\lambda'}\mathcal{H}(\mathfrak{l})_{\widehat{\lambda}}$.

Proof. A $(P, \mathrm{U}^{\lambda'})$-module is the same thing as a $\mathrm{U}^{\lambda'} * \mathrm{U}(\mathfrak{p})$-module such that $1 \otimes \mathfrak{p}$ acts integrably. Under the algebra isomorphism

$$\mathrm{U}^{\lambda'} * \mathrm{U}(\mathfrak{p}) \xrightarrow{\sim} \mathrm{U}^{\lambda'} \otimes \mathrm{U}(\mathfrak{p}), \qquad 1 \otimes x \mapsto 1 \otimes x + x \otimes 1,\ y \otimes 1 \mapsto y \otimes 1,$$

the latter modules are equivalent to the category of $\mathrm{U}^{\lambda'} \otimes \mathrm{U}(\mathfrak{p})$-modules on which the action of $\Delta\mathfrak{p}$ is integrable, where $\Delta : \mathfrak{p} \to \mathrm{U}^{\lambda'} \otimes \mathrm{U}(\mathfrak{p})$ is given by $\Delta x := x \otimes 1 + 1 \otimes x$.

The $\Delta\mathfrak{p}$-integrability is equivalent to $\Delta\mathfrak{l}$-integrability and that $\Delta\mathfrak{r}$ acts locally nilpotently. Thus $\mathrm{Mod}(\mathfrak{r}, P, \mathrm{U}^{\lambda'})$ is equivalent to the category of $\mathrm{U}^{\lambda'} \otimes \mathrm{U}(\mathfrak{l})$-modules such that the action of $\Delta\mathfrak{l}$ is integrable and $\mathfrak{r} \subset \mathrm{U}^{\lambda'}$ acts nilpotently. Thus, using the principal anti-involution of $\mathfrak{l}$ to identify $\mathrm{U}^{\lambda'} \otimes \mathrm{U}(\mathfrak{l})$-modules with $\mathrm{U}^{\lambda'}$-$\mathrm{U}(\mathfrak{l})$-bimodules, we get $\mathrm{Mod}(\mathfrak{r}, P, \mathrm{U}^{\lambda'}) \cong {}_{\lambda'}\mathcal{H}(\mathfrak{l})$. From this one deduces the lemma. □

7.3. Bi-equivariant $\mathcal{D}$-modules and category O

We want to describe blocks in category O in terms of bi-equivariant $\mathcal{D}_G$-modules. Let $\lambda \in \mathfrak{h}^*$. Throughout this section we assume that $\lambda' \in \mathfrak{h}^*$ is a regular dominant weight such that $\lambda - \lambda'$ is integral.

Denote by $\mathrm{Mod}(\lambda', \mathfrak{n}, B, \mathcal{D}_G, P, \mathfrak{r}, \widehat{\lambda})$ the full subcategory of $\mathrm{Mod}(\mathcal{D}_G, P, \mathfrak{r}, \widehat{\lambda})$ whose object M satisfies (1)–(3), $\widehat{(4)}$ from Section 4.2 and is in addition equipped with a left B-action $\tau : B \to \mathrm{Aut}(M)$ that commutes with $\rho : P \to \mathrm{Aut}(M)^{\mathrm{op}}$ and satisfies

(5) $d\tau(x)m = (\epsilon_l(x) - \lambda'(x))m$ for $m \in M$ and $x \in \mathfrak{b}$.

(Strictly speaking, $\mathrm{Mod}(\lambda', \mathfrak{n}, B, \mathcal{D}_G, P, \mathfrak{r}, \widehat{\lambda})$ is obtained from $\mathrm{Mod}(\mathcal{D}_G, P, \mathfrak{r}, \widehat{\lambda})$ by adding a B-action, but since this B-action is determined by its differential, the former identifies with a subcategory of the latter.)

Lemma 7.2. *Assume that* λ *is* P*-regular. Then* $\mathrm{mod}(\lambda', \mathfrak{n}, B, \mathcal{D}_G, P, \mathfrak{r}, \widehat{\lambda}) \cong \mathrm{O}_{\widehat{\lambda}}$.

