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Group actions on monotone skew-product semiflows with applications

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Abstract. We discuss a general framework of monotone skew-product semiflows under a connected group action. In a prior work, a compact connected group G-action has been considered on a strongly monotone skew-product semiflow. Here we relax the strong monotonicity and compactness requirements, and establish a theory concerning symmetry or monotonicity properties of uniformly stable 1-cover minimal sets. We then apply this theory to show rotational symmetry of certain stable entire solutions for a class of nonautonomous reaction-diffusion equations on $\mathbb{R}^n$, as well as monotonicity of stable traveling waves of some nonlinear diffusion equations in time-recurrent structures including almost periodicity and almost automorphy.

Keywords. Monotone skew-product semiflows, group actions, rotational symmetry, reaction-diffusion equations, traveling waves

1. Introduction

In this article, we investigate monotone skew-product semiflows with certain symmetry such as ones with respect to rotation or translation. We will restrict our attention to solutions which are 'stable' in a certain sense and discuss the relation between stability and symmetry.

Historically, stability is in many cases known to imply some sort of symmetry. For autonomous (or time-periodic) parabolic equations, any stable equilibrium (or time-periodic) solution inherits the rotational symmetry of the domain Ω (see [3, 11] for bounded domains and [18, 19] for unbounded domains). In [18, 19], the symmetry of the stable solutions was also obtained for degenerate diffusion equations and systems of reaction-diffusion equations. Ni et al. [16] showed the spatially symmetric or monotonic structure of stable solutions in shadow systems as a limit of reaction-diffusion systems. It is now

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well known that parabolic equations and systems admitting the comparison principle define (strongly) monotone dynamical systems, whose concept was introduced in [8] (see [9, 24] for a comprehensive survey on the development of this theory). If the domain and the coefficients in such an equation or system exhibit some symmetry, then the dynamical system commutes with the action of some topological group G. Extensions and generalizations of group actions to a general framework of (strongly) monotone systems were given in [10, 13, 18, 19, 30].

Nonperiodic and nonautonomous equations have been attracting much attention recently. A unified framework to study nonautonomous equations is based on the so-called skew-product semiflows (see [25, 26]). In [32], a compact connected group G-action on a strongly monotone skew-product semiflow Π_t was considered. Assuming that a minimal set K of Π_t is stable, it was proved in [32] that K is residually symmetric, and moreover any uniformly stable orbit is asymptotically symmetric. In this article, motivated by work of Ogiwara and Matano [18, 19], we relax the strong monotonicity of Π_t, as well as the compactness of G. To formulate our results precisely, we let K be a uniformly stable 1-cover of the base flow. Under the assumption that Π_t is only monotone and G is only connected, we establish the globally topological structure of the group orbit GK of K, where $GK = \{(g \cdot x, \omega) : g \in G \text{ and } (x, \omega) \in K\}$ (see Theorem B). Roughly speaking, the group orbit GK either coincides with K (which entails that K is G-symmetric), or is a 1-dimensional continuous subbundle over the base, each fiber being totally ordered and homeomorphic to $\mathbb{R}$. In particular, when the second case holds, the uniform stability of K will imply its asymptotic uniform stability (see Theorem D).

Our main theorems are extensions of symmetry results in [18, 19] on stable equilibria (resp. fixed points) for continuous-time (resp. discrete-time) monotone systems. This enables us to investigate the symmetry of stable entire solutions of nonlinear reaction-diffusion equations in *time-recurrent structures* (see Definition 2.6) on a symmetric domain. This is satisfied, for instance, when the reaction term is a *uniformly almost periodic* or, more generally, a *uniformly almost automorphic* function in t (see Section 2 for more details).

Since strong monotonicity of the skew-product semiflow is weakened, we are able to deal with the time-recurrent parabolic equation on an unbounded symmetric domain such as the entire space $\mathbb{R}^n$. For nonautonomous parabolic equations, radial symmetry has been shown to be a consequence of positivity of the solutions (see e.g. [1, 7, 21, 22] and the references therein). For nonautonomous parabolic equations on $\mathbb{R}^n$, we also refer to a series of very recent works by Poláčik [20, 21, 23] on this topic and its applications. In particular, in [21] he proved that, under some symmetry conditions, any positive bounded entire solution decaying to zero at spatial infinity uniformly with respect to time is radially symmetric. However, as far as we know, symmetry properties of stable entire (possibly sign-changing) solutions of nonautonomous parabolic equations on $\mathbb{R}^n$ have been poorly studied. By applying our abstract results mentioned above, we initiate research on this subject. More precisely, we show (see Theorem 7.1) that any uniformly stable entire solution is radially symmetric provided that it satisfies a certain module containment (see Definition 2.7) and decays to zero at spatial infinity uniformly with respect to time.

Note also that we have relaxed the requirement of compactness of the acting group G. This will allow us to discuss symmetry or monotonicity properties with respect to the translation group. Based on this, we can investigate monotonicity of uniformly stable traveling waves for time-recurrent bistable reaction-diffusion equations or systems. Traveling waves in time-almost periodic nonlinear evolution equations governed by bistable nonlinearities were first established in a series of pioneer works by Shen [27]–[29]. In [27, 28], she proved the existence of such almost periodic traveling waves, and showed that any such monotone traveling wave is uniformly stable. By using our abstract Theorem B, on the other hand, we provide a converse theorem (see Theorem 7.6): any uniformly stable almost periodic traveling wave is monotone. In particular, this implies that solitary waves, i.e., waves connecting the same stable spatially homogeneous almost periodic solution, are not uniformly stable. We shall further show (in Theorem 7.7) that any uniformly stable almost periodic traveling wave is uniformly stable with asymptotic phase. Although a result similar to Theorem 7.7 can also be found in Shen [27], our approach (by Theorem D) is introduced in a very general framework, and hence it can be applied in a rather general context and to wider classes of equations with little modification.

This paper is organized as follows. In Section 2, we present some basic concepts and preliminary results from the theory of skew-product semiflows and almost periodic (automorphic) functions which will be important to our proofs. We state our main results in Section 3, where we also give standing assumptions characterizing our general framework. Sections 4–6 contain the proofs of our main results. In Section 7, we apply our abstract theorems to obtain symmetry properties of certain stable entire (possibly sign-changing) solutions of nonautonomous parabolic equations on $\mathbb{R}^n$, as well as the monotonicity of stable almost periodic traveling waves for time-recurrent reaction-diffusion equations.

2. Notation and preliminary results

In this section, we summarize some preliminary material to be used in later sections. First, we summarize some lifting properties of compact dynamical systems. We then collect definitions and basic facts concerning monotone skew-product semiflows and order-preserving group actions. Finally, we give a brief review of uniformly almost periodic and uniformly almost automorphic functions and flows.

Let Ω be a compact metric space with metric d_Ω, and $\sigma : \Omega\times\mathbb{R} \to \Omega$ be a continuous flow on Ω, denoted by (Ω, σ) or $(\Omega, \mathbb{R})$. As has become customary, we denote the value of σ at (ω, t) alternatively by $\sigma_t(\omega)$ or $\omega \cdot t$. By definition, $\sigma_0(\omega) = \omega$ and $\sigma_{t+s}(\omega) = \sigma_t(\sigma_s(\omega))$ for all $t, s \in \mathbb{R}$ and $\omega \in \Omega$. A subset $S \subset \Omega$ is *invariant* if $\sigma_t(S) = S$ for every $t \in \mathbb{R}$. A nonempty compact invariant set $S \subset \Omega$ is called *minimal* if it contains no nonempty, proper and closed invariant subset. We say that the continuous flow $(\Omega, \mathbb{R})$ is *minimal* if Ω itself is a minimal set. Let $(Z, \mathbb{R})$ be another continuous flow. A continuous map $p : Z \to \Omega$ is called a *flow homomorphism* if $p(z \cdot t) = p(z) \cdot t$ for all $z \in Z$ and $t \in \mathbb{R}$. A flow homomorphism which is onto is called a *flow epimorphism*, and a one-to-one flow epimorphism is referred as a *flow isomorphism*. We note that a homomorphism of minimal flows is already an epimorphism.

We say that a Banach space $(V, \|\cdot\|)$ is *ordered* if it contains a closed convex cone, that is, a nonempty closed subset $V_+ \subset V$ satisfying $V_+ + V_+ \subset V_+$, $\alpha V_+ \subset V_+$ for all $\alpha \geq 0$, and $V_+ \cap (-V_+) = \{0\}$. The cone V_+ induces an *ordering* on V via $x_1 \leq x_2$ if $x_2 - x_1 \in V_+$. We write $x_1 < x_2$ if $x_2 - x_1 \in V_+ \setminus \{0\}$. Given $x_1, x_2 \in V$, the set $[x_1, x_2] = \{x \in V : x_1 \leq x \leq x_2\}$ is called a *closed order interval* in V, and we write $(x_1, x_2) = \{x \in V : x_1 < x < x_2\}$.

A subset U of V is said to be *order convex* if for any $a, b \in U$ with $a < b$, the segment $\{a + s(b-a) : s \in [0, 1]\}$ is contained in U. And U is called *lower-bounded* (resp. *upper-bounded*) if there exists an element $a \in V$ such that $a \leq U$ (resp. $a \geq U$). Such an a is said to be a *lower bound* (resp. an *upper bound*) for U. A lower bound a_0 is said to be the *greatest lower bound* (g.l.b.) if any other lower bound a satisfies $a \leq a_0$. Similarly, we can define the *least upper bound* (l.u.b.).

Let $X = [a, b]_V$ with $a < b$ $(a, b \in V)$ or $X = V_+$, or let X be a closed order convex subset of V. Throughout this paper, we always assume that, for any $u, v \in X$, the greatest lower bound of $\{u, v\}$, denoted by $u \wedge v$, exists and that $(u, v) \mapsto u \wedge v$ is a continuous mapping from $X \times X$ into X.

Let $\mathbb{R}^+ = \{t \in \mathbb{R} : t \geq 0\}$. We consider a continuous *skew-product semiflow* $\Pi : \mathbb{R}^+ \times X \times \Omega \to X \times \Omega$ defined by

$$\Pi_t(x, \omega) = (u(t, x, \omega), \omega \cdot t), \quad \forall (t, x, \omega) \in \mathbb{R}^+ \times X \times \Omega, \tag{2.1}$$

satisfying (1) $\Pi_0 = \text{Id}$; (2) the *cocycle property*: $u(t+s, x, \omega) = u(s, u(t, x, \omega), \omega \cdot t)$ for each $(x, \omega) \in X \times \Omega$ and $s, t \in \mathbb{R}^+$. A subset $A \subset X \times \Omega$ is *positively invariant* if $\Pi_t(A) \subset A$ for all $t \in \mathbb{R}^+$; and *totally invariant* if $\Pi_t(A) = A$ for all $t \in \mathbb{R}^+$. The *forward orbit* of any $(x, \omega) \in X \times \Omega$ is $O^+(x, \omega) = \{\Pi_t(x, \omega) : t \geq 0\}$, and the *omega-limit set* of (x, ω) is $\mathcal{O}(x, \omega) = \{(\hat{x}, \hat{\omega}) \in X \times \Omega : \Pi_{t_n}(x, \omega) \to (\hat{x}, \hat{\omega})\ (n \to \infty)$ for some sequence $t_n \to \infty\}$. Clearly, if $O^+(x, \omega)$ is relatively compact, then $\mathcal{O}(x, \omega)$ is a nonempty, compact and totally invariant subset in $X \times \Omega$ for Π_t.

Let $P : X \times \Omega \to \Omega$ be the natural projection. A compact positively invariant set $K \subset X \times \Omega$ is called a 1-*cover* of the base flow if $P^{-1}(\omega) \cap K$ contains a unique element for every $\omega \in \Omega$. In this case, we denote this element by $(c(\omega), \omega)$ and write $K = \{(c(\omega), \omega) : \omega \in \Omega\}$, where $c : \Omega \to X$ is continuous with

$$\Pi_t(c(\omega), \omega) = (c(\omega \cdot t), \omega \cdot t), \quad \forall t \geq 0,$$

and hence $K \cap P^{-1}(\omega) = \{(c(\omega), \omega)\}$ for every $\omega \in \Omega$.

Next, we introduce some definitions concerning the stability of the skew-product semiflow Π_t. A forward orbit $O^+(x_0, \omega_0)$ of Π_t is said to be *uniformly stable* if for every $\varepsilon > 0$ there is a $\delta = \delta(\varepsilon) > 0$ such that if $s \geq 0$ and $\|u(s, x_0, \omega_0) - x\| \leq \delta(\varepsilon)$ for some $x \in X$, then for each $t \geq 0$, $\|u(t+s, x_0, \omega_0) - u(t, x, \omega_0 \cdot s)\| < \varepsilon$. We now define uniform stability for a compact positively invariant set $K \subset X \times \Omega$:

Definition 2.1 (Uniform stability for K). A compact positively invariant set K is said to be *uniformly stable* if for any $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$, called the *modulus of uniform stability*, such that if $(x, \omega) \in K$ and $(y, \omega) \in X \times \Omega$ are such that $\|x - y\| \leq \delta(\varepsilon)$, then

$$\|u(t, x, \omega) - u(t, y, \omega)\| < \varepsilon \quad \text{for all } t \geq 0.$$

Remark 2.2. It is easy to see that all the trajectories in a uniformly stable set are uniformly stable. Conversely, if a trajectory is uniformly stable, its omega-limit set inherits this property: that is, if $O^+(x_0, \omega_0)$ is relatively compact and uniformly stable, then $\mathcal{O}(x_0, \omega_0)$ is uniformly stable with the same modulus of uniform stability (see [17, 25]).

