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The geometry of dented pentagram maps

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Abstract. We propose a new family of natural generalizations of the pentagram map from 2D to higher dimensions and prove their integrability on generic twisted and closed polygons. In dimension d there are $d-1$ such generalizations called dented pentagram maps, and we describe their geometry, continuous limit, and Lax representations with a spectral parameter. We prove algebraic-geometric integrability of the dented pentagram maps in the 3D case and compare the dimensions of invariant tori for the dented maps with those for the higher pentagram maps constructed with the help of short diagonal hyperplanes. When restricted to corrugated polygons, the dented pentagram maps coincide with one another and with the corresponding corrugated pentagram map. Finally, we prove integrability for a variety of pentagram maps for generic and partially corrugated polygons in higher dimensions.

Keywords. Pentagram maps, space polygons, Lax representation, discrete integrable system, KdV hierarchy, Boussinesq equation, algebraic-geometric integrability

Contents

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Introduction

The pentagram map was originally defined in [10] as a map on plane convex polygons considered up to projective equivalence, where a new polygon is spanned by the shortest diagonals of the initial one (see Figure 1). This map is the identity for pentagons, it is an involution for hexagons, while for polygons with more vertices it was shown to exhibit quasi-periodic behaviour under iterations. The pentagram map was extended to the case of twisted polygons and its integrability in 2D was proved in [9] (see also [12]).

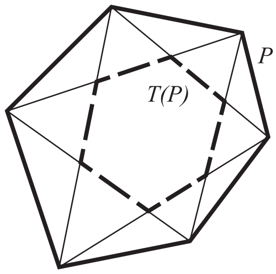

Fig. 1. The image $T(P)$ of a hexagon P under the 2D pentagram map.

While this map is in a sense unique in 2D, its generalizations to higher dimensions seem to allow more freedom. A natural requirement for such generalizations, though, is their integrability. In [4] we observed that there is no natural generalization of this map to polyhedra and suggested a natural integrable generalization of the pentagram map to generic twisted space polygons (see Figure 2). This generalization in any dimension was defined via intersections of “short diagonal” hyperplanes, which are symmetric higher-dimensional analogs of polygon diagonals (see Section 1 below). This map turned out to be scale invariant (see [9] for 2D, [4] for 3D, [8] for higher D) and integrable in any dimension as it admits a Lax representation with a spectral parameter [4].

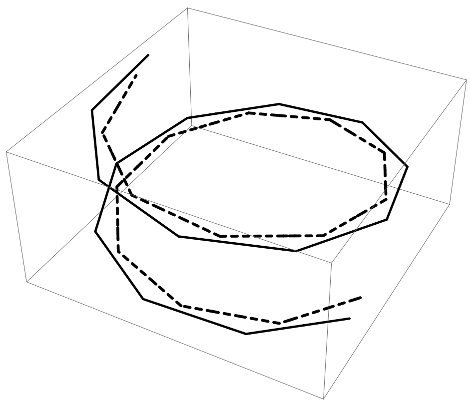

Fig. 2. A space pentagram map is applied to a twisted polygon in 3D.

A different integrable generalization to higher dimensions was proposed in [2], where the pentagram map was defined not on generic, but on so-called corrugated polygons. These are piecewise linear curves in $\mathbb{RP}^d$ whose pairs of edges with indices differing by d lie in one two-dimensional plane. It turned out that the pentagram map on corrugated polygons is integrable and it admits an explicit description of the Poisson structure, a cluster algebra structure, and other interesting features [2].

In this paper we present a variety of integrable generalized pentagram maps, which unifies these two approaches. "Primary integrable maps" in our construction are called the dented pentagram maps. These maps are defined for generic twisted polygons in $\mathbb{RP}^d$. It turns out that the pentagram maps for corrugated polygons considered in [2] are a particular case (more precisely, a restriction) of these dented maps. We describe in detail how to perform such a reduction in Section 5.

To define the dented maps, we propose a definition of a "dented diagonal hyperplane" depending on a parameter $m = 1, \dots, d-1$, where d is the dimension of the projective space. The parameter m marks the skipped vertex of the polygon, and in dimension d there are $d-1$ different dented integrable maps. The vertices in the "dented diagonal hyperplanes" are chosen in a nonsymmetric way (as opposed to the unique symmetric choice in [4]). We would like to stress that in spite of the nonsymmetric choice, the integrability property is preserved, and each of the dented maps can be regarded as a natural generalization of the classical 2D pentagram map of [10]. We describe the geometry and Lax representations of the dented maps and their generalizations, the deep-dented pentagram maps, and prove their algebraic-geometric integrability in 3D. In a sense, from now on a new challenge would be to find examples of nonintegrable Hamiltonian maps of pentagram type (cf. [5]).

We emphasize that throughout the paper we understand *integrability* as the existence of a Lax representation with a spectral parameter corresponding to scaling invariance of a given dynamical system. We show how it is used to prove algebraic-geometric integrability for the primary maps in $\mathbb{CP}^3$. In any dimension, the Lax representation provides first integrals (as the coefficients of the corresponding spectral curve) and allows one to use algebraic-geometric machinery to prove various integrability properties. We also note that while most of the paper deals with n-gons satisfying the condition $\gcd(n, d+1) = 1$, the results hold in full generality and we show how they are adapted to the general setting in Section 2.2. While most of the definitions below work both over $\mathbb{R}$ and $\mathbb{C}$, throughout the paper we describe the geometric features of pentagram maps over $\mathbb{R}$, while their Lax representations over $\mathbb{C}$.

Here are the main results of the paper.

• We define generalized pentagram maps $T_{I,J}$ on (projective equivalence classes of) twisted polygons in $\mathbb{RP}^d$, associated with $(d-1)$-tuples I and J of numbers: the tuple I defines which vertices to take in the definition of the diagonal hyperplanes, while J determines which of the hyperplanes to intersect in order to get the image point. In Section 1 we prove a duality between such pentagram maps:

$$T_{I,J}^{-1} = T_{J^*,I^*} \circ Sh,$$

where I^* and J^* stand for the $(d-1)$-tuples taken in the opposite order, and *Sh* is any shift in the indices of polygon vertices.

• The *dented pentagram maps* T_m on polygons (v_k) in $\mathbb{RP}^d$ are defined by intersecting d consecutive diagonal hyperplanes. Each hyperplane P_k passes through all vertices but one from v_k to v_{k+d} by skipping only the vertex v_{k+m}. The main theorem on such maps is the following (cf. Theorem 2.6):

Theorem 0.1. *The dented pentagram map T_m on both twisted and closed n-gons in any dimension d and any $m = 1, \ldots, d-1$ is an integrable system in the sense that it admits a Lax representation with a spectral parameter.*

We also describe the dual dented maps, prove their scale invariance (see Section 3), and study their geometry in detail. Theorem 2.15 shows that in dimension 3 the algebraic-geometric integrability follows from the proposed Lax representation for both dented pentagram maps and the short-diagonal pentagram map.

• The continuous limit of any dented pentagram map T_m (and more generally, of any generalized pentagram map) in dimension d is the $(2, d+1)$-KdV flow of the Adler–Gelfand–Dickey hierarchy on the circle (see Theorem 4.1). For 2D this is the classical Boussinesq equation on the circle, $u_{tt} + 2(u^2)_{xx} + u_{xxxx} = 0$, which appears as the continuous limit of the 2D pentagram map [9, 11].

• Consider the space of corrugated polygons in $\mathbb{RP}^d$, i.e., twisted polygons whose vertices v_{k-1}, v_k, v_{k+d-1}, and v_{k+d} span a projective two-dimensional plane for every $k \in \mathbb{Z}$, following [2]. It turns out that the pentagram map $T_{\rm cor}$ on them can be viewed as a particular case of the dented pentagram map (see Theorem 5.3):

Theorem 0.2. *This pentagram map $T_{\rm cor}$ is a restriction of the dented pentagram map T_m for any $m = 1, \ldots, d-1$ from generic n-gons $\mathcal{P}_n$ in $\mathbb{RP}^d$ to corrugated ones $\mathcal{P}_n^{\rm cor}$ (or differs from it by a shift in vertex indices). In particular, these restrictions for different m coincide modulo an index shift.*

We also describe the algebraic-geometric integrability for corrugated pentagram map in $\mathbb{CP}^3$ (see Section 5.2).

• Finally, we provide an application of dented pentagram maps. The latter can be regarded as "primary" objects, simplest integrable systems of pentagram type. By considering more general diagonal hyperplanes, such as "deep-dented diagonals", i.e., those skipping more than one vertex, one can construct new integrable systems (see Theorem 6.2):

Theorem 0.3. *The deep-dented pentagram maps in $\mathbb{RP}^d$ are restrictions of integrable systems to invariant submanifolds and have Lax representations with a spectral parameter.*

The main tool to prove integrability in this more general setting is an introduction of the corresponding notion of *partially corrugated polygons*, occupying an intermediate position between corrugated and generic ones (see Section 6). The pentagram map on such polygons also turns out to be integrable. This work brings about the following question, which manifests the change of perspective on generalized pentagram maps:

Problem 0.4. Is it possible to choose the diagonal hyperplane so that the corresponding pentagram map is nonintegrable?

Some numerical evidence in this direction is presented in [5].

1. Duality of pentagram maps in higher dimensions

We start with the notion of a twisted n-gon in dimension d.

Definition 1.1. A *twisted n-gon* in a projective space $\mathbb{RP}^d$ with monodromy $M \in SL_{d+1}(\mathbb{R})$ is a map $\phi : \mathbb{Z} \to \mathbb{RP}^d$ such that $\phi(k+n) = M \circ \phi(k)$ for each $k \in \mathbb{Z}$ and where M acts naturally on $\mathbb{RP}^d$. Two twisted n-gons are *equivalent* if there is a transformation $g \in SL_{d+1}(\mathbb{R})$ such that $g \circ \phi_1 = \phi_2$.

We assume that the vertices $v_k := \phi(k)$, $k \in \mathbb{Z}$, are in general position (i.e., no $d+1$ consecutive vertices lie in the same hyperplane in $\mathbb{RP}^d$), and we denote by $\mathcal{P}_n$ the space of generic twisted n-gons considered up to the above equivalence. Define general pentagram maps as follows.

Definition 1.2. Let $I = (i_1, \ldots, i_{d-1})$ and $J = (j_1, \ldots, j_{d-1})$ be $(d-1)$-tuples of numbers $i_\ell, j_m \in \mathbb{N}$. For a generic twisted n-gon in $\mathbb{RP}^d$ one can define an *I-diagonal hyperplane* P_k as the one passing through d vertices of the n-gon by taking every i_ℓth vertex starting from v_k, i.e.,

$$P_k := (v_k, v_{k+i_1}, v_{k+i_1+i_2}, \ldots, v_{k+i_1+\cdots+i_{d-1}})$$

(see Figure 3).

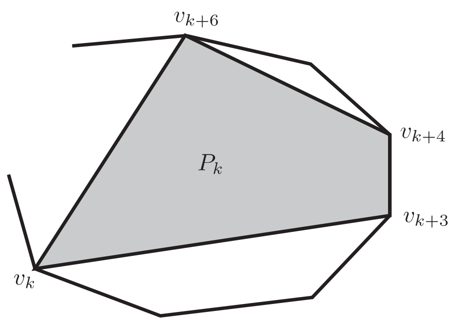

Fig. 3. The diagonal hyperplane for the jump tuple $I = (3, 1, 2)$ in $\mathbb{RP}^4$.

The image of v_k under the *generalized pentagram map* $T_{I,J}$ is defined by intersecting every j_mth out of the I-diagonal hyperplanes starting with P_k:

$$T_{I,J}v_k := P_k \cap P_{k+j_1} \cap P_{k+j_1+j_2} \cap \cdots \cap P_{k+j_1+\cdots+j_{d-1}}.$$

(Thus I defines the structure of the diagonal hyperplane, while J governs which of them to intersect.) The corresponding map $T_{I,J}$ is considered (and is generically defined) on the space $\mathcal{P}_n$ of equivalence classes of n-gons in $\mathbb{RP}^d$. As usual, we assume that the vertices are in "general position," and any d hyperplanes P_i intersect in one point in $\mathbb{RP}^d$.

Example 1.3. Consider the case of $I = (2, \ldots, 2)$ and $J = (1, \ldots, 1)$ in $\mathbb{RP}^d$. This choice of I corresponds to "short diagonal hyperplanes", i.e., every I-diagonal hyperplane passes through d vertices by taking every other vertex of the twisted polygon. The choice of J corresponds to taking intersections of d consecutive hyperplanes. This recovers the definition of the short-diagonal (or higher) pentagram maps from [4]. Note that the classical 2D pentagram map has I and J each consisting of one number: $I = (2)$ and $J = (1)$.

Denote by $I^* = (i_{d-1}, \ldots, i_1)$ the $(d-1)$-tuple I taken in the opposite order and by Sh the operation of any index shift on the sequence of vertices.

Theorem 1.4 (Duality). *There is the following duality for the generalized pentagram maps $T_{I,J}$:*

$$T_{I,J}^{-1} = T_{J^*,I^*} \circ Sh,$$

where Sh stands for some shift in indices of vertices.

Proof. To prove this theorem we introduce the following duality maps (cf. [9]).

Definition 1.5. Given a generic sequence of points $\phi(j) \in \mathbb{RP}^d$, $j \in \mathbb{Z}$, and a $(d-1)$-tuple $I = (i_1, \ldots, i_{d-1})$, we define the following *sequence of hyperplanes* in $\mathbb{RP}^d$:

$$\alpha_I(\phi(j)) := (\phi(j), \phi(j+i_1), \ldots, \phi(j+i_1+\cdots+i_{d-1})),$$

which is regarded as a sequence of points in the dual space: $\alpha_I(\phi(j)) \in (\mathbb{RP}^d)^*$.

The generalized pentagram map $T_{I,J}$ can be defined as a composition of two such maps up to a shift of indices: $T_{I,J} = \alpha_I \circ \alpha_J \circ Sh$.

Note that for a special $I = (p, \ldots, p)$ the maps α_I are involutions modulo index shifts (i.e., $\alpha_I^2 = Sh$), but for general I the maps α_I are no longer involutions. However, one can see from their construction that they have the following duality property: $\alpha_I \circ \alpha_{I^*} = Sh$, and they commute with index shifts: $\alpha_I \circ Sh = Sh \circ \alpha_I$.

