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Self-similar Lie algebras

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Abstract. We give a general definition of branched, self-similar Lie algebras, and show that important examples of Lie algebras fall into that class. We give sufficient conditions for a self-similar Lie algebra to be nil, and prove in this manner that the self-similar algebras associated with Grigorchuk's and Gupta–Sidki's torsion groups are nil as well as self-similar. We derive the same results for a class of examples constructed by Petrogradsky, Shestakov and Zelmanov.

Keywords. Groups acting on trees, Lie algebras, wreath products

1. Introduction

Since its origins, mankind has been divided into hunters and gatherers. This paper is resolutely of the latter kind, and brings together Caranti et al.'s Lie algebras of maximal class [9–12, 24], self-similar Lie algebras associated with self-similar groups from [4], the self-similar associative algebras from [3], and Petrogradsky et al.'s nil Lie algebras [34, 35, 40]. Contrary to tradition [1, Gen 4.8], we do not proclaim superiority of gatherers; yet we reprove, in what seems a more natural language, the main results of these last papers. In particular, we extend their criteria for growth (Propositions 2.17 and 3.12) and nillity (Corollary 2.9).

The fundamental notion we consider is that of a *self-similar algebra* (Definition 2.1). For $\mathscr{L}$ a Lie algebra and X a commutative algebra, whose Lie algebra of derivations is written $\mathfrak{Der}\, X$, their *wreath product* $\mathscr{L} \wr \mathfrak{Der}\, X$ is the Lie algebra $X \otimes \mathscr{L} \rtimes \mathfrak{Der}\, X$. A *self-similar Lie algebra* is then a Lie algebra endowed with a map $\psi : \mathscr{L} \to \mathscr{L} \rtimes \mathfrak{Der}\, X$.

Every Lie algebra is self-similar via the map $\psi(a) = 1 \otimes a$; *branched* self-similar Lie algebras, on the contrary, are algebras such that ψ is an isomorphism up to finite codimension (see Definition 3.3 for the precise definition). We offer "branchness" as the important unifying concept relating the examples mentioned above.

We show how, starting from a self-similar group such as the Grigorchuk and Gupta–Sidki group, we get natural examples of self-similar nil Lie algebras of polynomial growth. The main connections we develop are summarized as follows:

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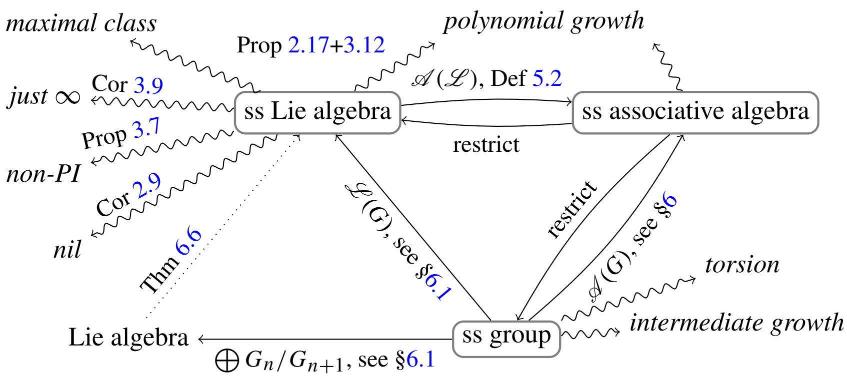

We give in §§2–3 a sufficient condition for a self-similar Lie algebra $\mathscr{L}$ to be nil, that is, for $\mathrm{ad}(x)$ to be a nilpotent endomorphism of $\mathscr{L}$ for all $x \in \mathscr{L}$. Our condition is then applied in §4 to known examples, and provides systematic proofs of their nillity. We also prove, by different means, the nillity of a Lie algebra associated with Grigorchuk's group (Theorem 4.12).

In particular, we show that the Lie algebras in [34, 40] are contained in finitely generated self-similar Lie algebras, a fact hinted at, but never explicitly stated or used in these papers. The self-similar Lie algebra we consider has more properties than the original one, e.g. it is just infinite.

Golod [15, 16] constructed infinite-dimensional, finitely generated nil associative algebras. These algebras have exponential growth—indeed, this is how they are proved to be infinite-dimensional—and are quite intractable. In searching for more examples, Small asked whether there existed such examples with finite Gelfand–Kirillov dimension (i.e., roughly speaking, of polynomial growth). That question was answered positively by Lenagan and Smoktunowicz [28].

The question may also be asked for Lie algebras; one then has the concrete constructions described in §4. Alas, none of their enveloping algebras seems to be nil, though they may be written as the sum of two nil subalgebras (Proposition 5.3); in particular, all their homogeneous elements are nil.

In §5 we construct a natural self-similar associative algebra from a self-similar Lie algebra. In §6 we construct a natural self-similar Lie algebra from a self-similar group acting "cyclically" on an alphabet of prime order. We relate in this manner the examples from §4 to the well-studied Grigorchuk [18] and Gupta–Sidki [21] groups:

Theorem 1.1. *The Lie algebra associated with the Grigorchuk group, after quotienting by its centre; the Lie algebra associated with the Gupta–Sidki group; and the extended Petrogradsky–Shestakov–Zelmanov Lie algebras enjoy the following properties:*

They are self-similar, nil, just infinite, not PI, and of finite Gelfand–Kirillov dimension.

Furthermore, the first one is of maximal class. Their descriptions as self-similar Lie algebras are given in §§4.2, 4.1 and 4.5 respectively.

We might venture the following

Conjecture 1.2. *If G is a torsion group, then its associated Lie algebra is nil.*

1.1. Preliminaries

All our algebras (Lie or associative) are over a commutative domain $\Bbbk$. If $\Bbbk$ is of positive characteristic p, a Lie algebra $\mathscr{L}$ over $\Bbbk$ may be *restricted*, in the sense that it admits a semilinear map $x \mapsto x^p$ satisfying the usual axioms of raising-to-the-pth-power, e.g. $(\xi a)^2 = \xi^2 a^2$ and $(a+b)^2 = a^2 + b^2 + [a,b]$ if $p = 2$. The centre of $\mathscr{L}$ is denoted by $\zeta(\mathscr{L})$. Note that if $\mathscr{L}$ is centreless, then the p-mapping is unique if it exists. Unless otherwise stated (see e.g. §4.3), we will assume that our algebras are restricted. We will also, by convention, say that algebras in characteristic 0 are restricted. See [23] or [22, Chapter V] for an introduction to restricted algebras.

If $\mathscr{L}, \mathscr{M}$ are Lie algebras, with $\mathscr{M}$ acting on $\mathscr{L}$ by right derivations, their *semidirect product* $\mathscr{L} \rtimes \mathscr{M}$ is $\mathscr{L} \oplus \mathscr{M}$ qua $\Bbbk$-module, with Lie bracket $[\ell_1 + m_1, \ell_2 + m_2] = [\ell_1, \ell_2] + [m_1, m_2] + \ell_1 \cdot m_2 - \ell_2 \cdot m_1$. In other words, the original Lie brackets of $\mathscr{L}$ and $\mathscr{M}$ are kept, and the bracket of $\mathscr{L}$ with $\mathscr{M}$ is given by the action.

The tensor algebra over a $\Bbbk$-module V is written $T(V)$. It is the free associative algebra generated by V, and is, qua $\Bbbk$-module, $\bigoplus_{n\geq 0} V^{\otimes n}$. Every Lie algebra has a *universal enveloping algebra*, unique up to isomorphism. In characteristic 0, or for unrestricted Lie algebras, it is the associative algebra

$$\mathbb{U}(\mathscr{L}) = T(\mathscr{L})/\langle a \otimes b - b \otimes a - [a,b] \text{ for all } a, b \in \mathscr{L}\rangle.$$

If however $\mathscr{L}$ is restricted, it is the associative algebra

$$\mathbb{U}(\mathscr{L}) = T(\mathscr{L})/\langle a \otimes b - b \otimes a - [a,b] \text{ for all } a, b \in \mathscr{L}, a^{\otimes p} - a^p \text{ for all } a \in \mathscr{L}\rangle.$$

No confusion should arise, because it is always the latter algebra that is meant in this text if $\Bbbk$ has positive characteristic.

Wreath products of Lie algebras have appeared in various places in the literature [9, 13, 24, 43, 44, 46]. We use them as a fundamental tool in describing and constructing Lie algebras; though they are essentially (at least, in the case of a wreath product with the trivial Lie algebra $\Bbbk$) equivalent to the inflation/deflation procedures of [9].

2. Self-similar Lie algebras

Just as self-similar sets contain many "shrunk" copies of themselves, a self-similar Lie algebra is a Lie algebra containing embedded, "infinitesimal" copies of itself:

Definition 2.1. Let X be a commutative ring with 1, and let $\mathfrak{Der}\, X$ denote the Lie algebra of derivations of X. A Lie algebra $\mathscr{L}$ is *self-similar* if it is endowed with a homomorphism

$$\psi : \mathscr{L} \to X \otimes \mathscr{L} \rtimes \mathfrak{Der}\, X =: \mathscr{L} \wr \mathfrak{Der}\, X,$$

in which the derivations act on $X \otimes \mathscr{L}$ by deriving the first coordinate.

If emphasis is needed, X is called the *alphabet* of $\mathscr{L}$, and ψ is its *self-similarity structure*.

We note that, under this definition, every Lie algebra is self-similar—though, probably, not interestingly so; indeed, the definition does not forbid $\psi = 0$. The condition below, on faithfulness of the action (1), will make it clear in which sense an example is interesting or not.

Running example. The following simple example is sufficiently rich to explain the main concepts, and will be used throughout this section. Consider $X = \mathbb{F}_p[x]/(x^p)$ and $\mathscr{L} = \mathbb{F}_p a$, the one-dimensional Lie algebra. Set $\psi(a) = x^{p-1} \otimes a + d/dx$.

If $\Bbbk$ has characteristic p, then $\mathscr{L}$ may be a *restricted* Lie algebra (see §1.1). Note that $\mathfrak{Der}\, X$ is naturally a restricted Lie algebra, and so is $X \otimes \mathscr{L}$ for the p-mapping $(x \otimes a)^p = x^p \otimes a^p$. A restricted algebra is *self-similar* if furthermore ψ preserves the p-mapping.

Running example. Consider the restricted Lie algebra $\mathscr{L}' = \bigoplus_{k \geq 0} \mathbb{F}_p a^{p^k}$ extending $\mathscr{L}$. Its Lie bracket is trivial (the algebra is abelian), and its p-mapping is $(a^{p^k})^p = a^{p^{k+1}}$. The homomorphism ψ is extended by $\psi(a^{p^{k+1}}) = 1 \otimes a^{p^k}$.

A self-similar Lie algebra $\mathscr{L}$ has a *natural action* on the direct sum $\bigoplus_{n \geq 0} X^{\otimes n}$, defined as follows: given $a \in \mathscr{L}$ and an elementary tensor $v = x_1 \otimes \cdots \otimes x_n \in X^{\otimes n}$, set $a \cdot v = 0$ if $n = 0$; otherwise, compute $\psi(a) = \sum y_i \otimes a_i + \delta \in \mathscr{L} \wr \mathfrak{Der}\, X$, and set

$$a \cdot v = \sum x_1 y_i \otimes (a_i \cdot x_2 \otimes \cdots \otimes x_n) + (\delta x_1) \otimes x_2 \otimes \cdots \otimes x_n. \tag{1}$$

Extend the action by linearity to $\bigoplus_{n \geq 0} X^{\otimes n}$. Note that this defines an action of $\mathscr{L}$ precisely because ψ is a homomorphism.

We insist that $\mathscr{L}$ does *not* act by derivations of $T(X)$ qua free associative algebra, but only by endomorphisms of the underlying $\Bbbk$-module.

Running example. We may view $X^{\otimes n}$ as truncated polynomials in n variables $x_0, \ldots, x_{n-1}$. The elementary tensor $x^{\alpha_0} \otimes \cdots \otimes x^{\alpha_{n-1}}$ is then written $x_0^{\alpha_0} \cdots x_{n-1}^{\alpha_{n-1}}$. The action of a is given by $a \cdot 1 = 0$ and

$$a \cdot (x_i^{\alpha_i} \cdots x_{n-1}^{\alpha_{n-1}}) = \alpha_i x_0^{p-1} \cdots x_{i-1}^{p-1} x_i^{\alpha_i - 1} x_{i+1}^{\alpha_{i+1}} \cdots x_{n-1}^{\alpha_{n-1}},$$

when $\alpha_i \neq 0$. In other words, if $\alpha_0 = \cdots = \alpha_{i-1} = 0 \neq \alpha_i$, then the action of a sets $\alpha_0, \ldots, \alpha_{i-1}$ to $p-1$ and decreases α_i by 1.

The element a^{p^k} acts similarly: the first $\alpha_0, \ldots, \alpha_{k-1}$ are ignored. If $\alpha_k = \cdots = \alpha_{i-1} = 0 \neq \alpha_i$, then $\alpha_k, \ldots, \alpha_{i-1}$ are set to $p-1$ and α_i is decreased by 1.

A self-similar Lie algebra is *faithful* if its natural action is faithful. From now on, we will tacitly assume that all our Lie algebras satisfy this condition.

There are natural embeddings $X^{\otimes n} \to X^{\otimes n+1}$, given by $v \mapsto v \otimes 1$; and for all $a \in \mathscr{L}$ we have

$$\begin{array}{ccc} X^{\otimes n} & \longrightarrow & X^{\otimes n+1} \\ {\scriptstyle a}\big\downarrow & & \big\downarrow{\scriptstyle a} \\ X^{\otimes n} & \longrightarrow & X^{\otimes n+1} \end{array} \tag{2}$$

We define $R(X) = \bigcup_{n\geq 0} X^{\otimes n}$ under these embeddings; or, what is the same, as the quotient $\bigoplus_{n\geq 0} X^{\otimes n}/\langle u\otimes 1 - u : u \in \bigoplus_{n\geq 0} X^{\otimes n}\rangle$. Note that $\mathscr{L}$ acts on $R(X)$; more precisely, the action of $\mathscr{L}$ on $R(X)$ restricts to the action of $\mathscr{L}$ on $X^{\otimes n}$ given in (1).

Running example. $R(X)$ is the ring of truncated polynomials in countably many variables $x_0, x_1, \dots$.

Self-similar Lie algebras may be *defined* by considering $\mathscr{F}$ a free Lie algebra, and $\psi : \mathscr{F} \to \mathscr{F} \wr \mathfrak{Der}\, X$ a homomorphism. The self-similar Lie algebra defined by these data is the quotient of $\mathscr{F}$ that acts faithfully on $R(X)$, namely, the quotient of $\mathscr{F}$ by the kernel of the action homomorphism $\mathscr{F} \to \mathfrak{Der}\, R(X)$. It is the largest quotient of $\mathscr{F}$ on which ψ induces an injective homomorphism.

We may iterate a self-similarity structure, and in this manner obtain a self-similarity structure with a larger alphabet. If in $\mathscr{L}\wr\mathfrak{Der}\, X$ we apply $1\otimes\psi$ to the "$X\otimes\mathscr{L}$" summand, we obtain a map to $X\otimes(X\otimes\mathscr{L}\rtimes\mathfrak{Der}\, X)\rtimes\mathfrak{Der}\, X$; now both $X\otimes\mathfrak{Der}\, X$ and $\mathfrak{Der}\, X$ are Lie subalgebras of $\mathfrak{Der}(X^{\otimes 2})$, so the map $(1\otimes\psi)\psi$ has range in $X^{\otimes 2}\otimes\mathscr{L}\rtimes\mathfrak{Der}(X^{\otimes 2})$. More generally, there is a map, which by abuse of notation we denote by ψ^n, from $\mathscr{L}$ to $X^{\otimes n}\otimes\mathscr{L}\rtimes\mathfrak{Der}(X^{\otimes n}) = \mathscr{L}\wr\mathfrak{Der}(X^{\otimes n})$.

2.1. Matrix recursions

We assume now that X is finite-dimensional, with basis $\{x_1, \dots, x_d\}$. For $x \in X$, consider the $d\times d$ matrix m_x describing multiplication by x on X. Similarly, for a derivation $\delta\in\mathfrak{Der}\, X$, consider the $d\times d$ matrix m_δ describing its action on X. Recall that $\mathbb{U}(\mathscr{L})$ denotes the universal enveloping algebra of $\mathscr{L}$. Consider the map

$$\psi' : \begin{cases} \mathscr{L} \to \mathrm{Mat}_{d\times d}(\mathscr{L}\oplus\Bbbk) \subset \mathrm{Mat}_{d\times d}(\mathbb{U}(\mathscr{L})),\\ a \mapsto \sum m_{y_i} a_i + m_\delta \text{ if } \psi(a) = \sum y_i\otimes a_i + \delta. \end{cases} \tag{3}$$

We will see in Proposition 5.1 that ψ' is a homomorphism, and extends to a homomorphism again written $\psi : \mathbb{U}(\mathscr{L}) \to \mathrm{Mat}_{d\times d}(\mathbb{U}(\mathscr{L}))$. We may therefore use this convenient matrix notation to define self-similar Lie algebras, and study their related enveloping algebras.

Running example. In the basis $\{x^i/i! \mid i = 0, \dots, p-1\}$ of divided powers, the matrix $\psi'(a)$ is the permutation matrix with 1's just above the diagonal and a in the lower left corner, and $\psi'(a^{p^{k+1}})$ is the scalar matrix with a^{p^k} on its diagonal:

$$a \mapsto \begin{pmatrix} 0 & 1 & \cdots & 0\\ \vdots & \ddots & \ddots & \vdots\\ \vdots & & \ddots & 1\\ -a & \cdots & \cdots & 0 \end{pmatrix}, \quad a^{p^{k+1}} \mapsto \begin{pmatrix} a^{p^k} & 0 & \cdots & 0\\ 0 & \ddots & \ddots & \vdots\\ \vdots & \ddots & \ddots & 0\\ 0 & \cdots & 0 & a^{p^k} \end{pmatrix}.$$

2.2. *Gradings*

Assume that X is graded, say by an abelian group Λ. Then, as a module, $T(X)$ is $\Lambda[\lambda]$-graded, where the degree of $x_1 \otimes \cdots \otimes x_n$ is $\sum_{i=1}^{n} \deg(x_i)\lambda^{i-1}$ for homogeneous $x_1, \dots, x_n$. Here λ is a formal parameter, called the *dilation* of the grading; though we sometimes force it to take a value in $\mathbb{R}$. Similarly, $R(X)$ is $\Lambda[\lambda]$-graded.

