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High-order phase transitions in the quadratic family

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Abstract. We give the first example of a transitive quadratic map whose real and complex geometric pressure functions have a high-order phase transition. In fact, we show that this phase transition resembles a Kosterlitz–Thouless singularity: Near the critical parameter the geometric pressure function behaves as $x \mapsto \exp(-x^{-2})$ near $x = 0$, before becoming linear. This quadratic map has a non-recurrent critical point, so it is non-uniformly hyperbolic in a strong sense.

Keywords. Quadratic family, thermodynamic formalism, phase transition

1. Introduction

This paper is concerned with the thermodynamic formalism of smooth dynamical systems. This study was initiated by Sinaĭ, Ruelle, and Bowen [Sin72, Bow75, Rue76] in the context of uniformly hyperbolic diffeomorphisms and Hölder continuous potentials. In the last decades there have been important efforts to extend the theory beyond the uniformly hyperbolic setting, specially in real and complex dimension 1 where a complete picture is emerging; see for example [BT09, MS00, MS03, PS08, PRL11, PRL13] and references therein. See also [Sar11, UZ09, VV10] and references therein for (recent) results in higher dimensions.

For a smooth map f in real or complex dimension 1 and a real parameter t, we consider the pressure of f with respect to the geometric potential $-t \log |Df|$ (see §1.1 for the details). The function of t so defined is the *geometric pressure function* of f. It is closely related to several multifractal spectra and large deviation rate functions associated with f; see for example [BMS03, Lemma 2], [GPR10], [IT11], [KN92, Theorems 1.2 and 1.3], [PRL11, Appendix B], and references therein.

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We exhibit a transitive quadratic map whose geometric pressure function behaves, for some constants $A > 0$ and $\chi > 0$ and for t near a certain parameter t_*, as the function

$$t \mapsto \begin{cases} -t\chi + \exp(-A(t_* - t)^{-2}) & \text{if } t < t_*, \\ -t\chi & \text{if } t \geq t_* \end{cases}$$

(see the Main Theorem in §1.1). Thus, the geometric pressure function of this map has a phase transition at $t = t_*$ that resembles a Kosterlitz–Thouless type singularity (see for example [DGM08] or [LBMB04, §7.6]). It is the first example of a transitive smooth dynamical system having a second-order phase transition that does not correspond to a power law singularity. This example is also robust: Every family of sufficiently regular unimodal maps that is close to the quadratic family has a member with the same property.

The quadratic map we study has a non-recurrent critical point, so it is non-uniformly hyperbolic in a strong sense. Thus, roughly speaking, lack of expansion is not responsible for the phase transition. Instead, it is the irregular behavior of the critical orbit that is one of the mechanisms behind the phase transition. Considering a different behavior of the critical orbit, in the companion paper [CRL12] we gave the first example of a quadratic map having a phase transition at a large value of t, that is, of a "low-temperature phase transition"; see also [MS03, §5] for some conformal Cantor sets with similar properties. In contrast with the example studied here, the geometric pressure function of the quadratic map studied in [CRL12] is not differentiable at the phase transition, that is, it is a phase transition of *first order*.

Another interesting feature of the quadratic map we study is that it has no equilibrium state at the phase transition. At a low-temperature phase transition there can be at most one equilibrium state,[1] and in the companion paper [CRL12] we provide an example of a quadratic map having one.

There are various examples in the literature of transitive smooth maps whose geometric pressure function has a first-order phase transition. This includes quadratic maps that have an absolutely continuous invariant measure, and that do not satisfy the Collet–Eckmann condition.[2] By the work of Makarov and Smirnov [MS00], this also includes those phase transitions in the complex case that occur at a parameter in $(-\infty, 0)$. See also [DGR11, DGR14, LOR11] for examples of first-order phase transitions of some transitive 3-dimensional diffeomorphisms.

The only known phase transitions that are not of first order are those related to the existence of a neutral periodic point. For a given $\alpha \geq 1$, consider the map f_α given by $x \mapsto x(1 + x^\alpha) \bmod 1$, which has a neutral fixed point at $x = 0$. It is the prototypical example of an interval map with a neutral periodic point. The geometric pressure function of f_α is studied in [Lop93]. It has a phase transition at $t = 1$ with a power law singularity

[1] See [Dob15, Theorem 6] in the real setting, and [Dob12, Theorem 8] in the complex setting.

[2] For such a map, the geometric pressure function is identically zero after its first zero (see [NS98, Theorem A] or [RL12, Corollary 1.3] in the real case and [PRLS03, Main Theorem] in the complex case). On the other hand, since every absolutely continuous invariant measure has a strictly positive Lyapunov exponent, the existence of such a measure easily implies that the geometric pressure function is not differentiable at its first zero.

of exponent α. Thus, it is natural to expect that for a quadratic map f having a periodic point p of period $n \geq 1$ satisfying $Df^n(p) = \pm 1$, the geometric pressure function of f has a unique phase transition, and that this phase transition corresponds to a power law singularity.[3]

1.1. *Statement of results*

For $c, z \in \mathbb{C}$, write $f_c(z) := z^2 + c$. We consider a set of real parameters c close to -2 such that $f_c(c) > c$, the interval $I_c := [c, f_c(c)]$ of $\mathbb{R}$ is invariant by f_c, and f_c is topologically exact on I_c. We consider two dynamical systems associated to f_c: the interval map $f_c|_{I_c}$ and the complex quadratic polynomial f_c acting on its Julia set J_c.

For such c, define

$$\chi_{\mathrm{crit}}(c) := \liminf_{m \to \infty} \frac{1}{m} \log |Df_c^m(c)|,$$

and denote by $\mathscr{M}_c^{\mathbb{R}}$ the space of Borel probability measures supported on I_c that are invariant by f_c. For a measure μ in $\mathscr{M}_c^{\mathbb{R}}$ denote by $h_\mu(f_c)$ the measure-theoretic entropy of f_c with respect to μ, and for each t in $\mathbb{R}$ set

$$P_c^{\mathbb{R}}(t) := \sup\left\{ h_\mu(f_c) - t \int \log |Df_c| \, d\mu \;\middle|\; \mu \in \mathscr{M}_c^{\mathbb{R}} \right\},$$

which is finite. The function $P_c^{\mathbb{R}} : \mathbb{R} \to \mathbb{R}$ so defined is called the *geometric pressure function of* $f_c|_{I_c}$; it is convex and nonincreasing.

Similarly, denote by $\mathscr{M}_c^{\mathbb{C}}$ the space of Borel probability measures supported on J_c that are invariant by f_c, and for a measure μ in $\mathscr{M}_c^{\mathbb{C}}$, denote by $h_\mu(f_c)$ the measure-theoretic entropy of f_c with respect to μ. Then the *geometric pressure function* $P_c^{\mathbb{C}} : \mathbb{R} \to \mathbb{R}$ *of* f_c is defined by

$$P_c^{\mathbb{C}}(t) := \sup\left\{ h_\mu(f_c) - t \int \log |Df_c| \, d\mu \;\middle|\; \mu \in \mathscr{M}_c^{\mathbb{C}} \right\}.$$

Main Theorem. *There is a real parameter c such that the critical point of f_c is non-recurrent, for some $t_* > 0$ and every $t \geq t_*$ we have*

$$P_c^{\mathbb{R}}(t) = P_c^{\mathbb{C}}(t) = -t \chi_{\mathrm{crit}}(c)/2,$$

and for some constants $A, B^+, B^- > 0$ and every t in $(0, t_*)$ close to t_*,

$$-t \chi_{\mathrm{crit}}(c)/2 + 2^{-\left(\frac{A}{t_* - t} + B^-\right)^2} \leq P_c^{\mathbb{R}}(t) \leq P_c^{\mathbb{C}}(t) \leq -t \chi_{\mathrm{crit}}(c)/2 + 2^{-\left(\frac{A}{t_* - t} - B^+\right)^2}.$$

In particular, both $P_c^{\mathbb{R}}$ and $P_c^{\mathbb{C}}$ are of class C^2 at $t = t_*$, but neither is real analytic at $t = t_*$.

[3] Notice that when $Df^n(p) = 1$ (resp. $Df^n(p) = -1$), the function $x \mapsto f^n(x) - x$ (resp. $x \mapsto f^{2n}(x) - x$) is of the order of $(x - p)^2$ (resp. $(x - p)^3$) near p (see for example [CG93, Mil06]).

We show in addition that for each $t \geq t_*$ there is no equilibrium state of f_c for the potential $-t \log |Df_c|$, there is a unique associated conformal measure, and this measure is dissipative and purely atomic (supported on the backward orbit of the critical point);[4] see §3.1 for the relevant definitions and for a strengthened version of the Main Theorem. It can also be shown that if for each t in $(0, t_*)$ we denote by ν_t the unique equilibrium state of f_c for the potential $-t \log |Df_c|$, then the measure ν_t converges as $t \to t_*^-$ to the invariant probability measure supported on a certain periodic point of period 3 of f_c. Thus, roughly speaking, the quadratic map in the Main Theorem is a low-temperature analog of the quadratic map that has a physical measure supported on a repelling fixed point, studied in [HK90].

Since the critical point of a map f_c as in the Main Theorem is non-recurrent, it follows that f_c satisfies the Collet–Eckmann condition: $\chi_{\text{crit}}(c) > 0$ (see [Mis81] for the real case and [Mañ93] for the complex case). So, t_* in the Main Theorem is strictly larger than the first zero of the geometric pressure function of f_c, that is, f_c has a "low-temperature" phase transition at $t = t_*$ in the sense of [CRL12].

1.2. Notes and references

For complex rational maps, Makarov and Smirnov showed that every phase transition occurring at a negative parameter is removable, in the sense that the geometric pressure function has a real analytic continuation to all of $(-\infty, 0)$ [MS00, Theorem B]. In contrast, the geometric pressure function of a map as in the Main Theorem cannot admit a real analytic continuation beyond the phase transition.

For a map as in the Main Theorem, the non-existence of equilibrium states also follows from [IRRL12, Corollary 1.3].

For a quadratic map having a phase transition at the first zero of the pressure function, that is, a *high-temperature phase transition*, the number of ergodic equilibrium states can be arbitrary; see [CRL10, Corollaries 2 and 3], and also [BK98, Example 5.4] and [BT06, Corollary 2] for an example having no equilibrium state.

Bruin and Todd [BT15] study certain piecewise linear models (with an infinite number of break points) of the smooth unimodal maps having a wild attractor in [BKNvS96]. They show that for a large value of the order of the critical point, the piecewise linear model has a high-order phase transition. Notice that no quadratic map can have a wild attractor (see [Lyu94]).

1.3. Strategy and organization

To prove the Main Theorem, we consider the set of parameters introduced in [CRL12]. For each parameter c in this set, the critical value is eventually mapped to an expanding Cantor set, denoted by Λ_c. For such a parameter, the behavior of the geometric pressure function at low temperatures is intimately related to the derivatives of the map along the

[4] In contrast, for each $t < t_*$ there is a unique equilibrium state of f_c for the potential $-t \log |f_c|$, there is a unique associated conformal measure, and this last measure is supported on the conical Julia set (see [PRL13] for the real case and [PRL11] for the complex case).

critical orbit (Proposition 5.6). As a first approximation we use the multipliers of the two periodic orbits of period 3 of f_c to estimate these derivatives. However, the distortion constants in these estimates are too big to achieve the level of precision needed to prove the Main Theorem. To achieve a higher precision, we estimate these distortion constants in terms of the total distortion along certain homoclinic orbits connecting the two periodic orbits of period 3 (Proposition 3.1 in §3.2).

We now proceed to describe the organization of the paper more precisely.

After some preliminaries in §2, we state a strengthened version of the Main Theorem in §3.1, as the "Main Technical Theorem". In §3.3 we introduce an abstract two-variable series that captures the behavior of the geometric pressure function at low temperatures (Proposition A). Its definition is based on an approximation of the derivatives at the critical value in terms of its itinerary in Λ_c (Proposition 3.1), as mentioned above.

In §4, which is independent of the rest of the paper, we study in an abstract setting the two-variable series for a specific class of itineraries. We show that this series has a phase transition with an asymptotic behavior as in the Main Theorem. The itineraries are defined in §4.1, and the estimates of the corresponding two-variable series are made in §4.2.

The proof of the Main Technical Theorem is given in §5. After some general results about conformal measures in §5.1, we make some technical estimates in §5.2. The proof of the Main Technical Theorem is in §5.3, after recalling a few results from [CRL12].

2. Preliminaries

We use $\mathbb{N}$ to denote the set of integers greater than or equal to 1, and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

2.1. *Quadratic polynomials, Green's functions, and Böttcher coordinates*

In this subsection and the next we recall some basic facts about the dynamics of complex quadratic polynomials; see for instance [CG93] or [Mil06] for references

For c in $\mathbb{C}$ we denote by f_c the complex quadratic polynomial

$$f_c(z) = z^2 + c,$$

and by K_c the *filled Julia set* of f_c, that is, the set of all points z in $\mathbb{C}$ whose forward orbit under f_c is bounded in $\mathbb{C}$. The set K_c is compact and its complement is the connected set consisting of all points whose orbit converges to infinity in the Riemann sphere. Furthermore, $f_c^{-1}(K_c) = K_c$ and $f_c(K_c) = K_c$. The boundary J_c of K_c is the *Julia set* of f_c.

For a parameter c in $\mathbb{C}$, the *Green's function* of K_c is the function $G_c : \mathbb{C} \to [0, \infty)$ that is identically 0 on K_c, and that for z outside K_c is given by the limit

$$G_c(z) = \lim_{n\to\infty} \frac{1}{2^n} \log |f_c^n(z)| > 0. \tag{2.1}$$

The function G_c is continuous, subharmonic, satisfies $G_c \circ f_c = 2G_c$ on $\mathbb{C}$, and it is harmonic and strictly positive outside K_c. On the other hand, the critical values of G_c are bounded from above by $G_c(0)$, and the open set

$$U_c := \{z \in \mathbb{C} \mid G_c(z) > G_c(0)\}$$

is homeomorphic to a punctured disk. Notice that $G_c(c) = 2G_c(0)$, thus U_c contains c.

By Böttcher's Theorem there is a unique conformal representation

$$\varphi_c : U_c \to \{z \in \mathbb{C} \mid |z| > \exp(G_c(0))\},$$

and this map conjugates f_c to $z \mapsto z^2$. It is called *the Böttcher coordinate* of f_c and satisfies $G_c = \log|\varphi_c|$.

