JEMS

Radu Laza

The KSBA compactification for the moduli space of degree two $K3$ pairs

Abstract. Inspired by the ideas of the minimal model program, Shepherd-Barron, Kollár, and Alexeev have constructed a geometric compactification for the moduli space of surfaces of log general type. In this paper, we discuss one of the simplest examples that fits into this framework: the case of pairs (X, H) consisting of a degree two $K3$ surface X and an ample divisor H. Specifically, we construct and explicitly describe a geometric compactification $\overline{\mathcal{P}}_2$ for the moduli space of degree two $K3$ pairs. This compactification has a natural forgetful map to the Baily–Borel compactification of the moduli space $\mathcal{F}_2$ of degree two $K3$ surfaces. Using this map and the modular meaning of $\overline{\mathcal{P}}_2$, we obtain a better understanding of the geometry of the standard compactifications of $\mathcal{F}_2$.

Keywords. $K3$ surfaces, moduli space of $K3$ surfaces, KSBA

Introduction

The search for geometric compactifications for moduli spaces is one of the central problems in algebraic geometry. After the successful constructions of compactifications for the moduli spaces of curves (Deligne–Mumford), and abelian varieties (Mumford, Namikawa, Alexeev, and others), a case that attracted a great deal of interest was that of polarized $K3$ surfaces (see e.g. [FM83b]). Similar to the case of abelian varieties, the moduli space of polarized $K3$ surfaces is a locally symmetric variety and as such it has several compactifications, the most commonly studied being the Baily–Borel and toroidal compactifications. Unfortunately, very little is known about the geometric meaning of those. The best understood situation is that of low degree $K3$ surfaces where algebraic constructions for the moduli space are available via GIT. Namely, for degree 2 (and similarly for degree 4), Shah constructed a compactification $\widehat{\mathcal{M}}$ for the moduli of degree 2 $K3$ surfaces which has several good properties (see Thm. 1.6). For instance, $\widehat{\mathcal{M}}$ is an Artin stack with weak modular meaning (in the sense of GIT): $\widehat{\mathcal{M}}$ parameterizes degenerations of $K3$ surfaces that are Gorenstein and have at worst semi-log-canonical singularities.

The space $\widehat{\mathcal{M}}$ was constructed by Shah [Sha80] as a partial Kirwan desingularization of the GIT quotient $\overline{\mathcal{M}}$ for sextic curves (see also [KL89]). Alternatively, for any

R. Laza: Stony Brook University, Department of Mathematics, Stony Brook, NY 11794, USA;
e-mail: rlaza@math.sunysb.edu

degree, the moduli space of polarized $K3$ surfaces is isomorphic to a locally symmetric variety $\mathcal{D}/\Gamma_d$. The space $\mathcal{D}/\Gamma_d$ has a natural compactification, the Baily–Borel compactification $(\mathcal{D}/\Gamma_d)^*$. For degree 2, as shown by Looijenga [Loo86], Shah's model $\widehat{\mathcal{M}}$ is a small partial resolution of $(\mathcal{D}/\Gamma_2)^*$, and is in fact a semitoric compactification in the sense of [Loo03] (see Thm. 1.9). Thus, $\widehat{\mathcal{M}}$ has a dual description which gives complementary information: the GIT construction provides some geometric meaning to the boundary, and, on the other hand, the semitoric construction gives a rich structure, which can be further exploited in applications. Arguably $\widehat{\mathcal{M}}$ is the "best" compactification for the moduli space $\mathcal{F}_2$ of degree 2 $K3$ surfaces known at this point.

The issue is that $\widehat{\mathcal{M}}$ is not modular in the usual sense: it fails to be separated at the boundary. While one might hope that some toroidal compactification $\overline{\mathcal{D}/\Gamma_2}^\Sigma$ (refining $\widehat{\mathcal{M}}$ and $(\mathcal{D}/\Gamma_2)^*$) would give a modular compactification for $\mathcal{F}_2$, as in the case of abelian varieties (see [Nam80], [Ale02]), this is not known and seems out of reach (see however [Ols04] and Rem. 6.5). In this paper, we go in a different direction. Namely, we modify the moduli problem and construct a modular compactification $\overline{\mathcal{P}}_2$ of the corresponding moduli space which admits a forgetful map $\overline{\mathcal{P}}_2 \to (\mathcal{D}/\Gamma_2)^*$ (generically a $\mathbb{P}^2$-fibration). In other words, we obtain a fibration with modular meaning over some compactification of $\mathcal{F}_2$. We note that $\overline{\mathcal{P}}_2$ sheds further light on the geometric meaning of the standard compactifications (e.g. GIT, Baily–Borel) of $\mathcal{F}_2$ and we expect it to play an important role in the elusive search for a geometric compactification for the moduli space of $K3$ surfaces.

Remark. For clarity, let us comment on the meaning of "modular" or "geometric" used in this paper. First of all, most of the spaces considered here (e.g. $\widehat{\mathcal{M}}$) are constructed via GIT and consequently have some weak geometric meaning, e.g. over the stable locus there is (locally) a universal family, and each point corresponds to a unique polystable (i.e. semistable and closed) orbit. From a stack perspective, these spaces are (separated) coarse moduli spaces associated to Artin stacks, and they have good properties (see Alper [Alp13] for a formalization of these properties, and the corresponding notion of "good moduli"). Here, we would like to obtain something more geometric; ideally, we would like to obtain a coarse moduli space associated to a proper and separated Deligne–Mumford stack. Unfortunately, we obtain somewhat less: Namely, as usually in the KSBA framework, there is a Deligne–Mumford stack, but the associated coarse moduli space has multiple components. Our goal is to describe the objects parameterized by the main component $\overline{\mathcal{P}}_2$ of smoothable pairs. When we reduce to $\overline{\mathcal{P}}_2$ (essentially, considering the reduced closure of the smooth locus), we loose the functorial meaning. Nonetheless, we still say that $\overline{\mathcal{P}}_2$ is geometric; for $K3$'s there might be some workarounds (see e.g. Rem. 2.15), but this is of secondary concern for us.

Concretely, we consider the moduli space $\mathcal{P}_2$ of pairs (X, H) consisting of a degree 2 $K3$ surface and an ample divisor of degree 2. There is a natural forgetful map $\mathcal{P}_2 \to \mathcal{F}_2$ given by $(X, H) \to (X, \mathcal{O}_X(H))$, that makes $\mathcal{P}_2$ a $\mathbb{P}^2$-bundle over the moduli space of degree 2 $K3$ surfaces. We compactify $\mathcal{P}_2$ using the framework introduced by Kollár–Shepherd-Barron [KSB88] and Alexeev [Ale96] (called KSBA in what follows) and the ϵ-coefficient approach pioneered by Hacking [Hac04]. The main idea of this approach is

to view a degree 2 pair as a log general type pair $(X, \epsilon H)$ and to compactify by allowing stable pairs (i.e. require $(X, \epsilon H)$ to have slc singularities and H to be ample). Then a geometric compactification for $\mathcal{P}_2$ exists by general principles in the minimal model program (MMP). In fact, the same is true for all degrees, and thus one obtains geometric compactifications $\overline{\mathcal{P}}_d$ for all degrees $d \in 2\mathbb{Z}_+$ (see Cor. 2.12). The issue is that it is very difficult to understand $\overline{\mathcal{P}}_d$ directly. The main result of the paper is to construct $\overline{\mathcal{P}}_2$ explicitly and to describe the boundary pairs. We summarize the main result as follows:

Main Theorem. *Let $\mathcal{F}_2$ and $\mathcal{P}_2$ be the moduli spaces of degree* 2 *$K3$ surfaces and of degree* 2 *pairs respectively. There exists a geometric compactification $\overline{\mathcal{P}}_2$ of $\mathcal{P}_2$ parameterizing stable degree* 2 *pairs (Def. 2.3) and a natural map $\overline{\mathcal{P}}_2 \to (\mathcal{D}/\Gamma_2)^*$ to the Baily–Borel compactification extending the forgetful map $\mathcal{P}_2 \to \mathcal{F}_2$. Furthermore, there exist six irreducible boundary components for $\overline{\mathcal{P}}_2$ of dimensions* 3, 4, 10, 12, 13, *and* 19 *respectively. The geometric meaning of these components is described in Table* 1 (*see Thms.* 6.1 *and* 7.1 *and Table* 2 *for further details*).

Table 1. Boundary components of $\overline{\mathcal{P}}_2$.

	Description (generic point)	dim	Type II case	Type III
1	$X = V_1 \cup_E V_2$, $V_i \cong \mathbb{P}^2$	3	A_{17}	E nodal
2	$X = V_1 \cup_E V_2$, V_i deg 1 del Pezzo's	19	$E_8^2 + A_1$ (A)	E nodal
3	X^ν a quadric in $\mathbb{P}^3$, E a double curve	4	$D_{16} + A_1$	E nodal, or $E = C_1 \cup C_2$
4	X^ν a deg 2 del Pezzo, E a double curve	10	$E_7 + D_{10}$ (A)	E nodal
5	X rational with an $\widetilde{E}_8$ singularity	12	$E_8^2 + A_1$ (B)	$T_{2,3,7}$
6	X rational with an $\widetilde{E}_7$ singularity	13	$E_7 + D_{10}$ (B)	$T_{2,4,5}$

We recall that the Baily–Borel compactification $(\mathcal{D}/\Gamma_2)^*$ is obtained by adding four rational curves to $\mathcal{F}_2$ (see Thm. 1.1 and Fig. 2). Each of the six boundary components of $\overline{\mathcal{P}}_2$ will map to one of the four Baily–Borel boundary components, giving them a fibration structure over $\mathbb{P}^1$. For instance, the three-dimensional boundary component of $\overline{\mathcal{P}}_2$ corresponding to the first case of Table 1 is a $\mathbb{P}^2$-fibration over $\mathbb{P}^1$ (the closure of the Type II Baily–Borel boundary component $\mathrm{II}_{A_{17}}$). For further details see the following remark and Sections 6 and 7.

Remark. Here we make some comments on the content of Table 1. The boundary components are labeled by the cases given by the classification of degree two 0-surfaces (cf. [SB83b]) of Proposition 3.14. The second column describes the generic stable pair (X, H) parameterized by a boundary component. The class of the polarizing divisor H is easily determined in each case, and we omit it from the description. In the table, E refers to an anticanonical divisor on some (normalized) component of X. The map sending a boundary component in $\overline{\mathcal{P}}_2$ to a Baily–Borel boundary component (which is isomorphic to $\mathbb{P}^1$) is given by taking the j-invariant of E. The division into Type II (i.e. E smooth) cases

is discussed in Section 6. The column labeled Type III describes the generic degeneracy condition to get a Type III case (see Section 7). Note that in case 3 there are two (codimension 1) possibilities for the degenerations of E: either a nodal quartic curve in $\mathbb{P}^3$ or a union of two hyperplane sections of a quadric in $\mathbb{P}^3$.

Our approach to understanding $\overline{\mathcal{P}}_2$ is to relate this space to a GIT quotient for pairs. Specifically, we first construct a GIT quotient $\widehat{\mathcal{P}}_2$ and a natural forgetful map $\widehat{\mathcal{P}}_2 \to \widehat{\mathcal{M}}$ (see Thm. 4.1) by including the GIT analysis of Shah [Sha80] into a larger VGIT problem that takes into account the polarization divisor as well. This VGIT set-up is quite similar to that of [Laz09]. To get an idea of the set-up and of why considering divisors instead of line bundles is relevant, we recommend the reader to see first the example discussed in §4.1.

The GIT space $\widehat{\mathcal{P}}_2$ is not the same as $\overline{\mathcal{P}}_2$, but they agree over the stable locus in $\widehat{\mathcal{P}}_2$. We show $\overline{\mathcal{P}}_2$ is a flip (at least at a set-theoretic level, but likely in a VGIT sense) of $\widehat{\mathcal{P}}_2$ along the semistable locus (see Thm. 5.1). The main point in comparing the GIT and KSBA compactifications is a good understanding of the GIT boundary pairs and the results on linear systems on anticanonical pairs of Friedman [Fri83b] and Harbourne [Har97a, Har97b] (see especially Prop. 3.14).