If $\mathscr{L}$ is both a graded Lie algebra and a self-similar Lie algebra, its grading and self-similarity structures are compatible if, for homogeneous $a \in \mathscr{L}$, one has $\deg(a) = \deg(y_i) + \lambda \deg(a_i) = \deg(\delta)$ for all i, where $\psi(a) = \sum y_i \otimes a_i + \delta$. In other words, ψ is a degree-preserving map.

We prefer to grade the ring X negatively, so that $\mathscr{L}$, acting by derivations, is graded in positive degree. This convention is of course arbitrary.

Running example. Set $\deg(a^{p^n}) = p^n$. This turns $\mathscr{L}$ and $\mathscr{L}'$ into self-similar graded algebras, if one sets $\lambda = p$. Indeed then $x_0^{\alpha_0} \cdots x_{n-1}^{\alpha_{n-1}}$ has degree $-\sum_{i=0}^{n-1} \alpha_i p^i$, and this degree is increased by p^k under the action of a^{p^k}.

2.3. *The full self-similar algebra*

Assume that X is finite-dimensional. Recall that every self-similar Lie algebra $\mathscr{L}$ with alphabet X admits an action on $\bigoplus_{n\ge 0} X^{\otimes n}$; namely, it admits for all $n \in \mathbb{N}$ a homomorphism $\mathscr{L} \to \mathfrak{Der}(X^{\otimes n})$, whose image actually lies in $X^{\otimes n-1} \otimes \mathfrak{Der}\, X$. Via the embedding (2), the algebra $\mathscr{L}$ acts on $R(X)$. On the other hand, $X^{\otimes n} \otimes \mathfrak{Der}\, X$ also acts on $R(X)$, by

$$(x_1 \otimes \cdots \otimes x_n \otimes \delta) \cdot y_1 \otimes \cdots \otimes y_m = x_1 y_1 \otimes \cdots x_n y_n \otimes \delta(y_{n+1}) \otimes y_{n+2} \otimes \cdots \otimes y_m,$$

in which $m > n$ is ensured by appending $1 \otimes \cdots \otimes 1$ if necessary to $y_1 \otimes \cdots \otimes y_m$.

We now define a self-similar algebra $\mathscr{W}(X)$ acting on $R(X)$, maximal in the sense that it contains all images of self-similar algebras as above. Qua $\mathbb{k}$-module,

$$\mathscr{W}(X) = \prod_{n=0}^{\infty} (X^{\otimes n} \otimes \mathfrak{Der}\, X). \tag{4}$$

Elements of $\mathscr{W}(X)$ are written $(a_0, a_1, \dots)$, with $a_n \in X^{\otimes n} \otimes \mathfrak{Der}\, X$.

The Lie bracket and p-mapping on $\mathscr{W}(X)$ can be described explicitly as follows. Consider $a, b \in \mathscr{W}(X)$, and assume first that all coordinates a_n, b_n are trivial except $a_m = x_1 \otimes \cdots \otimes x_m \otimes \delta$ and $b_n = y_1 \otimes \cdots \otimes y_n \otimes \epsilon$. Then all coordinates of $[a, b]$ are trivial except

$$[a,b]_{\max(m,n)} = \begin{cases} x_1 y_1 \otimes \cdots \otimes x_m y_m \otimes \delta y_{m+1} \otimes \cdots \otimes y_n \otimes \epsilon & \text{if } m < n, \\ x_1 y_1 \otimes \cdots \otimes x_m y_m \otimes [\delta, \epsilon] & \text{if } m = n, \\ -x_1 y_1 \otimes \cdots \otimes x_n y_n \otimes \epsilon x_{n+1} \otimes \cdots \otimes x_m \otimes \delta & \text{if } m > n, \end{cases} \tag{5}$$

and

$$a^p = x_1^p \otimes \cdots \otimes x_m^p \otimes \delta^p.$$

Since coordinate m of $[a, b]$ only depends on coordinates $\leq m$ of a and b, this definition extends to the infinite product (4). Likewise, for any $a \in \mathscr{W}(X)$ and any $y \in X^{\otimes n} \subset R(X)$, at most one coordinate of a acts non-trivially on y, so $\mathscr{W}(X)$ acts on $R(X)$.

The algebra $\mathscr{W}(X)$ is (tautologically) self-similar. Indeed, choose a basis $\{x_1, \dots, x_d\}$ of X. Given $(a_0, a_1, \dots) \in \mathscr{W}(X)$, write each a_{n+1} as $\sum_{i=1}^d x_i \otimes b_{n,i}$ for some $b_{n,i} \in \mathscr{W}(X)$; then the self-similarity structure of $\mathscr{W}(X)$ is defined by

$$\psi(a_0, a_1, \dots) = \sum_{i=1}^{d} x_i \otimes (b_{0,i}, b_{1,i}, \dots) + a_0, \tag{6}$$

or more compactly $\psi(a_0, a_1, \dots) = (a_1, a_2, \dots) + a_0$ with, on the right-hand side of the equation, a_{i+1} now seen as an element of $X \otimes (X^{\otimes i} \otimes \mathfrak{Der}\, X) \leq X \otimes \mathscr{W}(X)$. Note that (6) is a bijection. The algebra $\mathscr{W}(X)$ is maximal in the sense that every self-similar Lie algebra with alphabet X is a subalgebra of $\mathscr{W}(X)$. Also note that $\mathscr{W}(X)$ is restricted.

We note that if X is a Λ-graded ring, then $\mathscr{W}(X)$ is a $\Lambda[\lambda]$-graded self-similar Lie algebra; for homogeneous $x_1, \dots, x_n \in X$ and $\delta \in \mathfrak{Der}\, X$, we set

$$\deg(x_1 \otimes \cdots \otimes x_n \otimes \delta) = \sum_{i=1}^{n} \deg(x_i)\lambda^{i-1} + \lambda^n \deg(\delta).$$

We shall consider here subalgebras of $\mathscr{W}(X)$ that satisfy a finiteness condition. The most important is the following:

Definition 2.2. An element $a \in \mathscr{W}(X)$ is *finite state* if there exists a finite-dimensional subspace S of $\mathscr{W}(X)$ containing a such that the self-similarity structure $\psi : \mathscr{W}(X) \to \mathscr{W}(X) \wr \mathfrak{Der}\, X$ restricts to a map $S \to X \otimes S \oplus \mathfrak{Der}\, X$.

More generally, define a self-map $\widehat{\psi}$ on subspaces of $\mathscr{W}(X)$ by

$$\widehat{\psi}(V) = \bigcap \{W \mid W \leq \mathscr{W}(X) \text{ with } \psi(V) \leq X \otimes W \oplus \mathfrak{Der}\, X\}.$$

Then $a \in \mathscr{W}(X)$ is finite state if and only if $\sum_{n\geq 0} \widehat{\psi}^n(\Bbbk a)$ is finite-dimensional.

If X is finite-dimensional, a finite state element may be described by a finite amount of data in $\Bbbk$, as follows. Choose a basis (e_i) of S, and write a as well as the coordinates of $\psi(e_i)$ in that basis, for all i.

Running example. The element a is finite state: take $S = \mathscr{L}$, which is 1-dimensional. More generally, a^{p^k} is finite state: take $S = \langle a^{p^j} : 0 \leq j \leq k \rangle$.

2.4. *Hausdorff dimension*

Consider a self-similar Lie algebra $\mathscr{L}$ with self-similarity structure $\psi : \mathscr{L} \to \mathscr{L} \wr \mathfrak{Der}\, X$. As noted in §2.3, $\mathscr{L}$ is a subalgebra of $\mathscr{W}(X)$; and both act on $X^{\otimes n}$ for all $n \in \mathbb{N}$.

We wish to measure "how much of $\mathscr{W}(X)$" is "filled in" by $\mathscr{L}$. We essentially copy, and translate to Lie algebras, the definitions from [3, §3.2].

Let $\mathscr{L}_n$, respectively $\mathscr{W}(X)_n$, denote the image of $\mathscr{L}$, respectively $\mathscr{W}(X)$, in $\mathfrak{Der}(X^{\otimes n})$. We compute

$$\dim \mathscr{W}(X)_n = \sum_{i=0}^{n-1}(\dim X)^i \dim \mathfrak{Der}\, X = \frac{(\dim X)^n - 1}{\dim X - 1} \dim \mathfrak{Der}\, X,$$

and define the *Hausdorff dimension* of $\mathscr{L}$ by

$$\mathrm{Hdim}(\mathscr{L}) = \liminf_{n\to\infty} \frac{\dim \mathscr{L}_n}{\dim \mathscr{W}(X)_n} = \liminf_{n\to\infty} \frac{\dim \mathscr{L}_n}{(\dim X)^n - 1} \frac{\dim X - 1}{\dim \mathfrak{Der}\, X}.$$

Furthermore, there may exist a subalgebra $\mathscr{P}$ of $\mathfrak{Der}\, X$ such that $\psi : \mathscr{L} \to \mathscr{L} \wr \mathscr{P}$; our typical examples will have the form $X = \Bbbk[x]/(x^d)$, and $\mathscr{P} = \Bbbk d/dx$ a one-dimensional Lie algebra; in that case, $\mathscr{L}$ is called an *algebra of special derivations*. We then define the *relative Hausdorff dimension* of $\mathscr{L}$ by

$$\mathrm{Hdim}_{\mathscr{P}}(\mathscr{L}) = \liminf_{n\to\infty} \frac{\dim \mathscr{L}_n}{(\dim X)^n - 1} \frac{\dim X - 1}{\dim \mathscr{P}}.$$

2.5. *Bounded Lie algebras*

We now define a subalgebra of $\mathscr{W}(X)$, important because it contains many interesting examples, yet gives control on the nillity of the algebra's p-mapping. We suppose throughout this subsection that $\Bbbk$ is a ring of characteristic p, so that $\mathscr{W}(X)$ is a restricted Lie algebra.

We suppose that X is an *augmented* algebra: there is a homomorphism $\varepsilon : X \to \Bbbk$ with kernel ϖ. This gives a splitting $X^{\otimes n+1} \to X^{\otimes n}$ of (2), given by $v \otimes x_{n+1} \mapsto \varepsilon(x_{n+1})v$. The algebra $X^{\otimes n}$ also admits an augmentation ideal,

$$\varpi_n = \ker(\varepsilon \otimes \cdots \otimes \varepsilon) = \sum X \otimes \cdots \otimes \varpi \otimes \cdots \otimes X.$$

The union of the ϖ_n defines an augmentation ideal, again written ϖ, in $R(X)$.

There is a natural action of $R(X)$ on $\mathscr{W}(X)$: given $a = x_1 \otimes \cdots \otimes x_m \in R(X)$ and $b = y_1 \otimes \cdots \otimes y_n \otimes \delta \in \mathscr{W}(X)$, first replace a by $a \otimes 1 \otimes \cdots \otimes 1$ with enough 1's so that $m \geq n$; then set

$$a \cdot b = \varepsilon(x_{n+1}) \cdots \varepsilon(x_m) x_1 y_1 \otimes \cdots \otimes x_n y_n \otimes \delta.$$

Definition 2.3. An element $a \in \mathscr{W}(X)$ is *bounded* if there exists a constant m such that $\varpi^m a = 0$. Writing $a = (a_0, a_1, \dots)$, this means $\varpi^m a_i = 0$ for all i. The set of bounded elements is written $M(X)$.

The *bounded norm* $\|a\|$ of a is then the minimal such m.

Running example. The alphabet X is augmented, with $\varpi = xX$ the set of polynomials without constant term. The element a is bounded, of bounded norm 1. Indeed $a = (x_0^{p-1} \cdots x_i^{p-1} \otimes d/dx)_{i\geq 0}$, so $x_i a = 0$ for all $i \in \mathbb{N}$, so $\varpi a = 0$. More generally, a^{p^k} is bounded of norm $(p-1)k + 1$, since $a^{p^k} = (0, \dots, 0, x_k^{p-1} \cdots x_i^{p-1} \otimes d/dx)_{i\geq 0}$, and every monomial in $\varpi^{(p-1)k+1}$ contains a non-zero power of some x_j with $j \geq k$.

The following statement is inspired by [40, Lemma 1].

Lemma 2.4. *The set $M = M(X)$ of bounded elements forms a restricted Lie subalgebra of $\mathscr{W}(X)$. More precisely, the bounded norm of $[a, b]$ is at most* $\max\{\|a\|, \|b\|\} + 1$*, and* $\|a^p\| \le \|a\| + p - 1$.

Proof. Consider first $f \in X^{\otimes n}$ and $\delta \in \mathfrak{Der}\, X$. If $\varpi^m f = 0$, then $\varpi^{m+1}\delta f \subseteq \delta(\varpi^{m+1} f) - \delta(\varpi^{m+1}) f = 0$ since $\delta(\varpi^{m+1}) \subseteq \varpi^m$. Consider now elementary tensors $a = f \otimes \delta \in M$ of bounded norm m and $b = g \otimes \epsilon \in M$ of bounded norm n. It follows from (5) that the bounded norm of $[a, b]$ is at most $\max\{m, n\} + 1$. The same estimate then holds for arbitrary $a, b \in M$, by linearity.

Consider next $a = (a_0, a_1, \dots) \in M$, of bounded norm m. Write each $a_i = \sum f_i \otimes \delta_i$. Then $a_i^p = \sum f_i^p \otimes \delta_i^p$+commutators of weight p; now $\varpi^m f_i^p \otimes \delta_i^p = 0$ and, by the first paragraph, commutators are annihilated by ϖ^{m+p-1}. □

We now suppose for simplicity that the alphabet has the form $X = \Bbbk[x]/(x^p)$. Its augmentation ideal is $\varpi = xX$, and satisfies $\varpi^p = 0$. We seek conditions on elements $a \in M(X)$ that ensure that they are *nil*, that is, there exists $n \in \mathbb{N}$ such that $a^{p^n} = (((a^p)^p)\cdots)^p = 0$. The standard derivation d/dx of X is written ∂_x.

Definition 2.5. An element $a \in \mathscr{W}(X)$ is *ℓ-evanescent*, for $\ell \in \mathbb{N}$, if when we write $a = (a_0, a_1, \dots)$, each a_i has the form $\sum b_i \otimes c_i \otimes \partial_x$ with $b_i \in X^{\otimes \max\{i-\ell,0\}}$, $c_i \in X^{\otimes \min\{i,\ell\}}$, and $\deg(c_i) < (p-1)\ell$.

An element is *evanescent* if it is ℓ-evanescent for some $\ell \in \mathbb{N}$.

In words, a is ℓ-evanescent if, in all coordinates a_i of a, the derivation is ∂_x and the maximal degree is *never* reached in each of the last ℓ alphabet variables.

Running example. The element a is not evanescent, but the element b defined by $\psi(b) = x^{p-1} \otimes b + 1 \otimes d/dx$ is 1-evanescent. Indeed, $b = (x_0^{p-1} \cdots x_{i-1}^{p-1} x_i^0 \otimes \partial_x)_{i \ge 0}$.

Lemma 2.6. *The set of ℓ-evanescent elements forms a restricted Lie subalgebra of $\mathscr{W}(X)$.*

Proof. Clearly linear combinations of ℓ-evanescent elements are ℓ-evanescent. Consider then ℓ-evanescent elements $a, b \in \mathscr{W}(X)$, and without loss of generality assume $a = x_1 \otimes \cdots \otimes x_m \otimes \partial_x$, $b = y_1 \otimes \cdots \otimes y_n \otimes \partial_x$ and $m \le n$. If $m = n$, then $[a, b] = 0$, while if $m < n$ then $[a, b] = x_1 y_1 \otimes \cdots \otimes \partial_x y_{m+1} \otimes \cdots \otimes y_n \otimes \partial_x$. If $n - m > \ell$ then there is nothing to do, while if $n - m \le \ell$ then $\deg(\delta y_{m+1}) < p - 1$, so the total degree in the last ℓ alphabet variables of $[a, b]$ is $< (p-1)\ell$.

Finally, the p-mapping is trivial on elementary tensors $x_1 \otimes \cdots \otimes x_n \otimes \partial_x$ because $\partial_x^p = 0$ in $\mathfrak{Der}\, X$. □

The following statement is inspired by [40, Lemma 2].

Lemma 2.7. *Let $a \in M(X)$ be evanescent. Then there exists $s \ge 1$ such that $a^{p^s} \in \varpi^2 \mathscr{W}(X)$.*

Proof. We put an $\mathbb{R}$-grading on $\mathscr{W}(X)$, by using the natural $\{1-p,\dots,0\}$-grading of X and choosing for λ the largest positive root of $f(\lambda)=\lambda^{\ell+1}-p\lambda^{\ell}+\lambda-1$. We note that $f(+\infty)=+\infty$ and $f(0)=-1<0$, so f has one or three positive roots. Next, $f'(0)>0$ and f' has at most two positive roots in $\mathbb{R}_+$, while $f(1)=1-p\le f(0)$, so f has at most one extremum in $(1,\infty)$. Finally, $f(p)=p-1>0$, so f has a unique zero in $(1,p)$ and we deduce $\lambda\in(1,p)$.