Proof. We recall that since λ is P-regular, restriction defines an equivalence of categories $res : \mathrm{Mod}^{\widehat{\lambda}}(\widetilde{\mathrm{U}}^{\mathcal{W}_P}) \xrightarrow{\sim} \mathrm{Mod}^{\widehat{\lambda}}(\mathrm{U})$. Now $(\widehat{4})$, the two lines preceding it and Theorem 5.2 give the equivalence

$$\mathrm{Mod}(\mathcal{D}_G, P, \mathfrak{r}, \widehat{\lambda}) \cong \mathrm{Mod}^{\widehat{\lambda}}(\mathrm{U}), \qquad V \mapsto res(V^P).$$

From this we deduce that the full subcategory $\mathrm{O}_{\widehat{\lambda}} = \mathrm{mod}^{\widehat{\lambda}}(\lambda', \mathfrak{n}, B, \mathrm{U})$ of $\mathrm{Mod}^{\widehat{\lambda}}(\mathrm{U})$ is equivalent to $\mathrm{mod}(\lambda', \mathfrak{n}, B, \mathcal{D}_G, P, \mathfrak{r}, \widehat{\lambda})$. □

By using the inversion on G, left B-action and right P-action become right B-action and left P-action, so $\mathrm{mod}(\lambda', \mathfrak{n}, B, \mathcal{D}_G, P, \mathfrak{r}, \widehat{\lambda})$ is equivalent to a full subcategory of $\mathrm{Mod}(\mathcal{D}_G, B, \mathfrak{n}, \lambda')$ that we denote by

$$\mathrm{mod}(\widehat{\lambda}, \mathfrak{r}, P, \mathcal{D}_G, B, \mathfrak{n}, \lambda') \tag{7.3}$$

and whose definition is obvious. Since λ' is dominant and regular we deduce from Beilinson–Bernstein localization that $\mathrm{Mod}(\mathcal{D}_G, B, \mathfrak{n}, \lambda') \cong \mathrm{Mod}(\mathrm{U}^{\lambda'})$. This induces an equivalence between (7.3) and $\mathrm{mod}(\widehat{\lambda}, \mathfrak{r}, P, \mathrm{U}^{\lambda'})$. (This is not the parabolic-singular Koszul duality of [BGS96].)

Similarly, if we do not pass to global sections on $\mathcal{B}$, we find that (7.3) is equivalent to the category $\mathrm{mod}(\widehat{\lambda}, \mathfrak{r}, P, \mathcal{D}^{\lambda'}_{\mathcal{B}})$, whose definition is also obvious.

Summarizing we get

Proposition 7.3. $\mathrm{O}_{\widehat{\lambda}} \cong \mathrm{mod}(\widehat{\lambda}, \mathfrak{r}, P, \mathrm{U}^{\lambda'}) \cong \mathrm{mod}(\widehat{\lambda}, \mathfrak{r}, P, \mathcal{D}^{\lambda'}_{\mathcal{B}})$ *for* λ *dominant and* P*-regular.*

Thus, by Lemma 7.1 we obtain

Corollary 7.4. $\mathrm{O}_{\widehat{\lambda}} \cong {}_{\lambda'}\mathcal{H}(\mathfrak{l})_{\widehat{\lambda}}$.

Similarly, one shows that $\mathrm{O}_{\lambda} \cong \mathrm{mod}(\lambda, \mathfrak{r}, P, \mathrm{U}^{\lambda'}) \cong \mathrm{mod}(\lambda, \mathfrak{r}, P, \mathcal{D}^{\lambda'}_{\mathcal{B}}) \cong {}_{\lambda'}\mathcal{H}(\mathfrak{l})_{\lambda}$.

Example 7.5. Let $P = B$ and let $\lambda \in \mathfrak{h}^*$ be regular and dominant. Then $\mathrm{O}_{\widehat{\lambda}} \cong \mathrm{mod}(\widehat{\lambda}, \mathfrak{n}, B, \mathrm{U}^{\lambda'})$, which is the category of left $\mathrm{U}^{\lambda'}$-modules which are locally finite over $\mathfrak{b}$ (so the $\mathfrak{h}$-action need not be semisimple). This equivalence was first established in [Soe86].

