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On an Abstract Differential Equation and Its Application to Positive Eigenvalues of Schrödinger Operators

By

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Abstract

A second order differential equation in Hilbert space is shown to have only trivial solutions. This functional analytic result is then used to derive an upper bound for the set of positive eigenvalues of the one-body Schrödinger operator.

§ 1. Introduction

The theory of the Schrödinger operator $-\Delta+q(x)$, where Δ is the Laplacian and $q(x)$ a measurable function on $\boldsymbol{R}^n$, initiated by Kato in [7], has become a highly developed discipline in mathematical analysis. The spectral theory of the Schrödinger operator is by now fairly well understood.

In [12], Wigner and von Neumann have constructed a function $q(x)$ such that $-\Delta+q$ has the positive number 1 as an eigenvalue. This phenomenon suggests that a natural spectral problem for the operator $-\Delta+q(x)$ is to give a good upper bound for the set of positive eigenvalues under reasonable assumptions on $q(x)$ in a neighbourhood of infinity. This problem has been studied by Agmon [1, 2], Kato [8], Odeh [9] and Simon [11] among others. An exposition of the contributions by these authors can be found in Reed and Simon [10], and most recently, Eastham and Kalf [5]. A discussion on the hypotheses and conclusions of the papers of Agmon and Simon can be found in Jansen and Kalf [6]. Eastham [4] has obtained an upper bound for the set of positive eigenvalues of the Sturm-Liouville operator $-d^2/dx^2+q(x)$ on $(0, \infty)$.

The object of this paper is to combine the techniques of Eastham in [4] and Jansen and Kalf in [6] to construct a second order differential equation in Hilbert space with only trivial solutions. The details are given in Sections 2 and 3. As an application, we derive an upper bound (similar to that of Eastham) for the set of positive eigenvalues of the Schrödinger operator $-\Delta+q(x)$. For technical reasons, we assume that the function $q(x)$ is so smooth that the regularity and

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unique continuation properties for the solutions of elliptic partial differential equations can be applied. Detailed assumptions on $q(x)$ are given in Section 4. Since the proofs are quite complicated, an attempt is made to supply with full details. In this respect the paper is sufficiently selfcontained.

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§ 2. An Abstract Differential Equation

Let $\mathcal{H}$ be a complex Hilbert space with inner product and norm denoted by $\langle\, ,\, \rangle$ and $\|\ \|$ respectively. Denote by $L^2(\mathcal{H})$ the space of all functions

$$u : (0, \infty) \longrightarrow \mathcal{H}$$

such that $u(t)$ is strongly measurable on $(0, \infty)$ and

$$\int_0^\infty \|u(t)\|^2 dt < \infty .$$

We denote by $C^k(\mathcal{H})$, $1 \leqq k < \infty$, the space of all functions

$$u : (0, \infty) \longrightarrow \mathcal{H}$$

such that its first k strong derivatives are strongly continuous on $(0, \infty)$.

Let $B(\mathcal{H})$ denote the set of all bounded linear operators on $\mathcal{H}$. With the usual definition of operator norm, $B(\mathcal{H})$ is a Banach space. The derivative of a function

$$T : (0, \infty) \longrightarrow B(\mathcal{H})$$

will also be taken in the strong sense.

Let A be a nonnegative, unbounded linear operator with domain $\mathcal{D}(A)$ dense in $\mathcal{H}$. Let $T_0(t)$ and $T_1(t)$ be self-adjoint bounded linear operators defined on $\mathcal{H}$ satisfying the following conditions:

(i) The derivative $T_0'(t)$ of $T_0(t)$ exists and is a self-adjoint bounded linear operator on $\mathcal{H}$ for t large enough, say for $t \geqq \xi_1$.

(ii) For any $\varepsilon > 0$, there is a $\xi_2 > 0$ such that for all functions $u : (0, \infty) \to \mathcal{H}$ and $v : (0, \infty) \to \mathcal{H}$, we have

$$|\langle T_0 u, v\rangle| \leqq \varepsilon \|u\| \|v\| , \tag{2.1}$$

$$|\langle T_1 u, v\rangle| \leqq \varepsilon \|u\| \|v\| , \tag{2.2}$$

whenever $t \geqq \xi_2$.

