J. Eur. Math. Soc. 17, 2843–2886

DOI 10.4171/JEMS/573

Serge Bouc

The Roquette category of finite p-groups

Received April 21, 2013 and in revised form September 26, 2013

Abstract. Let p be a prime number. This paper introduces the *Roquette category* $\mathcal{R}_p$ of finite p-groups, which is an additive tensor category containing all finite p-groups among its objects. In $\mathcal{R}_p$, every finite p-group P admits a canonical direct summand ∂P, called the *edge* of P. Moreover P splits uniquely as a direct sum of edges of *Roquette p-groups*, and the tensor structure of $\mathcal{R}_p$ can be described in terms of such edges.

The main motivation for considering this category is that the additive functors from $\mathcal{R}_p$ to abelian groups are exactly the *rational p-biset functors*. This yields in particular very efficient ways of computing such functors on arbitrary p-groups: this applies to the representation functors R_K, where K is any field of characteristic 0, but also to the functor of units of Burnside rings, or to the torsion part of the Dade group.

Keywords. p-group, Roquette, rational, biset, genetic

1. Introduction

Let p be a prime number. This article introduces the *Roquette category* $\mathcal{R}_p$ of finite p-groups, which is an additive tensor category with the following properties:

- Every finite p-group can be viewed as an object of $\mathcal{R}_p$. The tensor product of two finite p-groups P and Q in $\mathcal{R}_p$ is the direct product $P \times Q$.
- In $\mathcal{R}_p$, any finite p-group has a direct summand ∂P, called the *edge* of P, such that

  $$P \cong \bigoplus_{N \trianglelefteq P} \partial(P/N).$$

  Moreover, if the center of P is not cyclic, then $\partial P = 0$.
- In $\mathcal{R}_p$, every finite p-group P decomposes as a direct sum

  $$P \cong \bigoplus_{R \in \mathcal{S}} \partial R,$$

  where $\mathcal{S}$ is a finite sequence of *Roquette groups*, i.e. p-groups of normal p-rank 1, and such a decomposition is essentially unique. Given the group P, such a decomposition

S. Bouc: CNRS-LAMFA, Université de Picardie – Jules Verne, 33 rue St Leu, 80039 Amiens Cedex 1, France; e-mail: serge.bouc@u-picardie.fr

Mathematics Subject Classification (2010): 18B99, 19A22, 20C99, 20J15

can be obtained explicitly from the knowledge of a *genetic basis* of P (Theorem 3.11 and Proposition 5.14).

- The tensor product $\partial P \times \partial Q$ of the edges of two Roquette p-groups P and Q is isomorphic to a direct sum of a certain number $\nu_{P,Q}$ of copies of the edge $\partial(P \diamond Q)$ of another Roquette group (where both $\nu_{P,Q}$ and $P \diamond Q$ are known explicitly—see Theorem 4.20 and Corollary 4.24).
- The additive functors from $\mathcal{R}_p$ to the category of abelian groups are exactly the *rational p-biset functors* introduced in [Bou05].

The latter is the main motivation for considering this category: any structural result on $\mathcal{R}_p$ will provide for free some information on such rational functors for p-groups, e.g. the representation functors R_K, where K is a field of characteristic 0 (see [Bou96], [Bou04], and L. Barker's article [Bar08]), the functor of units of Burnside rings [Bou07], or the torsion part of the Dade group [Bou06].

In particular, the above results on $\mathcal{R}_p$ yield isomorphisms describing the structure of some p-groups as objects of this category, and this is enough to compute the evaluations of rational p-biset functors. For example (Example 3.13), an elementary abelian p-group of rank n splits as

$$(C_p)^n \cong \mathbf{1} \oplus \frac{p^n - 1}{p - 1} \partial C_p$$

in $\mathcal{R}_p$. Similarly (equation (5.25)), in the category $\mathcal{R}_2$, the product of n copies of a dihedral group of order 8 splits as

$$(D_8)^n \cong \mathbf{1} \oplus (5^n - 1) \cdot \partial C_2.$$

More generally, Proposition 5.40 gives a formula for $(D_{2^m})^n$. A straightforward consequence, applying the functor $R_{\mathbb{Q}}$, is the following

Example 1.1. For any $n \in \mathbb{N}$, the group $(D_8)^n$ has 5^n conjugacy classes of cyclic subgroups.

Another important by-product of the above result giving the tensor structure of $\mathcal{R}_p$ is the explicit description of a genetic basis of a direct product $P \times Q$, in terms of a genetic basis of P and a genetic basis of Q (Theorem 5.20). This allows in particular for a quick computation of the torsion part of the Dade group of some p-groups, e.g. (Theorem 5.36(1) & (3)):

Example 1.2.
- Let P be an arbitrary finite direct product of groups of order 2 and dihedral 2-groups. Then the Dade group of any factor group of P is torsion free.
- Let n be a positive integer. For any integer $m \geq 4$, let $P = SD_{2^m}$ be a semidihedral group of order 2^m, and let P^{*n} denote the central product of n copies of P. Then the torsion part of the Dade group of P^{*n} is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{2^{(n-1)(m-3)}}$.

This also yields similar results on groups of units of Burnside rings of these groups (Remark 5.39), or on representations of central products of p-groups, as in Examples 5.34 and 5.35:

Example 1.3. Let p be a prime, let X be an extraspecial p-group, and let Q be a non-trivial p-group. Let K be a field of characteristic 0. Then Q has the same number (possibly 0) of isomorphism classes of faithful irreducible representations over K as any central product $X * Q$.

Another possibly interesting phenomenon is that some non-isomorphic p-groups may become isomorphic in the category $\mathcal{R}_p$. This means that some non-isomorphic p-groups cannot be distinguished using only rational p-biset functors. When $p = 2$, there are even examples where this occurs for groups of different orders (Example 5.16). When $p > 2$, saying that the p-groups P and Q are isomorphic in $\mathcal{R}_p$ is equivalent to saying that the group algebras $\mathbb{Q}P$ and $\mathbb{Q}Q$ have isomorphic centers (Proposition 5.17).

The category $\mathcal{R}_p$ is built as follows: consider first the category $\mathcal{R}_p^\sharp$, which is the quotient category of the biset category of finite p-groups (in which objects are finite p-groups and morphisms are virtual bisets) obtained by killing a specific element δ in the Burnside group of the Sylow p-subgroup of $\mathrm{PGL}(3, \mathbb{F}_p)$. Then take idempotent completion, and additive completion of the resulting category.

In particular, this construction relies on bisets, and related functors. Consequently, the paper is organized as follows: Section 2 is a (not so) quick summary of the background on biset functors, Roquette groups, genetic bases of p-groups, and rational p-biset functors. The category $\mathcal{R}_p$ is introduced in Section 3, and in Section 4, its tensor structure is described. Finally Section 5 gives some examples and applications.

2. Rational p-biset functors

2.1. Biset functors. The *biset category* $\mathcal{C}$ of finite groups is defined as follows:

- The objects of $\mathcal{C}$ are the finite groups.
- Let G and H be finite groups. Then

$$\mathrm{Hom}_{\mathcal{C}}(G, H) = B(H, G),$$

  where $B(H, G)$ denotes the Grothendieck group of the category of finite (H, G)-bisets, i.e. the Burnside group of the group $H \times G^{\mathrm{op}}$.
- Let G, H, and K be finite groups. The composition of morphisms

$$B(K, H) \times B(H, G) \to B(K, G)$$

  in the category $\mathcal{C}$ is the linear extension of the product induced by the product of bisets $(V, U) \mapsto V \times_H U$, where V is a (K, H)-biset, and U is an (H, G)-biset.
- The identity morphism of the finite group G is the image in $B(G, G)$ of the set G, endowed with its (G, G)-biset structure given by left and right multiplication.

Definition 2.2. A *biset functor* is an additive functor from $\mathcal{C}$ to the category of abelian groups. A *morphism* of biset functors is a natural transformation of functors.

Morphisms of biset functors can be composed, and the resulting category of biset functors is denoted by $\mathcal{F}$. It is an abelian category.

Example 2.3. 1. The correspondence B sending a finite group G to its Burnside group $B(G)$ is a biset functor, called the *Burnside functor*: indeed, $B(G) = \mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, G)$, so B is in fact the Yoneda functor $\mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, -)$.

2. The formalism of bisets gives a single framework for the usual operations of induction, restriction, inflation, deflation, and transport by isomorphism via the following correspondences:

- If H is a subgroup of G, then let $\mathrm{Ind}_H^G \in B(G, H)$ denote the set G with left action of G and right action of H by multiplication.
- If H is a subgroup of G, then let $\mathrm{Res}_H^G \in B(H, G)$ denote the set G with left action of H and right action of G by multiplication.
- If $N \trianglelefteq G$ and $H = G/N$, then let $\mathrm{Inf}_H^G \in B(G, H)$ denote the set H with left action of G by projection and multiplication, and right action of H by multiplication.
- If $N \trianglelefteq G$ and $H = G/N$, then let $\mathrm{Def}_H^G \in B(H, G)$ denote the set H with left action of H by multiplication, and right action of G by projection and multiplication.
- If $\varphi : G \to H$ is a group isomorphism, then let $\mathrm{Iso}_G^H = \mathrm{Iso}_G^H(\varphi) \in B(H, G)$ denote the set H with left action of H by multiplication, and right action of G by taking image under φ, and then multiplying in H.
- When H is a subgroup of G, let $\mathrm{Defres}_{N_G(H)/H}^G \in B(N_G(H)/H, G)$ denote the set $H\backslash G$ viewed as an $(N_G(H)/H, G)$-biset. It is equal to the composition $\mathrm{Def}_{N_G(H)/H}^{N_G(H)} \circ \mathrm{Res}_{N_G(H)}^G$.
- When H is a subgroup of G, let $\mathrm{Indinf}_{N_G(H)/H}^G \in B(G, N_G(H)/H)$ denote the set G/H viewed as a $(G, N_G(H)/H)$-biset. It is equal to the composition $\mathrm{Ind}_{N_G(H)}^G \circ \mathrm{Inf}_{N_G(H)/H}^{N_G(H)}$.

2.4. p-biset functors. From now on, the symbol p will denote a prime number.

Definition and Notation 2.5.
- The *biset category* $\mathcal{C}_p$ of finite p-groups is the full subcategory of $\mathcal{C}$ consisting of finite p-groups.
- A p-biset functor is an additive functor from $\mathcal{C}_p$ to the category of abelian groups. A morphism of p-biset functors is a natural transformation of functors.
- The p-biset functors form an abelian category $\mathcal{F}_p$.

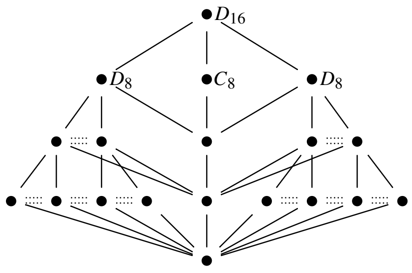
2.6. Roquette p-groups

Definition 2.7. A finite group G is called a *Roquette group* if it has normal rank 1, i.e. all the normal abelian subgroups of G are cyclic.

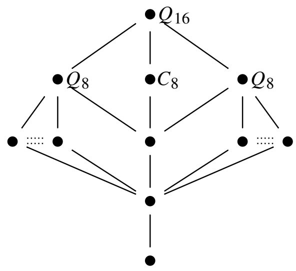
The Roquette p-groups have been first classified by… Roquette [Roq58] (see also [Gor68]): these are the cyclic groups if $p > 2$, and Roquette 2-groups are the cyclic groups, the generalized quaternion groups, the dihedral and semidihedral groups of order at least 16.

More generally, the p-hyperelementary Roquette groups have been classified by Hambleton, Taylor, and Williams [HTW90, Theorem 3.A.6].

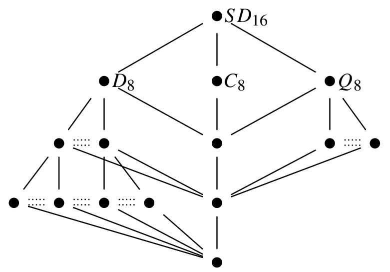
The following schematic diagram represents the lattice of subgroups of the dihedral group D_{16}, the quaternion group Q_{16}, and the semidihedral group SD_{16} (a horizontal dotted link between two vertices means that the corresponding subgroups are conjugate):



D_{16}



Q_{16}



SD_{16}

These diagrams give a good idea of the general case.

Definition 2.8. Let G be a finite group, of exponent e. An *axis* of G is a cyclic subgroup of order e in G. An *axial* subgroup of G is a subgroup of an axis of G.

With these definitions, let us recall without proof the following properties of Roquette p-groups:

Lemma 2.9. *Let P be a non-trivial Roquette p-group, of exponent e_P.*

(1) *The center of P is cyclic, hence P admits a unique central subgroup Z_P of order p.*
(2) *There exists a non-trivial subgroup Q of P such that $Q \cap Z(P) = \mathbf{1}$ if and only if $p = 2$ and P is dihedral or semidihedral. In this case moreover $|Q| = 2$ and $N_P(Q) = QZ_P$.*
(3) *If P is not cyclic, then $p = 2$ and $e_P = |P|/2$.*
(4) *There is a unique axis in P, except in the case $P \cong Q_8$, where there are three of them. Any axis of P is normal in P.*
(5) *If R is a non-trivial axial subgroup of P, then $R \geq Z_P$ and $R \trianglelefteq P$. If moreover $|R| \geq p^2$, then $C_P(R)$ is the only axis of P containing R.*

Let us also recall the following:

Lemma 2.10 ([Bou10, Proposition 9.3.5]). *Let P be a finite Roquette p-group. Then there is a unique simple faithful $\mathbb{Q}P$-module Φ_P, up to isomorphism.*

Example 2.11. Let P be a cyclic group of order p^m, and suppose first that $m \geq 1$. The algebra $\mathbb{Q}P$ is isomorphic to the algebra $A = \mathbb{Q}[X]/(X^{p^m} - 1)$. As

$$X^{p^m} - 1 = (X^{p^{m-1}} - 1)\Psi_{p^m}(X),$$

where Ψ_{p^m} denotes the p^m-th cyclotomic polynomial, it follows that there is a split exact sequence of A-modules

$$0 \to \mathbb{Q}[X]/(\Psi_{p^m}) \to A \to \mathbb{Q}[X]/(X^{p^{m-1}} - 1) \to 0,$$

which can be viewed as a sequence of $\mathbb{Q}P$-modules

$$0 \to \Phi_P \to \mathbb{Q}P \to \mathbb{Q}(P/Z) \to 0,$$

where Z is the unique subgroup of order p of P. It follows that there is an isomorphism of $\mathbb{Q}$-algebras

$$\mathrm{End}_{\mathbb{Q}P}\,\Phi_P \cong \mathbb{Q}[X]/(\Psi_{p^m}) \cong \mathbb{Q}(\zeta_{p^m}),$$

where ζ_{p^m} is a primitive p^m-th root of unity in $\mathbb{C}$.

Now if $m = 0$, then $P = \mathbf{1}$, $\Phi_P = \mathbb{Q}$, and $\mathrm{End}_{\mathbb{Q}P}\,\Phi_P \cong \mathbb{Q}$ too.

2.12. Expansive and genetic subgroups

Definition 2.13. A subgroup H of a group G is called *expansive* if for any $g \in G$ such that $H^g \neq H$, the group $(H^g \cap N_G(H))H/H$ contains a non-trivial normal subgroup of $N_G(H)/H$, i.e.

$$g \in G - N_G(H) \;\Rightarrow\; \bigcap_{n \in N_G(H)} (H^{gn} \cap N_G(H))H > H.$$

Example 2.14. If $H \trianglelefteq G$, then H is expansive in G. More generally, if $N_G(H) \trianglelefteq G$, then H is expansive in G. Indeed, $N_G(H^g) = N_G(H)$ for any $g \in G$. Hence for g in $G - N_G(H)$,

$$\bigcap_{n \in N_G(H)} (H^{gn} \cap N_G(H))H = (H^g \cap N_G(H))H = H^g \cdot H > H.$$

Notation 2.15. When H is a subgroup of the group G, denote by $Z_G(H)$ the subgroup of $N_G(H)$, containing H, defined by

$$Z_G(H)/H = Z(N_G(H)/H).$$

The following is an easy consequence of well known properties of p-groups:

Lemma 2.16 ([Bou10, Lemma 9.5.2]). *Let Q be a subgroup of a finite p-group P. Then Q is expansive in P if and only if*

$$\forall g \in P, \quad Q^g \cap Z_P(Q) \leq Q \Rightarrow Q^g = Q.$$

Example 2.17. Let P be a p-group, and let

$$\Delta(P) = \{(x, x) \in P \times P \mid x \in P\}$$

denote the diagonal subgroup of $P \times P$. Then $N_{P\times P}(\Delta(P))/\Delta(P) \cong Z(P)$, and $\Delta(P)$ is expansive in $P \times P$ if and only if

$$\forall x \in P, \quad [P, x] \cap Z(P) = \{1\} \Rightarrow x \in Z(P),$$

where $[P, x]$ is the *set* of commutators $[y, x] = y^{-1}x^{-1}yx = y^{-1}y^x$ for $y \in P$.

In particular, if $[P, P] \leq Z(P)$, then $\Delta(P)$ is expansive in $P \times P$.

Proof. Indeed, $N_{P\times P}(\Delta(P))$ consists of pairs $(a, b) \in P \times P$ such that $ab^{-1} \in Z(P)$. This shows that $N_{P\times P}(\Delta(P))/\Delta(P) \cong Z(P)$, and that

$$Z_{P\times P}(\Delta(P)) = N_{P\times P}(\Delta(P)) = (\mathbf{1} \times Z(P))\Delta(P).$$

Now $\Delta(P)^{(u,v)} = \Delta(P)^{(1,x)}$ for any $(u, v) \in P \times P$, where $x = u^{-1}v$, and

$$\Delta(P)^{(1,x)} \cap Z_{P\times P}(\Delta(P)) = \{(t, t^x) \mid t \in P,\ t^{-1}t^x \in Z(P)\}.$$

Hence $\Delta(P)^{(1,x)} \cap Z_{P\times P}(\Delta(P)) \leq \Delta(P)$ if and only if for any $t \in P$, the assumption $t^{-1}t^x \in Z(P)$ implies $t = t^x$, i.e. $[t, x] = 1$, in other words if $[P, x]\cap Z(P) = \{1\}$. Hence $\Delta(P)$ is expansive in $P \times P$ if and only if for any $x \in P$, the assumption $[P, x]\cap Z(P) = \{1\}$ implies $(1, x) \in N_{P\times P}(\Delta(P))$, i.e. $x \in Z(P)$, as claimed. The last assertion follows trivially. □

Definition 2.18. Let Q be a subgroup of the finite p-group P. Then Q is called a *genetic* subgroup of P if Q is expansive in P and $N_P(Q)/Q$ is a Roquette group.

Definition 2.19. Define a relation $\frown_P$ on the set of subgroups of the finite p-group P by

$$Q \frown_P R \Leftrightarrow \exists g \in P,\ Q^g \cap Z_P(R) \leq R \text{ and } {}^gR \cap Z_P(Q) \leq Q.$$

Lemma 2.20. *Let P be a finite p-group. If Q and R are subgroups of P such that $N_P(Q) = N_P(R) \trianglelefteq P$, then*

$$Q \frown_P R \Leftrightarrow Q =_P R.$$

Proof. Indeed, since P is a p-group, saying that $Q^g \cap Z_P(R) \leq R$ is equivalent to saying that the subgroup $(Q^g \cap N_P(R))R/R$ of $N_P(R)/R$ contains no non-trivial normal subgroup of $N_P(R)/R$, i.e.

$$\bigcap_{n\in N_P(R)} (Q^{gn} \cap N_P(R))R = R.$$

But if $N_P(Q) \trianglelefteq P$, then $N_P(Q) = N_P(Q^g) = N_P(R)$ for any $g \in G$. Hence

$$\bigcap_{n \in N_P(R)} (Q^{gn} \cap N_P(R))R = Q^g \cdot R.$$

This is equal to R if and only if $Q^g \leq R$. Similarly ${}^gR \cap Z_P(Q) \leq Q$ if and only if ${}^gR \leq Q$. Hence $Q^g = R$. □

Definition 2.21. Let G be a group.