Consider a homogeneous component $h=x_1\otimes\cdots\otimes x_n\otimes\partial_x$ of a. Because $\deg(\partial_x)=1$ and $-\deg(x_i)\le p-1$ for all i and $-\deg(x_m)\le p-2$ for some $m\in\{n-\ell+1,\dots,n\}$, we have

$$\begin{aligned}\deg(h)&\ge\lambda^n-(p-1)(1+\lambda+\cdots+\lambda^{n-1})+\lambda^m\\&\ge\lambda^n-(p-1)\frac{\lambda^n-1}{\lambda-1}+\lambda^{n-\ell}=\frac{\lambda^{n-\ell}}{\lambda-1}f(\lambda)+\frac{p-1}{\lambda-1}=\frac{p-1}{\lambda-1}>1;\end{aligned}$$

so every homogeneous component of a^{p^s} has degree $\ge p^s$.

We now use the assumption $a\in M(X)$, say $\|a\|=m$. Then $\|a^{p^s}\|\le m+(p-1)s$, by Lemma 2.4. Write $a^{p^s}=(b_0,b_1,\dots)\in\mathscr{W}(X)$; then $\varpi_i^{m+(p-1)s}b_i=0$ for all $i\ge 0$.

Let $j\le i$, and assume that b_i does not belong to $\varpi_j^2X^{\otimes i}\otimes\mathfrak{Der}\,X$. Then $\varpi_i^{(p-1)j-1}b_i\neq 0$, and this can only happen if $(p-1)j-1<m+(p-1)s$. In other words, for every $s\in\mathbb{N}$ there exists $j\in\mathbb{N}$ such that $b_i\in\varpi_j^2\otimes X^{\otimes i-j}$ for all $i\ge j$.

Consider then the b_i with $i<j$. The degree of a homogeneous component in such a b_i is at most λ^j; on the other hand, since it is a homogeneous component of a^{p^s}, it has degree at least p^s. Now the inequalities

$$p^s\le\lambda^j,\qquad(p-1)j-1<m+(p-1)s$$

cannot simultaneously be satisfied for arbitrarily large s, because $\lambda<p$. It follows that, at least for s large enough, $b_i=0$ for all $i<j$, and therefore $a^{p^s}\in\varpi_j^2M(X)$. □

To make this text self-contained, we reproduce almost *verbatim* the proof of the following statement by Shestakov and Zelmanov:

Lemma 2.8 ([40, Lemma 5]). *Assume that $\varpi\subset X$ is nilpotent. Then the associative subalgebra of* $\operatorname{End}_{\Bbbk}R(X)$ *generated by $\varpi^2\mathscr{W}(X)$ is locally nilpotent.*

Proof. By assumption $X^d=0$ for some $d\in\mathbb{N}$. Consider a finite collection of elements $a_i=x_i'x_i''b_i\in\varpi^2\mathscr{W}(X)$ for $i\in\{1,\dots,r\}$, with $x_i',x_i''\in\varpi$ and $b_i\in\mathscr{W}(X)$. Let A be the subalgebra of ϖ generated by $x_1',x_1'',\dots,x_r',x_r''$. From $(x_i')^d=(x_i'')^d=0$ follows $A^s=0$ with $s=2r(d-1)+1$. Now $a_{i_1}\cdots a_{i_s}=\sum y_1\cdots y_{2s}d_{j_1}\cdots d_{j_q}$, with $q\le s$ and the y_j obtained from the x_i',x_i'' by $(q-s)$-fold application of the derivations b_k. Therefore $2s-q\ge s$ of the y_j belong to $\{x_1',x_1'',\dots,x_r',x_r''\}$, so $a_{i_1}\cdots a_{i_s}=0$. □

Corollary 2.9. *If $\mathscr{L}$ is a subalgebra of $\mathscr{W}(X)$ that is generated by bounded, ℓ-evanescent elements for some $\ell\in\mathbb{N}$, then $\mathscr{L}$ is nil.*

Proof. It follows from Lemmata 2.4 and 2.6 that every element $a\in\mathscr{L}$ is bounded and ℓ-evanescent; and then from Lemmata 2.7 and 2.8 that a is nil. □

2.6. Growth and contraction

Let $\mathscr{A}$ denote an algebra, not necessarily associative. For a finite-dimensional subspace $S \leq \mathscr{A}$, let S^n denote the span in $\mathscr{A}$ of n-fold products of elements of S; if $p \mid n$ and $\mathscr{A}$ is a restricted Lie algebra in characteristic p, then S^n also contains the pth powers of elements of $S^{n/p}$. Define

$$\mathrm{GKdim}(\mathscr{A}, S) = \limsup \frac{\log \dim S^n}{\log n}, \qquad \mathrm{GKdim}(\mathscr{A}) = \sup_{S \leq \mathscr{A}} \mathrm{GKdim}(\mathscr{A}, S),$$

the (upper) *Gelfand–Kirillov dimension* of $\mathscr{A}$. Note that if $\mathscr{A}$ is generated by the finite-dimensional subspace S, then $\mathrm{GKdim}(\mathscr{A}) = \mathrm{GKdim}(\mathscr{A}, S)$. We give here conditions for a self-similar Lie algebra $\mathscr{L}$ to have finite Gelfand–Kirillov dimension.

Definition 2.10. Let $\mathscr{L}$ be a self-similar Lie algebra with self-similarity structure $\psi : \mathscr{L} \to \mathscr{L} \wr \mathfrak{Der}\, X$. It is *contracting* if there exists a finite-dimensional subspace $N \leq \mathscr{L}$ with the following property: for every $a \in \mathscr{L}$, there exists $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$, we have $\psi^n(a) \in X^{\otimes n} \otimes N \oplus \mathfrak{Der}(X^{\otimes n})$.

The minimal such N, if it exists, is called the *nucleus* of $\mathscr{L}$.

In words, the nucleus is the minimal subspace of $\mathscr{L}$ such that, for every $a \in \mathscr{L}$, if one applies often enough the map ψ to it and its coordinates (discarding the term in $\mathfrak{Der}\, X$), one obtains only elements of N.

Note that elements of a contracting self-similar algebra are finite state. The following test is useful in practice to prove that an algebra is contracting, and leads to a simple algorithm:

Lemma 2.11. *Let the self-similar Lie algebra $\mathscr{L}$ be generated by the finite-dimensional subspace S, and consider a finite-dimensional subspace $N \leq \mathscr{L}$. Then N contains the nucleus of $\mathscr{L}$ if and only if there exist $m_0, n_0 \in \mathbb{N}$ such that $\psi^m(S + N + [N, S]) \leq X^{\otimes m} \otimes N \oplus \mathfrak{Der}(X^{\otimes m})$ and (for restricted algebras) $\psi^n((N + S)^p) \leq X^{\otimes n} \otimes N \oplus \mathfrak{Der}(X^{\otimes n})$ for all $m \geq m_0$ and $n \geq n_0$.*

Proof. Note first that if N contains the nucleus N_0, then the coordinates of $\psi^m([N + S, S])$ and $\psi^n((N + S)^p)$ will be contained in N_0, so *a fortiori* in N for all sufficiently large m, n.

Conversely, consider $a \in \mathscr{L}$, say a product of ℓ elements of S. Then applying $\ell - 1$ times the map ψ^m, we get $\psi^{m(\ell-1)}(a) \in X^{\otimes m(\ell-1)} \otimes N + \mathfrak{Der}(X^{\otimes m(\ell-1)})$, so N contains the nucleus. □

The following algorithm computes the nucleus of a self-similar algebra in case it is finite-dimensional.

Input: A generating set S
Output: The nucleus N
$N \leftarrow 0$;
repeat
  $N' \leftarrow N$;
  $B \leftarrow N + S + [N, S]$;
  $N \leftarrow \bigcap_{i \geq 0} \sum_{j \geq i} \widehat{\psi}^j(B)$;
until $N = N'$;

Definition 2.12. Let $\mathscr{L}$ be a self-similar algebra; assume that the alphabet X admits an augmentation $\varepsilon : X \to \Bbbk$, and denote by π the projection $\mathscr{L} \to \mathfrak{Der}\, X$. We say $\mathscr{L}$ is *recurrent* if the $\Bbbk$-linear map $(\varepsilon \otimes 1)\psi : \ker(\pi) \to \mathscr{L}$ is onto.

Lemma 2.13. *Let $\mathscr{L}$ be a finitely generated, contracting and recurrent self-similar Lie algebra. Then $\mathscr{L}$ is generated by its nucleus.*

Proof. Let S be a generating finite-dimensional subspace, and let N denote the nucleus of $\mathscr{L}$. Because S is finite-dimensional, there exists $n \in \mathbb{N}$ such that $\psi^n(S) \le X^{\otimes n} \otimes N \oplus \mathfrak{Der}(X^{\otimes n})$. Let $\mathscr{M}$ denote the subalgebra generated by N; then, for every $a \in \mathscr{L}$, we have $\psi^n(a) \in \mathscr{M} \wr \mathfrak{Der}(X^{\otimes n})$; so, because $\mathscr{L}$ is recurrent, $\mathscr{M} = \mathscr{L}$. □

Lemma 2.14. *Let $\mathscr{L}$ be a contracting, finitely generated self-similar Lie algebra, with X finite-dimensional. Then there exists a finite-dimensional generating subspace N of $\mathscr{L}$ such that*

$$\mathscr{L} \le \sum_{n\ge 0} X^{\otimes n} \otimes N.$$

Proof. Let N_0 be the nucleus of $\mathscr{L}$, and let S generate $\mathscr{L}$. Enlarge S, keeping it finite-dimensional, so that $\pi(N) = \pi(\mathscr{L}) \le \mathfrak{Der}\, X$. Set finally $N = N_0 + \sum_{n\ge 0} \widehat{\psi}(S)$.

By the definition of nucleus, $\mathscr{L}$ is contained in $\sum_{n\ge 0} X^{\otimes n} \otimes N \oplus \mathfrak{Der}(X^{\otimes n})$. Now using the fact that $\pi(N) = \pi(\mathscr{L})$, we can eliminate the $\mathfrak{Der}(X^{\otimes n})$ terms while still staying in $X^{\otimes n} \otimes N$. □

Lemma 2.15. *Let $\mathscr{L}$ be an $\mathbb{R}_+$-graded self-similar Lie algebra, with dilation $\lambda > 1$, and generated by finitely many elements, all of positive degree. Let X have bounded degree. Then $\mathscr{L}$ is contracting.*

Proof. Suppose that $\deg(x) > -K$ for all homogeneous $x \in X$. Consider a homogeneous $a \in \mathscr{L}$, and write $\phi(a) = \sum y_i \otimes a_i + \delta$; then $0 < \deg(a_i) < (\deg(a) + K)/\lambda$. Let N denote the linear span of all elements of $\mathscr{L}$ of degree $\le K/(\lambda - 1)$; then N is finite-dimensional and contains the nucleus. □

Lemma 2.16. *Let $\mathscr{L}$ be a contracting, $\mathbb{R}_+$-graded self-similar Lie algebra, with dilation λ, and let N be as in Lemma 2.14. Consider a homogeneous $a \in \mathscr{L}$ with $\deg(a) = d$. Then $a \in \sum_{j=0}^{n} X^{\otimes j} \otimes N$ with*

$$n \ge \log(d/m)/\log|\lambda| \quad \text{where} \quad m = \max_{n\in N} \deg(n). \tag{7}$$

If furthermore $\lambda > 1$ and $\mathscr{L}$ is generated by finitely many positive-degree elements, then there exists $\epsilon > 0$ such that every $a \in \mathscr{L}$ satisfies

$$n \le \log(d/\epsilon)/\log\lambda. \tag{8}$$

Proof. Consider $a \in X^{\otimes n} \otimes N \cap \mathscr{L}$. Then $\deg(a) \le \lambda^n \max\deg(N)$, and this proves (7).

Now, if $\mathscr{L}$ is generated by finitely many positive-degree elements, then there exists $m \in \mathbb{N}$ such that all elements of $X^{\otimes m} \otimes N \cap \mathscr{L}$ have degree $> -\min_{x\in X}\deg(x)/(\lambda - 1)$.

Let then $\epsilon > 0$ be such that all these elements have degree $\geq B := \lambda^m\epsilon - \min_{x\in X}\deg(x)/(\lambda-1)$.

We return to our a, which we write as $a = \sum f \otimes b$ with $f \in X^{\otimes m-n}$ and $b \in X^{\otimes m} \cap \mathscr{L}$. Then every homogeneous summand of b has degree at least B, while every homogeneous summand of f has degree at least $\min_{x\in X}\deg(x)(1+\lambda+\cdots+\lambda^{m-n-1})$; so every homogeneous summand of a has degree at least $\epsilon\lambda^n$. □

Proposition 2.17. *Let $\mathscr{L}$ be an $\mathbb{R}_+$-graded self-similar Lie algebra, with dilation $\lambda > 1$, that is generated by finitely many positive-degree elements. Then $\mathscr{L}$ has finite Gelfand–Kirillov dimension; more precisely,*

$$\mathrm{GKdim}(\mathscr{L}) \leq \frac{\log(\dim X)}{\log\lambda}.$$

Proof. Let $\mathscr{L}_d$ denote the span of homogeneous elements in $\mathscr{L}$ of degree $\leq d$. Consider $a \in \mathscr{L}_d$, and, up to expressing a as a sum, assume $a = f \otimes b$ with $f \in X^{\otimes n}$ and $b \in N$. By (8) we have $n \leq \log_\lambda(d/\epsilon)$, so

$$\dim\mathscr{L}_d \leq \sum_{j=0}^{n}(\dim X)^j \dim N \precsim d^{\log(\dim X)/\log\lambda}. \qquad \square$$

3. (Weakly) branched Lie algebras

We will concentrate here on some conditions on a self-similar Lie algebra that have consequences on its algebraic structure, and in particular on its possible quotients. We impose, in this section, restrictions that will be satisfied by all our examples: the alphabet X has the form $\Bbbk[x]/(x^d)$, and the image of $\mathscr{L}$ in $\mathfrak{Der}\,X$ is precisely $\Bbbk\partial_x$. Let $\theta = x^{d-1}$ denote the top-degree element of X.

We consider self-similar Lie algebras $\mathscr{L}$ with self-similarity structure $\psi : \mathscr{L} \to \mathscr{L} \wr \Bbbk\partial_x$. Let $\pi : \mathscr{L} \to \Bbbk\partial_x$ denote the natural projection. We recall that $\mathscr{L}$ acts on $X^{\otimes n}$ for all n. We denote by $\mathbb{U}(\mathscr{L})$ the universal enveloping algebra of $\mathscr{L}$; then $\mathbb{U}(\mathscr{L})$ also acts on $X^{\otimes n}$, and acts on $\mathscr{L}$ by derivations. For $u = u_1\cdots u_n \in \mathbb{U}(\mathscr{L})$ and $a \in \mathscr{L}$, we write $[[u,a]] = [u_1,[u_2,\dots,[u_n,a]\cdots]]$ for this action.

We also identify $\mathscr{L}$ with $\psi(\mathscr{L})$, so as to write, e.g., "$\theta \otimes a \in \mathscr{L}$" when we mean "$\theta \otimes a \in \psi(\mathscr{L})$".

Definition 3.1. The self-similar algebra $\mathscr{L}$ is *transitive* if for all $v \in X^{\otimes n}$ there exists $u \in \mathbb{U}(\mathscr{L})$ with $u \cdot (\theta^{\otimes n}) = v$.

We note immediately that if $\mathscr{L}$ is transitive, then it is infinite-dimensional.

Lemma 3.2. *If $\mathbb{U}(\mathscr{L}) \cdot \theta = X$ and $\mathscr{L}$ is recurrent, then $\mathscr{L}$ is transitive.*

Proof. We proceed by induction on n, the case $n = 0$ being obvious. Write $M = \mathbb{U}(\mathscr{L}) \cdot \theta^{\otimes n+1}$. Because $\mathscr{L}$ is recurrent and by the inductive hypothesis, $\mathbb{U}(\ker\pi) \cdot \theta^{\otimes n+1} = \theta \otimes X^{\otimes n}$. Then, by hypothesis, there exists in $\mathscr{L}$ an element a of the form $\partial_x + \sum x_i \otimes a_i$;

so $a\cdot\theta^{\otimes n+1} \in x^{d-2}\otimes\theta^{\otimes n}+\theta\otimes X^{\otimes n}$. Because M contains $\theta\otimes X^{\otimes n}$, it also contains $x^{d-2}\otimes \theta^{\otimes n}$, and again acting with $\mathbb{U}(\ker\pi)$ shows that it contains $x^{d-2}\otimes X^{\otimes n}$. Continuing in this manner, we conclude that M contains $x^i \otimes X^{\otimes n}$ for all i, and therefore equals $X^{\otimes n+1}$. □

Definition 3.3. Let $\mathscr{L}$ be a recurrent, transitive, self-similar Lie algebra. It is

- *weakly branched* if for every $n \in \mathbb{N}$ there exists a non-zero $a \in \mathscr{W}(X)$ such that $\theta^{\otimes n} \otimes a \in \mathscr{L}$;
- *branched* if for every $n \in \mathbb{N}$ the ideal K_n generated by all $\theta^{\otimes n} \otimes a \in \mathscr{L}$ has finite codimension in $\mathscr{L}$;
- *regularly weakly branched* if there exists a non-trivial ideal $\mathscr{K} \triangleleft \mathscr{L}$ such that $X \otimes \mathscr{K} \le \psi(\mathscr{K})$;
- *regularly branched* if furthermore there exists such a $\mathscr{K}$ of finite codimension in $\mathscr{L}$.

In the last two cases, we say that $\mathscr{L}$ is regularly [weakly] branched *over* $\mathscr{K}$.

