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Jörg Brüdern · Trevor D. Wooley

Cubic moments of Fourier coefficients and pairs of diagonal quartic forms

Abstract. We establish the non-singular Hasse principle for pairs of diagonal quartic equations in 22 or more variables. Our methods involve the estimation of a certain entangled two-dimensional 21st moment of quartic smooth Weyl sums via a novel cubic moment of Fourier coefficients.

1. Introduction

A consideration of disjoint systems of diagonal Diophantine equations lends credibility to the guiding principle that the number of variables required to solve a system should in general grow in proportion to the number of its equations. That such systems are no harder to analyse is made evident in an elegant paper of Cook [6]. There is remarkably little work available in which systems of entangled equations have been successfully analysed in circumstances where the average number of variables per equation is smaller than that accessible for a single equation. These investigations have been limited almost exclusively to equations of degree at most three, and exploit the interaction between component equations by means of restricted moments of Fourier coefficients of unconventional type (see [3, 4, 5]). In this paper we add to this small stock of examples, analysing pairs of diagonal quartic forms through a novel cubic moment of certain Fourier coefficients of arithmetic origin.

In order to describe our results, we introduce some notation. When s is a natural number, let a_j, b_j ($1 \le j \le s$) be fixed rational integers, and consider the pair of simultaneous diagonal quartic equations

$$a_1 x_1^4 + a_2 x_2^4 + \cdots + a_s x_s^4 = b_1 x_1^4 + b_2 x_2^4 + \cdots + b_s x_s^4 = 0. \qquad (1.1)$$

Given a positive number P, we denote by $\mathcal{N}(P)$ the number of integral solutions $\mathbf{x}$ of (1.1) with $|x_j| \le P$ ($1 \le j \le s$).

J. Brüdern: Mathematisches Institut, Bunsenstrasse 3–5, D-37073 Göttingen, Germany;
e-mail: bruedern@uni-math.gwdg.de

T. D. Wooley: School of Mathematics, University of Bristol, University Walk,
Clifton, Bristol BS8 1TW, United Kingdom; e-mail: matdw@bristol.ac.uk

Theorem 1.1. *Let s be a natural number with $s \geq 22$. Suppose that $a_i, b_i \in \mathbb{Z}$ ($1 \leq i \leq s$) satisfy the condition that for any pair $(c, d) \in \mathbb{Z}^2 \setminus \{(0, 0)\}$, at least $s - 7$ of the numbers $ca_j + db_j$ ($1 \leq j \leq s$) are non-zero. Then provided that the system (1.1) has non-singular real and p-adic solutions for each prime number p, one has $\mathcal{N}(P) \gg P^{s-8}$.*

We pause at this point to discuss the various hypotheses of this theorem. First, the application of conventional technology (see [1, 12]) has the potential to establish a conclusion analogous to Theorem 1.1 only for $s \geq 24$. Whilst our conclusion saves two variables over this bound, one may safely conjecture that subject to appropriate rank conditions the constraint $s \geq 17$ should suffice. When $s \geq 21$, it follows from work of Godinho [8] that the system (1.1) has non-zero p-adic solutions whenever $p > 73$. Theorem 1.1 consequently implies a Hasse principle for pairs of equations of the shape (1.1) in a rather strong form. By combining Godinho's work with that of Poehler [10], meanwhile, one finds that the p-adic solubility of the system (1.1) is assured for $s \geq 49$.

The most novel feature of our proof of Theorem 1.1 involves a consideration of a suitable cubic moment of certain Fourier coefficients. For a continuous function

$$H : \mathbb{R} \to [0, \infty)$$

of period 1, let $c(n) = \int_0^1 H(\alpha)e(-n\alpha)\,d\alpha$, where as usual we write $e(z)$ for $e^{2\pi i z}$. It transpires that one may relate the moment

$$\sum_{n \in \mathbb{Z}} |c(n)|^3 \tag{1.2}$$

to the correlation

$$\int_0^1 \int_0^1 H(\alpha)H(\beta)H(\alpha + \beta)\,d\alpha\,d\beta. \tag{1.3}$$

We shall take $H(\alpha) = |\sum_{x \in \mathcal{A}} e(\alpha x^4)|^7$, in which $\mathcal{A}$ is a suitable set of smooth numbers. In this case, one may control the moment (1.2) by means of the large values estimates for Fourier coefficients provided in [9]. The mean value (1.3) may be viewed as an entangled 21st moment of smooth Weyl sums, and it is this that plays the leading role in our application of the Hardy–Littlewood (circle) method. We remark that with $H(\alpha)$ defined as above, the Fourier coefficient $c(n)$ has no direct arithmetic interpretation, despite its arithmetic origin. Indeed, this coefficient may be non-zero for arbitrarily large n, a complication reflected in technical difficulties that we encounter when relating (1.2) to (1.3) in §3. Although in principle our methods are applicable in wide generality, it would appear that in light of currently available mean value estimates for Weyl sums, new ideas are required for their application to Diophantine problems of higher degree.

Following some preliminary discussion of smooth Weyl sums, we announce the key 21st-moment estimate in §2. The aforementioned cubic moment of Fourier coefficients is analysed in §3, and then estimated by means of large values estimates in §4. In §5 we shift our focus to preparations for the application of the circle method, tackling the minor arcs in the Hardy–Littlewood dissection in §6, and concluding with the major arc analysis in §7.

Our basic parameter is P, a sufficiently large positive number. In this paper, implicit constants in Vinogradov's notation $\ll$ and $\gg$ may depend on s and ε, as well as ambient coefficients such as $\mathbf{a}$ and $\mathbf{b}$. Whenever ε appears in a statement, either implicitly or explicitly, we assert that the statement holds for each $\varepsilon > 0$.

