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Symmetry of minimizers with a level surface parallel to the boundary

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Abstract. We consider the functional

$$\mathcal{I}_\Omega(v) = \int_\Omega [f(|Dv|) - v]\,dx,$$

where Ω is a bounded domain and f is a convex function. Under general assumptions on f, Crasta [Cr1] has shown that if $\mathcal{I}_\Omega$ admits a minimizer in $W_0^{1,1}(\Omega)$ depending only on the distance from the boundary of Ω, then Ω must be a ball. With some restrictions on f, we prove that spherical symmetry can be obtained only by assuming that the minimizer has *one* level surface parallel to the boundary (i.e. it has only a level surface in common with the distance).

We then discuss how these results extend to more general settings, in particular to functionals that are not differentiable and to solutions of fully nonlinear elliptic and parabolic equations.

Keywords. Overdetermined problems, minimizers of integral functionals

1. Introduction

We consider a bounded domain Ω in $\mathbb{R}^N$ ($N \geq 2$) and, for $x \in \overline{\Omega}$, denote by $d(x)$ the distance of x from $\mathbb{R}^N \setminus \Omega$, that is,

$$d(x) = \min_{y \in \mathbb{R}^N \setminus \Omega} |x - y|, \qquad x \in \overline{\Omega};$$

d is Lipschitz continuous on $\overline{\Omega}$. For a positive number δ, we define the *parallel surface* to the boundary $\partial\Omega$ of Ω as

$$\Gamma_\delta = \{x \in \Omega : d(x) = \delta\}.$$

In this paper, we shall be concerned with minimizers of variational problems and solutions of quite general nonlinear elliptic and parabolic partial differential equations, which admit a single level surface that is parallel to $\partial\Omega$.

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A motivation is the work of G. Crasta [Cr1] on the minimizers of certain problems of the calculus of variations in the class of the so-called *web functions*, that is, functions that depend only on the distance from the boundary (see [CG] and [Ga], where the term web function was introduced for the first time). In [Cr1], it is proved that, if Ω is a smooth domain and the functional

$$\mathcal{I}_\Omega(v) = \int_\Omega [f(|Dv|) - v]\, dx \tag{1.1}$$

has a minimizer in the class of $W_0^{1,1}(\Omega)$-regular web functions, then Ω must be a ball. The assumptions on the lagrangian f are very general: f is merely required to be convex and the function $p \mapsto f(|p|)$ to be differentiable.

Related to Crasta's result, here, we consider the variational problem

$$\inf\{\mathcal{I}_\Omega(v) : v \in W_0^{1,\infty}(\Omega)\}, \tag{1.2}$$

under the following assumptions on $f : [0, \infty) \to \mathbb{R}$:

(f1) $f \in C^1([0,\infty))$ is a convex, nondecreasing function such that $f(0) = 0$ and

$$\lim_{s\to\infty} f(s)/s = \infty;$$

(f2) there exists $\sigma \geq 0$ such that $f'(s) = 0$ for every $0 \leq s \leq \sigma$, $f'(s) > 0$ for $s > \sigma$ and $f \in C^{2,\alpha}(\sigma, \infty)$ $(0 < \alpha < 1)$, with $f''(s) > 0$ for $s > \sigma$.

Also, we suppose that there exists a domain G such that

$$\overline{G} \subset \Omega,\ \partial G \in C^1 \text{ satisfies the interior sphere condition, and } \partial G = \Gamma_\delta, \tag{1.3}$$

for some $\delta > 0$.

The main result in this paper is the following.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and let f and G satisfy assumptions* (f1)–(f2) *and* (1.3)*, respectively. Let u be the solution of* (1.2) *and suppose u is C^1-smooth in a tubular neighborhood of Γ_δ. If*

$$u = c \quad \text{on } \Gamma_\delta \tag{1.4}$$

for some constant $c > 0$, then Ω must be a ball.

Thus, at the cost of requiring more restrictive growth and regularity assumptions on f, we can sensibly improve Crasta's theorem: indeed, if u is a web function, then *all* its level surfaces are parallel to $\partial\Omega$. We also point out that we make no (explicit) assumption on the regularity of $\partial\Omega$: we only require that the parallel surface Γ_δ has some special topology and is mildly smooth.

In Theorem 3.6, we will extend this result to a case in which the function $p \mapsto f(|p|)$ is no longer differentiable at $p = 0$. Our interest in this kind of functionals (not considered in [Cr1]) is motivated by their relevance in the study of complex-valued solutions of the *eikonal* equation (see [MT1]–[MT4] and [CeM]).

The reason why we need stricter assumptions on f is a different method of proof. While in [Cr1] one obtains symmetry by directly working with Euler's equation for $\mathcal{I}_\Omega$ (which only involves f'), in the proof of Theorem 1.1 we rely on the fact that minimizers of $\mathcal{I}_\Omega$ satisfy in a generalized sense a nonlinear equation of type

$$F(u, Du, D^2u) = 0 \quad \text{in } \Omega, \tag{1.5}$$

(which involves f''); moreover, to obtain symmetry, we use the *method of moving planes* that requires extra regularity for f''.