(iii) There exist nonnegative constants L and K such that for any $\varepsilon > 0$, there is a $\xi_3 > 0$ such that for all u, v as in (ii),

$$\langle tT_0'u,\ u\rangle \leqq (L+\varepsilon)\|u\|^2 \tag{2.3}$$

$$|\langle tT_1u,\ v\rangle| \leqq (K+\varepsilon)\|u\|\,\|v\| \tag{2.4}$$

whenever $t \geqq \xi_3$.

The abstract differential equation to be studied is of the following form:

$$v''(t)+\{\lambda-(t^{-2}A+T_0(t)+T_1(t))\}v(t)=0, \qquad 0<t<\infty. \tag{2.5}$$

where λ is a real number.

We assume that (2.5) satisfies the following unique continuation property:

(U. C. P.) If $v(t)\in C^2(\mathcal{H})\cap L^2(\mathcal{H})$ is a solution of (2.5) which is not identically zero, then there is a sequence $\{t_n\}$ of real numbers such that

$$\lim_{n\to\infty} t_n=\infty$$

and

$$\|v(t_n)\|>0,\ n=1,\ 2,\ \cdots.$$

Theorem 2.1. *The abstract differential equation* (2.5) *has no nonzero solutions in* $L^2(\mathcal{H})\cap C^2(\mathcal{H})$ *if*

$$\lambda>\frac{1}{2}\{K^2+L+K\sqrt{(2L+K^2)}\}.$$

Proof. We assume $K>0$ and $L>0$. Let $v\in C^2(\mathcal{H})\cap L^2(\mathcal{H})$ be a nonzero solution of (2.5). Define

$$h(t)=\|v'(t)\|^2+\langle(\lambda-t^{-2}A-T_0(t))v(t),\ v(t)\rangle, \quad t>\max(\xi_1,\ \xi_2,\ \xi_3). \tag{2.6}$$

Differentiating and applying (2.5), then for $t>\max(\xi_1,\ \xi_2,\ \xi_3)$,

$$h'(t)=-\langle T_0'(t)v(t),\ v(t)\rangle+2\mathrm{Re}\langle T_1(t)v(t),\ v'(t)\rangle+2t^{-3}\langle Av(t),\ v(t)\rangle. \tag{2.7}$$

By (2.3) and (2.4), the following inequality is valid:

$$h'(t)\geqq -t^{-1}\{2K_1\|v(t)\|\,\|v'(t)\|+L_1\|v(t)\|^2-2t^{-2}\langle Av(t),\ v(t)\rangle\} \tag{2.8}$$

where K_1 and L_1 lie within an arbitrary but fixed ε of K and L respectively and it is understood that (2.8) is valid for t large enough.

Let $c\in(-1,\ 1)$. Then

$$\begin{aligned} &-\{ct^{-1}\mathrm{Re}\langle v(t),\ v'(t)\rangle\}' \\ &=-\{ct^{-1}\mathrm{Re}\langle v(t),\ v''(t)\rangle+ct^{-1}\mathrm{Re}\langle v'(t),\ v'(t)\rangle-ct^{-2}\mathrm{Re}\langle v(t),\ v'(t)\rangle\} \\ &=-\{ct^{-1}\|v'(t)\|^2-ct^{-1}\langle v(t),\ (\lambda-t^{-2}A-T_0(t)-T_1(t))v(t)\rangle-ct^{-2}\mathrm{Re}\langle v(t),\ v'(t)\rangle\}. \end{aligned} \tag{2.9}$$

Adding (2.8) and (2.9), letting $\alpha>0$, we get

$$\begin{aligned} &\{h(t)-ct^{-1}\mathrm{Re}\langle v(t),\ v'(t)\rangle\}' \\ &\geqq -t^{-1}\{c\|v'\|^2+2K_2\|v\|\,\|v'\|+(L_2-c\lambda)\|v\|^2+(c-2)t^{-2}\langle Av,\ v\rangle\} \end{aligned} \tag{2.10}$$

$$\geqq -t^{-1}\{(c+\alpha K_2)\|v'\|^2+(L_2-c\lambda+\alpha^{-1}K_2)\|v\|^2+(c-2)t^{-2}\langle Av, v\rangle\}$$

where K_2 and L_2 are constants lying within ε of K and L respectively and it is understood that (2.10) is valid for t large enough.