Our paper builds on the work on $K3$ surfaces of Shah [Sha80, Sha79], Looijenga [Loo86, Loo03, Loo81], Friedman and Scattone [Fri84, FS85, Sca87], and on the work on compactifications of Kollár [Kol10], Shepherd-Barron [SB83a, SB83b], [KSB88], Alexeev [Ale96], and Hacking [Hac04]. We also note that some discussion of degenerations of degree 2 $K3$ surfaces from the perspective of the minimal model program was given recently by Thompson [Tho10] (see Thm. 2.16). The main difference from our paper is that [Tho10] never keeps track of the polarizing divisor H, and consequently it is not possible to fit the degenerations occurring in [Tho10] into a proper and separated moduli stack. We believe that one of the main contributions this paper provides for the general theory of moduli is to show concretely the importance of working with log general type: by considering polarizing divisors instead of polarizations, the boundary points are naturally separated and fit into a moduli space. The example of §4.1 clearly illustrates this point in a simple case. Related to this example, we note that the moduli space of weighted pointed curves considered by Hassett [Has03] is a one-dimensional analogue (especially for genus 1) of the moduli problem considered here. The geometric compactification for $\mathcal{A}_g$ constructed by Alexeev [Ale02] is closely related to the $K3$ case studied here, but the methods of understanding the boundary are different. Finally, Hacking–Keel–Tevelev [HKT09] is another application of the KSBA approach to compactifying moduli spaces of special classes of surfaces (del Pezzo in [HKT09]).

We close with some remarks about the general degree d case. First, a very similar analysis (involving GIT) can be carried out for other low degree cases. On the other hand, in general, the results of Section 2 establish the existence of a geometric compactification $\overline{\mathcal{P}}_d$ for the moduli space of degree d $K3$ pairs. By Hodge-theoretic considerations (see [Sha79], [KSS10], and [Usu06]), we also expect that this compactification maps to the Baily–Borel compactification (i.e. $\overline{\mathcal{P}}_d \to (\mathcal{D}/\Gamma_d)^*$). Then the results of Section 3 give a procedure for identifying the essential components (i.e. the "0-surfaces") of the central fiber in a degree d degeneration. In principle, for a given degree d, these techniques would allow one to identify the boundary components in $\overline{\mathcal{P}}_d$. However, as the de-

gree increases, the number of cases in the classification of 0-surfaces (analogue to Prop. 3.14) and the number of possible gluings of these 0-surfaces will grow very fast (roughly proportionally to the number of partitions of d), making an explicit classification unfeasible for large d. Finally, we note that the GIT approach (for small d) not only helps classify the boundary cases, but also gives a lot of structure to the fibration $\overline{\mathcal{P}}_d \to (\mathcal{D}/\Gamma_d)^*$.

We are also aware of some partial results and general approaches to the study of $\overline{\mathcal{P}}_d$ by other researchers (see e.g. [GHK15]). While we are considering only the degree two case here, our study is the first complete analysis of a geometric compactification for $K3$ pairs and one of the first in the KSBA framework for log general type surfaces (see however [HKT09]). We believe that our study is relevant to the general $\overline{\mathcal{P}}_d$ case and to the original compactification problem for $K3$ surfaces.

Organization

In Section 1, we review the standard compactifications for the moduli space of degree 2 $K3$'s and discuss the space $\widehat{\mathcal{M}}$. This material is standard, but rather scattered throughout the literature. Then, in Section 2, we introduce the KSBA compactification (based on [SB83b], [KSB88], [Ale96], and [Hac04]) and establish the existence of a modular compactification $\overline{\mathcal{P}}_d$. Next, in Section 3, we review and adapt some results on linear systems on anticanonical pairs of Friedman [Fri83b] and Harbourne [Har97a].

The actual construction of $\overline{\mathcal{P}}_2$ starts in Section 4, where we introduce the VGIT problem (generalizing [Sha80] to $K3$ pairs) and discuss the space $\widehat{\mathcal{P}}_2$. Then, in Section 5, we compare the GIT compactification $\widehat{\mathcal{P}}_2$ with the KSBA compactification $\overline{\mathcal{P}}_2$ for the moduli space of degree 2 $K3$ pairs. Finally, in Sections 6 and 7, we discuss in some detail the classification of Type II and Type III degenerations respectively. Here, we also discuss the connection to the standard compactifications (GIT, Baily–Borel, or partial toroidal) of $\mathcal{F}_2$.

1. Review of the standard compactifications of $\mathcal{F}_2$

In this section we review some facts about the moduli space $\mathcal{F}_2$ of degree 2 $K3$ surfaces and its compactifications. While all the results here are well known (see especially Shah [Sha80], Looijenga [Loo86], Friedman [Fri84], and Scattone [Sca87]), the presentation is somewhat new and adapted to the needs of the paper.

1.1. The Baily–Borel compactification

In general, the moduli space $\mathcal{F}_d$ of $K3$ surfaces of degree d is isomorphic to a locally symmetric variety $\mathcal{D}/\Gamma_d$, where $\mathcal{D}$ is a 19-dimensional Type IV domain and Γ_d is an arithmetic group acting on $\mathcal{D}$. Namely, $\mathcal{D} \cong \{\omega \in \mathbb{P}(\Lambda_d \otimes_{\mathbb{Z}} \mathbb{C}) \mid \omega.\omega = 0, \ \omega.\bar{\omega} > 0\}_0$ and Γ_d is a subgroup of finite index in $O(\Lambda_d)$, where $\Lambda_d \cong \langle -d \rangle \oplus E_8^{\oplus 2} \oplus U^{\oplus 2}$ is the primitive middle cohomology of a degree d $K3$ surface. By Baily–Borel theory, the space $\mathcal{D}/\Gamma_d$ is a quasi-projective algebraic variety and admits a projective compactification $(\mathcal{D}/\Gamma_d)^*$. For Type IV domains, the Baily–Borel compactification $(\mathcal{D}/\Gamma_d)^*$ is quite small: topologically, it is obtained by adding points (*Type III components*) and curves (*Type II components*), which are quotients of the upper half-space $\mathfrak{H}$ by modular groups.

The Baily–Borel compactifications for the moduli spaces of $K3$ surfaces were analyzed by Scattone [Sca87]. In particular, for degree 2, the following holds:

Theorem 1.1 (Scattone). *The boundary of $\mathcal{F}_2^* = (\mathcal{D}/\Gamma_2)^*$ consists of four curves* (*the closures of the Type II components*) *meeting in a single point* (*the unique Type III component*). *Furthermore, each Type II component is isomorphic to $\mathfrak{H}/\mathrm{SL}(2,\mathbb{Z})$.*

Proof. [Sca87, §6.2] and [Sca87, §5.5, esp. Fig. 5.5.7] for the second statement. $\square$

Remark 1.2. The Type II components are in one-to-one correspondence with the rank 2 isotropic sublattices E of Λ_d modulo Γ_d. Moreover, $E_{\Lambda_d}^\perp/E$ is a negative definite rank 18 lattice and a basic arithmetic invariant of E (and of the corresponding Type II component). The subroot lattice R contained in $E_{\Lambda_d}^\perp/E$ is another (coarser) arithmetic invariant. In many cases (e.g. degree 2), R uniquely determines the isometry class of E. Consequently, it is customary to label the Type II components by the root lattice R. For degree 2, the four Type II components correspond to the root lattices $2E_8 + A_1$, $E_7 + D_{10}$, $D_{16} + A_1$, and A_{17} respectively (see Fig. 2).

1.2. The GIT compactification

For low degree $K3$ surfaces (e.g. $d \leq 8$), an alternative (purely algebraic) construction for the moduli space $\mathcal{F}_d$ can be given via GIT. Additionally, GIT produces a compactification with some weak geometric meaning. Here, we review the results of Shah [Sha80] for degree 2 $K3$ surfaces. The connection to the Baily–Borel compactification is discussed in §1.4 below.

A generic $K3$ surface of degree 2 is a double cover of $\mathbb{P}^2$ branched along a plane sextic. Thus, a first approximation of the moduli space $\mathcal{F}_2$ of degree 2 $K3$ surfaces is the GIT quotient $\overline{\mathcal{M}} := \mathbb{P}H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(6)) /\!\!/ \mathrm{SL}(3)$ for plane sextics. This GIT quotient was described by Shah [Sha80, Thm. 2.4].

Theorem 1.3 (Shah). *Let $\overline{\mathcal{M}}$ be the GIT quotient of plane sextics.*

(1) *A sextic with ADE singularities is GIT stable. Thus, there exists an open subset $\mathcal{M} \subset \overline{\mathcal{M}}$ which is a coarse moduli space for sextics with ADE singularities* (*or equivalently non-unigonal degree 2 $K3$ surfaces*).

(2) *$\overline{\mathcal{M}} \setminus \mathcal{M}$ consists of seven strata* (*irreducible, locally closed, disjoint subsets*)*:*

   (Type II) *Z_1, Z_2, Z_3, Z_4 of dimensions 2, 1, 2, and 1 respectively* (*with Z_i corresponding to case* II(i) *of* [Sha80, *Thm.* 2.4])*;*
   (Type III) *τ and ζ of dimensions 1 and 0* (*cf.* III(1) *and* III(2) *of* [Sha80, *Thm.* 2.4])*;*
   (Type IV) *a point ω* (*cf.* IV *of* [Sha80, *Thm.* 2.4]).

(3) *The following is a complete list of adjacencies among the boundary strata:*

   (a) *$\zeta \in \overline{Z}_i$ for all $i \in \{1, \ldots, 4\}$;*
   (b) *$\overline{\tau} = \overline{Z}_1 \cap \overline{Z}_3$;*
   (c) *$\overline{\tau} = \tau \cup \{\zeta\} \cup \{\omega\}$*

   (*see Fig. 1*).

Remark 1.4. Each point of a boundary stratum corresponds to a unique minimal orbit. The singularities of $\overline{\mathcal{M}}$ along the boundary stratum depend on the stabilizers of these minimal orbits. For our situation, we have the following:

(i) The points parameterized by Z_3 and Z_4 are stable points. In particular, $\overline{\mathcal{M}}$ has finite quotient singularities along these strata.

(ii) The stabilizers of closed orbits parameterized by Z_1, Z_2, and τ are, up to finite index, $\mathbb{C}^*$.

(iii) The stabilizer of the closed orbit parameterized by ζ (equation $(x_0 x_1 x_2)^2$) is the standard diagonal 2-torus.

(iv) The stabilizer of the closed orbit parameterized by ω (equation $(x_0 x_2 - x_1^2)^3$) is SL(2).

In particular, note that $\overline{\mathcal{M}}$ has toric singularities everywhere except the point ω.

As noted above, the space $\mathcal{M}$ is a moduli space of curves with ADE singularities. The boundary $\overline{\mathcal{M}} \setminus \mathcal{M}$ is not strictly speaking a GIT boundary, but a boundary of non-ADE singularities. Shah has noted that except for the curves corresponding to the point ω the singularities that occur are "cohomologically insignificant" (see [Sha79]). In modern language, the cohomologically insignificant singularities are *du Bois singularities* (cf. [Ste81]). In the situation considered here, of two-dimensional hypersurfaces, these singularities are the same as the semi-log-canonical (slc) singularities of Kollár–Shepherd-Barron [KSB88] (see also [KK10] and [KSS10] for a more general discussion). Rephrasing the analysis of Shah (especially [Sha80, Thm. 3.2]) in modern language, we get the following key result:

Proposition 1.5. *Let C be a plane sextic, and X the double cover of $\mathbb{P}^2$ branched along C (not necessarily normal). Then X is slc iff C is GIT semistable and the closure of the orbit of C does not contain the orbit of the triple conic.*

Proof. Assume first that X is slc. This is equivalent to $\left(\mathbb{P}^2, \frac{1}{2}C\right)$ being a log canonical pair. Then C is GIT semistable by [KL04] and [Hac04, §10].