Example 7.6. Let $P = G$ and let $\lambda \in \mathfrak{h}^*$ be any weight. Since $\mathfrak{r}_G = 0$ we write for simplicity $\mathrm{Mod}(\widehat{\lambda}, G, \mathrm{U}^{\lambda'}) := \mathrm{Mod}(\widehat{\lambda}, \mathfrak{r}_G, G, \mathrm{U}^{\lambda'})$. Set $\mathrm{O}_{\widehat{\lambda+\Lambda}} := \bigoplus_{\mu\in\Lambda} \mathrm{O}_{\widehat{\lambda+\mu}}$. Then we have

$$\mathrm{O}_{\widehat{\lambda}} \xrightarrow{\sim} \mathrm{mod}(\widehat{\lambda}, G, \mathrm{U}^{\lambda'}) \quad \text{and} \quad \mathrm{O}_{\widehat{\lambda+\Lambda}} \xrightarrow{\sim} \mathrm{mod}(G, \mathrm{U}^{\lambda'}),$$

both given by $V \mapsto (\mathcal{O}_G \otimes V)^B$. Thus $\mathrm{O}_{\widehat{\lambda}} \cong {}_{\lambda'}\mathcal{H}_{\widehat{\lambda}}$. See [BeGe81], [Soe86].

Remark 7.7. $\mathrm{mod}(\widehat{\lambda}, \mathfrak{r}, P, \mathcal{D}^{\lambda'}_{\mathcal{B}})$ does *not* consist of holonomic $\mathcal{D}$-modules, unless $P = B$. For instance, if $\lambda = -\rho$, $P = G$ and $\lambda' = 0$, then $\mathrm{O}_{\widehat{-\rho}}$ consists of direct sums of copies of the simple Verma module $M_{-\rho}$. To $M_{-\rho}$ corresponds a non-holonomic submodule of the $\mathcal{D}_{\mathcal{B}}$-module $\mathcal{D}_{\mathcal{B}}$ (see (4.5)).

8. Whittaker modules

Let $f : \mathrm{U}(\mathfrak{n}) \to \mathbb{C}$ be an algebra homomorphism, $\Delta_f := \{\alpha \in \Delta : f(X_\alpha) \neq 0\}$ and $J_f := \operatorname{Ker} f$. Let $\widetilde{\mathcal{N}}_f := \widetilde{N}(\mathfrak{g})_f$ be the category of left U-modules on which J_f acts locally nilpotently, and let $\mathcal{N}_f$ be its subcategory of modules which are f.g. over U. Objects of $\mathcal{N}_f$ are called *Whittaker modules*. Replacing $\mathfrak{g}$ by $\mathfrak{l}$ and f by $f|_{\mathrm{U}(\mathfrak{n}\cap\mathfrak{l})}$ we get the category $\mathcal{N}_f(\mathfrak{l})$. For regular f, i.e. when $\Delta_f = \Delta$, it was studied by Kostant [K78]; he showed that $\mathcal{N}_f$ has the exceptionally simple description

$$\mathrm{Mod}(\mathrm{Z}) \xrightarrow{\sim} \mathcal{N}_f, \qquad M \mapsto M \otimes_{\mathrm{Z}} \mathrm{U} / \mathrm{U} \cdot J_f. \tag{8.1}$$

At the other extreme, when $f = 0$, $\mathcal{N}_f$ is O with the $\mathfrak{h}$-semisimplicity condition dropped and it has the same simple objects as O.

Our main result here is a new proof of Theorem 8.1 of [MS97]. It enables one to compute the characters of standard Whittaker modules by means of the Kazhdan–Lusztig conjectures. (For non-integral weights they were computed in [B97].)

Throughout this section we assume $\lambda \in \mathfrak{h}^*$ and $\Delta_P = \Delta_f = \Delta_\lambda$.