2.2. *External rays and equipotentials*

Let c be in $\mathbb{C}$. For $v > 0$ the *equipotential v of f_c* is by definition $G_c^{-1}(v)$. A *Green's line* of G_c is a smooth curve in the complement of K_c in $\mathbb{C}$ that is orthogonal to the equipotentials of G_c and that is maximal with this property. Given t in $\mathbb{R}/\mathbb{Z}$, the *external ray of angle t* of f_c, denoted by $R_c(t)$, is the Green's line of G_c containing

$$\{\varphi_c^{-1}(r\exp(2\pi i t)) \mid \exp(G_c(0)) < r < \infty\}.$$

By the identity $G_c \circ f_c = 2G_c$, for each $v > 0$ and each t in $\mathbb{R}/\mathbb{Z}$ the map f_c maps the equipotential v to the equipotential $2v$, and maps $R_c(t)$ to $R_c(2t)$. For t in $\mathbb{R}/\mathbb{Z}$ the external ray $R_c(t)$ *lands at a point z* if $G_c : R_c(t) \to (0,\infty)$ is a bijection and $G_c|_{R_c(t)}^{-1}(v)$ converges to z as v converges to 0 in $(0,\infty)$. By the continuity of G_c, every landing point is in $J_c = \partial K_c$.

The *Mandelbrot set* $\mathcal{M}$ is the subset of $\mathbb{C}$ of those parameters c for which K_c is connected. The function

$$\Phi : \mathbb{C} \setminus \mathcal{M} \to \mathbb{C} \setminus \mathrm{cl}(\mathbb{D}), \qquad c \mapsto \Phi(c) := \varphi_c(c),$$

is a conformal representation (see [DH84, VIII, Théorème 1]). For $v > 0$ the *equipotential v of $\mathcal{M}$* is by definition

$$\mathcal{E}(v) := \Phi^{-1}(\{z \in \mathbb{C} \mid |z| = v\}).$$

On the other hand, for t in $\mathbb{R}/\mathbb{Z}$ the set

$$\mathcal{R}(t) := \Phi^{-1}(\{r\exp(2\pi i t) \mid r > 1\})$$

is called the *external ray of angle t of $\mathcal{M}$*. We say that $\mathcal{R}(t)$ *lands at a point z* in $\mathbb{C}$ if $\Phi^{-1}(r\exp(2\pi i t))$ converges to z as $r \searrow 1$. When this happens, z belongs to $\partial\mathcal{M}$.

2.3. *The wake* 1/2

In this subsection we recall a few facts that can be found for example in [DH84] or [Mil00].

The external rays $\mathcal{R}(1/3)$ and $\mathcal{R}(2/3)$ of $\mathcal{M}$ land at the parameter $c = -3/4$, and these are the only external rays of $\mathcal{M}$ that land at this point (see for example [Mil00, Theorem 1.2]). In particular, the complement in $\mathbb{C}$ of the set

$$\mathcal{R}(1/3) \cup \mathcal{R}(2/3) \cup \{-3/4\}$$

has two connected components; we denote by $\mathcal{W}$ the connected component containing the point $c = -2$ of $\mathcal{M}$.

For each parameter c in $\mathcal{W}$ the map f_c has two distinct fixed points; one of them is the landing point of the external ray $R_c(0)$ and it is denoted by $\beta(c)$; the other is denoted by $\alpha(c)$. The only external ray landing at $\beta(c)$ is $R_c(0)$, and the only external ray landing at $-\beta(c)$ is $R_c(1/2)$.

Moreover, for every c in $\mathcal{W}$ the only external rays of f_c landing at $\alpha(c)$ are $R_c(1/3)$ and $R_c(2/3)$ (see for example [Mil00, Theorem 1.2]). The complement of $R_c(1/3) \cup R_c(2/3) \cup \{\alpha(c)\}$ in $\mathbb{C}$ has two connected components: one containing $-\beta(c)$ and $z = c$, and the other containing $\beta(c)$ and $z = 0$. On the other hand, the point $\alpha(c)$ has two preimages by f_c: itself and $\widetilde{\alpha}(c) := -\alpha(c)$. The only external rays landing at $\widetilde{\alpha}(c)$ are $R_c(1/6)$ and $R_c(5/6)$.

2.4. *Yoccoz puzzles and para-puzzles*

In this subsection we recall the definitions of Yoccoz puzzle and para-puzzle. We follow [Roe00].

Definition 2.1 (Yoccoz puzzles). Fix c in $\mathcal{W}$ and consider the open region $X_c := \{z \in \mathbb{C} \mid G_c(z) < 1\}$. The *Yoccoz puzzle* of f_c is the sequence $(I_{c,n})_{n=0}^{\infty}$ of graphs defined by

$$I_{c,0} := \partial X_c \cup (X_c \cap \mathrm{cl}(R_c(1/3)) \cap \mathrm{cl}(R_c(2/3))) \quad \text{and} \quad I_{c,n} := f_c^{-n}(I_{c,0}) \quad \text{for } n \geq 1.$$

The *puzzle pieces of depth n* are the connected components of $f_c^{-n}(X_c) \setminus I_{c,n}$. The puzzle piece of depth n containing a point z is denoted by $P_{c,n}(z)$.

Note that for a real parameter c, every puzzle piece intersecting the real line is invariant under complex conjugation. Since puzzle pieces are simply connected, it follows that the intersection of such a puzzle piece with $\mathbb{R}$ is an interval.

Definition 2.2 (Yoccoz para-puzzles[5]). Given an integer $n \geq 0$, set

$$J_n := \{t \in [1/3, 2/3] \mid 2^n t \pmod 1 \in \{1/3, 2/3\}\},$$

[5] In contrast with [Roe00], we only consider para-puzzles contained in $\mathcal{W}$.

let $\mathcal{X}_n$ be the intersection of $\mathcal{W}$ with the open region in the parameter plane bounded by the equipotential $\mathcal{E}(2^{-n})$ of $\mathcal{M}$, and define

$$\mathcal{I}_n := \partial \mathcal{X}_n \cup \left(\mathcal{X}_n \cap \bigcup_{t \in J_n} \mathrm{cl}(\mathcal{R}(t)) \right).$$

Then the *Yoccoz para-puzzle* of $\mathcal{W}$ is the sequence $(\mathcal{I}_n)_{n=0}^{\infty}$ of graphs. The *para-puzzle pieces of depth* n are the connected components of $\mathcal{X}_n \setminus \mathcal{I}_n$. The para-puzzle piece of depth n containing a parameter c is denoted by $\mathcal{P}_n(c)$.

Observe that there is only one para-puzzle piece of depth 0, and only one of depth 1; they are bounded by the same external rays but different equipotentials. Both contain $c = -2$.

Fix c in $\mathcal{P}_0(-2)$. There are precisely two puzzle pieces of depth 0: $P_{c,0}(\beta(c))$ and $P_{c,0}(-\beta(c))$. Each is bounded by the equipotential 1 and by the closures of the external rays landing at $\alpha(c)$. Furthermore, the critical value c of f_c is contained in $P_{c,0}(-\beta(c))$, and the critical point in $P_{c,0}(\beta(c))$. It follows that $f_c^{-1}(P_{c,0}(\beta(c)))$ is the disjoint union of $P_{c,1}(-\beta(c))$ and $P_{c,1}(\beta(c))$, so f_c maps each of the sets $P_{c,1}(-\beta(c))$ and $P_{c,1}(\beta(c))$ biholomorphically onto $P_{c,0}(\beta(c))$. Moreover, there are precisely three puzzle pieces of depth 1:

$$P_{c,1}(-\beta(c)), \quad P_{c,1}(0), \quad P_{c,1}(\beta(c));$$

$P_{c,1}(-\beta(c))$ is bounded by the equipotential $1/2$ and by the closures of the external rays that land at $\alpha(c)$; $P_{c,1}(\beta(c))$ is bounded by the equipotential $1/2$ and by the closures of the external rays that land at $\widetilde{\alpha}(c)$; and $P_{c,1}(0)$ is bounded by the equipotential $1/2$ and by the closures of the external rays that land at $\alpha(c)$ and at $\widetilde{\alpha}(c)$. In particular, the closure of $P_{c,1}(\beta(c))$ is contained in $P_{c,0}(\beta(c))$. It follows that for each integer $n \geq 1$ the map f_c^n maps $P_{c,n}(-\beta(c))$ biholomorphically onto $P_{c,0}(\beta(c))$.

The following is used several times (see [CRL12, Lemma 3.3]).

Lemma 2.3. *For each integer $n \geq 1$, the following properties hold.*

(i) *The para-puzzle piece $\mathcal{P}_n(-2)$ contains the closure of $\mathcal{P}_{n+1}(-2)$.*
(ii) *For each parameter c in $\mathcal{P}_n(-2)$ the critical value c of f_c is in $P_{c,n}(-\beta(c))$.*

2.5. The uniformly expanding Cantor set

For a parameter c in $\mathcal{P}_3(-2)$, the maximal invariant set Λ_c of f_c^3 in $P_{c,1}(0)$ plays an important role in the proof of the Main Theorem. After recalling some of the properties of Λ_c shown in [CRL12, §3.3], in this subsection we prove that f_c^3 is uniformly expanding on Λ_c and we make some distortion estimates for f_c^3 on Λ_c (Lemma 2.4).

Fix c in $\mathcal{P}_3(-2)$. There are precisely two connected components of $f_c^{-3}(P_{c,1}(0))$ contained in $P_{c,1}(0)$, which we denote by Y_c and $\widetilde{Y}_c$. The closures of these sets are disjoint and contained in $P_{c,1}(0)$. The sets Y_c and $\widetilde{Y}_c$ are distinguished by the fact that Y_c contains in its boundary the common landing point of the external rays $R_c(7/24)$ and $R_c(17/24)$, denoted $\gamma(c)$, while $\widetilde{Y}_c$ contains in its boundary the common landing point of the external rays $R_c(5/24)$ and $R_c(19/24)$. The map f_c^3 maps each of the sets Y_c and $\widetilde{Y}_c$ biholomorphically onto $P_{c,1}(0)$. Thus, if we set

$$g_c : Y_c \cup \widetilde{Y}_c \to P_{c,1}(0), \qquad z \mapsto g_c(z) := f_c^3(z),$$

then

$$\Lambda_c = \bigcap_{n \in \mathbb{N}} g_c^{-n}(\mathrm{cl}(P_{c,1}(0))).$$

The rest of this section is dedicated to proving the following lemma.

Lemma 2.4. *There are constants $C_0, \upsilon_0 > 0$ such that for every parameter c in $\mathcal{P}_5(-2)$, every ℓ in $\mathbb{N}$, and every connected component W of $g_c^{-\ell}(P_{c,1}(0))$, we have*

$$\mathrm{diam}(W) \leq C_0 \exp(-\upsilon_0 \ell);$$

furthermore, for all z and w in W we have

$$\left| \frac{Dg_c(z)}{Dg_c(w)} - 1 \right| \leq C_0 \exp(-\upsilon_0 \ell).$$

To prove this lemma, we recall some facts from [CRL12, §4.1]. For c in $\mathcal{P}_2(-2)$, the open disk $\widehat{U}_c$ containing $-\beta(c)$ and bounded by the equipotential 2 and by

$$R_c(7/24) \cup \{\gamma(c)\} \cup R_c(17/24)$$

contains the closure of $P_{c,0}(-\beta(c))$ and is disjoint from $P_{c,1}(\beta(c))$; the set $\widehat{W}_c := f_c^{-1}(\widehat{U}_c)$ contains the closure of $P_{c,1}(0)$ and varies continuously with c in $\mathcal{P}_3(-2)$.

Lemma 2.5. *For every parameter c in $\mathcal{P}_4(-2)$, each of the maps*

$$\psi_c := (g_c|_{Y_c})^{-1} \quad \textit{and} \quad \widetilde{\psi}_c := (g_c|_{\widetilde{Y}_c})^{-1}$$

extends biholomorphically to $\widehat{W}_c$. Moreover, the closures of $\psi_c(\widehat{W}_c)$ and $\widetilde{\psi}_c(\widehat{W}_c)$ are both included in $P_{c,1}(0)$.

Proof. Fix c in $\mathcal{P}_4(-2)$. To prove the first assertion, it is sufficient to show that for j in $\{0, 1, 2\}$ the critical value c is not in $f_c^{-j}(\widehat{W}_c)$. By Lemma 2.3(ii), c is in $P_{c,4}(-\beta(c))$. Then for i in $\{1, 2, 3\}$ the point $f_c^i(c)$ is in the set $P_{c,1}(\beta(c))$ disjoint from $\widehat{U}_c$. Using $\widehat{W}_c = f_c^{-1}(\widehat{U}_c)$, we conclude the proof of the extension.

To prove the second assertion, we use the fact that $f_c(Y_c) = f_c(\widetilde{Y}_c)$ and $f_c^2(Y_c) \subset P_{c,1}(\beta(c))$ (cf. [CRL12, proof of Lemma 3.5]). Denote by $\widetilde{U}_c$ the open disk containing 0 and bounded by the equipotential 2, the point $\widetilde{\alpha}(c)$ and the external rays landing at $\widetilde{\alpha}(c)$. Observe that $\widehat{U}_c \subset \widetilde{U}_c$ and thus $\widehat{W}_c$ is contained in the connected set $f_c^{-1}(\widetilde{U}_c)$. The latter is contained in the set containing $\beta(c)$ and bounded by the equipotential 1, by the preimage $\alpha_1(c)$ of $\widetilde{\alpha}(c)$ contained in $P_{c,1}(-\beta(c))$, and by the external rays $R_c(5/12)$ and $R_c(7/12)$ that land at $\alpha_1(c)$. In particular, $f_c^{-1}(\widetilde{U}_c)$ is disjoint from $P_{c,4}(-\beta(c))$. This implies that $f_c^{-2}(\widetilde{U}_c)$ has two connected components, one disjoint from $P_{c,1}(\beta(c))$ and the other containing $f_c^2(Y_c)$; the closure of the latter is contained in $P_{c,0}(\beta(c))$.