The following immediately follows from the definitions:

Lemma 3.4. *"Regularly branched" implies both "regularly weakly branched" and "branched", and each of these implies "weakly branched".*

Note, as a partial converse, that if $\mathscr{L}$ is weakly branched, then

$$\{a \in \mathscr{L} \mid \theta^{\otimes n} \otimes a \le \mathscr{L}\} = \{a \in \mathscr{L} \mid X^{\otimes n} \otimes a \le \mathscr{L}\} =: \mathscr{K}_n$$

is a non-trivial ideal in $\mathscr{L}$:

Lemma 3.5. *If $\mathscr{L}$ is weakly branched, then for every $v \in X^{\otimes n}$ there exists a non-zero $a \in \mathscr{W}(X)$ with $v \otimes a \in \mathscr{L}$.*

Proof. Let $0 \neq a \in \mathscr{W}(X)$ be such that $\theta^{\otimes n} \otimes a \in \mathscr{L}$; let $\mathscr{K}_n$ denote the ideal generated by a. Because $\mathscr{L}$ is recurrent, $\theta^{\otimes n} \otimes c \in \mathscr{L}$ for all $c \in \mathscr{K}$. Because $\mathscr{L}$ is transitive, there exists $u \in \mathbb{U}(\mathscr{L})$ with $u \cdot \theta^{\otimes n} = v$. Consider then $[[u, \theta^{\otimes n} \otimes a]]$. It is of the form $v \otimes a + \sum v' \otimes a'$ for some $a' \in \mathscr{K}$ and $v' > v$ in reverse lexicographic ordering. By induction on v in that ordering, $v \otimes a$ belongs to $\mathscr{L}$. □

The algebra $\mathscr{W}(X)$ itself is branched; indeed, as was noted above, (6) is a bijection. We remark in passing that if, more generally, $\mathscr{L}$ is recurrent and transitive and ψ has finite cokernel, then $\mathscr{L}$ is branched.

We are now ready to deduce some structural properties of (weakly) branched Lie algebras:

Proposition 3.6. *Let $\mathscr{L}$ be a weakly branched Lie algebra. Then the centralizer of $\mathscr{L}$ in $\mathscr{W}(X)$ is trivial. In particular, $\mathscr{L}$ has trivial centre.*

Proof. Consider a non-zero $a \in \mathscr{W}(X)$. There then exists $v \in X^{\otimes n}$ such that $a \cdot v \neq 0$; suppose that n is minimal with that property. Let $i \in \{0, \dots, d-1\}$ be maximal such that $1^{\otimes n-1} \otimes \varpi^i(a \cdot v) \neq 0$. Since $\mathscr{L}$ is weakly branch, there exists a non-zero element $b = 1^{\otimes n-1} \otimes x^{i+1} \otimes b' \in \mathscr{L}$; and because $b \neq 0$, there exists $w \in X^{\otimes m}$ with $b \cdot x \neq 0$.

Consider $c = [a, b]$; the claim is that $c \neq 0$. Indeed, $c \cdot (v \otimes w) = a \cdot b \cdot (v \otimes w) - b \cdot a \cdot (vw)$; and $b \cdot (v \otimes w) = 0$, while $b \cdot a \cdot (v \otimes w) = (a \cdot v) \otimes (b \cdot w) \neq 0$. □

Recall that a (not necessarily associative) algebra $\mathscr{A}$ is PI ("Polynomial Identity") if there exists a non-zero polynomial expression $\Psi(X_1, \dots, X_n)$ in non-associative, non-commutative indeterminates such that $\Psi(a_1, \dots, a_n) = 0$ for all $a_i \in \mathscr{A}$.

Proposition 3.7. *Let $\mathscr{L}$ be a weakly branched Lie algebra. Then $\mathscr{L}$ is not PI.*

Proof. There should exist a purely Lie-theoretical proof of this fact, but it is shorter to note that if $\mathscr{L}$ is weakly branched, then its associative envelope $\mathscr{A}$ (see §5) is weakly branched in the sense of [3, §3.1.6]. Weakly branched associative algebras satisfy no polynomial identity by [3, Theorem 3.10], so the same must hold for any Lie subalgebra that generates $\mathscr{A}$. □

Recall that a Lie algebra is *just infinite* if it is infinite-dimensional, but all its proper quotients are finite-dimensional.

Proposition 3.8. *Let $\mathscr{L}$ be a regularly branched Lie algebra, with branching ideal $\mathscr{K}$. If furthermore $\mathscr{K}/[\mathscr{K}, \mathscr{K}]$ is finite-dimensional, then $\mathscr{L}$ is just infinite.*

(Note that the proposition's conditions are clearly necessary; otherwise, $\mathscr{L}/[\mathscr{K}, \mathscr{K}]$ would itself be a finite-dimensional proper quotient.)

Proof. Consider a non-zero ideal $\mathscr{I} \triangleleft \mathscr{L}$. Without loss of generality, $\mathscr{I} = \langle a \rangle$ is principal. Let $n \in \mathbb{N}$ be minimal such that $a \cdot X^{\otimes n} \neq 0$; so $a = x^{i_1} \otimes \cdots \otimes x^{i_{n-1}} \otimes \partial_x +$ higher-order terms. Because $\mathscr{L}$ is transitive, we can derive $i_1 + \cdots + i_{n-1}$ times a, by an element $u \in \mathbb{U}(\mathscr{L})$, to obtain $b = [[u, a]] \in \mathscr{I}$ of the form

$$b = 1^{\otimes n-1} \otimes \partial_x + 1^{\otimes n} \otimes b' + \text{higher-order terms}.$$

Consider now $c = \theta^n \otimes k \in \mathscr{K}$. Then $[b, c] = \theta^{n-1} \otimes x^{d-2} \otimes k + \theta^n \otimes [b', k] +$higher-order terms.

Consider next $d = 1^{\otimes n-1} \otimes x \otimes \ell \in \mathscr{K}$. Then $[[b, c], d] = \theta^n \otimes [k, \ell] \in \mathscr{I}$.

It follows that $\mathscr{I}$ contains $X^{\otimes n} \otimes [\mathscr{K}, \mathscr{K}]$, and so $\mathscr{L}/\mathscr{I}$ has finite dimension. □

Corollary 3.9. *Let $\mathscr{L}$ be a finitely generated, nil, regularly branched Lie algebra. Then $\mathscr{L}$ is just infinite.*

Proof. Let $\mathscr{K}$ denote the branching ideal of $\mathscr{L}$. Since $\mathscr{L}$ is finitely generated and $\mathscr{K}$ has finite index in $\mathscr{L}$, it is also finitely generated [27]. Since $\mathscr{L}$ is nil, the abelianization of $\mathscr{K}$ is finite and we may apply Proposition 3.8. □

Proposition 3.10. *Let $\mathscr{L}$ be a regularly branched Lie algebra. Then there does not exist a bound on its nillity.*

Proof. Assume that $\mathscr{K}$ contains a non-trivial nil element, say $a \in \mathscr{K}$ with $a^{p^m} = 0$ but $a^{p^{m-1}} \neq 0$. Let $b \in \mathscr{K}$ be such that $b = 1^{\otimes \ell} \otimes \partial_x +$higher terms. Construct then the following sequence of elements: $a_0 = a$ and $a_{n+1} = 1^{\otimes \ell-1} \otimes \theta \otimes a_n + b$. By induction, the element a_n has nillity exactly p^{m+n}. □

The following is essentially [3, Proposition 3.5]:

Proposition 3.11. *Let $\mathscr{L}$ be a regularly branched Lie algebra. Then its Hausdorff dimension is a rational number in* $(0, 1]$.

Proof. Suppose that $\mathscr{L}$ is regularly branched over $\mathscr{K}$. As in §2.4, let $\mathscr{L}_n$ denote the image of $\mathscr{L}$ in $\mathfrak{Der}(X^{\otimes n})$, with quotient map $\pi_n : \mathscr{L} \to \mathscr{L}_n$. Let M be large enough so that $\mathscr{L}/\psi^{-1}(X \otimes \mathscr{K})$ maps isomorphically onto its image in $\mathscr{L}_n$. We have, for all $n \geq M$,

$$\begin{aligned}\dim \mathscr{L}_n &= \dim(\mathscr{L}/\mathscr{K}) + \dim \pi_n(\mathscr{K})\\ &= \dim(\mathscr{L}/\mathscr{K}) + \dim(\psi\mathscr{K}/(X \otimes \mathscr{K})) + \dim X \dim \pi_{n-1}(\mathscr{K})\\ &= (1 - \dim X)\dim(\mathscr{L}/\mathscr{K}) + \dim(\psi\mathscr{K}/(X \otimes \mathscr{K})) + \dim X \dim \mathscr{L}_{n-1}.\end{aligned}$$

We write $\dim \mathscr{L}_n = \alpha(\dim X)^n + \beta$ for some α, β to be determined; we have

$$\begin{aligned}\alpha(\dim X)^n + \beta = (1 - \dim X)\dim(\mathscr{L}/\mathscr{K}) + \dim(\psi\mathscr{K}/(X \otimes \mathscr{K}))\\ + (\dim X)(\alpha(\dim X)^{n-1} + \beta),\end{aligned}$$

and so $\beta = \dim(\mathscr{L}/\mathscr{K}) - \dim(\psi\mathscr{K}/(X \otimes \mathscr{K}))/(\dim X - 1)$. Then we set $\alpha = (\dim \mathscr{L}_M - \beta)/(\dim X)^M$. We have solved the recurrence for $\dim \mathscr{L}_n$, and $\alpha > 0$ because $\mathscr{L}_n$ has unbounded dimension, since $\mathscr{L}$ is infinite-dimensional.

Now it suffices to note that $\mathrm{Hdim}(\mathscr{L}) = \alpha$ to obtain $\mathrm{Hdim}(\mathscr{L}) > 0$. Furthermore only linear equations with integer coefficients were involved, so $\mathrm{Hdim}(\mathscr{L})$ is rational. □

Note that if $\mathscr{L} \to \mathscr{L} \wr \mathscr{P}$ is the self-similarity structure, then $\mathrm{Hdim}_{\mathscr{P}}(\mathscr{L})$ is also positive and rational.

Proposition 3.12. *Let $\mathscr{L}$ be a regularly weakly branched self-similar Lie algebra. Suppose $\mathscr{L}$ is graded with dilation $\lambda > 1$. Then the Gelfand–Kirillov dimension of $\mathscr{L}$ is at least* $\log(\dim X)/\log\lambda$.

Proof. Suppose $\mathscr{L}$ is weakly branched over $\mathscr{K}$; consider a non-zero $a \in \mathscr{K}$. Let ϵ be the degree of a. Then $\mathscr{L}$ contains for all $n \in \mathbb{N}$ the subspace $X^{\otimes n} \otimes \Bbbk a$; and the maximal degree of these elements is $\lambda^n\epsilon$. Let $\mathscr{L}_d$ denote the span of elements of degree $\leq d$; it follows that $\dim \mathscr{L}_d \geq (\dim X)^n$ whenever $\lambda^n\epsilon \leq d$, and therefore

$$\dim \mathscr{L}_d \geq (d/\epsilon)^{\log(\dim X)/\log\lambda}.$$ □

4. Examples

We begin by two examples of Lie algebras, inspired and related to group-theoretical constructions. The links between the groups and the Lie algebras will be explored in §6. We then phrase in our language of self-similar algebras an example by Petrogradsky, later generalized by Shestakov and Zelmanov.

4.1. The Gupta–Sidki Lie algebra

Inspired by the self-similarity structure (15), we consider $X = \mathbb{F}_3[x]/(x^3)$ and a Lie algebra $\mathscr{L}_{GS} = \mathscr{L}$ generated by a, t with self-similarity structure

$$\psi : \begin{cases} \mathscr{L} \to \mathscr{L} \wr \mathfrak{Der}\, X, \\ a \mapsto \partial_x, \quad t \mapsto x \otimes a + x^2 \otimes t. \end{cases} \tag{9}$$

We put a grading on $\mathscr{L}$ such that the generators a, t are homogeneous. The ring X is $\mathbb{Z}$-graded with $\deg(x) = -1$, so $\deg(a) = 1$, and

$$\deg(t) = -2 + \lambda \deg(t) = -1 + \lambda \deg(a),$$

so $(1-\lambda)^2 = 2$ and $\deg(t) = \sqrt{2}$; the Lie algebra $\mathscr{L}$ is $\mathbb{Z}[\lambda]/(\lambda^2 - 2\lambda - 1)$-graded. We repeat the construction using our matrix notation. For that purpose, we take the divided powers $\{1, t, t^2/2 = -t^2\}$ as a basis of X; the self-similarity structure is then

$$a \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad t \mapsto \begin{pmatrix} 0 & 0 & 0 \\ a & 0 & 0 \\ -t & -a & 0 \end{pmatrix}.$$

Proposition 4.1. *The Lie algebra $\mathscr{L}$ is regularly branched on its ideal $[\mathscr{L}, \mathscr{L}]$ of codimension* 2.

Proof. First, $\mathscr{L}$ is recurrent: indeed, $(\varepsilon \otimes 1)\psi[a, t] = a$ and $(\varepsilon \otimes 1)\psi[a, [a, t]] = t$. Then, by Lemma 3.2, $\mathscr{L}$ is transitive.

The ideal $[\mathscr{L}, \mathscr{L}]$ is generated by $c = [a, t]$; to prove that $\mathscr{L}$ is branched on $[\mathscr{L}, \mathscr{L}]$, it suffices to exhibit $c' \in [\mathscr{L}, \mathscr{L}]$ with $\psi(c') = x^2 \otimes c$. A direct calculation shows that $c' = [[a, t], t]$ will do.

Clearly $\mathscr{L}/[\mathscr{L}, \mathscr{L}]$ is the commutative algebra $\Bbbk^2$ generated by a, t. □

Theorem 4.2. *The Lie algebra $\mathscr{L}$ is nil, of unbounded nillity, but not nilpotent.*

Proof. The ideal $\langle t \rangle$ in $\mathscr{L}$ has codimension 1; and it is generated by $t, [t, a], [t, a, a]$. Each of these elements is bounded and 1-evanescent, so Corollary 2.9 applies. Then $\mathscr{L}$ itself is nil, because $a^3 a = 0$.

That the nillity in unbounded follows from Proposition 3.10. Clearly $\mathscr{L}$ is not nilpotent, since by Proposition 3.7 it is not even PI. □

Proposition 4.3. *The relative Hausdorff dimension of $\mathscr{L}$ with respect to $\mathscr{P} = \Bbbk\partial_x$ is*

$$\mathrm{Hdim}_{\mathscr{P}}(\mathscr{L}) = 4/9.$$

Proof. We follow the proof of Proposition 3.11. We may take $M = 2$, and readily compute $\dim(\mathscr{L}/\mathscr{K}) = 2 = \dim(\mathscr{K}/(X \otimes \mathscr{K}))$, the latter having basis $\{[a, t], [a, [a, t]]\}$. Letting $\mathscr{L}_n$ denote the image of $\mathscr{L}$ in $\mathfrak{Der}(X^{\otimes n})$, we find $\dim \mathscr{L}_2 = 3$. This gives

$$\dim \mathscr{L}_n = 2 \cdot 3^{n-2} + 1,$$

and the claimed result. □

Lemma 4.4. *The algebra $\mathscr{L}$ is contracting.*

Proof. Its nucleus is $\Bbbk\{a, t\}$. □

Corollary 4.5. *The Gelfand–Kirillov dimension of $\mathscr{L}$ is $\log 3/\log\lambda$.*

Proof. This follows immediately from Propositions 2.17 and 3.12. □

In fact, thanks to Theorem 6.8, a much stronger result holds:

Proposition 4.6 ([4, Corollary 3.9]). *Set $\alpha_1 = 1$, $\alpha_2 = 2$, and $\alpha_n = 2\alpha_{n-1} + \alpha_{n-2}$ for $n \geq 3$. Then, for $n \geq 2$, the dimension of the degree-n component of $\mathscr{L}_{\mathrm{GS}}$ is the number of ways of writing $n-1$ as a sum $k_1\alpha_1 + \cdots + k_t\alpha_t$ with all $k_i \in \{0, 1, 2\}$.*

The Gupta–Sidki Lie algebra generalizes to arbitrary characteristic p, with now $\psi(t) = x \otimes a + x^{p-1} \otimes t$. We will explore in §6.4 the connections between $\mathscr{L}_{\mathrm{GS}}$ and the Gupta–Sidki group.

4.2. *The Grigorchuk Lie algebra*

Again inspired by the self-similarity structure (14), we consider $X = \mathbb{F}_2[x]/(x^2)$, a Lie algebra $\mathscr{L}_{\mathrm{G}}$, and a restricted Lie algebra ${}_2\mathscr{L}_{\mathrm{G}}$. Both are generated by a, b, c, d with $b + c + d = 0$, and have the same self-similarity structure

$$\psi : \begin{cases} \mathscr{L}_{\mathrm{G}} \to \mathscr{L}_{\mathrm{G}} \wr \mathfrak{Der}(X) \text{ respectively } {}_2\mathscr{L}_{\mathrm{G}} \to {}_2\mathscr{L}_{\mathrm{G}} \wr \mathfrak{Der}(X), \\ a \mapsto \partial_x, \\ b \mapsto x \otimes (a+c), \\ c \mapsto x \otimes (a+d), \\ d \mapsto x \otimes b. \end{cases} \tag{10}$$

We seek gradings for these two Lie algebras that make the generators homogeneous. Again X is $\mathbb{Z}$-graded, with $\deg(x) = -1$, so $\deg(a) = 1$, and $\deg(b) = \deg(c) = \deg(d) = \deg(a) = 1$, so $\lambda = 2$. In other words, $\mathscr{L}_{\mathrm{G}}$ and ${}_2\mathscr{L}_{\mathrm{G}}$ are no more than $\mathbb{Z}$-graded. Using our matrix notation:

$$a \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad b \mapsto \begin{pmatrix} 0 & 0 \\ a+c & 0 \end{pmatrix}, \quad c \mapsto \begin{pmatrix} 0 & 0 \\ a+d & 0 \end{pmatrix}, \quad d \mapsto \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix}.$$

In contrast to §4.1, there are important differences between $\mathscr{L}_{\mathrm{G}}$ and ${}_2\mathscr{L}_{\mathrm{G}}$. Their relationship is as follows: ${}_2\mathscr{L}_{\mathrm{G}}$ is an extension of $\mathscr{L}_{\mathrm{G}}$ by the abelian algebra $\Bbbk\{1^{\otimes n} \otimes [a, b]^2\}$.