Now we see that

$$T_{I,J} \circ T_{J^*,I^*} = (\alpha_I \circ \alpha_J \circ Sh) \circ (\alpha_{J^*} \circ \alpha_{I^*} \circ Sh) = Sh,$$

as required. □

Remark 1.6. For d-tuples $I = (p, \ldots, p)$ and $J = (r, \ldots, r)$ the generalized pentagram maps correspond to the general pentagram maps $T_{p,r} = T_{I,J}$ discussed in [4], and they exhibit the following duality: $T_{p,r}^{-1} = T_{r,p} \circ Sh$.

Note that in [7] one considered the intersection of the hyperplane P_k with a chord joining two vertices, which leads to a different generalization of the pentagram map and for which an analog of the above duality is yet unknown.

In [4] we studied the case $T_{2,1}$ of short diagonal hyperplanes: $I = (2, \ldots, 2)$ and $J = (1, \ldots, 1)$, which is a very symmetric way of choosing the hyperplanes and their intersections. In this paper we consider the general, nonsymmetric choice of vertices.

Theorem 1.7. *If $J = J^*$ (i.e., α_J is an involution), then modulo a shift in indices*

(i) *the pentagram maps $T_{I,J}$ and T_{J,I^*} are inverses of each other;*

(ii) *the pentagram maps $T_{I,J}$ and $T_{J,I}$ (and hence $T_{I,J}$ and $T_{I^*,J}^{-1}$) are conjugate to each other, i.e., the map α_J conjugates the map $T_{I,J}$ on n-gons in $\mathbb{RP}^d$ to the map $T_{J,I}$ on n-gons in $(\mathbb{RP}^d)^*$.*

In particular, all four maps $T_{I,J}$, $T_{I^*,J}$, $T_{J,I}$ and T_{J,I^*} are integrable or nonintegrable simultaneously. Whenever they are integrable, their integrability characteristics, e.g. the dimensions of invariant tori, the periods of the corresponding orbits, etc., coincide.

Proof. The statement (i) follows from Theorem 1.4. To prove (ii) we note that for $J = J^*$ one has $\alpha_J^2 = Sh$ and therefore

$$\alpha_J \circ T_{I,J} \circ \alpha_J^{-1} = \alpha_J \circ (\alpha_I \circ \alpha_J \circ Sh) \circ \alpha_J = (\alpha_J \circ \alpha_I \circ Sh) \circ \alpha_J^2 = T_{J,I} \circ Sh.$$

Hence modulo index shifts, the pentagram map $T_{I,J}$ is conjugate to $T_{J,I}$, while by (i) they are also inverses of T_{J,I^*} and $T_{I^*,J}$ respectively. This proves the theorem. □

2. Dented pentagram maps

2.1. Integrability of dented pentagram maps

From now on we consider the case of $J = \mathbf{1} := (1, \ldots, 1) = J^*$ for different I's, i.e., we take the intersection of *consecutive* I-diagonal hyperplanes.

Definition 2.1. Fix an integer parameter $m \in \{1, \ldots, d-1\}$ and for the $(d-1)$-tuple I set $I = I_m := (1, \ldots, 1, 2, 1, \ldots, 1)$, where the only 2 is at the mth place. This choice of I corresponds to the diagonal plane P_k which passes through consecutive vertices $v_k, v_{k+1}, \ldots, v_{k+m-1}$, then skips vertex v_{k+m} and continues passing through consecutive vertices $v_{k+m+1}, \ldots, v_{k+d}$:

$$P_k := (v_k, v_{k+1}, \ldots, v_{k+m-1}, v_{k+m+1}, v_{k+m+2}, \ldots, v_{k+d}).$$

We call such a plane P_k a *dented* (or *m-dented*) *diagonal plane*, as it is "dented" at the vertex v_{k+m} (see Figure 4). We define the *dented pentagram map* T_m by intersecting d consecutive planes P_k:

$$T_m v_k := P_k \cap P_{k+1} \cap \cdots \cap P_{k+d-1}.$$

In other words, the dented pentagram map is $T_m := T_{I_m,\mathbf{1}}$, i.e. $T_{I_m,J}$ where $J = \mathbf{1}$.

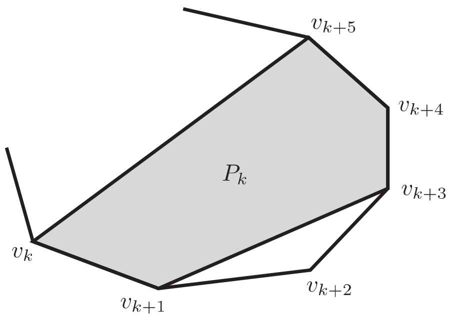

Fig. 4. The dented diagonal hyperplane P_k for $m = 2$ in $\mathbb{RP}^5$.

Corollary 2.2. *The dented map T_m is conjugate (by means of the involution $\alpha_{\mathbf{1}}$) to T^{-1}_{d-m} modulo shifts.*

Proof. Indeed, $I_m = I^*_{d-m}$ and hence, due to Theorem 1.7, one has $\alpha_{\mathbf{1}} \circ T_m \circ \alpha_{\mathbf{1}} = \alpha_{\mathbf{1}} \circ T_{I_m,\mathbf{1}} \circ \alpha_{\mathbf{1}} = T_{\mathbf{1},I_m} \circ Sh = T^{-1}_{I^*_m,\mathbf{1}} \circ Sh = T^{-1}_{I_{d-m},\mathbf{1}} \circ Sh = T^{-1}_{d-m} \circ Sh$, where $\alpha_{\mathbf{1}}$ stands for α_J with $J = (1, \ldots, 1)$. □

One can also see that for $m = 0$ or $m = d$ all the vertices defining the hyperplane P_k are taken consecutively, and T_m is the identity modulo a shift in the indices of v_k.

For 2D the only option for the dented map is $I = (2)$ and $J = (1)$, and we have $m = 1$, so T_m coincides with the classical pentagram transformation $T = T_{2,1}$ in 2D. Thus the above definition of T_m for various m is another natural higher-dimensional generalization of the 2D pentagram map. Unlike the definition of the short-diagonal pentagram map $T_{2,1}$ in $\mathbb{RP}^d$, the dented pentagram map is not unique for each dimension d, but also has one more integer parameter $m = 1, \ldots, d-1$.

It turns out that the dented pentagram map T_m defined this way, i.e., as $T_{I_m,\mathbf{1}}$ for $I_m = (1, \ldots, 1, 2, 1, \ldots, 1)$ and $\mathbf{1} = (1, 1, \ldots, 1)$, has a special scaling invariance. To describe it we need to introduce coordinates on the space $\mathcal{P}_n$ of twisted n-gons.

Now we complexify the setting and consider the spaces and maps over $\mathbb{C}$.

Remark-Definition 2.3. One can show that there exists a lift of the vertices $v_k = \phi(k) \in \mathbb{CP}^d$ to vectors $V_k \in \mathbb{C}^{d+1}$ satisfying $\det(V_j, V_{j+1}, \ldots, V_{j+d}) = 1$ and $V_{j+n} = MV_j$, $j \in \mathbb{Z}$, where $M \in SL_{d+1}(\mathbb{C})$, provided that $\gcd(n, d+1) = 1$. The corresponding lifted vectors satisfy difference equations of the form

$$V_{j+d+1} = a_{j,d}V_{j+d} + a_{j,d-1}V_{j+d-1} + \cdots + a_{j,1}V_{j+1} + (-1)^d V_j, \quad j \in \mathbb{Z}, \qquad (1)$$

with coefficients n-periodic in j. This allows one to introduce *coordinates* $\{a_{j,k}, 0 \le j \le n-1, 1 \le k \le d\}$ on the space of twisted n-gons in $\mathbb{CP}^d$. In the theorems below we assume $\gcd(n, d+1) = 1$ whenever we use explicit formulas in the coordinates $\{a_{j,k}\}$. However, the statements hold in full generality and we discuss how the corresponding formulas can be adapted in Section 2.2. (Strictly speaking, the lift from vertices to vectors is not unique, because it is defined up to simultaneous multiplication of all vectors by ε, where $\varepsilon^{d+1} = 1$, but the coordinates $\{a_{j,k}\}$ are well-defined as they have the same values for all lifts.)[1]

Theorem 2.4 (Scaling invariance). *The dented pentagram map T_m on twisted n-gons in $\mathbb{CP}^d$ with hyperplanes P_k defined by taking the vertices in a row but skipping the mth vertex is invariant with respect to the following scaling transformations:*

$$a_{j,1} \to s^{-1}a_{j,1}, \quad a_{j,2} \to s^{-2}a_{j,2}, \quad \dots, \quad a_{j,m} \to s^{-m}a_{j,m},$$
$$a_{j,m+1} \to s^{d-m}a_{j,m+1}, \quad \dots, \quad a_{j,d} \to sa_{j,d},$$

for all $s \in \mathbb{C}^*$.

For $d = 2$ this is the case of the classical pentagram map (see [9]). We prove this theorem in Section 3.2. The above scale invariance implies the Lax representation, which opens up the possibility to establish algebraic-geometric integrability of the dented pentagram maps.

Remark 2.5. Recall that a discrete Lax equation with a spectral parameter is a representation of a dynamical system in the form

$$L_{j,t+1}(\lambda) = P_{j+1,t}(\lambda)L_{j,t}(\lambda)P_{j,t}^{-1}(\lambda), \tag{2}$$

where t stands for the discrete time variable, j refers to the vertex index, and λ is a complex spectral parameter. It is a discrete version of the classical zero curvature equation $\partial_t L - \partial_x P = [P, L]$.

Theorem 2.6 (Lax form). *The dented pentagram map T_m on both twisted and closed n-gons in any dimension d and for any $m = 1, \dots, d-1$ is an integrable system in the sense that it admits a Lax representation with a spectral parameter. In particular, for* $\gcd(n, d+1) = 1$ *the Lax matrix is*

$$L_{j,t}(\lambda) = \left(\begin{array}{cccc|c} 0 & 0 & \cdots & 0 & (-1)^d \\ \hline & & & & a_{j,1} \\ & & D(\lambda) & & a_{j,2} \\ & & & & \cdots \\ & & & & a_{j,d} \end{array}\right)^{-1},$$

with the diagonal $d \times d$ matrix $D(\lambda) = \mathrm{diag}(1, \dots, 1, \lambda, 1, \dots 1)$, where the spectral parameter λ is at the $(m+1)$th place, and an appropriate matrix $P_{j,t}(\lambda)$.

[1] Note also that over $\mathbb{R}$ for odd d to obtain the lifts of n-gons from $\mathbb{RP}^d$ to $\mathbb{R}^{d+1}$ one might need to switch the sign of the monodromy matrix: $M \to -M \in SL_{d+1}(\mathbb{R})$, since the field is not algebraically closed. These monodromies in $SL_{d+1}(\mathbb{R})$ correspond to the same projective monodromy in $PSL_{d+1}(\mathbb{R})$.

Proof. Rewrite the difference equation (1) in matrix form. It is equivalent to the relation $(V_{j+1}, \ldots, V_{j+d+1}) = (V_j, \ldots, V_{j+d})N_j$, where the transformation matrix N_j is

$$N_j := \left(\begin{array}{ccc|c} 0 & \cdots & 0 & (-1)^d \\ \hline & & & a_{j,1} \\ & \mathrm{Id} & & \cdots \\ & & & a_{j,d} \end{array}\right),$$

and where Id stands for the identity $d \times d$ matrix.

It turns out that the monodromy M for twisted n-gons is always conjugate to the product $\tilde{M} := N_0 \ldots N_{n-1}$ (see Remark 2.7 below). Note that the pentagram map defined on classes of projective equivalence preserves the conjugacy class of M and hence that of $\tilde{M}$. Using the scaling invariance of the pentagram map T_m, replace $a_{j,k}$ by $s^* a_{j,k}$ for all k in the right column of N_j to obtain a new matrix $N_j(s)$. The pentagram map preserves the conjugacy class of the new monodromy $\tilde{M}(s) := N_0(s) \ldots N_{n-1}(s)$ for any s, that is, the monodromy can only change to a conjugate one during its pentagram evolution: $\tilde{M}_{t+1}(s) = P_t(s)\tilde{M}_t(s)P_t^{-1}(s)$. Then $N_j(s)$ (or, more precisely, $N_{j,t}(s)$ to emphasize its dependence on t), being a discretization of the monodromy $\tilde{M}$, could be taken as a Lax matrix $L_{j,t}(s)$. The gauge transformation $L_{j,t}^{-1}(\lambda) := (g^{-1}N_j(s)g)/s$ for $g = \mathrm{diag}(s^{-1}, s^{-2}, \ldots, s^{-m-1}, s^{d-m-1}, \ldots, s, 1)$ and $\lambda \equiv s^{-d-1}$ simplifies the formulas and gives the required matrix $L_{j,t}(\lambda)$.

Closed polygons are subvarieties defined by polynomial relations on the coefficients $a_{j,k}$. These relations ensure that the monodromy $\tilde{M}(s)$ has an eigenvalue of multiplicity $d+1$ at $s = 1$. □

Remark 2.7. Define the *current monodromy* $\tilde{M}_j$ for twisted n-gons by the relation

$$(V_{j+n}, \ldots, V_{j+n+d}) = (V_j, \ldots, V_{j+d})\tilde{M}_j,$$

i.e., as the product $\tilde{M}_j := N_j \ldots N_{j+n-1}$. Note that $\tilde{M}_j$ acts on matrices by multiplication on the right, whereas in Definition 1.1 the monodromy M acts on vectors V_j on the left. The theorem above uses the following fact:

Lemma 2.8. *All current monodromies $\tilde{M}_j$ lie in the same conjugacy class in $SL_{d+1}(\mathbb{C})$ as M.*

Proof. All products $\tilde{M}_j := N_j \ldots N_{j+n-1}$ are conjugate: $\tilde{M}_{j+1} = N_j^{-1}\tilde{M}_j N_j$ for all $j \in \mathbb{Z}$, since $N_j = N_{j+n}$. Furthermore,

$$\begin{aligned}(V_j, \ldots, V_{j+d})\tilde{M}_j(V_j, \ldots, V_{j+d})^{-1} &= (V_{j+n}, \ldots, V_{j+n+d})(V_j, \ldots, V_{j+d})^{-1} \\ &= M(V_j, \ldots, V_{j+d})(V_j, \ldots, V_{j+d})^{-1} = M.\end{aligned}$$ □

To prove the scale invariance of dented pentagram maps we need to introduce the appropriate notion of the dual map.