By (2.6) and (2.10), it follows that for t large enough, say for $t\geqq T=T(\varepsilon)$,

$$\{h(t)-ct^{-1}\operatorname{Re}\langle v(t), v'(t)\rangle\}'\geqq -\gamma t^{-1}\{h(t)-ct^{-1}\operatorname{Re}\langle v(t), v'(t)\rangle\}$$

provided that

$$\gamma>c+\alpha K_3;\ \lambda\gamma>L_3-c\lambda+\alpha^{-1}K_3;\ 2-c>\gamma; \tag{2.11}$$

where K_3 and L_3 are constants lying within ε of K and L respectively.

(2.11) is valid with $0<\gamma<1$ provided that

$$(1-c)>\alpha K;\ \lambda-L+c\lambda>\alpha^{-1}K. \tag{2.12}$$

The inequalities in (2.12) are valid for some $\alpha>0$ if

$$(1-c)(\lambda-L+c\lambda)>K^2. \tag{2.13}$$

Using (2.13), the inequalities in (2.12) are valid for some $\alpha>0$ if

$$\lambda>K^2(1-c^2)^{-1}+L(1+c)^{-1}. \tag{2.14}$$

Considering $K^2(1-c^2)^{-1}+L(1+c)^{-1}$ as a function of $c\in(-1, 1)$ the minimum value is

$$\frac{1}{2}\{K^2+L+K\sqrt{(2L+K^2)}\} \tag{2.15}$$

and is attained at

$$c_0=L^{-1}\{K^2+L-K\sqrt{(2L+K^2)}\}. \tag{2.16}$$

Hence by (2.11), $\cdots$, (2.15) and (2.16), we have proved that there is a $\gamma\in(0, 1)$ such that

$$\{h(t)-c_0t^{-1}\operatorname{Re}\langle v(t), v'(t)\rangle\}'\geqq -\gamma t^{-1}\{h(t)-c_0t^{-1}\operatorname{Re}\langle v(t), v'(t)\rangle\} \tag{2.17}$$

for $t\geqq T=T(\varepsilon)$, provided that

$$\lambda>\frac{1}{2}\{K^2+L+K\sqrt{(2L+K^2)}\}.$$

CLAIM: *There is an $\eta\in(0, \infty)$ such that*

$$h(t)-c_0t^{-1}\operatorname{Re}\langle v(t), v'(t)\rangle>0,\quad t\geqq\eta. \tag{2.18}$$

The proof of (2.18) will be given in Section 3.

Let $\tau=\max(\eta, T)$. Then (2.17) and (2.18) imply that

$$\int_\tau^t \frac{\{h(s)-c_0 s^{-1}\mathrm{Re}\langle v(s),\ v'(s)\rangle\}'}{h(s)-c_0 s^{-1}\mathrm{Re}\langle v(s),\ v'(s)\rangle}\,ds \geqq -\gamma\int_\tau^t \frac{ds}{s}\,.$$

Hence

(2.19) $$h(t)-c_0 t^{-1}\mathrm{Re}\langle v(t),\ v'(t)\rangle \geqq c' t^{-\gamma},\quad t>\tau$$

where c' is a positive constant.

Recalling the definition of $h(t)$, we have

(2.20) $$\|v'(t)\|^2+\langle(\lambda-t^{-2}A-T_0(t))v(t),\ v(t)\rangle-c_0 t^{-1}\mathrm{Re}\langle v(t),\ v'(t)\rangle \geqq c' t^{-\gamma},\quad t>\tau.$$

By (2.1),

(2.21) $$\|v'(t)\|^2+\left\langle\left(\lambda_1-\frac{1}{2}t^{-2}A\right)v(t),\ v(t)\right\rangle \geqq C_1 t^{-\gamma}$$

where C_1 and λ_1 are positive constants and (2.21) holds for t large enough, say for $t\geqq\sigma>\tau$.

Since $v(t)\in L^2(\mathcal{H})$, we have

$$\int_0^\infty \|v(t)\|^2 dt<\infty\,.$$

Hence the function $\|v(t)\|^2$ on $(0,\ \infty)$ cannot be monotone increasing on any interval of the form $(\beta,\ \infty)$, $\beta\geqq 0$.