We note that $\mathscr{L}_{\mathrm{G}}$ is not recurrent. However, let us define $e = a + c$, $f = a + d$, and set $\mathscr{L}' = \langle b, e, f, f^2 \rangle$:

$$b \mapsto \begin{pmatrix} 0 & 0 \\ e & 0 \end{pmatrix}, \quad e \mapsto \begin{pmatrix} 0 & 1 \\ f & 0 \end{pmatrix}, \quad f \mapsto \begin{pmatrix} 0 & 1 \\ b & 0 \end{pmatrix}.$$

Proposition 4.7. *$\mathscr{L}'$ is an ideal of codimension 1 in $\mathscr{L}_{\mathrm{G}}$. It is recurrent, transitive, and regularly branched on its ideal $\mathscr{K} = \langle [b, e] \rangle$ of codimension 3.*

Proof. First, $\mathscr{L}'$ is recurrent: indeed, $(\varepsilon \otimes 1)\psi f^2 = b$ and $(\varepsilon \otimes 1)\psi[b, e] = e$; note the relation $b + e + f = 0$, so $\mathscr{L}'$ is 2-generated. By Lemma 3.2, $\mathscr{L}'$ is transitive.

To show that $\mathscr{L}'$ is branched on $\mathscr{K}$, it suffices to note $\psi[[b, e], b] = x \otimes [b, e]$, so $\psi(\mathscr{K})$ contains $X \otimes \mathscr{K}$. Finally, $\mathscr{L}'/\mathscr{K}$ has basis $\{b, e, e^2\}$, as a direct calculation shows. □

We note, however, that the corresponding subalgebra ${}_2\mathscr{L}'$ is not regularly branched on the restricted ideal ${}_2\mathscr{K} = \langle [b, e] \rangle$; indeed, as we noted above, ${}_2\mathscr{K}$ contains $1^{\otimes n} \otimes [b, e]^2$ for all $n \in \mathbb{N}$, yet does not contain $x \otimes [b, e]^2$.

Proposition 4.8. *The relative Hausdorff dimensions of $\mathscr{L}_{\mathrm{G}}$ and ${}_2\mathscr{L}_{\mathrm{G}}$ with respect to $\mathscr{P} = \Bbbk\partial_x$ are*

$$\mathrm{Hdim}_{\mathscr{P}}(\mathscr{L}_{\mathrm{G}}) = \mathrm{Hdim}_{\mathscr{P}}({}_2\mathscr{L}_{\mathrm{G}}) = 1/2.$$

Proof. We follow the proof of Proposition 3.11. We may take $M = 3$, and readily compute $\dim(\mathscr{L}_{\mathrm{G}}/\mathscr{K}) = 4$ while $\dim(\mathscr{K}/(X \otimes \mathscr{K})) = 1$, the latter having basis $\{[a, b]\}$. Letting $\mathscr{L}_n$ denote the image of $\mathscr{L}_{\mathrm{G}}$ in $\mathfrak{Der}(X^{\otimes n})$, we find $\dim \mathscr{L}_3 = 7$. This gives

$$\dim \mathscr{L}_n = 2^{n-1} + 3,$$

and the claimed result. The same arguments apply to ${}_2\mathscr{L}_{\mathrm{G}}$. □

Note, on the other hand, that $\dim ({}_2\mathscr{L}_{\mathrm{G}})_n = 2^{n-1} + n$, by the same calculation but taking into account the $1^{\otimes n} \otimes [a, b]^2$.

Lemma 4.9. *The algebra $\mathscr{L}_{\mathrm{G}}$ is contracting.*

Proof. Its nucleus is $\Bbbk\{a, b, d\}$. □

Corollary 4.10. *The Gelfand–Kirillov dimension of $\mathscr{L}_{\mathrm{G}}$ and ${}_2\mathscr{L}_{\mathrm{G}}$ is 1.*

Proof. This follows immediately from Propositions 2.17 and 3.12. □

In fact (see also Theorem 6.6), a much stronger result holds, namely $\mathscr{L}_{\mathrm{G}}$ and ${}_2\mathscr{L}_{\mathrm{G}}$ have bounded width:

Proposition 4.11. *Keeping the notation $e = a + c$ and $f = a + d$, a basis of $\mathscr{L}_{\mathrm{G}}$ is*

$$\left\{a, 1 \otimes f, x^{i_1} \otimes \cdots \otimes x^{i_n} \otimes e \,\middle|\, n \in \mathbb{N}, i_k \in \{0, 1\}\right\}.$$

The element a has degree 1, the element $1 \otimes f$ has degree 2, and the element $x^{i_1} \otimes \cdots \otimes x^{i_n} \otimes e$ has degree $2^n - \sum 2^{k-1} i_k$.

A basis of ${}_2\mathscr{L}_{\mathrm{G}}$ consists of the above basis, with in addition the elements $1^{\otimes n} \otimes f$ of degree 2^n for all $n \geq 2$.

In particular, in $\mathscr{L}_{\mathrm{G}}$ there is a one-dimensional subspace of degree n for all $n \geq 3$, while in ${}_2\mathscr{L}_{\mathrm{G}}$ there is a two-dimensional subspace of degree n for all $n \geq 2$ a power of two.

Proof. Follows from $b = x \otimes e$ and $[b, e] = [b, f] = [e, f] = 1 \otimes e$ and $e^2 = 1 \otimes f$ and $f^2 = 1 \otimes b$ and $b^2 = 0$. □

Let similarly ${}_2\mathscr{L}'$ denote the restricted Lie subalgebra of ${}_2\mathscr{L}_{\mathrm{G}}$ generated by b, e, f. Since $b + e + f = 0$, the algebra ${}_2\mathscr{L}'$ is 2-generated, so cannot have maximal class, because it is infinite-dimensional [37]. It could actually be that the growth of ${}_2\mathscr{L}'$ is minimal among infinite restricted Lie algebras over a field of characteristic 2.

Theorem 4.12. *If $\Bbbk = \mathbb{F}_2$, the Lie algebra ${}_2\mathscr{L}_{\mathrm{G}}$ is nil; while if $\Bbbk$ contains $\mathbb{F}_4$, then ${}_2\mathscr{L}_{\mathrm{G}}$ is not nil. In all cases, ${}_2\mathscr{L}_{\mathrm{G}}$ has unbounded nillity, but is not nilpotent.*

Proof. We first prove that ${}_2\mathscr{L}_{\mathrm{G}}$ is graded nil when $\Bbbk = \mathbb{F}_2$, that is, homogeneous elements are nil (alternatively, this follows from Theorem 6.6). Let $u \in {}_2\mathscr{L}_{\mathrm{G}}$ be homogeneous of degree n. If $n \geq 2$, then $\psi(u) = 1 \otimes v$ or $\psi(u) = x \otimes v$, depending on whether n is even or odd, with v of lower degree, so we are done by induction. We are left with proving that degree-1 elements are nil; but they belong to $\{a, b, c, d\}$, of nillity 2, or $f, e, a + b$ of respective nillities 4, 8, 16 because $\psi(f^4) = \psi([a, d]^2) = 1 \otimes b^2 = 0$ and $\psi(e^8) = \psi([a, c]^4) = 1 \otimes f^4 = 0$ and $\psi(a + b)^{16} = \psi([a, b]^8) = 1 \otimes e^8 = 0$.

On the other hand, if $\Bbbk$ contains $\mathbb{F}_4 = \{0, 1, \omega, \omega^2\}$ with $\omega^3 = 1$, then consider $u = a + b + \omega c + \omega^2 d$. Then

$$\psi(u) = \partial_x + x \otimes ((1 + \omega)a + c + \omega d + \omega^2 b) = \partial_x + \omega^2 x \otimes u,$$

so $\psi(u^2) = \omega^2 \otimes u$. If we had $u^n = 0$ for some $n \in \mathbb{N}$, that would imply $u^{\lceil n/2 \rceil} = 0$ and eventually $u = 0$, a contradiction. Therefore, ${}_2\mathscr{L}_{\mathrm{G}} \otimes \mathbb{F}_4$ is not graded nil.

Consider now $u \in {}_2\mathscr{L}_{\mathrm{G}}$, and let n denote the maximal degree of its homogeneous components. Write $\psi(u) = \alpha\partial_x + 1 \otimes v + x \otimes w$ for $v, w \in {}_2\mathscr{L}_{\mathrm{G}}$ and $\alpha \in \Bbbk$; then $\psi(u^2) = 1 \otimes (v^2 + \alpha w) + x \otimes [v, w]$. Therefore,

$$u \text{ is nil } \Leftrightarrow u^2 \text{ is nil } \Leftrightarrow u' := v^2 + \alpha w \text{ is nil.}$$

The maximal degree of a homogeneous component of u' is $\leq n$; and if $n \geq 2$, the degree-n part of u' is the square of the degree-n part of u, again because the degree-n component of ${}_2\mathscr{L}_{\mathrm{G}}$ is one-dimensional. We proceed with u' in lieu of u, and (because the homogeneous component of degree n is nil) eventually obtain an element of maximal degree $\leq n - 1$. Proceeding further, we obtain u homogeneous of degree 1, which is nil by the first paragraph.

That the nillity in unbounded follows from Proposition 3.10. Clearly ${}_2\mathscr{L}_{\mathrm{G}}$ is not nilpotent, since by Proposition 3.7 it is not even PI. □

We will explore in §6.3 the connections between $\mathscr{L}_{\mathrm{G}}$, ${}_2\mathscr{L}_{\mathrm{G}}$ and the Grigorchuk group.

4.3. *Grigorchuk Lie algebras*

We generalize the previous example $\mathscr{L}'$ as follows. We fix a field $\Bbbk$ of characteristic p, the alphabet $X = \Bbbk[x]/(x^p)$, and an infinite sequence $\omega = \omega_0\omega_1 \cdots \in \mathbb{P}^1(\Bbbk)^\infty$. Choose a projective lift $\mathbb{P}^1(\Bbbk) \to \mathrm{Hom}(\Bbbk^2, \Bbbk)$, and apply it to ω. Consider also the shift map $\sigma : \omega_0\omega_1 \ldots \mapsto \omega_1 \ldots$.

Define then a Lie algebra $\mathscr{L}_\omega$ acting on $R(X)$, generated by $\Bbbk^2$, with (non-self!-)similarity structure

$$\psi : \begin{cases} \mathscr{L}_\omega \to \mathscr{L}_{\sigma\omega} \wr \mathfrak{Der}\, X, \\ \Bbbk^2 \ni a \mapsto x^{p-1} \otimes a + \omega_0(a)\partial_x. \end{cases}$$

It is a $\mathbb{Z}$-graded algebra, with dilation $\lambda = p$ and $\deg(a) = 1$ for all $a \in \Bbbk^2$. To see better the connection to the Grigorchuk example, consider $\Bbbk = \mathbb{F}_2$ and $\omega = (\omega_0\omega_1\omega_2)^\infty$ with $\omega_0, \omega_1, \omega_2$ the three non-trivial maps $\Bbbk^2 \to \Bbbk$. The three non-trivial elements of $\Bbbk^2$ are b, e, f generating the subalgebra $\mathscr{L}'$ from the previous subsection. Using our matrix notation,

$$b \mapsto \begin{pmatrix} 0 & 0 \\ e & 0 \end{pmatrix}, \quad e \mapsto \begin{pmatrix} 0 & 1 \\ f & 0 \end{pmatrix}, \quad f \mapsto \begin{pmatrix} 0 & 1 \\ b & 0 \end{pmatrix}.$$

We summarize the findings of the previous subsection in this more general context. Recall from [9] that a graded algebra $\mathscr{L}$ is of *maximal class* if it is generated by the two-dimensional subspace $\mathscr{L}_1$ and $\dim \mathscr{L}_n = 1$ for all $n \geq 2$. In particular, it has Gelfand–Kirillov dimension at most 1.

Proposition 4.13. *The algebra $\mathscr{L}_\omega$ is branched and of maximal class.*

The construction of $\mathscr{L}_\omega$ is modelled on that of the Grigorchuk groups G_ω introduced in [17]. We now detail the connection to the algebras of maximal class studied by Caranti et al., and recall their definition of *inflation* [9, §6]. Let $\mathscr{L}$ be a Lie algebra of maximal class, and let $\mathscr{M}$ be a codimension-1 ideal, specified by $\omega \in \mathbb{P}^1(\Bbbk)$: $a \in \mathscr{L}$ belongs to $\mathscr{M}$ if and only if $\omega(a_1) = 0$, where a_1 denotes the degree-1 part of a.

Choose $s \in \mathscr{L} \setminus \mathscr{M}$; then s acts as a derivation on $\mathscr{M}$. The corresponding *inflated* algebra is ${}^{\mathscr{L}}\!\mathscr{M} = \mathscr{M} \otimes \Bbbk[\varepsilon]/(\varepsilon^p) \rtimes \Bbbk$, where $\Bbbk$ acts by the derivation $s' = 1 \otimes \partial_\varepsilon - s \otimes \varepsilon^{p-1}$. Note $(s')^p = s$. It is shown in [9] that ${}^{\mathscr{L}}\!\mathscr{M}$ is again an algebra of maximal class. The main results of [10, 24] are that every infinite-dimensional Lie algebra of maximal class is obtained through a (possibly infinite) number of steps from elementary building blocks such as the Albert–Franks algebras.

In fact, ${}^{\mathscr{L}}\!\mathscr{M}$ may also be described as the subalgebra of $\mathscr{L} \wr \Bbbk\partial_\varepsilon$ generated by $\mathscr{M}$ and s'. The algebra ${}^{\mathscr{L}}\!\mathscr{M}$ is independent (up to isomorphism) of the choice of s, and therefore solely depends on the choice of ω.

The algebras $\mathscr{L}_\omega$ presented in this subsection are examples of Lie algebras of maximal class that fall into the "infinitely iterated inflations" subclass [9, §9]. However, they do not appear here as inverse limits, but rather as countable-dimensional vector spaces, dense in the algebras constructed by Caranti et al.

If $\mathscr{L}'$ was obtained from $\mathscr{L}$ through inflation, then $\mathscr{L}$ may be recovered from $\mathscr{L}'$ through *deflation*: choose $s' \in \mathscr{L}'$ of degree 1 that does not commute with $\mathscr{L}'_2$, and set $(\mathscr{L}')^\downarrow = \Bbbk(s')^p \oplus \bigoplus_{n\geq 1}(\mathscr{L}')_{pn}$. Then $\mathscr{L} \cong (\mathscr{L}')^\downarrow$.

4.4. Fabrykowski–Gupta Lie algebras

Again inspired by the self-similarity structure of the Fabrykowski–Gupta group [14], we consider $X = \mathbb{F}_p[x]/(x^p)$ and a Lie algebra $\mathscr{L}$ generated by a, t with self-similarity structure

$$\psi : \begin{cases} \mathscr{L} \to \mathscr{L} \wr \mathfrak{Der}\, X, \\ a \mapsto \partial_x, \quad t \mapsto x^{p-1} \otimes (a+t). \end{cases}$$

We seek a grading for $\mathscr{L}$ that makes the generators homogeneous. Again X is $\mathbb{Z}$-graded with $\deg(x) = -1$, so $\deg(a) = 1$, and $\deg(t) = \deg(a) = 1$, while $\lambda \deg(t) = p - 1 + \deg(t)$, so $\lambda = p$. In other words, the Lie algebra $\mathscr{L}$ is no more than $\mathbb{Z}$-graded. In our matrix notation,

$$a \mapsto \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 \\ 0 & \cdots & \cdots & 0 \end{pmatrix}, \quad t \mapsto \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ -a-t & \cdots & 0 & 0 \end{pmatrix}.$$

Theorem 4.14. *The Lie algebra $\mathscr{L}$ is not nil.*

Proof. Consider the element $x = a + t$. A direct calculation gives $\psi(x^p) = -1 \otimes x$. It follows that if $x^{p^s} = 0$ for some $s > 0$, then $-1 \otimes x^{p^{s-1}} = 0$ so $x^{p^{s-1}} = 0$. Since $x \neq 0$, it is not nil. □

4.5. Petrogradsky–Shestakov–Zelmanov algebras

We consider a field $\Bbbk$ of characteristic p, and $X = \Bbbk[x]/(x^p)$. We fix an integer $m \geq 2$, and consider the Lie algebra $\mathscr{L}_{m,\Bbbk}$ with generators $d_1, \dots, d_{m-1}, v$ and self-similarity structure

$$\psi : \begin{cases} \mathscr{L}_{m,\Bbbk} \to \mathscr{L}_{m,\Bbbk} \wr \mathfrak{Der}\, X, \\ d_1 \mapsto \partial_x, \\ d_{n+1} \mapsto 1 \otimes d_n \text{ for } n = 1, \dots, m-2, \\ v \mapsto 1 \otimes d_{m-1} + x^{p-1} \otimes v. \end{cases}$$

In the special case $m = 2$, $\Bbbk = \mathbb{F}_2$, Petrogradsky [34] actually considered the subalgebra $\langle v, [d_1, v] \rangle$, of codimension 1 in $\mathscr{L}_{2,\mathbb{F}_2}$. Shestakov and Zelmanov [40] consider the case $m = 2$; see also [35]. We repeat for clarity that last example (with $d = d_1$) using our matrix notation. To this end, we take the divided powers $\{1, x, x^2/2, \dots, x^{p-1}/(p-1)!\}$ as a basis of X. The endomorphisms m_x and m_{∂_x} are respectively

$$m_x = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 1 & \ddots & \ddots & \vdots \\ \vdots & 2 & \ddots & 0 \\ 0 & \cdots & p-1 & 0 \end{pmatrix}, \quad m_{\partial_x} = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 \\ 0 & \cdots & \cdots & 0 \end{pmatrix}.$$

It follows that the matrix decompositions of d, v are

$$d \mapsto \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 \\ 0 & \cdots & \cdots & 0 \end{pmatrix}, \quad v \mapsto \begin{pmatrix} d & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ -v & \cdots & 0 & d \end{pmatrix}.$$

We seek a grading that makes the generators homogeneous. The ring X is $\mathbb{Z}$-graded by setting $\deg(x) = -1$, so $R(X)$ is $\mathbb{Z}[\lambda]$-graded. If that grading is to be compatible with the self-similarity structure, however, we must impose $\deg(d_n) = \lambda^{n-1}$ and

$$\deg(v) = \lambda \deg(d_{m-1}) = -(p-1) + \lambda \deg(v) = \lambda^{m-1},$$

so $\lambda^m - \lambda^{m-1} + 1 - p = 0$. We therefore grade $R(X)$ and $\mathscr{L}_{m,\Bbbk}$ by the abelian group

$$\Lambda = \mathbb{Z}[\lambda]/(\lambda^m - \lambda^{m-1} + 1 - p).$$

For simplicity, we state the following result only in the case $m = 2$:

Proposition 4.15. *The Lie algebra $\mathscr{L}_{2,\Bbbk}$ is regularly branched on its ideal $\langle [v, [d, v]] \rangle$ of codimension $p + 1$.*

Proof. First, $\mathscr{L}_{2,\Bbbk}$ is recurrent: indeed, $(\varepsilon \otimes 1)\psi(v) = d$ and $(\varepsilon \otimes 1)\psi[[d^{p-1}, v]] = v$. Then, by Lemma 3.2, $\mathscr{L}_{2,\Bbbk}$ is transitive.