Definition 2.9. The *dual dented pentagram map* $\widehat{T}_m$ for twisted polygons in $\mathbb{RP}^d$ or $\mathbb{CP}^d$ is defined as $\widehat{T}_m := T_{\mathbf{1},I_m^*}$ for $I_m^* = (1,\dots,1,2,1,\dots,1)$ where 2 is at the $(d-m)$th place and $\mathbf{1} = (1,\dots,1)$. In this case the diagonal planes P_k are defined by taking d consecutive vertices of the polygon starting from the vertex v_k, but to define $\widehat{T}_m v_k$ one takes the intersection $P_k \cap \dots \cap P_{k+d-m-1} \cap P_{k+d-m+1} \cap \dots \cap P_{k+d}$ of all but one consecutive planes by skipping only the plane P_{k+d-m}.

Remark 2.10. According to Theorem 1.7, the dual map satisfies $\widehat{T}_m = T_m^{-1} \circ Sh$. In particular, $\widehat{T}_m$ is also integrable and has the same scaling properties and the Lax matrix as T_m. The dynamics for $\widehat{T}_m$ is obtained by reversing time in the dynamics of T_m. Moreover, the map $\widehat{T}_m$ is conjugate to T_{d-m} (modulo shifts) by means of the involution $\alpha_{\mathbf{1}}$.

Example 2.11. In dimension $d = 3$ one has the following explicit Lax representations. For the case of T_1 (i.e., $m = 1$) one sets $D(\lambda) = (1, \lambda, 1)$. The dual map $\widehat{T}_1$, being the inverse of T_1, has the same Lax form and scaling.

For the map T_2 (where $m = 2$) one has $D(\lambda) = (1, 1, \lambda)$. Similarly, $\widehat{T}_2$ is the inverse of T_2. Note that the maps T_1 and T_2^{-1} are conjugate by means of $\alpha_{\mathbf{1}}$. They have the same dimensions of invariant tori, but their Lax forms differ.

Example 2.12. In dimension $d = 4$ one has two essentially different cases, according to whether the dent is on a side of the diagonal plane or in its middle. Namely, the map T_2 is the case where the diagonal hyperplane is dented in the middle point, i.e., $m = 2$ and $I_m = (1, 2, 1)$. In this case $D(\lambda) = (1, 1, \lambda, 1)$.

For the side case consider the map T_1 (i.e., $m = 1$ and $I_m = (2, 1, 1)$), where $D(\lambda) = (1, \lambda, 1, 1)$. The dual map $\widehat{T}_1$ is the inverse of T_1 and has the same Lax form. The map T_3 has the Lax form with $D(\lambda) = (1, 1, 1, \lambda)$ and is conjugate to the inverse T_1^{-1} (see Corollary 2.2).

2.2. Coordinates in the general case

In this section we describe how to introduce coordinates on the space of twisted polygons for any n. If $\gcd(n, d + 1) \neq 1$ one can use quasiperiodic coordinates $a_{j,k}$ subject to a certain equivalence relation, instead of periodic ones (cf. [4, Section 5.3]).

Definition 2.13. Call d sequences of coordinates $\{a_{j,k},\ k = 1,\dots,d,\ j \in \mathbb{Z}\}$ *n-quasi-periodic* if there is a $(d + 1)$-periodic sequence t_j, $j \in \mathbb{Z}$, satisfying $t_j \dots t_{j+d} = 1$ and such that $a_{j+n,k} = a_{j,k} \cdot t_j / t_{j+k}$ for each $j \in \mathbb{Z}$.

This definition arises from the fact that there are different lifts of vertices $v_j \in \mathbb{CP}^d$ to vectors $V_j \in \mathbb{C}^{d+1}$, $j \in \mathbb{Z}$, so that $\det(V_j,\dots,V_{j+d}) = 1$ and $v_{j+n} = M v_j$ for $M \in SL_{d+1}(\mathbb{C})$ and $j \in \mathbb{Z}$. (The latter monodromy condition on the vertices v_j is weaker than the condition $V_{j+n} = M V_j$ on the lifted vectors in Definition 2.3.) We take arbitrary lifts $V_0,\dots,V_{d-1}$ of the first d vertices $v_0,\dots,v_{d-1}$ and then obtain $V_{j+n} = t_j M V_j$, where $t_j \dots t_{j+d} = 1$ and $t_{j+d+1} = t_j$ for all $j \in \mathbb{Z}$ (see details in [9, 4]). This way twisted n-gons are described by quasiperiodic coordinate sequences $a_{j,k}$, $k = 1,\dots,d$,

$j \in \mathbb{Z}$, with the equivalence furnished by different choices of t_j, $j \in \mathbb{Z}$. Indeed, the defining relation (1) after adding n to all indices j becomes

$$t_j V_{j+d+1} = a_{j,d} V_{j+d} t_{j+d} + a_{j,d-1} V_{j+d-1} t_{j+d-1} + \cdots + a_{j,1} V_{j+1} t_{j+1} + (-1)^d V_j t_j, \quad j \in \mathbb{Z},$$

which is consistent with the quasi-periodicity condition on $\{a_{j,k}\}$.

In the case when n satisfies $\gcd(n, d+1) = 1$, one can choose the parameters t_j in such a way that the sequences $\{a_{j,k}\}$ are n-periodic in j. For a general n, from n-quasi-periodic sequences $\{a_{j,k}, k = 1, \ldots, d, j \in \mathbb{Z}\}$ one can construct n-periodic ones (in j) as follows:

$$\tilde{a}_{j,k} = \frac{a_{j+1,k-1}}{a_{j,k} a_{j+1,d}}$$

for $j \in \mathbb{Z}$ and $k = 1, \ldots, d$, where one sets $a_{j,0} = 1$ for all j. These new n-periodic coordinates $\{\tilde{a}_{j,k}, 0 \le j \le n-1, 1 \le k \le d\}$ are well-defined coordinates on twisted n-gons in $\mathbb{CP}^d$ (i.e., they do not depend on the choice of the lift coefficients t_j). The periodic coordinates $\{\tilde{a}_{j,k}\}$ are analogs of the cross-ratio coordinates x_j, y_j in [9] and x_j, y_j, z_j in [4].

Theorem 2.14 (= 2.6′). *The dented pentagram map T_m on n-gons in any dimension d and any $m = 1, \ldots, d-1$ is an integrable system. In the coordinates $\{\tilde{a}_{j,k}\}$ its Lax matrix is*

$$\widetilde{L}_{j,t}(\lambda) = \left(\begin{array}{cccc|c} 0 & 0 & \cdots & 0 & (-1)^d \\ \hline & & & & 1 \\ & & A(\lambda) & & 1 \\ & & & & \cdots \\ & & & & 1 \end{array} \right)^{-1},$$

where $A(\lambda) = \mathrm{diag}(\tilde{a}_{j,1}, \ldots, \tilde{a}_{j,m}, \lambda \tilde{a}_{j,m+1}, \tilde{a}_{j,m+2}, \ldots, \tilde{a}_{j,d})$.

Note that the Lax matrices $\widetilde{L}$ and L are related as follows: $\widetilde{L}_{j,t}(\lambda) = a_{j+1,d}(h_{j+1}^{-1} L_{j,t}(\lambda) h_j)$ for $h_j = \mathrm{diag}(1, a_{j,1}, \ldots, a_{j,d})$.

2.3. Algebraic-geometric integrability of pentagram maps in 3D

The key ingredient responsible for algebraic-geometric integrability of the pentagram maps is a Lax representation with a spectral parameter. It allows one to construct the direct and inverse spectral transforms, which imply that the dynamics of the maps takes place on invariant tori, the Jacobians of the corresponding spectral curves. The proofs in 3D for the short-diagonal pentagram map $T_{2,1}$ are presented in detail in [4] (see also [12] for the 2D case). In dimension 3 we consider two dented pentagram maps T_1 and T_2, where the diagonal hyperplane P_k is dented on *different sides*, as opposed to the short-diagonal pentagram map $T_{2,1}$, where the diagonal hyperplane is dented on *both sides* (see Figure 5).

The proofs for the maps T_1 and T_2 follow the same lines as in [4], so in this section we present only the main statements and outline the necessary changes.

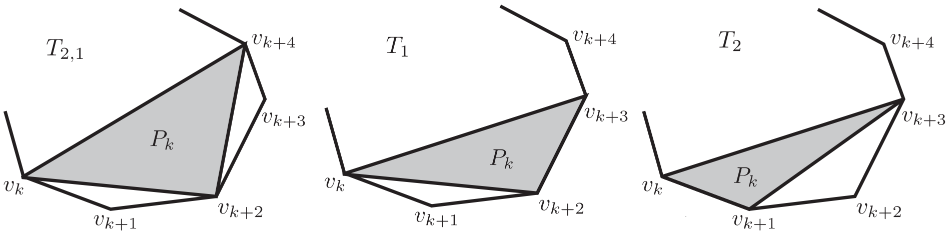

Fig. 5. Different diagonal planes in 3D: for $T_{2,1}$, T_1, and T_2.

For simplicity, in this section we assume that n is odd, which is equivalent to the condition $\gcd(n, d+1) = 1$ for $d = 3$ (this condition may not appear for a different choice of coordinates, but the results of [4] show that the dimensions of tori may depend on the parity of n). In this section we consider twisted polygons in the complex space $\mathbb{CP}^3$.

Theorem 2.15. *In dimension* 3 *the dented pentagram maps on twisted n-gons generically are fibered into* (*Zariski open subsets of*) *tori of dimension* $3\lfloor n/2 \rfloor - 1$ *for n odd and divisible by* 3 *and of dimension* $3\lfloor n/2 \rfloor$ *for n odd and not divisible by* 3.

Recall that for the short-diagonal pentagram map in 3D the torus dimension is equal to $3\lfloor n/2 \rfloor$ for any odd n (see [4]).

Proof. To prove this theorem we need the notion of a spectral curve. Recall that the product of the Lax functions $L_j(\lambda)$, $0 \le j \le n-1$, gives the monodromy operator $T_0(\lambda)$, which determines the spectral function $R(k, \lambda) := \det(T_0(\lambda) - k\,\mathrm{Id})$. The zero set $R(k, \lambda) = 0$ is an algebraic curve in $\mathbb{C}^2$. A standard procedure (of adding the infinite points and normalization with a few blow-ups) makes it into a compact Riemann surface, which we call the *spectral curve* and denote by Γ. Its genus equals the dimension of the corresponding complex torus, its Jacobian, and Proposition 2.16 below shows how to find this genus.

As is always the case with integrable systems, the spectral curve Γ is an invariant of the map and the dynamics takes place on its Jacobian. To describe the dynamics, one introduces a *Floquet–Bloch solution* which is formed by eigenvectors of the monodromy operator $T_0(\lambda)$. After a certain normalization it becomes a uniquely defined meromorphic vector function ψ_0 on the spectral curve Γ. Other Floquet–Bloch solutions are defined as the vector functions $\psi_{i+1} = L_i \dots L_1 L_0 \psi_0$, $0 \le i \le n-1$. Theorem 2.15 is based on the study of Γ and Floquet–Bloch solutions, which we summarize in the tables below.

In each case, the analysis starts with an evaluation of the spectral function $R(k, \lambda)$. Then we provide Puiseux series for the singular points at $\lambda = 0$ and at $\lambda = \infty$. They allow us to find the genus of the spectral curve and the symplectic leaves for the corresponding Krichever–Phong universal formula. Then we describe the divisors of the Floquet–Bloch solutions, which are essential for constructing the inverse spectral transform.

We start by reproducing the corresponding results for the short-diagonal map $T_{2,1}$ for odd n, obtained in [4]. We set $q := \lfloor n/2 \rfloor$. The tables below contain information on the Puiseux series of the spectral curve, Casimir functions of the pentagram dynamics, and divisors $(\psi_{i,k})$ of the components of the Floquet–Bloch solutions ψ_i (we refer to [4] for more details).