- A *section* of G is a pair (T, S) of subgroups of G such that $S \trianglelefteq T$. The quotient T/S is called the corresponding *subquotient* of G.
- Two sections (T, S) and (Y, X) of G are said to be *linked* (notation $(T, S) \;\text{—}\; (Y, X)$) if

$$S(T \cap Y) = T, \qquad X(T \cap Y) = Y, \qquad T \cap X = S \cap Y.$$

  They are said to be *linked modulo* G (notation $(T, S) \;\text{—}_G\; (Y, X)$) if there exists $g \in G$ such that $(T, S) \;\text{—}\; ({}^gY, {}^gX)$.

Observe in particular that if $(T, S) \;\text{—}_G\; (Y, X)$, then the corresponding subquotients T/S and Y/X are isomorphic.

Theorem 2.22 ([Bou10, Theorem 9.6.1]). *Let P be a finite p-group.*

(1) *If S is a genetic subgroup of P, then the module*

$$V(S) = \mathrm{Ind}_{N_P(S)}^P \mathrm{Inf}_{N_P(S)/S}^{N_P(S)} \Phi_{N_P(S)/S}$$

is a simple $\mathbb{Q}P$-module. Moreover, the functor $\mathrm{Ind}_{N_P(S)}^P \mathrm{Inf}_{N_P(S)/S}^{N_P(S)}$ induces an isomorphism of $\mathbb{Q}$-algebras

$$\mathrm{End}_{QP}\, V(S) \cong \mathrm{End}_{\mathbb{Q}N_P(S)/S}\, \Phi_{N_P(S)/S}.$$

(2) *If V is a simple $\mathbb{Q}P$-module, then there exists a genetic subgroup S of P such that $V \cong V(S)$.*

(3) *If S and T are genetic subgroups of P, then*

$$V(S) \cong V(T) \;\Leftrightarrow\; S \,\widehat{=}_P\, T \;\Leftrightarrow\; (N_P(S), S) \;\text{—}_P\; (N_P(T), T).$$

In particular, if $S \,\widehat{=}_P\, T$, then $N_P(S)/S \cong N_P(T)/T$. Moreover, the relation $\widehat{=}_P$ is an equivalence relation on the set of genetic subgroups of P, and the corresponding set of equivalence classes is in one-to-one correspondence with the set of isomorphism classes of simple $\mathbb{Q}P$-modules.

Definition 2.23. Let P be a finite p-group. A *genetic basis* of P is a set of representatives of equivalence classes of genetic subgroups of P for the relation $\widehat{=}_P$.

2.24. Faithful elements (cf. [Bou10, Sections 6.2 and 6.3]). Let G be a finite group. If N is a normal subgroup of G, recall from Example 2.3 that $\mathrm{Inf}_{G/N}^G$ denotes the set G/N viewed as a $(G, G/N)$-biset for the actions given by (projection to the factor group and) multiplication in G/N. Similarly $\mathrm{Def}_{G/N}^G$ denotes the same set G/N considered as a $(G/N, G)$-biset.

There is an isomorphism of $(G/N, G/N)$-bisets

$$\mathrm{Id}_{G/N} \cong \mathrm{Def}_{G/N}^G \circ \mathrm{Inf}_{G/N}^G. \tag{2.25}$$

More generally, if M and N are normal subgroups of G, there is an isomorphism of $(G/M, G/N)$-bisets

$$\mathrm{Def}_{G/M}^G \circ \mathrm{Inf}_{G/N}^G = \mathrm{Inf}_{G/MN}^{G/M} \circ \mathrm{Def}_{G/MN}^{G/N}. \tag{2.26}$$

It follows that if j_N^G is defined by

$$j_N^G = \mathrm{Inf}_{G/N}^G \circ \mathrm{Def}_{G/N}^G,$$

then $j_M^G \circ j_N^G = j_{MN}^G$. In particular j_N^G is an idempotent of $B(G, G)$. Moreover, by a standard orthogonalization procedure, the elements f_N^G defined for $N \trianglelefteq G$ by

$$f_N^G = \sum_{N \leq M \trianglelefteq G} \mu_{\trianglelefteq G}(N, M) j_M^G,$$

where $\mu_{\trianglelefteq G}(N, M)$ is the Möbius function of the poset of normal subgroups of G, are orthogonal idempotents of $B(G, G)$, and their sum is equal to $\mathrm{Id}_{G/N}$. The idempotent $f_{\mathbf{1}}^G$ is of special importance:

Lemma 2.27. *Let G be a finite group, and N be a normal subgroup of G.*

(1) $f_N^G = \mathrm{Inf}_{G/N}^G \circ f_{\mathbf{1}}^{G/N} \circ \mathrm{Def}_{G/N}^G$ *in* $B(G, G)$.
(2) *If* $N \neq 1$, *then* $\mathrm{Def}_{G/N}^G \circ f_{\mathbf{1}}^G = 0$ *in* $B(G/N, G)$, *and* $f_{\mathbf{1}}^G \circ \mathrm{Inf}_{G/N}^G = 0$ *in* $B(G, G/N)$.

Proof. Assertion (1) is [Bou10, Remark 6.2.9], and (2) is a special case of Proposition 6.2.6. □

If F is a biset functor, the set $\partial F(G)$ of *faithful elements* of $F(G)$ is defined by

$$\partial F(G) = F(f_{\mathbf{1}}^G) F(G).$$

It can be shown [Bou10, Lemma 6.3.2] that

$$\partial F(G) = \bigcap_{\mathbf{1} < N \trianglelefteq G} \mathrm{Ker}\, F(\mathrm{Def}_{G/N}^G).$$

Example 2.28. Let $F = R_K$ be the representation functor over a field K of characteristic 0. Then for a finite group G, the group $\partial R_K(G)$ is the direct summand of $R_K(G)$ with basis the set of (isomorphism classes of) faithful irreducible KG-modules.

Lemma 2.29. *Let G be a group, and S be a subgroup of G such that $S \cap Z(G) \neq \mathbf{1}$. Then* $\mathrm{Defres}^G_{N_G(S)/S} f^G_{\mathbf{1}} = 0$ *in* $B(N_G(S)/S, G)$.

Proof. Set $N = S \cap Z(G)$, $\overline{G} = G/N$, and $\bar{S} = S/N$. Then

$$\mathrm{Defres}^G_{N_G(S)/S} = \mathrm{Defres}^{\overline{G}}_{N_{\overline{G}}(\bar{S})/\bar{S}} \mathrm{Def}^G_{G/N} .$$

Now $\mathrm{Def}^G_{G/N} f^G_{\mathbf{1}} = 0$ if $N \neq \mathbf{1}$, by Lemma 2.27. □

Theorem 2.30 ([Bou10, Theorem 10.1.1]). *Let P be a finite p-group, and $\mathcal{B}$ be a genetic basis of P. Then, for any p-biset functor F, the map*

$$\mathcal{I}_{\mathcal{B}} = \bigoplus_{S \in \mathcal{B}} \mathrm{Indinf}^P_{N_P(S)/S} : \bigoplus_{S \in \mathcal{B}} \partial F(N_P(S)/S) \to F(P)$$

is split injective. A left inverse is the map

$$\mathcal{D}_{\mathcal{B}} = \bigoplus_{S \in \mathcal{B}} f^{N_P(S)/S}_{\mathbf{1}} \circ \mathrm{Defres}^P_{N_P(S)/S} : F(P) \to \bigoplus_{S \in \mathcal{B}} \partial F(N_P(S)/S).$$

One can show [Bou10, Lemma 10.1.2] that if $\mathcal{B}$ and $\mathcal{B}'$ are genetic bases of P, the map $\mathcal{I}_{\mathcal{B}}$ is an isomorphism if and only if $\mathcal{I}_{\mathcal{B}'}$ is. This motivates the following definition:

Definition 2.31. A p-biset functor F is called *rational* if for any finite p-group P, there exists a genetic basis $\mathcal{B}$ of P such that the map $\mathcal{I}_{\mathcal{B}}$ is an isomorphism.

So F is rational if and only if for any finite p-group P and *any* genetic basis $\mathcal{B}$ of P, the map $\mathcal{I}_{\mathcal{B}}$ is an isomorphism.

Example 2.32. • The functor $R_{\mathbb{Q}}$ of rational representations, which sends the finite p-group P to the group $R_{\mathbb{Q}}(P)$, is a rational p-biset functor. This example is of course the reason for calling the p-biset functors of Definition 2.31 *rational*. This choice has proved rather unfortunate, since the p-biset functor $R_{\mathbb{C}}$ of *complex* representations is also a rational functor... More generally, if K is a field of characteristic 0, then the functor R_K is a rational p-biset functor.

• The functor of units of the Burnside ring, sending a p-group P to the group of units $B^{\times}(P)$ of its Burnside ring, is a rational p-biset functor (see [Bou07]).

• Let k be a field of characteristic p. The correspondence sending a finite p-group P to the torsion part $D^t_k(P)$ of the Dade group of P over k is not a biset functor in general, because of phenomena of *Galois twists*, but still the maps $\mathcal{I}_{\mathcal{B}}$ and $\mathcal{D}_{\mathcal{B}}$ can be defined for D^t_k, and Theorem 2.30 holds (see [Bou06]).

3. The Roquette category

Notation 3.1. Let π be a projective plane over $\mathbb{F}_p$, and let X denote a Sylow p-subgroup of $\mathrm{Aut}(\pi) \cong \mathrm{PGL}(3, \mathbb{F}_p)$. Let $\mathbb{L}$ be the set of lines of π, and let $\mathbb{P}$ be the set of points of π, both viewed as elements of $B(X)$. Let $\delta = \mathbb{L} - \mathbb{P} \in B(X)$. Equivalently

$$\delta = (X/I - X/IZ) - (X/J - X/JZ),$$

where I and J are non-conjugate non-central subgroups of order p of X, and Z is the center of X.

Let B_δ denote the p-biset subfunctor of B generated by δ.

Remark 3.2. When $p = 2$, the group X is dihedral of order 8, and δ is well defined up to sign. When $p > 2$, the group X is an extraspecial p-group of order p^3 and exponent p, and there are several possible choices for the element δ. However, in any case, the functor B_δ does not depend on the choice of δ.

Definition 3.3. The *Roquette category* $\mathcal{R}_p$ of finite p-groups is defined as the *idempotent additive completion* of the category $\mathcal{R}_p^\sharp$, quotient of the biset category $\mathcal{C}_p$, defined as follows:

- The objects of $\mathcal{R}_p^\sharp$ are the finite p-groups.
- If P and Q are finite p-groups, then

  $$\mathrm{Hom}_{\mathcal{R}_p^\sharp}(P, Q) = (B/B_\delta)(Q, P)$$

  is the quotient of $B(Q \times P^{\mathrm{op}})$ by $B_\delta(Q \times P^{\mathrm{op}})$.
- The composition in $\mathcal{R}_p^\sharp$ is induced by the composition of bisets.
- The identity morphism of the finite p-group P in $\mathcal{R}_p^\sharp$ is the image of Id_P in the group $(B/B_\delta)(P, P)$.

Remark 3.4. It was shown in [Bou08] that $\mathcal{R}_p^\sharp$ is indeed a category. It was also shown there that if $p > 2$, the functor B_δ is equal to the kernel K of the linearization morphism $B \to R_\mathbb{Q}$. It follows that in this case, for any two finite p-groups P and Q,

$$\mathrm{Hom}_{\mathcal{R}_p^\sharp}(P, Q) \cong R_\mathbb{Q}(Q \times P^{\mathrm{op}})$$

is isomorphic to the Grothendieck group of $(\mathbb{Q}Q, \mathbb{Q}P)$-bimodules, or equivalently, by the Ritter–Segal theorem, to the Grothendieck group of the subcategory of $(\mathbb{Q}Q, \mathbb{Q}P)$-permutation bimodules. In other words, in this case the category $\mathcal{R}_p^\sharp$ is the full subcategory of the category considered by Barker [Bar08], consisting of finite p-groups. The construction of the category $\mathcal{R}_p^\sharp$ is also very similar to the construction of the category $\mathbb{Q}G$-Morita by Hambleton, Taylor, and Williams [HTW90, Definition 1.A.4].

In the case $p = 2$, the situation is more complicated: the functor B_δ is a proper subfunctor of the kernel K, and there is a short exact sequence

$$0 \to K/B_\delta \to B/B_\delta \to R_\mathbb{Q} \to 0$$

of p-biset functors. Moreover, for each p-group P, the group $(K/B_\delta)(P)$ is a finite elementary abelian 2-group of rank equal to the number of groups S in a genetic basis of P for which $N_P(S)/S$ is dihedral.

Lemma 3.5. *The direct product* $(P, Q) \mapsto P \times Q$ *of* p*-groups induces a well defined symmetric monoidal structure on* $\mathcal{R}_p^\sharp$.

Proof. Let P, P', Q, and Q' be finite p-groups. If U is a finite (P', P)-biset and V is a finite (Q', Q)-biset, then $U \times V$ is a $(P' \times Q', P \times Q)$-biset. This induces a bilinear map

$$\pi : B(P', P) \times B(Q', Q) \to B(P' \times Q', P \times Q),$$

and this clearly induces a symmetric monoidal structure on the biset category $\mathcal{C}_p$. The latter induces a monoidal structure on the quotient category if

$$\pi(B_\delta(P', P), B(Q', Q)) \subseteq B_\delta(P' \times Q', P \times Q).$$

But this is a consequence of the following. Let X be as defined in Notation 3.1, let U be a finite $(P', P \times X)$-set, let D be an X-set, and V be a finite (Q', Q)-biset. Clearly, there is an isomorphism of $(P' \times Q', P \times Q)$-sets

$$(U \times_X D) \times V \cong (U \times V) \times_X D,$$

where the right action of X on $U \times V$ is defined in the obvious way

$$\forall (u, v) \in U \times V,\ \forall x \in X, \quad (u, v)x = (ux, v).$$

The lemma follows. □

3.6. Recall that the objects of the idempotent additive completion $\mathcal{R}_p$ are by definition formal sums $\bigoplus_{(P,e)\in\mathcal{P}}(P, e)$, where $\mathcal{P}$ is a finite sequence of pairs (P, e) consisting of a finite p-group P and an idempotent e in the endomorphism ring $\mathrm{Hom}_{\mathcal{R}_p^\sharp}(P, P) = (B/B_\delta)(P, P)$. A morphism

$$\varphi : \bigoplus_{(P,e)\in\mathcal{P}} (P, e) \to \bigoplus_{(Q,f)\in\mathcal{Q}} (Q, f)$$

in $\mathcal{R}_p$ is a matrix indexed by $\mathcal{P} \times \mathcal{Q}$, where the coefficient $\varphi_{(P,e),(Q,f)}$ indexed by the pair $((P, e), (Q, f))$ belongs to $f\,\mathrm{Hom}_{\mathcal{R}_p^\sharp}(P, Q)e$. The composition of morphisms is given by matrix multiplication. In particular:

Definition and Notation 3.7. Let P be a finite p-group.

- The object (P, Id_P) of $\mathcal{R}_p$ is denoted by P. Similarly, when Q is a finite p-group and $f \in B(Q, P)$, the corresponding morphism from (P, Id_P) to (Q, Id_P) in the category $\mathcal{R}_p$ is simply denoted by f.
- The *edge* ∂P of P is the object $(P, f_{\mathbf{1}}^P)$ of $\mathcal{R}_p$.

The category of additive functors from $\mathcal{R}_p$ to abelian groups is equivalent to the category of additive functors from $\mathcal{R}_p^\sharp$ to abelian groups. It was shown in [Bou08] that the latter is exactly the category of rational p-biset functors. If $F^\sharp$ is such a functor, then $F^\sharp$ extends to a functor F on $\mathcal{R}_p$ defined as follows:

$$F\Big(\bigoplus_{(P,e)\in\mathcal{P}}(P,e)\Big)=\bigoplus_{(P,e)\in\mathcal{P}}F^\sharp(e)(F^\sharp(P)),$$

with the obvious definition of $F(\varphi)$ for a morphism φ in the category $\mathcal{R}_p$. In particular, with the above notation,

$$F(\partial P)=\partial F^\sharp(P).$$

We will use the same symbol for F and $F^\sharp$, writing in particular $F(\partial P)=\partial F(P)$.

Proposition 3.8. *Let P be a finite p-group. Then*

$$P\cong\bigoplus_{N\trianglelefteq P}\partial(P/N)\quad \text{in the category } \mathcal{R}_p.$$

Proof. Let

$$a:P\to\bigoplus_{N\trianglelefteq P}\partial(P/N)$$

be the direct sum of the morphisms induced by the elements $f_{\mathbf{1}}^{P/N}\operatorname{Def}^P_{P/N}$ of $B(P/N,P)$, and let

$$b:\bigoplus_{N\trianglelefteq P}\partial(P/N)\to P$$

be defined similarly from the elements $\operatorname{Inf}^P_{P/N}f_{\mathbf{1}}^{P/N}$ of $B(P,P/N)$.

By Lemma 2.27,

$$\sum_{N\trianglelefteq P}\operatorname{Inf}^P_{P/N}f_{\mathbf{1}}^{P/N}\operatorname{Def}^P_{P/N}=\sum_{N\trianglelefteq P}f_N^P=\operatorname{Id}_P$$

in $B(P,P)$, thus $a\circ b$ is equal to the identity morphism of P in $\mathcal{R}_p$. Conversely, for normal subgroups N and M of P,

$$f_{\mathbf{1}}^{P/N}\operatorname{Def}^P_{P/N}\operatorname{Inf}^P_{P/M}f_{\mathbf{1}}^{P/M}=f_{\mathbf{1}}^{P/N}\operatorname{Inf}^{P/N}_{P/NM}\operatorname{Def}^{P/M}_{P/NM}f_{\mathbf{1}}^{P/M},$$

by (2.25). This is equal to 0 if $N\neq M$, by Lemma 2.27. And if $N=M$, this is equal to $f_{\mathbf{1}}^{P/N}$. It follows that $b\circ a$ is equal to the identity morphism of $\bigoplus_{N\trianglelefteq P}\partial(P/N)$, and this completes the proof. □

Corollary 3.9. *If P is non-trivial, with cyclic center, then*

$$P\cong\partial P\oplus(P/Z)\quad \text{in } \mathcal{R}_p,$$

where Z is the unique central subgroup of order p in P.

Proof. Indeed, if N is a non-trivial normal subgroup of P, then $N\geq Z$. Thus

$$P\cong\partial P\oplus\bigoplus_{N\geq Z}\partial(P/N)\cong\partial P\oplus(P/Z)\quad \text{in } \mathcal{R}_p.$$

□

Remark 3.10. More generally, let P be a finite p-group, and let N be a normal subgroup of P. Since

$$P/N \cong \bigoplus_{N \le M \trianglelefteq P} \partial\big((P/N)\big/(M/N)\big) \cong \bigoplus_{N \le M \trianglelefteq P} \partial(P/M) \quad \text{in } \mathcal{R}_p,$$

it follows that P/N is isomorphic to a direct summand of P in the category $\mathcal{R}_p$.