Write $c = [v, [d, v]]$ and $\mathscr{K} = \langle c \rangle$. To prove that $\mathscr{L}_{2,\Bbbk}$ is branched on $\mathscr{K}$, it suffices to exhibit $c' \in \mathscr{K}$ with $\psi(c') = x^{p-1} \otimes c$. If $p = 2$, then $c' = [v, c]$ will do, while if $p \geq 3$ then take $c' = [[d, v], [[d^{p-3}, c]]]$.

A direct computation shows that $\mathscr{L}_{2,\Bbbk}/\mathscr{K}$ is finite-dimensional, with basis $\{d, v, [d, v], \dots, [[d^{p-1}, v]]\}$. □

We are now ready to reprove the following main result from Shestakov and Zelmanov:

Theorem 4.16 ([40, Example 1]). *The algebra $\mathscr{L}_{m,\Bbbk}$ is nil but not nilpotent.*

Proof. The ideal $\langle v \rangle$ has finite codimension and is generated by m-evanescent elements, so Corollary 2.9 applies to it. We conclude by noting that $\mathscr{L}_{m,\Bbbk}/\langle v \rangle$ is abelian and hence nil. □

We concentrate again on $m = 2$ in the following results:

Proposition 4.17. *The relative Hausdorff dimension of $\mathscr{L}_{2,\Bbbk}$ with respect to $\mathscr{P} = \Bbbk\partial_x$ is*

$$\operatorname{Hdim}_{\mathscr{P}}(\mathscr{L}_{2,\Bbbk}) = (p-1)^2/p^3.$$

Proof. We follow the proof of Proposition 3.11, using the notation $\mathscr{L} = \mathscr{L}_{2,\Bbbk}$. We may take $M = 3$, and readily compute $\dim(\mathscr{L}/\mathscr{K}) = p + 1$ with basis $\{d, [[d^i, v]]\}$ and $\dim(\mathscr{K}/(X \otimes \mathscr{K})) = (p-1)^2$ with basis $\{[[d^i, v^j, c]] \mid i, j \in \{0, \dots, p-2\}$. Letting $\mathscr{L}_n$ denote the image of $\mathscr{L}$ in $\mathfrak{Der}(X^{\otimes n})$, we find $\dim \mathscr{L}_3 = p + 1$. This gives

$$\dim \mathscr{L}_n = (p-1) \cdot p^{n-3} + 2,$$

and the claimed result. □

Lemma 4.18. *The algebra $\mathscr{L}_{m,\Bbbk}$ is contracting.*

Proof. Its nucleus is $\Bbbk\{d_1, \dots, d_{m-1}, v\}$. □

Corollary 4.19. *The Gelfand–Kirillov dimension of $\mathscr{L}_{m,\Bbbk}$ is* $\log p / \log \lambda$.

Proof. This follows immediately from Propositions 2.17 and 3.12. □

Note, however, that careful combinatorial calculations give a much sharper result. For example, again for $m = p = 2$, the dimension of the span of commutators of length $\le n$ in $\mathscr{L}_{2,\mathbb{F}_2}$ is, for $F_{k-1} < n \le F_k$ and (F_k) the Fibonacci numbers, the number of manners of writing $F_k - n$ as a sum of distinct Fibonacci numbers among $F_1, \dots, F_{k-4}$. In particular, for $n = F_k$ at least 5, there is precisely one commutator of length n, namely $1^{\otimes k-2} \otimes v$.

5. Self-similar associative algebras

At least three associative algebras may be associated with a self-similar Lie algebra $\mathscr{L}$. The first is the universal enveloping algebra $\mathbb{U}(\mathscr{L})$, which maps onto the other two. The second is the adjoint algebra of $\mathscr{L}$, that is, the associative subalgebra $\mathfrak{Adj}(\mathscr{L})$ of $\operatorname{End}(\mathscr{L})$ generated by the derivations $[a, -] : \mathscr{L} \to \mathscr{L}$ for all $a \in \mathscr{L}$. The third one is the *thinned algebra* $\mathscr{A}(\mathscr{L})$, defined as follows.

Let $d = \dim X$. Recall from (3) that the self-similarity structure $\psi : \mathscr{L} \to \mathscr{L} \wr \mathfrak{Der}\, X$ gives rise to a linear map $\psi' : \mathscr{L} \to \operatorname{Mat}_d(\mathscr{L} \oplus \Bbbk)$. We extend this map multiplicatively to an algebra homomorphism $\psi' : T(\mathscr{L}) \to \operatorname{Mat}_d(T(\mathscr{L}))$. Now, for $a = \sum x_i \otimes a_i + \delta$ and $b = \sum y_j \otimes b_j + \epsilon$ in $\mathscr{L}$, we have $[a, b] = \sum\sum x_i y_j \otimes [a_i, b_j] + \sum \delta y_j \otimes b_j - \sum \epsilon x_i \otimes a_i + [\delta, \epsilon]$, and therefore

$$\begin{aligned}\psi'(ab - ba - [a, b]) = &\sum\sum m_{x_i y_j}(a_i b_j - b_j a_i - [a_i, b_j])\\ &+ \sum (m_{x_i} m_\epsilon - m_\epsilon m_{x_i} - m_{\epsilon x_i}) a_i\\ &- \sum (m_{y_j} m_\delta - m_\delta m_{y_j} - m_{\delta y_j}) b_j\\ &+ m_\delta m_\epsilon - m_\epsilon m_\delta - m_{[\delta,\epsilon]};\end{aligned}$$

the last three summands are zero, so all entries of $\psi'(ab - ba - [a, b])$ lie in the ideal generated by the $a'b' - b'a' - [a', b']$. We deduce:

Proposition 5.1. *The map ψ' induces an algebra homomorphism*

$$\psi' : \mathbb{U}(\mathscr{L}) \to \operatorname{Mat}_d(\mathbb{U}(\mathscr{L})).$$

Now we note that even though $\psi : \mathscr{L} \to \mathscr{L} \wr \mathfrak{Der}\, X$ may be injective, this does not imply that ψ' is injective. As a simple example, consider the Grigorchuk Lie algebra from §4.2. We have $bc \ne 0$ in $\mathbb{U}(\mathscr{L})$, but $\psi'(bc) = 0$.

Since $\mathscr{L}$ acts on $R(X)$ by $\Bbbk$-linear maps, the universal enveloping algebra $\mathbb{U}(\mathscr{L})$ also acts on $R(X)$. Quite clearly, the kernel of the action of $\mathbb{U}(\mathscr{L})$ on $R(X)$ contains the kernel of ψ'.

Definition 5.2. Let $\mathscr{L}$ be a self-similar Lie algebra. The *thinned enveloping algebra* of $\mathscr{L}$ is the quotient $\mathscr{A}(\mathscr{L})$ of $\mathbb{U}(\mathscr{L})$ by the kernel of its natural action on $R(X)$.

Rephrasing the results from §4, we deduce that, for $\mathscr{L}$ = the Gupta–Sidki Lie algebra, the Grigorchuk Lie algebra, and the Petrogradsky–Shestakov–Zelmanov algebras, the image of $\mathscr{L}$ in $\mathbb{U}(\mathscr{L})$ is nil; the same property then holds for the adjoint algebra and the thinned algebra of $\mathscr{L}$. Apart from this information, the structure of $\mathbb{U}(\mathscr{L})$ or $\mathfrak{Adj}(\mathscr{L})$ seems mysterious.

In case $\mathscr{L}$ admits two gradings with different dilations, it is possible to deduce that $\mathbb{U}(\mathscr{L})$ is a sum of locally nilpotent subalgebras. Assume therefore that $\mathscr{L}$ admits two degree functions, written $\deg_\lambda$ and $\deg_\mu$. For simplicity, assume also $\lambda > |\mu| > 1$, although the case $|\mu| \leq 1$ can also be handled as a limit, or by a small change in the argument; indeed increasing $|\mu|$ only makes inequalities tighter in what follows. The following is drawn from [35, Theorem 2.1]:

Proposition 5.3. *Let $\mathscr{L}$ be a graded self-similar Lie algebra, in characteristic $p > 0$, with gradings $\deg_\lambda$, $\deg_\mu$, such that $\mathscr{L}$ is generated by finitely many positive-degree elements with respect to $\deg_\lambda$. Then there is a decomposition as a sum of subalgebras*

$$\mathbb{U}(\mathscr{L}) = \mathbb{U}_+ \oplus \mathbb{U}_0 \oplus \mathbb{U}_-,$$

in which $\mathbb{U}_+, \mathbb{U}_0, \mathbb{U}_-$ are the spans of homogeneous elements a with $\deg_\mu(a)$ positive, zero and negative respectively; and $\mathbb{U}_+$ and $\mathbb{U}_-$ are locally nilpotent. In particular, homogeneous elements with $\deg_\mu \neq 0$ are nil.

Proof. We first deduce from (7) and (8) that for appropriate constants C, D we have, for all $a \in X^{\otimes n} \otimes N$,

$$\log_{|\mu|}(\deg_\mu(a)/C) \leq n \leq \log_\lambda(\deg_\lambda(a)/D).$$

Setting $\theta = \log|\mu|/\log\lambda \in (0,1)$, we get, for a fresh constant C,

$$\deg_\mu(a) \leq C \deg_\lambda(a)^\theta.$$

We seek a similar inequality for $\mathbb{U}(\mathscr{L})$. For that purpose, choose a homogeneous basis $(a_1, a_2, \dots)$ of $\mathscr{L}$, and recall that $\mathbb{U}(\mathscr{L})$ has a basis

$$\Big(\prod_{i\geq 1} a_i^{n_i} \;\Big|\; n_i \in \mathbb{F}_p,\ \text{almost all } 0\Big).$$

Each a_i belongs to $X^{\otimes m_i} \otimes N$ for some minimal $m_i \in \mathbb{N}$. For $j \in \mathbb{N}$, let ℓ_j denote the number of m_i with $m_i = j$. Then $\ell_j \leq (\dim X)^j \dim N$.

Consider now $u = \prod_{i\geq 1} a_i^{n_i} \in \mathbb{U}(\mathscr{L})$. We have

$$\deg_\lambda(u) = \sum_{i\geq 1} n_i \lambda^{m_i}, \quad |\deg_\mu(u)| \leq \sum_{i\geq 1} n_i |\mu|^{m_i},$$

and we wish to study cases in which $|\deg_\mu(u)|$ is as large as possible for given $\deg_\lambda(u)$. Because $|\mu| < \lambda$ and by convexity of the functions $\lambda^x, |\mu|^x$, this will occur when

$n_i = p-1$ whenever m_i is small, say $m_i < K$, and $n_i = 0$ whenever $m_i > K$; with intermediate values whenever $m_i = K$. We have

$$\deg_\lambda(u) \approx (p-1)\sum_{j=0}^{K} \ell_j \lambda^j \approx (\lambda \dim X)^K,$$
$$\deg_\mu(u) \precapprox (p-1)\sum_{j=0}^{K} \ell_j |\mu|^j \approx (|\mu| \dim X)^K,$$

so

$$\deg_\mu(u) < C \deg_\lambda(u)^{\theta'}, \tag{11}$$

for a constant C and $\theta' = \log(|\mu| \dim X)/\log(\lambda \dim X) \in (0,1)$.

It is clear that we have a decomposition $\mathbb{U}(\mathscr{L}) = \mathbb{U}_+ \oplus \mathbb{U}_0 \oplus \mathbb{U}_-$. We now show that $\mathbb{U}_+$ is locally nilpotent, the same argument applying to $\mathbb{U}_-$. For that purpose, consider $u_1, \dots, u_k \in \mathbb{U}_+$, spanning a subspace V, and set

$$D = \max_{i\in\{1,\dots,k\}} \frac{\deg_\lambda(u_i)}{\deg_\mu(u_i)}.$$

For $s \in \mathbb{N}$, consider a non-zero homogeneous element u in the s-fold product V^s. Then $\deg_\lambda(u) \le D \deg_\mu(u)$; combining this with (11) we get

$$\deg_\mu(u) \le C(D \deg_\mu(u))^{\theta'},$$

so $\deg_\mu(u)$ is bounded and therefore s is also bounded. □

Note that if $\mathscr{L}$ is generated by a set S such that the $\deg_\mu(s)$ with $s \in S$ are linearly independent in $\mathbb{R}$, then $\mathbb{U}_0 = \mathbb{k}$ and therefore homogeneous elements except scalars are nil in $\mathbb{U}(\mathscr{L})$.

On the other hand, note that even the quotient algebra $\mathscr{A}(\mathscr{L})$ tends to have transcendental elements. For example, $\mathscr{A}({}_2\mathscr{L}_{\mathrm{G}})$ contains $a+b+ad$, which is transcendental by [3, Theorem 4.20] and Theorem 6.7. It seems that the element a^2+t of $\mathscr{A}(\mathscr{L}_{\mathrm{GS}})$ is transcendental.

Lemma 5.4. *If $p = \alpha^m - \alpha^{m-1} + 1$ for some $\alpha \in \mathbb{N}$, then $\mathscr{A}(\mathscr{L}_{m,\mathbb{F}_p})$ is not nil.*

Note that the condition actually says that the maximal dilation factor of the grading is an integer; in light of Proposition 5.3, the non-nil element will have degree 0.

Proof. Write $\beta = \alpha - 1$, and consider the element $x = d_1^{\alpha^{m-2}\beta} d_2^{\alpha^{m-3}\beta} \cdots d_{m-1}^{\beta} v$. Then $\psi(x^\alpha)$ is a lower triangular matrix, with $-x$ at position (p,p). Because $x \neq 0$, it follows that $x^{\alpha^n} \neq 0$ for all $n \in \mathbb{N}$, so x is not a nil element. □

There does not seem to exist such a simple argument for arbitrary primes; for instance, for $p = m = 2$ it seems that $x = v + v^2 + vuv$ has infinite order (its order is at least 2^{10}), while for $p = 5$ and $m = 2$ it seems that $x = v + d^2 - d^4$ has infinite order (its order is at least 5^3).

5.1. *Thinned algebras*

The “thinned algebra” from Definition 5.2 is an instance of a *self-similar algebra*, as defined in [3]. We recall a basic notion: a *self-similar associative algebra* is an algebra $\mathscr{A}$ endowed with a homomorphism $\phi : \mathscr{A} \to \mathrm{Mat}_d(\mathscr{A})$. If furthermore $\mathscr{A}$ is augmented (by $\varepsilon : \mathscr{A} \to \Bbbk$), then $\mathscr{A}$ acts on $R(\Bbbk^d)$: given $a \in \mathscr{A}$ and an elementary tensor $v = x_1 \otimes \cdots \otimes x_n \in R(\Bbbk^d)$, set $a \cdot 1 = \varepsilon(a)$ and recursively

$$a(v) = \sum_{i,j=1}^{d} \langle e_i | x_1 \rangle e_j \otimes a_{ij}(x_2 \otimes \cdots \otimes x_n)$$

if $\psi(a) = (a_{ij})$. Conversely, specifying $\phi(s), \varepsilon(s)$ for generators s of $\mathscr{A}$ defines at most one self-similar algebra acting faithfully on $R(\Bbbk^d)$.

The first example of self-similar algebra (though not couched in that language) is due to Sidki [42]. He constructed a primitive ring $\mathscr{A}$ containing both the Gupta–Sidki torsion group (see §6.4) and a transcendental element.

Another example [3] contains both the Grigorchuk group (see §6.3) and an invertible transcendental element, and has quadratic growth. It is also primitive, although the opposite was erroneously claimed in [3, Theorem 4.29].

The fundamental idea of “linearizing” the definition of a self-similar group (see §6) already appears in [48].

5.2. *Bimodules*

The definition of self-similar associative algebra, given above, has the defect of imposing a specific choice of basis. The following more abstract definition is essentially equivalent.

Definition 5.5. An associative algebra $\mathscr{A}$ is *self-similar* if it is endowed with a *covering bimodule*, that is, an $\mathscr{A}$-$\mathscr{A}$-bimodule $\mathscr{M}$ that is free qua right $\mathscr{A}$-module.