8.1. *Equivalence between a block of $\mathcal{N}_f$ and of singular* O

Fix a character $f : \mathrm{U}(\mathfrak{n}) \to \mathbb{C}$. For $\mu \in \mathfrak{h}^*$ we set

$${}_\mu\mathcal{N}_f := \{M \in \mathcal{N}_f : I_\mu M = 0\}, \quad {}_{\widehat{\mu}}\mathcal{N}_f := \{M \in \mathcal{N}_f : I_\mu \text{ acts locally nilpotently on } M\}.$$

(The categories ${}_\mu\widetilde{\mathcal{N}}_f$ and ${}_{\widehat{\mu}}\widetilde{\mathcal{N}}_f$ are defined similarly.) Our aim is to prove

Theorem 8.1. *Assume that $\lambda, \lambda' \in \Lambda$ are such that $\Delta_f = \Delta_\lambda$ and λ' is regular dominant. Then $\mathrm{O}_{\widehat{\lambda}} \cong {}_{\lambda'}\mathcal{N}_f$.*

Before proving this we establish some preliminary results.

Lemma 8.2. (i) *For any $\mu, \lambda \in \mathfrak{h}^*$ with μ dominant and such that $\mathcal{W}_\mu \subseteq \mathcal{W}_\lambda$, ${}_\mu\mathcal{H}_{\widehat{\lambda}}$ identifies with a finite length subcategory of $\mathcal{O}_{\widehat{\lambda}}$ which is non-zero iff $\lambda - \mu$ is integral (analogous statements hold with μ and/or λ replaced by $\widehat{\mu}$ and/or $\widehat{\lambda}$).*
(ii) ${}_\mu\mathcal{H}_{\widehat{-\rho}} \cong \mathrm{mod}(\mathbb{C})$ *and* ${}_\mu\widetilde{\mathcal{H}}_{\widehat{-\rho}} \cong \mathrm{Mod}(\mathbb{C})$*, for μ integral.*
(iii) $\mathcal{H}_{\mathrm{Z}\text{-fin}}$ *is a finite length category.*

Proof. That ${}_\mu\mathcal{H}_{\widehat{\lambda}} = 0$ if $\mu - \lambda$ is not integral is a consequence of the fact that every G-module is a sum of G-modules with integral central characters.

On the other hand, let $\mu - \lambda$ be integral and E be an irreducible G-module with extremal weight $\mu - \lambda$. For $M \in \mathcal{H}_\lambda$ we have $E \otimes M \in \mathcal{H}_\lambda$ with respect to the diagonal left U-action and the right U-action on the second factor. Thus, $T_\lambda^\mu M = pr_{\widehat{\mu}}(E \otimes M) \in {}_{\widehat{\mu}}\mathcal{H}_\lambda$ (and similarly with λ replaced by $\widehat{\lambda}$).

Now $\mathrm{U}^\lambda \in {}_\lambda\mathcal{H}_\lambda$ with its natural bimodule structure. Since $\mathcal{W}_\mu \subseteq \mathcal{W}_\lambda$ it is known that T_λ^μ is faithful. Hence $0 \neq T_\lambda^\mu(\mathrm{U}^\lambda) \in {}_{\widehat{\mu}}\mathcal{H}_\lambda$. Thus, also ${}_\mu\mathcal{H}_\lambda$ and ${}_\mu\mathcal{H}_{\widehat{\lambda}}$ are non-zero. We have

$$
\begin{aligned}
{}_\mu\mathcal{H}_{\widehat{\lambda}} &\cong \mathrm{mod}(\lambda, G, \mathrm{U}^\mu) \xrightarrow{\mathcal{L}} \mathrm{mod}(\lambda, G, \mathcal{D}_G, B, \mu) \\
&\qquad\qquad \cong \mathrm{mod}(\mu, B, \mathcal{D}_G, G, \widehat{\lambda}) \cong \mathrm{mod}^{\widehat{\lambda}}(\mu, B, \mathrm{U}) = \mathrm{O}_{\widehat{\lambda}}.
\end{aligned}
$$

Since μ is dominant we have $\Gamma \circ \mathcal{L} = \mathrm{Id}$. Since $\mathrm{O}_{\widehat{\lambda}}$ is a finite length category this implies ${}_\mu\mathcal{H}_{\widehat{\lambda}}$ is as well. This proves (i). Moreover, the fact that $\mathcal{O}_{\widehat{-\rho}} \cong \mathrm{mod}(\mathbb{C})$ now implies ${}_\mu\mathcal{H}_{\widehat{-\rho}} \cong \mathrm{mod}(\mathbb{C})$. A similar argument shows ${}_\mu\widetilde{\mathcal{H}}_{\widehat{-\rho}} \cong \mathrm{Mod}(\mathbb{C})$. This proves (ii).