2. A 21st moment of biquadratic Weyl sums

In this section we introduce the novel ingredients utilised in our application of the circle method. Before announcing these unconventional mean value estimates, however, we must introduce some notation. When P and R are real numbers with $1 \leq R \leq P$, we define the set $\mathcal{A}(P, R)$ of smooth numbers by

$$\mathcal{A}(P, R) = \{n \in \mathbb{Z} \cap [1, P] : p \text{ prime and } p \mid n \Rightarrow p \leq R\}.$$

We then define the smooth Weyl sum $h(\alpha) = h(\alpha; P, R)$ by

$$h(\alpha; P, R) = \sum_{x \in \mathcal{A}(P,R)} e(\alpha x^4).$$

It is convenient to refer to an exponent Δ_t as *admissible* if there exists a positive number η such that, whenever $1 \leq R \leq P^\eta$, one has

$$\int_0^1 |h(\alpha; P, R)|^t \, d\alpha \ll P^{t-4+\Delta_t}. \tag{2.1}$$

Lemma 2.1. *The exponents* $\Delta_7 = 0.849408$, $\Delta_{10} = 0.213431$ *and* $\Delta_{12} = 0$ *are admissible.*

Proof. The desired conclusion follows from [2, Theorem 2] and the discussion surrounding the table of exponents on [2, page 393]. As explained in the proof of [2, Lemma 2.3], the upper bound (2.1) when $t = 12$ is a consequence of [12, Lemma 5.2]. □

Henceforth, we fix $R = P^\eta$ with $\eta > 0$ chosen in accordance with Lemma 2.1 and the upper bounds (2.1). When $a, b, c, d \in \mathbb{Z}$, we define the integral

$$I(a, b, c, d) = \int_0^1 \int_0^1 |h(a\alpha)h(b\beta)h(c\alpha + d\beta)|^7 \, d\alpha \, d\beta. \tag{2.2}$$

Our goal in §§3 and 4 is the proof of the following upper bound for this integral.

Theorem 2.2. *Suppose that* a, b, c, d *are non-zero integers. Then*

$$I(a, b, c, d) \ll P^{13 + \frac{1}{2}\Delta_{10} + \varepsilon}.$$

Previous authors would disentangle the mixed moment (2.2) via Hölder's inequality to obtain an estimate of the shape

$$I(a, b, c, d) \ll \left(\int_0^1 |h(\alpha)|^{21/2} \, d\alpha \right)^2 \ll P^{13 + \frac{3}{2}\Delta_{10}}.$$

The superiority of our new estimate is self-evident. We direct the curious reader to the explanation following the statement of [3, Theorem 3] for a related discussion. By following the argument of the proof of [3, Theorem 4], presented in §4 of the latter source, one obtains from Theorem 2.2 a superficially more general conclusion of similar type.

Theorem 2.3. *Suppose that c_i, d_i ($1 \le i \le 3$) are integers satisfying*

$$(c_1 d_2 - c_2 d_1)(c_1 d_3 - c_3 d_1)(c_2 d_3 - c_3 d_2) \ne 0.$$

Write $\Lambda_j = c_j \alpha + d_j \beta$ ($j = 1, 2, 3$). Then whenever $1 \le R \le P^\eta$, one has

$$\int_0^1 \int_0^1 |h(\Lambda_1)h(\Lambda_2)h(\Lambda_3)|^7 \, d\alpha \, d\beta \ll P^{13 + \frac{1}{2}\Delta_{10} + \varepsilon}.$$

3. Cubic moments of certain Fourier coefficients

An inspection of the mean value (2.2) suggests that Fourier coefficients associated with $|h(\alpha)|^7$ should play a prominent role in its estimation. The absence of any direct arithmetic interpretation forces us, however, to indulge in a far more detailed discussion of these Fourier coefficients than would ordinarily be the case. Define

$$\psi(n) = \int_0^1 |h(\alpha)|^7 e(-n\alpha) \, d\alpha. \tag{3.1}$$

Lemma 3.1. *The Fourier expansion*

$$|h(\alpha)|^7 = \sum_{n \in \mathbb{Z}} \psi(n) e(n\alpha) \tag{3.2}$$

is uniformly convergent in α, and one has $\psi(n) \ll P^{15}/n^2$.

Proof. We begin by observing that $h(\alpha)$ is an analytic function, and hence $|h(\alpha)|^7$ is a real valued function that is twice continuously differentiable. Note that when $h(\alpha) = 0$, then the first two derivatives of $|h(\alpha)|^7$ are also 0. By writing $|h(\alpha)|^7 = h(\alpha)^{7/2} h(-\alpha)^{7/2}$ when $h(\alpha) \ne 0$, and noting that

$$h'(\alpha) = 2\pi i \sum_{x \in \mathcal{A}(P,R)} x^4 e(\alpha x^4) \ll P^5, \quad h''(\alpha) = (2\pi i)^2 \sum_{x \in \mathcal{A}(P,R)} x^8 e(\alpha x^4) \ll P^9,$$

it follows that, uniformly in α, one has $\frac{d^2}{d\alpha^2}|h(\alpha)|^7 \ll P^{15}$. Consequently, by integrating by parts, one deduces that

$$\int_0^1 |h(\alpha)|^7 e(-n\alpha) \, d\alpha = \int_0^1 \frac{e(-n\alpha)}{(2\pi i n)^2} \frac{d^2}{d\alpha^2}|h(\alpha)|^7 \, d\alpha \ll \frac{P^{15}}{n^2}.$$

Finally, since $|h(\alpha)|^7$ is differentiable and the series on the right hand side of (3.2) is absolutely convergent, one finds from [7, Theorem 1.4.2] or [16, Theorem 8.14, page 60], for example, that the Fourier expansion (3.2) converges to $|h(\alpha)|^7$ uniformly in α. □

This lemma allows us to replace the mean value (2.2) by a cubic moment of $\psi(n)$, truncated both in terms of n and the magnitude of $\psi(n)$.