Conversely, assume C is GIT semistable and that its orbit closure does not contain the triple conic. By the semicontinuity of the log canonical threshold, we can assume without loss of generality that the orbit of C is closed. An inspection of the list of Shah [Sha80, Thm. 2.1] shows that the non-ADE singularities of C are either isolated singularities of type $\widetilde{E}_r$ (for $r = 7, 8$) or $T_{2,q,r}$, or non-isolated singularities that lead to normal crossings, pinch points, or degenerate cusp singularities for the double cover X. The conclusion follows (see e.g. [KSB88, Thm. 4.21]).

Finally, the triple conic gives a surface X which does not have slc singularities. It remains to see that the same is true for semistable curves C that degenerate to the triple conic. Such a curve C is of type $V((x_0 x_2 + x_1^2)^3 + f_6(x_0, x_1, x_2))$, where f_6 is a degree 6 polynomial which has negative degree with respect to the weights $(1, 0, -1)$. Passing to affine coordinates x, y around $(1, 0, 0)$ and after the change of coordinates $y' = y + x^2$, we find that C is given by

$$(y')^3 + f_6(1, x, y' - x^2) = (y')^3 + \alpha x^7 + \text{h.o.t.},$$

where the higher order terms are with respect to the weights $1/3$ and $1/7$ for y' and x respectively. If $\alpha \neq 0$, $(y')^3 + \alpha x^7$ defines a singularity of type E_{12} in Arnold's classification (cf. [AGZV85, §16.2.7$_2$]). Since this is a quasi-homogeneous singularity, the log canonical threshold does not depend on the higher order terms. We conclude that X is not log canonical. By semicontinuity, the same is true if $\alpha = 0$. $\square$

1.3. The blow-up of the point $\omega \in \overline{\mathcal{M}}$

By Mayer's Theorem, a degree 2 linear system $|H|$ on a $K3$ surface X is of one of the following types:

(NU) (Hyperelliptic case) $|H|$ is base point free, in which case X is a double cover of $\mathbb{P}^2$ branched along a plane sextic C with at worst ADE singularities.

(U) (Unigonal case) $|H|$ has a base-curve R, then $H = 2E + R$, where E is elliptic and R smooth rational. The free part of $|H|$ (i.e. $2E$) maps X to a plane conic, and gives an elliptic fibration on X. On the other hand, $|2H|$ is base point free and maps X two-to-one onto $\Sigma_4^0 \subset \mathbb{P}^5$, where Σ_4^0 is the cone over the rational normal curve in $\mathbb{P}^4$. The map $X \to \Sigma_4^0$ is ramified at the vertex and in a degree 12 curve B, which does not pass through the vertex. The curve B has at worst ADE singularities.

As discussed above, all degree 2 $K3$ surfaces of type (NU) correspond to stable points of $\overline{\mathcal{M}}$. On the other hand, all the surfaces of type (U) are mapped to the point $\omega \in \overline{\mathcal{M}}$. The blow-up $\widehat{\mathcal{M}}$ of ω will introduce all the unigonal surfaces and will give a compactification for $\mathcal{F}_2$. More precisely, we restate the main result of Shah [Sha80] as follows:

Theorem 1.6 (Shah). *The Kirwan blow-up $\widehat{\mathcal{M}}$ of the point $\omega \in \overline{\mathcal{M}}$ gives a projective compactification of the moduli space $\mathcal{F}_2$ of degree 2 $K3$ surfaces. The boundary strata of $\mathcal{F}_2 \subset \widehat{\mathcal{M}}$ are the strict transforms of the boundary strata of $\overline{\mathcal{M}}$ (cf. Thm. 1.3 and see Figs. 1 and 2). Furthermore, the boundary points of $\widehat{\mathcal{M}}$ correspond (in the sense of GIT) to degenerations of $K3$ surfaces of degree 2 that are double covers of $\mathbb{P}^2$ or Σ_4^0 and have at worst slc singularities.*

Remark 1.7. $\widehat{\mathcal{M}}$ is the blow-up of the most singular point of $\overline{\mathcal{M}}$ in the sense that $\omega \in \overline{\mathcal{M}}$ is the only point whose stabilizer is not almost abelian. It follows that $\widehat{\mathcal{M}}$ has only toric singularities. Kirwan–Lee [KL89] have constructed a full partial desingularization of $\overline{\mathcal{M}}$ (i.e. $\widehat{\mathcal{M}}$ blown-up along the strata with toric stabilizers). While this full desingularization is essential for cohomological computations on the moduli space, these extra blow-ups do not seem relevant here.

Remark 1.8. We note that the locus of unigonal $K3$ surfaces gives a divisor in $\mathcal{F}_2$. In fact, at the level of period domains $\mathcal{D}/\Gamma_2$, the unigonal $K3$ surfaces correspond to an irreducible Heegner divisor $\mathcal{H}_\infty/\Gamma_2$, where $\mathcal{H}_\infty$ is the hyperplane arrangement associated to the rank 2 lattice $\left(\begin{smallmatrix} 0 & 1 \\ 1 & -2 \end{smallmatrix}\right)$. Theorem 1.3 (combined with Mayer's result) gives the isomorphism $\mathcal{M} \cong (\mathcal{D} \setminus \mathcal{H}_\infty)/\Gamma_2$. Then, Theorem 1.6 identifies the unigonal divisor with (an open subset of) the exceptional divisor of $\widehat{\mathcal{M}} \to \overline{\mathcal{M}}$.
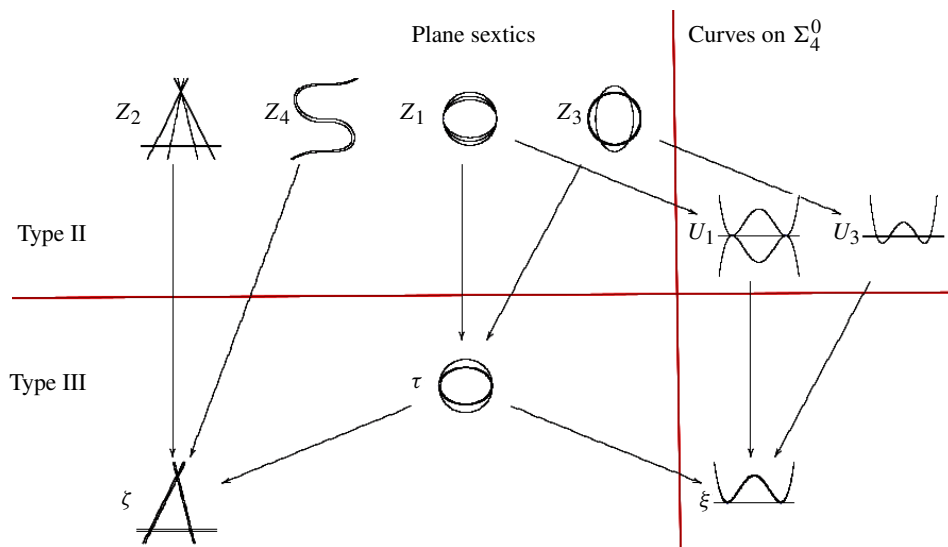
Fig. 1. The minimal orbits parameterized by $\partial\widehat{\mathcal{M}}$.

As stated above, the boundary components of $\mathcal{F}_2 \subset \widehat{\mathcal{M}}$ are the strict transforms $\widehat{Z}_i$ of the strata $\overline{Z}_i \subset \overline{\mathcal{M}}$ (i.e. closures of Z_i). Clearly, $\overline{Z}_2$ and $\overline{Z}_4$ are unaffected by the blow-up of ω. On the other hand, $\widehat{Z}_i \to \overline{Z}_i$ for $i = 1, 3$ are blow-ups of the point ω on the surfaces $\overline{Z}_i$. This introduces the exceptional divisors $\widehat{U}_i \subset \widehat{Z}_i$ (with open stratum U_i). The two exceptional divisors intersect the strict transform $\widehat{\tau}$ of $\overline{\tau}$ in a point ξ. We have the following correspondence with the strata of Shah (see also Thm. 4.12):

(i) U_1 corresponds to [Sha80, Thm. 4.3 Case 1(ii)], the minimal orbits parameterize three rational normal curves of degree 4 (hyperplane sections of Σ_4^0) tangent in two points, giving two $\widetilde{E}_8$ singularities.

(ii) U_3 corresponds to [Sha80, Thm. 4.3 Case 2(i)], the minimal orbits parameterize two rational normal curves of degree 4 meeting transversely, one of them counted with multiplicity 2. This case is in fact stable.

(iii) ξ corresponds to [Sha80, Thm. 4.3 Case 2(ii)], the minimal orbit parameterizes two rational normal curves tangent in two points, and one of them counted with multiplicity 2.

The geometry of the minimal orbits corresponding to the boundary of $\widehat{\mathcal{M}}$ is schematically summarized in Figure 1 (taken from [Loo86]).

1.4. Comparison of the GIT and Baily–Borel compactifications

As discussed above, there are two natural compactifications for the moduli space of degree 2 $K3$ surfaces: $\mathcal{F}_2 \subset \widehat{\mathcal{M}}$ (the Shah/Kirwan GIT construction) and $\mathcal{F}_2 \subset (\mathcal{D}/\Gamma_2)^*$ (the Baily–Borel compactification). Since the singularities of the surfaces corresponding to the boundary of $\widehat{\mathcal{M}}$ are slc (or "cohomologically insignificant"), Shah [Sha79, Sha80]

noted that there is a well-defined extended period map $\widehat{\mathcal{M}} \to (\mathcal{D}/\Gamma_2)^*$. A little later, Looijenga [Loo86, Loo03] gave a precise relationship between the two compactifications as summarized below.
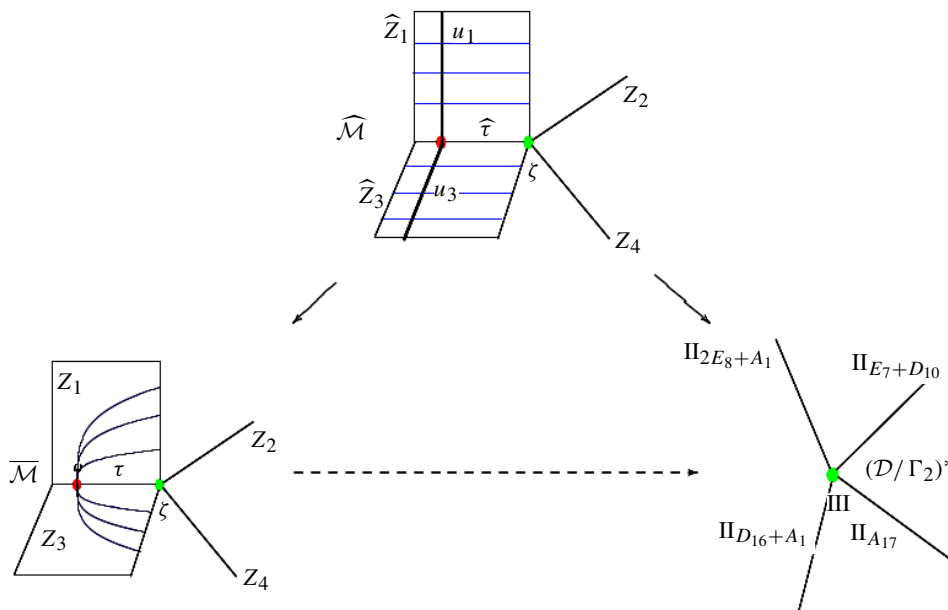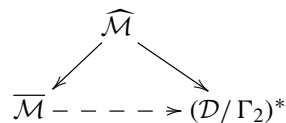

Fig. 2. The boundary strata of $\widehat{\mathcal{M}}$.