By (7.2), $\mathcal{H}_{\text{Z-fin}} = {}_{\text{Z-fin}}\mathcal{H}_{\text{Z-fin}}$. Since ${}_\mu\mathcal{H}_\lambda$ is a finite length category for all $\mu, \lambda \in \mathfrak{h}^*$, a devissage implies (iii). □

Lemma 8.3. *Let* $\mu \in \Lambda$. *The functors* $\Theta_\mu := (\ ) \otimes_{\mathrm{U}(\mathfrak{n}\cap\mathfrak{l})} \mathbb{C}_f : {}_\mu\widetilde{\mathcal{H}}(\mathfrak{l}, \mathfrak{l})_{\widehat{\lambda}} \to {}_\mu\widetilde{\mathcal{N}}(\mathfrak{l})_f$ *and* $\Theta_{\widehat{\mu}} := (\ ) \otimes_{\mathrm{U}(\mathfrak{n}\cap\mathfrak{l})} \mathbb{C}_f : {}_{\widehat{\mu}}\widetilde{\mathcal{H}}(\mathfrak{l}, \mathfrak{l})_{\widehat{\lambda}} \to {}_{\widehat{\mu}}\widetilde{\mathcal{N}}(\mathfrak{l})_f$ *are equivalences of categories.*

Proof. This certainly holds for $\mathfrak{l} = \mathfrak{h}$ and from that we immediately reduce to the case $\mathfrak{g} = \mathfrak{l}$, $\Delta_f = \Delta$ and $\lambda = -\rho$. We must then show that the functor

$$\Theta_\mu : {}_\mu\widetilde{\mathcal{H}}_{\widehat{-\rho}} \to {}_\mu\widetilde{\mathcal{N}}_f, \qquad M \mapsto M \otimes_{\mathrm{U}(\mathfrak{n})} \mathbb{C}_f,$$

is an equivalence of categories. It follows from Kostant's equivalence (8.1) that ${}_\mu\widetilde{\mathcal{N}}_f$ is equivalent to $\mathrm{Mod}(\mathbb{C})$ (for all $\mu \in \mathfrak{h}^*$). By Lemma 8.2(ii) also ${}_\mu\widetilde{\mathcal{H}}_{\widehat{-\rho}} \cong \mathrm{Mod}(\mathbb{C})$; hence it suffices to show that Θ_μ takes simples to simples. The Θ_μ's commute with translation functors, so since $\mathrm{U}^{-\rho} \in {}_{-\rho}\mathcal{H}_{\widehat{-\rho}}$ we get

$$\Theta_\mu T_{-\rho}^\mu(\mathrm{U}^{-\rho}) = T_{-\rho}^\mu \Theta_{-\rho}(\mathrm{U}^{-\rho}) = T_{-\rho}^\mu(U^{-\rho} \otimes_{\mathrm{U}(\mathfrak{n})} \mathbb{C}_f).$$

By [K78] the latter is simple. This implies both that $T_{-\rho}^\mu(\mathrm{U}^{-\rho})$ is a simple generator for ${}_\mu\widetilde{\mathcal{H}}_{\widehat{-\rho}}$ and that Θ_μ takes simples to simples. Thus Θ_μ is an equivalence.

A devissage using Lemma 8.4 now shows that $\Theta_{\widehat{\mu}}$ is an equivalence. □

Lemma 8.4. *Each* $M \in \widetilde{\mathcal{H}}_{\widehat{-\rho}}$ *which is countably generated as a left* U*-module is faithfully flat as a right* $\mathrm{U}(\mathfrak{n})$*-module.*

Proof. Assume first that M is simple. Then it follows from Schur's lemma that $M \in {}_\mu\mathcal{H}_{\widehat{-\rho}}$ for some integral $\mu \in \mathfrak{h}^*$. By Lemma 8.2 we know that ${}_\mu\mathcal{H}_{\widehat{-\rho}} \cong \mathrm{mod}(\mathbb{C})$. Hence, $M \cong T_{-\rho}^\mu(\mathrm{U}^{-\rho})$ as this is simple (and hence a simple generator for ${}_\mu\mathcal{H}_{\widehat{-\rho}}$) by the proof of Lemma 8.3. By an adjunction argument, M is projective as a right $\mathrm{U}^{-\rho}$-module. By Kostant's separation of variables theorem [K63], $\mathrm{U}^{-\rho}$ is free over $\mathrm{U}(\mathfrak{n})$. Hence M is projective over $\mathrm{U}(\mathfrak{n})$.