Theorem 3.11. (1) *The Roquette category $\mathcal{R}_p$ is an additive tensor category.*
(2) *Let P be a finite p-group, and $\mathcal{B}$ be a genetic basis of P. Then*

$$P \cong \bigoplus_{S \in \mathcal{B}} \partial \overline{N}_P(S) \quad \textit{in } \mathcal{R}_p,$$

where $\overline{N}_P(S) = N_P(S)/S$.
(3) *Let P be a finite p-group, and $\mathcal{B}$ be a genetic basis of P. Then*

$$\partial P \cong \bigoplus_{\substack{S \in \mathcal{B} \\ S \cap Z(P) = \mathbf{1}}} \partial \overline{N}_P(S) \quad \textit{in } \mathcal{R}_p.$$

Proof. Assertion (1) results from standard results: in particular, the tensor product of the objects $\bigoplus_{(P,e)\in\mathcal{P}}(P,e)$ and $\bigoplus_{(Q,f)\in\mathcal{Q}}(Q,f)$ is defined by

$$\Big(\bigoplus_{(P,e)\in\mathcal{P}}(P,e)\Big) \times \Big(\bigoplus_{(Q,f)\in\mathcal{Q}}(Q,f)\Big) = \bigoplus_{\substack{(P,e)\in\mathcal{P} \\ (Q,f)\in\mathcal{Q}}} (P \times Q, e \times f).$$

For (2), by [Bou10, Proposition 10.7.2], if F is a rational p-biset functor, the functor F_P obtained from F by the Yoneda–Dress construction at P is also a rational p-biset functor. This applies in particular to the functor $Y = B/B_\delta$, so the functor Y_P is rational. Hence, if Q is any finite p-group and $\mathcal{B}_Q$ is a genetic basis of Q, there are mutually inverse isomorphisms

$$Y_P(Q) \underset{\mathcal{I}_Q}{\overset{\mathcal{D}_Q}{\rightleftarrows}} \bigoplus_{S \in \mathcal{B}_Q} \partial Y_P(\overline{N}_Q(S))$$

where $\mathcal{I}_Q = \bigoplus_{S\in\mathcal{B}_Q} \mathrm{Indinf}_{\overline{N}_P(S)}^P$ and $\mathcal{D}_Q = \bigoplus_{S\in\mathcal{B}_Q} f_{\mathbf{1}}^{\overline{N}_P(S)} \circ \mathrm{Defres}_{\overline{N}_P(S)}^P$.

Thus for any $f \in Y_P(Q)$,

$$f = \Big(\sum_{S \in \mathcal{B}_Q} \mathrm{Indinf}_{\overline{N}_P(S)}^P \, f_{\mathbf{1}}^{\overline{N}_P(S)} \, \mathrm{Defres}_{\overline{N}_P(S)}^P\Big) \circ f.$$

Applying this to $Q = P$, $\mathcal{B}_Q = \mathcal{B}$, and $f = \mathrm{Id}_P$ gives $\mathcal{I}_P \circ \mathcal{D}_P = \mathrm{Id}_P$. On the other hand, by [Bou10, Proposition 6.4.4 and Theorem 9.6.1], for $S, T \in \mathcal{B}$, the composition

$$f_{\mathbf{1}}^{\overline{N}_P(T)} \, \mathrm{Defres}_{\overline{N}_P(T)}^P \circ \mathrm{Indinf}_{\overline{N}_P(S)}^P \, f_{\mathbf{1}}^{\overline{N}_P(S)}$$

is equal to $f_{\mathbf{1}}^{\overline{N}_P(S)}$ in $B(\overline{N}_P(S), \overline{N}_P(S))$ if $T = S$, and to 0 if $T \neq S$. It follows that $\mathcal{D}_P \circ \mathcal{I}_P$ is also equal to the identity map of the direct sum $\bigoplus_{S\in\mathcal{B}} \partial Y_P(\overline{N}_P(S))$ in the category $\mathcal{R}_p$.

For (3), observe that by Lemma 2.29,

$$f_{\mathbf{1}}^{\overline{N}_P(S)} \operatorname{Defres}^P_{\overline{N}_P(S)} f_1^P = 0$$

if $S \cap Z(P) \neq \mathbf{1}$. Taking opposite bisets gives also

$$f_1^P \operatorname{Indinf}^P_{\overline{N}_P(S)} f_{\mathbf{1}}^{\overline{N}_P(S)} = 0,$$

so the isomorphism of (2) restricts to an isomorphism

$$\partial P \cong \bigoplus_{\substack{S \in \mathcal{B} \\ S \cap Z(P) = \mathbf{1}}} \partial \overline{N}_P(S),$$

as the diagram

$$\begin{array}{ccc}
P & \xrightarrow{\cong} & \bigoplus_{S \in \mathcal{B}} \partial \overline{N}_P(S) \\
{\scriptstyle f_{\mathbf{1}}^P} \uparrow & & \uparrow \\
\partial P & \longrightarrow & \bigoplus_{S \in \mathcal{B},\, S \cap Z(P) = \mathbf{1}} \partial \overline{N}_P(S)
\end{array}$$

is commutative. □

Corollary 3.12. *Let P be a finite p-group. If $Z(P)$ is non-cyclic, then $\partial P = 0$ in $\mathcal{R}_p$.*

Proof. This follows from Theorem 3.11(3). Suppose indeed that there exists a genetic subgroup S of P such that $S \cap Z(P) = \mathbf{1}$. Then the group $Z(P)$ maps injectively in the center of the Roquette group $\overline{N}_P(S)$, which is cyclic. Hence $Z(P)$ is cyclic. □

Example 3.13. 1. Let $P = D_8$ be the dihedral group of order 8. Let A, B, and C be the subgroups of index 2 in P, and let I be a non-central subgroup of order 2 in P. Then the set $\{P, A, B, C, I\}$ is a genetic basis of P, and there is an isomorphism

$$P \cong \mathbf{1} \oplus 4 \cdot \partial C_2 \quad \text{in } \mathcal{R}_2,$$

where $4 \cdot \partial C_2$ denotes the direct sum of four copies of ∂C_2: indeed, for $S \in \{A, B, C, I\}$, the group $\overline{N}_P(S)$ is isomorphic to C_2.

2. Let $P = Q_8$ be the quaternion group of order 8. Let A, B, and C be the subgroups of index 2 in P. Then the set $\{P, A, B, C, \mathbf{1}\}$ is a genetic basis of P (such a basis is unique in this case), and there is an isomorphism

$$P \cong \mathbf{1} \oplus 3 \cdot \partial C_2 \oplus \partial Q_8 \quad \text{in } \mathcal{R}_2.$$

3. Let $P = (C_p)^n$ be an elementary abelian p-group of rank n. Then P has a unique genetic basis, consisting of P and all its subgroups of index p. Hence

$$P \cong \mathbf{1} \oplus \frac{p^n - 1}{p - 1} \cdot \partial C_p \quad \text{in } \mathcal{R}_p.$$

4. The tensor structure

Notation 4.1. Let G and H be groups. When L is a subgroup of $G \times H$, set

$$\begin{aligned} p_1(L) &= \{g \in G \mid \exists h \in H,\ (g,h) \in L\},\\ p_2(L) &= \{h \in H \mid \exists g \in G,\ (g,h) \in L\},\\ k_1(L) &= \{g \in G \mid (g,1) \in L\},\\ k_2(L) &= \{h \in H \mid (1,h) \in L\}. \end{aligned}$$

Recall [Bou10, 2.3.18 and 2.3.21] that $k_i(L) \trianglelefteq p_i(L)$ for $i \in \{1,2\}$, and the direct product $k_1(L) \times k_2(L)$ is normal in L. Moreover, setting $q(L) = L/(k_1(L) \times k_2(L))$, we have canonical group isomorphisms

$$q(L) \cong p_1(L)/k_1(L) \cong p_2(L)/k_2(L).$$

Definition 4.2. Let G and H be groups. A subgroup L of $G \times H$ will be called *diagonal* if

$$L \cap (G \times \mathbf{1}) = L \cap (\mathbf{1} \times H) = \mathbf{1},$$

i.e. $k_1(L) = \mathbf{1}$ and $k_2(L) = \mathbf{1}$.

The subgroup L will be called *centrally diagonal* if

$$L \cap (Z(G) \times \mathbf{1}) = L \cap (\mathbf{1} \times Z(H)) = \mathbf{1},$$

i.e. $k_1(L) \cap Z(G) = \mathbf{1}$ and $k_2(L) \cap Z(H) = \mathbf{1}$.

Notation 4.3. Let G and H be groups. When K is a subgroup of G, and $\varphi : K \to H$ is a group homomorphism, set

$$\begin{aligned} \overrightarrow{\Delta}_\varphi(K) &= \{(x, \varphi(x)) \mid x \in K\} \leq G \times H,\\ \overleftarrow{\Delta}_\varphi(K) &= \{(\varphi(x), x) \mid x \in K\} \leq H \times G. \end{aligned}$$

Remark 4.4. The subgroup L of $G \times H$ is diagonal if and only if there exists a subgroup $K \leq G$ and an *injective* group homomorphism $\varphi : K \hookrightarrow H$ such that $L = \overrightarrow{\Delta}_\varphi(K)$.

Lemma 4.5. *Let P and Q be p-groups, and let L and L' be genetic subgroups of $P \times Q$ such that $L \frown_{P\times Q} L'$. Then L is centrally diagonal in $P \times Q$ if and only if L' is centrally diagonal in $P \times Q$.*

Proof. Let $(x, y) \in P \times Q$ be such that $L'^{(x,y)} \cap Z_{P\times Q}(L) \leq L$. Since $Z(P) \times Z(Q) \leq Z_{P\times Q}(L)$, it follows that

$$\begin{aligned} L' \cap (Z(P) \times \mathbf{1}) = (L' \cap (Z(P) \times \mathbf{1}))^{(x,y)} &= L'^{(x,y)} \cap (Z(P) \times \mathbf{1})\\ &= L'^{(x,y)} \cap Z_{P\times Q}(L) \cap (Z(P) \times \mathbf{1}) \leq L \cap (Z(P) \times 1) = \mathbf{1}. \end{aligned}$$

A similar argument shows that $L' \cap (\mathbf{1} \times Z(Q)) = \mathbf{1}$. □

Recall from Definition 2.8 that an *axial* subgroup of a finite group G is a *subgroup* of a cyclic subgroup of maximal order of G:

Theorem 4.6. *Let P and Q be non-trivial Roquette p-groups, let e_P (resp. e_Q) denote the exponent of P (resp. Q), and let Z_P (resp. Z_Q) denote the central subgroup of order p in P (resp. in Q).*

(1) *Let L be a centrally diagonal genetic subgroup of $P \times Q$. Then $L = \overrightarrow{\Delta}_\varphi(H)$, where $H \leq P$ and $\varphi : H \hookrightarrow Q$ is an injective group homomorphism. Moreover, either $P \cong Q \cong Q_8$ and $H = P$, or H is an axial subgroup of P of order $\min(e_P, e_Q)$ such that $\varphi(H)$ is an axial subgroup of Q.*

(2) *Conversely:*
  (a) *If $P \cong Q \cong Q_8$, let $L = L_\varphi = \overrightarrow{\Delta}_\varphi(P)$, where $\varphi : P \to Q$ is a group isomorphism. Then L is a centrally diagonal genetic subgroup of $P \times Q$ and $N_{P\times Q}(L)/L \cong C_2$.*
  (b) *In all other cases, let $L = L_\varphi = \overrightarrow{\Delta}_\varphi(H)$, where H is an axial subgroup of P of order $\min(e_P, e_Q)$ and $\varphi : H \hookrightarrow Q$ is an injective group homomorphism such that $\varphi(H)$ is an axial subgroup of Q. Then L is a centrally diagonal genetic subgroup of $P \times Q$. Moreover, the isomorphism class of the group $N_{P\times Q}(L)/L$ depends only on P and Q.*

Proof. • Observe first that the group $(Z_P \times Z_Q)L/L$ is a central subgroup of the Roquette group $N_{P\times Q}(L)/L$, hence it is cyclic. Hence $(Z_P \times Z_Q) \cap L \neq \mathbf{1}$. Since both $(Z_P \times \mathbf{1}) \cap L$ and $(\mathbf{1} \times Z_Q) \cap L$ are trivial, it follows that $(Z_P \times Z_Q) \cap L$ is equal to

$$\overrightarrow{\Delta}_\psi(Z_P) = \{(z, \psi(z)) \mid z \in Z_P\}, \qquad (\mathbf{4.7})$$

where $\psi : Z_P \xrightarrow{\cong} Z_Q$ is some group isomorphism. In particular $p_1(L)$ contains Z_P, and $p_2(L)$ contains Z_Q.

• Let us prove now that L is diagonal, i.e. there exists a subgroup H of P and an injective group homomorphism $\varphi : H \hookrightarrow Q$ such that

$$L = \overrightarrow{\Delta}_\varphi(H) = \{(h, \varphi(h)) \mid h \in H\}.$$

Otherwise, at least one of the groups $k_1(L)$ or $k_2(L)$ is non-trivial. But the assumption $L \cap (Z(P) \times \mathbf{1}) = \mathbf{1}$ is equivalent to $L \cap (Z_P \times \mathbf{1}) = \mathbf{1}$, i.e. $k_1(L) \cap Z_P = \mathbf{1}$, and similarly the assumption $L \cap (\mathbf{1} \times Z(Q)) = \mathbf{1}$ is equivalent to $k_2(L) \cap Z_Q = \mathbf{1}$. But if there exists a non-trivial subgroup X of P such that $X \cap Z_P = \mathbf{1}$, then $p = 2$, X has order 2, and P is dihedral or semidihedral (Lemma 2.9). So if L is not diagonal, then $p = 2$, and at least one of P or Q is dihedral or semidihedral.

Up to exchanging P and Q, one can assume that the group $C = k_1(L)$ is non-trivial, hence non-central of order 2 in P. Set $A = p_1(L)$. Since $A \leq N_P(C) = CZ_P$ (Lemma 2.9), it follows that $q(L) = A/C$ has order 1 or 2.

If $q(L) = \mathbf{1}$, then $A = C$ and $L = C \times D$, where $D = k_2(L) = p_2(L)$. In this case

$$N_{P\times Q}(L)/L = (N_P(C)/C) \times (N_Q(D)/D) \cong C_2 \times (N_Q(D)/D)$$

cannot be a Roquette group, since $N_Q(D)/D$ is non-trivial (as $D \cap Z_Q = \mathbf{1}$).

And if $|q(L)| = 2$, then $A = CZ_P$. If $(a, b) \in N_{P\times Q}(L)$, then in particular

$$a \in N_P(A, C) = A.$$

Thus $N_{P\times Q}(L) \le A \times Q$. Now $N_P(A)$ is a proper subgroup of P, since A is elementary abelian of rank 2, and P is a Roquette group. Choose $x \in P - N_P(A)$, whence $A^x \cap A = Z_P$. If $(a, b) \in L^{(x,1)} \cap (A\times Q)$, then $a \in A^x \cap A = Z_P$, hence $(a, b) = {}^{(x,1)}(a, b) \in L$. Thus $L^{(x,1)} \cap (A \times Q) \le L$, and

$$L^{(x,1)} \cap Z_{P\times Q}(L) \le L^{(x,1)} \cap N_{P\times Q}(L) \le L^{(x,1)} \cap (A \times Q) \le L$$

but $L^{(x,1)} \ne L$, since $A^x \ne A$. It follows that L is not expansive in $P \times Q$, hence L is not a genetic subgroup of $P \times Q$.

• Hence L is diagonal in $P \times Q$, i.e.

$$L = \overrightarrow{\Delta}_\varphi(H) = \{(h, \varphi(h)) \mid h \in H\}$$

for some subgroup $H \ge Z_P$ of P and some $\varphi : H \hookrightarrow Q$ such that $\varphi(H) \ge Z_Q$. Then

$$N_{P\times Q}(L) = \{(x, y) \in N_P(H) \times N_Q(\varphi(H)) \mid \forall h \in H,\ \varphi({}^x h) = {}^y\varphi(h)\}.$$

The unique central subgroup of order p of the Roquette group $N_{P\times Q}(L)/L$ is equal to Z/L, where

$$Z = (Z_P \times \mathbf{1})L = (\mathbf{1} \times Z_Q)L. \tag{4.8}$$

For any $(x, y) \in (P \times Q)$, saying that $L^{(x,y)} \cap Z_{P\times Q}(L)$ is contained in L is equivalent to saying that the group $I = L^{(x,y)} \cap Z$ is contained in L. In particular, for $y = 1$,

$$\begin{aligned} I &= L^{(x,1)} \cap (Z_P \times \mathbf{1})L \\ &= \{(h^x, \varphi(h)) \mid h \in H,\ \exists z \in Z_P,\ \exists h' \in H,\ (h^x, \varphi(h)) = (zh', \varphi(h'))\} \\ &= \{(h^x, \varphi(h)) \mid h \in H,\ h^{-1}h^x \in Z_P\}, \end{aligned}$$

since $\varphi(h) = \varphi(h')$ implies $h = h'$, and since Z_P is central in P. Denoting by $[h, x] = h^{-1}h^x$ the commutator of h and x, it follows that $I \le L$ if and only if

$$\forall h \in H, \quad [h, x] \in Z_P \Rightarrow h^x \in H,\ (h^x, \varphi(h)) = (h^x, \varphi(h^x)).$$

In other words $[h, x] \in Z_P$ implies $h^x = h$. Thus $I \le L$ if and only if

$$\forall h \in H, \quad [h, x] \in Z_P \Rightarrow [h, x] = 1.$$

Equivalently $[H, x] \cap Z_P = \{1\}$, where $[H, x]$ denotes the *set* of commutators $[h, x]$ for $h \in H$.

Since $L = \overrightarrow{\Delta}_\varphi(H)$ is expansive in $P \times Q$ it follows that

$$[H, x] \cap Z_P = \{1\} \Rightarrow L^{(x,1)} = L.$$

Now $(x, 1)$ normalizes L if and only if $x \in C_P(H)$, i.e. if $[H, x] = \{1\}$. Hence

$$[H, x] \cap Z_P = \{1\} \Rightarrow [H, x] = \{1\}. \quad (\mathbf{4.9})$$

• Let us show now that unless $H = P \cong Q \cong Q_8$, the group H is an axial subgroup of P, and the subgroup $\varphi(H)$ is an axial subgroup of Q.

Let X be a cyclic subgroup of P of order e_P, let x be a generator of X, and suppose that $H \nleq X$. Then in particular P is not cyclic, so $p = 2$, the group X is a (normal) subgroup of index 2 of P (Lemma 2.9), and X is equal to its centralizer in P. Moreover $|H : H \cap X| = 2$ since $H \cdot X = P$. The set $[H, x]$ is equal to $\{1, x^2\}$ if P is cyclic or generalized quaternion, or to $\{1, x^{2+2^{n-2}}\}$ if P is semidihedral: indeed, the image of H in the group of automorphisms of X has order 2, as $H \cap X$ centralizes X, and H does not. Since Z_P is generated by $x^{2^{n-2}}$, it follows that $[H, x] \cap Z_P = \{1\}$ if $n \geq 4$, i.e. if $|P| \geq 16$. But $[H, x] \neq \{1\}$, hence L is not expansive in $P \times Q$ if $P \geq 16$.

So if $H \nleq X$, then $p = 2$, and P is non-cyclic, of order at most 8. Hence $P \cong Q_8$. If $H \neq P$, then H is cyclic, and $H \nleq X$. Thus $|H| = 4 = e_P$, and in particular H is an axis of P. Since H embeds into Q, it follows that $|H| = \min(e_P, e_Q)$. The same argument applied to $\varphi(H)$ shows that $\varphi(H)$ is an axial subgroup of Q, as claimed.