Indeed, given $\psi : \mathscr{A} \to \mathrm{Mat}_d(\mathscr{A})$, define $\mathscr{M} = \Bbbk^d \otimes \mathscr{A}$, with natural right action, and left action

$$a \cdot (e_i \otimes b) = \sum_{j=1}^{d} e_j \otimes a_{ij} b \quad \text{for } \psi(a) = (a_{ij}).$$

Conversely, if $\mathscr{M}$ is free, choose an isomorphism $\mathscr{M}_{\mathscr{A}} \cong X \otimes \mathscr{A}$ for a $\Bbbk$-module X, and choose a basis (e_i) of X; then write $\psi(a) = (a_{ij})$ where $a \cdot (e_i \otimes 1) = \sum e_j \otimes a_{ij}$ for all i.

Note that we had no reason to require X to be finite-dimensional (of dimension d) in Definition 5.5, though all our examples are of that form.

The natural action of $\mathscr{A}$ may be defined without explicit reference to a basis X of $\mathscr{M}_{\mathscr{A}}$: one simply lets $\mathscr{A}$ act on the left on

$$\bigoplus_{n \geq 0} \mathscr{M} \otimes_{\mathscr{A}} \cdots \otimes_{\mathscr{A}} \mathscr{M} \otimes_{\mathscr{A}} \Bbbk.$$

It is possible to define self-similar Lie algebras $\mathscr{L}$ in a basis-free manner, by requiring that the universal enveloping algebra $\mathbb{U}(\mathscr{L})$ be endowed with a covering bimodule. We shall not follow that approach here, because it is not (yet) justified by any applications.

5.3. Growth

Under the assumptions of Propositions 2.17 and 3.12, we may estimate the growth of the associative algebras $\mathbb{U}(\mathscr{L})$ and $\mathscr{A}(\mathscr{L})$. We begin by $\mathbb{U}(\mathscr{L})$, for which analytic number theory methods are useful. The following is an adaptation of [31, Theorem 1]. Note that, with a little more care, Nathanson obtained the same bounds $\Phi_1 = \Phi_2$, for $\alpha = 1/2$.

Lemma 5.6. *Let $f(x) = \sum_{n\geq 0} a_n x^n$ be a power series with positive coefficients, and consider $\alpha \in (0, 1)$.*

(1) *There exists a homeomorphism $\Phi_1 : [0, \infty] \to [0, \infty]$ such that*

$$\liminf_{n\to\infty} \frac{\log a_n}{n^\alpha} \geq L \quad \textit{implies} \quad \liminf_{x\to 1^-} (1-x)^{\alpha/(1-\alpha)} \log f(x) \geq \Phi_1(L).$$

(2) *There exists a homeomorphism $\Phi_2 : [0, \infty] \to [0, \infty]$ such that*

$$\limsup_{n\to\infty} \frac{\log a_n}{n^\alpha} \leq L \quad \textit{implies} \quad \limsup_{x\to 1^-} (1-x)^{\alpha/(1-\alpha)} \log f(x) \leq \Phi_2(L).$$

Proof. (1) For every $\epsilon > 0$ there are arbitrarily large $n \in \mathbb{N}$ with $a_n \geq e^{(L-\epsilon)n^\alpha}$. Consider $x = e^{-t}$ with $t \in \mathbb{R}_+$. Then $f(x) \geq a_n x^{-n} \geq e^{(L-\epsilon)n^\alpha - tn}$. This expression is maximized at $t = \alpha(L-\epsilon)n^{\alpha-1}$, with $t \to 0$ as $n \to \infty$. Therefore, $\log f(x) \geq (1-\alpha)(L-\epsilon) \times \left(\frac{t}{\alpha(L-\epsilon)}\right)^{\alpha/(\alpha-1)} = K t^{\alpha/(\alpha-1)}$ for a function $K(L,\epsilon)$. Now, for $t \to 0$, we have $t \approx 1-x$, so $(1-x)^{\alpha/(1-\alpha)} \log f(x) \geq K$ for x near 1^-. Thus $\Phi_1(L) := \lim_{\epsilon\to 0} K(L,\epsilon)$ satisfies (1).

(2) For every $\epsilon > 0$ there is $N_0 \in \mathbb{N}$ such that $a_n \leq e^{(L-\epsilon)n^\alpha}$ for all $n \geq N_0$. Consider x near 1^-, and write again $x = e^{-t}$. Set $N_1 = \left(\frac{t}{\alpha(L+\epsilon)}\right)^{1/(\alpha-1)}$. Write

$$f(x) = \sum_{n<N_0} a_n x^n + \sum_{n=N_0}^{2N_1} a_n x^n + \sum_{n\geq 2N_1} a_n x^n.$$

The first summand is bounded by a function $K_1(\epsilon)$. The second is bounded by $2N_1 e^{(L+\epsilon)N_1^\alpha - N_1 t}$. For $n \geq 2N_1$ we have $(d/dn)((L+\epsilon)n^\alpha - tn) \leq (2^{\alpha-1}-1) \times \alpha(L+\epsilon)N_1^{\alpha-1} < 0$, so the third summand is bounded by the geometric series $\sum_{i\geq 0} e^{(L+\epsilon)(2N_1)^\alpha - t(2N_1)} e^{(2^{\alpha-1}-1)\alpha(L+\epsilon)N_1^{\alpha-1} i}$. Collecting all three summands into a bound $\Phi_2(L)$ yields (2). □

Lemma 5.7. *Consider the series $f(x) = \prod_{n\geq 1}(1-x^n)^{-n^\beta}$. Then $(1-x)^{\beta+1}\log f(x)$ is bounded away from $\{0,\infty\}$ as $x \to 1^-$.*

Proof. We have $\log f(x) = -\sum_{n\geq 1} n^\beta \log(1-x^n)$; and $-\log(1-x^n) \geq x^n$ so $\log f(x) \geq \sum_{n\geq 1} n^\beta x^n \approx (1-x)^{-\beta-1}$; therefore $L_f := (1-x)^{\beta+1}\log f(x)$ is bounded away from 0.

Conversely, it makes no difference to consider $f(x)$ or $g(x) := f(x)/f(x^p)$, because $L_g = (1-p^{-1})L_f$. Now $-\log((1-x^n)/(1-x^{pn})) = \sum_{i\geq 1}(x^{ni} - x^{pni})/i \leq px^n$, so $\log g(x) \precapprox p(1-x)^{-\beta-1}$ is bounded away from ∞. □

The following is an improvement on the bounds given in [33, Proposition 1].

Theorem 5.8. *Let $\mathscr{L}$ be an $\mathbb{R}_+$-graded self-similar Lie algebra, with dilation $\lambda > 1$, that is generated by finitely many positive-degree elements. Then the universal enveloping algebra $\mathbb{U}(\mathscr{L})$ has subexponential growth; more precisely, the growth of $\mathbb{U}(\mathscr{L})$ is bounded as*

$$\dim(\mathbb{U}(\mathscr{L}))_{\leq d} \precapprox \exp(Cd^{\operatorname{GKdim}(\mathscr{L})/(\operatorname{GKdim}(\mathscr{L})+1)}) \quad \text{for some } C > 0. \tag{12}$$

If furthermore $\mathscr{L}$ is regularly weakly branched, then the exponent in (12) is sharp.

Proof. We denote, for an algebra $\mathscr{A}$, by $\mathscr{A}_n$ the homogeneous summand of degree n. Let $f(x) = \sum_{n\geq 0} \dim \mathbb{U}(\mathscr{L})_n x^n$ denote the Poincaré series of $\mathbb{U}(\mathscr{L})$. By the Poincaré–Birkhoff–Witt Theorem, we have $f(x) = f_0(x) = \prod_{n\geq 1}(1-x^n)^{-\dim \mathscr{L}_n}$ in characteristic 0, and $f(x) = f_0(x)/f_0(x^p)$ in characteristic p. By Proposition 2.17, we have

$$\sum_{n=\lambda^m}^{\lambda^{m+1}-1} \dim \mathscr{L}_n \precapprox \lambda^{\theta m} \quad \text{with } \theta = \operatorname{GKdim}(\mathscr{L}). \tag{13}$$

The estimate (12) only depends on the asymptotics of $f(x)$ near 1, and these change only by a factor of λ between the extreme cases when the sum in (13) is concentrated in its first or its last terms. It therefore makes no harm to assume $\dim \mathscr{L}_n \precapprox n^{\theta-1}$, so $(1-x)^\theta \log(x)$ is bounded away from ∞ by Lemma 5.7; and the upper bound in (12) follows from Lemma 5.6(1).

On the other hand, if $\mathscr{L}$ is regularly weakly branched then by Proposition 3.12 we have $\sum_{n=\lambda^m}^{\lambda^{m+1}-1} \dim \mathscr{L}_n \succapprox \lambda^{\theta m}$ and the lower bound in (12) follows from Lemma 5.6(2). □

We now show that the thinned algebra $\mathscr{A}(\mathscr{L})$ has finite Gelfand–Kirillov dimension, double that of $\mathscr{L}$, under the same hypotheses:

Theorem 5.9. *Let $\mathscr{L}$ be an $\mathbb{R}_+$-graded self-similar Lie algebra, with dilation $\lambda > 1$, that is generated by finitely many positive-degree elements. Then*

$$\operatorname{GKdim} \mathscr{A}(\mathscr{L}) \leq 2\frac{\log \dim X}{\log \lambda}.$$

On the other hand, if $\mathscr{L}$ is regularly weakly branched, then

$$\operatorname{GKdim} \mathscr{A}(\mathscr{L}) \geq 2\frac{\log \dim X}{\log \lambda}.$$

Proof. Let $\mathscr{A}_d$ denote the span of homogeneous elements in $\mathscr{A}(\mathscr{L})$ of degree $\leq d$. Consider $a \in \mathscr{A}_d$, and express it in $\mathrm{Mat}_{X^{\otimes n} \times X^{\otimes n}}(N)$. By (8) we have $n \leq \log_\lambda(d/\epsilon)$, so

$$\dim \mathscr{A}_d \leq \sum_{j=0}^{n} (\dim X)^{2j} \dim N \precapprox d^{2\log(\dim X)/\log\lambda}.$$

If $\mathscr{L}$ is regularly weakly branched then, following the proof of Proposition 3.12, we have

$$\dim \mathscr{A}_d \geq (d/\epsilon)^{2\log(\dim X)/\log\lambda}. \qquad \square$$

There is yet another notion of growth, which we just mention in passing. For $v \in R(X)$, consider the *orbit growth* function

$$\gamma_v(d) = \dim(\mathscr{L}_d v),$$

where $\mathscr{L}_d$ denotes the span of homogeneous elements in $\mathscr{A}(\mathscr{L})$ of degree $\leq d$; and consider also

$$\gamma(d) = \sup_{v \in R(X)} \gamma_v(d).$$

It seems that $\gamma(d)$ is closely related to $\dim \mathscr{L}_d$, but that may be an artefact of the simplicity of the examples yet considered.

5.4. Sidki's monomial algebra

Sidki [41] considered a self-similar associative algebra $\mathscr{A}$ on two generators s, t, given by the self-similarity structure

$$s \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad t \mapsto \begin{pmatrix} 0 & t \\ 0 & s \end{pmatrix}.$$

He gives a presentation for $\mathscr{A}$, all of whose relators are monomial, and shows that monomials are nil in $\mathscr{A}$ while $s + t$ is transcendental.

The self-similarity structure of $\mathscr{A}$ is close both to that of $\mathscr{A}(\mathscr{L}_2)$, the difference being the "s" at position $(1, 1)$ of $\psi(t)$; and to the thinned ring $\mathscr{A}(I_4)$ of a semigroup considered in [6], the difference being the "1" at position $(1, 2)$ in $\psi(s)$. Note also that $\mathscr{A}$ and $\mathscr{A}(I_4)$ have the same Gelfand–Kirillov dimension. One is led to wonder whether $\mathscr{A}$ may be obtained as an associated graded of $\mathscr{A}(I_4)$.

6. Self-similar groups

My starting point, in studying self-similar Lie algebras, was the corresponding notion for groups:

Definition 6.1. Let X be a set, called the *alphabet*. A group G is *self-similar* if it is endowed with a homomorphism

$$\psi : G \to G^X \rtimes \operatorname{Sym} X =: G \wr \operatorname{Sym} X,$$

called its *self-similarity structure*.

The first occurrence of this definition seems to be [39, p. 310]; it has also appeared in the context of groups generated by automata [47].

A self-similar group naturally acts on the set X^* of words over the alphabet X: given $g \in G$ and $v = x_1 \dots x_n$, define recursively

$$g(v) = \pi(x_1) g_{x_1}(x_2 \dots x_n) \quad \text{where} \quad \psi(g) = ((g_x)_{x \in X}, \pi).$$

Conversely, if $\psi(g)$ is specified for the generators of a group G, this defines at most one self-similar group acting faithfully on X^*.

Note that we have no reason to require X to be finite, though all our examples are of that form.

Nekrashevych [32] introduced a more abstract, essentially equivalent notion:

Definition 6.2. A group G is *self-similar* if it is endowed with a *covering biset*, that is, a G-G-biset M that is free qua right G-set.

Indeed, given $\psi : G \to G \wr \operatorname{Sym} X$, define $M = X \times G$, with natural right action, and left action

$$g(x, h) = (\pi(x), g_x h) \quad \text{for } \psi(g) = ((g_x)_{x \in X}, \pi).$$

Conversely, if M is free, choose an isomorphism $M_G \cong X \times G$ for a set X, and write $\psi(g) = ((g_x), \pi)$ where $g(x, 1) = (\pi(x), g_x)$ for all $x \in X$.

The natural action of G may be defined without explicit reference to a "basis" X of M: one simply lets G act on the left on

$$\bigsqcup_{n \geq 0} M \times_G \cdots \times_G M \times_G \{*\},$$

where the fibred product of bisets is $M \times_G N = M \times N / (mg, n) = (m, gn)$.

If G is a self-similar group, with self-similarity structure $\psi : G \to G \wr \operatorname{Sym} X$, and $\Bbbk$ is a ring, then its *thinned algebra* is a self-similar associative algebra $\mathscr{A}(G)$ with alphabet $\Bbbk X$. It is defined as the quotient of the group ring $\Bbbk G$ acting faithfully on $R(\Bbbk X)$, for the self-similarity structure $\psi' : \Bbbk G \to \operatorname{Mat}_X(\Bbbk G)$ given by

$$\psi'(g) = \sum_{x \in X} \mathbb{1}_{x, \pi(x)} g_x \quad \text{for } \psi(g) = ((g_x), \pi),$$

where $\mathbb{1}_{x,y}$ is the elementary matrix with a 1 at position (x, y).

The definition is even simpler in terms of bisets and bimodules: if G has a covering biset M, then $\Bbbk G$ has a covering module $\Bbbk M$, turning it into a self-similar associative algebra.

6.1. From groups to Lie algebras

We shall consider two methods of associating a Lie algebra to a discrete group. The first one, quite general, is due to Magnus [29]. Given a group G, consider its *lower central series* $(\gamma_n)_{n\geq 1}$, defined by $\gamma_1 = G$ and $\gamma_n = [G, \gamma_{n-1}]$ for $n \geq 2$. Form then

$$\mathscr{L}^{\mathbb{Z}}(G) = \bigoplus_{n\geq 1} \gamma_n/\gamma_{n+1}.$$

This is a graded abelian group; and the bracket $[g\gamma_{n+1}, h\gamma_{m+1}] = [g, h]\gamma_{n+m+1}$, extended bilinearly, gives it the structure of a graded Lie ring.

Consider now a field $\Bbbk$ of characteristic p, and define the *dimension* series $(\gamma_n^p)_{n\geq 1}$ of G by $\gamma_1^p = G$ and $\gamma_n^p = [G, \gamma_{n-1}^p](\gamma_{\lceil n/p\rceil}^p)^p$. The corresponding abelian group

$$\mathscr{L}^{\Bbbk}(G) = \bigoplus_{n\geq 1} \gamma_n^p/\gamma_{n+1}^p \otimes_{\mathbb{F}_p} \Bbbk$$

is now a Lie algebra over $\Bbbk$, which furthermore is restricted, with the p-mapping defined by $(x\gamma_{n+1}^p \otimes \alpha)^p = x^p\gamma_{pn+1}^p \otimes \alpha^p$.

Note the following alternative definition [36]: The group ring $\Bbbk G$ is filtered by powers of its augmentation ideal ϖ. The corresponding graded ring $\overline{\Bbbk G} = \bigoplus_{n\geq 0} \varpi^n/\varpi^{n+1}$ is a Hopf algebra, because ϖ is a Hopf ideal in $\Bbbk G$. Then $\mathscr{L}^{\Bbbk}(G)$ is the Lie algebra of primitive elements in $\overline{\Bbbk G}$, which itself is the universal enveloping algebra of $\mathscr{L}^{\Bbbk}(G)$.

There is a natural graded map $\mathscr{L}^{\mathbb{Z}}(G) \to \mathscr{L}^{\Bbbk}(G)$, given by $g\gamma_{n+1} \mapsto g\gamma_{n+1}^p \otimes 1$. Furthermore, its kernel $\mathscr{K}_1$ consists of elements of the form $(g\gamma_{n+1})^p$. There is a p-linear map $\mathscr{K}_1 \to \mathscr{L}^{\Bbbk}(G)$, given by $(g\gamma_{n+1})^p \mapsto g^p\gamma_{pn+1}^p \otimes 1$, and similarly for higher kernels $\mathscr{K}_m$ with $m \geq 2$.