Theorem 1.9 (Looijenga). *The open embeddings $\mathcal{F}_2 \subset \widehat{\mathcal{M}}$ and $\mathcal{F}_2 \subset (\mathcal{D}/\Gamma_2)^*$ extend to a diagram* (*with regular maps*)

$$\widehat{\mathcal{M}}$$
$$\overline{\mathcal{M}} - - - - \to (\mathcal{D}/\Gamma_2)^*$$

such that

(i) $\widehat{\mathcal{M}} \to \overline{\mathcal{M}}$ *is the partial Kirwan blow-up of $\omega \in \overline{\mathcal{M}}$;*
(ii) $\widehat{\mathcal{M}} \to (\mathcal{D}/\Gamma_2)^*$ *is the Looijenga modification of the Baily–Borel compactification associated to the hyperplane arrangement $\mathcal{H}_\infty$ (see [Loo03]); more intrinsically, it is a small modification of $(\mathcal{D}/\Gamma_2)^*$ such that the closure of the Heegner divisor $\mathcal{H}_\infty/\Gamma_2$ becomes $\mathbb{Q}$-Cartier;*
(iii) *the exceptional divisor of $\widehat{\mathcal{M}} \to \overline{\mathcal{M}}$ maps to the unigonal divisor;*
(iv) *the boundary components are mapped as in Figure* 2.

Remark 1.10. Shah's results [Sha80] give a set-theoretic extension of the period map from $\widehat{\mathcal{M}}$ to $(\mathcal{D}/\Gamma_2)^*$ without matching the strata. Scattone [Sca87] has computed the Baily–Borel boundary strata. The first matching of the strata (without any extension claim

for the period map) is due to Friedman [Fri84, Rem. 5.6]. Finally, Looijenga's results [Loo86, Loo03] imply that the map $\widehat{\mathcal{M}} \to (\mathcal{D}/\Gamma_2)^*$ is analytic (and thus algebraic). Additionally, it follows that:

(i) For $i = 2, 4$: $Z_i \cong \mathfrak{H}/\mathrm{SL}(2, \mathbb{Z}) \cong \mathbb{A}^1$, the map being given by the j-invariant associated to the minimal orbits $x_0^2 f_4(x_1, x_2)$ for Z_2 and $f_3(x_0, x_1, x_2)^2$ for Z_4. The Z_4 case is stable, thus the orbits are in one-to-one correspondence with the points of Z_4, but for Z_2 many orbits degenerate to the same minimal orbit. Even in this case, the j-invariant is well defined. Namely, Z_2 parameterizes two different cases: a sextic containing a double line meeting the residual quartic in four distinct points, or a curve with $\widetilde{E}_7$ singularities; there is an obvious j-invariant in both cases.

(ii) For $i = 1, 3$, there are rational maps $\overline{Z}_i \dashrightarrow \mathbb{P}^1$, which are given by j-invariants; they are undefined at ω. After the blow-up of ω, we get regular maps $\widehat{Z}_i \to \mathbb{P}^1$ (essentially $\mathbb{P}^1$-fibrations). The fibers correspond to configurations of conics such that the j-invariant is unchanged. For example, for Z_3, fix the double conic and four points on it (this fixes the j-invariant), then the corresponding fiber of $\widehat{Z}_3 \to \mathbb{P}^1$ is the pencil of conics passing through these four points.

2. The KSBA compactification for log $K3$ surfaces

The Shah–Looijenga compactification $\widehat{\mathcal{M}}$ for $\mathcal{F}_2$ has several good properties including that the boundary points correspond (in the sense of GIT) to Gorenstein surfaces with slc singularities (higher dimensional analogues of the nodal curves). However, $\widehat{\mathcal{M}}$ is not geometric in the sense of moduli theory. In order to obtain a geometric moduli space, we need to rigidify the moduli problem and change the perspective to the so called *KSBA approach* (cf. Kollár–Shepherd-Barron [KSB88] and Alexeev [Ale96]) to compactifying moduli spaces of varieties of (log) general type. Related ideas appear frequently in the recent literature on moduli spaces of $K3$ surfaces (especially work of Gross, Hacking, Keel, and Alexeev) and abelian varieties [Ale02]. The main new content of our paper is to explicitly study this KSBA approach in the degree 2 case and to relate it to the GIT and Hodge-theoretic approaches.

The basic set-up of the KSBA approach was outlined in [KSB88] and [Ale96]. Subsequently several subtle technical difficulties were settled (see e.g. [Kol08]), giving a rather complete theoretical understanding of the dimension 2 case (a subcase of which we are studying here) (see [Kol]). Roughly speaking, any time there is a moduli space of varieties (pairs) of (log) general type, there is a natural geometric compactification given by KSBA (generalizing the Deligne–Mumford compactification of M_g). The case of pairs is more subtle, but nonetheless the situation considered here (floating coefficients for two-dimensional pairs) is well understood (see [Ale13, Ch. 1] for a recent survey).

The purpose of this section is to specialize these general results to the case of $K3$ surfaces. Specifically, we will obtain better behavior (than predicted by the general theory) for the boundary points, and an explicit control of these boundary points.

2.1. Moduli of $K3$ pairs

In order to apply the KSBA compactifying approach, we need to change the moduli problem from $K3$ surfaces to varieties of log general type. A natural solution is to consider instead of $\mathcal{F}_d$ the *moduli stack* $\mathcal{P}_d$ of pairs (X, H) consisting of $K3$ surfaces (possibly with ADE singularities) together with an ample divisor H of degree d; we call such pairs *degree d $K3$ pairs*. The two moduli functors are related by the natural forgetful map

$$\mathcal{P}_d \to \mathcal{F}_d, \quad (X, H) \mapsto (X, \mathcal{O}_X(H)),$$

which realizes $\mathcal{P}_d$ as a $\mathbb{P}^g$-fibration (with $d = 2g - 2$) over $\mathcal{F}_d$.

Proposition 2.1. *With notation as above, both $\mathcal{F}_d$ and $\mathcal{P}_d$ are smooth Deligne–Mumford stacks. Furthermore, the forgetful map $\mathcal{P}_d \to \mathcal{F}_d$ is smooth and proper with fibers isomorphic to $\mathbb{P}^g$.*

Proof. The smoothness of the moduli functor $\mathcal{F}_d$ is well known. For a big and nef divisor H on a $K3$ surface, $h^i(\mathcal{O}_X(H)) = 0$ for $i > 0$, and

$$h^0(\mathcal{O}_X(H)) = 2 + H^2/2 = p_a(H) + 1 = g + 1.$$

The smoothness of the forgetful map $\mathcal{P}_d \to \mathcal{F}_d$ follows from the fact that $H^1(\mathcal{O}_X(H)) = 0$, which implies that every section of $L := \mathcal{O}_X(H)$ extends to a first-order deformation $(\mathcal{X}, \mathcal{L})$ of (X, L) (see [Ser06, Prop. 3.3.14]; see also [Bea04, §5]). Finally, since the automorphism group (as a polarized variety) of a $K3$ surface is finite (and the characteristic is 0), it follows that $\mathcal{F}_d$ and $\mathcal{P}_d$ are Deligne–Mumford stacks. □

Remark 2.2. We note that $H^1(X, L) = 0$ for all degenerations of $K3$ surfaces considered in this paper (see Def. 2.3 below). Thus, the forgetful map $\mathrm{Def}(X, H) \to \mathrm{Def}(X, L)$ is always smooth in our situation (here H is an ample Cartier divisor and $L = \mathcal{O}_X(H)$ is the associated invertible sheaf). Specifically, Kodaira vanishing ($H^i(X, L^{-1}) = 0$ for L ample and $i = 0, 1$) holds if X is a demi-normal (i.e. X satisfies S_2 and is normal crossing in codimension 1) projective surface (see [AJ89, Thm. 3.1], and also [KSS10]). By definition, an slc variety is demi-normal. In our situation, we are also assuming X is Gorenstein with $\omega_X \cong \mathcal{O}_X$. Thus, by duality, we get $H^i(X, L) = 0$ for $i = 1, 2$. By flatness, we also get $h^0(X, L) = g + 1$.

2.2. Stable $K3$ pairs

Since the pairs (X, H) are of log general type, the KSBA theory gives a natural compactification for $\mathcal{P}_d$ by allowing degenerations that satisfy a condition on the singularities of the pair (i.e. slc singularities) and a stability condition (i.e. ampleness for the polarization). More precisely, one has some flexibility in the definition of the moduli points by allowing a coefficient for the polarizing divisor H (see [Has03] for a similar situation in dimension 1); only some choices for the coefficients give compactifications for $\mathcal{P}_d$ (see Rem. 2.6). In our situation, we want the KSBA compactification of $\mathcal{P}_d$ to be closely related to some compactifications of $\mathcal{F}_d$. Thus, we would like the choice of a divisor in a

linear system to be mostly irrelevant. This is achieved by working with the moduli space of pairs with $0 < \epsilon \ll 1$ coefficients as in Hacking [Hac04]. By adapting the general KSBA framework to our situation (see Rem. 2.5 and [Ale13, §1.5]), we define the limit objects in a compactified moduli stack $\overline{\mathcal{P}}_d$ to be *stable pairs* as follows:

Definition 2.3. Let X be a surface, H an effective divisor on X and $d = 2g - 2$ an even positive integer. We say that the pair (X, H) is a *stable K3 pair of degree d* if the following conditions are satisfied:

(1) X is Gorenstein with $\omega_X \cong \mathcal{O}_X$.
(2) H is an ample Cartier divisor.
(3) The pair $(X, \epsilon H)$ is semi-log-canonical (slc) for all small $\epsilon > 0$.
(4) There exists a flat deformation $(\mathcal{X}, \mathcal{H})/T$ of (X, H) over the germ of a smooth curve such that the general fiber (X_t, H_t) is a degree d $K3$ pair. Additionally, it is assumed that $\mathcal{H}$ is a relative effective Cartier divisor.

Remark 2.4. Clearly, $(X, \epsilon H)$ slc implies that X is slc. Conversely, if X is slc, then $(X, \epsilon H)$ being slc for small ϵ is equivalent to saying that H does not pass through a log canonical center. In our situation (X slc and Gorenstein), this means that H does not contain a component of the double locus of X and does not pass through a simple elliptic or cusp (possibly degenerate) singularity (see [KSB88, Thm. 4.21]). By working with ϵ coefficients, the singularities of the divisor H are irrelevant.

Remark 2.5. The previous definition is standard in a log general type situation with the exception of the requirements that X is Gorenstein and H Cartier. In fact, the standard requirement in the KSBA approach is that $K_X + \epsilon H$ is $\mathbb{Q}$-Cartier (see e.g. [Hac12, Def. 6.1]). If this condition holds for ϵ in an interval (or equivalently we have generic coefficients), then both K_X and H have to be $\mathbb{Q}$-Cartier and thus X is $\mathbb{Q}$-Gorenstein (see also [Ale13, §1.5.3]). Moreover, for degenerations of $K3$ surfaces, it follows easily that X has to be Gorenstein with K_X trivial (see e.g. [Hac04, Lem. 2.7] or Shepherd-Barron's Theorem 2.8). Finally, the Cartier condition is justified by the observation that in the set-up of Theorem 2.8 it follows that $\overline{\mathcal{L}} = \rho_* \mathcal{L}$ is a relatively ample Cartier divisor on $\overline{\mathcal{X}}$ (see the arguments for Theorem 2.9). In other words, for $K3$ surfaces we can assume the stronger conditions of Gorenstein (vs. $\mathbb{Q}$-Gorenstein) in (1) and Cartier (vs. $\mathbb{Q}$-Cartier) in (2) and still get a proper moduli space.