In this case moreover, the group Q cannot be isomorphic to Q_8. Indeed, otherwise one can assume that $P = Q$ and $L = \Delta(H)$ is the diagonal embedding. Then

$$N_{P\times P}(L) = \{(a, b) \mid a^{-1}b \in H\}.$$

The group $N_{P\times P}(L)/L$ has order 8, generated by the cyclic subgroup

$$C = \{(a, 1)L \mid a \in H\}$$

of index 2, and the involution $(b, b)L$, where $b \in P - H$. Hence $N_{P\times P}(L)/L \cong D_8$ is not a Roquette group, and L is not a genetic subgroup of $P \times P$.

If H is non-cyclic, then $H = P$, and the same argument applied to $\varphi(H)$ shows that $Q \cong Q_8$. And indeed $\overrightarrow{\Delta}_\varphi(P)$ is a genetic subgroup of $P \times Q$: this follows from Example 2.17, since the map $(x, y) \mapsto (x, \varphi^{-1}(y))$ is a group isomorphism from $P \times Q$ to $P \times P$, sending $\overrightarrow{\Delta}_\varphi(P)$ to $\Delta(P)$. Moreover $[P, P] \leq Z(P)$, and $Z(P)$ has order 2. In particular

$$N_{P\times Q}(L)/L \cong C_2 \quad (\mathbf{4.10})$$

does not depend on φ, up to isomorphism. This proves (2)(a).

• In the remaining cases $L = \overrightarrow{\Delta}_\varphi(H)$, where H is a non-trivial axial subgroup of P, and $\varphi : H \hookrightarrow Q$ is such that $\varphi(H)$ is an axial subgroup of Q. In particular H is cyclic and non-trivial. As $H \cong \varphi(H) \leq Q$, it follows that $|H| \leq \min(e_P, e_Q)$.

Let C_{e_P} be an axis of P containing H, and C_{e_Q} be an axis of Q containing $\varphi(Q)$. Then $(C_{e_P} \times C_{e_Q})/\Delta_\varphi(H)$ is an abelian normal subgroup of the Roquette group $N_{P\times Q}(L)/L$, hence it is cyclic. In particular $L = \Delta_\varphi(H)$ is not contained in the Frattini subgroup $C_{e_P/p} \times C_{e_Q/p}$ of $C_{e_P} \times C_{e_Q}$. Thus $p_1(L) = C_{e_P}$ or $p_2(L) = C_{e_Q}$. In other words $H = C_{e_P}$ or $\varphi(H) = C_{e_Q}$, hence $|H| = \min(e_P, e_Q)$. This completes the proof of (1).

• Now assume that P or Q is not isomorphic to Q_8. Assume also that $e_P \le e_Q$, and let H be an axis of P; then H is unique if $P \ncong Q_8$, and there are three possibilities for H if $P \cong Q_8$ (Lemma 2.9). In any case $H \trianglelefteq P$. Let K denote an axial subgroup of Q of order e_P. Such a group is unique except if $p = 2$, $e_P = 4$, and $Q \cong Q_8$ (thus $P \cong C_4$ as P has exponent 4, and is not isomorphic to Q_8). In any case $K \trianglelefteq Q$.

Let $\varphi : H \xrightarrow{\cong} K$ be any group isomorphism, and set $L_\varphi = \overrightarrow{\Delta}_\varphi(H) \le P \times Q$. Then L_φ is obviously centrally diagonal, and

$$N_{P\times Q}(L_\varphi) = \{(a, b) \in P \times Q \mid \forall h \in H,\ \varphi({}^a h) = {}^b\varphi(h)\}.$$

Since H is cyclic of order e_P, the map

$$\pi_H : (\mathbb{Z}/e_P\mathbb{Z})^\times \ni r \mapsto (x \mapsto x^r) \in \mathrm{Aut}(H)$$

is a canonical group isomorphism. Similarly, the map

$$\pi_K : (\mathbb{Z}/e_P\mathbb{Z})^\times \ni r \mapsto (x \mapsto x^r) \in \mathrm{Aut}(K)$$

is a canonical group isomorphism.

Let $\alpha : P \to \mathrm{Aut}(H) \xrightarrow{\pi_H^{-1}} (\mathbb{Z}/e_P\mathbb{Z})^\times$ (resp. $\beta : Q \to \mathrm{Aut}(K) \xrightarrow{\pi_K^{-1}} (\mathbb{Z}/e_P\mathbb{Z})^\times$) denote the group homomorphism obtained from the action of P on its normal subgroup H by conjugation (resp. from the action of Q on its normal subgroup K). Then

$$N_{P\times Q}(L_\varphi) = \{(a, b) \in P \times Q \mid \alpha(a) = \beta(b)\}.$$

Now the group $(\mathbb{Z}/e_P\mathbb{Z})^\times$ is abelian. The map

$$\Theta : P \times Q \ni (a, b) \mapsto \beta(b)^{-1} \cdot \alpha(a) \in (\mathbb{Z}/e_P\mathbb{Z})^\times$$

is a group homomorphism, and $N_{P\times Q}(L_\varphi) = \mathrm{Ker}\,\Theta$. In particular, it is a normal subgroup of $P \times Q$ which does not depend on φ once H and $K = \varphi(H)$ are fixed. In particular L_φ is an expansive subgroup of $P \times Q$, by Example 2.14. Moreover if we set $I_{P,Q} = \mathrm{Im}(\alpha) \cap \mathrm{Im}(\beta)$, there is an exact sequence

$$\textbf{(4.11)} \qquad \mathbf{1} \to C_P(H) \times C_Q(K) \to N_{P\times Q}(L_\varphi) \xrightarrow{\Psi} I_{P,Q} \to \mathbf{1},$$

where $\Psi(a, b) = \alpha(a) = \beta(b)$ for $(a, b) \in N_{P\times Q}(L_\varphi)$.

Suppose first that $e_P = p$, i.e. $P \cong C_p$. In this case $H = Z_P = P$, and $K = Z_Q$, so L_φ is central in $P \times Q$. Moreover

$$\textbf{(4.12)} \qquad N_{P\times Q}(L_\varphi)/L_\varphi = (P \times Q)/L_\varphi = (P \times Q)/\overrightarrow{\Delta}_\varphi(P) \cong Q$$

is a Roquette group, independent of φ up to isomorphism. In particular L_φ is a genetic subgroup of $P \times Q$.

Assume from now on that $e_P \ge p^2$. Then $C_P(H) \cong C_{e_P}$ and $C_Q(K) \cong C_{e_Q}$, by Lemma 2.9.

If $p > 2$, then P and Q are cyclic, hence $H = P$, and

$$N_{P\times Q}(L_\varphi)/L_\varphi \cong (P\times Q)/\overrightarrow{\Delta}_\varphi(P) \cong Q \tag{4.13}$$

as above. It is a Roquette group, independent of φ up to isomorphism. In particular L_φ is genetic in $P \times Q$.

Assume now that $p = 2$. The image of α has order $|P : C_P(H)|$, which is equal to 1 if P is cyclic, and to 2 otherwise. Similarly, the image of β has order $|Q : C_Q(K)|$, which is equal to 1 if K is central in Q, i.e. if Q is cyclic (since $|K| = e_P \geq 4$ by assumption), and to 2 otherwise. Set $I_{P,Q} = \mathrm{Im}(\alpha) \cap \mathrm{Im}(\beta)$. Then $I_{P,Q}$ has order 1 or 2, and there is an exact sequence

$$1 \to C_{e_P} \times C_{e_Q} \to N_{P\times Q}(L_\varphi) \to I_{P,Q} \to 1. \tag{4.14}$$

Note that $I_{P,Q}$ does not depend on $\varphi : H \xrightarrow{\cong} K$. More precisely,

$$\mathrm{Im}(\alpha) = \begin{cases} \{1\} & \text{if } P \text{ is cyclic,} \\ \{1,-1\} & \text{if } P \text{ is dihedral or generalized quaternion,} \\ \{1, e_P/2-1\} & \text{if } P \text{ is semidihedral.} \end{cases}$$

So $\mathrm{Im}(\alpha)$ only depends on the type of P.

Similarly

$$\mathrm{Im}(\beta) = \begin{cases} \{1\} & \text{if } Q \text{ is cyclic,} \\ \{1,-1\} & \text{if } Q \text{ is dihedral or generalized quaternion,} \\ \{1, e_Q/2-1\} & \text{if } Q \text{ is semidihedral.} \end{cases}$$

Moreover if Q is semidihedral and $e_Q > e_P$, then $e_Q/2 - 1 \equiv -1 \pmod{e_P}$, hence $\mathrm{Im}(\beta) = \{1, -1\}$. In other words, the group $I_{P,Q}$ is trivial in each of the following cases:

- P or Q is cyclic,
- P is dihedral or generalized quaternion, Q is semidihedral, and $e_P = e_Q$, (i.e. equivalently $|P| = |Q|$),
- P is semidihedral, and Q is dihedral or generalized quaternion,
- P and Q are semidihedral, and $e_P < e_Q$ (i.e. equivalently $|P| < |Q|$),

and the group $I_{P,Q}$ has order 2 in all other cases, i.e. in each of the following cases:

- P and Q are dihedral or generalized quaternion,
- P is dihedral or generalized quaternion, Q is semidihedral, and $|Q| > |P|$,
- P and Q are semidihedral, and $P \cong Q$.

As $L_\varphi \leq C_{e_P} \times C_{e_Q}$, the exact sequence (4.14) yields the exact sequence

$$1 \to (C_{e_P} \times C_{e_Q})/L_\varphi \to N_{P\times Q}(L_\varphi)/L_\varphi \to I_{P,Q} \to 1. \tag{4.15}$$

Case 1: If $I_{P,Q}$ is trivial, then $N_{P\times Q}(L_\varphi) \cong C_{e_P} \times C_{e_Q}$, and

$$N_{P\times Q}(L_\varphi)/L_\varphi \cong C_{e_Q}, \tag{4.16}$$

which is a Roquette group, independent of φ up to isomorphism. In particular L_φ is a genetic subgroup of $P \times Q$.

Case 2: Suppose now that $I_{P,Q}$ has order 2, i.e. $\mathrm{Im}(\alpha) = \mathrm{Im}(\beta) = \{1, \epsilon\}$, where ϵ is either -1 or $e_Q/2-1$ (in the case where P and Q are semidihedral and isomorphic). One can choose an element $u \in P$, of order 2 if P is dihedral or semidihedral, and of order 4 if P is generalized quaternion, such that $\alpha(u) = \epsilon$. Similarly, one can choose an element $v \in Q$, of order 2 if Q is dihedral or semidihedral, and of order 4 if Q is generalized quaternion, such that $\beta(v) = \epsilon$. These choices imply that $(u, v) \in N_{P\times Q}(L_\varphi)$.

In the exact sequence (4.15),

$$1 \to (C_{e_P} \times C_{e_Q})/L_\varphi \to N_{P\times Q}(L_\varphi)/L_\varphi \to \{1, \epsilon\} \to 1,$$

the group $C = (C_{e_P} \times C_{e_Q})/L_\varphi$ is cyclic, isomorphic to C_{e_Q}. The element $\pi = (u, v)L_\varphi$ of $N_{P\times Q}(L_\varphi)/L_\varphi$ acts on C in the same way that v acts on the subgroup C_{e_Q} of Q, namely by inversion if Q is dihedral or generalized quaternion, and by raising elements to the power $e_Q/2 - 1$ if Q is semidihedral. Finally $\pi^2 = (u^2, v^2)L_\varphi = L_\varphi$ if none of P and Q is generalized quaternion. If P is generalized quaternion and Q is not, then $\pi^2 = (z_P, 1)L_\varphi \in C - \{1\}$, where z_P is a generator of Z_P. Similarly, if Q is generalized quaternion and P is not, then $\pi^2 = (1, z_Q)L_\varphi \in C - \{1\}$, where z_Q is a generator of Z_Q. In these two cases

$$\pi^4 = (1, 1)L_\varphi = L_\varphi,$$

so π has order 4 in $N_{P\times Q}(L_\varphi)/L_\varphi$. And finally, if both P and Q are generalized quaternion, then $\pi^2 = (z_P, z_Q)L_\varphi = L_\varphi$, since $\varphi(Z_P) = Z_Q$.

It follows that the group $N_{P\times Q}(L_\varphi)/L_\varphi$ has order $2e_Q = |Q|$, and that it is:

(**4.17**)
- dihedral if P and Q are both dihedral, or both generalized quaternion,
- generalized quaternion if one of P, Q is generalized quaternion, and the other is dihedral,
- semidihedral if Q is semidihedral.

So $N_{P\times Q}(L_\varphi)/L_\varphi$ is a Roquette group, independent of φ up to isomorphism. In particular, L_φ is a genetic subgroup of $P \times Q$. This completes the proof of Theorem 4.6. □

Notation 4.18. • Let P and Q be Roquette p-groups. If P and Q are non-trivial, set $P \diamond Q = N_{P\times Q}(L)/L$, where L is a centrally diagonal genetic subgroup of $P \times Q$. Set moreover $\mathbf{1} \diamond P = P \diamond \mathbf{1} = P$.

• Let P and Q be Roquette p-groups. If P and Q are non-trivial, let $\nu_{P,Q}$ denote the number of equivalence classes of centrally diagonal genetic subgroups of $P \times Q$ for the relation $\frown_{P\times Q}$. Set moreover $\nu_{\mathbf{1},P} = \nu_{P,\mathbf{1}} = 1$.

Remark 4.19. If $P = \mathbf{1}$ and Q is a Roquette p-group, then $P \times Q \cong Q$, and the centrally diagonal genetic subgroups of $P \times Q$ are the subgroups $\mathbf{1} \times R$, where R is a genetic subgroup of Q such that $R \cap Z(Q) = \mathbf{1}$. The only such subgroup is $R = \mathbf{1}$, so $N_Q(R)/R \cong Q \cong N_{P\times Q}(\mathbf{1} \times R)/(\mathbf{1} \times R)$. Hence the above definition of $P \diamond Q$ and $\nu_{P,Q}$ is consistent in the case $P = \mathbf{1}$.

Theorem 4.20. *Let P and Q be Roquette p-groups, of exponents e_P and e_Q, respectively. Suppose $e_P \le e_Q$, and set $q = |Q|$. Then*

$$P \diamond Q \cong \begin{cases} Q & \text{if } P = \mathbf{1} \text{ or } P \cong C_p, \\ C_2 & \text{if } P \cong Q \cong Q_8, \\ D_q & \text{if } q \ge 16 \text{ and } P \text{ and } Q \text{ are both dihedral,} \\ & \quad \text{or both generalized quaternion,} \\ Q_q & \text{if one of } P, Q \text{ is dihedral,} \\ & \quad \text{and the other one is generalized quaternion,} \\ SD_q & \text{if } Q \text{ is semidihedral, and either} \\ & \bullet\ P \text{ is dihedral or generalized quaternion,} \\ & \quad \text{and } |P| < |Q|, \text{ or} \\ & \bullet\ P \cong Q, \\ C_{e_Q} & \text{otherwise.} \end{cases}$$

Proof. The case $P = \mathbf{1}$ is trivial, the case $P \cong C_p$ follows from (4.12), the case $P \cong Q \cong Q_8$ follows from (4.10), the three next cases in the list follow from (4.17), and the last case follows from (4.13) and (4.16). □

Theorem 4.21. *Let P and Q be Roquette p-groups, of exponent e_P and e_Q, respectively, and let $m = \min(e_P, e_Q)$.*

(1) *If $p = 2$ and one of the groups P or Q is isomorphic to Q_8, then $L \frown_{P\times Q} L'$ for any centrally diagonal genetic subgroups L and L' of $P \times Q$. In other words $\nu_{P,Q} = 1$.*
(2) *In all other cases, if L and L' are centrally diagonal genetic subgroups of $P \times Q$, then $L \frown_{P\times Q} L'$ if and only if L and L' are conjugate in $P \times Q$. In particular*

$$\nu_{P,Q} = \phi(m)m\frac{|P \diamond Q|}{|P|\,|Q|},$$

where ϕ is the Euler function.

Proof. Assume $e_P \le e_Q$, without loss of generality.

• If $P = \mathbf{1}$, then $\nu_{P,Q} = 1$, and $P \diamond Q = Q$ by definition, and $e_P = 1$, so there is nothing to prove.

• If $p = 2$ and $P \cong Q \cong Q_8$, then by Theorem 4.6, a genetic centrally diagonal subgroup L of $P \times Q$ is of the form $L_\varphi = \overrightarrow{\Delta}_\varphi(P)$, where $\varphi : P \to Q$ is a group isomorphism. Moreover $N_{P\times Q}(L)/L \cong C_2$, so $P \diamond Q \cong C_2$.

Now let $\varphi, \psi : P \to Q$ be two group isomorphisms. Then $L_\varphi \frown_{P\times Q} L_\psi$ if and only if there exists $(x, y) \in (P \times Q)$ such that

$$L_\varphi^{(x,y)} \cap Z_{P\times Q}(L_\psi) \le L_\psi, \qquad {}^{(x,y)}L_\psi \cap Z_{P\times Q}(L_\varphi) \le L_\varphi.$$

These conditions depend only on the double coset $N_{P\times Q}(L_\varphi)(x, y)N_{P\times Q}(L_\psi)$, which admits a representative of the form $(u, 1)$.

Now the condition $L_\varphi^{(u,1)} \cap Z_{P\times Q}(L_\psi) \leq L_\psi$ is equivalent to

$$\forall h \in P, \quad \varphi(h)^{-1}\psi(h^u) \in Z_Q \;\Rightarrow\; \varphi(h)^{-1}\psi(h^u) = 1,$$

and similarly the condition ${}^{(u,1)}L_\psi \cap Z_{P\times Q}(L_\varphi) \leq L_\varphi$ is equivalent to

$$\forall h \in P, \quad \psi(h)^{-1}\varphi({}^u h) \in Z_Q \;\Rightarrow\; \psi(h)^{-1}\varphi({}^u h) = 1.$$

If we apply ψ^{-1} to the first condition and φ^{-1} to the second one, and set $\theta = \psi^{-1}\varphi$, these two conditions become

$$\begin{aligned} &\forall h \in P, \quad \theta(h)^{-1}h^u \in Z_P \;\Rightarrow\; \theta(h) = h^u,\\ &\forall h \in P, \quad \theta^{-1}(h)^{-1}\,{}^u h \in Z_P \;\Rightarrow\; \theta^{-1}(h) = {}^u h. \end{aligned}$$

Since $h^u h^{-1} \in [P,P] = Z_P$, there are equivalences

$$\begin{aligned} 2\theta(h)^{-1}h^u \in Z_P \;&\Leftrightarrow\; \theta(h)^{-1}h \in Z_P \;\Leftrightarrow\; h^{-1}\theta(h) \in Z_P,\\ \theta^{-1}(h)^{-1}\,{}^u h \in Z_P \;&\Leftrightarrow\; \theta^{-1}(h)^{-1}h \in Z_P \;\Leftrightarrow\; h^{-1}\theta(h) \in Z_P. \end{aligned}$$

Hence, to prove that $L_\varphi \frown_{P\times Q} L_\psi$ for any $\varphi, \psi : P \xrightarrow{\cong} Q$, it is enough to show that

$$(\mathbf{4.22}) \qquad \forall \theta \in \mathrm{Aut}(P),\ \exists u \in P,\ \forall h \in P, \quad \theta(h)h^{-1} \in Z_P \;\Rightarrow\; \begin{cases} \theta(h) = h^u,\\ \theta^{-1}(h) = {}^u h. \end{cases}$$

If h has order 1 or 2, then $\theta(h) = h = h^u$ for any $u \in P$, hence the conditions $\theta(h) = h^u$ and $\theta^{-1}(h) = {}^u h$ only have to be checked for $|h| = 4$. Now the group $\mathrm{Aut}(P)$ permutes the three cyclic subgroups of order 4 of P, and this gives an exact sequence

$$1 \to \mathrm{Inn}(P) \to \mathrm{Aut}(P) \to S_3 \to 1,$$

where S_3 is the symmetric group on three symbols. Saying that $\theta(h)h^{-1} \in Z_P$ is equivalent to saying that $\theta(\langle h\rangle) = \langle h\rangle$. Hence, there are two possibilities:

Either θ stabilizes the three subgroups of order 4 of P, and in this case θ is inner, hence there exists $u \in P$ such that $\theta(h) = h^u$ for any $h \in P$, hence $\theta^{-1}(h) = {}^u h$ for any $h \in P$.