Since

$$\frac{d}{dt}\|v(t)\|^2=2\,\mathrm{Re}\langle v(t),\ v'(t)\rangle,$$

it follows that there exists a sequence of real numbers $\{t_n\}$, $t_n\geqq\sigma$, such that

(2.22) $$\lim_{n\to\infty} t_n=\infty$$

and

(2.23) $$\mathrm{Re}\langle v(t_n),\ v'(t_n)\rangle \leqq 0\,.$$

Hence by (2.5), we get

$$\begin{aligned}
&-\mathrm{Re}\langle v(t_n),\ v'(t_n)\rangle+\mathrm{Re}\langle v(\sigma),\ v'(\sigma)\rangle \\
&=-\int_\sigma^{t_n}\frac{d}{dt}\{\mathrm{Re}\langle v(t),\ v'(t)\rangle\}\,dt \\
&=-\int_\sigma^{t_n}\{\|v'(t)\|^2+\mathrm{Re}\langle v(t),\ v''(t)\rangle\}\,dt \\
&=-\int_\sigma^{t_n}\|v'(t)\|^2 dt+\int_\sigma^{t_n}\mathrm{Re}\langle v(t),\ (\lambda-t^{-2}A-T_0(t)-T_1(t))v(t)\rangle dt\,.
\end{aligned}$$

Hence by (2.1) and (2.23), we get

$$(2.24) \qquad -\int_{\sigma}^{t_n}\|v'(t)\|^2dt+\int_{\sigma}^{t_n}\langle v(t),\ (\lambda_1-t^{-2}A)v(t)\rangle dt \geqq \operatorname{Re}\langle v(\sigma),\ v'(\sigma)\rangle .$$

Integrating (2.21), we get

$$(2.25) \qquad \int_{\sigma}^{t_n}\|v'(t)\|^2dt+\int_{\sigma}^{t_n}\left\langle v(t),\ \left(\lambda_1-\frac{1}{2}t^{-2}A\right)v(t)\right\rangle dt \geqq C_2t_n^{1-\gamma}-C_2\sigma^{1-\gamma}$$

where C_2 is a positive constant.

Hence by (2.24) and (2.25), we have

$$\int_{\sigma}^{t_n}\left\langle v(t),\ \left(2\lambda_1-\frac{3}{2}t^{-2}A\right)v(t)\right\rangle dt \geqq C_2t_n^{1-\gamma}-C_2\sigma^{1-\gamma}+\operatorname{Re}\langle v(\sigma),\ v'(\sigma)\rangle .$$

Using $0<\gamma<1$ and (2.22), we get

$$(2.26) \qquad \lim_{n\to\infty}\int_{\sigma}^{t_n}\left\langle v(t),\ \left(2\lambda_1-\frac{3}{2}t^{-2}A\right)v(t)\right\rangle dt=\infty .$$

Since A is a nonnegative operator, (2.26) implies that

$$(2.27) \qquad \lim_{n\to\infty}\int_{\sigma}^{t_n}\|v(t)\|^2dt=\infty .$$

But (2.27) contradicts the fact that $v\in L^2(\mathscr{H})$. This proves the theorem.

Remark. By slightly modifying the proof of the theorem, the cases when $K=0$ or $L=0$ can be covered. We omit the details.

§3. Continuation of the Proof of Theorem 2.1

In this section we prove the claim in (2.18).

Proposition 3.1. *There exists an* $\eta\in(0,\ \infty)$ *such that*

$$\|v'(t)\|^2+\langle(\lambda-t^{-2}A-T_0(t))v(t),\ v(t)\rangle-c_0t^{-1}\operatorname{Re}\langle v(t),\ v'(t)\rangle>0$$

for $t\geqq\eta$, *provided that*

$$\lambda>\frac{1}{2}\{K^2+L+K\sqrt{(2L+K^2)}\}.$$

Proof. For $t\geqq T=T(\varepsilon)$, define

$$F(t)=\|v'(t)\|^2+\langle(\lambda-t^{-2}A-T_0(t))v(t),\ v(t)\rangle+(1-d)t^{-1}\operatorname{Re}\langle v(t),\ v'(t)\rangle,$$

where $d=1+c_0$. Then using (2.5),

$$\begin{aligned}(tF(t))'&=(2-d)\|v'\|^2+(2-d)t^{-2}\langle Av,\ v\rangle+2\operatorname{Re}\langle tT_1v,\ v'\rangle\\&\quad+\langle\{d(\lambda-T_0)-tT_0'+(1-d)T_1\}v,\ v\rangle\\&\geqq(2-d)\|v'\|^2+2\operatorname{Re}\langle tT_1v,\ v'\rangle+\langle\{d(\lambda-T_0)-tT_0'+(1-d)T_1\}v,\ v\rangle.\end{aligned}$$

We have used the nonnegativity of A to obtain the above inequality. Let $\varepsilon>0$.