Lemma 3.2. *When a, b, c, d are non-zero integers, one has*

$$I(a, b, c, d) \ll P^9 + \sum_{\substack{|n| \leq P^9 \\ |\psi(n)| > 1}} |\psi(n)|^3.$$

Proof. Our initial step is to accommodate non-zero integral coefficients l by extracting from (3.2) the relation

$$|h(l\alpha)|^7 = \sum_{n \in \mathbb{Z}} \psi(n) e(nl\alpha) = \sum_{m \in \mathbb{Z}} \psi_l(m) e(m\alpha),$$

where

$$\psi_l(m) = \begin{cases} \psi(m/l) & \text{when } l \mid m, \\ 0 & \text{otherwise.} \end{cases} \tag{3.3}$$

Using the uniform convergence of the respective Fourier expansions of $|h(a\alpha)|^7$, $|h(b\beta)|^7$ and $|h(c\alpha + d\beta)|^7$, one finds from (2.2) that $I(a, b, c, d)$ is equal to

$$\sum_{n_1, n_2, n_3 \in \mathbb{Z}} \int_0^1 \int_0^1 \psi_a(n_1) \psi_b(n_2) \psi(n_3) e(n_1 \alpha + n_2 \beta - n_3(c\alpha + d\beta)) \, d\alpha \, d\beta.$$

By orthogonality, the double integral here can be non-zero only when $n_1 = cn_3$ and $n_2 = dn_3$, whence

$$I(a, b, c, d) = \sum_{n \in \mathbb{Z}} \psi_a(cn) \psi_b(dn) \psi(n).$$

Thus, on noting that (3.3) delivers the estimate

$$\sum_{n \in \mathbb{Z}} |\psi_a(cn)|^3 \leq \sum_{m \in \mathbb{Z}} |\psi_a(m)|^3 = \sum_{k \in \mathbb{Z}} |\psi(k)|^3,$$

we deduce by Hölder's inequality and symmetry that

$$I(a, b, c, d) \leq \sum_{n \in \mathbb{Z}} |\psi(n)|^3.$$

We therefore conclude from Lemma 3.1 that

$$I(a, b, c, d) \ll \sum_{|n| \leq P^9} |\psi(n)|^3 + \sum_{|n| > P^9} (P^{15}/n^2)^3 = \sum_{\substack{|n| \leq P^9 \\ |\psi(n)| > 1}} |\psi(n)|^3 + O(P^9),$$

and the proof of the lemma is complete. $\qquad\square$

A dyadic dissection simplifies the discussion to come. When $T > 0$, write

$$M(T) = \sum_{\substack{|n| \leq P^9 \\ T < |\psi(n)| \leq 2T}} |\psi(n)|^3.$$

By applying the triangle inequality to (3.1), the estimates available from Lemma 2.1 furnish the bound $\psi(n) \leq \psi(0) \ll P^{3+\Delta_7}$, and hence

$$\sum_{\substack{|n| \leq P^9 \\ |\psi(n)| > 1}} |\psi(n)|^3 \leq \sum_{\substack{l=0 \\ 2^l \leq P^{3+\Delta_7} \log P}}^{\infty} M(2^l).$$

Consequently, for some positive number T with $1 \leq T \leq P^{3+\Delta_7} \log P$, we may conclude from Lemma 3.2 that

$$I(a, b, c, d) \ll P^9 + (\log P)M(T). \tag{3.4}$$

4. The arithmetic harmonic analysis

The moment has come to deliver the proof of Theorem 2.2. We may suppose that $\eta > 0$ is small enough that the estimates implicit in (2.1) hold. We bound the quantity $M(T)$ when $1 \leq T \leq P^{3+\Delta_7} \log P$. Define $\mathcal{Z}$ to be the set of integers n with $|n| \leq P^9$ such that $T < |\psi(n)| \leq 2T$, and write $Z = \operatorname{card}(\mathcal{Z})$. For each $n \in \mathcal{Z}$, we take $\omega_n = 1$ when $\psi(n) > 0$, and $\omega_n = -1$ when $\psi(n) < 0$, and then put

$$K(\alpha) = \sum_{n \in \mathcal{Z}} \omega_n e(-n\alpha).$$

Then we find from (3.1) that

$$\int_0^1 |h(\alpha)|^7 K(\alpha) \, d\alpha = \sum_{n \in \mathcal{Z}} \omega_n \int_0^1 |h(\alpha)|^7 e(-n\alpha) \, d\alpha = \sum_{n \in \mathcal{Z}} |\psi(n)| > TZ. \tag{4.1}$$

We bound $M(T)$ by estimating the integral on the left hand side of (4.1), controlling the frequency of large Fourier coefficients $\psi(n)$.

Before proceeding further, we recall that the estimate

$$\int_0^1 |h(\alpha)^4 K(\alpha)^2| \, d\alpha \ll P^3 Z + P^{2+\varepsilon} Z^{3/2} \tag{4.2}$$

is an immediate consequence of [9, Lemma 2.1].

Lemma 4.1. *One has the estimates*

$$Z \ll P^{\frac{28}{3} + \frac{1}{3}\Delta_{10}} T^{-2} + P^{13 + \frac{1}{2}\Delta_{10} + \varepsilon} T^{-3}, \quad Z \ll P^{9 + \Delta_{10}} T^{-2} + P^{16 + 2\Delta_{10} + \varepsilon} T^{-4}.$$

Proof. An application of Hölder's inequality shows that

$$\int_0^1 |h(\alpha)|^7 K(\alpha) \, d\alpha \leq \left(\int_0^1 |h(\alpha)^4 K(\alpha)^2| \, d\alpha \right)^{1/3} \left(\int_0^1 |K(\alpha)|^2 \, d\alpha \right)^{1/6}$$

$$\times \left(\int_0^1 |h(\alpha)|^{10} \, d\alpha \right)^{1/6} \left(\int_0^1 |h(\alpha)|^{12} \, d\alpha \right)^{1/3}.$$

Recalling Lemma 2.1, Parseval's identity and (4.2), we deduce from (4.1) that

$$TZ \ll (P^3 Z + P^{2+\varepsilon} Z^{3/2})^{1/3} (Z)^{1/6} (P^{6+\Delta_{10}})^{1/6} (P^8)^{1/3}$$
$$\ll P^{\frac{14}{3} + \frac{1}{6}\Delta_{10}} Z^{1/2} + P^{\frac{13}{3} + \frac{1}{6}\Delta_{10} + \varepsilon} Z^{2/3},$$

and the first of the claimed estimates follows by disentangling this inequality.