Or there exists a unique subgroup C of order 4 of P such that $\theta(C) = C$. Then either $\theta(h) = h$ for any $h \in C$, or $\theta(h) = h^{-1}$ for any $h \in C$. In the first case, take $u = 1$, and in the second case take $u \in P - \langle h\rangle$; then $\theta(h) = h^u$ for any $h \in C$, hence $h = \theta^{-1}(h)^u$ for $h \in C$, since $C = \theta(C)$. Thus (4.22) holds. This completes the proof in this case.

• If $P \cong Q_8$ and $Q \ncong Q_8$, then a centrally diagonal genetic subgroup of $P \times Q$ is of the form $L = \overrightarrow{\Delta}_\varphi(H)$, where H is one of the three subgroups of order 4 of P, and φ is some isomorphism from H to the unique axial subgroup K of order 4 of Q. Moreover $Z_{P\times Q}(L) = L(\mathbf{1} \times Z_Q)$.

Let H and H' be subgroups of order 4 of P. Let $\varphi : H \to K$ and $\varphi' : H' \to K$ be group isomorphisms, and set $L = \overrightarrow{\Delta}(\varphi)$ and $L' = \overrightarrow{\Delta}_{\varphi'}(H')$. Suppose first that $H \neq H'$, and let $(a,b) \in L' \cap Z_{P\times Q}(L) = L(\mathbf{1} \times Z_Q)$. This means that $a \in H' \cap H = Z_P$, and there exists $z \in Z_Q$ such that $\varphi'(a) = \varphi(a)z$. But the restrictions of φ and φ' to Z_P are equal, so $\varphi'(a) = \varphi(a)$, hence $z = 1$. It follows that $L' \cap Z_{P\times Q}(L) \leq L$, hence $L \cap Z_{P\times Q}(L') \leq L'$ by symmetry, so $L \frown_{P\times Q} L'$ in this case.

Now if $H = H'$, choose a subgroup H'' of order 4 in P, different from H, and a group isomorphism $\varphi'' : H'' \to K$. Set $L'' = \overrightarrow{\Delta}_{\varphi''}(H'')$. Then $L \frown_{P\times Q} L'' \frown_{P\times Q} L'$ by the previous argument, thus $L \frown_{P\times Q} L'$.

Hence $\nu_{P,Q} = 1$ in this case, as was to be shown.

• If there are several choices for K, i.e. $Q \cong Q_8$ and K has order $4 = \min(e_P, e_Q)$, it follows that $P \cong C_4$, since $P \ncong Q_8$. In this case, we can exchange P and Q, and use the previous argument. Hence $\nu_{P,Q} = 1$ in this case as well.

• In all other cases, by Theorem 4.6, a centrally diagonal genetic subgroup L of $P \times Q$ is of the form $\overrightarrow{\Delta}_\varphi(H)$, where $H \leq P$ is the unique axis of P, and $\varphi : H \hookrightarrow Q$ is a group isomorphism onto the unique axial subgroup K of order e_P of Q. The normalizer of L in $P \times Q$ does not depend on φ, by (4.11), so it does not depend on L, since H and K are also unique.

Let L and L' be two such centrally diagonal genetic subgroups of $P \times Q$. Then in particular $N_{P\times Q}(L) = N_{P\times Q}(L')$, thus $L \frown_{P\times Q} L'$ if and only if L and L' are conjugate in $P \times Q$, by Lemma 2.20. Moreover, it follows from the definition of $P \diamond Q$ that

$$|N_{P\times Q}(L)| = |L|\,|P \diamond Q| = e_P|P \diamond Q|,$$

so the conjugacy class of L in $P \times Q$ has cardinality $\frac{|P|\,|Q|}{e_P|P\diamond Q|}$. Since there are $\phi(e_P)$ possible choices for the isomorphism $\varphi : H \to K$, i.e. $\phi(e_P)$ centrally diagonal subgroups of $P \times Q$, it follows that

$$\nu_{P,Q} = \phi(e_P)e_P\frac{|P \diamond Q|}{|P|\,|Q|},$$

as was to be shown. □

Remark 4.23. Suppose that $P \cong Q_8$ and $Q \ncong Q_8$. Then $|P \diamond Q| = |Q|$, by Theorem 4.20. Hence

$$\phi(e_P)e_P\frac{|P \diamond Q|}{|P|\,|Q|} = 2 \times 4 \times \frac{|Q|}{8|Q|} = 1 = \nu_{P,Q},$$

so the formula for $\nu_{P,Q}$ holds in this case. The only case where $\nu_{P,Q}$ is not equal to $\phi(m)m\frac{|P\diamond Q|}{|P|\,|Q|}$ (where $m = \min(e_P, e_Q)$) is when $P \cong Q \cong Q_8$: in this case $\nu_{P,Q} = 1$, but

$$\phi(m)m\frac{|P \diamond Q|}{|P|\,|Q|} = 2 \times 4 \times \frac{2}{8 \times 8} = \frac{1}{4}.$$

Corollary 4.24. *Let P and Q be Roquette p-groups. Then, in the category $\mathcal{R}_p$,*

$$\partial P \times \partial Q \cong \nu_{P,Q} \cdot \partial(P \diamond Q).$$

In other words, if P has exponent e_P, Q has order q and exponent e_Q, and $e_P \leq e_Q$, then

$$\partial P \times \partial Q = \begin{cases} \partial Q & \text{if } P = \mathbf{1} \text{ or } P \cong C_2, \\ \partial C_2 & \text{if } P \cong Q \cong Q_8, \\ \frac{\phi(e_P)}{2} \cdot \partial D_q & \text{if } q \geq 16 \text{ and } P \text{ and } Q \text{ are both dihedral,} \\ & \quad \text{or both generalized quaternion,} \\ \frac{\phi(e_P)}{2} \cdot \partial Q_q & \text{if one of } P, Q \text{ is generalized quaternion,} \\ & \quad \text{and the other one is dihedral,} \\ \frac{\phi(e_P)}{2} \cdot \partial SD_q & \text{if } Q \text{ is semidihedral, and either} \\ & \bullet\ P \text{ is dihedral or generalized} \\ & \quad \text{quaternion, and } |P| < |Q|, \text{ or} \\ & \bullet\ P \cong Q, \\ \frac{\phi(e_P)e_Pe_Q}{|P||Q|} \cdot \partial C_{e_Q} & \text{otherwise.} \end{cases}$$

Proof. Let $\mathcal{B}$ be a genetic basis of the group $R = P \times Q$. In the category $\mathcal{R}_p$, the product $\partial P \times \partial Q$ is equal to $(P \times Q, f_{\mathbf{1}}^P \times f_{\mathbf{1}}^Q)$, and it is a summand of R. By Theorem 3.11, there are mutually inverse isomorphisms

$$R \underset{\mathcal{I}}{\overset{\mathcal{D}}{\rightleftarrows}} \bigoplus_{S \in \mathcal{B}} \partial \overline{N}_R(S),$$

where $\mathcal{I}$ is the direct sum of the maps $\mathrm{Indinf}_{\overline{N}_R(S)}^R$, and $\mathcal{D}$ is the direct sum of the maps $f_{\mathbf{1}}^{\overline{N}_R(S)} \mathrm{Defres}_{\overline{N}_R(S)}^R$. Corollary 4.24 follows from the fact that

$$f_{\mathbf{1}}^{\overline{N}_R(S)} \mathrm{Defres}_{\overline{N}_R(S)}^R (f_{\mathbf{1}}^P \times f_{\mathbf{1}}^Q) = 0 \tag{4.25}$$

unless S is centrally diagonal in $R = P \times Q$. Indeed, $\mathrm{Defres}_{\overline{N}_R(S)}^R$ is given by the action of the $(\overline{N}_R(S), R)$-biset $S \backslash R$. On the other hand

$$\begin{aligned} f_{\mathbf{1}}^P \times f_{\mathbf{1}}^Q &= (P/\mathbf{1} - P/Z_P) \times (Q/\mathbf{1} - Q/Z_Q) \\ &= R/(\mathbf{1} \times \mathbf{1}) - R/(\mathbf{1} \times Z_Q) - R/(Z_P \times \mathbf{1}) + R/(Z_P \times Z_Q), \end{aligned}$$

hence $\mathrm{Defres}_{\overline{N}_R(S)}^R (f_{\mathbf{1}}^P \times f_{\mathbf{1}}^Q)$ is equal to

$$S \backslash R/(\mathbf{1} \times \mathbf{1}) - S \backslash R/(\mathbf{1} \times Z_Q) - S \backslash R/(Z_P \times \mathbf{1}) + S \backslash R/(Z_P \times Z_Q),$$

which is

$$S \backslash R - S(\mathbf{1} \times Z_Q) \backslash R - S(Z_P \times \mathbf{1}) \backslash R + S(Z_P \times Z_Q) \backslash R. \tag{4.26}$$

If S is not centrally diagonal in $R = P \times Q$, then either $S = S(\mathbf{1} \times Z_Q)$ or $S = S(Z_P \times \mathbf{1})$. In each case the sum (4.26) vanishes.

And if S is centrally diagonal in R, then

$$S(\mathbf{1} \times Z_Q) = S(Z_P \times \mathbf{1}) = S(Z_P \times Z_Q),$$

since the image of these groups in the Roquette group $\overline{N}_R(S)$ is equal to its unique central subgroup $\widehat{S}/S$ of order p. In this case

$$\mathrm{Defres}^R_{\overline{N}_R(S)}(f_{\mathbf{1}}^P \times f_{\mathbf{1}}^Q) = S\backslash R - \widehat{S}\backslash R.$$

Since $f_{\mathbf{1}}^{\overline{N}_R(S)} = N_R(S)/S - N_R(S)/\widehat{S}$, it follows that $\mathrm{Defres}^R_{\overline{N}_R(S)}(f_{\mathbf{1}}^P \times f_{\mathbf{1}}^Q)$ is invariant under composition with $f_{\mathbf{1}}^{\overline{N}_R(S)}$.

Conversely, if S is not centrally diagonal in $P \times Q$, then

$$(f_{\mathbf{1}}^P \times f_{\mathbf{1}}^Q)\,\mathrm{Indinf}^R_{\overline{N}_R(S)}\, f_{\mathbf{1}}^{\overline{N}_R(S)} = 0,$$

as can be seen by taking opposite bisets in (4.25). And if S is centrally diagonal, then

$$(f_{\mathbf{1}}^P \times f_{\mathbf{1}}^Q)\,\mathrm{Indinf}^R_{\overline{N}_R(S)}\, f_{\mathbf{1}}^{\overline{N}_R(S)} = R/S - R/\widehat{S} = \mathrm{Indinf}^R_{\overline{N}_R(S)}\, f_{\mathbf{1}}^{\overline{N}_R(S)}.$$

Hence the isomorphisms $\mathcal{D}$ and $\mathcal{I}$ restrict to mutually inverse isomorphisms between the product $\partial P \times \partial Q$ and the direct sum of the edges $\partial \overline{N}_R(S)$, where S is a centrally diagonal genetic subgroup of R. But for all such subgroups S, the group $\overline{N}_R(S)$ is isomorphic to $P \diamond Q$, and there are $\nu_{P,Q}$ centrally diagonal subgroups in a genetic basis of $R = P \times Q$. This completes the proof. □

5. Examples and applications

5.1. Suppose first that p is odd. Then the Roquette p-groups are just the cyclic groups C_{p^n} for $n \geq 0$. The "multiplication rule" of the edges ∂C_{p^n} is the following:

$$\text{(5.2)} \qquad \forall m, \forall n \in \mathbb{N}, \quad \partial C_{p^m} \times \partial C_{p^n} = \phi(p^{\min(m,n)})\partial C_{p^{\max(m,n)}},$$

where ϕ is the Euler function (thus $\phi(p^k) = p^{k-1}(p-1)$ if $k > 0$, and $\phi(1) = 1$).

5.3. Some surprising phenomena occur when $p = 2$:

Proposition 5.4. *In $\mathcal{R}_2$, the edge ∂C_2 is isomorphic to the trivial group* $\mathbf{1}$ (*or its edge* $\partial\mathbf{1}$).

Proof. Indeed, Corollary 3.12 implies that if $E \cong (C_2)^2$, then $\partial E = 0$ in $\mathcal{R}_2$. Let X, Y, and Z denote the subgroups of order 2 of E. The element

$$u = \mathrm{Res}^E_X \times_E f_{\mathbf{1}}^E \times_E \mathrm{Ind}^E_Y$$

of $B(X, Y)$ can be viewed as a morphism from Y to X in the category $\mathcal{R}_2$, which factors through ∂E. So this morphism is equal to 0. Since

$$f_{\mathbf{1}}^E = E/\mathbf{1} - E/X - E/Y - E/Z + 2E/E,$$

it follows that

$$u = \mathrm{Ind}^X_{\mathbf{1}}\,\mathrm{Res}^Y_{\mathbf{1}} - \mathrm{Inf}^X_{\mathbf{1}}\,\mathrm{Res}^Y_{\mathbf{1}} - \mathrm{Ind}^X_{\mathbf{1}}\,\mathrm{Def}^Y_{\mathbf{1}} - \mathrm{Iso}(\varphi) + 2\,\mathrm{Inf}^X_{\mathbf{1}}\,\mathrm{Def}^Y_{\mathbf{1}},$$

where φ is the unique group isomorphism from Y to X. Thus

$$0 = u \ \mathrm{Iso}(\varphi^{-1}) = \mathrm{Ind}_{\mathbf{1}}^X \mathrm{Res}_{\mathbf{1}}^X - \mathrm{Inf}_{\mathbf{1}}^X \mathrm{Res}_{\mathbf{1}}^X - \mathrm{Ind}_{\mathbf{1}}^X \mathrm{Def}_{\mathbf{1}}^X - \mathrm{Id}_X + 2\,\mathrm{Inf}_{\mathbf{1}}^X \mathrm{Def}_{\mathbf{1}}^X .$$

Hence in the category $\mathcal{R}_2$,

$$\mathrm{Id}_X = (\mathrm{Ind}_{\mathbf{1}}^X - \mathrm{Inf}_{\mathbf{1}}^X)(\mathrm{Res}_{\mathbf{1}}^X - \mathrm{Def}_{\mathbf{1}}^X) + \mathrm{Inf}_{\mathbf{1}}^X \mathrm{Def}_{\mathbf{1}}^X .$$

It follows that

$$f_{\mathbf{1}}^X = f_{\mathbf{1}}^X (\mathrm{Ind}_{\mathbf{1}}^X - \mathrm{Inf}_{\mathbf{1}}^X)(\mathrm{Res}_{\mathbf{1}}^X - \mathrm{Def}_{\mathbf{1}}^X) f_{\mathbf{1}}^X . \tag{5.5}$$

But on the other hand $f_{\mathbf{1}}^X = \mathrm{Id}_X - \mathrm{Inf}_{\mathbf{1}}^X \mathrm{Def}_{\mathbf{1}}^X$, so

$$f_{\mathbf{1}}^X (\mathrm{Ind}_{\mathbf{1}}^X - \mathrm{Inf}_{\mathbf{1}}^X) = f_{\mathbf{1}}^X \mathrm{Ind}_{\mathbf{1}}^X = \mathrm{Ind}_{\mathbf{1}}^X - \mathrm{Inf}_{\mathbf{1}}^X \mathrm{Def}_{\mathbf{1}}^X \mathrm{Ind}_{\mathbf{1}}^X = \mathrm{Ind}_{\mathbf{1}}^X - \mathrm{Inf}_{\mathbf{1}}^X .$$

It follows that

$$\begin{aligned}(\mathrm{Res}_{\mathbf{1}}^X - \mathrm{Def}_{\mathbf{1}}^X) f_{\mathbf{1}}^X (\mathrm{Ind}_{\mathbf{1}}^X - \mathrm{Inf}_{\mathbf{1}}^X) &= (\mathrm{Res}_{\mathbf{1}}^X - \mathrm{Def}_{\mathbf{1}}^X)(\mathrm{Ind}_{\mathbf{1}}^X - \mathrm{Inf}_{\mathbf{1}}^X)\\ &= 2\,\mathrm{Id}_{\mathbf{1}} - \mathrm{Id}_{\mathbf{1}} - \mathrm{Id}_{\mathbf{1}} + \mathrm{Id}_{\mathbf{1}} = \mathrm{Id}_{\mathbf{1}} .\end{aligned}$$

Thus, if we set

$$\begin{aligned} a &= f_{\mathbf{1}}^X (\mathrm{Ind}_{\mathbf{1}}^X - \mathrm{Inf}_{\mathbf{1}}^X) \in \mathrm{Hom}_{\mathcal{R}_2}(\mathbf{1}, \partial X),\\ b &= (\mathrm{Res}_{\mathbf{1}}^X - \mathrm{Def}_{\mathbf{1}}^X) f_{\mathbf{1}}^X \in \mathrm{Hom}_{\mathcal{R}_2}(\partial X, \mathbf{1}),\end{aligned}$$

then the composition $b \circ a$ is equal to $\mathrm{Id}_{\mathbf{1}}$, and (5.5) shows that $a \circ b$ is equal to the identity of ∂X. So a and b are mutually inverse isomorphisms between $\mathbf{1}$ and ∂X. □

Corollary 5.6. *Let F be a rational 2-biset functor. Then for any finite 2-group P,*

$$F(C_2 \times P) \cong F(P) \oplus F(P), \qquad F(D_8 \times P) \cong F(P)^{\oplus 5}.$$

Proof. Indeed, rational p-biset functors are exactly those p-biset functors which factor through the category $\mathcal{R}_p$. And in $\mathcal{R}_2$, by Theorem 3.11, there is an isomorphism

$$C_2 \cong \mathbf{1} \oplus \partial C_2 \cong \mathbf{1} \oplus \mathbf{1}.$$

Thus $C_2 \times P \cong P \oplus P$, and the first assertion follows. The second one follows from Example 3.13, which shows that in $\mathcal{R}_2$,

$$D_8 \cong \mathbf{1} \oplus 4 \cdot \partial C_2 \cong 5 \cdot \mathbf{1}.$$

Hence $D_8 \times P \cong 5 \cdot P$, thus $F(D_8 \times P) \cong F(P)^{\oplus 5}$. □

Proposition 5.7. *The edge ∂Q_8 is an involution: more precisely,*

$$\partial Q_8 \times \partial Q_8 = \partial C_2 \cong \mathbf{1}.$$

Proof. Indeed, $Q_8 \diamond Q_8 = C_2$, and $\nu_{Q_8, Q_8} = 1$ by Theorem 4.21. □

Remark 5.8. The "action" of this involution on the edges of other Roquette 2-groups (that is, different from $\mathbf{1}$, C_2, and Q_8) is as follows: it stabilizes cyclic and semidihedral groups, and exchanges dihedral and generalized quaternion groups. More precisely, it follows from Corollary 4.24 that

$$\begin{aligned}
\forall n \geq 2, &\quad \partial Q_8 \times \partial C_{2^n} &= \partial C_{2^n},\\
\forall n \geq 4, &\quad \partial Q_8 \times \partial D_{2^n} &= \partial Q_{2^n},\\
\forall n \geq 4, &\quad \partial Q_8 \times \partial Q_{2^n} &= \partial D_{2^n},\\
\forall n \geq 4, &\quad \partial Q_8 \times \partial SD_{2^n} &= \partial SD_{2^n}.
\end{aligned}$$

5.9. By Theorem 3.11, any finite p-group is isomorphic to a direct sum of edges of Roquette p-groups in the category $\mathcal{R}_p$. The following result shows that the summands of such an arbitrary direct sum are unique up to group isomorphism, with the possible exception of the isomorphism $\mathbf{1} = \partial\mathbf{1} \cong \partial C_2$ of Proposition 5.4:

Proposition 5.10. *Let $\mathcal{S}$ and $\mathcal{T}$ be finite sequences of Roquette p-groups such that there exists an isomorphism*

$$\bigoplus_{S\in\mathcal{S}} \partial S \cong \bigoplus_{T\in\mathcal{T}} \partial T \quad \text{in the category } \mathcal{R}_p. \tag{5.11}$$

If $p = 2$, replace any occurrence of C_2 in $\mathcal{S}$ and $\mathcal{T}$ by the trivial group, which does not change the existence of the isomorphism (5.11), by Proposition 5.4. Then there exists a bijection $\varphi : \mathcal{S} \to \mathcal{T}$ such that the groups S and $\varphi(S)$ are isomorphic for any $S \in \mathcal{S}$.