Then for t large enough, we have

$$(tF(t))' \geqq (2-d)\|v'\|^2 - 2(K+\varepsilon)\|v\|\|v'\| + \{d\lambda - L - (d+2)\varepsilon\}\|v\|^2.$$

The discriminant of the quadratic expression

$$(2-d)\|v'\|^2 - 2(K+\varepsilon)\|v\|\|v'\| + \{d\lambda - L - (d+2)\varepsilon\}\|v\|^2$$

is given by

$$4(K+\varepsilon)^2 - 4(2-d)\{d\lambda - L - (d+2)\varepsilon\}$$
$$= 4\{K^2 - (1-c_0^2)\lambda + (1-c_0)L + f(\varepsilon)\}$$

where

$$f(\varepsilon) = 2K\varepsilon + \varepsilon^2 + (1-c_0)(3+c_0)\varepsilon.$$

Since

$$\lambda > \frac{1}{2}\{K^2 + L + K\sqrt{(2L+K^2)}\},$$

we can choose ε small enough so that

$$4\{K^2 - (1-c_0^2)\lambda + (1-c_0)L + f(\varepsilon)\}$$
$$< 4\left\{K^2 - \frac{1}{2}(1-c_0^2)[K^2 + L + K\sqrt{(2L+K^2)}] + (1-c_0)L\right\} = 0.$$

Hence for t large enough,

$$(tF(t))' \geqq 0. \tag{3.1}$$

Let $t \geqq 0$, $m \geqq 0$ and $\rho > 0$. Following Jansen and Kalf [4], we define

$$F(m, \rho, t) = \|v_m'\|^2 - t^{-2}\langle v_m, Av_m\rangle + \langle\{\lambda - T_0 - \rho t^{-1} + t^{-2}m(m+1)\}v_m, v_m\rangle$$

where

$$v_m = t^m v.$$

Using (2.5), we get

$$v_m'' = 2mt^{-1}v_m' - m(m+1)t^{-2}v_m + t^{-2}Av_m - (\lambda - T_0 - T_1)v_m.$$

Hence

$$\{t^2F(m, \rho, t)\}' = t\{2(2m+1)\|v_m'\|^2 + 2\operatorname{Re}\langle(tT_1-\rho)v_m, v_m'\rangle$$
$$+ \langle[2(\lambda - T_0) - tT_0' - \rho t^{-1}]v_m, v_m\rangle\}.$$

Let $\varepsilon > 0$. Then for t large enough,

$$\{t^2F(m, \rho, t)\}' \geqq t\{2(2m+1)\|v_m'\|^2 - 2(K+\varepsilon+\rho)\|v_m\|\|v_m'\| + (2\lambda - L - 3\varepsilon - \rho\varepsilon)\|v_m\|^2\}.$$

Choose a fixed ρ such that

$$(K+\rho)^2 - 2\{K^2 + L + K\sqrt{(2L+K^2)}\} + 2L < 0.$$

Hence for ε small enough,

$$(K+\varepsilon+\rho)^2-2(2m+1)(2\lambda-L-3\varepsilon-\rho\varepsilon)\leqq(K+\varepsilon+\rho)^2-2(2\lambda-L-3\varepsilon-\rho\varepsilon)<0.$$

Hence there is a $t'>0$ such that

$$\{t^2F(m,\ \rho,\ t)\}'\geqq 0,\quad t\geqq t',\ m\geqq 0. \tag{3.2}$$

By (U.C.P.), let $t_1\geqq t'$ be such that

$$\|v(t_1)\|>0.$$

Then

$$\begin{aligned}&t_1^{2-2m}F(m,\ \rho,\ t_1)\\&\quad\geqq-\langle v(t_1),\ Av(t_1)\rangle+m(m+1)\|v(t_1)\|^2+\langle t_1^2(\lambda-T_0(t_1)-\rho t_1^{-1})v(t_1),\ v(t_1)\rangle.\end{aligned}$$