Since $f_c^2(P_{c,1}(0)) \supset P_{c,0}(\beta(c))$, we conclude that the closures of the connected components of $f_c^{-4}(\widetilde{U}_c)$ containing Y_c and $\widetilde{Y}_c$ are both contained in $P_{c,1}(0)$. This proves that the closures of $\psi_c(\widehat{W}_c)$ and $\widetilde{\psi}_c(\widehat{W}_c)$ are both contained in $P_{c,1}(0)$. □

Proof of Lemma 2.4. By Lemma 2.3(i), the closure of $\mathcal{P}_5(-2)$ is a compact set included in $\mathcal{P}_4(-2)$. Since $P_{c,1}(0)$ and $\widehat{W}_c$ vary continuously with c in $\mathcal{P}_4(-2)$ (cf. [CRL12, Lemma 2.5]), the same holds for

$$W_c := \psi_c(\widehat{W}_c) \quad \text{and} \quad \widetilde{W}_c := \widetilde{\psi}_c(\widehat{W}_c).$$

Therefore, by Lemma 2.5 we have

$$\begin{aligned}
A &:= \inf_{c\in\mathcal{P}_5(-2)} \min\left\{\operatorname{mod}(\widehat{W}_c \setminus \operatorname{cl}(W_c)), \operatorname{mod}(\widehat{W}_c \setminus \operatorname{cl}(\widetilde{W}_c))\right\} > 0,\\
\Xi_0 &:= \inf_{c\in\mathcal{P}_5(-2)} \operatorname{dist}(\partial \widehat{W}_c, P_{c,1}(0)) > 0,\\
\Xi_1 &:= \sup_{c\in\mathcal{P}_5(-2)} \operatorname{diam}(P_{c,1}(0)) < \infty,\\
\Xi_2 &:= \sup_{c\in\mathcal{P}_5(-2)} \sup_{z\in\mathbb{C}, |z|\le 2\Xi_1} |Df_c^3(z)| < \infty.
\end{aligned}$$

For an open topological disk U in $\mathbb{C}$, denote by dist_U the Poincaré distance on U. Note that there is a constant $\widehat{C} > 0$, which only depends on Ξ_0, such that for every c in $\mathcal{P}_5(-2)$ the Euclidean and Poincaré distances on $\widehat{W}_c$ are comparable with a factor of $\widehat{C}$ on $P_{c,1}(0)$ (see for example [Mil06, Lemma A.8]). On the other hand, by Pick's Theorem (see for instance [Mil06]), for every c in $\mathcal{P}_4(-2)$ the maps ψ_c and $\widetilde{\psi}_c$ are isometries for the Poincaré distances on $\widehat{W}_c$ and on W_c and $\widetilde{W}_c$, respectively. Again by Pick's Theorem, the inclusion maps from W_c and $\widetilde{W}_c$ into $\widehat{W}_c$ are each contractions for the corresponding Poincaré distances. It follows that there is $\upsilon_0 > 0$, only depending on A, such that each of these inclusions contracts by a factor of at least $\exp(-\upsilon_0)$. Thus, for every c in $\mathcal{P}_5(-2)$ and all x and y in $\widehat{W}_c$, we have

$$\begin{aligned}
\operatorname{dist}_{\widehat{W}_c}(\psi_c(x), \psi_c(y)) &\le \exp(-\upsilon_0)\operatorname{dist}_{\widehat{W}_c}(x, y),\\
\operatorname{dist}_{\widehat{W}_c}(\widetilde{\psi}_c(x), \widetilde{\psi}_c(y)) &\le \exp(-\upsilon_0)\operatorname{dist}_{\widehat{W}_c}(x, y).
\end{aligned}$$

Let $\ell \ge 1$ be an integer and W a connected component of $g_c^{-\ell}(P_{c,1}(0))$. Note that $(g_c^\ell|_W)^{-1}$ extends to a holomorphic map ψ defined on $\widehat{W}_c$ that can be written as the composition of ℓ maps in $\{\psi_c, \widetilde{\psi}_c\}$. Thus,

$$\operatorname{diam}(W) = \operatorname{diam}(\psi(P_{c,1}(0))) \le \widehat{C}^2 \exp(-\upsilon_0\ell)\operatorname{diam}(P_{c,1}(0)).$$

This proves the first desired estimate with $C_0 = \widehat{C}^2\Xi_1$.

To prove the remaining estimates, note that for each point w in $Y_c \cup \widetilde{Y}_c$ and every z in $\mathbb{C}$ satisfying $|z| = 2\Xi_1$, we have

$$|z - w| \ge \Xi_1 \quad \text{and} \quad |Df_c^3(z) - Df_c^3(w)| \le 2\Xi_2.$$

So for each w in $Y_c \cup \widetilde{Y}_c$ the maximum principle applied to the holomorphic function

$$z \mapsto \frac{Df_c^3(z) - Df_c^3(w)}{z - w}$$

and to $\{z \in \mathbb{C} \mid |z| \leq 2\Xi_1\}$ gives, for every z in $Y_c \cup \widetilde{Y}_c$,

$$|Dg_c(z) - Dg_c(w)| = |Df_c^3(z) - Df_c^3(w)| \leq 2\Xi_2\Xi_1^{-1}|z - w|.$$

On the other hand, since each of the maps ψ_c and $\widetilde{\psi}_c$ is a contraction for the Poincaré distance on $\widehat{W}_c$, by the definition of $\widehat{C}$ for every w in $Y_c \cup \widetilde{Y}_c$ we have $|Dg_c(w)|^{-1} \leq \widehat{C}^2$. We conclude that for all z and w in Y_c or in $\widetilde{Y}_c$,

$$\left|\frac{Dg_c(z)}{Dg_c(w)} - 1\right| \leq 2\widehat{C}^2\Xi_2\Xi_1^{-1}|z - w|.$$

Together with the first estimate of the lemma, this implies the second estimate with $C_0 = (2\widehat{C}^2\Xi_2\Xi_1^{-1})(\widehat{C}^2\Xi_1)$. □

2.6. Parameters

The parameter we use to prove the Main Theorem is chosen from a set introduced in [CRL12, Proposition 3.1]. In this subsection we recall the definition of this parameter set, and give some dynamical properties of the corresponding maps.

Given an integer $n \geq 3$, let $\mathcal{K}_n$ be the set of all those real parameters $c < 0$ such that

$$f_c(c) > f_c^2(c) > \cdots > f_c^{n-1}(c) > 0 \quad \text{and} \quad f_c^n(c) \in \Lambda_c.$$

Note that for c in $\mathcal{K}_n$, the critical point of f_c cannot be asymptotic to a non-repelling periodic point. This implies that all the periodic points of f_c in $\mathbb{C}$ are hyperbolic repelling, and therefore $K_c = J_c$ (see [Mil06]). On the other hand, $f_c(c) > c$ and the interval $I_c = [c, f_c(c)]$ is invariant by f_c. This implies that $I_c \subset J_c$, and hence $P_c^{\mathbb{R}}(t) \leq P_c^{\mathbb{C}}(t)$ for every real t. Note also that $f_c|_{I_c}$ is not renormalizable, so f_c is topologically exact on I_c (see for example [dMvS93, Theorem III.4.1]).

Since for c in $\mathcal{K}_n$ the critical point of f_c is not periodic, for every integer $k \geq 0$ we have $f_c^{n+3k}(c) \neq 0$. Thus, we can define a sequence $\iota(c)$ in $\{0, 1\}^{\mathbb{N}_0}$ for each $k \geq 0$ by

$$\iota(c)_k := \begin{cases} 0 & \text{if } f_c^{n+3k}(c) \in Y_c, \\ 1 & \text{if } f_c^{n+3k}(c) \in \widetilde{Y}_c. \end{cases}$$

Proposition 2.6. *For each integer $n \geq 3$, the set $\mathcal{K}_n$ is a compact subset of*

$$\mathcal{P}_n(-2) \cap (-2, -3/4),$$

and for every sequence $\underline{x}$ in $\{0, 1\}^{\mathbb{N}_0}$ there is a unique parameter c in $\mathcal{K}_n$ such that $\iota(c) = \underline{x}$. Finally, for each $\delta > 0$ there is $n_0 \geq 3$ such that $\mathcal{K}_n \subset (-2, -2 + \delta)$ for each $n \geq n_0$.

Given an open subset G of $\mathbb{C}$ and a univalent map $f : G \to \mathbb{C}$, we say that the *distortion* of f on a subset C of G is

$$\sup_{x,y\in C} |Df(x)|/|Df(y)|.$$

The following is a uniform distortion bound for parameters as in the previous proposition.

Lemma 2.7 ([CRL12, Lemma 4.3]). *There is a constant $\Delta_0 > 1$ such that for each integer $n \geq 4$ and each parameter c in $\mathcal{K}_n$ the following properties hold: For each integer $m \geq 1$ and each connected component W of $f_c^{-m}(P_{c,1}(0))$ on which f_c^m is univalent, f_c^m maps a neighborhood of W biholomorphically onto $\widehat{W}_c$ and the distortion of this map on W is bounded by Δ_0.*

2.7. *Induced map and pressure function*

Let $n \geq 5$ be an integer and c a parameter in $\mathcal{K}_n$. Throughout this subsection we set $\widehat{V}_c := P_{c,4}(0)$. Note that the critical value c of f_c is in $P_{c,n}(-\beta(c))$ (Lemma 2.3(ii) and Proposition 2.6), so the closure of

$$V_c := P_{c,n+1}(0) = f_c^{-1}(P_{c,n}(-\beta(c)))$$

is contained in $\widehat{V}_c = f_c^{-1}(P_{c,3}(-\beta(c)))$ (cf. [CRL12, part 1 of Lemma 3.2]).

Set $D_c := \{z \in V_c \mid f_c^m(z) \in V_c \text{ for some } m \geq 1\}$. For z in D_c write $m_c(z) := \min\{m \in \mathbb{N} \mid f_c^m(z) \in V_c\}$, and call it the *first return time of z to V_c*. The *first return map to V_c* is defined by

$$F_c : D_c \to V_c, \qquad z \mapsto F_c(z) := f_c^{m_c(z)}(z).$$

It is easy to see that D_c is a disjoint union of puzzle pieces; so each connected component of D_c is a puzzle piece. Note furthermore that in each puzzle piece W, the return time function m_c is constant; denote its value by $m_c(W)$.

Denote by $\mathfrak{D}_c$ the collection of connected components of D_c and by $\mathfrak{D}_c^{\mathbb{R}}$ the sub-collection of $\mathfrak{D}_c$ of those sets intersecting $\mathbb{R}$. For each W in $\mathfrak{D}_c$ denote by $\phi_W : \widehat{V}_c \to V_c$ the extension of $F_c|_W^{-1}$ given by [CRL12, Lemma 6.1]. Given an integer $\ell \geq 1$ we denote by $E_{c,\ell}$ (resp. $E_{c,\ell}^{\mathbb{R}}$) the set of all words of length ℓ in the alphabet $\mathfrak{D}_c$ (resp. $\mathfrak{D}_c^{\mathbb{R}}$). Again by [CRL12, Lemma 6.1], for each integer $\ell \geq 1$ and each word $W_1 \cdots W_\ell$ in $E_{c,\ell}$ the composition

$$\phi_{W_1 \cdots W_\ell} = \phi_{W_1} \circ \cdots \circ \phi_{W_\ell}$$

is defined on $\widehat{V}_c$. We also set

$$m_c(W_1 \cdots W_\ell) := m_c(W_1) + \cdots + m_c(W_\ell).$$

For t, p in $\mathbb{R}$ and an integer $\ell \geq 1$ define

$$Z_\ell(t,p) := \sum_{\underline{W} \in E_{c,\ell}} \exp(-m_c(\underline{W})p)(\sup\{|D\phi_{\underline{W}}(z)| \mid z \in V_c\})^t$$

$$Z_\ell^{\mathbb{R}}(t,p) := \sum_{\underline{W} \in E_{c,\ell}^{\mathbb{R}}} \exp(-m_c(\underline{W})p)(\sup\{|D\phi_{\underline{W}}(z)| \mid z \in V_c\})^t.$$

For a fixed t and p in $\mathbb{R}$ the sequence

$$\left(\frac{1}{\ell}\log Z_\ell(t,p)\right)_{\ell=1}^{\infty} \quad \left(\text{resp. } \left(\frac{1}{\ell}\log Z_\ell^{\mathbb{R}}(t,p)\right)_{\ell=1}^{\infty}\right)$$

converges to the pressure function of F_c (resp. $F_c|_{D_c\cap\mathbb{R}}$) for the potential $-t\log|DF_c| - pm_c$; we denote the limit by $\mathscr{P}_c^{\mathbb{C}}(t,p)$ (resp. $\mathscr{P}_c^{\mathbb{R}}(t,p)$). On the set where it is finite, the function $\mathscr{P}_c^{\mathbb{C}}$ (resp. $\mathscr{P}_c^{\mathbb{R}}$) so defined is strictly decreasing in each of its variables.

3. The two-variable series

We start this section by stating a stronger version of the Main Theorem in §3.1. The rest of this section is dedicated to estimating, for a real parameter c in $\bigcup_{n=6}^{\infty}\mathcal{K}_n$ satisfying some mild hypotheses, a certain "postcritical series" in terms of an abstract two-variable series (Proposition A in §3.3). The postcritical series is used in §5 to estimate the geometric pressure function. The definition of the two-variable series is based on an approximation of the derivatives $(Df_c^n(c))_{n=1}^{\infty}$, using the derivatives of g_c at its fixed points $p(c)$ and $\widetilde{p}(c)$. This approximation, which is more precise than a direct application of the Koebe principle, incorporates an estimate of the corresponding distortion constants (Proposition 3.1 in §3.2). This estimate is given in terms of the total distortion of the two homoclinic orbits of g_c connecting $p(c)$ and $\widetilde{p}(c)$.

3.1. Main Technical Theorem

In this subsection we state the Main Technical Theorem from which the Main Theorem follows directly. The rest of the paper is dedicated to the proof of the Main Technical Theorem.

Let c be a parameter in $\bigcup_{n=6}^{\infty}\mathcal{K}_n$. An invariant probability measure supported on I_c (resp. J_c) is said to be an *equilibrium state of* $f_c|_{I_c}$ (resp. f_c) *for the potential* $-\log|Df_c|$ if the supremum defining $P_c^{\mathbb{R}}(t)$ (resp. $P_c^{\mathbb{C}}(t)$) is attained at this measure. Given $t>0$ and a real number p we say a measure μ is (t,p)*-conformal for* $f_c|_{I_c}$ (resp. f_c) if for every subset U of I_c (resp. J_c) on which $f_c|_{I_c}$ (resp. f_c) is injective we have

$$\mu(f_c|_{I_c}(U)) = \exp(p)\int_U |Df_c|^t\,d\mu \quad \left(\text{resp. } \mu(f_c(U)) = \exp(p)\int_U |Df_c|^t\,d\mu\right).$$

In the case where $P_c^{\mathbb{R}}(t)=0$ (resp. $P_c^{\mathbb{C}}(t)=0$), a $(t,0)$-conformal measure is simply called *conformal*.

For each c in $\mathcal{P}_3(-2)$ denote by $p(c)$ the unique fixed point of g_c in Y_c and by $\widetilde{p}(c)$ the unique fixed point of g_c in $\widetilde{Y}_c$. Each of the functions

$$p:\mathcal{P}_3(-2)\to\mathbb{C} \quad \text{and} \quad \widetilde{p}:\mathcal{P}_3(-2)\to\mathbb{C}$$

so defined is holomorphic.