Theorem 1.1 (and also Theorems 3.6, 4.1 and 4.2) works out an idea used by the last two authors in the study of the so-called stationary surfaces of solutions of (non-degenerate) fast-diffusion parabolic equations (see [MS1]). A *stationary surface* is a surface $\Gamma \subset \Omega$ of codimension 1 such that, for some function $a : (0, T) \to \mathbb{R}$, $u(x, t) = a(t)$ for every $(x, t) \in \Gamma \times (0, T)$. In fact, in [MS1], it is proved that if the initial-boundary value problem

$$u_t - \Delta\phi(u) = 0 \quad \text{in } \Omega \times (0, T),$$
$$u = 0 \quad \text{on } \Omega \times \{0\}, \qquad u = 1 \quad \text{on } \partial\Omega \times (0, T)$$

(here ϕ is a nonlinearity with derivative ϕ' bounded from below and above by positive constants) admits a solution that has a stationary surface, then Ω must be a ball.

The crucial arguments used in [MS1] are two: one is the discovery that a stationary surface must be parallel to the boundary; the other is the application of the method of moving planes. This method was created by A. D. Aleksandrov to prove the spherical symmetry of embedded surfaces with constant mean curvature or, more generally, of surfaces whose principal curvatures satisfy certain constraints and, ever since, it has been successfully employed to prove spherical symmetry in many a situation: the theorems of Serrin for overdetermined boundary value problems [Se2] and those of Gidas, Ni and Nirenberg for ground states [GNN] are the most celebrated. Here, we will use that method to prove Theorem 1.1 (and also Theorems 3.6, 4.1 and 4.2). Arguments similar to those used in [MS1] were recently used in [Sh].

Let us now comment on the connections between the problem considered in Theorem 1.1, the one studied in [Cr1] (both with $f(p) = \frac{1}{2}|p|^2$) and (the simplest instance of) Serrin's overdetermined problem:

$$-\Delta u = 1 \quad \text{in } \Omega, \tag{1.6}$$

$$u = 0 \quad \text{on } \partial\Omega, \qquad \frac{\partial u}{\partial \nu} = \text{constant} \quad \text{on } \partial\Omega. \tag{1.7}$$

It is clear that being a web function is a stronger condition, since it implies both (1.4) and (1.7). Moreover, even if the constraints (1.4) and (1.7) are not implied by each other, we observe the following: (i) if (1.4) is satisfied for two positive sequences $\{\delta_n\}_{n\in\mathbb{N}}$ and $\{c_n\}_{n\in\mathbb{N}}$ with $\delta_n \to 0$ as $n \to \infty$, then (1.7) holds true; (ii) conversely, from (1.7) we can conclude that the oscillation $\max_{\Gamma_\delta} u - \min_{\Gamma_\delta} u$ is $O(\delta^2)$ as $\delta \to 0$. All in all, it seems that the constraint (1.4) is weaker than (1.7).

Another important remark is in order: the method of moving planes is applied to prove our symmetry results in a much simplified manner than that used for (1.6)–(1.7); indeed, since the overdetermination takes place in Ω (and not on $\partial\Omega$) we need not use Serrin's corner lemma (in other words, property (B) in [Se2] for (1.5) is not required). A further benefit is that no regularity requirement is made on $\partial\Omega$, thanks to assumption (1.3).

In Section 2 we will present our results on the problem proposed by Crasta (for the proof of Theorem 1.1, see Subsection 2.2); in Section 3 we will extend them to some cases which involve nondifferentiable lagrangians. In Section 4 we will discuss how these results extend to fairly general settings, in particular to solutions of fully nonlinear elliptic and parabolic equations.

We mention that a stability version of Theorem 1.1 (for the semilinear equation $\Delta u = f(u)$) is obtained in the companion paper [CMS].

2. Minima of convex differentiable functionals

We first introduce some notation and prove some preliminary result.

2.1. Uniqueness and comparison results

Let f satisfy (f1)–(f2). The functional $\mathcal{I}_\Omega$ is differentiable and a critical point u of $\mathcal{I}_\Omega$ satisfies

$$\begin{cases} -\operatorname{div}\left(\dfrac{f'(|Du|)}{|Du|}Du\right) = 1 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

in the weak sense, i.e.

$$\int_\Omega \frac{f'(|Du|)}{|Du|} Du \cdot D\phi \, dx = \int_\Omega \phi \, dx \quad \text{for every } \phi \in C_0^1(\Omega). \tag{2.2}$$

It will be useful to have at hand the solution of (2.1) when Ω is the ball of given radius R (centered at the origin): it is given by

$$u_R(x) = \int_{|x|}^R g'\left(\frac{s}{N}\right) ds, \tag{2.3}$$

where

$$g(t) = \sup\{st - f(s) : s \geq 0\}$$

is the Fenchel conjugate of f. For future use, we notice that $|Du_R(x)| > \sigma$ for $x \neq 0$.

It is clear that, when $\sigma = 0$, (1.2) has a unique solution, since f is strictly convex. When $\sigma > 0$, proving the uniqueness for (1.2) needs some more work. In Theorem 2.3 we shall prove such a result as a consequence of Lemmas 2.1 and 2.2 below.

Lemma 2.1. *Let Ω be a bounded domain and let u be a solution of* (1.2)*, where f satisfies* (f1) *and* (f2)*, with $\sigma > 0$. Then $u \geq 0$ and $\mathcal{I}_\Omega(u) < 0$.*

Proof. Since $u \in W_0^{1,\infty}(\Omega)$, also $|u| \in W_0^{1,\infty}(\Omega)$. If $u < 0$ on some open subset of Ω (u is continuous), then $\mathcal{I}_\Omega(|u|) < \mathcal{I}_\Omega(u)$—a contradiction.