Proof. By [Bou04], the simple biset functors $S_{R,\mathbb{F}_p}$, where R is a Roquette p-group different from C_p, are rational biset functors. Moreover, if $|R| \geq p^2$, then for any finite p-group P, the dimension of $S_{R,\mathbb{F}_p}(P)$ is equal to the number of groups S in a genetic basis of P such that $\overline{N}_P(S) \cong R$. On the other hand, the $\mathbb{F}_p$-dimension of $S_{\mathbf{1},\mathbb{F}_p}(P)$ is equal to the number of groups S in a genetic basis of P such that $|\overline{N}_P(S)| \leq p$.

The functor $S_{R,\mathbb{F}_p}$ extends to an additive functor from $\mathcal{R}_p$ to the category of $\mathbb{F}_p$-vector spaces, and the value of this functor at the edge ∂P is by definition equal to $\partial S_{R,\mathbb{F}_p}(P)$. Let $\mathcal{B}$ be a genetic basis of P. When $N \trianglelefteq P$ and $N \leq S \leq P$, set $\overline{S} = S/N$, and note that $\overline{N}_{\overline{P}}(\overline{S}) \cong \overline{N}_P(S)$. Then the set $\mathcal{B}_N = \{\overline{S} \mid S \in \mathcal{B},\ S \geq N\}$ is a genetic basis of $\overline{P} = P/N$.

Thus for any Roquette group R,

$$\begin{aligned}
|\{S \in \mathcal{B} \mid \overline{N}_P(S) \cong R\}| &= \sum_{N \trianglelefteq P} \Big|\Big\{S \in \mathcal{B} \;\Big|\; \overline{N}_P(S) \cong R,\ \bigcap_{g\in P} S^g = N\Big\}\Big|\\
&= \sum_{N \trianglelefteq P} \Big|\Big\{\overline{S} \in \mathcal{B}_N \;\Big|\; \overline{N}_{\overline{P}}(\overline{S}) \cong R,\ \bigcap_{\overline{g}\in \overline{P}} \overline{S}^{\overline{g}} = \mathbf{1}\Big\}\Big|\\
&= \sum_{N \trianglelefteq P} |\{\overline{S} \in \mathcal{B}_N \mid \overline{N}_{\overline{P}}(\overline{S}) \cong R,\ \overline{S} \cap Z(\overline{P}) = \mathbf{1}\}|.
\end{aligned}$$

It follows easily from Proposition 3.8 that if $|R| \geq p^2$, then the $\mathbb{F}_p$-dimension of $\partial S_{R,\mathbb{F}_p}(P)$ is equal to the number of groups S in a genetic basis of P such that

$\overline{N}_P(S) \cong R$ and $S \cap Z(P) = \mathbf{1}$. In particular, if P itself is a Roquette group, then

$$\dim_{\mathbb{F}_p} S_{R,\mathbb{F}_p}(\partial P) = \dim_{\mathbb{F}_p} \partial S_{R,\mathbb{F}_p}(P) = \begin{cases} 1 & \text{if } P \cong R, \\ 0 & \text{otherwise.} \end{cases}$$

By applying the functor $S_{R,\mathbb{F}_p}$ to the isomorphism (5.11), this implies that the number of terms in the sequence $\mathcal{S}$ which are isomorphic to R is equal to the corresponding number in $\mathcal{T}$.

Similarly, for any finite p-group P, the $\mathbb{F}_p$-dimension of $\partial S_{\mathbf{1},\mathbb{F}_p}(P)$ is equal to the number of groups S in a genetic basis of P such that $\overline{N}_P(S) \leq p$ and $S \cap Z(P) = \mathbf{1}$. If P itself is a Roquette group, this gives

$$\dim_{\mathbb{F}_p} S_{\mathbf{1},\mathbb{F}_p}(\partial P) = \dim_{\mathbb{F}_p} \partial S_{\mathbf{1},\mathbb{F}_p}(P) = \begin{cases} 1 & \text{if } P \cong C_p, \\ 1 & \text{if } P \cong \mathbf{1}, \\ 0 & \text{otherwise.} \end{cases}$$

Hence the number of terms in $\mathcal{S}$ which are isomorphic to $\mathbf{1}$ or C_p is equal to the corresponding number in $\mathcal{T}$. If $p = 2$, there are no S in $\mathcal{S} \cup \mathcal{T}$ such that $S \cong C_2$, by assumption. It follows that for any Roquette p-group R, the number of terms in $\mathcal{S}$ which are isomorphic to R is equal to the corresponding number in $\mathcal{T}$. The proposition follows in this case.

If $p > 2$, the above argument shows that

$$\bigoplus_{\substack{S \in \mathcal{S} \\ |S| \geq p^2}} \partial S \cong \bigoplus_{\substack{T \in \mathcal{T} \\ |T| \geq p^2}} \partial T.$$

Let M denote this direct sum. The isomorphism (5.11) can be rewritten as

$$m_{\mathbf{1}}\mathbf{1} \oplus m_{C_p}\partial C_p \oplus M \cong n_{\mathbf{1}}\mathbf{1} \oplus n_{C_p}\partial C_p \oplus M \tag{5.12}$$

for some integers $m_{\mathbf{1}}, m_{C_p}, n_{\mathbf{1}}, n_{\mathbb{C}_p}$ such that $m_{\mathbf{1}} + m_{C_p} = n_{\mathbf{1}} + n_{C_p}$.

Now, let ζ be a primitive (i.e. non-trivial, since p is prime) character $(\mathbb{Z}/p\mathbb{Z})^\times \to \mathbb{C}$. Such a character exists since $p > 2$. The functor $S_{C_p,\zeta}$ is a rational p-biset functor, as it is a summand of $\mathbb{C}R_{\mathbb{C}}$ [Bou10, Corollary 7.3.5]. Applying this functor to the isomorphism 5.12 and taking dimensions gives

$$m_{C_p} + \dim_{\mathbb{C}} S_{C_p,\zeta}(M) = n_{C_p} + \dim_{\mathbb{C}} S_{C_p,\zeta}(M),$$

since $S_{C_p,\zeta}(\mathbf{1}) = 0$ and $S_{C_p,\zeta}(C_p) = \partial S_{C_p,\zeta}(C_p) \cong \mathbb{C}$.

It follows that $m_{C_p} = n_{C_p}$, hence $m_{\mathbf{1}} = n_{\mathbf{1}}$, which completes the proof. □

Corollary 5.13. *Let X, Y, and Z be objects of $\mathcal{R}_p$, isomorphic to direct sums of edges of Roquette p-groups.*

(1) *If $X \oplus Z \cong Y \oplus Z$ in $\mathcal{R}_p$, then $X \cong Y$.*
(2) *If n is a positive integer, and if $n \cdot X \cong n \cdot Y$ in $\mathcal{R}_p$, then $X \cong Y$.*

Proof. Decompose X as $X \cong \bigoplus_R n_R(X) \cdot \partial R$, where R runs through the set of isomorphism classes of Roquette p-groups, and the function $R \mapsto n_R(X) \in \mathbb{N}$ has finite support. Choose similar decompositions $Y \cong \bigoplus_R n_R(Y) \cdot \partial R$ and $Z \cong \bigoplus_R n_R(Z) \cdot \partial R$.

For (1), if $p > 2$, it follows from Proposition 5.10 that

$$n_R(X) + n_R(Z) = n_R(Y) + n_R(Z)$$

for each R. Thus $n_R(X) = n_R(Y)$ for each R, hence $X \cong Y$ in $\mathcal{R}_p$.

If $p = 2$, and if R is a Roquette p-group different from $\mathbf{1}$ and C_2, Proposition 5.10 shows that $n_R(X) + n_R(Z) = n_R(Y) + n_R(Z)$, hence $n_R(X) = n_R(Y)$. Proposition 5.10 also implies that

$$n_{\mathbf{1}}(X) + n_{C_2}(X) + n_{\mathbf{1}}(Z) + n_{C_2}(Z) = n_{\mathbf{1}}(Y) + n_{C_2}(Y) + n_{\mathbf{1}}(Z) + n_{C_2}(Z),$$

whence $n_{\mathbf{1}}(X) + n_{C_2}(X) = n_{\mathbf{1}}(Y) + n_{C_2}(Y)$, and $X \cong Y$ in $\mathcal{R}_2$ again, since $\mathbf{1} \cong C_2$.

The proof of (2) is similar: if $p > 2$, Proposition 5.10 shows that

$$n n_R(X) = n n_R(Y)$$

for any R, thus $n_R(X) = n_R(Y)$, and $X \cong Y$. And if $p = 2$, the conclusion $n_R(X) = n_R(Y)$ is valid for R different from $\mathbf{1}$ and C_2. Moreover

$$n(n_{\mathbf{1}}(X) + n_{C_2}(X)) = n(n_{\mathbf{1}}(Y) + n_{C_2}(Y)),$$

hence $n_{\mathbf{1}}(X) + n_{C_2}(X) = n_{\mathbf{1}}(Y) + n_{C_2}(Y)$, and $X \cong Y$, since $\mathbf{1} \cong C_2$. □

In the case of the decomposition of a p-group as a direct sum of edges of Roquette groups, the above isomorphism $\partial C_2 \cong \partial \mathbf{1}$ does not matter, and the decomposition is unique:

Proposition 5.14. *Let P and Q be finite p-groups. The following assertions are equivalent:*

(1) *P and Q are isomorphic in the category $\mathcal{R}_p$.*
(2) *There exist genetic bases $\mathcal{B}_P$ and $\mathcal{B}_Q$ of P and Q, respectively, and a bijection $\sigma : \mathcal{B}_P \to \mathcal{B}_Q$ such that*

$$\forall S \in \mathcal{B}_P, \quad N_Q(\sigma(S))/\sigma(S) \cong N_P(S)/S. \tag{5.15}$$

(3) *For any genetic bases $\mathcal{B}_P$ and $\mathcal{B}_Q$ of P and Q, respectively, there exists a bijection $\sigma : \mathcal{B}_P \to \mathcal{B}_Q$ such that (5.15) holds.*

Proof. (2) implies (1) by Theorem 3.11. Now suppose that (1) holds. Then in particular $F(P) \cong F(Q)$ for any rational p-biset functor F. Let $\mathcal{B}_P$ and $\mathcal{B}_Q$ be genetic bases of P and Q, respectively. If R is a Roquette p-group, set

$$m_P(R) = |\{S \in \mathcal{B} \mid N_P(S)/S \cong R\}|,$$

and define similarly $m_Q(R)$ for the group Q. The integers $m_P(R)$ and $m_Q(R)$ do not depend on the choices of the genetic bases $\mathcal{B}_P$ and $\mathcal{B}_Q$.

If R is not isomorphic to C_p, then the simple functor $S_{R,\mathbb{F}_p}$ is rational. Moreover, the $\mathbb{F}_p$-dimension of $S_{R,\mathbb{F}_p}(P)$ is equal to $m_P(R)$ if $|R| > p$, and to $1 + m_P(C_p)$ if $R = \mathbf{1}$. Since P is the only element S of $\mathcal{B}$ such that $N_P(S)/S = \mathbf{1}$, it follows that $m_P(\mathbf{1}) = 1$; then $m_P(R) = m_Q(R)$ for any Roquette p-group R, and (2) follows. The equivalence of (2) and (3) follows from Theorem 2.22. □

Example 5.16. • Let $p > 2$, and let X^+ (resp. X^-) denote the extraspecial p-group of order p^3 and exponent p (resp. p^2). Then $X^+ \cong X^-$ in $\mathcal{R}_p$, for if P is one of these groups, then each genetic basis of P consists of $S = P$, for which $N_P(S)/S = \mathbf{1}$, of the $p+1$ subgroups S of index p in P, for which $N_P(S) = P/S \cong C_p$, and an additional non-normal genetic subgroup S such that $N_P(S)/S \cong C_p$. In other words

$$X^+ \cong X^- \cong \mathbf{1} \oplus (p+2) \cdot \partial C_p \quad \text{in } \mathcal{R}_p.$$

• Similar examples exist for $p = 2$: if P is one of the groups labelled 6 or 7 in the GAP list of groups of order 32 (see [GAP13]), with respective structure $((C_4 \times C_2) \rtimes C_2) \rtimes C_2$ and $(C_8 \rtimes C_2) \rtimes C_2$, then in any genetic basis of P, there is a unique group S ($= P$) such that $N_P(S)/S = \mathbf{1}$, there are six groups S such that $N_P(S)/S \cong C_2$, and two groups S such that $N_P(S)/S \cong C_4$.

• Some 2-groups *with different orders* may become isomorphic in the category $\mathcal{R}_2$: using GAP, one can show that the elementary abelian group of order 16 is isomorphic to each of the groups labelled 134, 138, and 177 in GAP's list of groups of order 64. These groups have respective structures

$$((C_4 \times C_4) \rtimes C_2) \rtimes C_2, \quad \big(((C_4 \times C_2) \rtimes C_2) \rtimes C_2\big) \rtimes C_2, \quad \text{and} \quad (C_2 \times D_{16}) \rtimes C_2.$$

I could not find any similar example for $p > 2$. In this case however, the following result characterizes those p-groups which become isomorphic in the category $\mathcal{R}_p$:

Proposition 5.17. *Let p be a prime number, and let P and Q be finite p-groups.*

(1) *If $P \cong Q$ in the category $\mathcal{R}_p$, then the $\mathbb{Q}$-algebras $Z\mathbb{Q}P$ and $Z\mathbb{Q}Q$ are isomorphic.*
(2) *If $p > 2$, and if $Z\mathbb{Q}P$ and $Z\mathbb{Q}Q$ are isomorphic $\mathbb{Q}$-algebras, then $P \cong Q$ in $\mathcal{R}_p$.*

Proof. Let $\mathcal{G}$ be a genetic basis of P. For $S \in \mathcal{G}$, let $V(S)$ denote the corresponding simple $\mathbb{Q}P$-module, defined by

$$V(S) = \mathrm{Indinf}_{\overline{N}_P(S)}^P \Phi_{\overline{N}_P(S)}.$$

The multiplicity v_S of $V(S)$ in the $\mathbb{Q}P$-module $\mathbb{Q}P$ is equal to

$$v_S = \frac{\dim_{\mathbb{Q}} V(S)}{\dim_{\mathbb{Q}} \mathrm{End}_{\mathbb{Q}P}(V(S))}.$$

As S is a genetic subgroup of P, there is an isomorphism of (skew-)fields

$$\mathrm{End}_{\mathbb{Q}P}(V(S)) \cong \mathrm{End}_{\mathbb{Q}\overline{N}_P(S)}(\Phi_{\overline{N}_P(S)}).$$

It follows that there is an isomorphism of $\mathbb{Q}$-algebras

$$\mathbb{Q}P \cong \prod_{S\in\mathcal{G}} M_{v_S}(\mathrm{End}_{\mathbb{Q}\overline{N}_P(S)}(\Phi_{\overline{N}_P(S)})).$$

Hence

$$Z\mathbb{Q}P \cong \prod_{S\in\mathcal{G}} Z(\mathrm{End}_{\mathbb{Q}\overline{N}_P(S)}(\Phi_{\overline{N}_P(S)})).$$

This shows that the isomorphism type of the $\mathbb{Q}$-algebra $Z\mathbb{Q}P$ depends only on the genetic basis $\mathcal{G}$: more precisely, it is determined by the isomorphism type of P in $\mathcal{R}_p$. This proves (1).

Now if $p > 2$, the group $\overline{N}_P(S)$ is cyclic, of order p^{m_S} say. By Example 2.11, there is an isomorphism of (skew-)fields

$$\mathrm{End}_{\mathbb{Q}\overline{N}_P(S)}(\Phi_{\overline{N}_P(S)}) \cong \mathbb{Q}(\zeta_{p^{m_S}}),$$

where $\zeta_{p^{m_S}}$ is a primitive root of unity of order p^{m_S}. Hence

$$Z\mathbb{Q}P \cong \prod_{S\in\mathcal{G}} \mathbb{Q}(\zeta_{p^{m_S}}).$$

Similarly, if $\mathcal{H}$ is a genetic basis of Q then

$$Z\mathbb{Q}Q \cong \prod_{T\in\mathcal{H}} \mathbb{Q}(\zeta_{p^{n_T}}),$$

where $p^{n_T} = |\overline{N}_Q(T)|$.

Let l be an integer greater than all the m_S's for $S \in \mathcal{G}$, and all the n_T's for $T \in \mathcal{H}$. Set $K = \mathbb{Q}(\zeta_{p^l})$, and let G be the Galois group of K over $\mathbb{Q}$. By Galois theory [Sza09, Theorem 1.5.4 and Remark 1.5.5], the $\mathbb{Q}$-algebras $Z\mathbb{Q}P$ and $Z\mathbb{Q}Q$ are isomorphic if and only if there is an isomorphism of G-sets

$$\mathrm{Hom}_{\mathrm{alg}}(Z\mathbb{Q}P, K) \cong \mathrm{Hom}_{\mathrm{alg}}(Z\mathbb{Q}Q, K).$$

When $r \leq l$ is an integer, let G_r denote the Galois group of K over $\mathbb{Q}(\zeta_{p^r})$. Then the G-set $\mathrm{Hom}_{\mathrm{alg}}(Z\mathbb{Q}P, K)$ is isomorphic to

$$\bigsqcup_{S\in\mathcal{G}} G/G_{n_S}.$$

The isomorphism $Z\mathbb{Q}P \cong Z\mathbb{Q}Q$ implies that for any $r \leq l$, the number of $S \in \mathcal{G}$ such that $\overline{N}_P(S)$ has order p^r is equal to the number of $T \in \mathcal{H}$ such that $\overline{N}_Q(T)$ has order p^r. Now (2) follows from Proposition 5.14. □

Remark 5.18. Proposition 5.17(2) is not true for $p = 2$. Let $P = D_8$ and $Q = Q_8$ denote a dihedral group of order 8 and a quaternion group of order 8, respectively. By Example 3.13, in a genetic basis of P, there is one group S such that $N_P(S)/S = \mathbf{1}$ (namely $S = P$), and four subgroups S such that $N_P(S)/S \cong C_2$ (the three subgroups of index 2 in P, and a non-central subgroup of order 2 of P). It follows easily that

$$\mathbb{Q}D_8 \cong \mathbb{Q} \oplus \mathbb{Q} \oplus \mathbb{Q} \oplus \mathbb{Q} \oplus M_2(\mathbb{Q}).$$

On the other hand, a genetic basis of Q contains one subgroup S such that $N_Q(S)/S = \mathbf{1}$ (namely $S = Q$), three subgroups S such that $N_Q(S)/S \cong C_2$ (the three subgroups of index 2 in Q), and one subgroup S such that $N_Q(S)/S \cong Q_8$ (the trivial subgroup of Q). Hence

$$\mathbb{Q}Q_8 \cong \mathbb{Q} \oplus \mathbb{Q} \oplus \mathbb{Q} \oplus \mathbb{Q} \oplus \mathbb{H}_{\mathbb{Q}},$$

where $\mathbb{H}_{\mathbb{Q}}$ is the field of quaternions over $\mathbb{Q}$. Then

$$Z\mathbb{Q}D_8 \cong \mathbb{Q}^5 \cong Z\mathbb{Q}Q_8.$$

But D_8 and Q_8 are not isomorphic in $\mathcal{R}_2$, by Proposition 5.14.