The following lemma is due to Novo et al. [17, Proposition 3.6]:

Lemma 2.3. *Assume that* $(\Omega, \mathbb{R})$ *is minimal. Let* $O^+(x, \omega)$ *be a forward orbit of* Π_t *which is relatively compact. If* $\mathcal{O}(x, \omega)$ *contains a minimal set* K *which is uniformly stable, then* $\mathcal{O}(x, \omega) = K$.

For skew-product semiflows, we always use the order relation on each fiber $P^{-1}(\omega)$. We write $(x_1, \omega) \leq_\omega [<_\omega] (x_2, \omega)$ if $x_1 \leq x_2$ $[x_1 < x_2]$. Without any confusion, we will drop the subscript "ω". One can also use similar definitions and notations in $P^{-1}(\omega)$ as in X, such as order intervals, the greatest lower bound, the least upper bound, etc.

Let A, B be two compact subsets of X. We define their Hausdorff distance by

$$d_H(A, B) = \max\left\{\sup_{x\in A} d(x, B), \sup_{y\in B} d(y, A)\right\},$$

where $d(x, B) = \inf_{y\in B} \|x - y\|$. Similarly, we can also define the Hausdorff distance $d_{H,\omega}(A(\omega), B(\omega))$ for any two compact subset $A(\omega), B(\omega)$ of $P^{-1}(\omega)$. Again without any confusion, we drop the subscript "ω" and write $d_H(A(\omega), B(\omega))$. For any compact subsets K_1, K_2 of $X \times \Omega$, their Hausdorff distance is defined as

$$d(K_1, K_2) = \sup_{\omega\in\Omega} d_H(K_1(\omega), K_2(\omega)).$$

Let K_1, K_2 be positively invariant compact subsets of $X \times \Omega$. We write $K_1 \preceq K_2$ (resp. $K_1 \prec K_2$) if and only if for any $(x, \omega) \in K_1$ there exists $(y, \omega) \in K_2$ such that $(x, \omega) \leq (y, \omega)$ (resp. $(x, \omega) < (y, \omega)$), and for any $(y, \omega) \in K_2$ there exists some $(x, \omega) \in K_1$ such that $(x, \omega) \leq (y, \omega)$ (resp. $(x, \omega) < (y, \omega)$). $K_1 \succeq K_2$ (resp. $K_1 \succ K_2$) is similarly defined.

It is worth pointing out that if either K_1 or K_2 is a 1-cover of the base flow, then the binary relation "$\preceq$" between K_1 and K_2 is antisymmetric, i.e., $K_1 \preceq K_2$ and $K_1 \succeq K_2$ imply $K_1 = K_2$. In fact, assume without loss of generality that $K_1 = \{(c(\omega), \omega) : \omega \in \Omega\}$ is a 1-cover of the base flow. Then $K_1 \preceq K_2$ (resp. $K_1 \succeq K_2$) entails that

$$(c(\omega), \omega) \leq (y, \omega) \quad \text{(resp. } (c(\omega), \omega) \geq (y, \omega))$$

for any $(y, \omega) \in K_2$. Thus $K_1 \preceq K_2$ and $K_1 \succeq K_2$ imply that $K_1 = K_2$. Therefore, in this case, the binary relation "$\preceq$" between K_1 and K_2 is antisymmetric.

Definition 2.4. The skew-product semiflow Π is *monotone* if

$$\Pi_t(x_1, \omega) \leq \Pi_t(x_2, \omega) \quad \text{whenever } (x_1, \omega) \leq (x_2, \omega) \text{ and } t \geq 0.$$

Let G be a metrizable topological group with unit element e. We say that G acts on the ordered space X if there exists a continuous mapping $\gamma : G \times X \to X$ such that $a \mapsto \gamma(a, \cdot)$ is a group homomorphism of G into $\mathrm{Hom}(X)$, the group of homeomorphisms of X onto itself. For brevity, we write $\gamma(a, x) = ax$ for $x \in X$ and identify $a \in G$ with its action $\gamma(a, \cdot)$. The group action γ is said to be *order-preserving* if, for each $a \in G$, the mapping $\gamma(a, \cdot) : X \to X$ is increasing, i.e. $x_1 \le x_2$ in X implies $ax_1 \le ax_2$. We say that γ *commutes* with the skew-product semiflow Π if

$$au(t, x, \omega) = u(t, ax, \omega) \quad \text{for any } (x, \omega) \in X \times \Omega, t \ge 0 \text{ and } a \in G.$$

For $x \in X$ the *group orbit* of x is the set $Gx = \{ax : a \in G\}$. A point $(x, \omega) \in X \times \Omega$ is said to be *symmetric* if $(Gx, \omega) = \{(x, \omega)\}$.

Since the G-action commutes with Π_t, one has the following direct lemma:

Lemma 2.5. *For any* $(x_0, \omega_0) \in X \times \Omega$ *and* $g \in G$,

$$g\mathcal{O}(x_0, \omega_0) = \mathcal{O}(gx_0, \omega_0),$$

where $g\mathcal{O}(x_0, \omega_0) = \{(gx, \omega) : (x, \omega) \in \mathcal{O}(x_0, \omega_0),\ \omega \in \Omega\}$.

Proof. Fix $\omega \in \Omega$. Then for any $(x, \omega) \in \mathcal{O}(x_0, \omega_0)$, there exists a sequence $t_n \to \infty$ such that $\Pi_{t_n}(x_0, \omega_0) = (u(t_n, x_0, \omega_0), \omega_0 \cdot t_n) \to (x, \omega)$ as $n \to \infty$. So for any $g \in G$, we have $u(t_n, gx_0, \omega_0) = gu(t_n, x_0, \omega_0) \to gx$ as $n \to \infty$, and hence $(gx, \omega) \in \mathcal{O}(gx_0, \omega_0) \cap P^{-1}(\omega)$. Therefore, $g\mathcal{O}(x_0, \omega_0) \subset \mathcal{O}(gx_0, \omega_0)$.

Conversely, for any $(y, \omega) \in \mathcal{O}(gx_0, \omega_0)$, choose a sequence $s_n \to \infty$ such that $\Pi_{s_n}(gx_0, \omega_0) = (u(s_n, gx_0, \omega_0), \omega_0 \cdot s_n) \to (y, \omega)$ as $n \to \infty$. Thus, $gu(s_n, x_0, \omega_0) = u(s_n, gx_0, \omega_0) \to y$ as $n \to \infty$. Without loss of generality, we may assume that $u(s_n, x_0, \omega_0) \to x$ as $n \to \infty$. Then $(x, \omega) \in \mathcal{O}(x_0, \omega_0)$ and $y = gx$, which implies that $(y, \omega) \in g\mathcal{O}(x_0, \omega_0)$. So we have proved $\mathcal{O}(gx_0, \omega_0) \subset g\mathcal{O}(x_0, \omega_0)$. By the arbitrariness of $\omega \in \Omega$, we directly derive the result. □

We finish this section with the definitions of almost periodic and almost automorphic functions and flows.

A function $f \in C(\mathbb{R}, \mathbb{R}^n)$ is *almost periodic* if, for any $\varepsilon > 0$, the set $T(\varepsilon) := \{\tau : |f(t+\tau) - f(t)| < \varepsilon \text{ for all } t \in \mathbb{R}\}$ is relatively dense in $\mathbb{R}$; and f is *almost automorphic* if for any $\{t_n'\} \subset \mathbb{R}$ there is a subsequence $\{t_n\}$ and a function $g : \mathbb{R} \to \mathbb{R}^n$ such that $f(t + t_n) \to g(t)$ and $g(t - t_n) \to f(t)$ hold pointwise.

Let D be a subset of $\mathbb{R}^m$. A continuous function $f : \mathbb{R} \times D \to \mathbb{R}^n$, $(t, u) \mapsto f(t, u)$, is said to be *admissible* if $f(t, u)$ is bounded and uniformly continuous on $\mathbb{R} \times K$ for any compact subset $K \subset D$. A function $f \in C(\mathbb{R} \times D, \mathbb{R}^n)(D \subset \mathbb{R}^m)$ is *uniformly almost periodic* [*automorphic*] *in t* if f is both admissible and almost periodic [automorphic] in $t \in \mathbb{R}$.

Let $f \in C(\mathbb{R} \times D, \mathbb{R}^n)$ $(D \subset \mathbb{R}^m)$ be admissible. Then $H(f) = \mathrm{cl}\{f \cdot \tau : \tau \in \mathbb{R}\}$ is called the *hull* of f, where $f \cdot \tau(t, \cdot) = f(t + \tau, \cdot)$ and the closure is taken in the compact-open topology. Moreover, $H(f)$ is compact and metrizable under the compact-open topology. The time translation $g \cdot t$ of $g \in H(f)$ induces a natural flow on $H(f)$.

Definition 2.6. An admissible function $f \in C(\mathbb{R} \times D, \mathbb{R}^n)$ is called *time-recurrent* if $H(f)$ is minimal.

$H(f)$ is always minimal if f is uniformly almost periodic automorphic in t. Moreover, $H(f)$ is an almost periodic automorphic minimal flow when f is a uniformly almost periodic automorphic function in t (see e.g. [25, 26]).

Let $f \in C(\mathbb{R} \times D, \mathbb{R}^n)$ be uniformly almost periodic [automorphic], and

$$f(t,x) \sim \sum_{\lambda \in \mathbb{R}} a_\lambda(x) e^{i\lambda t} \tag{2.2}$$

be the Fourier series of f (see [26, 31] for the definition and the existence of Fourier series). Then $\mathcal{S}(f) = \{\lambda : a_\lambda(x) \not\equiv 0\}$ is called the *Fourier spectrum* of f associated to the Fourier series (2.2).

Definition 2.7. $\mathcal{M}(f)$ is the smallest additive subgroup of $\mathbb{R}$ containing $\mathcal{S}(f)$; it is called the *frequency module of* f.

Let $f, g \in C(\mathbb{R} \times \mathbb{R}^m, \mathbb{R}^n)$ be two uniformly almost periodic [automorphic] functions in t. We have the module containment $\mathcal{M}(f) \subset \mathcal{M}(g)$ if and only if there exists a flow epimorphism from $H(g)$ to $H(f)$ (see [4] or [26, Section 1.3.4]). In particular, $\mathcal{M}(f) = \mathcal{M}(g)$ if and only if $(H(g), \mathbb{R})$ is isomorphic to the flow $(H(f), \mathbb{R})$.

3. Main results

In this section our standing assumptions are as follows:

(A1) Ω is minimal;
(A2) G is a connected group acting on X and the action is order-preserving;
(A3) G commutes with the monotone skew-product semiflow Π_t.

In what follows we will denote by K a minimal set of Π_t in $X \times \Omega$, which is a uniformly stable 1-cover of Ω. In this context, we also write $K = \{(\bar{u}_\omega, \omega) : \omega \in \Omega\}$, and $gK = \{(g\bar{u}_\omega, \omega) : \omega \in \Omega\}$ if an element $g \in G$ acts on K. The *group orbit* of K is defined as

$$GK = \{(g\bar{u}_\omega, \omega) \in X \times \Omega : g \in G \text{ and } \omega \in \Omega\}.$$

In this paper we will investigate the topological structure of GK.

For $\delta > 0$, we define the δ-neighborhood of K in $X \times \Omega$ by

$$B_\delta(K) = \{(u,\omega) \in X \times \Omega : \|u - \bar{u}_\omega\| < \delta\}.$$

Hereafter, we impose the following additional condition on K:

(A4) There exists a $\delta > 0$ such that

- (i) $O^+(x_0, \omega_0)$ is relatively compact for any $(x_0, \omega_0) \in B_\delta(K)$;
- (ii) if $\mathcal{O}(x_0, \omega_0) \subset B_\delta(K)$ and $\mathcal{O}(x_0, \omega_0) \prec hK$ (resp. $\mathcal{O}(x_0, \omega_0) \succ hK$) for some $h \in G$, then there is a neighborhood $B(e) \subset G$ of e such that $\mathcal{O}(x_0, \omega_0) \prec ghK$ (resp. $\mathcal{O}(x_0, \omega_0) \succ ghK$) for any $g \in B(e)$.

Remark 3.1. In the case where Π_t is strongly monotone, (A4)(ii) is automatically satisfied. Recall that Π_t is *strongly monotone* if $\Pi_t(x_1, \omega) \ll \Pi_t(x_2, \omega)$ whenever $(x_1, \omega) < (x_2, \omega)$ and $t > 0$. Here $\Pi_t(x_1, \omega) \ll \Pi_t(x_2, \omega)$ means that $u(t, x_2, \omega) - u(t, x_1, \omega) \in \operatorname{Int} V_+$, where $\operatorname{Int} V_+$ is the interior of a solid cone V_+ (i.e., $\operatorname{Int} V_+ \neq \emptyset$) (see e.g. [26]). To derive (ii) of (A4) under this assumption, note that the total invariance of $\mathcal{O}(x_0, \omega_0)$ implies that, for any $(x, \omega) \in \mathcal{O}(x_0, \omega_0)$, there exists a neighborhood $B_{(x,\omega)}(e) \subset G$ of e such that $(x, \omega) \prec ghK$ for any $g \in B_{(x,\omega)}(e)$. As $\mathcal{O}(x_0, \omega_0)$ is compact, one can find a neighborhood $B(e) \subset G$ such that $\mathcal{O}(x_0, \omega_0) \prec ghK$ for any $g \in B(e)$.

Remark 3.2. For continuous-time [discrete-time] monotone systems, assumption (A4) was first imposed by Ogiwara and Matano [18, 19] to investigate the monotonicity and convergence of the stable equilibria [fixed points]. We here give a general version in nonautonomous cases. At first glance, (A4) is just a local dynamical hypothesis near K. Accordingly, it should only yield a local total-ordering property of the group orbit GK near K (see Lemma A below). However, in what follows, we can see that it will surprisingly imply a globally topological characteristic of the whole GK (see Theorem B below), which is our main result in this paper.