Main Technical Theorem. *There is $n_1 \geq 6$ such that for every integer $n \geq n_1$ there are a parameter c in $\mathcal{K}_n$, an integer $q \geq 3$, and real numbers κ in $[1, 2]$ and $\Delta \geq 1$, such that the following properties are satisfied. Set*

$$t_* := \frac{2 \log 2}{\log \frac{|Dg_c(p(c))|}{|Dg_c(\widetilde{p}(c))|}} \quad \text{and} \quad t_0 := \frac{q-2}{q-1} \cdot t_*,$$

and define the functions $\delta^+, \delta^-, p^+, p^- : (t_0, \infty) \to \mathbb{R}$ by

$$\delta^+(t) := \begin{cases} \frac{2\log 2}{3} \cdot 2^{-q\left(\frac{\kappa t_*}{q(t_*-t)} - 1\right)^2} & \text{if } t \in (t_0, t_*), \\ 0 & \text{if } t \geq t_*, \end{cases}$$

$$\delta^-(t) := \begin{cases} \frac{\log 2}{3} \cdot 2^{-q\left(\frac{\kappa t_*}{q(t_*-t)} + \Delta\right)^2} & \text{if } t \in (t_0, t_*), \\ 0 & \text{if } t \geq t_*, \end{cases}$$

$$p^+(t) := -t\chi_{\mathrm{crit}}(c)/2 + \delta^+(t) \quad \text{and} \quad p^-(t) := -t\chi_{\mathrm{crit}}(c)/2 + \delta^-(t).$$

Then $\chi_{\mathrm{crit}}(c) > 0$, for $t > t_0$ we have

$$p^-(t) \leq P_c^{\mathbb{R}}(t) \leq P_c^{\mathbb{C}}(t) \leq p^+(t),$$

and for $t \geq t_*$ there is no equilibrium state of $f_c|_{I_c}$ (resp. $f_c|_{J_c}$) for the potential $-t \log |Df_c|$ and we have

$$\mathscr{P}_c^{\mathbb{R}}(t, -t\chi_{\mathrm{crit}}(c)/2) \leq \mathscr{P}_c^{\mathbb{C}}(t, -t\chi_{\mathrm{crit}}(c)/2) < 0.$$

Moreover, for $t \geq t_*$ and for p in $\mathbb{R}$ the following properties hold:

(i) *If $p \geq -t\chi_{\mathrm{crit}}(c)/2$, then there is a unique (t, p)-conformal probability measure for $f_c|_{I_c}$ (resp. f_c) supported on I_c (resp. J_c). Moreover, this measure is dissipative, purely atomic, and supported on the backward orbit of $z = 0$.*
(ii) *If $p < -t\chi_{\mathrm{crit}}(c)/2$, then there is no (t, p)-conformal probability measure for $f_c|_{I_c}$ (resp. f_c) supported on I_c (resp. J_c).*

3.2. Improved distortion estimate

The purpose of this subsection is to prove Proposition 3.1 below. To state it, define for each c in $\mathcal{P}_3(-2)$ the itinerary map

$$\iota_c : \Lambda_c \to \{0, 1\}^{\mathbb{N}_0},$$

for x in Λ_c and k in $\mathbb{N}_0$, by

$$\iota_c(x)_k := \begin{cases} 0 & \text{if } g_c^k(x) \in Y_c, \\ 1 & \text{if } g_c^k(x) \in \widetilde{Y}_c. \end{cases}$$

We recall from [CRL12, §3.3] that the map ι_c conjugates the action of g_c on Λ_c to the action of the shift map on $\{0, 1\}^{\mathbb{N}_0}$. For c in $\mathcal{K}_n$, the point $f_c^n(c)$ is in Λ_c and the sequence $\iota(c)$ defined in §2.6 is equal to $\iota_c(f_c^n(c))$. Finally, for each $\underline{x}$ in $\{0,1\}^{\mathbb{N}_0}$ define

$$I_{\underline{x}} : \mathcal{P}_3(-2) \to \mathbb{C}, \quad c \mapsto I_{\underline{x}}(c) := \iota_c^{-1}(\underline{x}).$$

By a normality argument the function $I_{\underline{x}}$ is holomorphic.

Proposition 3.1 (Improved distortion estimate). *There are analytic functions*

$$\zeta : \mathcal{P}_5(-2) \to (0,\infty) \quad \textit{and} \quad \widetilde{\zeta} : \mathcal{P}_5(-2) \to (0,\infty),$$

and constants $C_1, \upsilon_1 > 0$, such that for every integer $n \geq 5$ and every parameter c in $\mathcal{K}_n$ the following property holds: Let m and m' be positive integers and let

$$\underline{x} = (x_j)_{j=0}^{\infty} \quad (\textit{resp. } \underline{\widetilde{x}} = (\widetilde{x}_j)_{j=0}^{\infty})$$

be a sequence in $\{0,1\}^{\mathbb{N}_0}$ such that $x_j = 0$ (resp. $\widetilde{x}_j = 1$) for j in $\{0,\ldots,m-1\}$, and $x_{m+j} = 1$ (resp. $\widetilde{x}_{m+j} = 0$) for j in $\{0,\ldots,m'-1\}$. Then

$$\left|\log \frac{|Dg_c^m(I_{\underline{x}}(c))|}{|Dg_c(p(c))|^m} - \log \zeta(c)\right| \leq C_1 \exp(-\min\{m,m'\}\upsilon_1),$$

$$\left|\log \frac{|Dg_c^m(I_{\underline{\widetilde{x}}}(c))|}{|Dg_c(\widetilde{p}(c))|^m} - \log \widetilde{\zeta}(c)\right| \leq C_1 \exp(-\min\{m,m'\}\upsilon_1).$$

The proof of this proposition is at the end of this subsection.

For each integer ℓ in $\mathbb{N}_0$, let

$$\underline{x}^{\ell} = (x_j^{\ell})_{j=0}^{\infty} \quad \text{and} \quad \underline{\widetilde{x}}^{\ell} = (\widetilde{x}_j^{\ell})_{j=0}^{\infty}$$

be the sequences in $\{0,1\}^{\mathbb{N}_0}$ defined for each j in $\mathbb{N}_0$ by

$$x_j^{\ell} := \begin{cases} 0 & \text{if } j \leq \ell - 1, \\ 1 & \text{if } j \geq \ell, \end{cases} \qquad \widetilde{x}_j^{\ell} := \begin{cases} 1 & \text{if } j \leq \ell - 1, \\ 0 & \text{if } j \geq \ell. \end{cases}$$

Observe that for every c in $\mathcal{P}_3(-2)$ and every ℓ in $\mathbb{N}$, the points $I_{\underline{x}^{\ell}}(c)$ and $p(c)$ are in the same connected component of $g_c^{-\ell}(P_{c,1}(0))$, and the same holds for $I_{\underline{\widetilde{x}}^{\ell}}(c)$ and $\widetilde{p}(c)$. Thus, the following is a direct consequence of Lemma 2.4.

Corollary 3.2. *Let $C_0, \upsilon_0 > 0$ be the constants given by Lemma 2.4. Then, for all c in $\mathcal{P}_5(-2)$ and ℓ in $\mathbb{N}$, we have*

$$\left|\frac{Dg_c(I_{\underline{x}^{\ell}}(c))}{Dg_c(p(c))} - 1\right| \leq C_0 \exp(-\upsilon_0 \ell) \quad \textit{and} \quad \left|\frac{Dg_c(I_{\underline{\widetilde{x}}^{\ell}}(c))}{Dg_c(\widetilde{p}(c))} - 1\right| \leq C_0 \exp(-\upsilon_0 \ell).$$

Lemma 3.3 (Homoclinic distortion). *For every parameter c in $\mathcal{P}_5(-2)$, the limits*

$$\zeta(c) := \lim_{m\to\infty} \prod_{\ell=1}^{m} \frac{|Dg_c(I_{\underline{x}^{\ell}}(c))|}{|Dg_c(p(c))|} \quad \textit{and} \quad \widetilde{\zeta}(c) := \lim_{m\to\infty} \prod_{\ell=1}^{m} \frac{|Dg_c(I_{\underline{\widetilde{x}}^{\ell}}(c))|}{|Dg_c(\widetilde{p}(c))|}$$

exist and depend analytically on c in $\mathcal{P}_5(-2)$.

Proof. We prove the existence of the first limit and its analytic dependence on c; the proof of the analogous assertions for the second limit are similar.

Denote by log the logarithm defined in the open disk of $\mathbb{C}$ of radius 1 centered at $z = 1$. By Corollary 3.2, there is ℓ_0 in $\mathbb{N}$ such that for every $\ell \geq \ell_0$ and every c in $\mathcal{P}_5(-2)$ we have $\left|\frac{Dg_c(I_{\underline{x}^\ell})(c)}{Dg_c(p(c))} - 1\right| < 1$, so $\log \frac{Dg_c(I_{\underline{x}^\ell})(c)}{Dg_c(p(c))}$ is defined. Corollary 3.2 also implies that

$$\sum_{\ell=\ell_0}^{\infty} \log \frac{Dg_c(I_{\underline{x}^\ell})(c)}{Dg_c(p(c))}$$

exists and is a holomorphic function of c on $\mathcal{P}_5(-2)$. Exponentiating, we find that the infinite product $\prod_{\ell=\ell_0}^{\infty} \frac{Dg_c(I_{\underline{x}^\ell})(c)}{Dg_c(p(c))}$ exists and is holomorphic on $\mathcal{P}_5(-2)$. This implies that the infinite product starting from $\ell = 1$ also exists and is holomorphic on $\mathcal{P}_5(-2)$. Taking the modulus we conclude the proof. □

Proof of Proposition 3.1. Let C_0 and υ_0 be the constants given by Lemma 2.4 and let $\zeta : \mathcal{P}_5(-2) \to (0, \infty)$ and $\widetilde{\zeta} : \mathcal{P}_5(-2) \to (0, \infty)$ be the continuous functions given by Lemma 3.3. We only prove the first inequality, the other being similar. We have

$$\begin{aligned}
&\log \frac{|Dg_c^m(I_{\underline{x}}(c))|}{|Dg_c(p(c))|^m} - \log \zeta(c) \\
&= \sum_{j=0}^{m-1} \log \frac{|Dg_c(g_c^j(I_{\underline{x}}(c)))|}{|Dg_c(p(c))|} - \lim_{\widetilde{m}\to\infty} \sum_{\ell=1}^{\widetilde{m}} \log \frac{|Dg_c(I_{\underline{x}^\ell}(c))|}{|Dg_c(p(c))|} \\
&= \sum_{j=0}^{m-1} \log \frac{|Dg_c(g_c^j(I_{\underline{x}}(c)))|}{|Dg_c(p(c))|} - \sum_{\ell=1}^{m} \log \frac{|Dg_c(I_{\underline{x}^\ell}(c))|}{|Dg_c(p(c))|} - \lim_{\widetilde{m}\to\infty} \sum_{\ell=m+1}^{\widetilde{m}} \log \frac{|Dg_c(I_{\underline{x}^\ell}(c))|}{|Dg_c(p(c))|}.
\end{aligned}$$

Notice that for every j in $\{0, \ldots, m-1\}$ we have $g_c^j(I_{\underline{x}^m}(c)) = I_{\underline{x}^{m-j}}(c)$, and $g_c^j(I_{\underline{x}}(c))$ and $g_c^j(I_{\underline{x}^m}(c))$ are in the same connected component of $g_c^{-(m+m'-j)}(P_{c,1}(0))$. Using Lemma 2.4 repeatedly, we get

$$\begin{aligned}
&\left|\sum_{j=0}^{m-1} \log \frac{|Dg_c(g_c^j(I_{\underline{x}}(c)))|}{|Dg_c(p(c))|} - \sum_{\ell=1}^{m} \log \frac{|Dg_c(I_{\underline{x}^\ell}(c))|}{|Dg_c(p(c))|}\right| \\
&\qquad = \left|\sum_{j=0}^{m-1} \log \frac{|Dg_c(g_c^j(I_{\underline{x}}(c)))|}{|Dg_c(p(c))|} - \sum_{j=0}^{m-1} \log \frac{|Dg_c(I_{\underline{x}^{m-j}}(c))|}{|Dg_c(p(c))|}\right| \\
&\qquad = \left|\sum_{j=0}^{m-1} \log \frac{|Dg_c(g_c^j(I_{\underline{x}}(c)))|}{|Dg_c(I_{\underline{x}^{m-j}}(c))|}\right| \\
&\qquad = \left|\sum_{j=0}^{m-1} \log \frac{|Dg_c(g_c^j(I_{\underline{x}}(c)))|}{|Dg_c(g_c^j(I_{\underline{x}^m}(c)))|}\right|
\end{aligned}$$

$$\leq C_0 \sum_{j=0}^{m-1} \exp(-\upsilon_0(m+m'-j))$$
$$\leq \frac{C_0 \exp(-\upsilon_0)}{1-\exp(-\upsilon_0)} \exp(-\upsilon_0 m').$$

On the other hand, by Corollary 3.2 we have, for every integer $\widetilde{m} \geq m$,

$$\left| \sum_{\ell=m+1}^{\widetilde{m}} \log \frac{|Dg_c(I_{\underline{x}^\ell}(c))|}{|Dg_c(p(c))|} \right| \leq C_0 \sum_{\ell=m+1}^{\widetilde{m}} \exp(-\upsilon_0 \ell) \leq \frac{C_0 \exp(-\upsilon_0)}{1-\exp(-\upsilon_0)} \exp(-\upsilon_0 m).$$

Taking $C_1 := 2\frac{C_0 \exp(-\upsilon_0)}{1-\exp(-\upsilon_0)}$ and $\upsilon_1 := \upsilon_0$ we conclude the proof of the proposition. □

3.3. The two variable series

For each integer $n \geq 4$ and for each parameter c in $\mathcal{K}_n$, denote by

$$\iota(c) := \iota_c(f_c^n(c))$$

the itinerary for g_c in the Cantor set Λ_c of the point $x = f_c^n(c)$ (see §2.6). Furthermore, denote by $N_c : \mathbb{N}_0 \to \mathbb{N}_0$ the function defined by $N_c(0) := 0$ and

$$N_c(k) := \sharp\{j \in \{0, \dots, k-1\} \mid \iota(c)_j = 0\} \quad \text{for } k \geq 1,$$

and by $B_c : \mathbb{N}_0 \to \mathbb{N}_0$ the function defined by $B_c(0) := 0$, $B_c(1) := 1$, and

$$B_c(k) := 1 + \sharp\{j \in \{0, \dots, k-2\} \mid \iota(c)_j \neq \iota(c)_{j+1}\} \quad \text{for } k \geq 2.$$

Note that for k in $\mathbb{N}$ the function $B_c(k)$ is equal to the number of blocks of 0's and 1's in the sequence $(\iota(c)_j)_{j=0}^{k-1}$.

On the other hand, for each parameter c in $\mathcal{P}_5(-2)$, set

$$\theta(c) := \left| \frac{Dg_c(p(c))}{Dg_c(\widetilde{p}(c))} \right|^{1/2}, \qquad \xi(c) := -\frac{\log(\zeta(c)\widetilde{\zeta}(c))}{4 \log \theta(c)}, \tag{3.1}$$

and define a two-variable series Π_c on $[0, \infty) \times [0, \infty)$ by

$$\Pi_c(\tau, \lambda) := \sum_{k=0}^{\infty} 2^{-\lambda k - \tau N_c(k) + \tau \xi(c) B_c(k)}.$$

The purpose of this subsection is to prove the following proposition.