- For the (symmetric) pentagram map $T_{2,1}$ defined in [4] by means of short diagonal hyperplanes, we have $D(\lambda) = \mathrm{diag}(\lambda, 1, \lambda)$;

$$R(k, \lambda) = k^4 - \frac{k^3}{\lambda^n}\sum_{j=0}^{q} G_j\lambda^j + \frac{k^2}{\lambda^{q+n}}\sum_{j=0}^{q} J_j\lambda^j - \frac{k}{\lambda^{2n}}\sum_{j=0}^{q} I_j\lambda^j + \frac{1}{\lambda^{2n}} = 0;$$

$\lambda = 0$	$\lambda = \infty$
$O_1 : k_1 = 1/I_0 + \mathcal{O}(\lambda)$ $O_2 : k_{2,3} = \pm\sqrt{-I_0/G_0}\,\lambda^{-n/2}(1 + \mathcal{O}(\sqrt{\lambda}))$ $O_3 : k_4 = G_0\lambda^{-n}(1 + \mathcal{O}(\lambda))$	$W_{1,2} : k_{1,2,3,4} = k_\infty\lambda^{-n/2}(1 + \mathcal{O}(\lambda^{-1}))$, where $k_\infty^4 + J_q k_\infty^2 + 1 = 0$
$g = 3q$, the Casimirs are $I_0 := \prod_{j=0}^{n-1} a_{j,3}$; J_q; $G_0 := \prod_{j=0}^{n-1} a_{j,1}$	
$(\psi_{i,1}) \geq -D + O_2 - i(O_2 + O_3) + (i + 1)(W_1 + W_2)$ $(\psi_{i,2}) \geq -D + (1 - i)(O_2 + O_3) + i(W_1 + W_2)$ $(\psi_{i,3}) \geq -D - i(O_2 + O_3) + (i + 1)(W_1 + W_2)$ $(\psi_{i,4}) \geq -D + O_2 + (1 - i)(O_2 + O_3) + i(W_1 + W_2)$	

- For the dented map T_1, we have $D(\lambda) = \mathrm{diag}(1, \lambda, 1)$;

$$R(k, \lambda) = k^4 - \frac{k^3}{\lambda^q}\sum_{j=0}^{q} G_j\lambda^j + \frac{k^2}{\lambda^n}\sum_{j=0}^{\lfloor 2n/3\rfloor} J_j\lambda^j - \frac{k}{\lambda^n}\sum_{j=0}^{\lfloor n/3\rfloor} I_j\lambda^j + \frac{1}{\lambda^n} = 0;$$

n odd and not divisible by 3: $n = 6l + 1$ or $n = 6l + 5$	
$\lambda = 0$	$\lambda = \infty$
$O_{1,2} : k_{1,2} = c_0 + \mathcal{O}(\lambda)$, where $c_0^2 J_0 - c_0 I_0 + 1 = 0$ $O_3 : k_{3,4} = \pm\sqrt{-J_0}\,\lambda^{-n/2}(1 + \mathcal{O}(\sqrt{\lambda}))$	$W_1 : k_1 = G_q + \mathcal{O}(\lambda^{-1})$, $W_2 : k_{2,3,4} = G_q^{-1/3}\lambda^{-n/3}(1 + \mathcal{O}(\lambda^{-1/3}))$
$g = 3q$, the Casimirs are I_0; $J_0 := (-1)^n \prod_{j=0}^{n-1} a_{j,2}$; $G_q := \prod_{j=0}^{n-1} a_{j,1}$	
$(\psi_{i,1}) \geq -D + O_3 + 2W_2 + i(W_2 - O_3)$ $(\psi_{i,2}) \geq -D + W_1 + 2W_2 + i(W_2 - O_3)$ $(\psi_{i,3}) \geq -D + 2O_3 + i(W_2 - O_3)$ $(\psi_{i,4}) \geq -D + 2O_3 + W_2 + i(W_2 - O_3)$	

n odd and divisible by 3: $n = 6l + 3$	
$\lambda = 0$	$\lambda = \infty$
$O_{1,2} : k_{1,2} = c_0 + \mathcal{O}(\lambda)$, where $c_0^2 J_0 - c_0 I_0 + 1 = 0$ $O_3 : k_{3,4} = \pm\sqrt{-J_0}\,\lambda^{-n/2}(1 + \mathcal{O}(\sqrt{\lambda}))$	$W_1 : k_1 = G_q + \mathcal{O}(\lambda^{-1})$, $W_{2,3,4} : k_{2,3,4} = k_\infty\lambda^{-n/3} + \mathcal{O}(\lambda^{-1})$, where $G_q k_\infty^3 - J_{\lfloor 2n/3\rfloor} k_\infty^2 + I_{\lfloor n/3\rfloor} k_\infty - 1 = 0$
$g = 3q - 1$, the Casimirs are I_0; J_0; G_q; $J_{\lfloor 2n/3\rfloor}$; $I_{\lfloor n/3\rfloor}$	

- For the dented map T_2, we have $D(\lambda) = \mathrm{diag}(1, 1, \lambda)$;

$$R(k, \lambda) = k^4 - \frac{k^3}{\lambda^{\lfloor n/3 \rfloor}} \sum_{j=0}^{\lfloor n/3 \rfloor} G_j \lambda^j + \frac{k^2}{\lambda^{\lfloor 2n/3 \rfloor}} \sum_{j=0}^{\lfloor 2n/3 \rfloor} J_j \lambda^j - \frac{k}{\lambda^n} \sum_{j=0}^{q} I_j \lambda^j + \frac{1}{\lambda^n} = 0.$$

The analysis of the spectral curve proceeds similarly:

n odd and not divisible by 3: $n = 6l + 1$ or $n = 6l + 5$	
$\lambda = 0$	$\lambda = \infty$
$O_1 : k_1 = 1/I_0 + \mathcal{O}(\lambda)$ $O_2 : k_{2,3,4} = I_0^{1/3} \lambda^{-n/3}(1 + \mathcal{O}(\lambda^{1/3}))$	$W_{1,2} : k_{1,2} = c_1 + \mathcal{O}(\lambda^{-1})$, where $c_1^2 - c_1 G_{\lfloor n/3 \rfloor} + J_{\lfloor 2n/3 \rfloor} = 0$ $W_3 : k_{3,4} = \pm\sqrt{-1/J_{\lfloor 2n/3 \rfloor}}\, \lambda^{-n/2}(1 + \mathcal{O}(\lambda^{-1/2}))$
$g = 3q$, the Casimirs are $I_0 = \prod_{j=0}^{n-1} a_{j,3}$; $J_{\lfloor 2n/3 \rfloor} := (-1)^n \prod_{j=0}^{n-1} a_{j,2}$; $G_{\lfloor n/3 \rfloor}$	
$(\psi_{i,1}) \geq -D + 2O_2 + W_3 + i(W_3 - O_2)$ $(\psi_{i,2}) \geq -D + O_2 + W_3 + i(W_3 - O_2)$ $(\psi_{i,3}) \geq -D + W_1 + W_2 + W_3 + i(W_3 - O_2)$ $(\psi_{i,4}) \geq -D + 3O_2 + i(W_3 - O_2)$	

As an example, we show how to use these tables to find the genus of the spectral curve. As before, we assume that n is odd, $n = 2q + 1$. Recall that for the short-diagonal pentagram map $T_{2,1}$ the genus is $g = 3q$ for odd n (see [4]).

Proposition 2.16. *The spectral curves for the dented pentagram maps in $\mathbb{CP}^3$ generically have genus $g = 3q - 1$ for n odd and divisible by* 3, *and genus $g = 3q$ for n odd and not divisible by* 3.

Proof. Let us compute the genus for the dented pentagram map T_1. As follows from the definition of the spectral curve Γ, it is a ramified 4-fold cover of $\mathbb{CP}^1$, since the 4×4 matrix $\tilde{T}_{i,t}(\lambda)$ (or $T_{i,t}(\lambda)$) has four eigenvalues. By the Riemann–Hurwitz formula the Euler characteristic of Γ is $\chi(\Gamma) = 4\chi(\mathbb{CP}^1) - \nu = 8 - \nu$, where ν is the ramification index of the covering. In our setting, the index ν is equal to the sum of the orders of the branch points at $\lambda = 0$ and $\lambda = \infty$, plus the number $\bar{\nu}$ of branch points over $\lambda \neq 0, \infty$, where we assume that the latter points are all of order 1 generically. On the other hand, $\chi(\Gamma) = 2 - 2g$, and once we know ν it allows us to find the genus of the spectral curve Γ from the formula $2 - 2g = 8 - \nu$.

The number $\bar{\nu}$ of branch points of Γ on the λ-plane equals the number of zeros of the function $\partial_k R(\lambda, k)$ aside from the singular points $\lambda = 0$ or ∞. The function $\partial_k R(\lambda, k)$ is meromorphic on Γ, therefore the number of its zeros equals the number of its poles. One can see that for any $n = 2q+1$ the function $\partial_k R(\lambda, k)$ has poles of total order $5n$ at $z = 0$, and it has zeros of total order $2n$ at $z = \infty$. Indeed, substitute the local series for k in λ from the table to the expression for $\partial_k R(\lambda, k)$. (E.g., at O_1 one has $k = \mathcal{O}(1)$. The leading terms of $\partial_k R(\lambda, k)$ for the pole at $\lambda = 0$ are $4k^3, -3k^2 G_0 \lambda^{-q}, 2k J_0 \lambda^{-n}, -I_0 \lambda^{-n}$. The last two terms, being of order λ^{-n}, dominate and give the pole of order $n = 2q + 1$.) For

n odd and not divisible by 3, the corresponding orders of the poles and zeros of $\partial_k R(\lambda, k)$ on the curve Γ are summarized as follows:

pole	order	zero	order
O_1	n	W_1	0
O_2	n	W_2	$2n$
O_3	$3n$		

Therefore, for such n, the total order of poles is $n+n+3n = 5n$, while the total order of zeros is $0+2n = 2n$. Consequently, the number of zeros of $\partial_k R(\lambda, k)$ at nonsingular points $\lambda \neq \{0, \infty\}$ is $\bar{\nu} = 5n - 2n = 3n$, and so is the total number of branch points of Γ in the finite part of the (λ, k) plane (generically, all of them have order 1). For n odd and not divisible by 3 there is an additional branch point at $\lambda = 0$ of order 1 and another branch point at $\lambda = \infty$ of order 2 (see the table for T_1). Hence the ramification index is $\nu = \bar{\nu} + 3 = 3n + 3 = 6q + 6$. The identity $2 - 2g = 8 - \nu$ implies that $g = 3q$.

For n odd and divisible by 3, $n = 6l+3$, one has the same orders of poles O_j, W_1 is of order zero, while each of the three zeros $W_{2,3,4}$ is of order $4l+2$. Then the total order of zeros is still $3(4l+2) = 12l+6 = 2n$, and again $\bar{\nu} = 5n-2n = 3n$. However, there is no branch point at $\lambda = \infty$, and hence the ramification index is $\nu = \bar{\nu}+1 = 3n+1 = 6q+4$. Thus for such n we see from the identity $2 - 2g = 8 - \nu$ that $g = 3q - 1$.

Finally, note that T_1 and T_2^{-1} are conjugate by means of the involution $\alpha_{\mathbf{1}}$, and hence T_1 and T_2 have the same dimensions of invariant tori. Their spectral curves are related by a change of coordinates furnished by this involution and have the same genus. □

3. Dual dented maps

3.1. Properties of dual dented pentagram maps

It turns out that the pentagram dynamics of $\hat{T}_m$ has the following simple description. (We consider the geometric picture over $\mathbb{R}$.)

Proposition 3.1. *The dual pentagram map $\hat{T}_m$ in $\mathbb{RP}^d$ sends the vertex v_k into the intersection of the subspaces of dimensions m and $d-m$ spanned by the vertices:*

$$\hat{T}_m v_k = (v_{k+d-m-1}, \dots, v_{k+d-1}) \cap (v_{k+d}, \dots, v_{k+2d-m}).$$

Proof. As discussed above, the point $\hat{T}_m v_k$ is defined by taking the intersection of all but one consecutive hyperplanes:

$$\hat{T}_m v_k = P_k \cap \dots \cap P_{k+d-m-1} \cap P_{k+d-m+1} \cap \dots \cap P_{k+d}.$$

Note that this point can be described as the intersection of the subspace

$$L_1^m = P_k \cap \dots \cap P_{k+d-m-1}$$

of dimension m and the subspace

$$L_2^{d-m} = P_{k+d-m+1} \cap \cdots \cap P_{k+d}$$

of dimension $d - m$ in $\mathbb{RP}^d$. (Here the upper index stands for dimension.) Since each of the subspaces L_1 and L_2 is the intersection of several consecutive hyperplanes P_j, and each hyperplane P_j is spanned by consecutive vertices, we see that $L_1^m = (v_{k+d-m-1}, \dots, v_{k+d-1})$ and $L_2^{d-m} = (v_{k+d}, \dots, v_{k+2d-m})$, as required. □

Consider the shift Sh of vertex indices by $d - (m+1)$ to obtain the map

$$\widehat{T}_m v_k := (\hat{T}_m \circ Sh) v_k = (v_k, \dots, v_{k+m}) \cap (v_{k+m+1}, \dots, v_{k+d+1}),$$

which we will study from now on.

Example 3.2. For $d = 3$ and $m = 2$ we have the dual pentagram map $\widehat{T}_2$ in $\mathbb{RP}^3$ defined via intersection of the two-dimensional plane $L_1 = (v_k, v_{k+1}, v_{k+2})$ and the line $L_2 = (v_{k+3}, v_{k+4})$:

$$\widehat{T}_2 v_k = (v_k, v_{k+1}, v_{k+2}) \cap (v_{k+3}, v_{k+4})$$

(see Figure 6). This map is dual to the dented pentagram map T_m for $I = (1, 2)$ and $J = (1, 1)$.

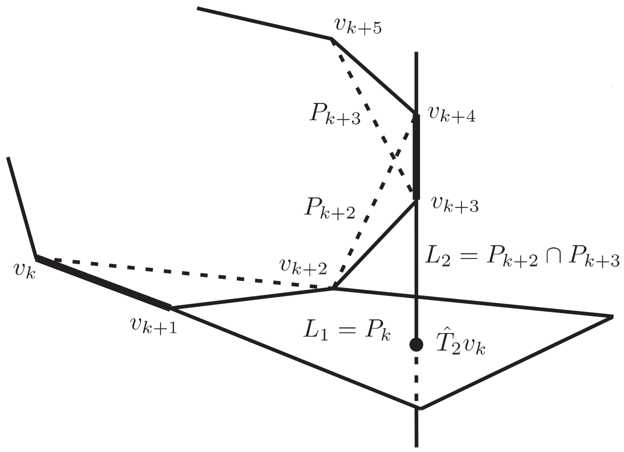

Fig. 6. The dual $\widehat{T}_2$ to the dented pentagram map T_m for $m = 2$ in $\mathbb{RP}^3$.

Let V_k be the lifts of the vertices v_k of a twisted n-gon from $\mathbb{RP}^d$ to $\mathbb{R}^{d+1}$. We assume that n and $d + 1$ are mutually prime and $\det(V_k, \dots, V_{k+d}) = 1$ with $V_{k+n} = MV_k$ for all $k \in \mathbb{Z}$ to provide lift uniqueness.

Proposition 3.3. *Given a twisted polygon (v_k) in $\mathbb{RP}^d$ with coordinates $a_{k,j}$, the image $\widehat{T}_m V_k$ in $\mathbb{R}^{d+1}$ under the dual pentagram map is proportional to the vector*

$$R_k = a_{k,m} V_{k+m} + a_{k,m-1} V_{k+m-1} + \cdots + a_{k,1} V_{k+1} + (-1)^d V_k$$

for all $k \in \mathbb{Z}$.