Lemma A (Local ordering property of GK near K)**.** *Assume that* (A1)–(A3) *hold. Let K be a uniformly stable* 1*-cover of Ω satisfying* (A4)*. Then there exists a neighborhood $B(e) \subset G$ of e such that $gK \preceq K$ or $gK \succeq K$, for any $g \in B(e)$.*

Theorem B (Global topological structure of GK)**.** *Assume that* (A1)–(A3) *hold and G is locally compact. Let K be a uniformly stable* 1*-cover of Ω satisfying* (A4)*. Then one of the following alternatives holds:*

(i) *$GK = K$, i.e., K is G-symmetric.*
(ii) *There is a continuous bijective mapping $H : \Omega \times \mathbb{R} \to GK \subset X \times \Omega$ satisfying:*

  (a) *for each $\alpha \in \mathbb{R}$, $H(\Omega, \alpha) = gK$ for some $g \in G$;*
  (b) *for each $\omega \in \Omega$, $H(\omega, \mathbb{R}) = (G\bar{u}_\omega, \omega)$;*
  (c) *H is strictly order-preserving with respect to $\alpha \in \mathbb{R}$, i.e., $H(\omega, \alpha_1) < H(\omega, \alpha_2)$ for any $\omega \in \Omega$ and any $\alpha_1, \alpha_2 \in \mathbb{R}$ with $\alpha_1 < \alpha_2$.*

Remark 3.3. Roughly speaking, Theorem B implies the following *dichotomy*: either K is G-symmetric, or its group orbit GK is a 1-dimensional continuous subbundle over the base with each fiber being totally ordered and homeomorphic to $\mathbb{R}$.

Based on Theorem B, one can further deduce the following two useful theorems on symmetry of K, as well as its uniform stability with asymptotic phase.

Theorem C. *Assume that all the hypotheses in Theorem B are satisfied. If G is a compact group, then K is G-symmetric.*

Theorem D (Uniform stability of K with asymptotic phase)**.** *Assume that all the hypotheses in Theorem B are satisfied. If $GK \neq K$, then there is a $\delta_* \in (0, \delta)$ such that if $(u, \omega) \in B_{\delta_*}(K)$, then $\mathcal{O}(u, \omega) = hK$ for some $h \in G$. Moreover,*

$$\|u(t, u, \omega) - h\bar{u}_{\omega \cdot t}\| \to 0 \quad \text{as } t \to \infty.$$

4. Global topological structure of GK

In this section, we shall prove Theorems B and C under the assumption that the conclusion of Lemma A holds. The proof of Lemma A will be given in Section 6. We first prove the following useful proposition.

Proposition 4.1. *For any $g \in G$, there exists a neighborhood $V_g \subset G$ of g such that $V_g K$ is totally ordered, i.e.,*

$$g_1 K \preceq g_2 K \quad \text{or} \quad g_1 K \succeq g_2 K, \qquad \forall g_1, g_2 \in V_g.$$

Proof. Since G is metrizable, one can write $B(e)$ in Lemma A as $B(e) = \{g \in G : \rho(g, e) < \delta\}$ for some $\delta > 0$, where ρ denotes the right-invariant metric on G (cf. [15, Section 1.22]) satisfying $\rho(g\sigma, h\sigma) = \rho(g, h)$ for all $g, h, \sigma \in G$. Thus for any $g_1, g_2 \in G$, it follows from (A2) and Lemma A that

$$g_2 K \preceq g_1 K \quad \text{or} \quad g_2 K \succeq g_1 K \quad \text{whenever } \rho(g_1^{-1} g_2, e) < \delta. \tag{4.1}$$

Now for any $g \in G$, let $V_g = \{h \in G : \rho(g^{-1}, h^{-1}) < \delta/2\}$. It is not difficult to see that V_g is a neighborhood of g. Hence if $g_1, g_2 \in V_g$, then

$$\begin{aligned}\rho(g_1^{-1} g_2, e) &\le \rho(g_1^{-1} g_2, g^{-1} g_2) + \rho(g^{-1} g_2, e) = \rho(g_1^{-1} g_2, g^{-1} g_2) + \rho(g^{-1} g_2, g_2^{-1} g_2) \\ &= \rho(g_1^{-1}, g^{-1}) + \rho(g^{-1}, g_2^{-1}) < \delta,\end{aligned}$$

because ρ is right-invariant. As a consequence, (4.1) implies the conclusion. □

Now we are in a position to prove our main result, Theorem B:

Proof of Theorem B. For any $g_1, g_2 \in G$, we write $g_1 \le g_2$ whenever $g_1 K \preceq g_2 K$. This induces a partial order in G. A subset $S \subset G$ is called totally ordered if any two distinct elements of S are related.

We first claim that G is totally ordered. To prove this, we define

$$\mathcal{F} = \{S \subset G : S \text{ is connected and totally ordered}\}.$$

By Lemma A, $V_g \in \mathcal{F} \neq \emptyset$. Note that $(\mathcal{F}, \subset)$ is a partially ordered set. It follows from Zorn's lemma that $\mathcal{F}$ has a maximal element, say M. We first show that M is a closed subset of G. Indeed, the closure $\bar{M}$ of M is connected. Now, for any $h_1, h_2 \in \bar{M}$, there exist sequences $\{g_n^1\}, \{g_n^2\} \subset M$ such that $g_n^1 \to h_1$ and $g_n^2 \to h_2$ as $n \to \infty$. For each $n \in \mathbb{N}$, $g_n^1 \le g_n^2$ or $g_n^1 \ge g_n^2$, because M is totally ordered. By taking a subsequence $\{n_k\}$ if necessary, we obtain

$$g_{n_k}^1 \le g_{n_k}^2, \ \forall k \in \mathbb{N} \quad \text{or} \quad g_{n_k}^1 \ge g_{n_k}^2, \ \forall k \in \mathbb{N}.$$

Letting $k \to \infty$, one has $h_1 \le h_2$ or $h_1 \ge h_2$, because the order "$\le$" is closed. Hence $\bar{M}$ is totally ordered. By maximality, we get $M = \bar{M}$, which means that M is closed.

To show that M is also open in G, we notice that for any $g \in M$, by Proposition 4.1, there is a neighborhood $V_g \subset G$ of g such that V_g is totally ordered and connected.

Suppose that M is not open. Then one can find $g \in M$ and $\{g_n\}_{n=1}^{\infty} \subset V_g \setminus M$ such that $g_n \to g$ as $n \to \infty$. Since V_g is totally ordered, we may assume that $g_n > g$ for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, we define

$$W_n^+ = \{h \in M \cap V_g : h \geq g_n\} \quad \text{and} \quad W_n^- = \{h \in M \cap V_g : h \leq g_n\}.$$

A direct examination yields (i) $M \cap V_g = W_n^+ \cup W_n^-$; (ii) $W_n^+ \cap W_n^- = \emptyset$ (since $g_n \notin M$); (iii) $W_n^- \neq \emptyset$ (since $g \in W_n^-$); and (iv) W_n^+, W_n^- are closed in $M \cap V_g$. By the connectedness of $M \cap V_g$, we have $W_n^+ = \emptyset$, and hence $W_n^- = M \cap V_g$. Since $g_n \notin M$, it entails that $M \cap V_g < g_n$ for each $n \in \mathbb{N}$. Letting $n \to \infty$, we obtain

$$M \cap V_g \leq g. \tag{4.2}$$

We assert that $M \leq g$. Otherwise, as $g \in M$ and M is totally ordered, there is an $f \in M$ such that $f > g$. Since M is also connected and locally compact, it follows from [18, Appendix, Proposition Y1, p. 434] that there is an order-preserving homeomorphism

$$\tilde{h} : [g, f]_M = \{h \in M : g \leq h \leq f\} \to [0, 1]$$

with $\tilde{h}(g) = 0$ and $\tilde{h}(f) = 1$. Thus by choosing $g_* \in \tilde{h}^{-1}(\delta)$ with $\delta > 0$ sufficiently small, one finds that $g_* \in (V_g \cap M) \setminus \{g\}$ and $g_* > g$, contrary to (4.2).

On the other hand, recall that $g_n \in V_g$ and $g_n > g$ for every $n \in \mathbb{N}$. Now we fix n. Since V_g is connected, totally ordered, and locally compact, [18, Appendix, Proposition Y1, p. 434] again implies that there is an order-preserving homeomorphism

$$\hat{h} : [g, g_n]_{V_g} = \{h \in V_g : g \leq h \leq g_n\} \to [0, 1]$$

with $\hat{h}(g) = 0$ and $\hat{h}(g_n) = 1$. Let $\hat{M} = M \cup [g, g_n]_{V_g}$. Then $\hat{M} \supsetneq M$. Due to the above assertion, $\hat{M}$ is connected and totally ordered. This contradicts the maximality of M. Accordingly, M is an open subset of G.

Since M is both open and closed in G, it follows from the connectedness of G that $G = M$. Thus we have proved that G is totally ordered.

Based on this claim, precisely one of the following three alternatives must occur:

(Alt_a) The least upper bound (l.u.b.) of G exists.
(Alt_b) The greatest lower bound (g.l.b.) of G exists.
(Alt_c) Neither l.u.b. nor g.l.b. of G exists.

If (Alt_a) holds, then one can find a $g_0 \in G$ such that

$$g\bar{u}_\omega \leq g_0\bar{u}_\omega \quad \text{for any } \omega \in \Omega \text{ and } g \in G.$$

In particular, $g_0^2\bar{u}_\omega \leq g_0\bar{u}_\omega$, and hence $g_0\bar{u}_\omega = g_0^{-1}(g_0^2\bar{u}_\omega) \leq g_0^{-1}(g_0\bar{u}_\omega) = \bar{u}_\omega \leq g_0\bar{u}_\omega$, which entails that $g_0\bar{u}_\omega = \bar{u}_\omega$ for any $\omega \in \Omega$. Consequently, $g^{-1}\bar{u}_\omega \leq \bar{u}_\omega$, and hence $\bar{u}_\omega = g(g^{-1}\bar{u}_\omega) \leq g\bar{u}_\omega \leq \bar{u}_\omega$, for any $g \in G$ and $\omega \in \Omega$. This implies that $GK = K$.

Similarly, one can obtain $GK = K$ provided that (Alt_b) is satisfied. Thus we have deduced statement (i) of Theorem B.

Finally, assume that (Alt$_c$) holds. Fix any $\omega \in \Omega$. Then $G\bar{u}_\omega$ is a connected, locally compact and totally ordered set in X. Moreover, $G\bar{u}_\omega$ has neither the l.u.b. nor the g.l.b. in X. It then follows from [18, Appendix, Proposition Y2, p. 434] that $G\bar{u}_\omega$ coincides with the image of a strictly order-preserving continuous path in X,

$$J_\omega : \mathbb{R} \to G\bar{u}_\omega \subset X. \tag{4.3}$$

Motivated by [2, Section 3], we choose an $\omega_0 \in \Omega$ and define the mapping

$$H : \Omega \times \mathbb{R} \to GK, \qquad (\omega, \alpha) \mapsto \mathcal{O}(J_{\omega_0}(\alpha), \omega_0) \cap P^{-1}(\omega), \tag{4.4}$$

where J_{ω_0} comes from (4.3) with ω replaced by ω_0. Then it is not hard to check (a)–(c) for H in statement (ii) of Theorem B. We only need to show that H is a bijective continuous map.

To see this, we first note that H is surjective. Indeed, for any $(g\bar{u}_\omega, \omega) \in GK$, let $\hat{\alpha} \in \mathbb{R}$ be such that $J_{\omega_0}(\hat{\alpha}) = g\bar{u}_{\omega_0}$. Then $\mathcal{O}(J_{\omega_0}(\hat{\alpha}), \omega_0) \cap P^{-1}(\omega) = (g\bar{u}_\omega, \omega)$, because gK is a uniformly stable 1-cover of Ω. Consequently, $H(\omega, \hat{\alpha}) = (g\bar{u}_\omega, \omega)$, which implies that H is surjective.

Next we choose any $(\omega_i, \alpha_i) \in \Omega \times \mathbb{R}$, $i = 1, 2$, with $H(\omega_1, \alpha_1) = H(\omega_2, \alpha_2)$. For each α_i, there is a $g_i \in G$ such that $J_{\omega_0}(\alpha_i) = g_i\bar{u}_{\omega_0}$ for $i = 1, 2$. Again by the 1-cover property of g_iK,

$$(g_1\bar{u}_{\omega_1}, \omega_1) = H(\omega_1, \alpha_1) = H(\omega_2, \alpha_2) = (g_2\bar{u}_{\omega_2}, \omega_2).$$

Combining this with (4.3), we find that $\omega_1 = \omega_2$ and $g_1 = g_2$, which implies that $\alpha_1 = \alpha_2$. Thus H is injective.