Proposition A. *There are constants $C_2 > 1$ and $\upsilon_2 > 0$ such that for every integer $n \geq 6$ the following property holds. Let c in $\mathcal{K}_n$ be such that $N_c(k)/k \to 0$ as $k \to \infty$. Denoting by $(m_j)_{j=0}^{\infty}$ the sequence of the lengths of blocks of 0's and 1's in the sequence $\iota(c)$, assume that the sum*

$$\sum_{j=0}^{\infty} \exp(-\min\{m_j, m_{j+1}\}\upsilon_2)$$

converges. Then for all $t > 0$ *and* $\delta \geq 0$*, we have*

$$C_2^{-t}\exp(-n\delta)\left(\frac{\exp(\chi_{\mathrm{crit}}(c))}{|Df_c(\beta(c))|}\right)^{tn/2}\Pi_c\left(t\frac{\log\theta(c)}{\log 2},\frac{3\delta}{\log 2}\right)$$
$$\leq \sum_{k=0}^{\infty}\exp(-(n+3k)(-t\chi_{\mathrm{crit}}(c)/2+\delta))|Df_c^{n+3k}(c)|^{-t/2}$$
$$\leq C_2^{t}\exp(-n\delta)\left(\frac{\exp(\chi_{\mathrm{crit}}(c))}{|Df_c(\beta(c))|}\right)^{tn/2}\Pi_c\left(t\frac{\log\theta(c)}{\log 2},\frac{3\delta}{\log 2}\right). \quad (3.2)$$

The proof of this proposition is at the end of this subsection.

Lemma 3.4. *Let* Δ_0 *be the constant given by Lemma 2.7 and let* $C_1, \upsilon_1 > 0$ *be the constants given by Proposition 3.1. Moreover, let* $n \geq 5$ *be an integer, let* c *be a parameter in* $\mathcal{K}_n$*, and denote by* $(m_j)_{j=0}^{\infty}$ *the sequence of the lengths of blocks of* 0*'s and* 1*'s in the sequence* $\iota(c)$*. Then for every* k *in* $\mathbb{N}$ *we have*

$$\Delta_0^{-1}\max\left\{\frac{\zeta(c)}{\widetilde{\zeta}(c)},\frac{\widetilde{\zeta}(c)}{\zeta(c)}\right\}^{-1/2}\exp\left(-C_1\sum_{j=0}^{B_c(k)-1}\exp(-\min\{m_j,m_{j+1}\}\upsilon_1)\right)$$
$$\leq \frac{|Dg_c^k(f_c^n(c))|}{|Dg_c(\widetilde{p}(c))|^k\cdot\theta(c)^{2N_c(k)}\cdot(\zeta(c)\widetilde{\zeta}(c))^{B_c(k)/2}}$$
$$\leq \Delta_0\max\left\{\frac{\zeta(c)}{\widetilde{\zeta}(c)},\frac{\widetilde{\zeta}(c)}{\zeta(c)}\right\}^{1/2}\exp\left(C_1\sum_{j=0}^{B_c(k)-1}\exp(-\min\{m_j,m_{j+1}\}\upsilon_1)\right).$$

Proof. If the first k entries of $\iota(c)$ are equal, then $B_c(k) = 1$ and the assertion follows from Lemma 2.7. Suppose otherwise, and let k_0 be maximal in $\{1,\ldots,k\}$ such that

$$\iota(c)_{k_0-1} \neq \iota(c)_{k_0}.$$

Moreover, denote by B and $\widetilde{B}$ the number of blocks of 0's and 1's, respectively, in the sequence $(\iota(c)_j)_{j=0}^{k_0-1}$. We have $B_c(k_0) = B + \widetilde{B}$, and

$$|B_c(k_0) - 2B| = |B_c(k_0) - 2\widetilde{B}| \leq 1. \quad (3.3)$$

Consider a block of 0's or 1's in $\iota(c)$ with initial position i and length m, and let m' be the length of the next block. By Proposition 3.1 we have the following two cases: If $\iota(c)_i = 0$, then

$$\left|\log\frac{|Dg_c^m(g_c^i(f_c^n(c)))|}{|Dg_c(p(c))|^m} - \log\zeta(c)\right| \leq C_1\exp(-\min\{m,m'\}\upsilon_1);$$

and if $\iota(c)_i = 1$, then

$$\left|\log\frac{|Dg_c^m(g_c^i(f_c^n(c)))|}{|Dg_c(\widetilde{p}(c))|^m} - \log\widetilde{\zeta}(c)\right| \leq C_1\exp(-\min\{m,m'\}\upsilon_1).$$

Applying these inequalities to each block of 0's and 1's in $(\iota(c)_j)_{j=0}^{k_0-1}$, we obtain

$$\begin{aligned}\exp\Bigg(-C_1\sum_{j=0}^{B_c(k_0)-1}\exp(-\min\{m_j,m_{j+1}\}\upsilon_1)\Bigg)\\ \le \frac{|Dg_c^{k_0}(f_c^n(c))|}{|Dg_c(p(c))|^{N_c(k_0)}|Dg_c(\widetilde{p}(c))|^{k_0-N_c(k_0)}\zeta(c)^B\widetilde{\zeta}(c)^{\widetilde{B}}}\\ \le \exp\Bigg(C_1\sum_{j=0}^{B_c(k_0)-1}\exp(-\min\{m_j,m_{j+1}\}\upsilon_1)\Bigg).\end{aligned} \tag{3.4}$$

Together with (3.3) this implies the desired chain of inequalities in the case where $k_0=k$. If $k_0\le k-1$, then by Lemma 2.7 we have

$$\Delta_0^{-1}\le\frac{|Dg_c^{k-k_0}(g_c^{k_0}(f_c^n(c)))|}{|Dg_c(\widetilde{p}(c))|^{k-k_0}\cdot\theta(c)^{2(N_c(k)-N_c(k_0))}}\le\Delta_0.$$

This, together with (3.3), (3.4), and $B_c(k)=B_c(k_0)+1$, implies the desired conclusion. □

Lemma 3.5. *Let $n\ge 4$ be an integer and let c in $\mathcal{K}_n$ be such that $N_c(k)/k\to 0$ as $k\to\infty$. Then*

$$\chi_{\mathrm{crit}}(c)=\tfrac{1}{3}\log|Dg_c(\widetilde{p}(c))|.$$

Proof. Set $\widehat{c}:=f_c^n(c)$. For all k in $\mathbb{N}$ and j in $\{0,1,2\}$, by the chain rule

$$Df_c^{3k+j}(c)=Df_c^j((f_c^{3k})(\widehat{c}))\cdot Df_c^{3k}(\widehat{c})\cdot Df_c^n(c)=Df_c^j(g_c^k(\widehat{c}))\cdot Dg_c^k(\widehat{c})\cdot Df_c^n(c).$$

Since $|Df_c^j((g_c^k)(\widehat{c}))|$ is bounded independently of k and j, we have

$$\chi_{\mathrm{crit}}(c)=\liminf_{m\to\infty}\frac{1}{m}\log|Df_c^m(c)|=\frac{1}{3}\liminf_{k\to\infty}\frac{1}{k}\log|Dg_c^k(\widehat{c})|. \tag{3.5}$$

On the other hand, by Lemma 2.7, there is a constant $\Delta_0>1$ such that for each k in $\mathbb{N}$,

$$\Delta_0^{-B_c(k)}\le\frac{|Dg_c^k(\widehat{c})|}{|Dg_c(\widetilde{p}(c))|^{k-N_c(k)}|Dg_c(p(c))|^{N_c(k)}}\le\Delta_0^{B_c(k)}.$$

Taking logarithms yields

$$\begin{aligned}-B_c(k)\log\Delta_0+N_c(k)\log\frac{|Dg_c(p(c))|}{|Dg_c(\widetilde{p}(c))|}\le\log|Dg_c^k(\widehat{c})|-k\log|Dg_c(\widetilde{p}(c))|\\ \le B_c(k)\log\Delta_0+N_c(k)\log\frac{|Dg_c(p(c))|}{|Dg_c(\widetilde{p}(c))|}.\end{aligned}$$

Since for each k in $\mathbb{N}$ we have $B_c(k)\le 2N_c(k)+1$, and using the hypothesis that $N_c(k)/k\to 0$ as $k\to\infty$, we conclude that

$$\lim_{k\to\infty}\frac{1}{k}\log|Dg_c^k(\widehat{c})|=\log|Dg_c(\widetilde{p}(c))|.$$

Combined with (3.5), this completes the proof of the lemma. □

Lemma 3.6 ([CRL12, Lemma 3.6]). *There is a constant $\Delta_1 > 1$ such that for each parameter c in $\mathcal{P}_2(-2)$, each integer $k \geq 2$, and each point y in $P_{c,k}(-\beta(c))$, we have*

$$\Delta_1^{-1}|Df_c(\beta(c))|^k \leq |Df_c^k(y)| \leq \Delta_1 |Df_c(\beta(c))|^k.$$

Proof of Proposition A. Let Δ_0 be the constant given by Lemma 2.7, let C_1 and υ_1 be the constants given by Proposition 3.1, and let Δ_1 be the constant given by Lemma 3.6. Note that by Proposition 3.1 and Lemma 2.3(i),

$$\Delta := \sup_{c \in \mathcal{P}_6(-2)} \max \left\{ \frac{\zeta(c)}{\widetilde{\zeta}(c)}, \frac{\widetilde{\zeta}(c)}{\zeta(c)} \right\} < \infty.$$

Let n, c, and $(m_j)_{j=0}^{\infty}$ be as in the statement of the proposition, and set

$$\sigma := C_1 \sum_{j=0}^{\infty} \exp(-\min\{m_j, m_{j+1}\}\upsilon_1) \quad \text{and} \quad \widehat{C}_2 := \Delta_0 \Delta_1 \Delta^{1/2} \exp(\sigma).$$

Then for every k in $\mathbb{N}$ and every $t > 0$, we have, using

$$Df_c^{n+3k}(c) = Dg_c^k(f_c^n(c)) \cdot Df_c^n(c)$$

and combining Lemmas 3.4 and 3.6,

$$\begin{aligned} \widehat{C}_2^{-t} \theta(c)^{-2tN_c(k)} (\zeta(c)\widetilde{\zeta}(c))^{-tB_c(k)/2} &\leq \frac{|Df_c^{n+3k}(c)|^{-t}}{|Dg_c(\widetilde{p}(c))|^{-tk} |Df_c(\beta(c))|^{-tn}} \\ &\leq \widehat{C}_2^t \theta(c)^{-2tN_c(k)} (\zeta(c)\widetilde{\zeta}(c))^{-tB_c(k)/2}. \end{aligned} \tag{3.6}$$

Since by Lemma 3.5,

$$\exp((n+3k)t\chi_{\text{crit}}(c)) = \exp(nt\chi_{\text{crit}}(c)) |Dg_c(\widetilde{p}(c))|^{tk},$$

if we multiply each term in (3.6) by

$$\left(\frac{\exp(\chi_{\text{crit}}(c))}{|Df_c(\beta(c))|} \right)^{tn},$$

we get

$$\begin{aligned} \widehat{C}_2^{-t} &\left(\frac{\exp(\chi_{\text{crit}}(c))}{|Df_c(\beta(c))|} \right)^{tn} \theta(c)^{-2tN_c(k)} (\zeta(c)\widetilde{\zeta}(c))^{-tB_c(k)/2} \\ &\leq \exp((n+3k)t\chi_{\text{crit}}(c)) |Df_c^{n+3k}(c)|^{-t} \\ &\leq \widehat{C}_2^t \left(\frac{\exp(\chi_{\text{crit}}(c))}{|Df_c(\beta(c))|} \right)^{tn} \theta(c)^{-2tN_c(k)} (\zeta(c)\widetilde{\zeta}(c))^{-tB_c(k)/2}. \end{aligned}$$

Taking square roots and then by multiplying by $\exp(-(n+3k)\delta)$, we obtain

$$\begin{aligned}
&\widehat{C}_2^{-t/2} \exp(-n\delta) \left(\frac{\exp(\chi_{\mathrm{crit}}(c))}{|Df_c(\beta(c))|} \right)^{tn/2} \exp(-3k\delta)\theta(c)^{-tN_c(k)}(\zeta(c)\widetilde{\zeta}(c))^{-tB_c(k)/4} \\
&\le \exp\big(-(n+3k)(-t\chi_{\mathrm{crit}}(c)/2+\delta)\big)|Df_c^{n+3k}(c)|^{-t/2} \\
&\le \widehat{C}_2^{t/2} \exp(-n\delta) \left(\frac{\exp(\chi_{\mathrm{crit}}(c))}{|Df_c(\beta(c))|} \right)^{tn/2} \exp(-3k\delta)\theta(c)^{-tN_c(k)}(\zeta(c)\widetilde{\zeta}(c))^{-tB_c(k)/4}.
\end{aligned}$$

Note that when $k = 0$ this chain of inequalities holds by Lemma 2.7 and our definition of $\widehat{C}_2$. Summing over $k \ge 0$, we obtain the proposition with $C_2 = \widehat{C}_2^{1/2}$. □

4. Estimating the two-variable series

This section is dedicated to estimating, in an abstract setting, the two-variable series defined in §3.3 for a certain itinerary defined in §4.1. Our main estimate is stated as Proposition B in §4.2.

4.1. The itinerary

Given an integer Ξ, let $q \ge 3$ be a sufficiently large integer such that $q + \Xi \ge 1$ and $2^{q-1} \ge q + 1 + \Xi$. Define the quadratic function

$$Q : \mathbb{R} \to \mathbb{R}, \qquad s \mapsto Q(s) := qs^2,$$

and for each real s in $[0, \infty)$ define the following intervals of $\mathbb{R}$:

$$\begin{aligned}
I_s &:= [2^{Q(s)}, 2^{Q(s)} + Q(s+1) - Q(s) + \Xi), \\
J_s &:= [2^{Q(s)} + Q(s+1) - Q(s) + \Xi, 2^{Q(s+1)}).
\end{aligned}$$

For integer values of s, the intervals I_s and J_s form a partition of $[1, \infty)$, which we use below to define a certain itinerary in $\{0, 1\}^{\mathbb{N}_0}$. For $s \in [0, \infty)$ that is not necessarily an integer, the interval J_s is used in the proof of Proposition B in §4.2.

Denote by $(x_j)_{j=0}^{\infty}$ the sequence in $\{0, 1\}^{\mathbb{N}_0}$ defined by the property that $x_j = 0$ if and only if there is an integer $s \ge 0$ such that $j + 1 \in I_s$. Note that the first $|I_0| = q + \Xi$ entries of $(x_j)_{j=0}^{\infty}$ are 0. Moreover, define $N : \mathbb{N}_0 \to \mathbb{N}_0$ by $N(0) := 0$ and

$$N(k) := \sharp\{j \in \{0, \ldots, k-1\} \mid x_j = 0\} \quad \text{for } k \ge 1,$$

and $B : \mathbb{N}_0 \to \mathbb{N}_0$ by $B(0) := 0$, $B(1) := 1$, and

$$B(k) := 1 + \sharp\{j \in \{0, \ldots, k-2\} \mid x_j \ne x_{j+1}\} \quad \text{for } k \ge 2.$$

Note that for $k \ge 1$ the number $B(k)$ is the number of blocks of 0's and 1's in the sequence $(x_j)_{j=0}^{k-1}$.