Proof. Since
$$\widehat{T}_m V_k \in (V_k, \dots, V_{k+m}) \cap (V_{k+m+1}, \dots, V_{k+d+1}),$$
the vector $W_k := \widehat{T}_m V_k$ can be represented as a linear combination of vectors from either of the groups:

$$W_k = \mu_k V_k + \dots + \mu_{k+m} V_{k+m} = \nu_{k+m+1} V_{k+m+1} + \dots + \nu_{k+d+1} V_{k+d+1}.$$

Normalize this vector by setting $\nu_{k+d+1} = 1$. Now recall that

$$V_{k+d+1} = a_{k,d} V_{k+d} + a_{k,d-1} V_{k+d-1} + \dots + a_{k,1} V_{k+1} + (-1)^d V_k$$

for $k \in \mathbb{Z}$. Replacing V_{k+d+1} by its expression via $V_k, \dots, V_{k+d}$ we obtain $\mu_k = (-1)^d$, $\mu_{k+1} = a_{k,1}, \dots, \mu_{k+m} = a_{k,m}$. Thus the vector

$$R_k = a_{k,m} V_{k+m} + a_{k,m-1} V_{k+m-1} + \dots + a_{k,1} V_{k+1} + (-1)^d V_k$$

belongs to both the subspaces, and hence spans their intersection. □

Note that the image $W_k := \widehat{T}_m V_k$ under the dual map is $\lambda_k R_k$, where the coefficients λ_k are determined by the condition $\det(W_k, \dots, W_{k+d}) = 1$ for all $k \in \mathbb{Z}$.

3.2. *Proof of scale invariance*

In this section we prove scaling invariance in any dimension d for any map $\widehat{T}_m$, $1 \le m \le d-1$, dual to the dented pentagram map T_m on twisted n-gons in $\mathbb{CP}^d$, whose hyperplanes P_k are defined by taking consecutive vertices, but skipping the mth vertex.

Theorem 3.4 (= 2.4$\widehat{\ }$). *The dual dented pentagram map $\widehat{T}_m$ on twisted n-gons in $\mathbb{CP}^d$ is invariant with respect to the following scaling transformations:*

$$a_{k,1} \to s^{-1} a_{k,1}, \quad a_{k,2} \to s^{-2} a_{k,2}, \quad \dots, \quad a_{k,m} \to s^{-m} a_{k,m},$$
$$a_{k,m+1} \to s^{d-m} a_{k,m+1}, \quad \dots, \quad a_{k,d} \to s a_{k,d},$$

for all $s \in \mathbb{C}^*$.

Proof. The dual dented pentagram map is defined by $W_k := \widehat{T}_m V_k = \lambda_k R_k$, where the coefficients λ_k are determined by the normalization condition $\det(W_k, \dots, W_{k+d}) = 1$ for all $k \in \mathbb{Z}$. The transformed coordinates are defined using the difference equation

$$W_{k+d+1} = \hat{a}_{k,d} W_{k+d} + \hat{a}_{k,d-1} W_{j+d-1} + \dots + \hat{a}_{k,1} W_{k+1} + (-1)^d W_k.$$

The corresponding coefficients $\hat{a}_{k,j}$ can be readily found using Cramer's rule:

$$\hat{a}_{k,j} = \frac{\lambda_{k+d+1}}{\lambda_{k+j}} \frac{\det(R_k, \dots, R_{k+j-1}, R_{k+d+1}, R_{k+j+1}, \dots, R_{k+d})}{\det(R_k, \dots, R_{k+d})}. \tag{3}$$

The normalization condition reads $\lambda_k \dots \lambda_{k+d} \det(R_k, \dots, R_{k+d}) = 1$ for all $k \in \mathbb{Z}$.

To prove the theorem, it is sufficient to prove that the determinants in (3) are homogenous in s, and to find their degrees of homogeneity.

Lemma 3.5. *The determinant* $\det(R_k, \dots, R_{k+d})$ *is homogeneous of degree zero in* s. *The determinant in the numerator of* (3) *has the same degree of homogeneity in* s *as* $a_{k,j}$.

The theorem immediately follows from this lemma since even if λ_k's have some nonzero degree of homogeneity, it does not depend on k anyway by the definition of scaling transformation, and it cancels out in the ratio. Hence the whole expression (3) for $\hat{a}_{k,j}$ transforms just like $a_{k,j}$, i.e., the dented pentagram map is invariant with respect to the scaling.

Proof of Lemma 3.5. Proposition 3.3 implies that the vector $R_k := (V_k, \dots, V_{k+d})\mathbf{r}_k$ has an expansion

$$\mathbf{r}_k = ((-1)^d, a_{k,1}, \dots, a_{k,m}, 0, \dots, 0)^t$$

in the basis $(V_k, \dots, V_{k+d})$, where t stands for the transposed matrix. Note that the vector R_{k+1} has a similar expression $\mathbf{r}_k = ((-1)^d, a_{k+1,1}, \dots, a_{k+1,m}, 0, \dots, 0)^t$ in the shifted basis $(V_{k+1}, \dots, V_{k+d+1})$, but in the initial basis $(V_k, V_{k+1}, \dots, V_{k+d})$ its expansion has the form $\mathbf{r}_{k+1} = N_k \mathbf{r}_k$ for the transformation matrix N_k (see its definition in the proof of Theorem 2.6), since the relation (1) implies

$$(V_{k+1}, \dots, V_{k+d+1}) = (V_k, \dots, V_{k+d}) N_k.$$

Note that formula (3) is independent of the choice of the basis used and we expand vectors R_k in the basis $(V_{k+m+1}, \dots, V_{k+m+1+d})$. It turns out that the corresponding expansions $\mathbf{r}_k, \dots, \mathbf{r}_{k+d+1}$ have a particularly simple form in this basis, which is crucial for the proof. We use hats, $\hat{\mathbf{r}}_k, \dots, \hat{\mathbf{r}}_{k+d+1}$, when the vectors $R_k, \dots, R_{k+d+1}$ are written in this new basis. Explicitly we obtain

$$\begin{aligned}
\hat{\mathbf{r}}_k &= (N_k \dots N_{k+m})^{-1}\mathbf{r}_k = (-a_{k,m+1}, -a_{k,m+2}, \dots, -a_{k,d}, 1, 0, \dots, 0)^t,\\
\hat{\mathbf{r}}_{k+1} &= (N_{k+1} \dots N_{k+m})^{-1}\mathbf{r}_{k+1}\\
&= (0, -a_{k+1,m+1}, -a_{k+1,m+2}, \dots, -a_{k+1,d}, 1, 0, \dots, 0)^t,\\
&\dots\\
\hat{\mathbf{r}}_{k+m} &= N_{k+m}^{-1}\mathbf{r}_{k+m} = (0, \dots, 0, -a_{k+m,m+1}, -a_{k+m,m+2}, \dots, -a_{k+m,d}, 1)^t,\\
\hat{\mathbf{r}}_{k+m+1} &= \mathbf{r}_{k+m+1} = ((-1)^d, a_{k+m+1,1}, \dots, a_{k+m+1,m}, 0, \dots, 0)^t,\\
\hat{\mathbf{r}}_{k+m+2} &= N_{k+m+1}\mathbf{r}_{k+m+2} = (0, (-1)^d, a_{k+m+2,1}, \dots, a_{k+m+2,m}, 0, \dots, 0)^t,\\
\hat{\mathbf{r}}_{k+m+3} &= N_{k+m+1}N_{k+m+2}\mathbf{r}_{k+m+3}\\
&= (0, 0, (-1)^d, a_{k+m+3,1}, \dots, a_{k+m+3,m}, 0, \dots, 0)^t,\\
&\dots\\
\hat{\mathbf{r}}_{k+d+1} &= N_{k+m+1}N_{k+m+2} \dots N_{k+d}\mathbf{r}_{k+d+1}\\
&= (0, \dots, 0, (-1)^d, a_{k+d+1,1}, \dots, a_{k+d+1,m})^t.
\end{aligned}$$

Consider the matrix $\mathbf{M} = (\hat{\mathbf{r}}_k, \dots, \hat{\mathbf{r}}_{k+d+1})$ of size $(d+1) \times (d+2)$, which is essentially the matrix of the system of linear equations determining $\hat{a}_{k,j}$. All its entries are homogeneous in s. Also note that the determinant $\det(R_k, \dots, R_{k+d}) = \det(\hat{\mathbf{r}}_k, \dots, \hat{\mathbf{r}}_{k+d})$ is the minor formed by the first $d+1$ columns, while the determinant in the numerator of formula (3) is up to a sign the $(j+1)$th minor, $0 \le j \le d$, formed by crossing out the $(j+1)$th column in $\mathbf{M}$.

For instance, for $d=6$ and $m=2$ this matrix has the form

$$\mathbf{M} = \begin{pmatrix} -a_{k,3} & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -a_{k,4} & -a_{k+1,3} & 0 & a_{k+3,1} & 1 & 0 & 0 & 0 \\ -a_{k,5} & -a_{k+1,4} & -a_{k+2,3} & a_{k+3,2} & a_{k+4,1} & 1 & 0 & 0 \\ -a_{k,6} & -a_{k+1,5} & -a_{k+2,4} & 0 & a_{k+4,2} & a_{k+5,1} & 1 & 0 \\ 1 & -a_{k+1,6} & -a_{k+2,5} & 0 & 0 & a_{k+5,2} & a_{k+6,1} & 1 \\ 0 & 1 & -a_{k+2,6} & 0 & 0 & 0 & a_{k+6,2} & a_{k+7,1} \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & a_{k+7,2} \end{pmatrix}.$$

Let us form the corresponding matrix $\mathbf{D}$ of the same size representing the homogeneity degrees of the entries of $\mathbf{M}$ given by the scaling transformations. One can assign an arbitrary degree to a zero entry, and we do this in such a way that within each column the degrees would change uniformly. Note that those degrees also change uniformly along all the rows, except for one simultaneous jump after the $(m+1)$th column. In the above example one has

$$\mathbf{D} = \begin{pmatrix} 4 & 5 & 6 & 0 & 1 & 2 & 3 & 4 \\ 3 & 4 & 5 & -1 & 0 & 1 & 2 & 3 \\ 2 & 3 & 4 & -2 & -1 & 0 & 1 & 2 \\ 1 & 2 & 3 & -3 & -2 & -1 & 0 & 1 \\ 0 & 1 & 2 & -4 & -3 & -2 & -1 & 0 \\ -1 & 0 & 1 & -5 & -4 & -3 & -2 & -1 \\ -2 & -1 & 0 & -6 & -5 & -4 & -3 & -2 \end{pmatrix}.$$

Then the determinant of the minors obtained by crossing any of the columns is homogeneous. Indeed, elementary transformations on the matrix rows by adding to one row another multiplied by a function homogeneous in s preserve this table of homogeneity degrees. On the other hand, producing the upper-triangular form by such transformations one can easily compute the homogeneity degree of the corresponding minor. For instance, the minor $\det(\hat{\mathbf{r}}_k, \hat{\mathbf{r}}_{k+1}, \dots, \hat{\mathbf{r}}_{k+d})$ formed by the first $d+1$ columns has zero degree. Indeed, we need to find the trace of the corresponding $(d+1)\times(d+1)$ degree matrix. It contains $m+1$ columns with the diagonal entries of degree $d-m$, as well as $d-m$ columns with the diagonal entries of degree $-m-1$, i.e., the total degree is $(d-m)\cdot(m+1)+(-m-1)\cdot(d-m)=0$. (In the example above it is $4\cdot 3+(-3)\cdot 4=0$ on the diagonal for the first seven columns.) Similarly one finds the degree of any jth minor of the matrix $\mathbf{M}$ for arbitrary d and m by calculating the difference of the degrees for the diagonal (j,j)-entry and the $(j,d+2)$-entry in the matrix $\mathbf{D}$. □

Remark 3.6. The idea of using Cramer's rule and a simple form of the vectors R_k was suggested in [8] to prove the scale invariance of the short-diagonal pentagram maps $T_{2,1}$. For the maps $\widehat{T_m}$ we employ this approach along with passing to the dual maps and the above "retroactive" basis change. This choice of basis in the proof of Theorem 3.4 also allows one to obtain explicit formulas for the pentagram map via the matrix $\mathbf{M}$.

4. Continuous limit of general pentagram maps

Consider the continuous limit of the dented pentagram maps on n-gons as $n \to \infty$. In the limit a generic twisted n-gon becomes a smooth nondegenerate quasi-periodic curve $\gamma(x)$ in $\mathbb{RP}^d$. Its lift $G(x)$ to $\mathbb{R}^{d+1}$ is defined by the conditions that the components of the vector function $G(x) = (G_1, \dots, G_{d+1})(x)$ provide the homogeneous coordinates for $\gamma(x) = (G_1 : \cdots : G_{d+1})(x)$ in $\mathbb{RP}^d$ and $\det(G(x), G'(x), \dots, G^{(d)}(x)) = 1$ for all $x \in \mathbb{R}$. Furthermore, $G(x + 2\pi) = MG(x)$ for a given $M \in SL_{d+1}(\mathbb{R})$. Then $G(x)$ satisfies a linear differential equation of order $d + 1$:

$$G^{(d+1)} + u_{d-1}(x)G^{(d-1)} + \cdots + u_1(x)G' + u_0(x)G = 0$$

with periodic coefficients $u_i(x)$, which is a continuous limit of the difference equations (1). Here $'$ stands for d/dx.

Fix a small $\epsilon > 0$ and let I be any $(d-1)$-tuple $I = (i_1, \dots, i_{d-1})$ of positive integers. For the I-diagonal hyperplane

$$P_k := (v_k, v_{k+i_1}, v_{k+i_1+i_2}, \dots, v_{k+i_1+\cdots+i_{d-1}})$$

its continuous analogue is the hyperplane $P_\epsilon(x)$ passing through the d points $\gamma(x)$, $\gamma(x + i_1\epsilon), \dots, \gamma(x + (i_1 + \cdots + i_{d-1})\epsilon)$ of the curve γ. In what follows we are going to make a parameter shift in x (equivalent to shift of indices) and define $P_\epsilon(x) := (\gamma(x + k_0\epsilon), \dots, \gamma(x + k_{d-1}\epsilon))$ for any real $k_0 < \cdots < k_{d-1}$ such that $\sum_l k_l = 0$.