Assume now that $M \in \mathcal{H}_{\widehat{-\rho}}$ is finitely generated. By Lemma 8.2, M has finite length and an induction on its length shows that M is again projective as a right $\mathrm{U}(\mathfrak{n})$-module.

For arbitrary M choose a filtration $M_0 \subseteq M_1 \subseteq \cdots \subseteq M$ of finitely generated submodules. Set $\overline{M_i} = M_i/M_{i-1}$. Since all M_i and $\overline{M_i}$ are projective, $M_i \cong \bigoplus_{j\leq i} M_j$ and thus

$$M = \varinjlim M_i \cong \varinjlim \bigoplus_{j\leq i} \overline{M_j} = \bigoplus_{i\in\mathbb{N}} \overline{M_i}$$

is projective, and therefore flat, as a right $\mathrm{U}(\mathfrak{n})$-module.

To see that M is faithful over $\mathrm{U}(\mathfrak{n})$, we observe that the above implies that M, as a right $\mathrm{U}(\mathfrak{n})$-module, is a direct sum of modules of the form $T^{\mu}_{-\rho}(\mathrm{U}^{-\rho})$, so it suffices to show that $T^{\mu}_{-\rho}(\mathrm{U}^{-\rho})$ is faithful over $\mathrm{U}(\mathfrak{n})$. Let $V \in \mathrm{Mod}(\mathrm{U}(\mathfrak{n}))$ be non-zero. We have

$$T^{\mu}_{-\rho}(U^{-\rho}) \otimes_{\mathrm{U}(\mathfrak{n})} V \cong T^{\mu}_{-\rho}(U^{-\rho} \otimes_{\mathrm{U}(\mathfrak{n})} V) \neq 0,$$

since $U^{-\rho} \otimes_{\mathrm{U}(\mathfrak{n})} V \neq 0$ and $T^{\mu}_{-\rho}$ is faithful (as $\mathcal{W}_\mu \subseteq \mathcal{W}_{-\rho}$). □

Lemma 8.5. *Let $\mu \in \Lambda$ and $M \in {}_{\widehat{\mu}}\mathcal{N}_f$. Then $M = \bigoplus_{\nu\in\Lambda} pr_{\mathfrak{l},\widehat{\nu}} M$.*

Proof. Note that M has a filtration $M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M$ such that each subquotient $\overline{M}_i := M_i/M_{i-1}$ is generated over U by a vector v_i such that $J_f \cdot v_i = I_\mu \cdot v_i = 0$. Thus each $\overline{M}_i$ is a quotient of a sum of copies of $\mathrm{U}^\mu / \mathrm{U}^\mu \cdot J_f$, and by [MS97] the latter has a filtration with subquotients of the form $\mathrm{U}^\mu / \mathrm{U}^\mu (I_{\mathfrak{l},w\cdot\mu} + J_f)$, $w \in \mathcal{W}$. These are in turn quotients of $\mathrm{U}^\mu / \mathrm{U}^\mu \cdot I_{\mathfrak{l},w\cdot\mu}$. Thus, it is enough to prove that

$$\mathrm{U}^\mu / \mathrm{U}^\mu \cdot I_{\mathfrak{l},w\cdot\mu} = \bigoplus_{\nu\in\Lambda} pr_{\mathfrak{l},\widehat{\nu}}\, \mathrm{U}^\mu / \mathrm{U}^\mu \cdot I_{\mathfrak{l},w\cdot\mu}, \qquad w \in \mathcal{W}.$$

Since ${}_{\widehat{\nu}}\mathcal{H}(\mathfrak{l},\mathfrak{l})_{w\cdot\mu} = 0$ for $\nu \notin w\cdot\mu + \Lambda = \Lambda$, and since $\mathrm{U}^\mu / \mathrm{U}^\mu \cdot I_{\mathfrak{l},w\cdot\mu} \in \widetilde{\mathcal{H}}(\mathfrak{l},\mathfrak{l})_{w\cdot\mu} = {}_{\mathrm{Z}(\mathfrak{l})\text{-fin}}\widetilde{\mathcal{H}}(\mathfrak{l},\mathfrak{l})_{w\cdot\mu}$, we are done. □