Observe that for each s in $\mathbb{N}_0$ the following numbers depend on s but not on the integer k in J_s:

$$N(k) = \sum_{j=0}^{s} |I_j| = \sum_{j=0}^{s}(Q(j+1) - Q(j) + \Xi) = Q(s+1) + \Xi \cdot (s+1), \tag{4.1}$$

$$B(k) = 2(s+1). \tag{4.2}$$

On the other hand, for each s in $\mathbb{N}_0$ and k in I_s, we have

$$N(k) = k - (2^{Q(s)} - 1) + Q(s) + \Xi s, \tag{4.3}$$

$$B(k) = 2s + 1. \tag{4.4}$$

Lemma 4.1. *The following properties hold for each real $s \geq 0$:*

(a) $2^{Q(s)} + Q(s+1) + \Xi \leq 2^{Q(s+1)-1}$.
(b) $|J_s| \geq 2^{Q(s+1)-1}$.

Proof. Part (a) with $s = 0$ is given by our hypothesis $2^{q-1} \geq q + 1 + \Xi$; and for $s > 0$, it follows from this and from the fact that the derivative of the function

$$s \mapsto 2^{Q(s+1)-1} - (2^{Q(s)} + Q(s+1) + \Xi)$$

is strictly positive on $[0, \infty)$. Part (b) follows easily from (a). □

4.2. Estimates

Let Ξ be a given integer and let q, N and B be as in the previous subsection. Given a real number ξ such that $1 \leq \Xi - 2\xi \leq 2$, define a two-variable series Π on $[0, \infty) \times [0, \infty)$ by

$$\Pi(\tau, \lambda) := \sum_{k=0}^{\infty} 2^{-\lambda k - \tau N(k) + \tau \xi B(k)}.$$

This subsection is devoted to proving the following proposition.

Proposition B. *For every $\tau \geq 1$ we have*

$$\Pi(\tau, 0) \leq 2(2^{\tau\xi} + 1).$$

Furthermore, for each τ in $\left(\frac{q-2}{q-1}, 1\right)$,

$$\Pi\left(\tau, 2 \cdot 2^{-q\left(\frac{\Xi - 2\xi}{q(1-\tau)} - 1\right)^2}\right) \leq 10 \cdot 2^{\tau\xi} + 101,$$

and for each $\Delta \geq 1$,

$$2^{\Delta - 4} \leq \Pi\left(\tau, 2^{-q\left(\frac{\Xi - 2\xi}{q(1-\tau)} + \Delta\right)^2}\right).$$

The proof of this proposition is at the end of this subsection.

For every real s in $[0, \infty)$, define

$$\lambda(s) := 1/|J_s|.$$

By Lemma 4.1(b) and the hypothesis $q \geq 3$, we have $0 < \lambda(s) \leq 1/4$.

Lemma 4.2. (i) *For $\tau \geq 1$ we have*

$$\Pi(\tau, 0) \leq 2(2^{\tau\xi} + 1).$$

(ii) *For every real s in $[0, \infty)$ and every τ in $(1/2, 1)$ satisfying $\tau > \frac{Q(s+1)-1}{Q(s+2)}$,*

$$\Pi(\tau, \lambda(s)) \leq 1 + 10 \cdot 2^{\tau\xi} + 5 \sum_{j=0}^{\lfloor s \rfloor + 1} 2^{(1-\tau)Q(j+1) - \tau(\Xi - 2\xi)(j+1)}.$$

(iii) *For every real s in $[0, \infty)$ and every $\tau > 0$,*

$$\tfrac{1}{8} \cdot 2^{(1-\tau)Q(\lfloor s \rfloor+1) - \tau(\Xi - 2\xi)(\lfloor s \rfloor + 1)} \leq \Pi(\tau, \lambda(s)).$$

Proof. For $\tau > 0$, $\lambda \geq 0$, and s in $\mathbb{N}_0$, define

$$I_s(\tau, \lambda) := \sum_{k \in I_s} 2^{-\lambda k - \tau N(k) + \tau\xi B(k)} \quad \text{and} \quad J_s(\tau, \lambda) := \sum_{k \in J_s} 2^{-\lambda k - \tau N(k) + \tau\xi B(k)},$$

so that $\Pi(\tau, \lambda) = 1 + \sum_{s=0}^{\infty} I_s(\tau, \lambda) + \sum_{s=0}^{\infty} J_s(\tau, \lambda)$.

(i) By (4.3), (4.4), and the hypothesis $\Xi - 2\xi \geq 1$, for all $\tau > 0$ and $\lambda \geq 0$ we have

$$\begin{aligned}
\sum_{s=0}^{\infty} I_s(\tau, \lambda) &\leq \sum_{s=0}^{\infty} \sum_{m=1}^{|I_s|} 2^{-\tau(Q(s) + \Xi s + m) + \tau\xi(2s+1)} \\
&= 2^{\tau\xi} \sum_{s=0}^{\infty} 2^{-\tau(Q(s) + (\Xi - 2\xi)s)} \sum_{m=1}^{|I_s|} 2^{-\tau m} \\
&\leq 2^{\tau\xi} \frac{2^{-\tau}}{1 - 2^{-\tau}} \sum_{s=0}^{\infty} 2^{-\tau(\Xi - 2\xi)s} \\
&\leq 2^{\tau\xi} \frac{2^{-\tau}}{(1 - 2^{-\tau})^2}.
\end{aligned} \tag{4.5}$$

On the other hand, using (4.1), (4.2), the hypothesis $\Xi - 2\xi \geq 1$, and the fact that for every $s \geq 0$ we have $|J_s| \leq 2^{Q(s+1)}$, we obtain, for every $\tau \geq 1$,

$$\begin{aligned}
\sum_{s=0}^{\infty} J_s(\tau, 0) &= \sum_{s=0}^{\infty} |J_s| 2^{-\tau(Q(s+1) + \Xi(s+1)) + 2\tau\xi \cdot (s+1)} \\
&\leq \sum_{s=0}^{\infty} 2^{-(\tau-1)Q(s+1) - \tau(\Xi - 2\xi)(s+1)} \\
&\leq \frac{2^{-\tau}}{1 - 2^{-\tau}}.
\end{aligned} \tag{4.6}$$

Combining inequalities (4.5) and (4.6) we get, for every $\tau \geq 1$,

$$\Pi(\tau, 0) \leq 1 + 2^{\tau\xi} \frac{2^{-\tau}}{(1 - 2^{-\tau})^2} + \frac{2^{-\tau}}{1 - 2^{-\tau}} \leq 2(2^{\tau\xi} + 1).$$

(ii) Fix s in $[0, \infty)$ and set $s_0 := \lfloor s \rfloor$. We use (4.5) to estimate $\Pi(\tau, \lambda(s))$. To estimate $\sum_{j=0}^{\infty} J_j(\tau, \lambda(s))$, note that by definition of $\lambda(s)$, for each integer ℓ satisfying $1 \leq \ell \leq |J_s|$ we have

$$1/2 \leq 2^{-\lambda(s)\ell} \leq 1.$$

On the other hand, the hypothesis $q \geq 3$ implies that the function $j \mapsto |J_j|$ is non-decreasing on $[0, \infty)$. Therefore, for each j in $\{0, \ldots, s_0\}$ we have $|J_j| \leq |J_s|$ and so

$$\frac{1}{2}|J_j| \leq \sum_{m=1}^{|J_j|} 2^{-\lambda(s)m} \leq |J_j|. \tag{4.7}$$

On the other hand,

$$\sum_{m=1}^{\infty} 2^{-\lambda(s)m} = \frac{1}{2^{\lambda(s)} - 1} \leq \frac{1}{\lambda(s) \log 2} \leq 2|J_s|. \tag{4.8}$$

Note also that, by (4.1), (4.2), and the hypothesis $q + \Xi \geq 1$, for every j in $\mathbb{N}_0$ we have, by (4.8) and $|J_s| \leq 2^{Q(s+1)}$,

$$\begin{aligned} J_j(\tau, \lambda(s)) &= 2^{-\tau(Q(j+1)+\Xi\cdot(j+1))+2\tau\xi\cdot(j+1)} \sum_{k \in J_j} 2^{-\lambda(s)k} \\ &\leq 2|J_s| 2^{-\tau(Q(j+1)+(\Xi-2\xi)(j+1))} \\ &\leq 2 \cdot 2^{Q(s+1)-\tau(Q(j+1)+(\Xi-2\xi)(j+1))}. \end{aligned} \tag{4.9}$$

Taking $j = s_0 + 1$ and using the inequality $Q(s+1) \leq Q(s_0+2)$, we obtain

$$J_{s_0+1}(\tau, \lambda(s)) \leq 2 \cdot 2^{(1-\tau)Q(s_0+2)-\tau(\Xi-2\xi)(s_0+2)}. \tag{4.10}$$

On the other hand, our hypothesis $\tau \geq \frac{Q(s+1)-1}{Q(s+2)}$ implies that for $j \geq s_0 + 2$,

$$Q(s+1) - \tau Q(j+1) \leq Q(s+1) - \tau Q(s+2) \leq 1.$$

So, using the hypothesis $\Xi - 2\xi \geq 1$ and summing (4.9) over $j \geq s_0 + 2$ yields

$$\sum_{j=s_0+2}^{\infty} J_j(\tau, \lambda(s)) \leq \sum_{j=s_0+2}^{\infty} 2^{2-\tau(\Xi-2\xi)(j+1)} \leq \frac{2^{2-3(\Xi-2\xi)\tau}}{1 - 2^{-\tau}}. \tag{4.11}$$

Now we complete the estimate of $\sum_{j=0}^{\infty} J_j(\tau, \lambda(s))$, by estimating the terms for which j is in $\{0, \ldots, s_0\}$. From (4.7), the first equality in (4.9), and $|J_j| \leq 2^{Q(j+1)}$, we deduce that for every j in $\{0, \ldots, s_0\}$ we have

$$J_j(\tau, \lambda(s)) \leq |J_j| \cdot 2^{-\tau(Q(j+1)+(\Xi-2\xi)(j+1))} \leq 2^{(1-\tau)Q(j+1)-\tau(\Xi-2\xi)(j+1)}.$$

Summing over j in $\{0, \ldots, s_0\}$ and using inequalities (4.10) and (4.11), we obtain

$$\sum_{j=0}^{\infty} J_j(\tau, \lambda(s))$$
$$\le \sum_{j=0}^{s_0} 2^{(1-\tau)Q(j+1)-\tau(\Xi-2\xi)(j+1)} + 2 \cdot 2^{(1-\tau)Q(s_0+2)-\tau(\Xi-2\xi)(s_0+2)} + \frac{2^{2-3(\Xi-2\xi)\tau}}{1-2^{-\tau}}$$
$$\le 2 \sum_{j=0}^{s_0+1} 2^{(1-\tau)Q(j+1)-\tau(\Xi-2\xi)(j+1)} + \frac{2^{2-3(\Xi-2\xi)\tau}}{1-2^{-\tau}}.$$

Together with (4.5) this implies

$$\Pi(\tau, \lambda(s)) \le 1 + \frac{2^{-\tau}}{(1-2^{-\tau})^2} 2^{\tau\xi} + 2 \sum_{j=0}^{s_0+1} 2^{(1-\tau)Q(j+1)-\tau(\Xi-2\xi)(j+1)} + \frac{2^{2-3(\Xi-2\xi)\tau}}{1-2^{-\tau}}. \tag{4.12}$$

Since τ is in $(1/2, 1)$, we have $2^{-\tau}/(1-2^{-\tau})^2 \le 10$. Using in addition the hypotheses $q \ge 3$ and $\Xi - 2\xi \ge 1$, we have

$$\frac{2^{2-3(\Xi-2\xi)\tau}}{1-2^{-\tau}} \le 3 \cdot 2^{3-(\Xi-2\xi+3)\tau}$$
$$\le 3 \cdot 2^{(1-\tau)Q(1)-(\Xi-2\xi)\tau}$$
$$\le 3 \sum_{j=0}^{s_0+1} 2^{(1-\tau)Q(j+1)-\tau(\Xi-2\xi)(j+1)}.$$

We obtain (ii) by combining these estimates with (4.12).

(iii) Fix s in $[0, \infty)$ and set $s_0 := \lfloor s \rfloor$. By Lemma 4.1(b) and the definition of $\lambda(s)$, for each s in $[0, \infty)$ we have

$$\lambda(s) = |J_s|^{-1} \le \frac{1}{2^{Q(s+1)-1}}.$$

From this inequality and from Lemma 4.1(a), for every integer j in $\{0, \ldots, s_0\}$ we have

$$\lambda(s)(2^{Q(j)} + Q(j+1) - Q(j) + \Xi - 1) \le \lambda(s)(2^{Q(j)} + Q(j+1) + \Xi)$$
$$\le \frac{2^{Q(s)} + Q(s+1) + \Xi}{2^{Q(s+1)-1}}$$
$$\le 1.$$

In view of Lemma 4.1(b), formulas (4.1) and (4.2), the first inequality of (4.7), the first equality in (4.9), and the hypothesis $q + \Xi \ge 1$, we deduce that for $j = s_0$,

$$\frac{1}{8} \cdot 2^{(1-\tau)Q(s_0+1)-\tau(\Xi-2\xi)(s_0+1)}$$
$$\le \frac{1}{4}|J_{s_0}| 2^{-\tau(Q(s_0+1)+\Xi\cdot(s_0+1))+2\tau\xi\cdot(s_0+1)}$$
$$\le \frac{1}{2}|J_{s_0}| 2^{-\lambda(s)(2^{Q(s_0)}+Q(s_0+1)-Q(s_0)+\Xi-1)-\tau(Q(s_0+1)+\Xi\cdot(s_0+1))+2\tau\xi\cdot(s_0+1)}$$
$$\le \left(\sum_{m=1}^{|J_{s_0}|} 2^{-\lambda(s)m}\right) 2^{-\lambda(s)(2^{Q(s_0)}+Q(s_0+1)-Q(s_0)+\Xi-1)-\tau(Q(s_0+1)+\Xi\cdot(s_0+1))+2\tau\xi\cdot(s_0+1)}$$
$$= J_{s_0}(\tau, \lambda(s)).$$

□

Define $s : (-\infty, 1) \to \mathbb{R}$ by

$$s(\tau) = \frac{\Xi - 2\xi}{q(1-\tau)}.$$

Lemma 4.3. *For every τ in $\left(\frac{q-2}{q-1}, 1\right)$, we have*

$$\Pi(\tau, \lambda(s(\tau) - 2)) \leq 10 \cdot 2^{\tau\xi} + 101,$$

and for every $\Omega \geq 0$,

$$2^{\Omega - 3} \leq \Pi(\tau, \lambda(s(\tau) + \Omega)).$$

Proof. Fix τ in $\left(\frac{q-2}{q-1}, 1\right)$ and $\Omega \geq 0$. Note that $\tau > \frac{q-2}{q-1}$ implies $\tau > \frac{q-4}{q}$, or equivalently

$$\tau > \frac{Q\left(\frac{2}{q(1-\tau)} - 1\right) - 1}{Q\left(\frac{2}{q(1-\tau)}\right)}.$$