To prove H is continuous, we choose any sequence $\{(\omega_k, \alpha_k)\}_{k=1}^{\infty} \subset \Omega \times \mathbb{R}$ with $(\omega_k, \alpha_k) \to (\omega_\infty, \alpha_\infty)$ as $k \to \infty$. Then, for each $k = 1, 2, \ldots, \infty$, we can find $g_k \in G$ such that $J_{\omega_0}(\alpha_k) = g_k\bar{u}_{\omega_0}$. As above, one can obtain

$$H(\omega_k, \alpha_k) = (g_k\bar{u}_{\omega_k}, \omega_k) \tag{4.5}$$

for $k = 1, 2, \ldots, \infty$. Since $\alpha_k \to \alpha_\infty$, we have $g_k\bar{u}_{\omega_0} \to g_\infty\bar{u}_{\omega_0}$ as $k \to \infty$. Note also that $g_\infty K$ is uniformly stable. Thus for any $\varepsilon > 0$, there exists an integer $N = N(\varepsilon) > 0$ such that $\|u(t, g_k\bar{u}_{\omega_0}, \omega_0) - u(t, g_\infty\bar{u}_{\omega_0}, \omega_0)\| \le \varepsilon/3$ for all $k \ge N$ and $t \ge 0$. Letting $t \to \infty$ implies that if $k \ge N$ then

$$\|g_k\bar{u}_\omega - g_\infty\bar{u}_\omega\| \le \varepsilon/3 \tag{4.6}$$

uniformly for all $\omega \in \Omega$. Moreover, for such ε and N (choose N larger if necessary), it is easy to see that

$$d_\Omega(\omega_k, \omega_\infty) < \varepsilon/3 \quad \text{and} \quad \|g_\infty\bar{u}_{\omega_k} - g_\infty\bar{u}_{\omega_\infty}\| < \varepsilon/3, \tag{4.7}$$

for all $k \ge N$. Let $\mathbf{d}$ be the metric on $X \times \Omega$ naturally induced by

$$\mathbf{d}((u_1, \omega_1), (u_2, \omega_2)) := \|u_1 - u_2\| + d_\Omega(\omega_1, \omega_2)$$

for any $(u_1, \omega_1), (u_2, \omega_2) \in X \times \Omega$. Then by (4.5)–(4.7), one has

$$\begin{aligned}\mathbf{d}(H(\omega_k, \alpha_k), H(\omega_\infty, \alpha_\infty)) &= \mathbf{d}((g_k \bar{u}_{\omega_k}, \omega_k), (g_\infty \bar{u}_{\omega_\infty}, \omega_\infty)) \\ &\leq d_\Omega(\omega_k, \omega_\infty) + \|g_k \bar{u}_{\omega_k} - g_\infty \bar{u}_{\omega_k}\| + \|g_\infty \bar{u}_{\omega_k} - g_\infty \bar{u}_{\omega_\infty}\| \\ &< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon\end{aligned}$$

for all $k \geq N$. We have proved that H is continuous. □

Proof of Theorem C. Since G is compact, both (Alt$_a$) and (Alt$_b$) are satisfied. Then we directly deduce that $GK = K$ from the proof above. □

5. Uniform stability of K with asymptotic phase

In this section, we will prove the asymptotic phase of the uniformly stable minimal set K, i.e., Theorem D in Section 3. We first present the following useful lemma:

Lemma 5.1. *Assume all the hypotheses in Theorem B are satisfied. Assume also that $GK \neq K$. Then there exists a $\delta_0 > 0$ such that if $(u, \omega) \in B_{\delta_0}(K)$ satisfies $\mathcal{O}(u, \omega) \preceq g_1 K$ for some $g_1 \in G$, then $\mathcal{O}(u, \omega) = g_2 K$ for some $g_2 \in G$. The same conclusion also holds if $(u, \omega) \in B_{\delta_0}(K)$ satisfies $\mathcal{O}(u, \omega) \succeq g_1 K$.*

Proof. We only prove the first statement. Suppose that there exists a sequence $\{(u_m, \omega_m)\}_{m=1}^\infty \subset X \times \Omega$ such that, for each $m \geq 1$,

(i) $(u_m, \omega_m) \in B_{1/m}(K)$;
(ii) $\mathcal{O}(u_m, \omega_m) \preceq g_m^1 K$ for some $g_m^1 \in G$; and
(iii) $\mathcal{O}(u_m, \omega_m) \neq gK$ for any $g \in G$.

By Lemma 2.3, (iii) implies that

$$gK \nsubseteq \mathcal{O}(u_m, \omega_m) \quad \text{for all } m \geq 1 \text{ and } g \in G. \tag{5.1}$$

Now we claim that

$$d(\mathcal{O}(u_m, \omega_m), K) \to 0 \quad \text{as } m \to \infty. \tag{5.2}$$

In fact, since K is uniformly stable, for any $\varepsilon > 0$ there exists a $\tilde{\delta}(\varepsilon) > 0$ such that, for any $\omega \in \Omega$, if $\|y - \bar{u}_\omega\| < \tilde{\delta}(\varepsilon)$ then $\|u(t, y, \omega) - u(t, \bar{u}_\omega, \omega)\| < \varepsilon$ for all $t \geq 0$. Therefore, for $(u_m, \omega_m) \in B_{1/m}(K)$ with m sufficiently large, one has $\|u_m - \bar{u}_{\omega_m}\| < 1/m < \tilde{\delta}(\varepsilon)$, and hence $\|u(t, u_m, \omega_m) - u(t, \bar{u}_{\omega_m}, \omega_m)\| < \varepsilon$ for all $t \geq 0$. By the minimality of Ω, it then follows that $\|z - \bar{u}_\omega\| \leq \varepsilon$ for any $(z, \omega) \in \mathcal{O}(u_m, \omega_m)$, proving the claim.

Now fix $m \in \mathbb{N}$. We define $A_m = \{g \in G : \mathcal{O}(u_m, \omega_m) \preceq gK\}$. Clearly, A_m is nonempty (because $g_m^1 \in A_m$ by (ii)) and closed in G. By (5.1) and (5.2), one obtains $A_m = \{g \in G : \mathcal{O}(u_m, \omega_m) \prec gK\}$, and moreover $\mathcal{O}(u_m, \omega_m) \subset B_\delta(K)$ for m sufficiently large. Here the δ comes from condition (A4) in Section 3.

As a consequence, (A4) entails that A_m is also open for all m sufficiently large. Since G is connected, $A_m = G$ for all m sufficiently large. This then implies that

$$\mathcal{O}(u_m, \omega_m) \preceq gK, \quad \forall g \in G,$$

for all m sufficiently large. By letting $m \to \infty$ in the above inequality, (5.2) yields $K \preceq gK$ for all $g \in G$. Replacing g with g^{-1} and applying g on both sides, we get $gK \preceq K$. Hence $gK = K$ for all $g \in G$, a contradiction. □

Proof of Theorem D. Let $\delta_0 > 0$ be as in Lemma 5.1. We take a $\delta_* \in (0, \min\{\delta, \delta_0\})$ such that $(u \wedge \bar{u}_\omega, \omega) \in B_{\delta_0}(K)$ whenever $(u, \omega) \in B_{\delta_*}(K)$. Since $u \wedge \bar{u}_\omega \le \bar{u}_\omega$, one has $\mathcal{O}(u \wedge \bar{u}_\omega, \omega) \preceq K$. It then follows from Lemma 5.1 that $\mathcal{O}(u \wedge \bar{u}_\omega, \omega) = g_* K$ for some $g_* \in G$. Note also that $u \wedge \bar{u}_\omega \le u$. Then $g_* K \preceq \mathcal{O}(u, \omega)$. Applying Lemma 5.1 again, we conclude that $\mathcal{O}(u, \omega) = gK$ for some $g \in G$. □

6. Proof of Lemma A

First we shall show that *there exists a neighborhood $B(e) \subset G$ of e such that for any $g \in B(e)$, one has $g\bar{u}_{\omega_0} \le \bar{u}_{\omega_0}$ or $g\bar{u}_{\omega_0} \ge \bar{u}_{\omega_0}$ for some $\omega_0 \in \Omega$.* Otherwise, one can find a sequence $\{g_n\}_{n=0}^{\infty} \subset G$ with $g_n \to e$ as $n \to \infty$ such that

$$g_n \bar{u}_\omega \nleq \bar{u}_\omega \quad \text{and} \quad g_n \bar{u}_\omega \ngeq \bar{u}_\omega, \tag{6.1}$$

for all $n \ge 0$ and $\omega \in \Omega$.

To deduce a contradiction, we fix an $\omega_0 \in \Omega$, and due to (A4)(i), we define $K_n = \mathcal{O}(g_n \bar{u}_{\omega_0} \wedge \bar{u}_{\omega_0}, \omega_0)$ for all n sufficiently large. Without loss of generality, one may also assume that K_n is defined for all $n \in \mathbb{N}$. Clearly, $K = \mathcal{O}(\bar{u}_{\omega_0}, \omega_0)$. Then one can obtain the following three facts, to be proved at the end of this section (see Propositions 6.1–6.3):

(F1) $K_n \prec K$ and $K_n \prec g_n K$ for all $n \in \mathbb{N}$.
(F2) $d(K_n, K) \to 0$ as $n \to \infty$.
(F3) Given $\delta > 0$ as in (A4), there exists a neighborhood $\hat{B}(e) \subset G$ of e and $N_0 \in \mathbb{N}$ such that

$$d(gK_n, K) \le \delta \quad \text{and} \quad d(g_n^{-1} g K_n, K) \le \delta,$$

for all $g \in \hat{B}(e)$ and $n \ge N_0$.

For $\hat{B}(e)$ and $N_0 \in \mathbb{N}$ as in (F3), we take a neighborhood $B(e) \subset G$ of e with $B(e) \subset \overline{B(e)} \subset \hat{B}(e)$, and define

$$A_n = \{g \in \overline{B(e)} : gK_n \preceq K \text{ and } g_n^{-1} g K_n \preceq K\}$$

for $n \ge N_0$. By (F1), $e \in A_n \ne \emptyset$. Moreover, A_n is closed in $\overline{B(e)}$. We assert that

$$A_n = \{g \in \overline{B(e)} : gK_n \prec K \text{ and } g_n^{-1} g K_n \prec K\}. \tag{6.2}$$

Indeed, for $g \in A_n$, suppose that there exists $(y, \tilde{\omega}) \in K_n$ such that $gy = \bar{u}_{\tilde{\omega}}$. Then from $g_n^{-1} g K_n \preceq K$ we have $g_n^{-1} g y \le \bar{u}_{\tilde{\omega}}$. This entails that $g_n^{-1} \bar{u}_{\tilde{\omega}} \le \bar{u}_{\tilde{\omega}}$, and hence $\bar{u}_{\tilde{\omega}} \le g_n \bar{u}_{\tilde{\omega}}$, contradicting (6.1). Similarly, for $g \in A_n$, suppose that there exists $(z, \hat{\omega}) \in K_n$ such that $g_n^{-1} g z = \bar{u}_{\hat{\omega}}$. Then from $gK_n \preceq K$ we have $gz \le \bar{u}_{\hat{\omega}}$, which yields $g_n^{-1} \bar{u}_{\hat{\omega}} \ge g_n^{-1} g z = \bar{u}_{\hat{\omega}}$, and hence $\bar{u}_{\hat{\omega}} \ge g_n \bar{u}_{\hat{\omega}}$, contradicting (6.1) again.

Now fix $n \geq N_0$ and $g \in A_n$, and write $v^0_{g,n} := g(g_n\bar{u}_{\omega_0} \wedge \bar{u}_{\omega_0})$ and $w^0_{g,n} := g_n^{-1}g(g_n\bar{u}_{\omega_0} \wedge \bar{u}_{\omega_0})$. Then by (F3) and Lemma 2.5,

$$d(\mathcal{O}(v^0_{g,n}, \omega_0), K) \leq \delta \quad \text{with} \quad \mathcal{O}(v^0_{g,n}, \omega_0) = gK_n \prec K,$$
$$d(\mathcal{O}(w^0_{g,n}, \omega_0), K) \leq \delta \quad \text{with} \quad \mathcal{O}(w^0_{g,n}, \omega_0) = g_n^{-1}gK_n \prec K.$$

Accordingly, (A4) implies that there exist neighborhoods $B_1(e), B_2(e) \subset G$ of e such that $gK_n = \mathcal{O}(v^0_{g,n}, \omega_0) \prec h_1K$ and $g_n^{-1}gK_n = \mathcal{O}(w^0_{g,n}, \omega_0) \prec h_2K$, for any $h_1 \in B_1(e)$ and $h_2 \in B_2(e)$. As a consequence,

$$(B_1(e))^{-1}gK_n \prec K \quad \text{and} \quad (B_2(e))^{-1}g_n^{-1}gK_n \prec K, \tag{6.3}$$

where $(B_i(e))^{-1} = \{h^{-1} \in G : h \in B_i(e)\}$ for $i = 1, 2$. Clearly, $(B_1(e))^{-1}g$ and $(B_2(e))^{-1}g_n^{-1}g$ are neighborhoods of g and $g_n^{-1}g$, respectively. Moreover, by the continuity of $g \mapsto g_n^{-1}g$, one can find a neighborhood V_g of g in G such that $g_n^{-1}V_g \subset (B_2(e))^{-1}g_n^{-1}g$. Thus by (6.3) we have $g_n^{-1}V_gK_n \prec K$. Now let $W_g := \overline{B(e)} \cap V_g \cap (B_1(e))^{-1}g$. Then by (6.3) again, W_g is a neighborhood of g in $\overline{B(e)}$ satisfying

$$W_gK_n \prec K \quad \text{and} \quad g_n^{-1}W_gK_n \prec K.$$

Therefore, $W_g \subset A_n$, which implies that A_n is also open in $\overline{B(e)}$. Thus by the connectedness of G (and hence of $\overline{B(e)}$), one has

$$A_n = \overline{B(e)}, \qquad \forall n \geq N_0.$$

Consequently,

$$\overline{B(e)}K_n \preceq K \quad \text{and} \quad g_n^{-1}\overline{B(e)}K_n \preceq K,$$

for all $n \geq N_0$. Letting $n \to \infty$, by (F2) we obtain

$$\overline{B(e)}K \preceq K. \tag{6.4}$$

Since $g_n \to e$ as $n \to \infty$, (6.4) implies that $g_n\bar{u}_\omega \leq \bar{u}_\omega$ for all $\omega \in \Omega$ and n sufficiently large, which contradicts (6.1).