Let $m_1>(1-d)/2$ be such that

$$t_1^{2-2m_1}F(m_1,\ \rho,\ t_1)>0.$$

Hence

$$t_1^2F(m_1,\ \rho,\ t_1)>0. \tag{3.3}$$

Then by (3.2) and (3.3), we have

$$t^2F(m_1,\ \rho,\ t)\geqq t_1^2F(m_1,\ \rho,\ t_1)>0,\quad t\geqq t_1.$$

Hence

$$t^{-2m_1}F(m_1,\ \rho,\ t)>0,\quad t\geqq t_1. \tag{3.4}$$

But

$$\begin{aligned}&t^{-2m_1}F(m_1,\ \rho,\ t)\\&=t^{-2m_1}\{\|v'_{m_1}\|^2-t^{-2}\langle v_{m_1},\ Av_{m_1}\rangle+\langle[\lambda-T_0-\rho t^{-1}+t^{-2}m_1(m_1+1)]v_{m_1},\ v_{m_1}\rangle\}\\&=\|v'\|^2+2m_1t^{-1}\operatorname{Re}\langle v',\ v\rangle-t^{-2}\langle v,\ Av\rangle+\langle(\lambda-T_0)v,\ v\rangle+(2m_1+1)m_1t^{-2}\|v\|^2-\rho t^{-1}\|v\|^2\\&\leqq\|v'\|^2+2m_1t^{-1}\operatorname{Re}\langle v',\ v\rangle-t^{-2}\langle v,\ Av\rangle+\langle(\lambda-T_0)v,\ v\rangle\end{aligned}$$

provided that we choose t so large that

$$(2m_1+1)m_1t^{-1}<\rho\ .$$

Hence there is a $t_2>t_1$ such that for $t\geqq t_2$,

$$\begin{aligned}&t^{-2m_1}F(m_1,\ \rho,\ t)\\&\quad\leqq\|v'\|^2+\langle(\lambda-t^{-2}A-T_0)v,\ v\rangle+(1-d)t^{-1}\operatorname{Re}\langle v',\ v\rangle+(d+2m_1-1)t^{-1}\operatorname{Re}\langle v',\ v\rangle\\&\quad=F(t)+(d+2m_1-1)t^{-1}\operatorname{Re}\langle v',\ v\rangle.\end{aligned} \tag{3.5}$$

Since $v\in L^2(\mathscr{H})$, $\|v\|^2$ cannot be monotone increasing on any interval of the form $(\beta,\ \infty)$, $\beta\geqq 0$. Hence there is a $t_3\geqq t_2$ such that

$$\mathrm{Re}\langle v'(t_3),\ v(t_3)\rangle \leqq 0\,.$$

By (3.4) and (3.5),

$$0 < t_3^{-2m_1} F(m_1,\ \rho,\ t_3) \leqq F(t_3)\,.$$

Hence

$$t_3 F(t_3) > 0\,. \tag{3.6}$$

By (3.1) and (3.6), we have

$$tF(t) > 0$$

for t large enough. This proves the proposition.

§ 4. The Positive Eigenvalues

In this section we use Theorem 2.1 to study the positive eigenvalues of the Schrödinger operator.

Theorem 4.1. *Let $q \in L^2_{\mathrm{loc}}(\boldsymbol{R}^n)$, $n \geqq 2$, be a real-valued function satisfying the following conditions:*

(1) *$q(x)$ is locally Hölder continuous on $\boldsymbol{R}^n$.*
(2) *$q(x) = q_0(x) + q_1(x)$, where $q_0(x)$ and $q_1(x)$ are real-valued continuous functions defined on $\boldsymbol{R}^n$.*
(3) *$\lim_{|x|\to\infty} q_0(x) = 0$ and $\lim_{|x|\to\infty} q_1(x) = 0$.*
(4) *The radial derivative $q_0'(x)$ of $q_0(x)$ exists.*
(5) *$\limsup_{|x|\to\infty} |x|\,|q_1(x)| = K$, $K < \infty$.*
(6) *$\limsup_{|x|\to\infty} |x|\, q_0'(x) = L$, $0 \leqq L < \infty$.*

Let P be a self-adjoint extension of the operator

$$-\Delta + q : C_0^\infty(\boldsymbol{R}^n) \longrightarrow L^2(\boldsymbol{R}^n)\,.$$

Then P has no eigenvalues in $(\Lambda,\ \infty)$, where

$$\Lambda = \frac{1}{2}\{K^2 + L + K\sqrt{(2L + K^2)}\}\,.$$

Remark. Following Agmon [1, 2], Jansen and Kalf [6] and Simon [11], we have assumed that the function $q(x)$ is fairly smooth. The smoothness conditions on $q(x)$ need only be imposed in a neighbourhood of infinity.