Let $\ell_\epsilon(x)$ be the envelope curve for the family of the hyperplanes $P_\epsilon(x)$ for a fixed ϵ. The envelope condition means that $P_\epsilon(x)$ are the osculating hyperplanes of the curve $\ell_\epsilon(x)$, that is, the point $\ell_\epsilon(x)$ belongs to the hyperplane $P_\epsilon(x)$, while the vector-derivatives $\ell'_\epsilon(x), \dots, \ell_\epsilon^{(d-1)}(x)$ span this hyperplane for each x. This means that the lift of $\ell_\epsilon(x)$ to $L_\epsilon(x)$ in $\mathbb{R}^{d+1}$ satisfies the system of d equations

$$\det\big(G(x + k_0\epsilon), \dots, G(x + k_{d-1}\epsilon), L_\epsilon^{(j)}(x)\big) = 0, \qquad j = 0, \dots, d-1.$$

A continuous limit of the pentagram map is defined as the evolution of the curve γ in the direction of the envelope ℓ_ϵ as ϵ changes. Namely, one can show that the expansion of $L_\epsilon(x)$ has the form

$$L_\epsilon(x) = G(x) + \epsilon^2 B(x) + \mathcal{O}(\epsilon^3),$$

where there is no term linear in ϵ due to the condition $\sum_l k_l = 0$. It satisfies the family of differential equations

$$L_\epsilon^{(d+1)} + u_{d-1,\epsilon}(x)L_\epsilon^{(d-1)} + \cdots + u_{1,\epsilon}(x)L'_\epsilon + u_{0,\epsilon}(x)L_\epsilon = 0, \quad \text{where} \quad u_{j,0}(x) = u_j(x).$$

The corresponding expansion of the coefficients, $u_{j,\epsilon}(x) = u_j(x) + \epsilon^2 w_j(x) + \mathcal{O}(\epsilon^3)$, defines the continuous limit of the pentagram map as a system of evolution differential equations $du_j(x)/dt = w_j(x)$ for $j = 0, \dots, d-1$. (This definition of limit assumes that we have the standard tuple $J = \mathbf{1} := (1, \dots, 1)$.)

Theorem 4.1 (Continuous limit). *The continuous limit of any generalized pentagram map $T_{I,J}$ for any $I = (i_1, \dots, i_{d-1})$ and $J = \mathbf{1}$ (and in particular, of any dented pentagram map T_m) in dimension d defined by the system $du_j(x)/dt = w_j(x)$, $j = 0, \dots, d-1$, for $x \in S^1$ is the $(2, d+1)$-KdV flow of the Adler–Gelfand–Dickey hierarchy on the circle.*

Remark 4.2. Recall that the $(n, d+1)$-KdV flow is defined on linear differential operators $L = \partial^{d+1} + u_{d-1}(x)\partial^{d-1} + \dots + u_1(x)\partial + u_0(x)$ of order $d+1$ with periodic coefficients $u_j(x)$, where ∂^k stands for d^k/dx^k. One can define the fractional power $L^{n/d+1}$ as a pseudo-differential operator for any positive integer n and take its pure differential part $Q_n := (L^{n/d+1})_+$. In particular, for $n = 2$ one has $Q_2 = \partial^2 + \frac{2}{d+1}u_{d-1}(x)$. Then the $(n, d+1)$-KdV equation is the evolution equation on (the coefficients of) L given by $dL/dt = [Q_n, L]$ (see [1]).

For $d = 2$ the $(2, 3)$-KdV equation is the classical Boussinesq equation on the circle: $u_{tt} + 2(u^2)_{xx} + u_{xxxx} = 0$, which appears as the continuous limit of the 2D pentagram map [9].

Proof of Theorem 4.1. By expanding in the parameter ϵ one can show that $L_\epsilon(x)$ has the form $L_\epsilon(x) = G(x) + \epsilon^2 C_{d,I}\big(\partial^2 + \frac{2}{d+1}u_{d-1}(x)\big)G(x) + \mathcal{O}(\epsilon^3)$ as $\epsilon \to 0$, for a certain nonzero constant $C_{d,I}$ (cf. [4, Theorem 4.3]). We obtain the following evolution of the curve $G(x)$ given by the ϵ^2-term of this expansion: $dG/dt = \big(\partial^2 + \frac{2}{d+1}u_{d-1}\big)G$, or, what is the same, $dG/dt = Q_2 G$.

We would like to find the evolution of the operator L tracing it. For any t, the curve G and the operator L are related by the differential equation $LG = 0$ of order $d+1$. Consequently, $d(LG)/dt = (dL/dt)G + L(dG/dt) = 0$.

Now note that if the operator L satisfies the $(2, d+1)$-KdV equation $dL/dt = [Q_2, L] := Q_2 L - L Q_2$, and G satisfies $dG/dt = Q_2 G$, we have the identity

$$\frac{dL}{dt}G + L\frac{dG}{dt} = (Q_2 L - L Q_2)G + L Q_2 G = Q_2 L G = 0.$$

In virtue of the uniqueness of the linear differential operator L of order $d+1$ for a given fundamental set of solutions G, we conclude that indeed the evolution of L is described by the $(2, d+1)$-KdV equation. ◻

5. Corrugated polygons and dented diagonals

5.1. Pentagram maps for corrugated polygons

In [2] pentagram maps were defined on spaces of corrugated polygons in $\mathbb{RP}^d$. These maps turned out to be integrable, while the corresponding Poisson structures are related to many interesting structures on such polygons. Below we describe how one can view integrability in the corrugated case as a particular case of the dented maps.

Let (v_k) be generic twisted n-gons in $\mathbb{RP}^d$ (here "generic" means that no $d+1$ consecutive vertices lie in a projective subspace). The space of equivalence classes of generic twisted n-gons in $\mathbb{RP}^d$ has dimension nd and is denoted by $\mathcal{P}_n$.

Definition 5.1. A twisted polygon (v_k) in $\mathbb{RP}^d$ is *corrugated* if for every $k \in \mathbb{Z}$ the vertices v_k, v_{k+1}, v_{k+d}, and v_{k+d+1} span a projective two-dimensional plane.

The projective group preserves the space of corrugated polygons. Denote by $\mathcal{P}_n^{\text{cor}} \subset \mathcal{P}_n$ the space of projective equivalence classes of generic corrugated n-gons. One can show that such polygons form a submanifold of dimension $2n$ in the nd-dimensional space $\mathcal{P}_n$.

The consecutive d-diagonals (the diagonal lines connecting v_k and v_{k+d}) of a corrugated polygon intersect pairwise, and the intersection points form the vertices of a new corrugated polygon: $T_{\text{cor}} v_k := (v_k, v_{k+d}) \cap (v_{k+1}, v_{k+d+1})$. This gives the definition of the pentagram map on (classes of projectively equivalent) corrugated polygons, $T_{\text{cor}} : \mathcal{P}_n^{\text{cor}} \to \mathcal{P}_n^{\text{cor}}$ (see [2]). In 2D one has $\mathcal{P}_n^{\text{cor}} = \mathcal{P}_n$, and this gives the definition of the classical pentagram map on $\mathcal{P}_n$.

Proposition 5.2 ([2]). *The pentagram map T_{cor} is well defined on $\mathcal{P}_n^{\text{cor}}$, i.e., it sends a corrugated polygon to a corrugated one.*

Proof. The image of the pentagram map T_{cor} is defined as the intersection of the diagonals in the quadrilateral $(v_{k-1}, v_k, v_{k+d-1}, v_{k+d})$. Consider the diagonal (v_k, v_{k+d}). It contains both vertices $T_{\text{cor}} v_{k-1}$ and $T_{\text{cor}} v_k$, as they are intersections of this diagonal with the diagonals (v_{k-1}, v_{k+d-1}) and (v_{k+1}, v_{k+d+1}) respectively. Similarly, both vertices $T_{\text{cor}} v_{k-d-1}$ and $T_{\text{cor}} v_{k-d}$ belong to the diagonal (v_{k-d}, v_k).

Hence we obtain two pairs of new vertices $T_{\text{cor}} v_{k-d-1}, T_{\text{cor}} v_{k-d}$ and $T_{\text{cor}} v_{k-1}, T_{\text{cor}} v_k$ for each $k \in \mathbb{Z}$ lying in one 2D plane passing through the old vertices (v_{k-d}, v_k, v_{k+d}). Note also that the indices of these new pairs differ by d. Thus they satisfy the corrugatedness condition. □

Theorem 5.3. *The pentagram map $T_{\text{cor}} : \mathcal{P}_n^{\text{cor}} \to \mathcal{P}_n^{\text{cor}}$ is the restriction of the dented pentagram map $T_m : \mathcal{P}_n \to \mathcal{P}_n$ for any $m = 1, \ldots, d-1$ from generic n-gons $\mathcal{P}_n$ in $\mathbb{RP}^d$ to corrugated ones $\mathcal{P}_n^{\text{cor}}$ (or differs from it by a shift in vertex indices).*

In order to prove this theorem we first show that the definition of a corrugated polygon in $\mathbb{RP}^d$ is equivalent to the following:

Proposition 5.4. *Fix any $\ell = 2, \ldots, d-1$. A generic twisted polygon (v_k) is corrugated if and only if the 2ℓ vertices $v_{k-(\ell-1)}, \ldots, v_k$ and $v_{k+d-(\ell-1)}, \ldots, v_{k+d}$ span a projective ℓ-space for every $k \in \mathbb{Z}$.*

Proof. The case $\ell = 2$ is the definition of a corrugated polygon. Denote the above projective ℓ-dimensional space by $Q_k^\ell = (v_{k-(\ell-1)}, \ldots, v_k, v_{k+d-(\ell-1)}, \ldots, v_{k+d})$.

Then for any $\ell > 2$ the intersection of the ℓ-spaces Q_k^ℓ and Q_{k+1}^ℓ is spanned by the vertices $(v_{k-(\ell-2)}, \ldots, v_k, v_{k+d-(\ell-2)}, \ldots, v_{k+d})$ and has dimension $\ell - 1$, i.e., is the space $Q_k^{\ell-1} = Q_k^\ell \cap Q_{k+1}^\ell$. This allows one to derive the condition on $(\ell-1)$-dimensional spaces from the condition on ℓ-dimensional spaces, and hence reduce everything to the case $\ell = 2$.

Conversely, start with the $(\ell-1)$-dimensional space $Q_k^{\ell-1}$ and consider the space Q_k^ℓ containing $Q_k^{\ell-1}$, as well as the vertices $v_{k-(\ell-1)}$ and $v_{k+d-(\ell-1)}$. We claim that after the addition of two extra vertices the new space has dimension ℓ, rather than $\ell+1$. Indeed, the four vertices $v_{k-(\ell-1)}, v_{k-(\ell-2)}, v_{k+d-(\ell-1)}, v_{k+d-(\ell-2)}$ lie in one two-dimensional plane according to the above reduction. Thus adding two vertices $v_{k-(\ell-1)}$ and $v_{k+d-(\ell-1)}$ to the space $Q_k^{\ell-1}$, which already contains $v_{k-(\ell-2)}$ and $v_{k+d-(\ell-2)}$, boils down to adding one more projective direction, because of the corrugated condition, and thus Q_k^ℓ has dimension ℓ for all $k \in \mathbb{Z}$. □

Proof of Theorem 5.3. Now we take a generic twisted n-gon in $\mathbb{RP}^d$ and consider the dented $(d-1)$-dimensional diagonal P_k corresponding to $m = 1$ and $I = (2, 1, \ldots, 1)$, i.e., the hyperplane passing through the d vertices $v_k, v_{k+2}, v_{k+3}, \ldots, v_{k+d}$.

For a corrugated n-gon in $\mathbb{RP}^d$, according to the proposition above, such a diagonal hyperplane will also pass through the vertices $v_{k-(\ell-1)}, \ldots, v_{k-1}$, i.e., it coincides with the space Q_k^ℓ for $\ell = d-1$:

$$P_k = Q_k^{d-1} = (v_{k-(d-2)}, \ldots, v_k, v_{k+2}, \ldots, v_{k+d})$$

(see Figure 7).

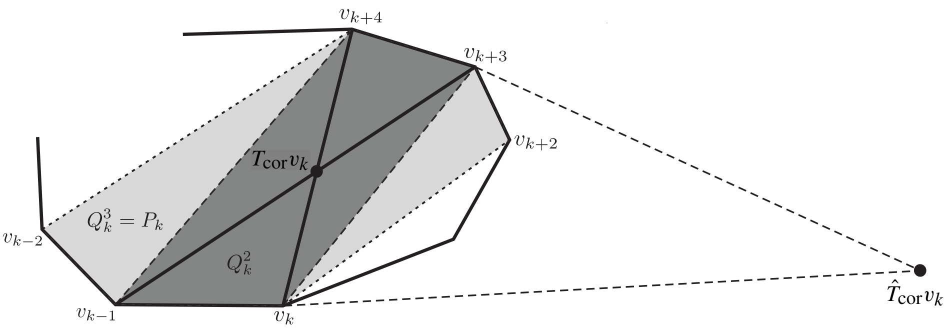

Fig. 7. The diagonal hyperplane P_k coincides with the hyperplane Q_k^3 in $\mathbb{RP}^4$. Definitions of the corrugated pentagram map and its dual.

Now the intersection of d consecutive hyperplanes $P_k \cap \cdots \cap P_{k+d-1}$, by the repeated use of the relation $Q_k^{\ell-1} = Q_k^\ell \cap Q_{k+1}^\ell$ for $\ell = d-1, d-2, \ldots, 3$, reduces to the intersection of $Q_k^2 \cap Q_{k+1}^2 \cap Q_{k+2}^2$. The latter is the intersection of the diagonals in Q_{k+1}^2, i.e., $(v_{k+1}, v_{k+d+1}) \cap (v_{k+2}, v_{k+d+2}) =: T_{\mathrm{cor}} v_{k+1}$. Thus the definition of the dented pentagram map T_m for $m = 1$ upon restriction to corrugated polygons reduces to the definition of the pentagram map T on the latter (modulo shifts).