Thus, we have proved that there exists a neighborhood $B(e) \subset G$ of e such that for any $g \in B(e)$, one has $g\bar{u}_{\omega_0} \leq \bar{u}_{\omega_0}$ or $g\bar{u}_{\omega_0} \geq \bar{u}_{\omega_0}$ for some $\omega_0 \in \Omega$.

Without loss of generality, we assume that $g\bar{u}_{\omega_0} \leq \bar{u}_{\omega_0}$. Then the monotonicity of Π_t implies $g\bar{u}_{\omega_0 \cdot t} \leq \bar{u}_{\omega_0 \cdot t}$ for any $t \geq 0$. Now for any $\omega \in \Omega$, we choose a sequence $t_n \to \infty$ such that $\omega_0 \cdot t_n \to \omega$ as $n \to \infty$. By the 1-cover property of K, one has $\bar{u}_{\omega_0 \cdot t_n} \to \bar{u}_\omega$ as $n \to \infty$. Thus, letting $n \to \infty$, we obtain $g\bar{u}_\omega \leq \bar{u}_\omega$ for any $\omega \in \Omega$. This implies that $gK \preceq K$ for any $g \in B(e)$. Similarly, one can obtain $K \preceq gK$ for any $g \in B(e)$ provided that $\bar{u}_{\omega_0} \leq g\bar{u}_{\omega_0}$. We conclude that for $K = \{(\bar{u}_\omega, \omega) : \omega \in \Omega\}$,

$$gK \preceq K \quad \text{or} \quad gK \succeq K, \qquad \forall g \in B(e).$$

This is the exact statement of Lemma A. □

Finally, it is only left to check (F1)–(F3) above. This will be done in the following three propositions.

Proposition 6.1. (F1) *holds, i.e., $K_n \prec K$ and $K_n \prec g_n K$ for all $n \in \mathbb{N}$.*

Proof. Note that $g_n\bar{u}_{\omega_0} \wedge \bar{u}_{\omega_0} < \bar{u}_{\omega_0}$ (and $< g_n\bar{u}_{\omega_0}$). The monotonicity of Π_t yields

$$\Pi_t(g_n\bar{u}_{\omega_0} \wedge \bar{u}_{\omega_0}, \omega_0) \le \Pi_t(\bar{u}_{\omega_0}, \omega_0) \text{ (and } < \Pi_t(g_n\bar{u}_{\omega_0}, \omega_0)) \tag{6.5}$$

for all $t \ge 0$. So, for any $(x, \omega) \in K_n$, one can find a sequence $t_k \to \infty$ ($k \to \infty$) such that $\Pi_{t_k}(g_n\bar{u}_{\omega_0} \wedge \bar{u}_{\omega_0}, \omega_0) \to (x, \omega)$ as $k \to \infty$. Since K is a 1-cover, one has $\Pi_{t_k}(\bar{u}_{\omega_0}, \omega_0) \to (\bar{u}_\omega, \omega)$. Then (6.5) implies that $(x, \omega) \le (\bar{u}_\omega, \omega)$. As a consequence, $K_n \preceq K$. Similarly, $K_n \preceq g_n K$ for every $n \in \mathbb{N}$.

Now we claim that $K_n \prec K$ (and $\prec g_n K$) for all $n \in \mathbb{N}$. Otherwise, there exist some $N \in \mathbb{N}$ and $(x, \tilde{\omega}) \in K_N$ such that

$$(x, \tilde{\omega}) = (\bar{u}_{\tilde{\omega}}, \tilde{\omega}) \text{ (or } = (g_N\bar{u}_{\tilde{\omega}}, \tilde{\omega})). \tag{6.6}$$

Choose a sequence $s_k \to \infty$ ($k \to \infty$) such that $\Pi_{s_k}(g_N\bar{u}_{\omega_0} \wedge \bar{u}_{\omega_0}, \omega_0) \to (x, \tilde{\omega})$ as $k \to \infty$. Since

$$\begin{aligned}\Pi_t(g_n\bar{u}_\omega \wedge \bar{u}_\omega, \omega) &\le \Pi_t(g_n\bar{u}_\omega, \omega) \wedge \Pi_t(\bar{u}_\omega, \omega)\\ &= (g_n\bar{u}_{\omega\cdot t}, \omega\cdot t) \wedge (\bar{u}_{\omega\cdot t}, \omega\cdot t) = (g_n\bar{u}_{\omega\cdot t} \wedge \bar{u}_{\omega\cdot t}, \omega\cdot t)\end{aligned}$$

for all $\omega \in \Omega$, $t \ge 0$ and $n \in \mathbb{N}$, it follows that

$$\Pi_{s_k}(g_N\bar{u}_{\omega_0} \wedge \bar{u}_{\omega_0}, \omega_0) \le (g_N\bar{u}_{\omega_0\cdot s_k} \wedge \bar{u}_{\omega_0\cdot s_k}, \omega_0\cdot s_k).$$

Letting $k \to \infty$, by the continuity of $\bar{u}_\omega$ with respect to $\omega \in \Omega$, we get

$$(x, \tilde{\omega}) \le (g_N\bar{u}_{\tilde{\omega}} \wedge \bar{u}_{\tilde{\omega}}, \tilde{\omega}) < (\bar{u}_{\tilde{\omega}}, \tilde{\omega}) \text{ (and } < (g_N\bar{u}_{\tilde{\omega}}, \tilde{\omega})),$$

where the last inequality is from (6.1). This contradicts (6.6). Thus we have proved that $K_n \prec K$ (and $\prec g_n K$) for all $n \in \mathbb{N}$. □

Proposition 6.2. (F2) *holds, i.e., $d(K_n, K) \to 0$ as $n \to \infty$.*

Proof. Note that $g_n\bar{u}_{\omega_0} \wedge \bar{u}_{\omega_0} \to \bar{u}_{\omega_0}$ as $n \to \infty$. Since K is a uniformly stable 1-cover of Ω, for any $\varepsilon > 0$ there is an $N_1 \in \mathbb{N}$ such that

$$\|u(t, g_n\bar{u}_{\omega_0} \wedge \bar{u}_{\omega_0}, \omega_0) - \bar{u}_{\omega_0\cdot t}\| < \varepsilon \tag{6.7}$$

for all $n \ge N_1$ and $t \ge 0$. Choose any $(x, \omega) \in K_n$. There exists a sequence $t_k \to \infty$ ($k \to \infty$) such that $\Pi_{t_k}(g_n\bar{u}_{\omega_0} \wedge \bar{u}_{\omega_0}, \omega_0) \to (x, \omega)$ as $k \to \infty$. By taking a subsequence if necessary, we get $\Pi_{t_k}(\bar{u}_{\omega_0}, \omega_0) \to (\bar{u}_\omega, \omega)$ as $k \to \infty$. Hence by (6.7), $\|x - \bar{u}_\omega\| \le \varepsilon$ for all $(x, \omega) \in K_n$ and $n \ge N_1$. Recall that $d(K_n, K) = \sup_{(x,\omega)\in K_n} \|x - \bar{u}_\omega\|$. Consequently, $d(K_n, K) \le \varepsilon$ for all $n \ge N_1$, which implies $d(K_n, K) \to 0$ as $n \to \infty$. □

Proposition 6.3. (F3) *holds, i.e., for $\delta > 0$ as in* (A4)*, there exists a neighborhood $\hat{B}(e) \subset G$ of e and $N_0 \in \mathbb{N}$ such that*

$$d(gK_n, K) \le \delta \quad \textit{and} \quad d(g_n^{-1}gK_n, K) \le \delta,$$

for all $g \in \hat{B}(e)$ and $n \ge N_0$.

Proof. Firstly, suppose that there exist a sequence $\{\tilde{g}_n\}_{n=0}^{\infty} \subset G$ with $\tilde{g}_n \to e$ and a subsequence of $\{K_n\}_{n=0}^{\infty}$, still denoted by $\{K_n\}_{n=0}^{\infty}$, such that

$$d(\tilde{g}_n K_n, K) = \sup_{(y,\omega)\in K_n} \|\tilde{g}_n y - \bar{u}_\omega\| > \delta$$

for all $n \in \mathbb{N}$. Then one can choose $(y_n, \omega_n) \in K_n$ such that

$$\|\tilde{g}_n y_n - \bar{u}_{\omega_n}\| > \delta. \tag{6.8}$$

We may assume that $\omega_n \to \omega$ in Ω as $n \to \infty$. Now we claim that $y_n \to \bar{u}_\omega$ as $n \to \infty$. Indeed, Proposition 6.2 shows that, for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\|z - \bar{u}_\omega\| < \varepsilon \quad \text{for all } (z, \omega) \in K_n \text{ and } n > N.$$

So $\|y_n - \bar{u}_{\omega_n}\| < \varepsilon$ for all $n > N$, because $(y_n, \omega_n) \in K_n$. By the continuity of $\bar{u}_\omega$ in $\omega \in \Omega$, one has

$$\|y_n - \bar{u}_\omega\| \le \|y_n - \bar{u}_{\omega_n}\| + \|\bar{u}_{\omega_n} - \bar{u}_\omega\| < \varepsilon + \varepsilon = 2\varepsilon, \quad \forall n > \bar{N},$$

for some $\bar{N} > N$. Thus, we have proved the claim.

Now by letting $n \to \infty$ in (6.8), we obtain $\|\bar{u}_\omega - \bar{u}_\omega\| = \|e\bar{u}_\omega - \bar{u}_\omega\| \ge \delta$, a contradiction. Hence one can find a neighborhood $B_1(e)$ of e and $N_1 \in \mathbb{N}$ such that $d(gK_n, K) \le \delta$ for all $g \in B_1(e)$ and $n \ge N_1$.

Secondly, suppose that there exist a sequence $\{h_n\}_{n=0}^{\infty} \subset G$ with $h_n \to e$ and a subsequence $\{K_{j_n}\}_{n=0}^{\infty}$ of $\{K_n\}_{n=0}^{\infty}$ such that

$$d(g_{j_n}^{-1} h_n K_{j_n}, K) = \sup_{(y,\omega)\in K_{j_n}} \|g_{j_n}^{-1} h_n y - \bar{u}_\omega\| > \delta \quad \text{for all } n \in \mathbb{N}.$$

Then there exists $(y_{j_n}, \omega_{j_n}) \in K_{j_n}$ such that $\|g_{j_n}^{-1} h_n y_{j_n} - \bar{u}_{\omega_{j_n}}\| > \delta$. Noticing $g_{j_n}^{-1} \to e$, one can repeat the argument above to deduce a contradiction. Thus, again one can find a neighborhood $B_2(e)$ of e and $N_2 \in \mathbb{N}$ such that $d(g_n^{-1} g K_n, K) \le \delta$ for all $g \in B_2(e)$ and $n \ge N_2$.

Finally, letting $\hat{B}(e) = B_1(e) \cap B_2(e)$ and $N_0 = \max\{N_1, N_2\}$ completes the proof of (F3). □

7. Applications to parabolic equations

In this section we give some examples of second order parabolic equations in time-recurrent structures which generate monotone skew-product semiflows satisfying (A1)–(A4).

7.1. Rotational symmetry

Assume that $\Omega \subset \mathbb{R}^n$ is a (possibly unbounded) rotationally symmetric domain with smooth boundary $\partial\Omega$. Let G be a connected closed subgroup of the rotation group $SO(n)$. The set Ω is called *G-symmetric* if it is G-invariant in the sense that $gx \in \Omega$ whenever $x \in \Omega$ and $g \in G$. A typical example of such a bounded domain is a ball, a spherical shell, a solid torus or any other body of rotation, while typical unbounded domains include a cylindrical domain or $\mathbb{R}^n$ itself. In [32], asymptotic symmetry has been investigated for bounded domains. In this section, we focus on unbounded domains and, for brevity, we will present the following example on $\mathbb{R}^n$. As a matter of fact, general unbounded G-symmetric domains can be dealt with as well.

Consider the following initial value problem on $\mathbb{R}^n$:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u + f(t,x,u), & x \in \mathbb{R}^n,\ t > 0, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^n. \end{cases} \tag{7.1}$$

Here the nonlinearity $f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is assumed to be C^1 and be almost periodic in t uniformly for $x \in \mathbb{R}^n$ and u in bounded sets of $\mathbb{R}$.

In what follows we assume that

(f 1) $f(t, gx, u) = f(t, x, u)$ for all $x \in \mathbb{R}^n,\ u \in \mathbb{R},\ g \in G$ and $t \in \mathbb{R}$;
(f 2) $f(t, x, 0) = 0$ for all $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$;
(f 3) there exist $\epsilon_0, R_0, \alpha > 0$ such that $\frac{\partial f}{\partial u}(t,x,u) \le -\alpha$ for all $|x| \ge R_0$, $|u| \le \epsilon_0$ and $t \in \mathbb{R}$.

Let X be defined to be

$$C_{\text{unif}}(\mathbb{R}^n) = \{u(\cdot) : u \ \text{ is bounded and uniformly continuous on } \mathbb{R}^n\}$$

with the L^∞-topology. Let $Y = H(f)$ be the hull of the nonlinearity f. Then, for any $g \in Y$, the function g is uniformly almost periodic in t and satisfies all the above assumptions (f 1)–(f 3). As a consequence, (7.1) gives rise to a family of equations associated to each $g \in Y$:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u + g(t,x,u), & x \in \mathbb{R}^n,\ t > 0, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^n. \end{cases} \tag{7.1_g}$$

By standard theory for parabolic equations (see [5, 6]), for every $u_0 \in X$ and $g \in H(f)$, equation (7.1_g) admits a (locally) unique classical solution $u(t, \cdot, u_0, g)$ in X with $u(0, \cdot, u_0, g) = u_0$. This solution also continuously depends on $g \in Y$ and $u_0 \in X$ (see e.g. [6, 14]). Therefore, (7.1_g) defines a (local) skew-product semiflow Π_t on $X \times Y$ with

$$\Pi_t(u_0, g) = (u(t, \cdot, u_0, g), g \cdot t), \qquad \forall (u_0, g) \in X \times Y,\ t \ge 0.$$

We define an order relation in X by $u \le v$ if $u(x) \le v(x)$ for all $x \in \mathbb{R}^n$. The action of G on $\mathbb{R}^n$ induces a group action on X by

$$a : u(\cdot) \mapsto u(a^{-1}\cdot).$$

Clearly, (A1)–(A3) in Section 3 are fulfilled.