Now observe that $\mathcal{I}_\Omega(v) < 0$ if $v \in W_0^{1,\infty}(\Omega)$ is any nonnegative function, $v \not\equiv 0$, with Lipschitz constant less than or equal to σ. Thus, $\mathcal{I}_\Omega(u) < 0$. □

In the following, for a given domain A, we shall denote by $\mathcal{I}_A$ the integral functional

$$\mathcal{I}_A(v) = \int_A [f(|Dv|) - v]\,dx;$$

a *local minimizer* of $\mathcal{I}_A$ means a function that minimizes $\mathcal{I}_A$ among all the functions with the same boundary values.

Lemma 2.2. *Let f satisfy* (f1) *and* (f2) *with $\sigma > 0$. Let A be a bounded domain and assume that $u_0, u_1 \in W^{1,\infty}(A)$ are local minimizers of $\mathcal{I}_A$ with $u_0 = u_1$ on ∂A. Next, define*

$$E_j = \{x \in A : |Du_j| > \sigma\}, \qquad j = 0, 1, \tag{2.4}$$

and assume that $|E_0 \cup E_1| > 0$. Then $u_0 \equiv u_1$.

Proof. Let $u = \frac{1}{2}(u_0 + u_1)$; since f is convex, it is clear that u is also a minimizer of $\mathcal{I}_A$ and $u = u_0 = u_1$ on ∂A. Thus, we have

$$\int_A \left[\tfrac{1}{2} f(|Du_0|) + \tfrac{1}{2} f(|Du_1|) - f(|Du|)\right] dx = 0, \tag{2.5}$$

and since f is convex,

$$\tfrac{1}{2} f(|Du_0|) + \tfrac{1}{2} f(|Du_1|) - f(|Du|) = 0 \quad \text{ a.e. in } A. \tag{2.6}$$

Assumption (f2) on f and (2.6) imply that

$$|(E_0 \cup E_1) \cap \{|Du_0| \neq |Du_1|\}| = 0, \tag{2.7}$$

since on $(E_0 \cup E_1) \cap \{|Du_0| \neq |Du_1|\}$ the convexity of f holds in the strict sense.

Thus, we have proven that $|Du_0| = |Du_1|$ a.e. in $E_0 \cup E_1$, and since

$$\int_A f(|Du_j|)\,dx = \int_{E_j} f(|Du_j|)\,dx, \qquad j = 0, 1,$$

we have

$$\int_A f(|Du_0|)\,dx = \int_A f(|Du_1|)\,dx.$$

Now, take $v = \max(u_0, u_1)$; then (2.7) implies that

$$\int_A f(|Dv|)\,dx = \int_A f(|Du_0|)\,dx = \int_A f(|Du_1|)\,dx, \tag{2.8}$$

and hence $\mathcal{I}_A(v) \le \mathcal{I}_A(u_0) = \mathcal{I}_A(u_1)$, since $v \ge u_0, u_1$. Thus, $\mathcal{I}_A(v) = \mathcal{I}_A(u_0) = \mathcal{I}_A(u_1)$; consequently,

$$\int_A (v - u_j)\,dx = 0$$

for $j = 0, 1$, and since $v \ge u_0, u_1$, we conclude that $v = u_0 = u_1$. □

Theorem 2.3. *Let f satisfy* (f1) *and* (f2) *with $\sigma > 0$ and assume that $u \in W_0^{1,\infty}(\Omega)$ is a solution of* (1.2). *Then:*

(i) $|\{x \in \Omega : |Du| > \sigma\}| > 0$;
(ii) *u is unique.*

Proof. (i) For contradiction, assume that $|Du| \leq \sigma$ a.e.; since u satisfies (2.2), we can easily infer that $\int_\Omega u\,dx = 0$. Thus, $I(u) = 0$, which contradicts Lemma 2.1.

(ii) The assertion follows from (i) and Lemma 2.2. □

As already mentioned in the introduction, our proof of Theorem 1.1 makes use of the method of moving planes. To apply this method, we need comparison principles for minimizers of $\mathcal{I}_\Omega$ (see also [Ci]).

Proposition 2.4 (Weak comparison principle). *Let A be a bounded domain and let f satisfy* (f1) *and* (f2). *Assume that $u_0, u_1 \in W^{1,\infty}(A)$ are local minimizers of $\mathcal{I}_A$ such that $u_0 \leq u_1$ on ∂A and suppose that $|E_0 \cup E_1| > 0$, where the sets E_0, E_1 are given by* (2.4). *Then $u_0 \leq u_1$ in $\overline{A}$.*

Proof. If $\sigma = 0$ in (f2), then the weak comparison principle is well established (see for instance [FGK, Lemma 3.7]).

Thus, in the rest of the proof, we assume that $\sigma > 0$. Assume for contradiction that $u_0 > u_1$ in a nonempty open subset B of A. We can suppose that B is connected (otherwise the argument can be repeated for each connected component of B). Observe that since $u_0 \leq u_1$ on ∂A and u_0 and u_1 are continuous, we have $u_0 = u_1$ on ∂B.