On the other hand, also by Hölder's inequality, one has similarly

$$\int_0^1 |h(\alpha)|^7 K(\alpha)\, d\alpha \le \left(\int_0^1 |h(\alpha)^4 K(\alpha)^2|\, d\alpha \right)^{1/2} \left(\int_0^1 |h(\alpha)|^{10}\, d\alpha \right)^{1/2}$$
$$\ll (P^3 Z + P^{2+\varepsilon} Z^{3/2})^{1/2} (P^{6+\Delta_{10}})^{1/2},$$

whence

$$TZ \ll P^{\frac{9}{2} + \frac{1}{2}\Delta_{10}} Z^{1/2} + P^{4 + \frac{1}{2}\Delta_{10} + \varepsilon} Z^{3/4},$$

and the second conclusion follows by further disentangling. $\square$

We are now equipped to establish Theorem 2.2. On the one hand, if one has $T \le P^{3 + \frac{3}{2}\Delta_{10}}$, then one finds from the first estimate of Lemma 4.1 that

$$M(T) \ll ZT^3 \ll P^{\frac{28}{3} + \frac{1}{3}\Delta_{10}} T + P^{13 + \frac{1}{2}\Delta_{10} + \varepsilon} \ll P^{13 + \frac{1}{2}\Delta_{10} + \varepsilon}.$$

On the other hand, when $P^{3 + \frac{3}{2}\Delta_{10}} < T \le P^{3 + \Delta_7} \log P$, one finds instead from the second estimate of Lemma 4.1 that

$$ZT^3 \ll P^{9 + \Delta_{10}} T + P^{16 + 2\Delta_{10} + \varepsilon} T^{-1} \ll P^{12 + \Delta_7 + \Delta_{10} + \varepsilon} + P^{13 + \frac{1}{2}\Delta_{10} + \varepsilon}.$$

Thus, on recalling (3.4), we deduce that

$$I(a, b, c, d) \ll P^9 + (\log P) P^{13 + \frac{1}{2}\Delta_{10} + \varepsilon} \ll P^{13 + \frac{1}{2}\Delta_{10} + 2\varepsilon}.$$

This concludes the proof of Theorem 2.2.

5. Preparations for the circle method

We suppose that the hypotheses of the statement of Theorem 1.1 are satisfied, so in particular $s \ge 22$. With the pairs $(a_j, b_j) \in \mathbb{Z}^2 \setminus \{(0, 0)\}$, we associate the binary forms

$$\Lambda_j = a_j \alpha + b_j \beta \quad (1 \le j \le s), \tag{5.1}$$

and the two linear forms $L_1(\boldsymbol{\theta})$ and $L_2(\boldsymbol{\theta})$ defined for $\boldsymbol{\theta} \in \mathbb{R}^s$ by

$$L_1(\boldsymbol{\theta}) = \sum_{j=1}^s a_j \theta_j \quad \text{and} \quad L_2(\boldsymbol{\theta}) = \sum_{j=1}^s b_j \theta_j. \tag{5.2}$$

We describe two forms Λ_i and Λ_j as *equivalent* when there exists a non-zero rational number λ with $\Lambda_i = \lambda \Lambda_j$. This notion defines an equivalence relation, and we refer to the number of elements in the equivalence class containing the form Λ_j as its *multiplicity*.

The hypotheses of Theorem 1.1 ensure that there is a non-singular real solution of the system (1.1). By invoking homogeneity, therefore, one finds that there exists a real solution $\mathbf{x} = \boldsymbol{\theta}$ in $[0, 1)^s$ satisfying the property that for some indices i and j with $1 \leq i < j \leq s$, one has

$$\det \begin{pmatrix} 4a_i \theta_i^3 & 4a_j \theta_j^3 \\ 4b_i \theta_i^3 & 4b_j \theta_j^3 \end{pmatrix} \neq 0.$$

By relabelling variables if necessary, there is no loss of generality in supposing that $i = 1$ and $j = 2$, and by taking suitable linearly independent linear combinations of the equations comprising (1.1), we may suppose further that $a_2 = b_1 = 0$ and $a_1 b_2 \neq 0$. Thus, since $\boldsymbol{\theta}$ is a non-singular solution, we have $\theta_1 \theta_2 \neq 0$, and then there is no loss of generality in supposing also that $\theta_1 > 0$ and $\theta_2 > 0$. An application of the inverse function theorem consequently confirms that whenever $\delta > 0$ is sufficiently small, the simultaneous equations

$$a_1 x_1^4 = -\sum_{i=3}^{s} a_i (\theta_i + \delta)^4 \quad \text{and} \quad b_2 x_2^4 = -\sum_{i=3}^{s} b_i (\theta_i + \delta)^4$$

remain soluble for x_1 and x_2, with $x_1 > 0$ and $x_2 > 0$. In this way we see that the system (1.1) possesses a non-singular real solution $\boldsymbol{\theta}$ satisfying $\boldsymbol{\theta} \in (0, 1)^s$.