Theorem 7.1 (Rotational symmetry). *Any uniformly L^∞-stable entire (possibly sign-changing) solution $\bar{u}_f(t,x)$ of (7.1), satisfying $\mathcal{M}(\bar{u}_f) \subset \mathcal{M}(f)$ and*

$$\sup_{t\in\mathbb{R}} |\bar{u}_f(t,x)| \to 0 \quad \text{as } |x| \to \infty, \tag{7.2}$$

is G-symmetric, i.e., $\bar{u}_f(t,gx) = \bar{u}_f(t,x)$ for all $t \in \mathbb{R}$, $x \in \mathbb{R}^n$ and $g \in G$.

For the entire solution $\bar{u}_f(t,x)$ given in Theorem 7.1, as we mentioned after Definition 2.7, the inclusion $\mathcal{M}(\bar{u}_f) \subset \mathcal{M}(f)$ will imply that there exists a flow epimorphism ϕ from the hull $H(f)$ to the hull $H(\bar{u}_f)$ defined by $\phi : H(f) \to H(\bar{u}_f)$, $g \mapsto \phi g$, such that $\phi f = \bar{u}_f$ and $\phi(g\cdot t) = \sigma_t(\phi g)$ for any $t \in \mathbb{R}$. Here σ_t is the time-translation flow on $H(\bar{u}_f)$. It is also easy to see that (7.2) holds uniformly for any element $v = v(\cdot,\cdot) \in H(\bar{u}_f)$. Let $E := \mathrm{cl}\{\bar{u}_f(t,\cdot) \in X : t \in \mathbb{R}\} \subset X$, and define

$$K = \{(v(0,\cdot), g) : g \in H(f) \text{ and } v = \phi g \in H(\bar{u}_f)\}.$$

Clearly, $K \subset E \times H(f)$ (and hence $K \subset X \times H(f)$). We now verify that K is positively invariant with respect to Π_t. In fact, for any $(v(0,\cdot), g) \in K$ (hence $v = \phi g \in H(\bar{u}_f)$), there is a sequence $t_n \to \infty$ such that $\bar{u}_f(t_n,\cdot) \to v(0,\cdot)$ in X and $f\cdot t_n \to g$ in $H(f)$. Then, for any $s \geq 0$, the solution $u(s,\cdot,v(0,\cdot),g)$ of (7.1_g) satisfies

$$u(s,\cdot,v(0,\cdot),g) = \lim_{n\to\infty} u(s,\cdot,\bar{u}_f(t_n,\cdot), f\cdot t_n) = \lim_{n\to\infty} \bar{u}_f(s+t_n,\cdot) = v(s,\cdot).$$

Meanwhile, it is clear that $v(s+\cdot,\cdot) = \sigma_s v(\cdot,\cdot) = \sigma_s(\phi g) = \phi(g\cdot s)(\cdot,\cdot)$. So, one obtains $u(s,\cdot,v(0,\cdot),g) = v(s,\cdot) = \phi(g\cdot s)(0,\cdot)$, which implies $(u(s,\cdot,v(0,\cdot),g), g\cdot s) \in K$. Thus, K is positively invariant with respect to Π_t. Moreover, K is a 1-cover of $H(f)$, because ϕ is a flow epimorphism and $K \cap P^{-1}(g) = \{((\phi g)(0,\cdot), g)\}$, which is a singleton on each fiber $P^{-1}(g)$ of $X \times H(f)$. In particular, $K \cap P^{-1}(f) = \{(\bar{u}_f(0,\cdot), f)\}$ because $\phi f = \bar{u}_f$. Moreover, since $\bar{u}_f$ is uniformly stable, it follows from Remark 2.2 that K is uniformly stable for Π_t.

For consistency with the fact $K \cap P^{-1}(f) = \{(\bar{u}_f(0,\cdot), f)\}$, we hereafter always rewrite for the convenience the 1-cover $K \subset X \times H(f)$ as

$$K := \{(\bar{u}_g(0,\cdot), g) : g \in H(f)\},$$

by letting $\bar{u}_g(0,\cdot) := (\phi g)(0,\cdot) \in E$ for any $g \in H(f)$. Following this notation of K, the entire solution $\bar{u}_f(s,\cdot)$ is $\bar{u}_{f\cdot s}(0,\cdot)$, for any $s \in \mathbb{R}$.

Recalling that the rotation group G is compact, in order to obtain the rotational symmetry of $\bar{u}_f(t,x)$, we only need to check (A4) in view of our abstract Theorem C. This will be done in Propositions 7.3 and 7.5 below. We first present the following useful lemma.

Lemma 7.2.

$$\sup_{g\in H(f)} \sup_{t\in\mathbb{R}} |\bar{u}_g(t,x)| \to 0 \quad \text{as } |x| \to \infty.$$

Proof. Since K is a 1-cover of $H(f)$, for any $g \in H(f)$ there exists a sequence $t_n \to \infty$ such that

$$\lim_{n\to\infty} \bar{u}_{f\cdot t_n}(t,x) = \lim_{n\to\infty} \bar{u}_{(f\cdot t_n)\cdot t}(0,x) = \bar{u}_{g\cdot t}(0,x) = \bar{u}_g(t,x)$$

uniformly in $t \in \mathbb{R}$ and $x \in \mathbb{R}^n$. Then for any $\varepsilon > 0$, it follows from (7.2) that there exists some $R_\varepsilon > 0$ such that

$$\begin{aligned} |\bar{u}_g(t,x)| &\le |\bar{u}_g(t,x) - \bar{u}_{f\cdot t_n}(t,x)| + |\bar{u}_{f\cdot t_n}(t,x)| \\ &= |\bar{u}_g(t,x) - \bar{u}_{f\cdot t_n}(t,x)| + |\bar{u}_f(t+t_n,x)| < \varepsilon/2 + \varepsilon/2 = \varepsilon \end{aligned}$$

for all $t \in \mathbb{R}$, $|x| > R_\varepsilon$, $g \in H(f)$ and n sufficiently large. This implies the conclusion. □

Proposition 7.3. *Let ϵ_0 be as in* (f 3). *Let also $(u_0, g_0) \in X \times H(f)$ be such that the omega-limit set $\mathcal{O}(u_0, g_0)$ is nonempty and satisfies*

$$\|v(\cdot) - \bar{u}_g(0,\cdot)\|_{L^\infty} < \epsilon_0/2 \quad \textit{for all } (v,g) \in \mathcal{O}(u_0,g_0),$$

with

$$(v(x), g) \le (\bar{u}_g(0,x), g),\ v(x) \not\equiv \bar{u}_g(0,x), \quad x \in \mathbb{R}^n,\ (v,g) \in \mathcal{O}(u_0,g_0).$$

Then there is a neighborhood $B(e) \subset G$ of e such that

$$(av(x), g) \le (\bar{u}_g(0,x), g), \quad av(x) \not\equiv \bar{u}_g(0,x),$$

for all $x \in \mathbb{R}^n$, $a \in B(e)$ and $(v,g) \in \mathcal{O}(u_0,g_0)$. The assertion remains true with $\le$ replaced by $\ge$.

Proof. We only prove the first assertion; the other is established similarly. Motivated by [18] and [19, Lemma 5.8], we let α, ϵ_0, R_0 be such that (f 3) holds. By Lemma 7.2, we choose some $R \ge R_0 > 0$ such that

$$|\bar{u}_g(t,x)| < \epsilon_0/4 \quad \text{for all } x \in \mathbb{R}^n \setminus B_R,\ g \in H(f) \text{ and } t \in \mathbb{R}, \tag{A}$$

where $B_R = \{x \in \mathbb{R}^n : |x| < R\}$. Moreover, for such an $\epsilon_0 > 0$, there exists a neighborhood $B_0(e) \subset G$ of e such that

$$|a\bar{u}_g(0,x) - \bar{u}_g(0,x)| < \epsilon_0/4 \quad \text{for all } x \in \mathbb{R}^n, a \in B_0(e) \text{ and } g \in H(f). \tag{7.3}$$

Recall that

$$\|v(\cdot) - \bar{u}_g(0,\cdot)\|_{L^\infty} < \epsilon_0/2 \quad \text{for all } (v,g) \in \mathcal{O}(u_0,g_0). \tag{7.4}$$

It follows from (7.3)–(7.4) and (A) that

$$\begin{aligned} |av(x)| &\le |av(x) - a\bar{u}_g(0,x)| + |a\bar{u}_g(0,x)| \\ &\le |v(a^{-1}x) - \bar{u}_g(0,a^{-1}x)| + |a\bar{u}_g(0,x) - \bar{u}_g(0,x)| + |\bar{u}_g(0,x)| \\ &< \epsilon_0/2 + \epsilon_0/4 + \epsilon_0/4 = \epsilon_0 \end{aligned} \tag{B}$$

for all $a \in B_0(e)$, $x \in \mathbb{R}^n \setminus B_R$ and $(v,g) \in \mathcal{O}(u_0,g_0)$.

Noticing that $(v(x), g) \leq (\bar{u}_g(0, x), g)$ and $v(x) \not\equiv \bar{u}_g(0, x)$ for $x \in \mathbb{R}^n$ and $(v, g) \in \mathcal{O}(u_0, g_0)$, the strong maximum principle yields

$$(u(t, x, v, g), g \cdot t) < (\bar{u}_{g \cdot t}(0, x), g \cdot t), \qquad \forall x \in \mathbb{R}^n,\ (v, g) \in \mathcal{O}(u_0, g_0),\ t > 0.$$

So, by the invariance of $\mathcal{O}(u_0, g_0)$, we obtain $(v(x), g) < (\bar{u}_g(0, x), g)$ for $x \in \mathbb{R}^n$ and $(v, g) \in \mathcal{O}(u_0, g_0)$. Since $\mathcal{O}(u_0, g_0)$ is compact in $X \times H(f)$, the continuity of $\bar{u}_g(0, \cdot)$ on g implies that there is an $\tilde{\epsilon} > 0$ such that

$$(v(x), g) < (\bar{u}_g(0, x) - \tilde{\epsilon}, g) \quad \text{ for all } x \in \overline{B_R} \text{ and } (v, g) \in \mathcal{O}(u_0, g_0).$$

As a consequence, there exists a smaller neighborhood $B(e) \subset B_0(e)$ of e such that

$$(av(x), g) < (\bar{u}_g(0, x), g) \quad \text{ for all } a \in B(e),\ x \in \overline{B_R} \text{ and } (v, g) \in \mathcal{O}(u_0, g_0). \tag{C}$$

Note also that $\bar{u}_g(0, \cdot) - av(\cdot) \geq \bar{u}_g(0, \cdot) - a\bar{u}_g(0, \cdot)$ for all $(v, g) \in \mathcal{O}(u_0, g_0)$. Then

$$\liminf_{|x| \to \infty}(\bar{u}_g(0, x) - av(x)) \geq \liminf_{|x| \to \infty}(\bar{u}_g(0, x) - a\bar{u}_g(0, x)) = 0 \tag{D}$$

for all $(v, g) \in \mathcal{O}(u_0, g_0)$ and $a \in B(e)$.

Now we claim that the proposition follows immediately from (A)–(D). Indeed, for any $(v, g) \in \mathcal{O}(u_0, g_0)$ and $\tau > 0$, one can find $(v_{-\tau}, g_{-\tau}) \in \mathcal{O}(u_0, g_0)$ such that $\Pi_\tau(v_{-\tau}, g_{-\tau}) = (v, g)$. Then for any $a \in B(e)$, by (A)–(D) and the invariance of $\mathcal{O}(u_0, g_0)$, we have

(i) $|\bar{u}_g(t, x)| < \epsilon_0$ for all $x \in \mathbb{R}^n \setminus B_R,\ g \in H(f)$ and $t \in \mathbb{R}$,
(ii) $|au(t, x, v_{-\tau}, g_{-\tau})| < \epsilon_0$ for all $t > 0$ and $x \in \mathbb{R}^n \setminus B_R$,
(iii) $au(t, x, v_{-\tau}, g_{-\tau}) < \bar{u}_{g_{-\tau} \cdot t}(0, x)$ for all $t > 0$ and $x \in \partial B_R$,
(iv) $\liminf_{|x| \to \infty}(\bar{u}_{g_{-\tau} \cdot t}(0, x) - au(t, x, v_{-\tau}, g_{-\tau})) \geq 0$ for all $t > 0$.