We now show that u_0 minimizes $\mathcal{I}_B$ among those functions v with $v - u_0 \in W_0^{1,\infty}(B)$. Indeed, if $\inf \mathcal{I}_B(v) < \mathcal{I}_B(u_0)$ for one such function, then the function w defined by

$$w(x) = \begin{cases} v, & x \in B, \\ u_0, & x \in \overline{A} \setminus B, \end{cases}$$

would belong to $W^{1,\infty}(A)$, be equal to u_0 on ∂A and have $\mathcal{I}_A(w) < \mathcal{I}_A(u_0)$—a contradiction. The same argument can be repeated for u_1, and hence we have proven that $\mathcal{I}_B(u_0) = \mathcal{I}_B(u_1)$ (since $u_0 = u_1$ on ∂A).

This last equality implies that

$$\int_B f(|Du_0|)\,dx > \int_B f(|Du_1|)\,dx \geq 0,$$

since $u_0 > u_1$ in B, and hence $|E_0 \cap B| > 0$.

By applying Lemma 2.2 to $\mathcal{I}_B$, we find that $u_0 \equiv u_1$ in B—a contradiction. □

Proposition 2.5 (Strong comparison principle). *Let A be a bounded domain and let f satisfy* (f1) *and* (f2). *Assume that $u_0, u_1 \in C^1(\overline{A})$ are local minimizers of $\mathcal{I}_A$ such that $u_0 \leq u_1$ in A and $|Du_0|, |Du_1| > \sigma$ in $\overline{A}$. Then either $u_0 \equiv u_1$, or $u_0 < u_1$ in A.*

Proof. As u_0 and u_1 are solutions of (2.2) with $|Du_0|, |Du_1| > \sigma$, the assertion easily follows from [Se1, Theorem 1], since by (f2) the needed uniform ellipticity is easily verified. □

Proposition 2.6 (Hopf comparison principle). *Let $u_0, u_1 \in C^2(\overline{A})$ satisfy the assumptions of Proposition 2.5. Assume that $u_0 = u_1$ at some point $P \in \partial A$ admitting an internally touching tangent sphere. Then either $u_0 \equiv u_1$ in A, or*

$$u_0 < u_1 \quad \text{in } A \quad \text{and} \quad \frac{\partial u_0}{\partial \nu} < \frac{\partial u_1}{\partial \nu} \quad \text{at } P;$$

here, ν denotes the inward unit normal to ∂A at P.

Proof. The assertion follows from [Se1, Theorem 2] by using an argument analogous to the one in the proof of Proposition 2.5. □

We conclude this subsection by giving a lower bound for $|Du|$ on Γ_δ when u is the function considered in Theorem 1.1.

Lemma 2.7. *Let Ω, G and f satisfy the assumptions of Theorem 1.1. Let $u \in W_0^{1,\infty}(\Omega)$ be a minimizer of (1.2) satisfying (1.4). For $x_0 \in \Gamma_\delta$, let $y_0 \in \partial\Omega$ be such that $\operatorname{dist}(y_0, \Gamma_\delta) = \delta$ and set $\nu = (x_0 - y_0)/\delta$; denote by $\rho = \rho(x_0)$ the radius of the optimal interior ball at x_0. Then*

$$\liminf_{t \to 0^+} \frac{u(x_0 + t\nu) - u(x_0)}{t} \geq g'\left(\frac{\rho}{N}\right), \tag{2.9}$$

where g is the Fenchel conjugate of f.

In particular, $\inf_{\Gamma_\delta} |Du| > \sigma$ in two cases:

(i) *if $u \in C^1(\Gamma_\delta)$;*
(ii) *if u is differentiable at every $x \in \Gamma_\delta$ and G satisfies the uniform interior sphere condition.*

Proof. Since $u - c$ minimizes the functional $\mathcal{I}_G$ among the functions vanishing on Γ_δ, Lemma 2.1 implies that $u \geq c$ in $\overline{G}$.

Let $B_\rho \subset G$ be the ball of radius ρ tangent to Γ_δ at x_0. The minimizer w of $\mathcal{I}_{B_\rho}$ with $w = c$ on ∂B_ρ is then $w = c + u_\rho$, where u_ρ given by (2.3) with $R = \rho$; notice that $u(x_0) = w(x_0) = c$.

Since $u \geq c \equiv w$ on ∂B_ρ, Proposition 2.4 yields $u \geq w$ in B_ρ, and thus

$$\liminf_{t \to 0^+} \frac{u(x_0 + t\nu) - c}{t} \geq \frac{\partial w}{\partial \nu}(x_0) = g'\left(\frac{\rho}{N}\right).$$

The last part of the lemma (assertions (i) and (ii)) is a straightforward consequence of (1.3) and (2.9). □

2.2. *The proof of Theorem 1.1*

We initially proceed as in [Se2] (see also [Fr]) and further introduce the necessary modifications as done in [MS1]–[MS2]. For $\xi \in \mathbb{R}^N$ with $|\xi| = 1$ and $\lambda \in \mathbb{R}$, we denote by $\mathcal{R}_\lambda x$ the reflection $x + 2(\lambda - x \cdot \xi)\,\xi$ of any point $x \in \mathbb{R}^N$ in the hyperplane

$$\pi_\lambda = \{x \in \mathbb{R}^N : x \cdot \xi = \lambda\},$$

and set

$$u^\lambda(x) = u(\mathcal{R}_\lambda x) \quad \text{for } x \in \mathcal{R}_\lambda(\Omega).$$