Remark 2.6. Theorem 2.8 (Shepherd-Barron) bounds the type for the polarized surface (X, L) that underlies a stable pair in the sense of Definition 2.3 (see also Thm. 2.16 for degree 2). Since H varies in a linear system ($L = \mathcal{O}_X(H)$), we also get a bounded type for (X, H). We conclude that there exists an $\epsilon_0 > 0$ (depending on d) such that for any $\epsilon \in (0, \epsilon_0)$ the stability condition (1) does not change. On the other hand, one can use other coefficients to compactify the moduli of pairs, say require $(X, \alpha H)$ to be slc for some fixed $\alpha \in (0, 1)$. If $\alpha < \epsilon_0$, we get the stable pairs of Definition 2.3. For larger α, typically the KSBA approach would modify even the interior of $\mathcal{P}_d$. For instance, Example 2.7 below shows that for $d = 2$ there exists a $K3$ surface X (with ADE singularities) and an ample Cartier divisor H such that $(X, \alpha H)$ is log canonical for no $\alpha > 1/8$ (in particular,

$\epsilon_0 \leq 1/8$ for $d = 2$). It would be interesting to determine the critical values of α (or even ϵ_0) for which the moduli problem changes. For $d = 2$, the GIT approach used in this paper can identify some of the critical values for α (see also [Laz09]). For some related discussion (for del Pezzo surfaces) from the perspective of MMP see [Che08] (N.B. even for del Pezzo's this is a delicate question, related to Tian's α-invariant).

Example 2.7. Consider the following special plane sextic: $C = L + Q$, where L is a line, Q is a quintic with an ordinary node at p, and L meets Q with multiplicity 5 at p. Then the associated double cover $\overline{X} \to \mathbb{P}^2$ will have a D_{10} singularity over p. Let $X \to \overline{X}$ be the minimal resolution, and $\pi : X \to \mathbb{P}^2$ the composite map. Let E_i be the exceptional (-2)-curves (giving a D_{10} graph), L' be the strict transform of L on X, and $H = \pi^*L$. Note that L' is also a (-2)-curve and meets only E_{10} (giving a $T_{2,3,8}$ graph; N.B. $D_{10} = T_{2,2,8}$). A simple computation shows that

$$H = \pi^*L = \sum_{k=1}^{8} kE_k + 4E_9 + 5E_{10} + 2L'.$$

We conclude that the degree 2 $K3$ pair $(X, \alpha H)$ (or equivalently $(\overline{X}, \alpha \overline{H})$) is log canonical iff $\alpha \leq 1/8$.

2.3. The limits of $K3$ pairs are stable pairs

As already mentioned, a key result that allows us to conclude that the stable pairs give a compactification for $\mathcal{P}_d$ is the following theorem of Shepherd-Barron [SB83b] (see also [KSB88] and [Kaw88]).

Theorem 2.8 (Shepherd-Barron [SB83b, Thm. 2]). *Let $\pi : \mathcal{X} \to \Delta$ be a semistable degeneration of $K3$ surfaces with $K_{\mathcal{X}} \equiv 0$ (i.e. a Kulikov degeneration). Assume $\mathcal{L} \in \mathrm{Pic}(\mathcal{X})$ is nef and $\mathcal{L}_{|X_t}$ is a polarization for all $t \in \Delta^*$. Then, for all $n \geq 4$, $\mathcal{L}^n$ is generated by $\pi_*\mathcal{L}^n$ and defines a birational morphism*

$$\rho : \mathcal{X} \to \overline{\mathcal{X}} = \mathrm{Proj}_\Delta \left(\bigoplus_n \pi_*\mathcal{L}^n \right)$$

defined over Δ such that

(a) *$\overline{\mathcal{X}}$ is Gorenstein with $K_{\overline{\mathcal{X}}} \equiv 0$;*
(b) *$\overline{\mathcal{X}}$ has canonical singularities;*
(c) *$\overline{X}_0$ is Gorenstein with slc singularities.*

As a direct application of the previous result (and [SB83b, Thm. 1]) we find that the limit of a one-parameter degeneration of $K3$ pairs can always be arranged to be a stable pair in the sense of Definition 2.3.

Theorem 2.9. *Let $(\mathcal{X}^*, \mathcal{H}^*)/\Delta^*$ be a flat family of degree d $K3$ pairs over the punctured disk. Then there exists a finite surjective base change $\Delta' \to \Delta$ and a family $(\mathcal{X}, \mathcal{H})/\Delta'$ of stable pairs extending the pull-back to Δ' of the original family $(\mathcal{X}^*, \mathcal{H}^*)$ such that $\mathcal{X}$ is Gorenstein with trivial $K_{\mathcal{X}}$ and $\mathcal{H}$ is an effective relative Cartier divisor. Furthermore, the family $(\mathcal{X}, \mathcal{H})$ is unique up to a further base change.*

Proof. Start with a one-parameter family $(\mathcal{X}^*, \mathcal{L}^*)/\Delta^*$ of polarized $K3$ surfaces, with $\mathcal{L}^* = \mathcal{O}(\mathcal{H}^*)$ with $\mathcal{H}^*$ a flat divisor. After a finite base change, one can assume a filling to a semistable family $\mathcal{X}/\Delta$ with $K_{\mathcal{X}} \equiv \mathcal{O}_{\mathcal{X}}$ (cf. Kulikov–Persson–Pinkham Theorem; alternatively this is a relative minimal model). By Shepherd-Barron [SB83b, Thm. 1], we can assume that the polarizing divisor extends to an effective relative Cartier divisor $\mathcal{H}$, which can then be assumed to be also nef. Let $\mathcal{L} = \mathcal{O}_{\mathcal{X}}(\mathcal{H})$. As before, we denote by X_t, L_t, and H_t the fiber over t (with $t = 0$ corresponding to the central fiber), the polarization of X_t, and the corresponding divisor respectively.

We consider the associated relative log canonical model $\overline{\mathcal{X}} = \mathrm{Proj}_\Delta(\bigoplus_n \pi_* \mathcal{L}^n)$ with associated $\overline{\mathcal{L}}$ and $\overline{\mathcal{H}}$. We note the following properties:

(1) $\overline{\mathcal{X}}$ *is a family of slc surfaces with* $K_{\overline{\mathcal{X}}} \equiv 0$. This is the second part of Theorem 2.8 (items a–c). In fact, this follows from the general KSBA framework, except for the Gorenstein property (a priori only $\mathbb{Q}$-Gorenstein) which is specific to $K3$'s.

(2) $\overline{\mathcal{L}}$ *is a relatively ample line bundle on* $\overline{\mathcal{X}}/\Delta$. This follows from Shepherd-Barron's Theorem 2.8: for $n \geq 4$, $\mathcal{L}^n$ defines a birational morphism $\mathcal{X} \to \overline{\mathcal{X}}$ and thus, by definition, $\overline{\mathcal{L}}^n$ is a relatively very ample line bundle for $\overline{\mathcal{X}}/\Delta$. Since this holds for all $n \geq 4$, it follows that $\overline{\mathcal{L}}$ is a relatively ample line bundle (and not only an orbifold-line bundle). Thus, we also get:

(2′) $\overline{\mathcal{H}}$ *is an effective relative Cartier divisor on* $\overline{\mathcal{X}}/\Delta$. As noted elsewhere, this property is specific for $K3$'s (due to Thm. 2.8); a priori $\overline{\mathcal{H}}$ is only $\mathbb{Q}$-Cartier.

Note that $\overline{\mathcal{X}}$ depends only on the choice of the line bundle $\mathcal{L}$, and not on the choice of the divisor $\mathcal{H}$. The choice of divisor is essential in order to obtain a separated moduli space (see e.g. Rem. 2.11). When taking into account the divisor $\mathcal{H}$ we distinguish two cases based on the following condition being satisfied or not:

($\star$) H_0 *does not contain any double curve or triple point of* X_0,

or equivalently, $(X_0, \epsilon H_0)$ is slc. If the condition ($\star$) holds, we find that

(3) $(\overline{X}_0, \epsilon \overline{H}_0)$ *is an slc pair.* From Theorem 2.8 above (or more generally from standard KSBA considerations), $\overline{X}_0$ is slc. Thus, we only need to see that H_0 does not pass through a log canonical center of $\overline{X}_0$, which in turn is guaranteed by ($\star$). Namely, by construction $\overline{X}_0$ is obtained from the normal crossing variety X_0 by contracting curves (and components) orthogonal to the polarization (cf. [SB83b]). It is clear that it suffices to consider the so-called 0-components (V_i, D_i) of X_0 in the terminology of [SB83b] (i.e. components that are not contracted, or equivalently those on which the polarization is big, and thus semiample). Denote by H_i the restriction of H_0 to V_i. The condition ($\star$) says that $(V_i, D_i + \epsilon H_i)$ is dlt, and then $K_{V_i} + (D_i + \epsilon H_i) = \epsilon H_i$ is big (since V_i is a 0-component). We conclude $(\overline{V}_i, \overline{D}_i + \epsilon \overline{H}_i)$ is log canonical (see e.g. [KM98, Thm. 7.10]), which in turn implies that $(\overline{X}_0, \epsilon \overline{H}_0)$ is slc.

In conclusion, if ($\star$) holds, conditions (1–4) of Definition 2.3 hold, and thus $(\overline{X}_0, \epsilon \overline{H}_0)$ is the KSBA limit for the family $(\mathcal{X}^*, \mathcal{H}^*)/\Delta^*$.

If the condition ($\star$) fails, we replace the semistable model $(\mathcal{X}, \mathcal{H})/\Delta$ by a model $(\mathcal{X}', \mathcal{H}'')/\Delta$ which satisfies ($\star$) and then proceed as before. The basic idea is to blow up

the double curves and triple points of X_0 which are contained in H_0 (and thus reduce their multiplicity in H_0). The process to achieve this is as follows: After a base change the new semistable (Kulikov) model $\mathcal{X}'/\Delta'$ will replace each double curve and triple point by a configuration of surfaces (e.g. a double curve in a Type II degeneration will be replaced by a chain of surfaces). On the components of X_0' corresponding to double loci for which $(\star)$ holds, the pull-back polarization will be either trivial or give a fibration, and thus they will be 1- or 2-surfaces, and will be contracted back. Thus, the only relevant components V_i' are those coming from double curves (and triple points) contained in H_0. On such V_i', the pull-back polarization H_0' vanishes identically (and thus $\mathcal{H}'$ will not be flat). By applying twists $\mathcal{H}'' := \mathcal{H}' \otimes \mathcal{O}(V_i')$ we achieve flatness. Note that the effect of twists is to decrease the multiplicity of a double curve (and similarly for a triple point; N.B. a triple point is effectively replaced by double curves after a base change) in H_0''. Thus, after possibly further (finitely many) base changes, we can assume that $\mathcal{H}''$ satisfies $(\star)$. At this point, after elementary modifications, we can assume $\mathcal{H}''$ is a polarization. These two conditions (nef and flat) can always be achieved (by the process described above), as proved by Shepherd-Barron [SB83b, Thm. 1]. Finally, it is easily checked that the elementary modifications preserve the condition $(\star)$, i.e. if $(\star)$ holds, it will hold after the elementary modifications necessary to make $\mathcal{H}''$ nef. This is clear for Type 0 and 1 modifications (those do not modify the incidence between the polarization and double curves). On the other hand, Type 2 modifications are not allowed, as the only curves that we need to flip are the components of H_0'' (but those by the assumption $(\star)$ are not components of the double locus). $\qquad\square$

Remark 2.10. The case of Type II degenerations is very similar to that of curves. Namely, let $\mathcal{C}/\Delta$ be a semistable family of curves, and $\mathcal{H} \subset \mathcal{C}$ be a flat divisor. The condition $(\star)$ is asking that $\mathcal{H}$ does not pass through any of the nodes of the central fiber C_0. The process described in the proof above is the usual way of achieving this via a base change (see [HM98, Prop. 3.49 (and related)] for further discussion).