For any $m = 1, \ldots, d-1$ we consider the dented diagonal hyperplane

$$P_{k-m+1} = (v_{k-m+1}, \ldots, v_k, v_{k+2}, v_{k+3}, \ldots, v_{k+d-m+1}).$$

For corrugated n-gons in $\mathbb{RP}^d$ this diagonal hyperplane coincides with the space Q_{k-m+1}^ℓ for $\ell = d-1$ since it passes through all vertices from $v_{k-(d-2)}$ to v_{k+d} except v_{k+1}:

$$P_{k-m+1} = Q_k^{d-1} = (v_{k-(d-2)}, \ldots, v_k, v_{k+2}, \ldots, v_{k+d}).$$

Thus the corresponding intersection of d consecutive dented diagonal hyperplanes starting with P_{k-m+1} will differ only by a shift of indices from the one for $m = 1$. □

Corollary 5.5. *For dented pentagram maps T_m with different values of m, their restrictions from generic to corrugated twisted polygons in $\mathbb{RP}^d$ coincide modulo an index shift.*

Note that the inverse dented pentagram map $\widehat{T_m}$ upon restriction to corrugated polygons also coincides with the inverse corrugated pentagram map $\widehat{T}_{\rm cor}$. The latter is defined as follows: for a corrugated polygon (v_k) in $\mathbb{RP}^d$ for every $k \in \mathbb{Z}$ consider the two-dimensional plane spanned by the vertices v_{k-1}, v_k, v_{k+d-1}, and v_{k+d}. In this plane take the intersection of (the continuations of) the sides of the polygon, i.e., the lines (v_{k-1}, v_k) and (v_{k+d-1}, v_{k+d}), and set

$$\widehat{T}_{\rm cor} v_k := (v_{k-1}, v_k) \cap (v_{k+d-1}, v_{k+d}).$$

Corollary 5.6. *The continuous limit of the pentagram map $T_{\rm cor}$ for corrugated polygons in $\mathbb{RP}^d$ is a restriction of the $(2, d+1)$-KdV equation.*

The continuous limit for dented maps is found by means of the general procedure described in Section 4. The restriction of the universal $(2, d+1)$-KdV system from generic to corrugated curves might lead to other interesting equations on the submanifold. (This phenomenon could be similar to the KP hierarchy on generic pseudo-differential operators $\partial + \sum_{j\geq 1} u_j(x)\partial^{-j}$, which when restricted to operators of the form $\partial + \psi(x)\partial^{-1}\psi^*(x)$ gives the NLS equation; see [6].)

Remark 5.7. One of applications of corrugated polygons is related to the fact that there is a natural map from generic polygons in 2D to corrugated polygons in any dimension (see [2] and Remark 5.8 below), which is generically a local diffeomorphism. Furthermore, this map commutes with the pentagram map, i.e., it takes deeper diagonals, which join vertices v_i and v_{i+p}, in 2D polygons to the intersecting diagonals of corrugated polygons in $\mathbb{RP}^p$. This way one obtains a representation of the deeper diagonal pentagram map $T_{p,1}$ in $\mathbb{RP}^2$ via the corrugated pentagram map in higher dimensions (see Figure 8).

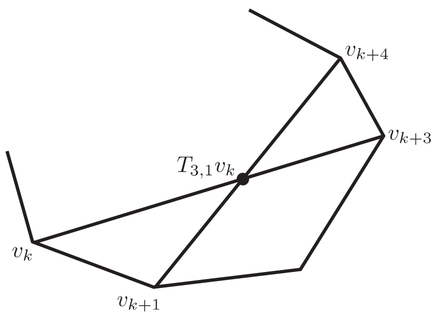

Fig. 8. Deeper pentagram map $T_{3,1}$ in 2D.

As a corollary, the deeper diagonal pentagram map $T_{p,1}$ in $\mathbb{RP}^2$ is also an integrable system [2]. Indeed, integrability of corrugated pentagram maps implies integrability of the pentagram map for deeper diagonals in 2D, since first integrals and other structures for the corrugated pentagram map in higher dimensions descend to those for the pentagram map on generic polygons in 2D thanks to the equivariant local diffeomorphism between them. Explicit formulas for the invariants seem to be complicated because of a nontrivial relation between coordinates for polygons in $\mathbb{RP}^2$ and in $\mathbb{RP}^p$.

5.2. *Integrability for corrugated polygons*

Generally speaking, the algebraic-geometric integrability of the pentagram map on the space $\mathcal{P}_n$ (see Theorem 2.15 for the 3D case) would not necessarily imply the algebraic-geometric integrability for a subsystem, the pentagram map on the subspace $\mathcal{P}_n^{\mathrm{cor}}$ of corrugated polygons.

However, a Lax representation with a spectral parameter for corrugated polygons naturally follows from that for generic ones. In this section, we perform its analysis in the 3D case (similarly to what has been done in Theorem 2.15), which implies the algebraic-geometric integrability for corrugated polygons in the 3D case. It exhibits some interesting features: the dynamics on the Jacobian depends on whether n is a multiple of 3, and if it is, it resembles a "staircase", but with shifts in three different directions. We also establish the equivalence of our Lax representation with that found in [2].

For simplicity, we assume that $\gcd(n, d+1) = 1$ (see Remark 2.3). In 3D this just means that n has to be odd. Note that this condition is technical, as one can get rid of it by using coordinates introduced in Section 2.2.[2]

Remark 5.8. The coordinates on the space $\mathcal{P}_n^{\mathrm{cor}}$ may be introduced using the same difference equation (1) for $\gcd(n, d+1) = 1$. Since corrugatedness means that the vectors V_j, V_{j+1}, V_{j+d} and V_{j+d+1} are linearly dependent for all $j \in \mathbb{Z}$, the subset $\mathcal{P}_n^{\mathrm{cor}}$ of corrugated polygons is singled out in the space of generic twisted polygons $\mathcal{P}_n$ by the relations $a_{j,l} = 0$, $2 \le l \le d-1$, in equation (1), i.e., they are defined by the equations

$$V_{j+d+1} = a_{j,d} V_{j+d} + a_{j,1} V_{j+1} + (-1)^d V_j, \qquad j \in \mathbb{Z}. \tag{4}$$

Furthermore, this relation also allows one to define a map ψ from generic twisted n-gons in $\mathbb{RP}^2$ to corrugated ones in $\mathbb{RP}^d$ for any dimension d (see [2]). Indeed, consider a lift of vertices $v_j \in \mathbb{RP}^2$ to vectors $V_j \in \mathbb{R}^3$ so that they satisfy (4) for all $j \in \mathbb{Z}$. Note that for $d \ge 3$ this is a nonstandard normalization of the lifts $V_j \in \mathbb{R}^3$, different from the one given in (1) for $d = 2$, since the vectors on the right-hand side are not consecutive. Now by considering solutions $V_j \in \mathbb{R}^{d+1}$ of (4) modulo the natural action of $SL_{d+1}(\mathbb{R})$ we obtain a polygon in $\mathbb{RP}^d$ satisfying the corrugatedness condition. The constructed map ψ commutes with the pentagram maps (since all operations are projectively invariant) and is a local diffeomorphism. Observe that the subset $\mathcal{P}_n^{\mathrm{cor}} \subset \mathcal{P}_n$ has dimension $2n$.

Now we return to considerations over $\mathbb{C}$. The above restriction $\gcd(n, d+1) = 1$ allows one to define a Lax function in a straightforward way. Here is an analogue of Theorem 2.15:

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[2] Another way to introduce the coordinates is by means of the difference equation $V_{j+d+1} = V_{j+d} + b_{j,d-1} V_{j+d-1} + \cdots + b_{j,1} V_{j+1} + b_{j,0} V_j$, used in [2].

Theorem 5.9. *In dimension* 3 *the subspace* $\mathcal{P}_n^{\mathrm{cor}} \subset \mathcal{P}_n$ *is generically fibred into* (*Zariski open subsets of*) *tori of dimension* $g = n - 3$ *if* $n = 3l$, *and* $g = n - 1$ *otherwise.*

Proof. The Lax function for the map T_2 restricted to the space $\mathcal{P}_n^{\mathrm{cor}}$ is

$$L_{j,t}^{-1}(\lambda) = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & a_{j,1} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \lambda & a_{j,3} \end{pmatrix}.$$

Now the spectral function has the form

$$R(k,\lambda) = k^4 - \frac{k^3}{\lambda^{\lfloor n/3 \rfloor}} \sum_{j=0}^{\lfloor n/3 \rfloor} G_j \lambda^j + \frac{k^2}{\lambda^{\lfloor 2n/3 \rfloor}} \sum_{j=0}^{N_0} J_j \lambda^j - \frac{k}{\lambda^n} \sum_{j=0}^{\lfloor n/3 \rfloor} I_j \lambda^j + \frac{1}{\lambda^n}$$

where $N_0 = \lfloor n/3 \rfloor - \lfloor \gcd(n-1,3)/3 \rfloor$. One can show that $G_{\lfloor n/3 \rfloor} = \prod_{j=0}^{n-1} a_{j,1}$ and $I_0 = \prod_{j=0}^{n-1} a_{j,3}$. Below we present the summary of relevant computations for the spectral functions, Casimirs, and the Floquet–Bloch solutions (cf. Section 2.3).

$n = 3l+1;\ n = 3l+2$	
$\lambda = 0$	$\lambda = \infty$
$O_1 : k_1 = 1/I_0 + \mathcal{O}(\lambda)$ $O_2 : k_{2,3,4} = I_0^{1/3} \lambda^{-n/3}(1 + \mathcal{O}(\lambda^{1/3}))$	$W_1 : k_1 = G_l(1 + \mathcal{O}(\lambda^{-1}))$ $W_2 : k_{2,3,4} = G_l^{-1/3} \lambda^{-n/3}(1 + \mathcal{O}(\lambda^{-1/3}))$
$g = n - 1$; there are $n+1$ first integrals; the Casimirs are I_0, G_l	
$(\psi_{i,1}) \geq -D + 2O_2 + W_2 + i(W_2 - O_2)$ $(\psi_{i,2}) \geq -D + O_2 + W_1 + W_2 + i(W_2 - O_2)$ $(\psi_{i,3}) \geq -D + W_1 + 2W_2 + i(W_2 - O_2)$ $(\psi_{i,4}) \geq -D + 3O_2 + i(W_2 - O_2)$	

$n = 3l$	
$\lambda = 0$	$\lambda = \infty$
$O_1 : k_1 = 1/I_0 + \mathcal{O}(\lambda)$ $O_{2,3,4} : k_{2,3,4} = c_1 \lambda^{-l}(1 + \mathcal{O}(\lambda))$, where $c_1^3 - c_1^2 G_0 + c_1 J_0 - I_0 = 0$	$W_1 : k_1 = G_l(1 + \mathcal{O}(\lambda^{-1}))$ $W_{2,3,4} : k_{2,3,4} = c_2 \lambda^{-l}(1 + \mathcal{O}(\lambda^{-1}))$, where $c_2^3 G_l - c_2^2 J_l + c_2 I_l - 1 = 0$
$g = n - 3$; there are $n+3$ first integrals; the Casimirs are $I_0, G_0, J_0, I_l, J_l, G_l$	
$(\psi_{i,1}) \geq -D + O_2 + O_3 + W_4 + i_2(W_2 - O_3) + i_1(W_3 - O_2) + i_0(W_4 - O_4)$ $(\psi_{i,2}) \geq -D + O_2 + W_1 + W_4 + i_2(W_2 - O_2) + i_1(W_3 - O_4) + i_0(W_4 - O_3)$ $(\psi_{i,3}) \geq -D + W_1 + W_2 + W_4 + i_2(W_3 - O_4) + i_1(W_4 - O_3) + i_0(W_2 - O_2)$ $(\psi_{i,4}) \geq -D + O_2 + O_3 + O_4 + i_2(W_4 - O_4) + i_1(W_2 - O_3) + i_0(W_3 - O_2)$ where $i_2 = \lfloor (i+2)/3 \rfloor$, $i_1 = \lfloor (i+1)/3 \rfloor$, $i_0 = \lfloor i/3 \rfloor$	

The genus of spectral curves found above exhibits the dichotomy $g = n-3$ or $g = n-1$ according to divisibility of n by 3. □

Remark 5.10. It is worth noting that the dimensions $g = n-3$ or $g = n-1$ of the Jacobians, and hence of the invariant tori, are consistent in the following sense:

- the sum of the genus of the spectral curve (which equals the dimension of its Jacobian) and the number of first integrals equals $2n$, i.e., the dimension of the system;
- the number of first integrals minus the number of Casimirs equals the genus of the curve. The latter also suggests that Krichever–Phong's universal formula provides a symplectic form for this system.

Also note that Lax functions corresponding to the maps T_1 and T_2 restricted to the subspace $\mathcal{P}_n^{\mathrm{cor}}$ lead to the same spectral curve in 3D, as one can check directly. This, in turn, is consistent with Corollary 5.5.

Proposition 5.11. *In any dimension the Lax function for the corrugated pentagram map T_1 in $\mathbb{CP}^d$ for* $\gcd(n, d+1) = 1$ *is*

$$L_{j,t}(\lambda) = \left(\begin{array}{cccc|c} 0 & 0 & \cdots & 0 & (-1)^d \\ \hline & & & & a_{j,1} \\ & & & & 0 \\ & & D(\lambda) & & \cdots \\ & & & & 0 \\ & & & & a_{j,d} \end{array}\right)^{-1},$$

with the diagonal $d \times d$ matrix $D(\lambda) = \mathrm{diag}(1, \lambda, 1, \ldots, 1)$. It is equivalent to the one found in [2].

Proof. The above Lax form follows from Remark 5.8 and Theorem 2.6. To show the equivalence we define the gauge matrix as follows:

$$g_j = \left(\begin{array}{cccc|c} 0 & 0 & \cdots & 0 & (-1)^d \\ \hline & & & & 0 \\ & & C_j & & \cdots \\ & & & & 0 \\ & & & & a_{j,d} \end{array}\right),$$

where C_j is the $d \times d$ diagonal matrix, and its diagonal entries are equal to $(C_j)_{ll} = \prod_{k=0}^{d-l} a_{j-k,d}$, $1 \le l \le d$. One can check that

$$\tilde{L}_{j,t}(\lambda) = \frac{g_j^{-1} L_{j,t}^{-1} g_{j+1}}{a_{j+1,d}} = \begin{pmatrix} 0 & 0 & 0 & \cdots & x_j & x_j + y_j \\ \lambda & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ & & & \cdots & & \\ 0 & 0 & 0 & \cdots & 1 & 1 \end{pmatrix},$$

where

$$x_j = \frac{a_{j,1}}{\prod_{l=0}^{d-1} a_{j-l,d}}, \qquad y_j = \frac{1}{\prod_{l=-1}^{d-1} a_{j-l,d}},$$

which agrees with [2, formula (10)]. □

Note that the corresponding corrugated pentagram map has a cluster interpretation [2] (see also [3] for the 2D case). On the other hand, it is a restriction of the dented pentagram map, which brings one to the following

Problem 5.12. Is it possible to realize the dented pentagram map T_m on generic twisted polygons in $\mathbb{P}^d$ as a sequence of cluster transformations?