5.19. Genetic bases of direct products. Theorem 4.21 yields a way to compute a genetic basis of a direct product of p-groups. More precisely:

Theorem 5.20. *Let P and Q be finite p-groups, let $\mathcal{B}_P$ be a genetic basis of P, and let $\mathcal{B}_Q$ be a genetic basis of Q.*

(1) *For each pair $(S, T) \in \mathcal{B}_P \times \mathcal{B}_Q$, let $\overline{R}$ be a centrally diagonal genetic subgroup of $\overline{N}_P(S) \times \overline{N}_Q(T)$, and let*

$$R = \{(x, y) \in N_P(S) \times N_Q(T) \mid (xS, yT) \in \overline{R}\}.$$

Then R is a genetic subgroup of $P \times Q$ such that

$$\overline{N}_{P\times Q}(R) \cong \overline{N}_P(S) \diamond \overline{N}_Q(T).$$

(2) *For $(S, T) \in \mathcal{B}_P \times \mathcal{B}_Q$, let $\mathcal{E}_{S,T}$ denote the set of subgroups R obtained in* (1) *when $\overline{R}$ runs through a set of representatives of centrally diagonal genetic subgroups of $\overline{N}_P(S) \times \overline{N}_Q(T)$ for the relation $\frown_{\overline{N}_P(S)\times\overline{N}_Q(T)}$, as described in Theorem 4.21. Then the sets $\mathcal{E}_{S,T}$ consist of mutually inequivalent genetic subgroups of $P \times Q$ for the relation $\frown_{P\times Q}$, and the* (*disjoint*) *union*

$$\mathcal{B}_{P\times Q} = \bigsqcup_{(S,T)\in\mathcal{B}_P\times\mathcal{B}_Q} \mathcal{E}_{S,T}$$

is a genetic basis of $P \times Q$.

Proof. (1) is straightforward if the group $\overline{N}_P(S)$ is trivial, i.e. $S = P$, or if the group $\overline{N}_Q(T)$ is trivial, i.e. $T = Q$. So we can assume that $S < P$ and $T < Q$.

By Theorem 4.6, the group $\overline{R}$ is diagonal in $\overline{N}_P(S) \times \overline{N}_Q(T)$. Hence $k_1(R) = S$, $k_2(R) = T$, and $\overline{R} = R/(S \times T)$. This implies that $N_{P\times Q}(R) \leq N_P(S) \times N_Q(T)$. More precisely

$$N_{P\times Q}(R) = \{(a, b) \in N_P(S) \times N_Q(T) \mid (aS, bT) \in N_{\overline{N}_P(S)\times\overline{N}_Q(T)}(\overline{R})\},$$

and the map $(a, b) \mapsto (aS, bT)$ induces a group isomorphism

$$N_{P\times Q}(R)/R \cong N_{\overline{N}_P(S)\times\overline{N}_Q(T)}(\overline{R})/\overline{R}.$$

It follows that $N_{P\times Q}(R)/R$ is a Roquette group, and by Theorem 4.6 again, and Notation 4.18,

$$N_{P\times Q}(R)/R \cong \overline{N}_P(S) \diamond \overline{N}_Q(T).$$

Let $\widehat{S} \geq S$ denote the subgroup of $N_P(S)$ such that $\widehat{S}/S$ is the unique central subgroup of order p of the Roquette group $\overline{N}_P(S)$. Define $\widehat{T} \geq T$ similarly, and let $\widehat{R}/R$ be the unique central subgroup of order p of $N_{P\times Q}(R)/R$. Then

$$\widehat{R} = (\widehat{S} \times \mathbf{1})R = (\mathbf{1} \times \widehat{T})R.$$

Let $(x, y) \in P \times Q$ be such that $R^{(x,y)} \cap \widehat{R} \leq R$. Intersecting this inclusion with $P \times \mathbf{1}$ gives

$$(S^x \cap \widehat{S}) \times \mathbf{1} \leq S \times \mathbf{1},$$

thus $S^x \cap \widehat{S} \leq S$, and it follows that $x \in N_P(S)$, since S is an expansive subgroup of P. Similarly, intersecting the inclusion $R^{(x,y)} \cap \widehat{R} \leq R$ with $\mathbf{1} \times Q$ gives $T^y \cap \widehat{T} \leq T$, hence $y \in N_Q(T)$.

Now $S \times T \leq R^{(x,y)} \cap \widehat{R} \leq R$, and taking the quotient by $S \times T$ gives

$$\overline{R}^{(xS,yT)} \cap (\widehat{R}/(S \times T)) \leq \overline{R}.$$

As $\overline{R}$ is a genetic subgroup of $\overline{N}_P(S) \times \overline{N}_Q(T)$, it follows that $\overline{R}^{(xS,yT)}$ is equal to $\overline{R}$, hence $R^{(x,y)} = R$. Thus R is an expansive subgroup of $P \times Q$. Since $N_{P\times Q}(R)/R$ is a Roquette group, the group R is a genetic subgroup of $P \times Q$, and this completes the proof of (1).

For (2), let (S, T) and (S', T') be in $\mathcal{B}_P \times \mathcal{B}_Q$, and let $R \in \mathcal{E}_{S,T}$ and $R' \in \mathcal{E}_{S',T'}$ be such that $R \frown_{P\times Q} R'$. This means that there exists $(x, y) \in P \times Q$ such that

$$R^{(x,y)} \cap \widehat{R'} \leq R', \qquad {}^{(x,y)}R' \cap \widehat{R} \leq R. \tag{5.21}$$

Intersecting these two inclusions with $P \times \mathbf{1}$ gives

$$S^x \cap \widehat{S'} \leq S', \qquad {}^{x}S' \cap \widehat{S} \leq S.$$

Hence $S' \frown_P S$, thus $S' = S$, since S and S' are in the same genetic basis of P. Moreover $x \in N_P(S)$. Similarly, intersecting (5.21) with $\mathbf{1} \times Q$ implies $T = T'$, and $y \in N_Q(T)$.

Quotienting the inclusions (5.21) by $S \times T$ gives that $\overline{R}' \frown_{\overline{N}_P(S)\times\overline{N}_Q(T)} \overline{R}$. Hence $R' = R$, as was to be shown.

Now setting

$$\mathcal{B}_{P\times Q} = \bigsqcup_{(S,T)\in\mathcal{B}_P\times\mathcal{B}_Q} \mathcal{E}_{S,T}$$

yields a set of genetic subgroups of $P \times Q$ which are inequivalent to each other for the relation $\frown_{P\times Q}$. But

$$|\mathcal{E}_{S,T}| = \nu_{\overline{N}_P(S),\overline{N}_Q(T)},$$

and $N_{P\times Q}(R)/R \cong \overline{N}_P(S) \diamond \overline{N}_Q(T)$ for any $R \in \mathcal{E}_{S,T}$. It follows that

$$\begin{aligned}\bigoplus_{R\in\mathcal{B}_{P\times Q}} \partial N_{P\times Q}(R)/R &\cong \bigoplus_{\substack{S\in\mathcal{B}_P\\ T\in\mathcal{B}_Q}} \nu_{\overline{N}_P(S),\overline{N}_Q(T)}\partial(\overline{N}_P(S) \diamond \overline{N}_Q(T))\\ &\cong \Big(\bigoplus_{S\in\mathcal{B}_P} \partial\overline{N}_P(S)\Big) \times \Big(\bigoplus_{T\in\mathcal{B}_Q} \partial\overline{N}_Q(T)\Big) \cong P \times Q.\end{aligned}$$

In particular, the rank $l_{\mathbb{Q}}(P \times Q)$ of the group $R_{\mathbb{Q}}(P \times Q)$ is equal to $|\mathcal{B}_{P\times Q}|$. Since $\mathcal{B}_{P\times Q}$ is contained in a genetic basis of $P \times Q$, which has cardinality $l_{\mathbb{Q}}(P \times Q)$, it follows that $\mathcal{B}_{P\times Q}$ is a genetic basis of $P \times Q$. □

Remark 5.22. Theorem 5.20 does not mean that any genetic subgroup of $P \times Q$ can be obtained by the construction of (1). For example, if $[P, P] \leq Z(P)$ and $Z(P)$ is cyclic, then the diagonal $R = \Delta(P)$ is a genetic subgroup of $P \times P$, by Example 2.17. But $k_1(R) = \mathbf{1}$ is not a genetic subgroup of P if P is not a Roquette group.

5.23. Example of application. As explained in Example 3.13, the dihedral group D_8 splits as

(5.24) $$D_8 \cong \mathbf{1} \oplus 4\partial C_2 \quad \text{in } \mathcal{R}_2.$$

Hence $D_8 \cong 5 \cdot \mathbf{1}$ in $\mathcal{R}_2$, by Proposition 5.4, and $(D_8)^n \cong 5^n \cdot \mathbf{1}$ for any $n \in \mathbb{N}$. In particular, if F is a rational 2-biset functor such that $F(\mathbf{1}) = \{0\}$, then $F((D_8)^n) = \{0\}$. Hence $F(P) = \{0\}$ for any quotient of a direct product of copies of D_8, by Remark 3.10.

Actually, one can be more precise: since $\partial C_2 \times \partial C_2 \cong \partial C_2$ by Corollary 4.24, it follows that for any $n \in \mathbb{N}$,

(5.25) $$(D_8)^n \cong \bigoplus_{i=0}^{n} \binom{n}{i} 4^i \cdot (\partial C_2)^i = \mathbf{1} \oplus \bigoplus_{i=1}^{n} \binom{n}{i} 4^i \cdot \partial C_2 \cong \mathbf{1} \oplus (5^n - 1) \cdot \partial C_2.$$

This means that a genetic basis of the group $P = (D_8)^n$ is made up of the group $S = P$, for which $N_P(S)/S = \mathbf{1}$, and of $5^n - 1$ subgroups S for which $N_P(S)/S \cong C_2$.

In particular, by [Bou06, Theorem 9.5] (or [Bou10, Corollary 12.10.3]), the Dade group of P is torsion free, and so is the Dade group of any factor group of P, by Remark 3.10 again. This shows that the Dade group of a central product of any number of copies of D_8 is torsion free (see Theorem 5.36 for a generalization of this result): this was proved by Nadia Mazza and myself [BM04, Theorem 9.2]. However, the above argument cannot be considered as a new proof of this result, since [Bou06, Theorem 9.5] relies on [BM04, Theorem 9.2].

5.26. Edges of central products. Let P and Q be non-trivial finite p-groups. We recall that a *central product* $P *_\varphi Q$ of P and Q is by definition a group of the form $(P \times Q)/\overrightarrow{\Delta}_\varphi(Z_P)$, where Z_P is a central subgroup of order p of P, and $\varphi : Z_P \hookrightarrow Z(Q)$ is some isomorphism from Z_P to some central subgroup Z_Q of Q.

In the case where $p = 2$ and the groups P and Q both have cyclic center, the group Z_P is unique, as also is the morphism φ, so the central product is simply denoted by $P * Q$ in this case.

Proposition 5.27. *Let p be a prime number, and let P and Q be non-trivial finite p-groups. Let Z_P (resp. Z_Q) denote a central subgroup of order p of P (resp. Q).*

(1) *If one of the groups $Z(P)$ or $Z(Q)$ is non-cyclic, or if $|Z(P)| > p$ and $|Z(Q)| > p$, then $\partial(P *_\varphi Q) = 0$ in $\mathcal{R}_p$ for any group isomorphism $\varphi : Z_P \to Z_Q$.*
(2) *If $Z(P)$ and $Z(Q)$ are cyclic, and if moreover $Z(P)$ or $Z(Q)$ has order p, then*

$$\bigoplus_{\varphi : Z_P \xrightarrow{\cong} Z_Q} \partial(P *_\varphi Q) \cong \partial P \times \partial Q \quad \text{in } \mathcal{R}_p.$$

Proof. The center of the group $P *_\varphi Q$ is equal to $Z(P) *_\varphi Z(Q)$. It is cyclic if and only if both $Z(P)$ and $Z(Q)$ are cyclic, and if one of them has order p. This proves (1).

For (2), suppose that $Z(P)$ and $Z(Q)$ are cyclic, and one of them has order p. Then the subgroups Z_P and Z_Q are uniquely determined, and there are $p-1$ group isomorphisms $\varphi : Z_P \to Z_Q$. For each of them, the only central subgroup Z_φ of order p of $P *_\varphi Q$ is equal to $(Z_P \times Z_Q)/\overrightarrow{\Delta}_\varphi(Z_P)$, and

$$(P *_\varphi Q)/Z_\varphi \cong (P \times Q)/(Z_P \times Z_Q) \cong \overline{P} \times \overline{Q},$$

where $\overline{P} = P/Z_P$ and $\overline{Q} = Q/Z_Q$.

By Proposition 3.8,

$$\begin{aligned}
P \times Q &\cong \bigoplus_{\mathbf{1} \leq N \trianglelefteq (P \times Q)} \partial((P \times Q)/N),\\
P *_\varphi Q &\cong \bigoplus_{\Delta_\varphi(Z_P) \leq N \trianglelefteq (P \times Q)} \partial((P \times Q)/N),\\
\overline{P} \times Q &\cong \bigoplus_{(Z_P \times \mathbf{1}) \leq N \trianglelefteq (P \times Q)} \partial((P \times Q)/N),\\
P \times \overline{Q} &\cong \bigoplus_{(\mathbf{1} \times Z_Q) \leq N \trianglelefteq (P \times Q)} \partial((P \times Q)/N),\\
\overline{P} \times \overline{Q} &\cong \bigoplus_{(Z_P \times Z_Q) \leq N \trianglelefteq (P \times Q)} \partial((P \times Q)/N).
\end{aligned}$$

Set

$$S = \partial(P \times Q) \oplus \Big(\bigoplus_{\varphi : Z_P \xrightarrow{\cong} Z_Q} (P *_\varphi Q) \Big) \oplus (\overline{P} \times Q) \oplus (P \times \overline{Q}). \tag{5.28}$$

Then S is equal to the direct sum of the edges $\partial((P \times Q)/N)$ for $N \trianglelefteq (P \times Q)$, with multiplicity $p+1$ if $N \geq Z_P \times Z_Q$, and multiplicity 1 otherwise. Hence

$$S \cong (P \times Q) + p \cdot (\overline{P} \times \overline{Q}). \tag{5.29}$$

Now $\partial(P \times Q) = 0$ by Corollary 3.12. Also, by Corollary 3.9, for each $\varphi : Z_P \xrightarrow{\cong} Z_Q$,

$$P *_\varphi Q = \partial(P *_\varphi Q) \oplus (\overline{P} \times \overline{Q}).$$

But $P \cong \partial P \oplus \overline{P}$ and $Q \cong \partial Q \oplus \overline{Q}$, by Corollary 3.9 again. Replacing P, Q, and $P *_\varphi Q$ by these values in (5.28) gives

$$S \cong \Big(\bigoplus_{\varphi : Z_P \xrightarrow{\cong} Z_Q} \partial(P *_\varphi Q) \Big) \oplus (p+1) \cdot (\overline{P} \times \overline{Q}) \oplus (\overline{P} \times (\partial Q)) \oplus ((\partial P) \times \overline{Q}). \tag{5.30}$$

But replacing P by $\partial P \oplus \overline{P}$ and Q by $\partial Q \oplus \overline{Q}$ in (5.29) gives

$$S \cong (\partial P \times \partial Q) \oplus (p+1) \cdot (\overline{P} \times \overline{Q}) \oplus (\overline{P} \times (\partial Q)) \oplus ((\partial P) \times \overline{Q}). \tag{5.31}$$

Comparing (5.30) and (5.31) gives

$$\bigoplus_{\varphi : Z_P \xrightarrow{\cong} Z_Q} \partial(P *_\varphi Q) \cong \partial P \times \partial Q,$$

by Corollary 5.13. This completes the proof. □

Corollary 5.32. (1) *Let P and Q be non-trivial finite 2-groups with cyclic center, and assume that $Z(P)$ or $Z(Q)$ has order 2. Then*

$$\partial(P * Q) \cong \partial P \times \partial Q \quad \textit{in the category } \mathcal{R}_2.$$

(2) *For each $i \in \{1, \dots, n\}$, let P_i be a finite 2-group with center Z_i of order 2. Let $\mathop{\ast}_{i=1}^{n} P_i = P_1 * \cdots * P_n$ denote the central product of the groups P_i. Then*

$$\mathop{\ast}_{i=1}^{n} P_i \cong \prod_{i=1}^{n} (\partial P_i) \oplus \prod_{i=1}^{n} (P_i/Z_i) \quad \textit{in the category } \mathcal{R}_2.$$

(3) *In particular, for any positive integer n, and any integer $m \geq 4$, there are isomorphisms*

$$\begin{aligned}
(D_{2^m})^{*n} &\cong 2^{(n-1)(m-3)} \cdot \partial D_{2^m} \oplus (D_{2^{m-1}})^n, \\
(SD_{2^m})^{*n} &\cong 2^{(n-1)(m-3)} \cdot \partial SD_{2^m} \oplus (D_{2^{m-1}})^n, \\
(Q_{2^m})^{*n} &\cong \begin{cases} 2^{(n-1)(m-3)} \cdot \partial D_{2^m} \oplus (D_{2^{m-1}})^n & \text{if } n \text{ is even}, \\ 2^{(n-1)(m-3)} \cdot \partial Q_{2^m} \oplus (D_{2^{m-1}})^n & \text{if } n \text{ is odd}, \end{cases}
\end{aligned}$$

in the category $\mathcal{R}_2$, where P^{*n} denotes the central product of n copies of P.

Proof. For (1), the assumptions imply that there is a unique group isomorphism $\varphi : Z_P \to Z_Q$. Hence there is only one term in the summation of Proposition 5.27.

(2) follows from Corollary 3.9, which gives an isomorphism

$$\mathop{\ast}_{i=1}^{n} P_i \cong \partial\Big(\mathop{\ast}_{i=1}^{n} P_i\Big) \oplus \Big(\Big(\mathop{\ast}_{i=1}^{n} P_i\Big)/Y\Big),$$

where Y is the unique central subgroup of order 2 in $\ast_{i=1}^{n} P_i$. Now an easy induction argument, using (1), shows that $\partial(\ast_{i=1}^{n} P_i) \cong \prod_{i=1}^{n}(\partial P_i)$, and that $(\ast_{i=1}^{n} P_i)/Y \cong \prod_{i=1}^{n}(P_i/Z_i)$.

Finally, when P is one of the groups D_{2^m}, SD_{2^m}, or Q_{2^m}, then $P/Z \cong D_{2^{m-1}}$. Now (3) follows from (2) and from an easy induction argument using Corollary 4.24. □

Remark 5.33. It follows from (3) that when n is even, the groups $(D_{2^m})^{*n}$ and $(Q_{2^m})^{*n}$ are isomorphic in the category $\mathcal{R}_2$; it is actually easy to check that they are isomorphic as *groups*.

Example 5.34. From Corollary 4.24 and (1), it follows that:

- $\partial((D_8)^{*n}) \cong \partial C_2$.
- $\partial((Q_8)^{*n}) \cong \begin{cases} \partial C_2 & \text{if } n \text{ is even,} \\ \partial Q_8 & \text{if } n \text{ is odd.} \end{cases}$
- $\partial((SD_{2^m})^{*n}) \cong 2^{(n-1)(m-3)} \cdot \partial SD_{2^m}$ for $m \geq 4$.