Remark 2.11. The results of Shepherd-Barron cited above established the existence of reasonable limits for degenerations of polarized $K3$ surfaces. In some sense, the Shah–Looijenga compactification $\widehat{\mathcal{M}}$ is a reflection of this fact. However, in the absence of a polarizing divisor, the limiting surfaces will not be separated in moduli. For example, the limiting surfaces might not have finite stabilizer, e.g. the standard tetrahedron in $\mathbb{P}^3$ is stabilized by a torus, leading to collapsing of orbits. The presence of a divisor giving a slc log general type pair eliminates such pathologies. In other words, the choice of a divisor (vs. line bundle) is essential in separating the boundary points and fitting everything together in a compact moduli space. The example discussed in §4.1 is a clear illustration of this point.

2.4. The moduli of stable pairs

It is well established now that there exists a good moduli functor (giving a proper Deligne–Mumford stack) for surface pairs with floating coefficients (see [Ale13, §1.5.3] and the references within). There are several differences from the standard KSBA moduli space

for pairs: we restrict to smoothable pairs, and then we assume Gorenstein and Cartier. The smoothability condition means that we restrict to the main component of the (coarse) moduli space. It is known that there are several issues in trying to define a functorial meaning for this main component (especially regarding the scheme structure at the boundary). We have nothing to add to this (see however Rem. 2.15)—for us $\overline{\mathcal{P}}_d$ will be a coarse moduli space with the reduced scheme structure. On the other hand, the restriction to Gorenstein and Cartier would give (a priori) an open subset of this main component, but for *K*3's, Theorem 2.9 says that this subset is everything. In other words, we obtain:

Corollary 2.12. *The coarse moduli space $\overline{\mathcal{P}}_d$ of stable pairs in the sense of Definition 2.3 is a geometric[1] compactification* (*proper algebraic space*) *of the moduli space of degree d K3 pairs.*

Proof. First, we note that the moduli functor for stable *K*3 pairs is bounded. This follows from Theorem 2.8 (see also Rem. 2.6), which in turn is based on very effective results on linear systems on anticanonical pairs (see Section 3). More generally, strong boundedness results for surfaces are known, due to Alexeev [Ale94].

Thus, one obtains a parameter space U for stable pairs (X, H) as a subset of the product of two Hilbert schemes $\mathrm{Hilb}_X \times \mathrm{Hilb}_H$ (where Hilb_Y denotes the appropriate Hilbert scheme parameterizing flat deformations of $Y \hookrightarrow \mathbb{P}^N$, for fixed $N \gg 0$). As a consequence of Kollár's results [Kol08], at least for KSBA stable surfaces or stable surface pairs (see e.g. [HK04] for a recent discussion), it follows that U is a locally closed subset (see Appendix to [HKT09] for details). Note that requiring X Gorenstein and H Cartier are locally closed conditions. We deduce that the quotient stack $\overline{\mathcal{P}}_d := [U/\mathrm{PGL}(N+1)]$ (with N fixed as before) is an algebraic stack of finite type.

Theorem 2.9 establishes that the main component $\overline{\mathcal{P}}_d$ of the moduli of stable pairs is separated and complete (via the usual valuative criterion arguments).

Finally, the fact that $\overline{\mathcal{P}}_d$ is a Deligne–Mumford stack follows from the standard statement that pairs of log general type have finite automorphisms (see e.g. [KSB88, p. 328]). Similarly, the existence of a coarse moduli space follows from [KM97]. $\qquad\square$

As mentioned in the remarks below, it is quite likely that further structural results on $\overline{\mathcal{P}}_d$ can be established (see also the series of papers of Gross–Hacking–Keel, e.g. [GHK15]), but this goes beyond the scope of our paper. Namely, our goal here is to explicitly construct (the coarse moduli space associated to) $\overline{\mathcal{P}}_2$ and to relate this to the GIT, Hodge theory, and KSBA points of view.

Remark 2.13. In general, we do not expect that $\overline{\mathcal{P}}_d$ is a smooth stack: as noted in Remark 2.2 the local structure of $\overline{\mathcal{P}}_d$ near (X, H) is controlled by the deformations of X as a polarized variety. Hacking and Keel informed us of some examples of degenerate *K*3 surfaces X (satisfying the assumptions of Definition 2.3) for which the local deformation space is singular. Such an example would show that $\overline{\mathcal{P}}_d$ is not smooth in general (for large d).

---

[1] See the discussion preceding the corollary and the first Remark of the introduction. Namely, every point of $\overline{\mathcal{P}}_d$ corresponds to a unique stable pair, but we do not know of a functor that picks up only the smoothable component of the moduli space of KSBA pairs.

Remark 2.14. On the other hand, it is likely that projectivity results for $\overline{\mathcal{P}}_d$ follow from the techniques of Kollár [Kol90] and the improvements of Fujino [Fuj12]. More precisely, Fujino has proved appropriate semipositivity results (e.g. [Fuj12, Theorems 1.12 and 1.13]) for moduli spaces of pairs. However, to our knowledge, Kollár's Ampleness Lemma [Kol90, 3.9] is not yet established in the case of pairs. It is possible that special arguments might give a version of the ampleness lemma in our situation and thus establish the projectivity of the coarse moduli space associated to $\overline{\mathcal{P}}_d$. Alternatively, more in the spirit of this paper, the quasi-projectivity of $\mathcal{P}_d$ might follow from GIT and the techniques of Viehweg [Vie95, Vie10].

Remark 2.15. We make some further comments on the deformation space of degenerate $K3$ surfaces X as considered in this paper. If X is a semistable Type III $K3$ (say with an ample polarization to fit the framework considered here), then $\mathrm{Def}(X)$ is very well behaved (cf. Friedman [Fri83a]): it is the union of two 19-dimensional strata[2] meeting transversely; one is the smoothing component (thus relevant for compactifying the moduli of $K3$'s), the other one is parameterizing non-smoothable Type III $K3$'s of the same combinatorial type as X. Furthermore, using the results of [KN94] (i.e. the smoothing direction corresponds to deformations of log structures), it might be possible to define a moduli functor that picks up only the smoothing component. On the other hand, if X is not semistable (it is obtained from a semistable model by contracting the curves orthogonal to the polarization), $\mathrm{Def}(X)$ is much less understood, and the known cases are fairly subtle. One case that was studied quite intensively (e.g. in the context of Looijenga's conjecture) is that of *pillow surfaces*: X is a union of $\mathbb{P}^2$'s (polarized by $\mathcal{O}(1)$) glued along triangles (i.e. triples of non-concurrent lines) as in a triangulation of S^2. Such surfaces naturally occur in the degeneration context, e.g. the "tetrahedron" in $\mathbb{P}^3$ (a degree 4 pillow) is a degeneration of quartic $K3$'s. The work of Gross–Hacking–Keel (e.g. [GHK15] and subsequent work) has deep and highly structural results on the deformations of pillow surfaces, but this is quite involved and not explicit. Our analysis covers the case of degree 2 pillows (see Remark 7.5) and gives a taste of the complexity of the situation.

2.5. Stable pairs in degree two

For degree 2 $K3$ surfaces, the possible central fibers $\overline{X}_0$ of relative log canonical models were identified by Thompson [Tho10]. Some discussion of (the components of) $\overline{X}_0$ for any d is given in the next section.

Theorem 2.16 ([Tho10, Thm. 1.1]). *Let $\mathcal{X}/\Delta$ be a Kulikov degeneration of $K3$ surfaces. Let $\mathcal{H}$ be a divisor on $\mathcal{X}$ that is effective, nef and flat over Δ. Suppose that $\mathcal{H}$ induces a polarization of degree two on the generic fiber X_t. Then the morphism $\phi : \mathcal{X} \to \overline{\mathcal{X}}$ taking $\mathcal{X}$ to the relative log canonical model of the pair $(\mathcal{X}, \epsilon\mathcal{H})$ maps the central fiber X_0 to a complete intersection of the following type:*

$$\overline{X}_0 = \{z^2 - f_6(x_i, y) = f_2(x_i, y) = 0\} \subset \mathbb{P}(1, 1, 1, 2, 3).$$

---

[2] In the polarized case (the unpolarized case is similar).

Remark 2.17. Thompson [Tho10] does not consider the polarizing divisor as part of the data so that it is not possible to fit the degenerations in a moduli space. In fact, no attempt of constructing a moduli space is made in [Tho10]. As explained, keeping track of the polarizing divisor allows us to construct a modular compactification for pairs. Also, even if one is only interested in $K3$ surfaces, by considering pairs one has a better understanding of how the various points of view—GIT [Sha80], Hodge-theoretic [Fri84], [FS85], or abstract MMP [Tho10]—interact (see Sections 6 and 7 for some concrete examples).

3. Classification of polarized anticanonical pairs in degree two

In order to understand the possible boundary points of $\overline{\mathcal{P}}_d$, we need to understand the possible central fibers $\overline{X}_0$ of relative log canonical models as in the previous section. We recall that $\overline{X}_0$ is a contraction of the central fiber X_0 of a Kulikov model. Then, depending on the index of nilpotency of the monodromy, the normal crossing variety $X_0 = \bigcup V_i$ is (see [FM83b, p. 11]) either

- of *Type II*, i.e. a chain of surfaces glued along elliptic curves with rational ends and elliptic ruled surfaces in the middle; or
- of *Type III*, rational surfaces such that the dual graph gives a triangulation of S^2, the double curves on each V_i form a cycle of rational curves, which is an anticanonical divisor on V_i.

Note also that $\overline{X}_0$ depends only on the polarized semistable model (X_0, L_0) and not on the degenerating family $(\mathcal{X}, \mathcal{L})$ (see [SB83b, Lem. 2.17], [Tho10, Lem. 4.1]). In fact, the analysis of Shepherd-Barron [SB83b] says that $\overline{X}_0$ can be essentially recovered from the 0-*surfaces* (V_i, L_i) in X_0 (with $L_i = L_{0|V_i}$), i.e. the components of X_0 that are mapped birationally onto the image (see [SB83b, Def. on p. 145]).

Thus, to understand the boundary points in $\overline{\mathcal{P}}_d$, it is essential to classify the possible 0-surfaces that can occur in degree d. Note that on a 0-surface V_i the polarization L_i is big and nef. Also, the degrees of the polarizations on all 0-surfaces of a polarized semistable X_0 satisfy $\sum (L_i)^2 = d$. Thus, we need to classify triples $(V, D; L)$, where (V, D) is an anticanonical pair and L is a big and nef divisor class with $1 \leq L^2 \leq d$. To fix the notation and terminology, we define the following:

Definition 3.1. A *polarized anticanonical surface* is a triple $(V, D; L)$ where

(i) V is a rational surface;
(ii) $D \in |-K_X|$ is a reduced anticanonical divisor;
(iii) $L \in \mathrm{Pic}(V)$ is a big and nef divisor class.

We say $(V, D; L)$ is *relatively minimal* if any (-1)-curve E on V satisfies $L.E > 0$. Additionally, we will be mostly concerned with the case where D is at worst nodal, in which case we say (V, D) is of *Type II* or *Type III* if D is a smooth (elliptic) curve or D is a cycle of rational curves respectively.

Remark 3.2. Any anticanonical pair (V, D) can be obtained by a series of blow-ups of a minimal anticanonical pair (a classification of such is [FM83a, Lem. 3.2]). Specifically, given an anticanonical pair (V', D'), the blow-up of a point $p \in D'$ gives another anticanonical pair $(V, D) \to (V', D')$, where $D = \pi^*D - E$. If p is a node of D' we say that such a blow-up is *toric*; if p is smooth on D', we call it *non-toric*. Consider a blow-up $\pi : (V, D) \to (V', D')$ of anticanonical pairs. Let L' be a big and nef divisor on (V', D') and $L = \pi^*L'$. Clearly, L is still big and nef and the following hold: $L^2 = (L')^2$, $L.D = L'.D'$, and $D^2 = (D')^2 - 1$.