Proof of Theorem 8.1. We have $\mathrm{O}_{\widehat{\lambda}} \cong {}_{\lambda'}\mathcal{H}(\mathfrak{l})_{\widehat{\lambda}}$, so we need to construct an equivalence

$$\Theta : {}_{\lambda'}\mathcal{H}(\mathfrak{l})_{\widehat{\lambda}} \xrightarrow{\sim} {}_{\lambda'}\mathcal{N}_f, \qquad M \mapsto M \otimes_{\mathrm{U}(\mathfrak{n}\cap\mathfrak{l})} \mathbb{C}_f. \tag{8.2}$$

Consider the restriction functor $res : {}_{\lambda'}\mathcal{H}(\mathfrak{l})_{\widehat{\lambda}} \to \widetilde{\mathcal{H}}(\mathfrak{l},\mathfrak{l})_{\widehat{\lambda}}$. A "reductive version" of Lemma 8.4 applied to $\mathfrak{l}$ shows that each object of $\mathcal{H}(\mathfrak{l},\mathfrak{l})_{\widehat{\lambda}}$ is faithfully flat as a right $\mathrm{U}(\mathfrak{n}\cap\mathfrak{l})$-module. Hence, Θ is faithful and exact.

Denote by Ψ the right adjoint of Θ. Thus

$$\Psi V = \mathrm{Hom}_{\mathbb{C}}(\varprojlim_i \mathrm{U}(\mathfrak{l})/(I_{\mathfrak{l},\lambda})^i \otimes_{\mathrm{U}(\mathfrak{n}\cap\mathfrak{l})} \mathbb{C}_f, V)^{\mathfrak{l}\text{-int}},$$

where $(\ )^{\mathfrak{l}\text{-int}}$ is the functor that assigns a maximal $\mathfrak{l}$-integrable subobject. (The left U-module structure on ΨV comes from the left U-action on V, and its right $\mathrm{U}(\mathfrak{l})$-module structure comes from the left $\mathrm{U}(\mathfrak{l})$-action on $\varprojlim_i \mathrm{U}(\mathfrak{l})/(I_{\mathfrak{l},\lambda})^i \otimes_{\mathrm{U}(\mathfrak{n}\cap\mathfrak{l})} \mathbb{C}_f$.)

In order to prove that Θ is an equivalence it is enough to show that the natural transformation $\Theta\circ\Psi \to \mathrm{Id}$ is an isomorphism. Take $V \in {}_{\lambda'}\mathcal{N}_f$ and set

$$K := \mathrm{Ker}\{\Theta\Psi V \to V\}, \qquad C := \mathrm{Coker}\{\Theta\Psi V \to V\}.$$

By Lemma 8.5 we have $K = \bigoplus_{\nu\in\Lambda} pr_{\mathfrak{l},\widehat{\nu}}K$ and $C = \bigoplus_{\nu\in\Lambda} pr_{\mathfrak{l},\widehat{\nu}}C$. Let $\Psi_{\widehat{\nu}}$ be the right adjoint of the functor $\Theta_{\widehat{\nu}}$ from Lemma 8.3. Note that $pr_{\mathfrak{l},\widehat{\nu}}V \in {}_{\widehat{\nu}}\widetilde{\mathcal{N}}(\mathfrak{l})_f$ and that $pr_{\mathfrak{l},\widehat{\nu}}K = \mathrm{Ker}\{\Theta_{\widehat{\nu}}\Psi_{\widehat{\nu}}pr_{\mathfrak{l},\widehat{\nu}}V \to pr_{\mathfrak{l},\widehat{\nu}}V\}$ and $pr_{\mathfrak{l},\widehat{\nu}}C = \mathrm{Coker}\{\Theta_{\widehat{\nu}}\Psi_{\widehat{\nu}}pr_{\mathfrak{l},\widehat{\nu}}V \to pr_{\mathfrak{l},\widehat{\nu}}V\}$.