More generally, if P is any central product of groups isomorphic to D_8 or Q_8, that is, if P is an extraspecial 2-group, then $\partial P \cong \partial C_2$ or $\partial P \cong \partial Q_8$. In particular (see Example 2.28), we recover the well known fact that P has a unique faithful rational irreducible representation. But more is true. Let Q be a non-trivial 2-group. If the center of Q is not cyclic, then the center of any central product $P * Q$ is not cyclic, hence $\partial Q = \partial(P * Q) = 0$ in $\mathcal{R}_2$. If the center of Q is cyclic, then there is a unique central product $P * Q$. By Theorem 3.11, there is a finite sequence $\mathcal{S}$ of Roquette 2-groups such that

$$\partial Q \cong \bigoplus_{R \in \mathcal{S}} \partial R \quad \text{in } \mathcal{R}_2.$$

By Corollary 5.32(1), it follows that

$$\partial(P * Q) \cong \bigoplus_{R \in \mathcal{S}} (\partial P \times \partial R).$$

Now $\partial P \cong \partial C_2$ or $\partial P \cong \partial Q_8$. In both cases, by Propositions 5.4 and 5.7, multiplication by ∂P is a permutation of the edges of the Roquette 2-groups. It follows that there is a sequence $\mathcal{S}'$ of Roquette 2-groups, of the same length as $\mathcal{S}$, such that

$$\partial(P * Q) \cong \bigoplus_{R \in \mathcal{S}'} \partial R.$$

In particular, for any field K of characteristic 0, the groups

$$R_K(\partial P) = \partial R_K(P) \quad \text{and} \quad R_K(\partial(P * Q)) = \partial R_K(P * Q)$$

are free of the same rank, equal to the length of $\mathcal{S}$ or $\mathcal{S}'$.

Hence in any case, the groups Q and $P * Q$ have the same number (possibly 0 if the center of Q is not cyclic) of faithful irreducible representations over K, up to isomorphism.

Similarly, the last example above means in particular that the group $(SD_{2^m})^{*n}$ admits $2^{(n-1)(m-3)}$ non-isomorphic faithful rational irreducible representations.

Example 5.35. Let p be an odd prime, and let $P = X^\epsilon$ (where $\epsilon \in \{\pm 1\}$) be one of the extraspecial groups of order p^3 considered in Example 5.16. Then $\partial P \cong \partial C_p$. Moreover $Z(P)$ has order p, and any automorphism of $Z(P) = Z_P$ can be extended to an automorphism of P. It follows that $P *_\varphi Q$ is independent (up to a group isomorphism) of the choice of an embedding $\varphi : P \hookrightarrow Z(Q)$, for any non-trivial p-group Q with cyclic center, so we can denote this group by $P * Q$.

Now if Q is a non-trivial p-group, and $\mathcal{B}$ is a genetic basis of Q, it follows from Theorem 3.11 that

$$\partial Q \cong \bigoplus_{\substack{S \in \mathcal{B} \\ S \cap Z(Q) = \mathbf{1}}} \partial \overline{N}_Q(S),$$

and the right hand side is a direct sum of edges of *non-trivial* Roquette p-groups. By (5.2), for any non-trivial Roquette p-group R,

$$\partial C_p \times \partial R \cong (p-1)\partial R.$$

Hence for any non-trivial p-group Q,

$$\partial C_p \times \partial Q \cong (p-1)\partial Q.$$

It follows that if Q is a non-trivial p-group with cyclic center, and if $P \cong X^\epsilon$, then

$$\bigoplus_{\varphi : Z_P \xrightarrow{\cong} Z_Q} \partial(P *_\varphi Q) \cong (p-1)\partial(P * Q) \cong \partial C_p \times \partial Q \cong (p-1)\partial Q,$$

hence $\partial(X^\epsilon * Q) \cong \partial Q$, by Corollary 5.13. Note that this is also true (for any central product of X^ϵ with Q) if the center of Q is non-trivial, since in this case the center of $X^\epsilon * Q$ is also non-trivial, and then $\partial(X^\epsilon * Q) \cong \partial Q \cong 0$ in $\mathcal{R}_p$, by Corollary 3.12.

It follows easily by induction that if P is any central product of groups isomorphic to X^+ or X^-, i.e. if P is an extraspecial p-group, then $\partial P \cong \partial C_p$. We thus recover the well known fact that for $p > 2$ too, extraspecial p-groups have a unique faithful rational irreducible representation. The same argument shows more generally that $\partial(P * Q) \cong \partial Q$ in $\mathcal{R}_p$ for any non-trivial p-group Q. In particular Q and $P * Q$ have the same number of faithful irreducible representations over a given field K of characteristic 0.

Theorem 5.36. (1) *Let P be an arbitrary finite direct product of groups of order 2 and dihedral 2-groups. Then the Dade group of any factor group of P is torsion free.*

(2) *Let $4 \le m_1 \le \cdots \le m_n$ be a non-decreasing sequence of integers. Set $s = \sum_{i=1}^{n-1}(m_i - 3)$. Then the torsion part of the Dade group of $\circledast_{i=1}^{n} SD_{2^{m_i}}$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{2^{s-1}}$ if $m_1 < m_n$, and to $(\mathbb{Z}/2\mathbb{Z})^{2^s}$ if $m_1 = m_n$.*

(3) *In particular, for any integers $n \ge 1$ and $m \ge 4$, the torsion part of the Dade group of $(SD_{2^m})^{*n}$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{2^{(n-1)(m-3)}}$.*

Proof. First, by Example 3.13,

$$D_8 \cong \mathbf{1} \oplus 4 \cdot \partial C_2 \quad \text{in } \mathcal{R}_2.$$

Now by Corollary 3.9, since the center Z of D_{16} has order 2, and since $D_{16}/Z \cong D_8$,

$$D_{16} \cong \mathbf{1} \oplus 4 \cdot \partial C_2 \oplus \partial D_{16}.$$

Since for $n \geq 3$, the group D_{2^n} has a center Z of order 2, and since $D_{2^n}/Z \cong D_{2^{n-1}}$, it follows by induction that

$$D_{2^n} \cong \mathbf{1} \oplus 4 \cdot \partial C_2 \oplus \bigoplus_{l=4}^{n} \partial D_{2^l}. \tag{5.37}$$

Now by Corollary 4.24, the product $\partial D_{2^l} \times \partial D_{2^m}$ for $l \leq m$ is isomorphic to $2^{l-3} \cdot \partial D_{2^m}$. Moreover $\partial C_2 \times \partial P \cong \partial P$ for any 2-group P, and since $\partial C_2 \cong \mathbf{1}$ by Proposition 5.4, it follows that any product of dihedral 2-groups and groups of order 2 is isomorphic to a direct sum of edges of the trivial group, of the edge of the group of order 2, and of edges of dihedral 2-groups.

It follows that if P is a direct product of dihedral 2-groups and groups of order 2, then P is isomorphic in $\mathcal{R}_2$ to the direct sum of the trivial group, and some copies of edges of the group of order 2 and the edge of dihedral 2-groups. In other words, if S is a genetic subgroup of P, then $N_P(S)/S$ is either trivial, or of order 2, or dihedral. Now the Dade group of dihedral 2-groups is torsion free (by [CT00, Theorem 10.3 (a)]), and the Dade groups of the trivial group and of the group of order 2 are trivial. It follows that the Dade group of P is torsion free, as also is the Dade group of any quotient of P, being a direct summand of the Dade group of P. This proves (1).

For (2), set $P = \mathop{\ast}_{i=1}^{n} SD_{2^{m_i}}$. By Corollary 5.32(2),

$$P \cong \prod_{i=1}^{m} \partial SD_{2^{m_i}} \oplus \prod_{i=1}^{n} D_{2^{m_i-1}} \quad \text{in } \mathcal{R}_2. \tag{5.38}$$

Now an easy induction on n, using Corollary 4.24, shows that

$$\prod_{i=1}^{n} \partial SD_{2^{m_i}} \cong \begin{cases} 2^{s-1} \cdot \partial C_{2^{m_n-1}} & \text{if } m_1 < m_n, \\ 2^s \cdot \partial SD_{2^{m_n}} & \text{if } m_1 = m_n, \end{cases}$$

where $s = \sum_{i=1}^{n-1} (m_i - 3)$.

By (5.38), this means that in a genetic basis $\mathcal{B}$ of P, there are 2^{s-1} or 2^s subgroups S such that $N_P(S)/S$ is semidihedral, depending on whether $m_1 < m_n$ or $m_1 = m_n$, and for the other $S \in \mathcal{B}$, the group $N_P(S)/S$ is trivial, of order 2, or dihedral.

The Dade group of a dihedral 2-group is torsion free, and the Dade groups of the trivial group and of C_2 are trivial. Moreover, the faithful torsion part $\partial D^t(C_{2^m})$ of the Dade group of C_{2^m} is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ if $m \geq 2$ (see [Bou10, Theorem 12.10.3]). Similarly, the faithful torsion part $\partial D^t(SD_{2^m})$ for $m \geq 4$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. This completes the proof of (2); and (3) is a particular case of (2). □

Remark 5.39. Let P be a finite product of groups of order 2 and dihedral 2-groups, as in (1), and let Q be a quotient of P. If T is a genetic subgroup of Q, then $N_Q(T)/T$ is either trivial, of order 2, or dihedral: indeed, T lifts to a genetic subgroup S of P such that $N_P(S)/S \cong N_Q(T)/T$. It follows in particular that the map

$$\bar{\epsilon}_Q : B^\times(Q) \to \mathrm{Hom}_{\mathbb{Z}}(R_{\mathbb{Q}}(Q), \mathbb{F}_2)$$

introduced in [Bou07, Notation 8.4] is a group isomorphism from the group of units of the Burnside ring of Q to the $\mathbb{F}_2$-dual of $R_{\mathbb{Q}}(Q)$. Indeed, there are non-negative integers a and b_i for $i \in \{4, \dots, m\}$ such that

$$Q \cong \mathbf{1} \oplus a \cdot \partial C_2 \oplus \bigoplus_{i=4}^{m} b_i \cdot \partial D_{2^i} \quad \text{in } \mathcal{R}_2.$$

Then $B^\times(Q) \cong (\mathbb{F}_2)^r$, where $r = 1 + a + \sum_{i=4}^m b_i$, by [Bou07, Theorem 8.5]. Similarly $R_{\mathbb{Q}}(Q) \cong \mathbb{Z}^r$ (hence r is equal to the number of conjugacy classes of cyclic subgroups of Q), so $\mathrm{Hom}_{\mathbb{Z}}(R_{\mathbb{Q}}(Q), \mathbb{F}_2) \cong (\mathbb{F}_2)^r$. As $\bar{\epsilon}$ is injective, it is an isomorphism.

Proposition 5.40. *Let $m \geq 3$ be an integer. Then for any integer n, there is an isomorphism*

$$(D_{2^m})^n \cong \mathbf{1} \oplus (5^n - 1) \cdot \partial C_2 \oplus \bigoplus_{l=4}^{m} \frac{(3+2^{l-2})^n - (3+2^{l-3})^n}{2^{l-3}} \cdot \partial D_{2^l} \quad \textit{in } \mathcal{R}_2. \tag{5.41}$$

Proof. Let $\mathcal{S}_p$ denote the full subcategory of $\mathcal{R}_p$ consisting of all finite direct sums of edges of Roquette p-groups, and let $\Gamma = K_0(\mathcal{S}_p)$ be the Grothendieck group of this category, for relations given by direct sum decomposition. Then Corollary 5.13 shows that Γ is a free abelian group, and that two objects of $\mathcal{S}_p$ have the same image in Γ if and only if they are isomorphic in $\mathcal{R}_p$. Moreover, by Corollary 4.24, the category $\mathcal{S}_p$ is a tensor subcategory of $\mathcal{R}_p$, and Γ is actually a commutative ring.

It follows that Γ identifies to a subring of the $\mathbb{Q}$-algebra $\mathbb{Q}\Gamma = \mathbb{Q} \otimes_{\mathbb{Z}} \Gamma$, and to prove the proposition, it suffices to check that the two sides of (5.41) have the same image in $\mathbb{Q}\Gamma$. Let c denote the image of ∂C_2 in Γ, and for $l \geq 4$, let d_l denote the image of ∂D_{2^l} in Γ. Note that $c = 1$ by Proposition 5.4. By (5.37), the image i_m of D_{2^m} in Γ is equal to

$$i_m = 1 + 4c + \sum_{l=4}^{m} d_l = 5 + \sum_{l=4}^{m} d_l.$$

By Corollary 4.24, for $4 \leq l \leq k$,

$$d_l \times d_k = 2^{l-3} d_k.$$

It follows that the elements $e_l = \frac{1}{2^{l-3}} d_l$ of $\mathbb{Q}\Gamma$, for $l \geq 4$, are such that

$$\forall l, k,\ 4 \leq l \leq k, \quad e_l \times e_k = e_k.$$

In particular e_l is an idempotent, and the elements

$$f_l = e_l - e_{l+1} \quad \text{for } 4 \le l < m, \quad \text{and} \quad f_m = e_m$$

are orthogonal idempotents of $\mathbb{Q}\Gamma$. With this notation, for $l \ge 4$,

$$e_l = f_l + f_{l+1} + \cdots + f_m,$$

and the element i_m can be written as

$$i_m = 5 + \sum_{l=4}^{m} d_l = 5 + \sum_{l=4}^{m} 2^{l-3}(f_l + f_{l+1} + \cdots + f_m) = 5 + \sum_{l=4}^{m} 2(2^{l-3} - 1) f_l.$$

Thus

$$\begin{aligned}
(i_m)^n &= 5^n + \sum_{j=1}^{n} \binom{n}{j} 5^{n-j} \Big(\sum_{l=4}^{m} 2(2^{l-3} - 1) f_l \Big)^j \\
&= 5^n + \sum_{j=1}^{n} \binom{n}{j} 5^{n-j} \sum_{l=4}^{m} 2^j (2^{l-3} - 1)^j f_l \\
&= 5^n + \sum_{l=4}^{m} \Big(\sum_{j=1}^{n} \binom{n}{j} 5^{n-j} 2^j (2^{l-3} - 1)^j \Big) f_l \\
&= 5^n + \sum_{l=4}^{m} ((5 + 2(2^{l-3} - 1))^n - 5^n) f_l \\
&= 5^n + \sum_{l=4}^{m} ((3 + 2^{l-2})^n - 5^n) f_l = 5^n + \sum_{l=4}^{m} ((3 + 2^{l-2})^n - (3 + 2^{l-3})^n) e_l.
\end{aligned}$$

The proposition follows, since $5^n = 1 + (5^n - 1)c$, and since $e_l = \frac{1}{2^{l-3}} d_l$. □

Remark 5.42. The isomorphism (5.41) is equivalent to saying that a genetic basis of the group $P = (D_{2^m})^n$ consists of one subgroup S such that $N_P(S)/S \cong \mathbf{1}$ (namely $S = P$), of $5^n - 1$ subgroups S such that $N_P(S)/S \cong C_2$, and, for $4 \le l \le m$, of $\frac{(3+2^{l-2})^n - (3+2^{l-3})^n}{2^{l-3}}$ subgroups S such that $N_P(S)/S \cong D_{2^l}$.

Together with Corollary 5.32(3), this also gives the structure of genetic bases of the groups $(D_{2^m})^{*n}$, $(SD_{2^m})^{*n}$, $(Q_{2^m})^{*n}$:

Corollary 5.43. *Let P be one of the groups D_{2^m}, SD_{2^m}, or Q_{2^m}, for $m \ge 4$. Then, for any positive integer n, any genetic basis of the group $Q = P^{*n}$ consists:*

- *of one group S such that $N_Q(S)/S = \mathbf{1}$ (namely $S = Q$);*
- *of $5^n - 1$ subgroups S such that $N_Q(S)/S \cong C_2$;*
- *for $4 \le l \le m - 1$, of $\frac{(3+2^{l-2})^n - (3+2^{l-3})^n}{2^{l-3}}$ subgroups S such that $N_Q(S)/S \cong D_{2^l}$;*

- *of* $2^{(n-1)(m-3)}$ *subgroups* S *such that* $N_Q(S)/S$ *is isomorphic to*

$$\begin{cases} D_{2^m} & \text{if } P = D_{2^m}, \\ SD_{2^m} & \text{if } P = SD_{2^m}, \\ D_{2^m} & \text{if } P = Q_{2^m} \text{ and } n \text{ is even}, \\ Q_{2^m} & \text{if } P = Q_{2^m} \text{ and } n \text{ is odd}. \end{cases}$$

Acknowledgments. Even though the idea of considering the category $\mathcal{R}_p$ was implicit in [Bou08], the present paper would not exist without the illuminating conversations I had with Paul Balmer in November 2010, during which I gradually understood I was a kind of *Monsieur Jourdain* of tensor categories… I wish to thank him for this revelation.

I also thank the referee for his/her careful reading, and useful comments and suggestions.

References

[Bar08] Barker, L.: Rhetorical biset functors, rational p-biset functors, and their semisimplicity in characteristic zero. J. Algebra **319**, 3810–3853 (2008) Zbl 1153.19001 MR 2407851

[Bou96] Bouc, S.: Foncteurs d'ensembles munis d'une double action. J. Algebra **183**, 664–736 (1996) Zbl 0858.19001 MR 1401673

[Bou04] Bouc, S.: The functor of rational representations for p-groups. Adv. Math. **186**, 267–306 (2004) Zbl 1055.20005 MR 2073907

[Bou05] Bouc, S.: Biset functors and genetic sections for p-groups. J. Algebra **284**, 179–202 (2005) Zbl 1062.19001 MR 2115011

[Bou06] Bouc, S.: The Dade group of a p-group. Invent. Math. **164**, 189–231 (2006) Zbl 1099.20004 MR 2207787

[Bou07] Bouc, S.: The functor of units of Burnside rings for p-groups. Comment. Math. Helv. **82**, 583–615 (2007) Zbl 1142.19001 MR 2314054

[Bou08] Bouc, S.: Rational p-biset functors. J. Algebra **319**, 1776–1800 (2008) Zbl 1149.19001 MR 2383066

[Bou10] Bouc, S.: Biset Functors for Finite Groups. Lecture Notes in Math. 1990, Springer (2010) Zbl 1205.19002 MR 2598185

[BM04] Bouc, S., Mazza, N.: The Dade group of (almost) extraspecial p-groups. J. Pure Appl. Algebra **192**, 21–51 (2004) Zbl 1055.20008 MR 2067187

[CT00] Carlson, J., Thévenaz, J.: Torsion endo-trivial modules. Algebras Represent. Theory **3**, 303–335 (2000) Zbl 0970.20004 MR 1808129

[GAP13] The GAP Group. GAP – Groups, Algorithms, and Programming, Version 4.6.4 (2013) http://www.gap-system.org

[Gor68] Gorenstein, W.: Finite Groups. Chelsea (1968) Zbl 0185.05701 MR 0231903

[HTW90] Hambleton, I., Taylor, L. R., Williams, E. B.: Detection theorems for K-theory and L-theory. J. Pure Appl. Algebra **63**, 247–299 (1990) Zbl 0718.18006 MR 1047584

[Roq58] Roquette, P.: Realisierung von Darstellungen endlicher nilpotenter Gruppen. Arch. Math. (Basel) **9**, 241–250 (1958) Zbl 0083.25002 MR 0097452

[Sza09] Szamuely, T.: Galois Groups and Fundamental Groups. Cambridge Stud. Adv. Math. 117, Cambridge Univ. Press (2009) Zbl 1189.14002 MR 2548205