The previous remark makes it clear that for a meaningful classification of the polarized anticanonical surfaces $(V, D; L)$ it is necessary to assume that they are relatively minimal. We note that the relative minimality is a purely numerical condition. Thus, if needed, we can assume that $(V, D; L)$ is relatively minimal (see also [Har97b, Lem. 2.12(a)]). More precisely, a standard application of Riemann–Roch and Hodge index shows that a class E with $E^2 = -1$, $E.D = 1$, and $E.L = 0$ is effective and contains a (-1)-curve (orthogonal to L) as component. After successive contractions of (-1)-curves orthogonal to the polarization, we obtain a relatively minimal surface $(V', D'; L')$ such that $\pi : (V, D) \to (V', D')$ is a composition of blow-ups as in the previous remark and $L = \pi^*L'$.

3.1. Basic remarks on polarized anticanonical surfaces

A nef divisor on an anticanonical pair is always effective (see e.g. [Har97b, Cor. 2.3]), and in many situations it is easy to compute the dimension of the corresponding linear system.

Proposition 3.3 ([Fri83b, Lem. 5], [Har97a, Thm. I.1]). *Let (V, D) be an anticanonical pair and L be a nef divisor.*

(a) *If $D.L > 0$, then $h^1(L) = 0$. Thus,*

$$h^0(L) = (L^2 + L.D)/2 + 1.$$

(b) *If $D.L = 0$ and $|L|$ contains a reduced connected member, then $h^1(L) = 1$. Thus,*

$$h^0(L) = (L^2 + L.D)/2 + 2.$$

As will see, in many cases it is possible to classify the polarized anticanonical surfaces $(V, D; L)$ based on the basic numerical invariants L^2, $L.D$, and D^2. As discussed, for degenerations of $K3$ surfaces occurring in degree d, we have $1 < L^2 \leq d$. The following lemmas establish some bounds for $L.D$ and D^2 in terms of L^2.

Lemma 3.4. *Let $(V, D; L)$ be a polarized anticanonical surface. Then*

(i) $L.D \equiv L^2 \bmod 2$;
(ii) $0 \leq L.D \leq L^2 + 2$.

Proof. The first part follows from the fact that the orthogonal complement in $\mathrm{Pic}(V)$ of the canonical class $K_V \ (= -D)$ is an even lattice.

For $L.D \geq 3$ the linear system L is base point free and defines a birational map (see e.g. [Har97b, Prop. 3.2]). Thus, to prove (ii), without loss of generality we can assume the general member of L is reduced and irreducible. Then $2p_a(L) - 2 = L^2 - L.D \geq -2$.   □

To control D^2, we distinguish two cases: either $L.D \leq L^2$ or $L.D = L^2 + 2$. To handle the first case the key observation is that it is possible to *twist the polarization*, i.e. replace L by $L - D$ (corresponding to replacing $\mathcal{L}$ by $\mathcal{L} \otimes \mathcal{O}_{\mathcal{X}}(V)$ for a family $(\mathcal{X}, \mathcal{L})/\Delta$ of polarized $K3$'s, with V a component of the central fiber X_0).

Lemma 3.5. *Let $(V, D; L)$ be a polarized anticanonical surface. Then $L - D$ is effective iff $L.D \leq L^2$.*

Proof. Note that $L_0 := L - D = L + K_V$ is an adjoint linear system with L big and nef. Thus, by Kodaira–Mumford vanishing, $h^i(L_0) = 0$. We conclude that $h^0(L_0) = 1 + \frac{1}{2}(L - D)L$; the claim follows.   □

Lemma 3.6. *Let $(V, D; L)$ be a polarized anticanonical surface. Assume additionally that $(V, D; L)$ is relatively minimal and $L.D \leq L^2$. Then*

(i) *$L - D$ is nef;*
(ii) *$2L.D - L^2 \leq D^2 \leq L.D$, and the right inequality is strict unless $L \sim D$.*

Proof. The first part is precisely [Har97a, Lem. III.9(c)] (use the fact that $L - D$ is effective by Lemma 3.5). Since $L - D$ is nef, we get $(L - D)^2 \geq 0$, which gives the first inequality above. The second follows from Hodge index: $D^2 \leq (L.D)^2/L^2 \ (\leq L.D)$.   □

In particular, we note the following classification result:

Corollary 3.7. *Let $(V, D; L)$ be a relatively minimal polarized anticanonical surface. Assume that $L.D = L^2$. Then V is a del Pezzo surface and $L \sim D$.*

Proof. From Lemma 3.6, we get

$$L^2 = 2L.D - L^2 \leq D^2 \leq L.D = L^2.$$

Thus, $D^2 = L.D = L^2$. From Hodge index applied to the classes L and D, we conclude $L \sim D$. It follows that V is a rational surface with a big and nef anticanonical divisor, thus a del Pezzo (possibly with ADE singularities).   □

It remains to consider the case $L.D = L^2 + 2$. Again, a classification is readily available.

Proposition 3.8. *Let $(V, D; L)$ be a polarized anticanonical surface. Assume additionally that $(V, D; L)$ is relatively minimal and $L.D = L^2 + 2$. Then either*

(i) *$V \cong \mathbb{P}^2$ with polarization $L = \ell$ or 2ℓ (where ℓ is the class of a line); or*
(ii) *(V, L) is the rational normal scroll, i.e. $V \cong \mathbb{F}_n$ and $L = \sigma + (n + k)f$ for some $k \geq 0$ (where σ is the class of the negative section, and f is the class of a fiber).*

Moreover, the pairs (V, D) with $V \cong \mathbb{P}^2$ or $\mathbb{F}_n$ are classified by [FM83a, Lem. 3.2].

Proof. Let $L^2 = n \geq 1$. Since $L.D = n + 2 \geq 3$, from [Har97b, Prop. 3.2] it follows that L is base point free defining a birational morphism from V to a normal surface $V' \subset \mathbb{P}^{n+1}$ (cf. Prop. 3.3(i)) of degree n. It follows that V' is a surface of minimal degree (see [GH94, p. 525]) and thus it is either the Veronese surface or the rational normal scroll (i.e. $\mathbb{F}_n$ embedded by $\sigma + (n + k)f$; the case $n = 1$, $k = 0$ gives $(\mathbb{P}^2, \ell)$). Finally, note that the morphism $V \to V'$ contracts the curves orthogonal to L and those curves are not (-1)-curves. The proposition follows.   □

Remark 3.9. Note $D^2 = K_V^2 \leq 9$ for all rational surfaces V. In fact, $b_2(V) = 10 - D^2$, and then D^2 is 9 or 8 only for $\mathbb{P}^2$ or $\mathbb{F}_n$ respectively. For Type III anticanonical pairs, one also considers the length $r(D)$ of the anticanonical cycle, and *the charge* $q(V, D) :=$ $12 - D^2 - r(D)$ (see e.g. [FM83a, §3]). Roughly, $9 - D^2$, $r(D) - 3$, and $q(D)$ count the total number of blow-ups, the number of toric blow-ups, and the number of non-toric blow-ups respectively. If (V, D) is a component of a Type III degeneration of $K3$ surfaces, then $0 \leq q(V, D) \leq 24$ (see e.g. [FM83a, §3]).

3.2. Linear systems on anticanonical surfaces

We now recall some results on the behavior of linear systems on anticanonical pairs analogous to Mayer's Theorem for $K3$ surfaces. Results on this topic were first obtained by Friedman [Fri83b], and then strengthened by Harbourne [Har97a, Har97b]. The following holds:

Theorem 3.10 (Harbourne [Har97b, Cor. 1.1]). *Let $(V, D; L)$ be a polarized anticanonical surface. Then $|3L|$ always defines a birational morphism from V onto the normal surface obtained by contracting all curves C on V orthogonal to L.*

Similar to $K3$'s, we have the following results on base loci of linear systems on anticanonical surfaces. For clarity, we separate the cases $L.D > 0$ and $L.D = 0$.

Theorem 3.11 (Friedman, Harbourne). *Let $(V, D; L)$ be a polarized anticanonical surface. Assume that $(V, D; L)$ is relatively minimal and $L.D > 0$.*

 (i) *If $L.D \geq 2$, then $|L|$ is base point free. Furthermore, if $L.D \geq 3$, then $|L|$ defines a birational morphism onto a normal surface.*
 (ii) *If $L.D = 1$ and L has no fixed component, then $|L|$ has a unique base point, which is on D.*
(iii) *If $L.D = 1$, then $|L|$ has a fixed component iff*

$$L = kE + R \quad \text{for some } k \geq 2,$$

where $E^2 = 0$, $E.D = 0$, $E.R = 1$, and R is a (-1)-curve.

Proof. The first two items follow directly from [Har97a, Thm. III.1(a, b)] and [Har97b, Prop. 3.2] (see also [Fri83b, Thm. 10]). The last statement follows also from [Har97a, Thm. III.1] after contracting the (-1)-curves orthogonal to L.   □

Theorem 3.12 (Friedman, Harbourne). *Let $(V, D; L)$ be a polarized anticanonical surface. Assume $L.D = 0$. Then either*

(i) *L has no fixed component, then L is base point free, $L \otimes \mathcal{O}_D$ is trivial, and $h^1(V, L) = 1$; or*

(ii) *the fixed part of L is a (-2)-curve R, and then*

$$L = kE + R \quad \text{for some } k \geq 2$$

with $E^2 = E.D = 0$, $E.R = 1$, and $R \otimes \mathcal{O}_D$ trivial; or

(iii) *$L \otimes D$ is non-trivial, which is equivalent to $F + K_V$ being an effective divisor, where F is the fixed part of L. In this situation, there exists a birational morphism $\pi : (V, D) \to (V', D')$ of anticanonical pairs with $(D')^2 < 0$ and such that $L = \pi^*(L' + D')$ for some nef divisor L' on V'.*

Proof. This is precisely [Har97a, Thm. III.1(c, d)] assuming L big. □

The items Thm. 3.11(iii) and Thm. 3.12(ii) correspond precisely to the unigonal case of Mayer's Theorem. Also, since we are considering only slc pairs, we can assume (if necessary) that L does not contain D as a fixed component.

Remark 3.13. The key fact that allows Harbourne [Har97a, Har97b] to strengthen the results of Friedman [Fri83b] is a precise control of Friedman's condition: *L has no fixed component which is also a component of the anticanonical cycle*. Namely, [Har97a, Cor. III.3] says: *If L is a nef divisor on an anticanonical pair (V, D), then either no fixed component of L is a component of any section of $-K_V$, or the fixed part of L contains an anticanonical divisor.* The latter situation can only occur if $L.D = 0$, but $L_{|D} \not\cong \mathcal{O}_D$ (see Thm. 3.12(iii) above).

3.3. The degree 2 case

We now restrict to the case where $(V, D; L)$ is a 0-surface in a degeneration of degree 2 $K3$ surfaces. The above discussion leads to the following simple classification of the possibilities.

Proposition 3.14. *Let $(V, D; L)$ be a relatively minimal polarized anticanonical surface with $L^2 \leq 2$. Then one of the following six cases holds:*

(A) *If $L^2 = 1$*

    (1) *and $L.D = 3$, then $V \cong \mathbb{P}^2$, $L \sim \ell$ (where ℓ is the class of a line);*

    (2) *and $L.D = 1$, then V is a degree 1 del Pezzo and $L \sim D \sim -K_V$.*

(B) *If $L^2 = 2$*

    (3) *and $L.D = 4$, then V is an irreducible reduced quadric in $\mathbb{P}^3$ and L the class of a hyperplane section;*

    (4) *and $L.D = 2$, then V is a degree 2 del Pezzo with $L \sim D \sim -K_V$;*

(5) *and $L.D = 0$ and $D^2 = -1$, then (V, D) is the resolution of a rational surface which has a unique non-ADE singularity, which is either a simple elliptic singularity of type $\widetilde{E}_8$, or a cusp singularity of type $T_{2,3,r}$ (with $7 \le r \le 16$);*
(6) *and $L.D = 0$ and $D^2 = -2$, then (V, D) is the resolution of a rational surface which has a unique non-ADE singularity, which is either a simple elliptic singularity of type $\widetilde{E}_7$, or a cusp singularity of type $T_{2,q,r}$ (with $q \ge 4$, $r \ge 5$, $q + r \le 19$).*

Proof. The possible values for $L.D$ and D^2 are determined by Lemmas 3.4 and 3.6. The first four items follow from Corollary 3.7 and Proposition 3.8. The statement about the type of singularities for cases (5) and (6) is standard. Finally, for the bounds on q and r, we note that the charge associated to a cusp lying on a rational surface is at most 21 (cf. [FM83a, Lem. 4.6]). For $T_{p,q,r}$ singularities the associated charge is $q(V, D) = p + q + r$. Thus, $q + r \le 19$ (or $r \le 16$). □

Remark 3.15. A precise analysis of the cusp singularities $T_{2,q,r}$ occurring in degree 2 can be made using [Wal99, §5] and [Wal99, §6] for $T_{2,3,r}$ and $T_{2,q,r}$ respectively.