We will address this problem in a future publication.

6. Applications: integrability of pentagram maps for deeper dented diagonals

In this section we consider in detail more general dented pentagram maps.

Definition 6.1. Fix an integer parameter $p \geq 2$ in addition to an integer parameter $m \in \{1, \dots, d-1\}$ and define the $(d-1)$-tuple $I = I_m^p := (1, \dots, 1, p, 1, \dots, 1)$, where the value p is at the mth place. This choice of I corresponds to the diagonal plane P_k which passes through m consecutive vertices $v_k, v_{k+1}, \dots, v_{k+m-1}$, then skips $p-1$ vertices $v_{k+m}, \dots, v_{k+m+p-2}$ (i.e., "jumps to the next pth vertex") and continues passing through the next $d-m$ consecutive vertices $v_{k+m+p-1}, \dots, v_{k+d+p-2}$:

$$P_k := (v_k, v_{k+1}, \dots, v_{k+m-1}, v_{k+m+p-1}, v_{k+m+p}, \dots, v_{k+d+p-2}).$$

We call such a plane P_k a *deep-dented diagonal (DDD) plane*, as the "dent" now is of depth p (see Figure 9). Now we intersect d consecutive planes P_k, to define the *deep-dented pentagram map* by

$$T_m^p v_k := P_k \cap \dots \cap P_{k+d-1}.$$

In other words, we keep the same definition of the $(d-1)$-tuple $J = \mathbf{1} := (1, \dots, 1)$ as before: $T_m^p := T_{I_m^p, \mathbf{1}}$.

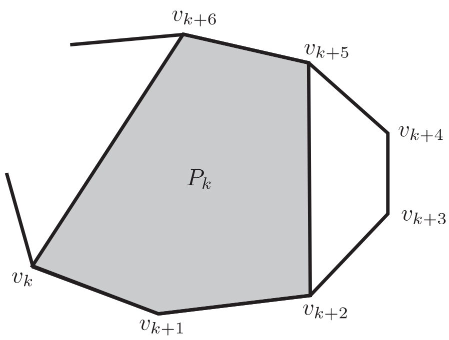

Fig. 9. The diagonal hyperplane for $I = (1, 1, 3, 1)$ in $\mathbb{RP}^5$.

Theorem 6.2. *The deep-dented pentagram map for both twisted and closed polygons in any dimension is the restriction of an integrable system to an invariant submanifold. Moreover, it admits a Lax representation with a spectral parameter.*

To prove this theorem we introduce spaces of partially corrugated polygons, occupying intermediate positions between corrugated and generic ones.

Definition 6.3. A twisted polygon (v_j) in $\mathbb{RP}^d$ is *partially corrugated* (or $(q, r; \ell)$-*corrugated*) if the diagonal subspaces P_j spanned by two clusters of q and r consecutive vertices v_j with a gap of $d - \ell$ vertices between them (i.e.,

$$P_j = (v_j, v_{j+1}, \ldots, v_{j+q-1}, v_{j+q+d-\ell}, v_{j+q+d-\ell+1}, \ldots, v_{j+q+d-\ell+r-1}),$$

see Figure 9) are subspaces of a fixed dimension $\ell \leq q + r - 2$ for all $j \in \mathbb{Z}$. The inequality $\ell \leq q + r - 2$ shows that indeed these vertices are not in general position, while $\ell = q + r - 1$ corresponds to a generic twisted polygon. We also assume that $q \geq 2$, $r \geq 2$, and $\ell \geq \max\{q, r\}$, so that the corrugatedness restriction would not be local, i.e., coming from one cluster of consecutive vertices, but would come from the interaction of two clusters of those.

Fix n and denote the space of partially corrugated twisted n-gons in $\mathbb{RP}^d$ (modulo projective equivalence) by $\mathcal{P}^{\text{par}}$. Note that "corrugated" of Definition 5.1 means "(2, 2; 2)-corrugated" in this terminology.

Proposition 6.4. *The definition of a $(q, r; \ell)$-corrugated polygon in $\mathbb{RP}^d$ is equivalent to the definition of a $(q + 1, r + 1; \ell + 1)$-corrugated polygon, i.e., one can add one extra vertex to (respectively, delete one extra vertex from) each of the two clusters of vertices, as well as to increase (respectively, decrease) by one the dimension of the subspace through them, as long as $q, r \geq 2$, $\ell \leq q + r - 2$, and $\ell \leq d - 2$.*

Proof. The proof of this fact is completely analogous to the proof of Proposition 5.4 by adding one vertex in each cluster. □

Define the *partially corrugated pentagram map* T_{par} on the space $\mathcal{P}^{\text{par}}$: to a partially corrugated twisted n-gon we associate a new one obtained by taking the intersections of $\ell + 1$ consecutive diagonal subspaces P_j of dimension ℓ.

Proposition 6.5. (i) *The partially corrugated pentagram map is well defined: by intersecting $\ell + 1$ consecutive diagonal subspaces one generically gets a point in the intersection.*
(ii) *This map takes a partially corrugated polygon to a partially corrugated one.*

Proof. Note that the gap of $d - \ell$ vertices between clusters is narrowing by one vertex at each step as the dimension ℓ increases by 1. Add the maximal number of vertices, so as to obtain a hyperplane (of dimension $d - 1$) passing through the clusters of q and r vertices with a gap of one vertex between them. This is a dented hyperplane. One can see that intersections of 2, 3, ... consecutive dented hyperplanes gives exactly the planes of dimensions $d-2$, $d-3$, ... obtained on the way while enlarging the clusters of vertices.

Then the intersection of d consecutive dented hyperplanes is equivalent to the intersection of $\ell+1$ consecutive diagonal subspaces of dimension ℓ for partially corrugated polygons, and is generically a point.

The fact that the image of a partially corrugated polygon is also partially corrugated can be proved similarly to the standard corrugated case (cf. Proposition 5.2). We demonstrate the necessary changes in the following example. Consider the (3, 2; 3)-corrugated polygon in $\mathbb{RP}^d$ (here $q = 3, r = 2, \ell = 3$), i.e., its vertices $(v_j, v_{j+1}, v_{j+2}, v_{j+d}, v_{j+d+1})$ form a 3D subspace P_j in $\mathbb{RP}^d$ for all $j \in \mathbb{Z}$. One can see that for the image polygon: (a) three new vertices will lie in the 2D plane obtained as the intersection $B_{j+1} := P_j \cap P_{j+1} = (v_{j+1}, v_{j+2}, v_{j+d+1})$ (since to get each of these three new vertices one needs to intersect these two planes with two more, and the corresponding intersections will always lie in this plane); (b) similarly, two new vertices will lie on a certain line passing through the vertex v_{j+d+1} (this line is the intersection of 2-planes: $l_{j+d} := B_{j+d} \cap B_{j+d+1}$). Hence the resulting five new vertices belong to one 3D plane spanned by B_{j+1} and l_{j+d}, and hence satisfy the (3, 2; 3)-corrugatedness condition for all $j \in \mathbb{Z}$. The general case of partial corrugatedness is proved similarly. □

Theorem 6.6 (= 6.2′). *The pentagram map on partially corrugated polygons in any dimension is an integrable system: it admits a Lax representation with a spectral parameter.*

Proof of Theorems 6.2 and 6.6. Now suppose that we are given a generic polygon in $\mathbb{RP}^c$ and the pentagram map constructed with the help of a deep-dented diagonal (of dimension $c-1$) with the $(c-1)$-tuple of jumps $I = (1, \dots, 1, p, 1, \dots, 1)$, which includes m and $c-m$ consecutive vertices before and after the gap respectively. Note that the corresponding gap between two clusters of points for such diagonals consists of $p-1$ vertices. Associate to this polygon a partially $(q, r; \ell)$-corrugated polygon in the higher-dimensional space $\mathbb{RP}^d$ with clusters of $q = m+1$ and $r = (c-m)+1$ vertices, the diagonal dimension $\ell = c$, and the space dimension $d = c+p-2$. Namely, in the partially corrugated polygon we add one extra vertex to each cluster, increase the dimension of the diagonal plane by one as well (without corrugatedness the diagonal dimension would increase by two after the addition of two extra vertices), while the gap between the two new clusters decreases by one: $(p-1)-1 = p-2$. Then the dimension d is chosen so that the gap between the two new clusters is $p-2 = d-\ell$, which implies that $d = \ell + p - 2 = c + p - 2$. (Example: for deeper p-diagonals in $\mathbb{RP}^2$ one has $c = 2, m = 1, q = r = \ell = 2$, and this way one obtains the space of corrugated polygons in $\mathbb{RP}^d$ for $d = p$.)

Consider the map ψ associating to a generic polygon in $\mathbb{RP}^c$ a partially corrugated twisted polygon in $\mathbb{RP}^d$, where $d = c+p-2$. (The map ψ is defined similarly to the one for corrugated polygons in Remark 5.8.) This map ψ is a local diffeomorphism and commutes with the pentagram map: the deep-dented pentagram map in $\mathbb{RP}^c$ is taken to the pentagram map T_{par} on partially corrugated twisted polygons in $\mathbb{RP}^d$. In turn, T_{par} is the restriction of the integrable dented pentagram map in $\mathbb{RP}^d$. Thus the deep-dented pentagram map on polygons in $\mathbb{RP}^c$ is the restriction to an invariant submanifold of an integrable map on partially corrugated twisted polygons in $\mathbb{RP}^d$, and hence it is a subsystem of an integrable system. The Lax form of T_{par} can be obtained by restricting the Lax form for dented maps from generic to partially corrugated polygons in $\mathbb{RP}^d$. We present this Lax form below. □

Remark 6.7. Now we describe coordinates on the subspace of partially corrugated polygons and a Lax form of the corresponding pentagram map T_{par} on them. Recall that on the space of generic twisted n-gons (v_j) in $\mathbb{RP}^d$ for $\gcd(n, d+1) = 1$ there are coordinates $a_{j,k}$ for $0 \le j \le n-1$, $0 \le k \le d-1$ defined by equation (1):

$$V_{j+d+1} = a_{j,d} V_{j+d} + \cdots + a_{j,1} V_{j+1} + (-1)^d V_j,$$

where the $V_j \in \mathbb{R}^{d+1}$ are lifts of the vertices $v_j \in \mathbb{RP}^d$. One can see that the submanifold of $(q, r; \ell)$-corrugated polygons in $\mathbb{RP}^d$ without loss of generality can be assumed to have the minimal number of vertices in clusters (see Proposition 6.4). In other words, in this case there is a positive integer m such that $q = m+1$, $r = (\ell - m) + 1$, while the gap between the clusters consists of $d - \ell$ vertices. Hence the corresponding twisted polygons are described by linear dependence of $q = m+1$ vertices $V_j, \ldots, V_{j+m}$ and r vertices $V_{j+d+m-\ell+1}, \ldots, V_{j+d+1}$. (Example: for $m = 1$, $\ell = 2$ implies a linear relation between V_j, V_{j+1} and V_{j+d}, V_{j+d+1}, which is the standard corrugatedness condition.) This relation can be written as

$$V_{j+d+1} = a_{j,d} V_{j+d} + \cdots + a_{j,d+m-\ell+1} V_{j+d+m-\ell+1} + a_{j,m} V_{j+m} + \cdots + a_{j,1} V_{j+1} + (-1)^d V_j$$

for all $j \in \mathbb{Z}$ by choosing an appropriate normalization of the lifts $V_j \in \mathbb{R}^{d+1}$. Thus the set of partially corrugated polygons is obtained by imposing the condition $a_{j,k} = 0$ for $m+1 \le k \le d+m-\ell$ and $0 \le j \le n-1$ in the space of generic twisted polygons given by equation (1). Note that the space of $(m+1, \ell-m+1; \ell)$-corrugated n-gons in $\mathbb{RP}^d$ has dimension $n\ell$, while the space of generic twisted n-gons has dimension nd.

In the complex setting, the Lax representation on such partially corrugated n-gons in $\mathbb{CP}^d$ or on generic n-gons in $\mathbb{CP}^c$ with deeper dented diagonals is described as follows.

Theorem 6.8. *The deep-dented pentagram map T_m^p on generic twisted and closed polygons in $\mathbb{CP}^c$ and the pentagram map T_{par} on the corresponding partially corrugated polygons in $\mathbb{CP}^d$ with $d = c + p - 2$ admits the following Lax representation with a spectral parameter: for* $\gcd(n, d+1) = 1$ *its Lax matrix is*

$$L_{j,t}(\lambda) = \left(\begin{array}{cccccc|c} 0 & 0 & \cdots & \cdots & 0 & 0 & (-1)^d \\ \hline & & & & & & a_{j,1} \\ & & & & & & \cdots \\ & & & & & & a_{j,m} \\ & & & D(\lambda) & & & 0 \\ & & & & & & \cdots \\ & & & & & & 0 \\ & & & & & & a_{j,d+m-\ell+1} \\ & & & & & & \cdots \\ & & & & & & a_{j,d} \end{array}\right)^{-1}, \qquad (5)$$

with the diagonal $d \times d$ *matrix* $D(\lambda) = \operatorname{diag}(1, \dots, 1, \lambda, 1, \dots 1)$*, where* λ *is at the* $(m+1)$th *place, and an appropriate matrix* $P_{j,t}(\lambda)$*.*

Proof. This follows from the fact that the partially corrugated pentagram map is the restriction of the dented map to the invariant subset of partially corrugated polygons, so the Lax form is obtained by the corresponding restriction as well (cf. Theorem 2.6). □

Note that the jump tuple $I = (2, 3)$ in $\mathbb{P}^3$ corresponds to the first case which is neither a deep-dented pentagram map, nor a short-diagonal one, and whose integrability is unknown. It would be very interesting if the corresponding pentagram map turned out to be nonintegrable. Some numerical evidence for nonintegrability in that case is presented in [5].

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