Therefore, Lemma 7.4 below implies that

$$\bar{u}_{g_{-\tau} \cdot t}(0, x) - au(t, x, v_{-\tau}, g_{-\tau}) = \bar{u}_{g_{-\tau} \cdot t}(0, x) - u(t, x, av_{-\tau}, g_{-\tau}) \geq -2\epsilon_0 e^{-\alpha t}$$

for all $x \in \mathbb{R}^n \setminus B_R$ and $t > 0$. In particular (let $t = \tau$),

$$\bar{u}_{g_{-\tau} \cdot \tau}(0, x) - au(\tau, x, v_{-\tau}, g_{-\tau}) \geq -2\epsilon_0 e^{-\alpha\tau} \quad \text{ for all } x \in \mathbb{R}^n \setminus B_R,$$

and hence

$$\bar{u}_g(0, x) - av(x) \geq -2\epsilon_0 e^{-\alpha\tau} \quad \text{ for all } x \in \mathbb{R}^n \setminus B_R.$$

Since $\tau > 0$ is arbitrarily chosen, by letting $\tau \to \infty$ we have $\bar{u}_g(0, x) \geq av(x)$ for all $x \in \mathbb{R}^n \setminus B_R$, $(v, g) \in \mathcal{O}(u_0, g_0)$ and $a \in B(e)$. Combining this with (C) completes the proof. ◻

Lemma 7.4. *Let* $\alpha,\ \epsilon_0,\ R_0$ *be such that* (f 3) *holds. Let* $R \geq R_0$ *be such that*

$$|\bar{u}_g(t,x)| < \epsilon_0 \quad \textit{for all } x \in \mathbb{R}^n \setminus B_R,\ g \in H(f) \textit{ and } t \in \mathbb{R}.$$

Let also $u(t,x,v_0,g)$ *be a solution of* (7.1$_g$) *satisfying*

$$|u(t,x,v_0,g)| < \epsilon_0 \quad \textit{for all } t > 0 \textit{ and } x \in \mathbb{R}^n \setminus B_R.$$

Assume that

$$\bar{u}_g(t,x) \geq u(t,x,v_0,g) \quad \textit{for all } x \in \partial B_R \textit{ and } t > 0$$

and

$$\liminf_{|x|\to\infty}(\bar{u}_g(t,x) - u(t,x,v_0,g)) \geq 0 \quad \textit{for all } t > 0.$$

Then

$$\bar{u}_g(t,x) - u(t,x,v_0,g) \geq -2\epsilon_0 e^{-\alpha t} \quad \textit{for all } x \in \mathbb{R}^n \setminus B_R \textit{ and } t > 0.$$

Proof. The proof is similar to that of [18, Lemma 5.9]; we give the details for completeness. For any $g \in H(f)$, the function $w(t,x) = \bar{u}_g(t,x) - u(t,x,v_0,g)$ is a solution of the linear parabolic equation

$$\frac{\partial w}{\partial t} = \Delta w + \xi(t,x)w, \qquad x \in \mathbb{R}^n \setminus \overline{B_R},\ t > 0, \tag{7.5}$$

under the boundary condition $w = \bar{u}_g - u \geq 0$ on ∂B_R, where

$$\xi(t,x) = \int_0^1 g'_u\big(t,x,\theta\bar{u}_g(t,x) + (1-\theta)u(t,x,v_0,g)\big)\, d\theta.$$

In view of our assumptions, it is easy to see that

$$|\theta\bar{u}_g(t,x) + (1-\theta)u(t,x,v_0,g)| < \epsilon_0 \quad \text{for all } x \in \mathbb{R}^n \setminus B_R \text{ and } t > 0.$$

Since $g \in H(f)$ satisfies (f 3), we have

$$\xi(t,x) \leq -\alpha \quad \text{for all } x \in \mathbb{R}^n \setminus B_R \text{ and } t > 0.$$

Let $\tilde{r}(t) = -2\epsilon_0 e^{-\alpha t}$. Then

$$\frac{\partial \tilde{r}}{\partial t} \leq \Delta\tilde{r} + \xi(t,x)\tilde{r} \quad \text{for all } x \in \mathbb{R}^n \setminus \overline{B_R} \text{ and } t > 0.$$

Clearly, $\tilde{r}(t) < 0 \leq w(t,x)$ on ∂B_R. Moreover,

$$\tilde{r}(0) = -2\epsilon_0 \leq \bar{u}_g(0,x) - v_0(x) = w(0,x) \quad \text{for all } x \in \mathbb{R}^n \setminus B_R,$$

and $\tilde{r}(t) < 0 \leq \liminf_{|x|\to\infty} w(t,x)$ for all $t > 0$. Then it follows from the comparison theorem that

$$\tilde{r}(t) \leq w(t,x) \quad \text{for all } x \in \mathbb{R}^n \setminus B_R \text{ and } t > 0,$$

which completes the proof. □

Proposition 7.5. *Let ϵ_0 be as in* (f 3). *Then for any solution $u(t,x,v_0,g)$ of* (7.1$_g$) *satisfying*

$$\sup_{t\geq 0}\|u(t,\cdot,v_0,g)-\bar{u}_g(t,\cdot)\|_{L^\infty}<\epsilon_0/4, \tag{7.6}$$

the forward orbit $O^+(v_0,g)$ is relatively compact in X.

Proof. Since

$$\sup_{t\in\mathbb{R}}|\bar{u}_g(t,x)|\to 0 \quad \text{as } |x|\to+\infty, \tag{7.7}$$

let $R>R_0$ be such that $\sup_{t\in\mathbb{R}}|\bar{u}_g(t,x)|\leq\epsilon_*$ for $x\in\mathbb{R}^n\setminus B_R$, where $B_R=\{x\in\mathbb{R}^n : |x|<R\}$ and $\epsilon_*=\epsilon_0/4$. In view of (7.6),

$$|u(t,x,v_0,g)|\leq 2\epsilon_* \quad \text{for all } t\geq 0 \text{ and } x\in\mathbb{R}^n\setminus B_R. \tag{7.8}$$

Furthermore, $u(t,x,v_0,g)$ satisfies the initial boundary value problem

$$\begin{cases}\dfrac{\partial w}{\partial t}=\Delta w+g(t,x,w), & x\in\mathbb{R}^n\setminus\overline{B_R},\ t>0,\\ w=u, & x\in\partial B_R,\ t>0,\\ w(0,x)=v_0(x), & x\in\mathbb{R}^n\setminus B_R.\end{cases} \tag{7.9}$$

Now let ϕ^+ satisfy

$$\begin{cases}\dfrac{\partial\phi^+}{\partial t}=\Delta\phi^+-\alpha\phi^+, & x\in\mathbb{R}^n\setminus\overline{B_R},\ t>0,\\ \phi^+=3\epsilon_*, & x\in\partial B_R,\ t>0,\\ \phi^+(0,x)=3\epsilon_*, & x\in\mathbb{R}^n\setminus B_R.\end{cases}$$

Then $\hat{u}:=\bar{u}_g+\phi^+$ satisfies

$$\begin{cases}\dfrac{\partial u}{\partial t}=\Delta u+g(t,x,\bar{u}_g)-\alpha\phi^+, & x\in\mathbb{R}^n\setminus\overline{B_R},\ t>0,\\ u=3\epsilon_*+\bar{u}_g, & x\in\partial B_R,\ t>0,\\ u(0,x)=3\epsilon_*+\bar{u}_g(0,x), & x\in\mathbb{R}^n\setminus B_R.\end{cases}$$

Note that

$$g(t,x,\hat{u})-g(t,x,\bar{u}_g)+\alpha\phi^+=\left[\int_0^1\frac{\partial g}{\partial u}(t,x,\bar{u}_g+\theta\phi^+)\,d\theta+\alpha\right]\cdot\phi^+. \tag{7.10}$$

Since $|\bar{u}_g(t,x)|\leq\epsilon_*$ and $|\theta\phi^+|\leq|\phi^+|\leq 3\epsilon_*$ on $\mathbb{R}^n\setminus B_R$, one has $|\bar{u}_g+\theta\phi^+|\leq\epsilon_0$. Thus by (f 3) (with f replaced by g), $\int_0^1\frac{\partial g}{\partial u}(t,x,\bar{u}_g+\theta\phi^+)\,d\theta\leq-\alpha$. Note also that $\phi^+>0$ on $\mathbb{R}^n\setminus\overline{B_R}$. It follows from (7.10) that $g(t,x,\hat{u})\leq g(t,x,\bar{u}_g)-\alpha\phi^+$, which implies that

$$\begin{cases}\dfrac{\partial\hat{u}}{\partial t}\geq\Delta\hat{u}+g(t,x,\hat{u}), & x\in\mathbb{R}^n\setminus\overline{B_R},\ t>0,\\ \hat{u}=3\epsilon_*+\bar{u}_g\geq 2\epsilon_*, & x\in\partial B_R,\ t>0,\\ \hat{u}(0,x)=3\epsilon_*+\bar{u}_g(0,x)\geq 2\epsilon_*, & x\in\mathbb{R}^n\setminus B_R.\end{cases}$$

Combined with (7.8) and (7.9), the comparison principle implies that

$$u(t,x,v_0,g) \le \bar{u}_g(t,x) + \phi^+(t,x) \quad \text{for all } t \ge 0 \text{ and } x \in \mathbb{R}^n \setminus B_R.$$

Similarly, we can construct ϕ^- satisfying

$$\begin{cases} \dfrac{\partial \phi^-}{\partial t} = \Delta\phi^- - \alpha\phi^-, & x \in \mathbb{R}^n \setminus \overline{B_R},\ t > 0, \\ \phi^- = -3\epsilon_*, & x \in \partial B_R,\ t > 0, \\ \phi^-(0,x) = -3\epsilon_*, & x \in \mathbb{R}^n \setminus B_R. \end{cases}$$

and obtain

$$u(t,x,v_0,g) \ge \bar{u}_g(t,x) + \phi^-(t,x) \quad \text{for all } t \ge 0 \text{ and } x \in \mathbb{R}^n \setminus B_R.$$

A direct estimate yields (see [12, p. 94])

$$\lim_{\substack{t\to+\infty \\ |x|\to+\infty}} \phi^\pm(t,x) = 0,$$

which implies that

$$\lim_{\substack{t\to+\infty \\ |x|\to+\infty}} |u(t,x,v_0,g) - \bar{u}_g(t,x)| = 0. \tag{7.11}$$

In order to prove the relative compactness of $\{u(t,\cdot,v_0,g)\}_{t\in[0,\infty)}$ in X, we note that, by (7.6)–(7.7), $u(t,x,v_0,g)$ is a bounded solution of (7.1_g) in X. Then the standard parabolic estimate shows that $u(t,\cdot,v_0,g)$ is bounded in $C^2_{\text{loc}}(\mathbb{R}^n)$. Combining (7.7), (7.11) and the Arzelà–Ascoli Theorem, we obtain the relative compactness of $\{u(t,\cdot,v_0,g)\}_{t\in[0,\infty)}$ in X. □

7.2. Traveling waves

In this subsection, we will utilize the abstract results in Section 3 to investigate the monotonicity of stable traveling waves for time-almost periodic reaction-diffusion equations with bistable nonlinearities. Our aim is to study such kind of problems from a general point of view. As a simple example, we consider the time-almost periodic reaction-diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial z^2} + f(t,u), \qquad z \in \mathbb{R},\ t > 0, \tag{7.12}$$

where the nonlinearity $f : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is C^1-admissible and uniformly almost periodic in t. Of course, our approach to (7.12) can be applied, with little modification, to monotonicity of stable traveling waves for various other types of equations (see e.g. [18, 19]) with bistable nonlinearities.

A solution $u(z,t)$ of (7.12) is called an *almost periodic traveling wave* (see e.g. [27, Section 2.2]) if there are $\phi \in C^1(\mathbb{R}\times\mathbb{R},\mathbb{R})$ and $c \in C^1(\mathbb{R},\mathbb{R})$ such that

$$u(z,t) = \phi(z - c(t), t),$$

where $\phi(x,t)$ (called the *wave profile*) is almost periodic in t uniformly with respect to x in bounded sets, and $c'(t)$ (called the *wave speed*) is almost periodic in t; and moreover, the frequency modules

$$\mathcal{M}(\phi(x,\cdot)) \text{ and } \mathcal{M}(c'(\cdot)) \text{ are contained in } \mathcal{M}(f).$$

We restrict our attention to traveling waves satisfying the connecting condition

$$\lim_{x\to\pm\infty} \phi(x,t) = u_{\pm}^{f}(t), \qquad \text{uniformly for } t \in \mathbb{R},$$

where $u_{\pm}^{f}(t)$ are spatially homogeneous time-almost periodic solutions of (7.12) with $\mathcal{M}(u_{\pm}^{f}(\cdot)) \subset \mathcal{M}(f)$. A traveling wave is called a *solitary wave* if $u_{+}^{f}(t) = u_{-}^{f}(t)$ for all $t \in \mathbb{R}$, and a *traveling front* if either $u_{-}^{f}(t) < u_{+}^{f}(t)$ for all $t \in \mathbb{R}$, or $u_{-}^{f}(t) > u_{+}^{f}(t)$ for all $t \in \mathbb{R}$.

In what follows we assume that

(F) there exist $\epsilon_0, \mu > 0$ such that

$$\frac{\partial f}{\partial u}(t,u) \le -\mu \qquad \text{for } |u - u_{\pm}^{f}(t)| < \epsilon_0 \text{ and } t \in \mathbb{R}.$$

Let $X = C_{\text{unif}}(\mathbb{R})$ denote the space of bounded and uniformly continuous functions on $\mathbb{R}$ endowed with the $L^{\infty}(\mathbb{R})$ topology. For any $u_0 \in X$, let $u(\cdot,t;u_0,f)$ be the solution of (7.12) with $u(\cdot,0;u_0,f) = u_0$.

A traveling wave $\phi(z - c(t), t)$ of (7.12) is called *uniformly stable* if for every $\varepsilon > 0$ there is a $\delta(\varepsilon) > 0$ such that, for every $u_0 \in X$, if $s \ge 0$ and $\|u(\cdot,s;u_0,f) - \phi(\cdot - c(s), s)\|_{L^\infty} \le \delta(\varepsilon)$ then

$$\|u(\cdot,t;u_0,f) - \phi(\cdot - c(t), t)\|_{L^\infty} < \varepsilon \qquad \text{for each } t \ge s.$$

Moreover, $\phi(z - c(t), t)$ is called *uniformly stable with asymptotic phase* if it is uniformly stable and there exists a $\delta > 0$ such that if $\|u_0 - \phi(\cdot - c(0), 0)\|_{L^\infty} < \delta$ then

$$\|u(\cdot,t;u_0,f) - \phi(\cdot - c(t) - \sigma, t)\|_{L^\infty} \to 0 \qquad \text{as } t \to \infty$$

for some $\sigma \in \mathbb{R}$. A traveling wave $\phi(z - c(t), t)$ is called *spatially monotone* if $\phi(x,t)$ is a nondecreasing or nonincreasing function of x for every $t \in \mathbb{R}$.