Then, for a fixed direction ξ, we define the *caps*

$$G_\lambda = \{x \in G : x \cdot \xi > \lambda\} \quad \text{and} \quad \Omega_\lambda = \{x \in \Omega : x \cdot \xi > \lambda\},$$

and set

$$\begin{aligned} \bar{\lambda} &= \inf\{\lambda \in \mathbb{R} : G_\lambda = \emptyset\}, \\ \lambda^* &= \inf\{\lambda \in \mathbb{R} : \mathcal{R}_\mu(G_\mu) \subset G \text{ for every } \mu \in (\lambda, \bar{\lambda})\}. \end{aligned}$$

As is well-known from Serrin [Se2], if we assume that Ω (and hence G) is not ξ-symmetric, then for $\lambda = \lambda^*$ at least one of the following two cases occurs:

(i) G_λ is internally tangent to ∂G at some point $P \in \partial G$ not in π_λ, or
(ii) π_λ is orthogonal to ∂G at some point Q.

Now, the crucial remark is given by the following lemma, whose proof is an easy adaptation of those of [MS2, Lemmas 2.1 and 2.2].

Lemma 2.8. *Let G satisfy assumption* (1.3). *Then:*

(i) $\Omega = G + B_\delta(0) = \{x + y : x \in G,\ y \in B_\delta(0)\}$;
(ii) *if $\mathcal{R}_\lambda(G_\lambda) \subset G$, then $\mathcal{R}_\lambda(\Omega_\lambda) \subset \Omega$.*

Let Ω'_λ denote the connected component of $\mathcal{R}_\lambda(\Omega_\lambda)$ whose closure contains P or Q. We notice that, since u is of class C^1 in a neighborhood of Γ_δ, Lemma 2.7 implies that $|Du|$ is bounded away from σ in the closure of a set $A_\delta \supset \Gamma_\delta$. This guarantees that Proposition 2.4 can be applied to the two (local) minimizers u and u^λ of $\mathcal{I}_{\Omega'_\lambda}$: since $u \geq u^\lambda$ on $\partial\Omega'_\lambda$ (and $|A_\delta \cap \Omega_\lambda|$, $|A_\delta \cap \Omega'_\lambda| > 0$), we have $u \geq u^\lambda$ in Ω'_λ.

If case (i) occurs, we apply Proposition 2.5 to u and u^λ in $A_\delta \cap \Omega'_\lambda$ and deduce that $u > u^\lambda$ in $A_\delta \cap \Omega'_\lambda$, since $u \not\equiv u^\lambda$ on $\Gamma_\delta \cap \Omega'_\lambda$. This is a contradiction, since P belongs to both $A_\delta \cap \Omega'_\lambda$ and $\Gamma_\delta \cap \mathcal{R}_\lambda(\Gamma_\delta)$, and hence $u(P) = u^\lambda(P)$.

Now, let us consider case (ii). Notice that ξ belongs to the tangent hyperplane to Γ_δ at Q. Since $u \in C^1(A_\delta)$ and $|Du|$ is bounded away from σ in the closure of A_δ, standard elliptic regularity theory (see [To1], [To2] and [GT]) implies that $u \in C^{2,\gamma}(A_\delta)$ for some $\gamma \in (0, 1)$. Thus, applying Proposition 2.6 to u and u^λ in $A_\delta \cap \Omega'_\lambda$ yields

$$\frac{\partial u}{\partial \xi}(Q) < \frac{\partial u^\lambda}{\partial \xi}(Q).$$

On the other hand, since Γ_δ is a level surface of u and u is differentiable at Q, we must have

$$\frac{\partial u}{\partial \xi}(Q) = \frac{\partial u^\lambda}{\partial \xi}(Q) = 0; \tag{2.10}$$

this gives the desired contradiction and concludes the proof of the theorem.

Remark 2.9. Notice that if we assume that $\sigma = 0$, then the assumption that u is of class C^1 in a neighborhood of Γ_δ can be removed from Theorem 1.1. Indeed, from elliptic regularity theory we find that $u \in C^{2,\gamma}(\Omega \setminus \{Du = 0\})$ for some $\gamma \in (0, 1)$; from Lemma 2.7 we know that $Du \neq 0$ on Γ_δ, and thus $u \in C^{2,\gamma}$ in an open neighborhood of Γ_δ. As far as we know, few regularity results are available in the literature for the case $\sigma > 0$ (see [Br], [BCS], [CM] and [SV]); since Hölder estimates for the gradient are missing, we have to assume that u is continuously differentiable in a neighborhood of Γ_δ.

Remark 2.10. We notice that (2.10) holds under the weaker assumption that u is Lipschitz continuous, as we readily show.

Let ξ and Q be as in the proof of Theorem 1.1. For $\varepsilon > 0$ small enough, we denote by $y(Q - \varepsilon\xi)$ the projection of $Q - \varepsilon\xi$ on Γ_δ. Since Γ_δ is a level surface of u,

$$u(Q - \varepsilon\xi) - u(Q) = u(Q - \varepsilon\xi) - u(y(Q - \varepsilon\xi)),$$

and, u being Lipschitz continuous, we have

$$|u(Q - \varepsilon\xi) - u(y(Q - \varepsilon\xi))| \le L|Q - \varepsilon\xi - y(Q - \varepsilon\xi)|$$

for a positive constant L independent of ε, ξ and Q. Since ξ is a vector belonging to the tangent hyperplane to Γ_δ at Q, we have

$$|Q - \varepsilon\xi - y(Q - \varepsilon\xi)| = o(\varepsilon)$$

as $\varepsilon \to 0^+$, and thus

$$\lim_{\varepsilon\to 0^+} \frac{u(Q - \varepsilon\xi) - u(Q)}{\varepsilon} = 0.$$

A similar argument applied to u^λ yields

$$\lim_{\varepsilon\to 0^+} \frac{u^\lambda(Q - \varepsilon\xi) - u^\lambda(Q)}{\varepsilon} = 0,$$

and hence (2.10) holds.