On the other hand, the function $s \mapsto \frac{Q(s-1)-1}{Q(s)}$ is strictly increasing on $\left(\frac{q-1}{q}, \infty\right)$. Since the inequalities $1 \leq \Xi - 2\xi$ and $\tau > \frac{q-2}{q-1}$ imply $s(\tau) > \frac{q-1}{q}$, using $\Xi - 2\xi \leq 2$ we deduce

$$\frac{Q(s(\tau) - 1) - 1}{Q(s(\tau))} \leq \frac{Q\left(\frac{2}{q(1-\tau)} - 1\right) - 1}{Q\left(\frac{2}{q(1-\tau)}\right)} < \tau.$$

So the hypotheses of Lemma 4.2(ii) are satisfied with $s = s(\tau) - 2$. Let $F : \mathbb{R} \to \mathbb{R}$ be the quadratic function defined by

$$F(\ell) := (1-\tau)Q(\ell) - \tau(\Xi - 2\xi)\ell.$$

Note that $F(0) = 0$,

$$F\left(\frac{s(\tau)}{2}\right) = \frac{(\Xi - 2\xi)^2}{2q} - \frac{\Xi - 2\xi}{2}\left(\frac{s(\tau)}{2}\right) \quad \text{and} \quad F(s(\tau)) = \frac{(\Xi - 2\xi)^2}{q}.$$

Since F is convex, for each ℓ in $[0, s(\tau)]$ we have

$$F(\ell) = (1-\tau)Q(\ell) - \tau(\Xi - 2\xi)\ell \leq \frac{(\Xi - 2\xi)^2}{q} - \frac{\Xi - 2\xi}{2}\min\{\ell, s(\tau) - \ell\}.$$

Therefore, setting $s^+ = s(\tau) - 2$ and using $1 \leq \Xi - 2\xi \leq 2$ and $q \geq 3$, we have

$$\begin{aligned}\sum_{j=0}^{\lfloor s^+ \rfloor + 1} 2^{(1-\tau)Q(j+1) - \tau(\Xi - 2\xi)(j+1)} &\leq 2\sum_{\ell=0}^{\lfloor s^+ \rfloor + 2} 2^{\frac{(\Xi-2\xi)^2}{q} - \frac{\Xi - 2\xi}{2}\ell} \\ &\leq 2 \cdot 2^{4/q}\frac{1}{1 - 2^{-1/2}} \leq 20.\end{aligned}$$

The first inequality of the lemma is then obtained using Lemma 4.2(ii) with $s = s^+$.

To prove the second inequality, note that

$$F(s(\tau)+\Omega) = \frac{(\Xi-2\xi)^2}{q} + (\Xi-2\xi)(2-\tau)\Omega + q(1-\tau)\Omega^2 \geq (\Xi-2\xi)\Omega \geq \Omega$$

and F is increasing on the interval $\left[\frac{\tau}{2}\frac{\Xi-2\xi}{q(1-\tau)}, \infty\right)$ containing $s(\tau)$. So, if we set $s^- = s(\tau)+\Omega$, then

$$\Omega \leq F(s^-) \leq F(\lfloor s^- \rfloor + 1) = (1-\tau)Q(\lfloor s^- \rfloor + 1) - \tau(\Xi-2\xi)(\lfloor s^- \rfloor + 1).$$

Together with Lemma 4.2(iii) with $s = s^-$, we obtain

$$2^{\Omega} \leq 2^{(1-\tau)Q(\lfloor s^- \rfloor+1) - \tau(\Xi-2\xi)(\lfloor s^- \rfloor+1)} \leq 8\Pi(\tau, \lambda(s^-)),$$

which is the second inequality of the lemma. □

Proof of Proposition B. The first inequality is Lemma 4.2(i). To prove the others, note that by the definition of $\lambda(s)$ we have $\lambda(s) \geq 2^{-Q(s+1)}$. On the other hand, by Lemma 4.1(b) we have $\lambda(s) \leq 2 \cdot 2^{-Q(s+1)}$. So, using the definition of the function s we see that for each τ in $(0,1)$ and $\Delta \geq 1$,

$$\lambda(s(\tau)-2) \leq 2 \cdot 2^{-q\left(\frac{\Xi-2\xi}{q(1-\tau)}-1\right)^2} \quad \text{and} \quad \lambda(s(\tau)+\Delta-1) \geq 2^{-q\left(\frac{\Xi-2\xi}{q(1-\tau)}+\Delta\right)^2}.$$

Then the desired inequalities are a direct consequence of Lemma 4.3 with $\Omega = \Delta - 1$ and of the fact that for a fixed τ the function $\lambda \mapsto \Pi(\tau, \lambda)$ is nonincreasing on the set where it is finite. □

5. Estimating the geometric pressure function

In this section we prove the Main Technical Theorem. In §5.1 we show a general result about conformal measures, and in §5.2 we make some technical estimates (Proposition 5.2). The proof of the Main Technical Theorem is in §5.3, after recalling a few results from [CRL12].

5.1. Conformal measures

Recall that, given an integer $n \geq 3$ and a parameter c in $\mathcal{K}_n$, the *conical* or *radial Julia set of $f_c|_{I_c}$* (resp. f_c) is the set of all points x in I_c (resp. J_c) for which the following property holds: There exists $r > 0$ and an unbounded sequence $(n_j)_{j=1}^{\infty}$ of positive integers such that for every j the map $f_c|_{I_c}^{n_j}$ (resp. $f_c^{n_j}$) maps a neighborhood of x in I_c (resp. J_c) diffeomorphically onto $B(f_c^{n_j}(x), r)$.

Proposition 5.1. *Let $n \geq 4$ be an integer, c a parameter in $\mathcal{K}_n$, and let $t > 0$ and p in $\mathbb{R}$ be given. Then there is at most one (t, p)-conformal probability measure of $f_c|_{I_c}$ (resp. f_c) supported on I_c (resp. J_c). If such a measure μ exists, then $p \geq P_c^{\mathbb{R}}(t)$ (resp. $p \geq P_c^{\mathbb{C}}(t)$), and either μ is supported on the backward orbit of 0 and dissipative, or μ is non-atomic and supported on the conical Julia set of $f_c|_{I_c}$ (resp. f_c). Furthermore, the former case holds precisely when the following series converges:*

$$\sum_{j=1}^{\infty} \exp(-jp) \sum_{y \in f_c|_{I_c}^{-j}(0)} |Df_c^j(y)|^{-t} \quad \left(\text{resp. } \sum_{j=1}^{\infty} \exp(-jp) \sum_{y \in f_c^{-j}(0)} |Df_c^j(y)|^{-t} \right). \tag{5.1}$$

Proof. By [Urb03, Theorem 4.2] the conical Julia set of f_c is the complement in J_c of the backward orbit of $z = 0$. This implies that the conical Julia set of $f_c|_{I_c}$ contains the complement in I_c of the backward orbit of $z = 0$ under $f_c|_{I_c}$. On the other hand, this last set is clearly disjoint from the conical Julia set of $f_c|_{I_c}$, so this proves that the conical Julia set of $f_c|_{I_c}$ is the complement in I_c of the backward orbit of $z = 0$.

Let μ be a (t, p)-conformal probability measure for $f_c|_{I_c}$ (resp. f_c) supported on I_c (resp. J_c). If μ is supported on the backward orbit of $z = 0$, then it is uniquely determined by the mass it assigns to $z = 0$, and therefore it is unique up to a scalar factor. Note moreover that in this case μ is dissipative, because it charges the wandering set $\{0\}$. If μ is not entirely supported on the backward orbit of $z = 0$, then it charges the conical Julia set, so μ is non-atomic, it is supported on the conical Julia set and it is the unique (t, p)-conformal measure of $f_c|_{I_c}$ (resp. f_c) supported on I_c (resp. J_c), up to a scalar factor (see [PRL11, Proposition 4.1] for the complex case; the proof of the uniqueness part of this result applies without change to the real case). This completes the proof that μ is unique.

To prove that in the complex case $p \geq P_c^{\mathbb{C}}(t)$, let $\delta > 0$ be so small that $B(0, 2\delta)$ is disjoint from the forward orbit of the critical point. It follows that there is a constant $K > 1$ such that for every integer $j \geq 1$ and every y in $f_c^{-j}(0)$, the map f_c^j maps a neighborhood W_y of y biholomorphically to $B(0, \delta)$ with distortion bounded by K. Therefore,

$$\mu(W_y) \geq K^{-t} \exp(-jp) |Df_c^j(y)|^{-t} \mu(B(0, \delta)).$$

So, if we set $C := K^{-1} \mu(B(0, \delta)) > 0$, then for every integer $j \geq 1$ we have

$$1 \geq \sum_{y \in f_c^{-j}(0)} \mu(W_y) \geq C \exp(-jp) \sum_{y \in f_c^{-j}(0)} |Df_c^j(y)|^{-t}.$$

Since by [PRLS04, Theorem A] we have

$$\lim_{j \to \infty} \frac{1}{j} \log \sum_{y \in f_c^{-j}(0)} |Df_c^j(y)|^{-t} = P_c^{\mathbb{C}}(t),$$

this proves $p \geq P_c^{\mathbb{C}}(t)$.

To prove that in the real case we have $p \geq P_c^{\mathbb{R}}(t)$, we note that the proof of [PRLS04, Proposition 2.1] can be adapted to show

$$\lim_{j\to\infty} \frac{1}{j} \log \sum_{y\in(f_c|_{I_c})^{-j}(0)} |Df_c^j(y)|^{-t} = P_c^{\mathbb{R}}(t),$$

using the fact that $z = 0$ is not in the closure of the orbit of the critical value of f_c. The rest of the proof of $p \geq P_c^{\mathbb{R}}(t)$ is similar to the proof above.

To prove the last statement, observe first that if there is a (t, p)-conformal measure for $f_c|_{I_c}$ (resp. f_c) that is supported on the backward orbit of $z = 0$, then its total mass is equal to (5.1) times the mass at $z = 0$. This proves that (5.1) is finite. Conversely, if (5.1) is finite, then

$$\delta_0 + \sum_{j=1}^{\infty} \sum_{y\in f_c|_{I_c}^{-j}(0)} \exp(-jp)|Df_c^j(y)|^{-t}\delta_y$$

$$\left(\text{resp. } \delta_0 + \sum_{j=1}^{\infty} \sum_{y\in f_c^{-j}(0)} \exp(-jp)|Df_c^j(y)|^{-t}\delta_y\right)$$

is finite and it is a (t, p)-conformal measure for $f_c|_{I_c}$ (resp. f_c) supported on I_c (resp. J_c). □

5.2. *Phase transition parameter*

Recall that for each parameter c in $\mathcal{P}_5(-2)$, we have set

$$\theta(c) = \left|\frac{Dg_c(p(c))}{Dg_c(\widetilde{p}(c))}\right|^{1/2} \quad \text{and} \quad \xi(c) = -\frac{\log(\zeta(c)\widetilde{\zeta}(c))}{4\log\theta(c)}.$$

Write $t(c) := \log 2/\log\theta(c)$ and for every integer $n \geq 5$ define

$$\xi_n := \sup_{c\in\mathcal{K}_n} \xi(c).$$

This subsection is dedicated to proving the following estimates, used in the proof of the Main Technical Theorem.

Proposition 5.2. *There is an integer $n_2 \geq 5$ such that for every integer $n \geq n_2$ and every c in $\mathcal{K}_n$, we have $\lceil 2\xi_n + 1\rceil - 2\xi(c) \leq 2$. Furthermore, for every constant $T > 0$ there is $n_3 \geq 5$ such that for every integer $n \geq n_3$ and every parameter c in $\mathcal{K}_n$, we have $t(c) \geq T$.*

This proposition is a consequence of the following sequence of lemmas.

Lemma 5.3 ([CRL12, Lemma A.1]). *We have*

$$\frac{\partial}{\partial c}|Df_c^3(p(c))|\bigg|_{c=-2} > \frac{\partial}{\partial c}|Df_c^3(\widetilde{p}(c))|\bigg|_{c=-2}.$$

Lemma 5.4. *We have* $\theta(-2) = 1$ *and* $D\theta(-2) > 0$.

Proof. For $c = -2$,

$$\{2\cos(2\pi/7), 2\cos(4\pi/7), 2\cos(6\pi/7)\}$$

and

$$\{2\cos(2\pi/9), 2\cos(4\pi/9), 2\cos(8\pi/9)\}$$

are the only orbits of minimal period 3 of f_{-2}. Thus,

$$|Df_{-2}^3(p(-2))| = |Df_{-2}^3(\widetilde{p}(-2))| = 8 \quad \text{and} \quad \theta(-2) = 1.$$

Together with Lemma 5.3, we obtain $D\theta(-2) > 0$. □

Lemma 5.5. *We have* $\zeta(-2) \cdot \widetilde{\zeta}(-2) = 1$*, and the function* $\xi(c)$ *is real analytic at* $c = -2$.

Proof. Let c be a parameter in $\mathcal{P}_5(-2)$. For every integer $m \geq 1$ denote by $p_m(c)$ the periodic point in Λ_c whose itinerary consists of the periodic sequence whose period is the concatenation of m consecutive 0's and of m consecutive 1's. By Proposition 3.1,

$$\frac{Dg_c^{2m}(p_m(c))}{(Dg_c(p(c))Dg_c(\widetilde{p}(c)))^m} \to \zeta(c)\widetilde{\zeta}(c) \quad \text{as } m \to \infty.$$

On the other hand, using the identity $f_{-2}(2\cos(x)) = 2\cos(2x)$ for x in $\mathbb{R}$, we obtain

$$\frac{Dg_{-2}^{2m}(p_m(-2))}{(Dg_{-2}(p(-2))Dg_{-2}(\widetilde{p}(-2)))^m} = 1.$$

This proves $\zeta(-2) \cdot \widetilde{\zeta}(-2) = 1$.

To prove that ξ is real analytic at $c = -2$, notice that θ, ζ, and $\widetilde{\zeta}$ are all real analytic at $c = -2$ (Proposition 3.1). Since $\zeta(-2) \cdot \widetilde{\zeta}(-2) = 1$ and $\theta(-2) = 1$ (Lemma 5.4), the functions A and B defined for c in $\mathcal{P}_5(-2)$ by

$$A(c) := \log(\zeta(c)\widetilde{\zeta}(c))/(c+2) \quad \text{and} \quad B(c) := \log\theta(c)/(c+2)$$

are also real analytic at $c = -2$. Moreover, $B(-2) \neq 0$ since $D\theta(-2) \neq 0$ by Lemma 5.4. Thus, the quotient $4\xi(c) = A(c)/B(c)$ is real analytic at $c = -2$. □

Proof of Proposition 5.2. By Lemma 5.4, there is $\delta > 0$ such that for every c in $(-2, -2+\delta)$ we have

$$1 < \theta(c) < 2^{1/T}.$$

On the other hand, by Proposition 2.6, there is $n_0 \geq 3$ such that $\mathcal{K}_n \subset (-2, -2+\delta)$ for every $n \geq n_0$. These assertions imply the second part of the proposition.