For any pair $(c, d) \in \mathbb{Z}^2 \setminus \{(0, 0)\}$, the linear form $cL_1(\boldsymbol{\xi}) + dL_2(\boldsymbol{\xi})$ necessarily possesses at least $s - 7$ non-zero coefficients. By choosing an appropriate subset $\mathcal{S}$ of $\{2, \ldots, s\}$ with $\mathrm{card}(\mathcal{S}) = 21$, therefore, we may ensure that the forms Λ_j with $j \in \mathcal{S}$ have multiplicity at most 7. Suppose that these 21 forms fall into t equivalence classes, and that the multiplicities of the representatives of these classes are $r_1, \ldots, r_t$. Then we may suppose that

$$7 \geq r_1 \geq \cdots \geq r_t \quad \text{and} \quad r_1 + \cdots + r_t = 21, \tag{5.3}$$

and hence also that $t \geq 3$. We relabel variables in (1.1), and likewise in (5.1) and (5.2), so that $\mathcal{S}$ becomes $\{2, \ldots, 22\}$, and for $1 \leq i \leq t$ the linear form Λ_{i+1} is in the ith equivalence class counted by r_i. We now fix a non-singular real solution $\boldsymbol{\theta} \in (0, 1)^s$ of (1.1) and a real number δ with $0 < \delta < \theta_1$. In addition, we fix $\eta > 0$ to be sufficiently small in the context of Lemma 2.1.

Next define the generating functions

$$g(\alpha) = \sum_{\delta P < x \leq P} e(\alpha x^4), \quad H_0(\alpha, \beta) = \prod_{j=2}^{22} h(\Lambda_j), \quad H(\alpha, \beta) = \prod_{j=2}^{s} h(\Lambda_j).$$

Then by orthogonality, one has

$$\mathcal{N}(P) \geq \int_0^1 \int_0^1 g(\Lambda_1) H(\alpha, \beta) \, d\alpha \, d\beta.$$

In order to define the Hardy–Littlewood dissection underlying our argument, we put $Q = (\log P)^{1/100}$, and when $a, b \in \mathbb{Z}$ and $q \in \mathbb{N}$ we define

$$\mathfrak{N}(q, a, b) = \{(\alpha, \beta) \in [0, 1)^2 : |\alpha - a/q| \le QP^{-4} \text{ and } |\beta - b/q| \le QP^{-4}\}.$$

We then take $\mathfrak{N}$ to be the union of the boxes $\mathfrak{N}(q, a, b)$ with $0 \le a, b \le q \le Q$ and $(q, a, b) = 1$. Finally, we put $\mathfrak{n} = [0, 1)^2 \setminus \mathfrak{N}$.

The contribution of the major arcs $\mathfrak{N}$ in this dissection satisfies

$$\iint_{\mathfrak{N}} g(\Lambda_1) H(\alpha, \beta) \, d\alpha \, d\beta \gg P^{s-8}, \tag{5.4}$$

a fact we confirm in §7. Meanwhile, in §6 we show that

$$\iint_{\mathfrak{n}} g(\Lambda_1) H(\alpha, \beta) \, d\alpha \, d\beta \ll P^{s-8}(\log \log P)^{-1}. \tag{5.5}$$

The desired conclusion $\mathcal{N}(P) \gg P^{s-8}$ is immediate from (5.4) and (5.5) on noting that $[0, 1)^2$ is the disjoint union of $\mathfrak{N}$ and $\mathfrak{n}$.

6. The minor arc treatment

The analysis of the minor arc contribution proceeds in two phases, one dominated by the use of Weyl's inequality, and a second in which pruning methods are deployed. We begin preparatory work for the first stage by deriving a consequence of Theorem 2.3.

Lemma 6.1. *One has*

$$\int_0^1 \int_0^1 |H_0(\alpha, \beta)| \, d\alpha \, d\beta \ll P^{13 + \frac{1}{2}\Delta_{10} + \varepsilon}.$$

Proof. Recall the discussion of the multiplicities associated with $\Lambda_2, \ldots, \Lambda_t$, and in particular the hypothesis (5.3). By applying [3, Lemma 5], just as in the deduction of [3, equation (4.5)], one finds that

$$\int_0^1 \int_0^1 |H_0(\alpha, \beta)| \, d\alpha \, d\beta \ll \int_0^1 \int_0^1 h_2^{r_2} \ldots h_t^{r_t} \, d\alpha \, d\beta, \tag{6.1}$$

where we have abbreviated $|h(\Lambda_j)|$ to h_j. Let ν be a non-negative integer, and suppose that $r_{t-1} = r_t + \nu < 7$. Then we may apply the argument of the proof of [3, Lemma 6] following equation (4.5) therein to obtain a bound of the shape (6.1), in which r_{t-1} and $r_t = r_{t-1} - \nu$ are replaced by $r_{t-1} + 1$ and $r_t - 1$, respectively, or else by $r_{t-1} - \nu - 1$ and $r_t + \nu + 1$. By relabelling if necessary, we thus derive a bound of the shape (6.1), subject to (5.3), wherein either r_t is reduced, or else t is reduced. By repeating this process, therefore, we ultimately arrive at the situation in which $t = 3$ and $(r_1, r_2, r_3) = (7, 7, 7)$. Then the desired estimate follows from (6.1) and Theorem 2.3. $\qquad\square$

In order to prepare for the pruning process, we introduce a conventional set of one-dimensional major arcs. Define $\mathfrak{M}$ to be the union of the intervals

$$\mathfrak{M}(q, a) = \{\alpha \in [0, 1) : |q\alpha - a| \le P^{-7/2}\}$$

with $0 \le a \le q \le P^{1/2}$ and $(a, q) = 1$, and put $\mathfrak{m} = [0, 1) \setminus \mathfrak{M}$. In addition, when $\lambda, A \in \mathbb{R}$, define the mean value $J(\lambda) = J(\lambda; A)$ by putting

$$J(\lambda; A) = \int_{\mathfrak{M}} |g(a_1\theta)|^{9/4} |h(A\theta + \lambda)|^4 \, d\theta. \tag{6.2}$$

Lemma 6.2. *For each $A \in \mathbb{Q} \setminus \{0\}$, one has $\sup_{\lambda \in \mathbb{R}} J(\lambda; A) \ll P^{9/4}$.*