Assume $\nu \in \Lambda$. Then $\Theta_{\widehat{\nu}}$ is an equivalence of categories, by Lemma 8.3, and hence $pr_{\mathfrak{l},\widehat{\nu}}K = pr_{\mathfrak{l},\widehat{\nu}}C = 0$. Thus $K = C = 0$, by Lemma 8.5, and consequently Θ is an equivalence. □

8.2. *Singular and parabolic case*

Let $Q \subseteq G$ be a parabolic, $\mathfrak{q} := Lie\, Q$, $\mathcal{Q} := G/Q$ and $I^{\mathfrak{q}} := \mathrm{Ker}\{\mathrm{U} \to \mathcal{D}(G/Q)\}$. It is known that $I^{\mathfrak{q}} = \mathrm{Ann}_{\mathrm{U}}(\mathrm{U} \otimes_{\mathrm{U}(\mathfrak{q})} \mathbb{C})$, $\mathrm{U}/I^{\mathfrak{q}} \xrightarrow{\sim} \mathcal{D}(\mathcal{Q})$, and there is a parabolic version of (regular) Beilinson–Bernstein localization: $\mathrm{Mod}(\mathcal{D}_G, Q, \mathfrak{q}) \cong \mathrm{Mod}(\mathcal{D}(\mathcal{Q}))$ [BoBr82]. Let $\mathrm{O}^{\mathfrak{q}} := \{M \in \mathrm{O} : \mathfrak{q}$ acts locally finitely on $M\}$ be $\mathfrak{q}$-parabolic category O, $\mathrm{O}^{\mathfrak{q}}_{\lambda} := \mathrm{O}^{\mathfrak{q}} \cap \mathrm{O}_{\lambda}$ and $\mathrm{O}^{\mathfrak{q}}_{\widehat{\lambda}} := \mathrm{O}^{\mathfrak{q}} \cap \mathrm{O}_{\widehat{\lambda}}$.

All results from Section 7 extend to these categories. We assume here for simplicity that λ is integral and so we can take $\lambda' := 0$. Then

$$\mathrm{O}^{\mathfrak{q}}_{\lambda} = \mathrm{mod}(\mathfrak{q}, Q, \mathrm{U}^{\lambda}), \quad \mathrm{O}^{\mathfrak{q}}_{\widehat{\lambda}} = \mathrm{mod}^{\widehat{\lambda}}(\mathfrak{q}, Q, \mathrm{U}). \tag{8.3}$$

As before we get (with self-explanatory notation)

$$\begin{aligned}\mathrm{O}^{\mathfrak{q}}_{\widehat{\lambda}} &\cong \mathrm{mod}(\mathfrak{q}, Q, \mathcal{D}_G, P, \mathfrak{r}_P, \widehat{\lambda}) \\ &\cong \mathrm{mod}(\widehat{\lambda}, \mathfrak{r}_{\mathfrak{p}}, P, \mathcal{D}_G, Q, \mathfrak{q}) \cong \mathrm{mod}(\widehat{\lambda}, \mathfrak{r}_{\mathfrak{p}}, P, \mathcal{D}(\mathcal{Q})) \cong \mathcal{H}(\mathcal{D}(\mathcal{Q}), \mathfrak{l}_P)_{\widehat{\lambda}}.\end{aligned}$$

Here $\mathcal{H}(\mathcal{D}(\mathcal{Q}), \mathfrak{l}_P)_{\widehat{\lambda}}$ is the category of $\mathcal{D}(\mathcal{Q})$-$\mathrm{U}(\mathfrak{l}_P)$-bimodules on which the adjoint $\mathfrak{l}_P$-action is integrable, $I_{\mathfrak{l},\lambda}$ acts locally nilpotently from the right, and $\mathfrak{r}_P$ acts locally nilpotently from the left. Let $\mathcal{N}^{\mathfrak{q}}_f := \{M \in \mathcal{N}_f : I^{\mathfrak{q}}M = 0\}$. Thus the equivalence of Theorem 8.1 induces an equivalence

Corollary 8.6 ([W09]). $\mathrm{O}^{\mathfrak{q}}_{\widehat{\lambda}} \cong \mathcal{N}^{\mathfrak{q}}_f$.

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