Based on our main abstract results, Theorems B and D of Section 3, we derive the following results:

Theorem 7.6. *Any uniformly stable traveling wave of* (7.12) *is spatially monotone. In particular, solitary waves of* (7.12) *are not uniformly stable.*

Theorem 7.7. *Any uniformly stable traveling wave of* (7.12) *is uniformly stable with asymptotic phase.*

Remark 7.8. Shen [27–29] first investigated traveling waves in time-almost periodic media. In [28, 29], she proved the existence of such traveling waves. In [27], she had shown that any spatially monotone time-almost periodic traveling wave is uniformly stable. Our Theorem 7.6 here is a converse of Shen's theorem of [27]. In particular, Theorem 7.6 implies that any solitary wave is not uniformly stable. Theorem 7.7 can also be found in Shen [27]. However, note that our approach (Theorem D) was introduced in a very general framework, and hence it can be applied to wider classes of equations with little modification.

Proof of Theorems 7.6 and 7.7. We first rewrite equation (7.12) with the moving coordinate $x = z - c(t)$:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + c'(t)\frac{\partial u}{\partial x} + f(t, u), \qquad x \in \mathbb{R},\ t > 0. \tag{7.13}$$

Obviously, $\phi(z - c(t), t)$ is an almost periodic traveling wave of (7.12) if and only if $\phi(x, t)$ is an almost periodic entire solution of (7.13) satisfying $\mathcal{M}(\phi(x, \cdot)) \subset \mathcal{M}(f)$. In the following, we rewrite $\phi(x, t)$ as $\phi^{y_0}(x, t)$ with $y_0 = (c', f)$. It is easy to see that

$$\lim_{x\to\pm\infty} \phi^{y_0}(x, t) = u^f_{\pm}(t) \quad \text{ uniformly in } t \in \mathbb{R}. \tag{7.14}$$

Let $Y = H(c', f)$ be the hull of the function $y_0 = (c', f)$. By the standard theory of reaction-diffusion systems (see e.g. [5, 6]), for every $v_0 \in X$ and $y = (d, g) \in Y$ the system

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + d(t)\frac{\partial u}{\partial x} + g(t, u), \qquad x \in \mathbb{R},\ t > 0, \tag{7.13$_y$}$$

admits a (locally) unique regular solution $v(\cdot, t; v_0, y)$ in X with $v(\cdot, 0; v_0, y) = v_0$. This solution also continuously depends on $y \in Y$ and $v_0 \in X$ (see e.g. [6, Sec. 3.4]). Therefore, (7.13$_y$) induces a (local) skew-product semiflow Π on $X \times Y$ with

$$\Pi_t(v_0, y) = (v(\cdot, t; v_0, y), y \cdot t), \qquad \forall (v_0, y) \in X \times Y,\ t \geq 0.$$

We define an order relation in X by $u \leq v$ if $u(x) \leq v(x)$ for all $x \in \mathbb{R}$. Let $G = \{a_\sigma : \sigma \in \mathbb{R}\}$ be the group of translations

$$a_\sigma : u(\cdot) \mapsto u(\cdot - \sigma)$$

acting on the space X. Then (A1)–(A3) are fulfilled.

Note that $\phi^{y_0}(x, t)$ is a uniformly almost periodic solution of (7.13) with $\mathcal{M}(\phi^{y_0}(x, \cdot)) \subset \mathcal{M}(f) = \mathcal{M}(y_0)$. So, the closure K of the orbit $\{(\phi^{y_0}(\cdot, t), y_0 \cdot t) : t \in \mathbb{R}\}$ of Π_t is a uniformly stable 1-cover of Y. As a consequence, K can be written as

$$K = \{(\phi^y(\cdot, 0), y) \in X \times Y : y = (d, g) \in Y\},$$

where the map $y \mapsto \phi^y(\cdot, 0) \in X$ is continuous and satisfies $\phi^{y_0}(\cdot, t) = \phi(\cdot, t)$ and $\phi^{y\cdot t}(\cdot, 0) = \phi^y(\cdot, t)$ for all $y \in Y$ and $t \in \mathbb{R}$. By (7.14), it is not difficult to see that

$$\lim_{x\to\pm\infty} \phi^y(x, t) = u^g_{\pm}(t) \quad \text{ uniformly for } y = (d, g) \in Y \text{ and } t \in \mathbb{R}, \tag{7.15}$$

where $\{(u_\pm^g(0), g) \in \mathbb{R} \times H(f) : g \in H(f)\}$ is a 1-cover of $H(f)$ and satisfies $u_\pm^{g\cdot t}(0) = u_\pm^g(t)$ for all $g \in H(f)$ and $t \in \mathbb{R}$. One can also easily see that, for any $g \in H(f)$, the function-pair $(g, u_\pm^g(t))$ satisfies condition (F), i.e.,

$(F)_g$ there exist $\epsilon_0, \mu > 0$ such that

$$\frac{\partial g}{\partial u}(t,u) \leq -\mu \quad \text{for } |u - u_\pm^g(t)| < \epsilon_0 \text{ and } t \in \mathbb{R}.$$

In order to apply Theorems B and D of Section 3, we have to check (A4). By (7.15) and $(F)_g$ above, (A4)(i) can be shown by repeating an analogue of Proposition 7.5, with $\bar{u}_g$ replaced by $\phi^y - u_\pm^g$ (see also similar arguments in [18, Lemma 5.6]). We omit the details.

As for (A4)(ii), we will deduce it from Proposition 7.9 below. Based on this, we can apply Theorem B to deduce that the group orbit GK of K is a 1-D subbundle of $X \times Y$. In particular, for fixed $y_0 \cdot t \in Y$, the fiber

$$GK_{y_0\cdot t} = G[\phi^{y_0\cdot t}(x,0)] = G[\phi^{y_0}(x,t)] = G[\phi(x,t)] = \{\phi(x-\sigma,t) : \sigma \in \mathbb{R}\}$$

is totally ordered, which implies that $\phi(x,t)$ is monotone in x for every $t \in \mathbb{R}$. Furthermore, it follows from Theorem D that the traveling wave $\phi(z - c(t), t)$ is uniformly stable with asymptotic phase. This completes the proof of Theorems 7.6 and 7.7.

Proposition 7.9. *Let ϵ_0 be as in* (F). *For $(u_0, y_0) \in X \times Y$, suppose that the omega-limit set $\mathcal{O}(u_0, y_0)$ is nonempty and satisfies*

$$\|v(\cdot) - \phi^y(\cdot, 0)\|_{L^\infty} < \epsilon_0/2 \quad \textit{for all } (v,y) \in \mathcal{O}(u_0,y_0), \tag{7.16}$$

as well as

$$(v(x), y) \leq (\phi^y(x-h,0), y), \quad v(x) \not\equiv \phi^y(x-h,0), \quad \forall (v,y) \in \mathcal{O}(u_0,y_0),\ x \in \mathbb{R}, \tag{7.17}$$

for some $h \in \mathbb{R}$. Then there exists some $\delta > 0$ such that

$$(v(x), y) \leq (\phi^y(x-h-\sigma,0), y), \quad v(x) \not\equiv \phi^y(x-h-\sigma,0),$$

for all $(v,y) \in \mathcal{O}(u_0,y_0)$, $x \in \mathbb{R}$ and $|\sigma| < \delta$. The assertion remains true with $\leq$ replaced by $\geq$.

Proof. We use similar arguments to those for Proposition 7.3. Let μ, ϵ_0 be such that (F) holds. By (7.15), we have

$$\lim_{x\to\pm\infty} \phi^y(x-h,0) = \lim_{x\to\pm\infty} \phi^y(x,0) = u_\pm^g(0) \quad \text{uniformly for } y = (d,g) \in Y.$$

Thus there exist some $R', R'' > 0$ such that

$$|\phi^y(x,0) - u_\pm^g(0)| < \epsilon_0/2 \quad \text{for all } |x| > R' \text{ and } y \in Y, \tag{7.18}$$

as well as

$$|\phi^y(x-h,0) - u_\pm^g(0)| < \epsilon_0/2 \quad \text{for all } |x| > R'' \text{ and } y \in Y. \tag{7.19}$$

Let $R = \max\{R', R''\}$, In view of (7.16), it follows from (7.18) that

$$|v(x) - u^g_\pm(0)| < \epsilon_0 \quad \text{for all } (v, y) \in \mathcal{O}(u_0, y_0) \text{ and } |x| > R. \tag{A$'$}$$

Moreover, together with (7.19), the continuity of the translation-group action on X implies that there exists a $\delta_0 > 0$ such that if $|\sigma| < \delta_0$ then

$$|\phi^y(x - h - \sigma, 0) - u^g_\pm(0)| < \epsilon_0 \quad \text{for all } |x| > R \text{ and } y \in Y. \tag{B$'$}$$

Due to the assumption (7.17), the strong maximum principle yields

$$(v(x, t; v, y), y\cdot t) < (\phi^{y\cdot t}(x-h, 0), y\cdot t) \quad \text{for all } (v, y) \in \mathcal{O}(u_0, y_0),\ x \in \mathbb{R} \text{ and } t > 0.$$

By the invariance of $\mathcal{O}(u_0, y_0)$, we get

$$(v(x), y) < (\phi^y(x - h, 0), y) \quad \text{for all } (v, y) \in \mathcal{O}(u_0, y_0) \text{ and } x \in \mathbb{R}.$$

Since $\mathcal{O}(u_0, g_0)$ is compact in $X \times Y$, it follows from the continuity of $\phi^y(\cdot, 0)$ in y that for a sufficiently small $\tilde{\epsilon} > 0$,

$$(v(x), y) < (\phi^y(x - h, 0) - \tilde{\epsilon}, y) \quad \text{for all } (v, y) \in \mathcal{O}(u_0, y_0) \text{ and } |x| \le R.$$

So one can find a $\delta > 0$ ($\delta \le \delta_0$) such that if $|\sigma| < \delta$ then

$$(v(x), y) < (\phi^y(x - h - \sigma, 0), y) \quad \text{for all } (v, y) \in \mathcal{O}(u_0, y_0) \text{ and } |x| \le R. \tag{C$'$}$$

Note also that $\phi^y(x - h - \sigma, 0) - v(x) \ge \phi^y(x - h - \sigma, 0) - \phi^y(x - h, 0)$ for all $(v, y) \in \mathcal{O}(u_0, y_0)$ and $x \in \mathbb{R}$. Then

$$\liminf_{|x|\to\infty}(\phi^y(x-h-\sigma, 0) - v(x)) \ge \liminf_{|x|\to\infty}(\phi^y(x-h-\sigma, 0) - \phi^y(x-h, 0)) = 0 \tag{D$'$}$$

for all $(v, y) \in \mathcal{O}(u_0, y_0)$ and $|\sigma| < \delta$.

Just as with (A)–(D) in the proof of Proposition 7.3, we can deduce from (A$'$)–(D$'$) that, for any $(v, y) \in \mathcal{O}(u_0, y_0)$ and $\tau > 0$, there exists $(v_{-\tau}, y_{-\tau}) \in \mathcal{O}(u_0, y_0)$ with $\Pi_\tau(v_{-\tau}, y_{-\tau}) = (v, y)$. Moreover, for any $|\sigma| < \delta$, the following statements hold true:

(i) $|v(x, t; v_{-\tau}, y_{-\tau}) - u^{g_{-\tau}\cdot t}_\pm(0)| < \epsilon_0$ for all $t > 0$ and $|x| > R$,
(ii) $|\phi^y(x - h - \sigma, t) - u^g_\pm(t)| < \epsilon_0$ for all $|x| > R$, $y \in Y$ and $t \in \mathbb{R}^+$,
(iii) $v(x, t; v_{-\tau}, y_{-\tau}) < \phi^{y_{-\tau}\cdot t}(x - h - \sigma, 0)$ for all $t > 0$ and $|x| \le R$,
(iv) $\liminf_{|x|\to\infty}(\phi^{y_{-\tau}\cdot t}(x - h - \sigma, 0) - v(x, t; v_{-\tau}, y_{-\tau})) \ge 0$ for all $t > 0$.

Therefore, by using an analogue of the last paragraph in the proof of Proposition 7.3 (the proof of this modified version of Lemma 7.4 is almost identical to that of Lemma 7.4), we obtain

$$\phi^{y_{-\tau}\cdot t}(x - h - \sigma, 0) - v(x, t; v_{-\tau}, y_{-\tau}) \ge -2\epsilon_0 e^{-\mu t} \quad \text{for all } |x| > R \text{ and } t > 0.$$

In particular, by letting $t = \tau$,

$$\phi^{y}(x-h-\sigma,0) - v(x) = \phi^{y_{-\tau}\cdot\tau}(x-h-\sigma,0) - v(x,\tau;v_{-\tau},y_{-\tau}) \geq -2\epsilon_0 e^{-\mu\tau}, \quad \forall |x| > R.$$

Since $\tau > 0$ is arbitrarily chosen, by letting $\tau \to \infty$ we have

$$\phi^{y}(x-h-\sigma,0) \geq v(x)$$

for all $|x| > R$, $(v,y) \in \mathcal{O}(u_0,y_0)$ and $|\sigma| < \delta$. Noting also (C′), we obtain the conclusion of the proposition. □

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