To prove the first part, notice that by Lemma 5.5 there is $\epsilon > 0$ such that $\xi(c)$ is uniformly continuous on $[-2, -2+\epsilon]$. By Proposition 2.6, for every sufficiently large integer n we have $\mathcal{K}_n \subset [-2, -2+\epsilon]$ and moreover the diameter of $\mathcal{K}_n$ converges to 0 as $n \to \infty$. Thus, for every sufficiently large n we have

$$\xi_n - \xi(c) < 1/2.$$

This implies the first assertion of the proposition and concludes the proof. □

5.3. *Proof of the Main Technical Theorem*

We start by recalling some results from [CRL12].

Proposition 5.6 ([CRL12, Proposition D]). *There is an integer $n_5 \geq 4$ and a constant $C_3 > 1$ such that for every integer $n \geq n_5$ and every parameter c in $\mathcal{K}_n$, the following properties hold for each $t \geq 3$:*

(i) *For p in $[-t\chi_{\text{crit}}(c)/2, 0)$ satisfying*

$$\sum_{k=0}^{\infty} \exp(-(n+3k)p)|Df_c^{n+3k}(c)|^{-t/2} \geq C_3^t,$$

we have $\mathscr{P}_c^{\mathbb{R}}(t,p) > 0$ and $P_c^{\mathbb{R}}(t) \geq p$.

(ii) *For $p \geq -t\chi_{\text{crit}}(c)/2$ satisfying*

$$\sum_{k=0}^{\infty} \exp(-(n+3k)p)|Df_c^{n+3k}(c)|^{-t/2} \leq C_3^{-t},$$

we have $\mathscr{P}_c^{\mathbb{C}}(t,p) < 0$ and $P_c^{\mathbb{C}}(t) \leq p$.

Lemma 5.7 ([CRL12, Proposition 6.2]). *For all $n \geq 5$, c in $\mathcal{K}_n$, and $t > 0$, we have*

$$P_c^{\mathbb{C}}(t) \geq P_c^{\mathbb{R}}(t) \geq -t\chi_{\text{crit}}(c)/2.$$

Lemma 5.8 ([CRL12, Lemma 6.5]). *There is $n_6 \geq 5$ such that for all $n \geq n_6$, c in $\mathcal{K}_n$, $t \geq 3$ and*

$$p \geq P_c^{\mathbb{R}}(t) \quad (\textit{resp. } p \geq P_c^{\mathbb{C}}(t))$$

satisfying $\mathscr{P}_c^{\mathbb{R}}(t,p) < 0$ (resp. $\mathscr{P}_c^{\mathbb{C}}(t,p) < 0$), the sum (5.1) is finite.

Proof of the Main Technical Theorem. Let C_2 and υ_2 be the constants given by Proposition A, n_5 and C_3 the constants given by Proposition 5.6, and n_6 the constant given by Lemma 5.8. Since for $c = -2$ we have

$$|Dg_{-2}(\widetilde{p}(-2))|^{1/3} = 2 \quad \text{and} \quad |Df_{-2}(\beta(-2))| = 4,$$

there is $\delta > 0$ such that for each c in $(-2, -2+\delta)$ we have

$$\frac{|Dg_c(\widetilde{p}(c))|^{1/3}}{|f_c(\beta(c))|} < \frac{2}{3}. \tag{5.2}$$

By Proposition 2.6 there is $n_0 \geq 3$ such that for all $n \geq n_0$ the set $\mathcal{K}_n$ is contained in $(-2, -2+\delta)$; thus for every c in $\mathcal{K}_n$ we have (5.2). Since the closure of $\mathcal{P}_6(-2)$ is contained in $\mathcal{P}_5(-2)$ (Lemma 2.3(i)), by Proposition 3.1 we have

$$Z := \sup_{c \in \mathcal{P}_6(-2)} -\frac{\log(\zeta(c)\widetilde{\zeta}(c))}{4\log 2} < \infty.$$

Fix $n \geq \max\{6, n_0, n_5, n_6\}$ large enough that

$$C_2\left(\tfrac{2}{3}\right)^{n/2}(10 \cdot 2^Z + 101) < C_3^{-1}. \tag{5.3}$$

In view of Proposition 5.2, we can take n larger if necessary so that for every c in $\mathcal{K}_n$ we have

$$t(c) = \frac{\log 2}{\log \theta(c)} \geq 6,$$

and if we set

$$\Xi := \lceil 2\xi_n + 1 \rceil,$$

then $\Xi - 2\xi(c) \leq 2$ for every c in $\mathcal{K}_n$. Consider the sequence $(x_j)_{j=0}^{\infty}$ in $\{0, 1\}^{\mathbb{N}_0}$ defined in §4.1 for this value of Ξ and for some integer $q \geq 3$ satisfying in addition

$$q + \Xi \geq 1 \quad \text{and} \quad 2^{q-1} \geq q + 1 + \Xi.$$

By Proposition 2.6, there is a parameter c in $\mathcal{K}_n$ such that $\iota(c) = (x_j)_{j=0}^{\infty}$. Finally, set $t_* := t(c)$, and fix $\Delta \geq 1$ sufficiently large such that

$$C_2^{-t_*} \exp(-n) \left(\frac{\exp(\chi_{\text{crit}}(c))}{|Df_c(\beta(c))|} \right)^{t_* n/2} 2^{\Delta - 4} > C_3^{t_*}. \tag{5.4}$$

Write

$$t_0 := \frac{q-2}{q-1} t_*, \qquad \xi := \xi(c), \qquad \kappa := \Xi - 2\xi,$$

and define functions $\delta^+, \delta^-, p^+, p^- : (t_0, \infty) \to \mathbb{R}$ as in the statement of the Main Technical Theorem. Taking Δ larger if necessary, assume that $p^-(t) < 0$ for every t in (t_0, ∞).

We start by showing that c satisfies the hypotheses of Proposition A. By (4.1) and (4.3),

$$N_c(k)/k \to 0 \quad \text{as } k \to \infty. \tag{5.5}$$

Denote by $(m_j)_{j=0}^{\infty}$ the sequence of the lengths of blocks of 0's and 1's in $\iota(c)$. So, using the notation in §4.1, for every integer $s \geq 0$ we have $m_{2s} = |I_s|$ and $m_{2s+1} = |J_s|$. By Lemma 4.1(a), for every integer $s \geq 0$ we have

$$\min\{m_{2s}, m_{2s+1}\} = \min\{|I_s|, |J_s|\} = |I_s| = q(2s+1) + \Xi,$$

and for every integer $s \geq 1$,

$$\min\{m_{2s+1}, m_{2s+2}\} = \min\{|J_s|, |I_{s+1}|\} = |I_{s+1}| = q(2s+3) + \Xi.$$

Thus

$$\sum_{j=2}^{\infty} \exp(-\min\{m_j, m_{j+1}\}\upsilon_2) \leq 2 \sum_{s=1}^{\infty} \exp(-(q(2s+1) + \Xi)\upsilon_2) < \infty.$$

This proves that c satisfies the hypotheses of Proposition A.

Note that by our choice of n and the hypothesis $q \geq 3$, we have

$$t_0 = \frac{q-2}{q-1} t_* \geq \frac{1}{2} t_* \geq 3.$$

On the other hand, by (5.5) and Lemma 3.5 we have

$$\exp(\chi_{\mathrm{crit}}(c)) = |Dg_c(\widetilde{p}(c))|^{1/3}. \tag{5.6}$$

In particular, $\chi_{\mathrm{crit}}(c) > 0$. Consider the two-variable series Π defined as in §4.2 for the above choices of Ξ, q, and ξ, and note that it coincides with the two-variable series Π_c defined in §3.3 for our choice of c.

To prove that $P_c^{\mathbb{R}}(t) \geq p^-(t)$ for every $t > t_0$, note first that when $t \geq t_*$, this is given by Lemma 5.7. On the other hand, from Propositions A and B, (5.2), (5.4), (5.6), and from the fact that $\delta_-(t) \leq 1$ for every t in (t_0, t_*), we deduce

$$\begin{aligned}
&\sum_{k=0}^{\infty} \exp(-(n+3k)p^-(t))|Df_c^{n+3k}(c)|^{-t/2} \\
&\quad = \sum_{k=0}^{\infty} \exp\big(-(n+3k)(-t\chi_{\mathrm{crit}}(c)/2 + \delta^-(t))\big)|Df_c^{n+3k}(c)|^{-t/2} \\
&\quad \geq C_2^{-t} \exp(-n\delta^-(t)) \left(\frac{\exp(\chi_{\mathrm{crit}}(c))}{|Df_c(\beta(c))|}\right)^{tn/2} \Pi\left(t\frac{\log\theta(c)}{\log 2}, \frac{3\delta^-(t)}{\log 2}\right) \\
&\quad \geq C_2^{-t} \exp(-n\delta^-(t)) \left(\frac{\exp(\chi_{\mathrm{crit}}(c))}{|Df_c(\beta(c))|}\right)^{tn/2} 2^{\Delta-4} \\
&\quad > C_3^{t_*} \\
&\quad \geq C_3^t.
\end{aligned}$$

Since $p^-(t) < 0$ for each t in (t_0, t_*), the inequality above combined with Proposition 5.6(i) implies that $P_c^{\mathbb{R}}(t) \geq p^-(t)$ for every t in (t_0, t_*) .

Now we turn to the proof that $P_c^{\mathbb{C}}(t) \leq p^+(t)$ for $t > t_0$ and $\mathscr{P}_c^{\mathbb{C}}(t, -t\chi_{\mathrm{crit}}(c)/2) < 0$ for $t \geq t_*$. Combining Propositions A and B, using the definition of $\xi = \xi(c)$ and Z, and using (5.2), (5.3) and (5.6), we deduce that for every $t \geq t_*$,

$$\begin{aligned}
&\sum_{k=0}^{\infty} \exp(-(n+3k)p^+(t))|Df_c^{n+3k}(c)|^{-t/2} \\
&\quad = \sum_{k=0}^{\infty} \exp\big(-(n+3k)(-t\chi_{\mathrm{crit}}(c)/2)\big)|Df_c^{n+3k}(c)|^{-t/2} \\
&\quad \leq C_2^t \left(\frac{\exp(\chi_{\mathrm{crit}}(c))}{|Df_c(\beta(c))|}\right)^{tn/2} \Pi\left(t\frac{\log\theta(c)}{\log 2}, 0\right) \\
&\quad \leq C_2^t \left(\frac{\exp(\chi_{\mathrm{crit}}(c))}{|Df_c(\beta(c))|}\right)^{tn/2} 2(2^{t\frac{\log\theta(c)}{\log 2}\xi} + 1) \\
&\quad \leq C_2^t \left(\frac{\exp(\chi_{\mathrm{crit}}(c))}{|Df_c(\beta(c))|}\right)^{tn/2} 2(2^{tZ} + 1) \\
&\quad < C_3^{-t},
\end{aligned}$$

and for every t in (t_0, t_*),

$$\begin{aligned}
\sum_{k=0}^{\infty} & \exp(-(n+3k)p^+(t))|Df_c^{n+3k}(c)|^{-t/2} \\
&= \sum_{k=0}^{\infty} \exp\big(-(n+3k)(-t\chi_{\mathrm{crit}}(c)/2 + \delta^+(t))\big)|Df_c^{n+3k}(c)|^{-t/2} \\
&\le C_2^t \exp(-n\delta^+(t)) \left(\frac{\exp(\chi_{\mathrm{crit}}(c))}{|Df_c(\beta(c))|}\right)^{tn/2} \Pi\left(t\frac{\log\theta(c)}{\log 2}, \frac{3\delta^+(t)}{\log 2}\right) \\
&\le C_2^t \exp(-n\delta^+(t)) \left(\frac{\exp(\chi_{\mathrm{crit}}(c))}{|Df_c(\beta(c))|}\right)^{tn/2} (10 \cdot 2^{t\frac{\log\theta(c)}{\log 2}\xi} + 101) \\
&\le C_2^t \exp(-n\delta^+(t)) \left(\frac{\exp(\chi_{\mathrm{crit}}(c))}{|Df_c(\beta(c))|}\right)^{tn/2} (10 \cdot 2^{tZ} + 101) \\
&< C_3^{-t}.
\end{aligned}$$

Since $p^+(t) \ge -t\chi_{\mathrm{crit}}(c)/2$ for $t > t_0$, applying Proposition 5.6(ii) we deduce that for $t > t_0$ we have $P_c^{\mathbb{C}}(t) \le p^+(t)$ and $\mathscr{P}_c^{\mathbb{C}}(t, -t\chi_{\mathrm{crit}}(c)/2) < 0$.

To prove the assertions concerning conformal measures, recall that we have proved that for $t \ge t_*$ we have

$$P_c^{\mathbb{R}}(t) = P_c^{\mathbb{C}}(t) = -t\chi_{\mathrm{crit}}(c)/2$$

and $\mathscr{P}_c^{\mathbb{C}}(t, -t\chi_{\mathrm{crit}}(c)/2) < 0$. This implies that for $p \ge -t\chi_{\mathrm{crit}}(c)/2$,

$$\mathscr{P}_c^{\mathbb{R}}(t, p) \le \mathscr{P}_c^{\mathbb{C}}(t, p) \le \mathscr{P}_c^{\mathbb{C}}(t, -t\chi_{\mathrm{crit}}(c)/2) < 0.$$

So the assertions about conformal measures follow from Proposition 5.1 and Lemma 5.8.

To prove the assertions about equilibrium states, let $t \ge t_*$ and suppose for contradiction that there is an equilibrium state ρ of $f_c|_{I_c}$ (resp. f_c) for the potential $-t \log |Df_c|$. Since f_c satisfies the Collet–Eckmann condition, the Lyapunov exponent of ρ is strictly positive (see [NS98, Theorem A] or [RL12, Main Theorem] for the real case and [PRLS03, Main Theorem] for the complex case). Then [Dob15, Theorem 6] in the real case and [Dob12, Theorem 8] in the complex case imply that ρ is absolutely continuous with respect to the $(t, -t\chi_{\mathrm{crit}}(c)/2)$-conformal measure for $f_c|_{I_c}$ (resp. f_c) that is supported on I_c (resp. J_c). This implies in particular that ρ is supported on the backward orbit of $z = 0$, and hence that ρ charges $z = 0$. This is impossible because this point is not periodic. This contradiction shows that there is no equilibrium state of $f_c|_{I_c}$ (resp. f_c) for the potential $-t \log |Df_c|$, and completes the proof of the Main Technical Theorem. □

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