Proof. Suppose that $\lambda \in \mathbb{R}$. Write $A = B/S$ with $B \in \mathbb{Z} \setminus \{0\}$, $S \in \mathbb{N}$ and $(B, S) = 1$. We define the modified set of major arcs $\mathfrak{W}$ by putting

$$\mathfrak{W} = \{\beta \in [0, 1) : S\beta \in \mathfrak{M}\}.$$

Then a change of variable yields

$$J(\lambda) = S \int_{\mathfrak{W}} |g(a_1 S\beta)|^{9/4} |h(B\beta + \lambda)|^4 \, d\beta. \tag{6.3}$$

It follows from the definition of $\mathfrak{M}$ that for each $\beta \in \mathfrak{W}$, there exist $c \in \mathbb{Z}$ and $r \in \mathbb{N}$ with $0 \le c \le r \le P^{1/2}$, $(c, r) = 1$ and $|S\beta r - c| \le P^{-7/2}$. Thus there exist $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ with $0 \le a \le q \le SP^{1/2}$, $(a, q) = 1$ and $|q\beta - a| \le P^{-7/2}$. Next, we define $\kappa(q)$ to be the multiplicative function defined for $q \in \mathbb{N}$ by taking, for prime numbers p and non-negative integers l,

$$\kappa(p^{4l}) = p^{-l}, \quad \kappa(p^{4l+1}) = 4p^{-l-1/2}, \quad \kappa(p^{4l+2}) = p^{-l-1}, \quad \kappa(p^{4l+3}) = p^{-l-1}.$$

Then as a consequence of [13, Theorem 4.1 and Lemmata 4.3, 4.4 and 6.2],

$$g(a_1 S\beta) \ll \kappa(q) P (1 + P^4 |\beta - a/q|)^{-1} + q^{1/2+\varepsilon} \ll \kappa(q) P (1 + P^4 |\beta - a/q|)^{-1/2}.$$

We therefore deduce from (6.3) that

$$J(\lambda) \ll \sum_{1 \le q \le SP^{1/2}} (P\kappa(q))^{9/4} \sum_{a=1}^{q} \int_{-\infty}^{\infty} \frac{|h(B(a/q + \gamma) + \lambda)|^4}{(1 + P^4|\gamma|)^{9/8}} \, d\gamma. \tag{6.4}$$

By orthogonality, we find that

$$\sum_{a=1}^{q} |h(B(a/q + \gamma) + \lambda)|^4 \le q \sum_{\substack{1 \le x_1, \ldots, x_4 \le P \\ q | B(x_1^4 + x_2^4 - x_3^4 - x_4^4)}} 1 \le |B|^4 (Pq^{-1} + 1)^4 q\rho(q), \tag{6.5}$$

where $\rho(q)$ denotes the number of solutions of the congruence

$$x_1^4 + x_2^4 \equiv x_3^4 + x_4^4 \pmod{q},$$

with $1 \leq x_i \leq q$ $(1 \leq i \leq 4)$. The argument of [14] leading to equation (5.8) of that paper shows that

$$q\rho(q) \ll q^4 \sum_{r|q} r\kappa(r)^4.$$

Hence, on substituting (6.5) into (6.4), we obtain

$$J(\lambda) \ll P^{25/4} \sum_{1 \leq q \leq SP^{1/2}} \kappa(q)^{9/4} \sum_{r|q} r\kappa(r)^4 \int_{-\infty}^{\infty} (1 + P^4|\gamma|)^{-9/8} \, d\gamma.$$

Observe that $\kappa(q)^{9/4} \leq q^{-1/20}\kappa(q)^2$, and hence the argument completing the proof of [14, Lemma 5.4] shows that for a suitable positive constant C, one has

$$J(\lambda) \ll P^{9/4} \sum_{1 \leq q \leq P} \kappa(q)^{9/4} \sum_{r|q} r\kappa(r)^4 \ll P^{9/4} \prod_{p} \left(1 + C \sum_{h=1}^{\infty} p^{-1-h/20}\right).$$

Thus we obtain $J(\lambda) \ll P^{9/4}$, and the proof of the lemma is complete. $\qquad\square$

When $\mathfrak{B} \subseteq [0,1)^2$ is measurable, define the auxiliary mean value

$$U(\mathfrak{B}) = \iint_{\mathfrak{B}} |g(\Lambda_1)H_0(\alpha, \beta)| \, d\alpha \, d\beta.$$

Lemma 6.3. *One has* $U(\mathfrak{n}) \ll P^{14}(\log\log P)^{-1}$.

Proof. Consider the auxiliary sets

$$\mathfrak{e} = \{(\alpha, \beta) \in \mathfrak{n} : \alpha \in \mathfrak{m}\} \quad \text{and} \quad \mathfrak{E} = \{(\alpha, \beta) \in \mathfrak{n} : \alpha \in \mathfrak{M}\}.$$

The treatment of the set $\mathfrak{e}$ is straightforward. On recalling that $\Lambda_1 = a_1\alpha$, one finds via an enhanced version of Weyl's inequality (see [11, Lemma 3]) that

$$\sup_{(\alpha,\beta)\in\mathfrak{e}} |g(\Lambda_1)| = \sup_{\alpha\in\mathfrak{m}} |g(a_1\alpha)| \ll P^{7/8+\varepsilon}.$$

Then from Lemma 6.1, one deduces that

$$U(\mathfrak{e}) \ll P^{7/8+\varepsilon} \int_0^1 \int_0^1 |H_0(\alpha, \beta)| \, d\alpha \, d\beta \ll P^{\frac{111}{8}+\frac{1}{2}\Delta_{10}+\varepsilon} \ll P^{14}Q^{-1}. \qquad (6.6)$$

We turn next to the complementary set $\mathfrak{E}$, handling this via Lemma 6.2. By applying the argument underlying the proof of Lemma 6.1, as in the discussion following the statement of [3, Lemma 10], one finds that for some indices l, m, n with $2 \leq l < m < n \leq 22$, one has

$$U(\mathfrak{E}) \ll \iint_{\mathfrak{E}} g_1 h_l^7 h_m^7 h_n^7 \, d\alpha \, d\beta.$$