3. A class of nondifferentiable functionals

In this section we consider the variational problem (1.2) and assume that f satisfies (f1) and

(f3) $f'(0) > 0$, $f \in C^{2,\alpha}(0, \infty)$ with $0 < \alpha < 1$, and $f''(s) > 0$ for every $s > 0$.

In this case, the function $s \mapsto f(|s|)$ is not differentiable at the origin, and a minimizer of (1.2) satisfies a variational inequality instead of an Euler–Lagrange equation.

By this nondifferentiability of $\mathcal{I}_\Omega$, it may happen that $u \equiv 0$ is the minimizer of (1.2) when Ω is "too small" (see Theorem 3.2); thus, it is clear that the symmetry result of Theorem 1.1 does not hold in the stated terms. In Theorem 3.6, we will state the additional conditions that enable us to extend Theorem 1.1 to this case.

We begin with a characterization of solutions to (1.2).

Proposition 3.1. *Let f satisfy* (f1) *and* (f3). *Then:*

(i) (1.2) *has a unique solution u;*
(ii) *u is characterized by the boundary condition, $u = 0$ on $\partial\Omega$, and the inequality*

$$\left| \int_{\Omega^\sharp} f'(|Du|) \frac{Du}{|Du|} \cdot D\phi \, dx - \int_\Omega \phi \, dx \right| \le f'(0) \int_{\Omega^0} |D\phi| \, dx \tag{3.1}$$

for any $\phi \in C_0^1(\Omega)$. Here

$$\Omega^0 = \{x \in \Omega : Du(x) = 0\} \quad \text{and} \quad \Omega^\sharp = \Omega \setminus \Omega^0. \tag{3.2}$$

Proof. Since f is strictly convex, the uniqueness of a minimizer follows easily.

Assume that u is a minimizer and let $\phi \in C_0^1(\Omega)$; then

$$\int_{\Omega^\sharp} \frac{f(|Du + \varepsilon D\phi|) - f(|Du|)}{\varepsilon} \, dx + \int_{\Omega^0} \frac{f(\varepsilon |D\phi|) - f(0)}{\varepsilon} \, dx - \int_\Omega \phi \, dx \ge 0$$

for all $\varepsilon > 0$. By taking the limit as $\varepsilon \to 0^+$, we obtain one of the two inequalities in (3.1). The remaining inequality is obtained by repeating the argument with $-\phi$.

Conversely, assume that u satisfies (3.1) and let $\phi \in C_0^1(\Omega)$. The convexity of the function $t \mapsto f(|Du + tD\phi|)$ yields

$$\begin{aligned} \mathcal{I}_\Omega(u+\phi) - \mathcal{I}_\Omega(u) &= \int_{\Omega^\sharp} [f(|Du + D\phi|) - f(|Du|)] \, dx + \int_{\Omega^0} f(|D\phi|) \, dx - \int_\Omega \phi \, dx \\ &\ge \int_{\Omega^\sharp} f'(|Du|) \frac{Du}{|Du|} \cdot D\phi \, dx + f'(0) \int_{\Omega^0} |D\phi| - \int_\Omega \phi \, dx \ge 0, \end{aligned}$$

where the last inequality follows from (3.1); thus, u is a minimizer of (1.2). □

Next, we recall the definition of the Cheeger constant $h(\Omega)$ of a set Ω (see [Ch] and [KF]):

$$h(\Omega) = \inf\{|\partial A|/|A| : A \subset \Omega, \ \partial A \cap \partial\Omega = \emptyset\}. \tag{3.3}$$

It is well-known (see [De] and [KF]) that an equivalent definition of $h(\Omega)$ is given by

$$h(\Omega) = \inf_{\phi \in C_0^1(\Omega)} \frac{\int_\Omega |D\phi| \, dx}{\int_\Omega |\phi| \, dx}. \tag{3.4}$$

Theorem 3.2. *Let u be the solution of* (1.2), *with f satisfying* (f1) *and* (f3). *Then $u \equiv 0$ if and only if*

$$f'(0) h(\Omega) \ge 1. \tag{3.5}$$

Proof. We first observe that

$$h(\Omega) = \inf_{\phi \in C_0^1(\Omega)} \frac{\int_\Omega |D\phi| \, dx}{|\int_\Omega \phi \, dx|}, \tag{3.6}$$

since we can always assume that the minimizing sequences in (3.4) consist of nonnegative functions.