We turn now to a second construction, specific for self-similar groups. We assume, further, that G's alphabet is $\mathbb{F}_p$, and that the image of $\psi : G \to G \wr \operatorname{Sym} X$ lies in $G \wr \mathbb{F}_p$, where $\mathbb{F}_p$ acts on itself by addition. Finally, we fix a generating set S of G, and assume that $\psi(S)$ lies in $S^X \times \mathbb{F}_p$. Let S' denote a subset of S such that every $s \in S$ is of the form $(s')^n$ for unique $s' \in S'$ and $n \in \mathbb{F}_p$.

Consider now the vector space $X' = \mathbb{F}_p[x]/(x^p)$, and the self-similar Lie algebra $\mathscr{L}(G)$ acting faithfully on $R(X')$, with generating set S', with the following self-similarity structure: set

$$\psi'(s') = \sum_{i\in\mathbb{F}_p} x^i(x+1)^{p-1-i} n_i \otimes s_i + n\partial_x \quad \text{for } \psi(s') = ((s_i^{n_i})_{i\in\mathbb{F}_p}, n).$$

Modify furthermore the self-similarity structure as follows, to obtain a graded algebra in which the elements of s' are homogeneous: if $\psi'(s') = \sum_{s\in S'} f_s(x) \otimes s + n\partial_x$, then let $\overline{f_s}$ denote the leading monomial of f_s, and set

$$\psi(s') = \sum_{s\in S'} \overline{f_s(x)} \otimes s + n\partial_x.$$

Conjecture 6.3. *Consider a self-similar group G as above, its thinned algebra $\mathscr{A}(G)$ with augmentation ideal ϖ, and the associated graded algebra $\overline{\mathscr{A}(G)} = \bigoplus_{n\geq 0} \varpi^n/\varpi^{n+1}$. Then the Lie algebra $\mathscr{L}(G)$ is a subalgebra of $\overline{\mathscr{A}(G)}$.*

Conjecture 6.4. *Consider a self-similar group G as above, and $\mathscr{L}^{\Bbbk}(G)$ its associated Lie algebra. Then $\mathscr{L}(G)$ is a quotient of $\mathscr{L}^{\Bbbk}(G)$.*

These conjectures should be proved roughly along the following construction: first, there is a natural map $\pi_0 : G \to G/[G,G] \to \mathscr{L}(G)$, sending $s \in S$ to its image in $\mathscr{L}(G)$. Then, given $g\gamma_{n+1}^p \in \mathscr{L}(G)$, let $k \in \mathbb{N}$ be maximal such that γ_n^p acts trivially on X^k, and consider $\psi^n(g) = (g_{0\dots0}, \dots, g_{p-1\dots p-1})$. Write then

$$\pi_k(g) = \sum_{i_1,\dots,i_k \in \mathbb{F}_p} x^{i_1}(x+1)^{p-1-i_1} \otimes \dots \otimes x^{i_k}(x+1)^{p-1-i_k} \otimes \pi_0(g_{i_1\dots i_k}).$$

These maps should induce a map $\mathscr{L}^{\Bbbk}(G) \to \mathscr{L}(G)$. A similar construction should relate $\varpi^n \le \mathscr{A}(G)$ and $\mathscr{L}(G)$.

Rather than pursuing this line in its generality, we shall now see, in specific examples, how $\mathscr{L}(G)$ and $\mathscr{L}^{\Bbbk}(G)$ are related, and lead from self-similar groups to the Lie algebras described in §4.

6.2. The p-Sylow subgroup of the infinite symmetric group

Kaloujnine [26] initiated the study of the Sylow p-subgroup of $\mathrm{Sym}(p^m)$, and its infinite generalization [25]. The Sylow subgroup of $\mathrm{Sym}(p^m)$ is an m-fold iterated wreath product of C_p, and these groups form a natural projective system; denote their inverse limit by W_p. Then $W_p \cong W_p \wr \mathbb{F}_p$, and if we denote this isomorphism by ψ then W_p is a self-similar group satisfying the conditions of the previous subsection.

Sushchansky and Netreba [45, 46] exploited Kaloujnine's representation of elements of W_p by "tableaux" to describe the Lie algebra $\mathscr{L}^{\mathbb{F}_p}(W_p)$ associated with W_p. This language is essentially equivalent to ours; Kaloujnine's "tableau" $x_1^{e_1}\dots x_n^{e_n}$ corresponds to our derivation $x^{p-1-e_1} \otimes \dots \otimes x^{p-1-e_n} \otimes \partial_x$. See also [4, §3.5] for more details, and [7, Theorem 11] for an embedding of $\mathscr{L}^{\mathbb{F}_p}(W_p)$ in the Poisson algebra of truncated polynomials in countably many variables. The upshot is

Proposition 6.5 ([4, Theorem 3.4]). *The Lie algebras $\mathscr{L}^{\mathbb{F}_p}(W_p)$, $\mathscr{L}^{\mathbb{Z}}(W_p)$, $\mathscr{L}(W_p)$ and $\overline{\mathscr{L}(W_p)}$ are isomorphic.*

6.3. The Grigorchuk groups

An essential example of a self-similar group was thoroughly investigated by Grigorchuk [18–20]. The "first" Grigorchuk group is defined as follows: it is self-similar; it acts faithfully on X^* for $X = \mathbb{F}_2$; it is generated by a, b, c, d; and it has self-similarity structure

$$\psi : \begin{cases} G \to G \wr \mathbb{F}_2, \\ a \mapsto ((1,1),1), \\ b \mapsto ((a,c),0), \\ c \mapsto ((a,d),0), \\ d \mapsto ((1,b),0). \end{cases} \tag{14}$$

The following notable properties of G stand out:

- It is an infinite, finitely generated torsion 2-group [2, 18], providing an accessible answer to a question by Burnside [8] about the existence of such groups.
- It has intermediate word-growth [19], namely the number of group elements expressible as a word of length $\le n$ in the generators grows faster than any polynomial, but slower than any exponential function. This answered a question by Milnor [30] on the existence of such groups.
- It has finite width [5, 38], that is, the ranks of the lower central factors γ_n/γ_{n+1} are bounded. This disproved a conjecture by Zelmanov [49].

Grigorchuk's construction was generalized, in [17], to an uncountable collection of groups G_ω, for $\omega \in \{0,1,2\}^\infty =: \Omega$. They are not self-similar anymore, but are related to each other by homomorphisms $\psi : G_\omega \to G_{\sigma\omega} \wr \mathbb{F}_2$, where $\sigma : \Omega \to \Omega$ is the one-sided shift. Interpret $0, 1, 2$ as the three non-trivial homomorphisms $\mathbb{F}_4 \to \mathbb{F}_2$. Then each G_ω is generated by $\mathbb{F}_4 \sqcup \{a\}$, and

$$\psi : \begin{cases} G_\omega \to G_{\sigma\omega} \wr \mathbb{F}_2, \\ a \mapsto ((1,1),1), \\ \mathbb{F}_4 \ni v \mapsto ((a^{\omega(v)}, v), 0). \end{cases}$$

The "first" Grigorchuk group is then G_ω for $\omega = (012)^\infty$.

The structure of the Lie algebra $\mathscr{L}^{\mathbb{Z}}(G)$, based on calculations in [38], and of $\mathscr{L}^{\mathbb{F}_2}(G)$, are described in [5], and more explicitly in [4, Theorem 3.5]. Note, however, some missing arrows in [4, Figure 2] between $\mathbb{1}^n(x^2)$ and $\mathbb{1}^{n+2}(x)$. Notice also that $\mathscr{L}^{\mathbb{F}_2}(G)$ is neither just infinite nor centreless: its centre is spanned by $\{W(x^2) \mid W \in \{\mathbb{0},\mathbb{1}\}^* \setminus \{\mathbb{1}\}^*\}$ and has finite codimension.

Recall the *upper central series* of a Lie algebra $\mathscr{L}$: it is defined inductively by $\zeta_0 = 0$, $\zeta_{n+1}/\zeta_n = \zeta(\mathscr{L}/\zeta_n)$, and $\zeta_\omega = \bigcup_{\alpha<\omega} \zeta_\alpha$. In particular, ζ_1 is the centre of $\mathscr{L}$.

Theorem 6.6. *The Lie algebras $\mathscr{L}(G)$, $\mathscr{L}^{\mathbb{F}_2}(G)/\zeta(\mathscr{L}^{\mathbb{F}_2}(G))$ and ${}_2\mathscr{L}_{\mathrm{G}}$ are isomorphic.*

The Lie algebras $\mathscr{L}^{\mathbb{Z}}(G)/\zeta_\omega(\mathscr{L}^{\mathbb{Z}}(G))$ and $\mathscr{L}_{\mathrm{G}}$ are isomorphic.

Proof. We recall from [4, Theorem 3.5] the following explicit description of $\mathscr{L}^{\mathbb{F}_2}(G)$. Write $e = [a,b]$. A basis of $\mathscr{L}^{\mathbb{F}_2}(G)$ is

$$\{a, b, d, [a,d]\} \cup \{W(e), W(e^2) \mid W \in \{\mathbb{0},\mathbb{1}\}^*\},$$

where a, b, d have degree 1; $[a,d]$ has degree 2; and $\deg(X_1 \dots X_n(e)) = 1 + \sum_{i=1}^n X_i 2^{i-1} + 2^n = \frac{1}{2}\deg(X_1 \dots X_n(e^2))$. The 2-mapping sends $W(e)$ to $W(e^2)$ and $\mathbb{1}^n(e^2)$ to $\mathbb{1}^{n+2}(e) + \mathbb{1}^{n+1}(e^2)$ and all other basis vectors to 0.

In particular, the centre of $\mathscr{L}^{\mathbb{F}_2}(G)$ is spanned by the set $\{W(e^2) \mid W \notin \{\mathbb{1}\}^*\}$, and $\mathscr{L}^{\mathbb{F}_2}(G)/\zeta(\mathscr{L}^{\mathbb{F}_2}(G))$ is centreless.

Note then the following isomorphism between $\mathscr{L}^{\mathbb{F}_2}(G)/\zeta(\mathscr{L}^{\mathbb{F}_2}(G))$ and ${}_2\mathscr{L}_{\mathrm{G}}$. It sends $a, b, d, [d,a]$ to $a, b, d, [d,a]$ respectively; and $X_1 \dots X_n(e^s)$ to $x^{1-X_1} \otimes \dots \otimes x^{1-X_n} \otimes e^s$.

It follows that $\mathscr{L}^{\mathbb{F}_2}(G)/\zeta(\mathscr{L}^{\mathbb{F}_2}(G))$ admits the injective self-similarity structure (10), and therefore equals ${}_2\mathscr{L}_{\mathrm{G}}$. On the other hand, direct inspection shows that the self-similarity structure of $\mathscr{L}(G)$ equals that of ${}_2\mathscr{L}_{\mathrm{G}}$.

In the description of $\mathscr{L}^{\mathbb{Z}}(G)$, the basis element $W(e^2)$ has degree $1+\sum_{i=1}^{n} X_i 2^{i-1}+2^{n+1}$. The centre of $\mathscr{L}^{\mathbb{Z}}(G)$ is spanned by the $\{\mathbb{1}^n(e^2)\}$, and more generally $\zeta_k(\mathscr{L}^{\mathbb{Z}}(G))$ is spanned by the $\{X_1 \ldots X_n(e^2) \mid \sum_{i=1}^{n} X_i 2^{i-1} \geq 2^n - k$. It then follows that $\mathscr{L}^{\mathbb{Z}}(G)/\zeta_\omega(\mathscr{L}^{\mathbb{Z}}(G))$ has a basis $\{a, b, d, [a, d]\} \cup \{W(e)\}$, and admits the injective self-similarity structure (10), so it equals $\mathscr{L}_{\mathrm{G}}$. □

Theorem 6.7. *The thinned algebras associated with G and $\mathscr{L}(G)$ are isomorphic: $\mathscr{A}(G) \cong \mathscr{A}(\mathscr{L}(G))$.*

Proof. The algebra $\mathscr{A}(G)$ is just infinite [3, Theorem 4.3], and has an explicit presentation

$$\mathscr{A}(G) = \langle A, B, C, D \mid \mathcal{R}_0, \sigma^n(CACACAC), \sigma^n(DACACAD) \text{ for all } n \geq 0\rangle,$$

where $\sigma : \{A, B, C, D\}^* \to \{A, B, C, D\}^*$ is the substitution

$$A \mapsto ACA, \quad B \mapsto D, \quad C \mapsto B, \quad D \mapsto C$$

and

$$\mathcal{R}_0 = \{A^2, B^2, C^2, D^2, B + C + D, BC, CB, BD, DB, CD, DC, DAD\}.$$

It is easy to check that $\psi(\sigma(w)) = \left(\begin{smallmatrix} 0 & 0 \\ w & 0 \end{smallmatrix}\right)$ for all words $w \in \{A, B, C, D\}$ of length at least 2 and starting and ending in $\{B, C, D\}$; moreover, the relations $\mathcal{R}_0 \cup \{CACACAC, DACACAD\}$ are satisfied in $\mathscr{A}(\mathscr{L}(G))$. There exists therefore a homomorphism $\mathscr{A}(G) \to \mathscr{A}(\mathscr{L}(G))$, which must be an isomorphism because its image has infinite dimension. □

6.4. The Gupta–Sidki group

Another important example of a self-similar group was studied by Gupta and Sidki [21]. This group is defined as follows: it is self-similar; it acts faithfully on X^* for $X = \mathbb{F}_3$; it is generated by a, t; and it has self-similarity structure

$$\psi : \begin{cases} G \to G \wr \mathbb{F}_3, \\ a \mapsto ((1, 1, 1), 1), \\ t \mapsto ((a, a^{-1}, t), 0). \end{cases} \tag{15}$$

Gupta and Sidki prove that G is an infinite, finitely generated torsion 3-group.

The Lie algebra $\mathscr{L}^{\mathbb{Z}}(G) = \mathscr{L}^{\mathbb{F}_3}(G)$ is described in [4, Theorem 3.8], where it is shown that $\mathscr{L}^{\mathbb{Z}}(G)$ has unbounded width.

Theorem 6.8. *The Lie algebras $\mathscr{L}(G)$, $\mathscr{L}^{\mathbb{F}_3}(G)$, $\mathscr{L}^{\mathbb{Z}}(G)$ and $\mathscr{L}_{\mathrm{GS}}$ are isomorphic.*

Proof. We recall from [4, Theorem 3.8] the following explicit description of $\mathscr{L}^{\mathbb{F}_p}(G)$. Write $c = [a, t]$ and $u = [a, c]$. Define the integers α_n by $\alpha_1 = 1$, $\alpha_2 = 2$, and $\alpha_n = 2\alpha_{n-1} + \alpha_{n-2}$ for $n \geq 3$. A basis of $\mathscr{L}^{\mathbb{F}_p}(G)$ is

$$\{a, t\} \cup \{W(c), W(u) \mid W \in \{\mathbb{0}, \mathbb{1}, \mathbb{2}\}^*\},$$

where a, t have degree 1, and where $\deg(X_1 \dots X_n(c)) = 1 + \sum_{i=1}^n X_i\alpha_i + \alpha_{n+1} = \deg(X_1 \dots X_n(u)) - \alpha_{n+1}$. In that basis, the 3-mapping sends $\mathbb{2}^n(c)$ to $\mathbb{2}^n\mathbb{0}^2(c) + \mathbb{2}^n\mathbb{1}(u)$ and all other basis vectors to 0.

In particular, $\mathscr{L}^{\mathbb{Z}}(G)$ and $\mathscr{L}^{\mathbb{F}_3}(G)$ are isomorphic.

Note then the following isomorphism between $\mathscr{L}^{\mathbb{Z}}(G)$ and ${}_2\mathscr{L}_{\mathrm{G}}$. It sends a, t to a, t respectively; and $X_1 \dots X_n(b)$ to $x^{2-X_1} \otimes \cdots \otimes x^{2-X_n} \otimes b$ for $b \in \{c, u\}$.

It follows that $\mathscr{L}^{\mathbb{Z}}(G)$ admits the injective self-similarity structure (9), and therefore equals $\mathscr{L}_{\mathrm{GS}}$. On the other hand, direct inspection shows that the self-similarity structure of $\mathscr{L}(G)$ equals that of $\mathscr{L}_{\mathrm{GS}}$. ◻

It is tempting to conjecture, in view of Theorem 6.7, that the associated graded $\bigoplus_{n\geq 0} \varpi^n/\varpi^{n+1}$ of $\mathscr{A}(G)$ is isomorphic to $\mathscr{A}(\mathscr{L})$; presumably, this could be proven by finding a presentation of $\mathscr{A}(G)$.

6.5. *From Lie algebras to groups*

We end with some purely speculative remarks. Although we gave a construction of a Lie algebra starting from a self-similar group, this construction depends on several choices, in particular of a generating set for the group. Could it be that the resulting algebra $\mathscr{L}(G)$, or $\overline{\mathscr{L}(G)}$, is independent of such choices? Is $\overline{\mathscr{L}(G)}$ always isomorphic to $\mathscr{L}^p(G)$?

On the other hand, I do not know of any “interesting” group to associate with a self-similar Lie algebra—that would, for example, be a torsion group if the Lie algebra is nil, or have subexponential growth if the Lie algebra has subexponential growth.

A naive attempt is the following. Consider the “Fibonacci” Lie algebra $\mathscr{L}_{2,\mathbb{F}_2}$ from §4.5. Its corresponding self-similar group should have alphabet $X = \mathbb{F}_2$, and generators a, t with self-similarity structure

$$\psi : \begin{cases} G \to G \wr \mathbb{F}_2, \\ a \mapsto ((1, 1), 1), \\ t \mapsto ((a, at), 0), \end{cases}$$

at least up to commutators. That group can easily be shown to be contracting, and also regularly weakly branched on the subgroup $\langle [a, t, t] \rangle^G$. It has Hausdorff dimension $1/3$, and probably exponential growth.

More generally, are there subgroups of the units in $\mathscr{A}(\mathscr{L})$ that are worth investigation, for $\mathscr{L}$ one of the Lie algebras from §4?

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