Here, we have abbreviated $|g(\Lambda_1)|$ to g_1. By relabelling variables if necessary, there is no loss of generality in supposing that $(l, m, n) = (2, 3, 4)$ and that Λ_3 and Λ_4 are

each pairwise linearly independent of Λ_1. Recall also that $\Lambda_2, \Lambda_3, \Lambda_4$ are assumed to be pairwise linearly independent. Define

$$U_{ij} = \iint_{\mathfrak{E}} g_1^{9/4} h_i^4 h_j^{12} \, d\alpha \, d\beta \quad \text{and} \quad V_{ij} = \iint_{\mathfrak{E}} h_i^{12} h_j^{12} \, d\alpha \, d\beta.$$

Then an application of Hölder's inequality yields the bound

$$U(\mathfrak{E}) \ll \left(\sup_{(\alpha,\beta)\in\mathfrak{E}} |h(\Lambda_2)h(\Lambda_3)h(\Lambda_4)| \right)^{5/27} (U_{23}U_{24})^{2/9} (V_{23}V_{24})^{17/81} V_{34}^{11/81}. \quad (6.7)$$

Let (i, j) be either $(2, 3)$ or $(2, 4)$. Recall that $\Lambda_1 = a_1\alpha$, and change variables from β to γ via the linear transformation $a_j\alpha + b_j\beta = b_j\gamma$. Note here that since Λ_1 and Λ_j are inequivalent, then necessarily $b_j \neq 0$. Write $A = a_i - b_i a_j/b_j$ and recall (6.2). Then in view of the definition of $\mathfrak{E}$, we may make use of the periodicity of the integrand to deduce that

$$U_{ij} \le \int_0^1 \int_{\mathfrak{M}} |g(\Lambda_1)|^{9/4} |h(A\alpha + b_i\gamma)|^4 |h(b_j\gamma)|^{12} \, d\alpha \, d\gamma \le W \sup_{\lambda\in\mathbb{R}} J(\lambda; A),$$

where

$$W = \int_0^1 |h(b_j\gamma)|^{12} \, d\gamma.$$

An application of Lemma 2.1 shows, via a change of variable, that $W = O(P^8)$, and so we deduce from Lemma 6.2 that $U_{23}U_{24} \ll (P^8)^2 (P^{9/4})^2 = P^{41/2}$. Since $\Lambda_2, \Lambda_3, \Lambda_4$ are pairwise linearly independent, when $2 \le i < j \le 4$ further changes of variable lead from Lemma 2.1 to the estimate

$$V_{ij} \ll \int_0^1 \int_0^1 |h(\theta_1)h(\theta_2)|^{12} \, d\theta_1 \, d\theta_2 = \left(\int_0^1 |h(\theta)|^{12} \, d\theta \right)^2 \ll (P^8)^2.$$

By substituting these estimates into (6.7), we deduce thus far that

$$U(\mathfrak{E}) \ll P^{14} \left(P^{-3} \sup_{(\alpha,\beta)\in\mathfrak{E}} |h(\Lambda_2)h(\Lambda_3)h(\Lambda_4)| \right)^{5/27}. \quad (6.8)$$

Our final task is to bound the second factor on the right hand side of (6.8). When θ is a real number with $|h(\theta)| \ge PQ^{-1/120}$, it follows from the proof of [15, Lemma 8.1] that there exist $a \in \mathbb{Z}$ and $q \in \mathbb{N}$ with $1 \le q \le Q^{1/10}$, $(a, q) = 1$ and $|q\theta - a| \le Q^{1/10}P^{-4}$. Consequently, if Λ_k and Λ_l are inequivalent linear forms, and $h_k h_l \ge P^2 Q^{-1/120}$, then for $\sigma = k, l$ there exist integers d_σ and q_σ with

$$1 \le q_\sigma \le Q^{1/10}, \quad (d_\sigma, q_\sigma) = 1 \quad \text{and} \quad |\Lambda_\sigma - d_\sigma/q_\sigma| \le q_\sigma^{-1} Q^{1/10} P^{-4}.$$

From here it follows as in the proof of [3, Lemma 10] that $(\alpha, \beta) \in \mathfrak{N}$. Thus

$$\sup_{(\alpha,\beta)\in\mathfrak{n}} |h(\Lambda_k)h(\Lambda_l)| \ll P^2 Q^{-1/120},$$

whence we obtain the estimate

$$\sup_{(\alpha,\beta)\in\mathfrak{E}} |h(\Lambda_2)h(\Lambda_3)h(\Lambda_4)| \ll P^3 Q^{-1/120}.$$

On substituting into (6.8), we conclude that $U(\mathfrak{E}) \ll P^{14}Q^{-1/648}$. The conclusion of the lemma therefore follows from (6.6) on recalling that $\mathfrak{n} = \mathfrak{e} \cup \mathfrak{E}$. □

A trivial estimate for the generating function $h(\theta)$ now leads from the conclusion of Lemma 6.3 to the estimate (5.5) by means of the relation

$$\iint_{\mathfrak{n}} g(\Lambda_1)H(\alpha,\beta)\,d\alpha\,d\beta \ll P^{s-22}U(\mathfrak{n}).$$

7. The major arcs analysis

Not only is the analysis of the major arcs largely standard, but it is also very similar to the work in [3, §7]. A brief sketch of the analysis therefore suffices on the present occasion. We begin with some additional notation. Define

$$S(q,a) = \sum_{r=1}^{q} e(ar^4/q), \quad T(q,c,d) = q^{-s}\prod_{j=1}^{s} S(q, a_j c + b_j d),$$

$$\mathfrak{A}(q) = \sum_{\substack{a=1 \\ (a,b,q)=1}}^{q}\sum_{b=1}^{q} T(q,a,b) \quad \text{and} \quad \mathfrak{S}(X) = \sum_{1\leq q \leq X} \mathfrak{A}(q).$$