Assume that $u = 0$ is a solution of (1.2); then from (3.1) and (3.6) we easily get (3.5). Conversely, if (3.5) holds, then thanks to (3.6), $u \equiv 0$ satisfies (3.1) and Proposition 3.1 implies that u is a solution of (1.2). □

Observe that if Ω is a ball of radius R, then its Cheeger constant is

$$h(\Omega) = N/R, \tag{3.7}$$

as seen in [KF]. Thus, Theorem 3.2 informs us that $u \equiv 0$ is the only minimizer of $\mathcal{I}_\Omega$ if and only if $R \le N f'(0)$, i.e. if Ω is small enough. In the following proposition we get an explicit expression of the solution of (1.2) in a ball. Notice that in this case the set Ω^0 always has positive Lebesgue measure.

Proposition 3.3. *Let f satisfy* (f1) *and* (f3) *and denote by g the Fenchel conjugate of f. Let $\Omega \subset \mathbb{R}^N$ be the ball of radius R centered at the origin, and let u_R be the solution of* (1.2). *Then*

$$u_R(x) = \int_{|x|}^{R} g'\left(\frac{s}{N}\right) ds, \qquad 0 \le |x| \le R. \tag{3.8}$$

Proof. Under very general assumptions on f, a proof of this proposition can be found in [Cr2]. In the following, we present a simpler *ad hoc* proof for the case we are considering.

As we have just noticed, if $R \le N f'(0)$, then $h(B_R) f'(0) \ge 1$, and hence Theorem 3.2 implies that the minimizer of $\mathcal{I}_\Omega$ must vanish everywhere. Thus, (3.8) holds, since we know that $g' = 0$ in the interval $[0, f'(0)]$ and hence in $[0, R/N]$.

Now, suppose that $R > N f'(0)$ and let $\phi \in C_0^1(B_R)$. We compute the number between the bars in (3.1) with $u = u_R$; since g is the Fenchel conjugate of f, we obtain

$$\begin{aligned}
&\int_{\Omega^\sharp} f'(|Du|)\frac{Du}{|Du|} \cdot D\phi\, dx - \int_\Omega \phi\, dx \\
&\qquad = -\int_{N f'(0) < |x| < R} \frac{x}{N} \cdot D\phi\, dx - \int_{|x|<R} \phi\, dx = \int_{|x| < N f'(0)} \frac{x}{N} \cdot D\phi\, dx,
\end{aligned}$$

after an application of the divergence theorem. Applying the Cauchy–Schwarz inequality to the last integrand, we obtain

$$\left| \int_{\Omega^\sharp} f'(|Du|)\frac{Du}{|Du|} \cdot D\phi\, dx - \int_\Omega \phi\, dx \right| \le f'(0) \int_{|x| < N f'(0)} |D\phi|\, dx,$$

that is, (3.1) holds; the conclusion then follows from Proposition 3.1. □

In the following two lemmas we derive the weak comparison principle and Hopf lemma that are necessary to prove our symmetry result.

Lemma 3.4 (Weak comparison principle). *Let f satisfy* (f1) *and* (f3) *and let A be a bounded domain. Assume that $u_0, u_1 \in W^{1,\infty}(A)$ are minimizers of $\mathcal{I}_A$ such that $u_0 \le u_1$ on ∂A. Then $u_0 \le u_1$ on $\overline{A}$.*

Proof. Let $B = \{x \in A : u_0(x) > u_1(x)\}$ and assume for contradiction that $B \ne \emptyset$. Since $u_0 \le u_1$, and u_0 and u_1 are both continuous, we have $u_0 = u_1$ on ∂B. Hence, u_0 and u_1 are two distinct solutions of the problem

$$\inf\left\{ \int_B [f(|Du|) - u]\, dx : u(x) = u_0(x) \text{ on } \partial B \right\},$$

which is a contradiction, on account of the uniqueness of the minimizer of $\mathcal{I}_B$. □

Lemma 3.5. *Let f satisfy* (f1) *and* (f3) *and denote by g the Fenchel conjugate of f. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and let G satisfy* (1.3)*. Assume that $u \in W_0^{1,\infty}(\Omega)$ is a minimizer of* (1.2) *satisfying* (1.4)*. For $x_0 \in \Gamma_\delta$, let $y_0 \in \partial\Omega$ be such that* $\mathrm{dist}(y_0, \Gamma_\delta) = \delta$ *and set $\nu = (x_0 - y_0)/\delta$; denote by $\rho = \rho(x_0)$ the radius of the optimal interior ball at x_0. Then*

$$\liminf_{t\to 0^+} \frac{u(x_0 + t\nu) - u(x_0)}{t} \geq g'\left(\frac{\rho}{N}\right).$$

In particular, $\inf_{\Gamma_\delta} |Du| > 0$ *in two cases:*

(i) *if $u \in C^1(\Gamma_\delta)$ and $\rho(x_0) > Nf'(0)$ for any $x_0 \in \Gamma_\delta$;*
(ii) *if u is differentiable at every $x \in \Gamma_\delta$, and G satisfies the uniform interior sphere condition with radius $\rho > Nf'(0)$.*

Proof. Notice that if $\rho > Nf'(0)$ then Proposition 3.3 implies that u_R given by (3.8) with $R = \rho$ is strictly positive in a ball of radius ρ. Then the proof can be easily adapted from the proof of Lemma 2.7. □

Finally, by repeating the argument for Theorem 1.1, we obtain the following theorem.

Theorem 3.6. *Let f, Ω and G satisfy the assumptions of Lemma* 3.5*. Assume that $u \in W_0^{1,\infty}(\Omega)$ is the minimizer of* (1.2) *and that* (1.4) *holds. If u is of class C^1 in a tubular neighborhood of Γ_δ and $\rho > Nf'(0)$, then Ω must be a ball.*

Proof. The proof follows the lines of the proof of Theorem 1.1.