Proof. We first consider the case when $n \geqq 3$. Let $\lambda > \Lambda$ be an eigenvalue. Let u be a corresponding eigenfunction. By hypothesis (1), u can be assumed to be in $C^2(\boldsymbol{R}^n)$. Introducing polar coordinates $t = |x|$, $\xi = t^{-1}x$; writing $v(t,\ \xi) = t^{(n-1)/2} u(t,\ \xi)$ and letting

$$S^{n-1} = \{x \in \boldsymbol{R}^n :\ |x| = 1\}\,,$$

it follows that the function

$$(0, \infty) \ni t \longmapsto v(t, \cdot) \in L^2(S^{n-1})$$

is in $L^2(\mathcal{H}) \cap C^2(\mathcal{H})$, where $\mathcal{H} = L^2(S^{n-1})$. $\mathcal{H}$ is a Hilbert space with inner product

$$\langle f, g\rangle = \int_{S^{n-1}} f(\xi)\overline{g(\xi)}d\xi, \quad f, g \in L^2(S^{n-1}). \tag{4.1}$$

We also have

$$v'' + \{\lambda - t^{-2}(A+\mu) - q_0 - q_1\} v = 0, \quad 0 < t < \infty \tag{4.2}$$

where $\mu = \frac{1}{4}(n-1)(n-3)$; A is the negative of the Laplace Beltrami operator and is hence a nonnegative, unbounded linear operator on $L^2(S^{n-1})$.

Define operators $T_0(t)$ and $T_1(t)$ on $\mathcal{H}$ for $t>0$ by

$$(T_0(t)f)(\xi) = q_0(t, \xi)f(\xi), \qquad f \in \mathcal{H} \tag{4.3}$$

and

$$(T_1(t)f)(\xi) = q_1(t, \xi)f(\xi), \qquad f \in \mathcal{H}. \tag{4.4}$$

Using the real-valuedness, continuity and hypothesis (3) of q_0 and q_1, it follows that $T_0(t)$ and $T_1(t)$ are self-adjoint bounded linear operators on $\mathcal{H}$ for $t>0$. Using (4.3) and hypothesis (6), for t large enough,

$$(T_0'(t)f)(\xi) = q_0'(t, \xi)f(\xi), \qquad f \in \mathcal{H} \tag{4.5}$$

and is a self-adjoint bounded linear operator on $\mathcal{H}$. Let $\varepsilon>0$. Then there is a $t_1>0$ such that for all functions $\alpha: (0, \infty) \to \mathcal{H}$ and $\beta: (0, \infty) \to \mathcal{H}$,

$$|\langle T_0\alpha, \beta\rangle| \leqq \varepsilon\|\alpha\|\|\beta\|, \tag{4.6}$$

$$|\langle T_1\alpha, \beta\rangle| \leqq \varepsilon\|\alpha\|\|\beta\|, \tag{4.7}$$

$$\langle tT_0'\alpha, \alpha\rangle \leqq (L+\varepsilon)\|\alpha\|^2, \tag{4.8}$$

$$|\langle tT_1\alpha, \beta\rangle| \leqq (K+\varepsilon)\|\alpha\|\|\beta\|, \tag{4.9}$$

whenever $t \geqq t_1$.

We have used hypotheses (3), (5) and (6) to obtain the above inequalities. Rewrite (4.2) in the form

$$v'' + \{\lambda - t^{-2}(A+\mu) - T_0(t) - T_1(t)\} v = 0, \qquad 0 < t < \infty. \tag{4.10}$$

The unique continuation property for the operator

$$-\Delta + q - \lambda$$

(see, for example, Aronszajn [3]), implies that the abstract differential equation (4.10) satisfies the (U.C.P.) property formulated in Section 2. Hence by Theorem 2.1,

$$v(t, \cdot)\notin L^2(\mathcal{H})\cap C^2(\mathcal{H}).$$

This is a contradiction. Hence P has no eigenvalues in (Λ, ∞).

For the case when $n=2$, the above proof can be modified by writing (4.2) in the form

$$v''+\{\lambda-t^{-2}A-q_0-\tilde{q}_1\}v=0, \qquad 0<t<\infty$$

where

$$\tilde{q}_1(t, \xi)=q_1(t, \xi)-\frac{1}{4}t^{-2}$$

and replacing $T_1(t)$ by $\tilde{T}_1(t)$ where

$$(\tilde{T}_1(t)f)(\xi)=\tilde{q}_1(t, \xi)f(\xi), \qquad f\in\mathcal{H}.$$

This completes the proof of the theorem.

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