Also, with λ_j as shorthand for $a_j\xi + b_j\zeta$, put

$$v(\theta) = \int_0^P e(\theta\gamma^4)\,d\gamma, \quad w(\theta) = \int_{\delta P}^P e(\theta\gamma^4)\,d\gamma, \quad V(\xi,\zeta) = w(a_1\xi)\prod_{j=2}^{s} v(\lambda_j),$$

and writing $\mathcal{B}(X) = [-XP^{-4}, XP^{-4}]^2$, define

$$\mathfrak{J}(X) = \iint_{\mathcal{B}(X)} V(\xi,\zeta)\,d\xi\,d\zeta.$$

Recall that $\Lambda_1 = a_1\alpha$. Then, as adapted to the current context, the argument leading to [3, equation (7.5)] shows that there is a positive number ρ having the property that whenever $(\alpha,\beta) \in \mathfrak{N}(q,a,b) \subseteq \mathfrak{N}$, one has

$$g(\Lambda_1)H(\alpha,\beta) - \rho T(q,a,b)V(\alpha - a/q, \beta - b/q) \ll P^s(\log P)^{-1/2}.$$

Integrating over $\mathfrak{N}$, we infer that

$$\iint_{\mathfrak{N}} g(\Lambda_1)H(\alpha,\beta)\,d\alpha\,d\beta = \rho\mathfrak{S}(Q)\mathfrak{J}(Q) + O(P^{s-8}(\log P)^{-1/4}). \quad (7.1)$$

Lemma 7.1. *Under the hypotheses of Theorem* 1.1, *the limit* $\mathfrak{S} = \lim_{X \to \infty} \mathfrak{S}(X)$ *exists, and one has* $\mathfrak{S} - \mathfrak{S}(X) \ll X^{-1}$. *Moreover, when the pair of equations* (1.1) *has a non-singular p-adic solution for all primes p, one has* $\mathfrak{S} \gg 1$.

Proof. This lemma is an adaptation of [3, Lemma 12] to our needs. We establish the estimate $\mathfrak{A}(q) = O(q^{-2})$ to replace the cognate bound [3, equation (7.14)]. Once this is confirmed, all the conclusions drawn in the lemma follow just as in the aforementioned work [3], and thus we may omit the details. We immitate the argument on [3, page 890] to establish the aforementioned bound. Note first that we may suppose that whenever $(c, d) \in \mathbb{Z}^2 \setminus \{(0, 0)\}$, then the linear form $cL_1(\boldsymbol{\theta}) + dL_2(\boldsymbol{\theta})$ contains at least $s - 7$ non-zero coefficients. Let $u_j = (q, ca_j + db_j)$ and apply [13, Theorem 4.2] to infer that

$$T(q, c, d) \ll q^{-11/2}(u_1 \ldots u_{22})^{1/4}.$$

Here, we have used the prearrangement of indices and removed dependence on potential indices with $22 < j \le s$ by the use of trivial estimates. Recall now the multiplicities $r_1, \ldots, r_t$ associated to the equivalence classes of the forms Λ_j. Following the analysis on [3, page 890], one finds that there is a natural number Δ depending only on a_j, b_j $(1 \le j \le s)$ such that

$$\mathfrak{A}(q) \ll q^{-7/2} \sum_{\substack{v_1, \ldots, v_t \\ v_1 \ldots v_t | \Delta q}} v_1^{(r_1 - 4)/4} \ldots v_t^{(r_t - 4)/4}.$$

The upper bound $r_l \le 7$ therefore leads to the estimate $\mathfrak{A}(q) \ll q^{\varepsilon - 11/4}$, so that in view of our earlier comments, the proof of the lemma is complete. $\qquad\square$

Lemma 7.2. *Under the hypotheses of Theorem* 1.1, *the limit* $\mathfrak{J} = \lim_{X \to \infty} \mathfrak{J}(X)$ *exists, and one has* $\mathfrak{J} - \mathfrak{J}(X) \ll P^{s-8} X^{-1}$. *Moreover, when the pair of equations* (1.1) *has a non-singular real solution, one has* $\mathfrak{J} \gg P^{s-8}$.

Proof. Write $\widehat{\mathcal{B}}(X)$ for $\mathbb{R}^2 \setminus \mathcal{B}(X)$, and recall the prearrangement of indices introduced in §5. Then a direct modification of the argument on [3, page 891] confirms that, for a suitable positive number $\Theta = \Theta(\mathbf{a}, \mathbf{b})$, one has

$$\iint_{\widehat{\mathcal{B}}(X)} |v(\lambda_2) \ldots v(\lambda_{22})| \, d\xi \, d\zeta \ll P^{21} \iint_{\widehat{\mathcal{B}}(\Theta X)} (1 + P^4 |\xi|)^{-21/8} (1 + P^4 |\zeta|)^{-21/8} \, d\xi \, d\zeta.$$

By applying trivial bounds for $w(a_1 \xi)$ and the additional factors $v(\lambda_j)$ for $j > 22$, we therefore conclude that

$$\iint_{\mathbb{R}^2 \setminus \mathcal{B}(X)} V(\xi, \zeta) \, d\xi \, d\zeta \ll P^{s-21}(P^{13} X^{-1}) \ll P^{s-8} X^{-1}.$$

This bound replaces [3, equation (7.18)], and the lemma now follows in the same manner as Lemma 13 was proved in [3]. $\qquad\square$

The conclusions of Lemmata 7.1 and 7.2 now combine with the asymptotic formula (7.1) to deliver the relation

$$\iint_{\mathfrak{N}} g(\Lambda_1) H(\alpha, \beta) \, d\alpha \, d\beta = \rho \mathfrak{S} \mathfrak{J} + O(P^{s-8} Q^{-1}) \gg P^{s-8},$$

thereby confirming the lower bound (5.4). In view of the discussion concluding §5, the lower bound $\mathcal{N}(P) \gg P^{s-8}$ that establishes Theorem 1.1 now follows.

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