In this case, the weak comparison principle (which has to be applied to u and u^λ in Ω'_λ) is given by Lemma 3.4.

Since u is of class C^1 in an open neighborhood of Γ_δ, Lemma 3.5 implies that $|Du|$ is bounded away from zero in an open set $A_\delta \supset \Gamma_\delta$. Proposition 3.1 then shows that u is a weak solution of

$$-\operatorname{div}\left\{ f'(|Du|)\frac{Du}{|Du|} \right\} = 1$$

in A_δ. Thus, a strong comparison principle and a Hopf comparison principle analogous to Propositions 2.5 and 2.6 apply.

Once these three principles are established, the proof can be completed by using the method of moving planes, as in Subsection 2.2. □

4. Symmetry results for fully nonlinear elliptic and parabolic equations

As already mentioned in the Introduction, the argument used in the proof of Theorem 1.1 applies to more general elliptic equations of the form (1.5). Following [Se2, properties (A)–(D), pp. 309–310], we state our assumptions on F in a very general form and to refer the reader to the vast literature for the relevant sufficient conditions.

Let u be a viscosity solution of (1.5) in Ω and assume that $u = 0$ on $\partial\Omega$. Let $A \subseteq \Omega$ denote an open connected set.

(WCP) We say that (1.5) enjoys the *Weak Comparison Principle* in A if, for any two viscosity solutions u and v of (1.5), $u \le v$ on ∂A extends to the inequality $u \le v$ on $\overline{A}$.

(SCP) We say that (1.5) enjoys the *Strong Comparison Principle* in A if for any two viscosity solutions u and v of (1.5), the inequality $u \le v$ on ∂A implies that either $u \equiv v$ in $\overline{A}$ or $u < v$ in A.

(BPP) Suppose ∂A contains a (relatively open) flat portion H. We say that (1.5) enjoys the *Boundary Point Property* at $P \in H$ if, for any two solutions u and v of (1.5), Lipschitz continuous in A and such that $u \le v$ in A, then the assumption $u(P) = v(P)$ implies that either $u \equiv v$ in $\overline{A}$, or $u < v$ in A and

$$\limsup_{\varepsilon \to 0^+} \frac{[v-u](P+\varepsilon\nu) - [v-u](P)}{\varepsilon} > 0.$$

Here, ν denotes the inward unit normal to ∂A at P.

We shall also suppose that

(IR) equation (1.5) is *invariant under reflections in any hyperplane;*

in other words, we require the following: for any ξ and λ, u is a solution of (1.5) in Ω if and only if u^λ is a solution of (1.5) in $\mathcal{R}_\lambda(\Omega)$ (here, we use the notation introduced in Subsection 2.2).

The following symmetry results hold. Here, we state our theorems for *continuous* viscosity solutions; however, the same arguments may be applied when the definitions of classical or weak solutions are considered.

Theorem 4.1. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and let G satisfy* (1.3)*. Let $u = u(x)$ be a nonnegative viscosity solution of* (1.5) *satisfying the homogeneous Dirichlet boundary condition $u = 0$ on $\partial\Omega$. Suppose there exist constants $c, \delta > 0$ such that* (1.4) *holds. Let F satisfy* (IR) *and*

(i) (WCP) *for $A = \Omega$;*
(ii) (SCP) *and* (BPP) *for some neighborhood A_δ of Γ_δ.*

Then Ω must be a ball.

The corresponding result for parabolic equations reads as follows.

Theorem 4.2. *Let F, Ω and G satisfy the assumptions of Theorem 4.1. Let $u = u(x,t)$ be a nonnegative viscosity solution of*

$$u_t - F(u, Du, D^2u) = 0 \quad \text{in } \Omega \times (0,T), \tag{4.1}$$

$$u = 0 \quad \text{on } \Omega \times \{0\}, \tag{4.2}$$

$$u = 1 \quad \text{on } \partial\Omega \times (0,T). \tag{4.3}$$

If there exist a time $t^* \in (0,T)$ and constants $c, \delta > 0$ such that

$$u = c \quad \text{on } \Gamma_\delta \times \{t^*\}, \tag{4.4}$$

then Ω must be a ball.

In the literature, there are a large number of results ensuring that (WCP), (SCP) and (BPP) hold provided sufficient structure conditions are assumed on F. In the following, we collect just a few of them.

For a *proper* equation (see [CIL] for the definition) of the form (1.5), a weak comparison principle is given in [KK2] (see also [KK1] and [BM]), where F is assumed to be *locally strictly elliptic* and to be *locally Lipschitz continuous* in the second variable (the one corresponding to Du). Under the additional assumption that F is *uniformly elliptic*, (SCP) and (BPP) are proved in [Tr]. The assumptions in [KK2] include some kind of mean curvature type equations and nonhomogeneous p-Laplace equations; however, they do not include the homogeneous p-Laplace equation and other degenerate elliptic equations.

We conclude this section by mentioning that, for classical or distributional solutions, the reader can refer to the monographs [PW], [GT], [Fr] and [PS]. More recent and interesting developments on comparison principles for classical and viscosity solutions can be found in [BB, Je, CLN1, CLN2, CLN3, DS, SS, Si].

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