For Type II degenerations, one might have to consider elliptic ruled components as 0-surfaces. Here, we note that, at least for degree 2, the elliptic ruled components can be viewed as degenerations of rational anticanonical surfaces.

Lemma 3.16. *Let $(V, D', D''; L)$ be a polarized anticanonical triple (i.e. V is elliptic ruled and D', D'' are sections with $D' + D'' \in |-K_V|$). Assume V is relatively minimal and $L^2 \in \{1, 2\}$. Furthermore, assume L has no fixed common component with $D' \cup D''$. Then one of the components, say D'', satisfies $-L^2 \le (D'')^2 < 0$ and $D''.L = 0$ and thus it can be contracted to an $\widetilde{E}_r$ ($r \in \{7, 8\}$) singularity at some point p on a normal surface $\overline{V}$. Moreover, there is a partial smoothing $(\mathcal{V}, \mathcal{D}, \mathcal{L})$ of p such that the central fiber is $(\overline{V}, D', L)$, while the general fiber $(V_t, D_t; L_t)$ is a polarized rational anticanonical surface.*

Proof. By [Tho10, Lem. 4.6 and pp. 23–24], we have a precise control of the surfaces $(V, D', D''; L)$ that can occur. The claim can be checked explicitly. For example, in the case of $\widetilde{E}_7$ non-unigonal, V is a double cover of $\mathbb{P}^2$ branched along the sextic $x_0^2 f_4(x_1, x_2)$, and a partial smoothing is given by $V(z^2 - x_0^2 F_4(x_0, x_1, t \cdot x_2)) \to \mathbb{A}_t^1$ for some homogeneous degree 4 polynomial F_4 with $F_4(x_0, x_1, 0) = f_4(x_0, x_1)$. □

4. A GIT construction for the moduli for pairs

In Section 2 we have shown that the moduli space of degree 2 $K3$ pairs has a geometric compactification $\overline{\mathcal{P}}_2$. While a rough classification of the degenerate degree 2 pairs is given by Proposition 3.14, a full classification of the geometric objects parameterized by the boundary of $\overline{\mathcal{P}}_2$ seems difficult to obtain by direct considerations. Instead, we study $\overline{\mathcal{P}}_2$ by using a related GIT space $\widehat{\mathcal{P}}_2$.

Namely, $\widehat{\mathcal{P}}_2$ is constructed by enhancing the GIT analysis of Shah (giving $\widehat{\mathcal{M}}$) to take into account a hyperplane section. This construction is closely related to that of [Laz09].

The main point here is that there is a choice of linearization involved in the construction of a GIT quotient for pairs, giving in fact a family of quotients $\widehat{\mathcal{P}}_2(\alpha)$ for $\alpha \in \mathbb{Q}_+$. As a limiting case, $\widehat{\mathcal{P}}_2(0)$ is still defined and $\widehat{\mathcal{P}}_2(0) \cong \widehat{\mathcal{M}}$. By general considerations from VGIT, one gets a natural forgetful map $\widehat{\mathcal{P}}_2(\epsilon) \to \widehat{\mathcal{P}}_2(0) \cong \widehat{\mathcal{M}}$ (for $0 < \epsilon \ll 1$), which is generically a $\mathbb{P}^2$-bundle. We define $\widehat{\mathcal{P}}_2 := \widehat{\mathcal{P}}_2(\epsilon)$ and note that (since $\epsilon \ll 1$) the stability conditions for $\widehat{\mathcal{P}}_2$ are essentially determined by Shah's stability conditions for $\widehat{\mathcal{M}}$. However, in $\widehat{\mathcal{P}}_2$ more orbits are separated than in $\widehat{\mathcal{M}}$. Finally, using Theorem 1.6, we conclude that $\widehat{\mathcal{P}}_2$ is closely related to the KSBA compactification $\overline{\mathcal{P}}_2$. We summarize the results of the section as follows:

Theorem 4.1. *The GIT quotient $\widehat{\mathcal{P}}_2$ (constructed in this section) compactifies the moduli space $\mathcal{P}_2$ of degree 2 pairs and has the following properties:*

(i) *$\widehat{\mathcal{P}}_2$ has a natural forgetful map $\widehat{\mathcal{P}}_2 \to \widehat{\mathcal{M}}$ (with generic fiber $\mathbb{P}^2$);*
(ii) *the (GIT) stable locus $\mathcal{P}_2^s \subset \widehat{\mathcal{P}}_2$ is a moduli space of KSBA stable degree 2 pairs (X, H) such that X is double cover of $\mathbb{P}^2$ or Σ_4^0 (and thus $\mathcal{P}_2^s$ is a common open subset of both $\widehat{\mathcal{P}}_2$ and $\overline{\mathcal{P}}_2$);*
(iii) *the strictly semistable locus $\widehat{\mathcal{P}}_2 \setminus \mathcal{P}_2^s$ is a surface $\widetilde{Z}_1$ that maps one-to-one to the closure of the stratum $\widehat{Z}_1 \subset \widehat{\mathcal{M}}$.*

The actual construction of $\widehat{\mathcal{P}}_2$ and the analysis of the stability conditions is the content of this section (see especially (4.3) and (4.11) for the construction, and 4.7 and 4.13 for the analysis of stability) after the introductory example discussed in §4.1.

4.1. A motivating example

We start by discussing a simple example that illustrates how the compactification procedure described in Section 2 works and also hints at the relevance of GIT/VGIT to the construction of $\overline{\mathcal{P}}_2$. Specifically, we consider the analogous one-dimensional compactification problem: the moduli space of pairs consisting of an elliptic curve E and a divisor D of degree d. Definition 2.3 can be easily adapted to this situation. The resulting analogue of $\overline{\mathcal{P}}_d$ is precisely the moduli space of weighted stable curves $\mathcal{M}_{1,\mathcal{A}}$ of Hassett [Has03] (or more precisely $\mathcal{M}_{1,\mathcal{A}}/\Sigma_d$ in the notation of loc. cit.) for the weight system $\mathcal{A} = (\epsilon, \ldots, \epsilon)$. Furthermore, for small ϵ, there is a natural forgetful map $\mathcal{M}_{1,\mathcal{A}} \to \overline{\mathcal{M}}_1$, where $\overline{\mathcal{M}}_1 \cong \mathbb{P}^1$ is the compactified j-line. The boundary points in $\mathcal{M}_{1,\mathcal{A}}$ (corresponding to the fiber over $\infty \in \overline{\mathcal{M}}_1$) are easily described: they are cycles C of rational curves such that each component contains at least one point of D (this is the ampleness condition of Def. 2.3); the points of D are allowed to coincide, but they should be distinct from the nodes of C (this is the slc condition of Def. 2.3).

When $d = 3$, the moduli of pairs as above can be constructed via GIT. Namely, an elliptic curve with a degree 3 polarization is a plane cubic C. If one considers instead an elliptic curve with a polarizing divisor, one gets a pair (C, L) consisting of a plane cubic and a line. A GIT quotient for such pairs (i.e. plane curves plus a line) was studied in [Laz09]. Namely, we have a one-parameter VGIT situation: the GIT quotient for pairs is $\mathcal{P}(\alpha) = \mathbb{P}H^0(\mathbb{P}^2, \mathcal{O}(3)) \times \check{\mathbb{P}}^2 /\!\!/_{\mathcal{O}(1,\alpha)} \mathrm{SL}(3)$ for $\alpha \in \mathbb{Q}_{\geq 0}$. Then $\mathcal{P}(\epsilon) \cong \mathcal{M}_{1,(\epsilon,\epsilon,\epsilon)}$ (for $0 < \epsilon \ll 1$) and $\mathcal{P}(0) \cong \overline{\mathcal{M}}_1 \cong \mathbb{P}^1$ (the GIT quotient for plane cubics). Furthermore,

by VGIT there is a natural forgetful morphism $\mathcal{P}(\epsilon) \to \mathcal{P}(0)$, which coincides with $\mathcal{M}_{1,(\epsilon,\epsilon,\epsilon)} \to \overline{\mathcal{M}}_1$ from the previous paragraph.

There are two advantages of using the GIT construction. First, the spaces $\mathcal{P}(\epsilon) \cong \mathcal{M}_{1,(\epsilon,\epsilon,\epsilon)}$ and $\mathcal{P}(0) \cong \overline{\mathcal{M}}_1$, and the forgetful morphism $\mathcal{P}(\epsilon) \to \mathcal{P}(0)$, are automatically projective (the same can be shown without GIT, but with more involved arguments). Also, the GIT description makes clear the difference between polarization and polarizing divisor. Namely, the GIT quotient $\mathcal{P}(0) \cong \mathbb{P}^1$ has a weak modular meaning: over $\mathbb{A}^1$ the quotient is modular (each point corresponding to a unique smooth cubic), but over ∞ three different orbits (the nodal cubic, the conic plus a line, and the triangle) are collapsed to the minimal orbit corresponding to the triangle in $\mathbb{P}^2$ (with $(\mathbb{C}^*)^2$ stabilizer). When one considers $\mathcal{P}(\epsilon)$, i.e. pairs (C, L) with the line given weight ϵ, essentially nothing changes over the stable locus $\mathbb{A}^1 \subset \mathcal{P}(0)$ (resulting in a $\mathbb{P}^2$-fibration), but over ∞ the three collapsing orbits are separated. The point is that a nodal cubic is strictly semistable, but when considered together with a line it becomes either stable (if the line does not pass through the node) or unstable (if the line passes through the node). Consequently, $\mathcal{P}(\epsilon)$ is modular, in contrast to the weakly modular space $\mathcal{P}(0)$. Also, it is easy to see that (up to finite stabilizer) $\mathcal{P}(\epsilon) \to \mathcal{P}(0)$ becomes a $\mathbb{P}^2$-fibration everywhere.

Remark 4.2. Note that $\mathcal{P}(0) \cong \mathbb{P}^1$ parameterizes smooth and nodal cubics (analogue to the slc condition from Thm. 1.6), and thus the only failure of the modularity is the non-separatedness at the boundary. Also, note that the limit procedure for a nodal cubic plus a line as the line approaches the node is to replace the nodal cubic by a conic plus a line (and then by a triangle); this illustrates one of the essential points of the proof of Thm. 2.9. Finally, some general connections between GIT stability and KSBA stability were noticed by Kim–Lee [KL04] and Hacking [Hac04, §10] (essentially appropriate KSBA stability implies GIT stability). This connection is the strongest for Calabi–Yau hypersurfaces. In some sense, this is what makes the example discussed in this section and the degree 2 $K3$ case work.

4.2. GIT for sextic pairs

The goal of the section is to construct a GIT moduli space $\widehat{\mathcal{P}}_2$ for degree 2 pairs together with a forgetful map $\widehat{\mathcal{P}}_2 \to \widehat{\mathcal{M}}$. Following Shah [Sha80], we carry out this construction in two steps. First in this subsection, we handle the non-unigonal case: we obtain an open subset $\mathcal{P}_2^{\mathrm{nu}} \subset \widehat{\mathcal{P}}_2$ and a forgetful map $\mathcal{P}_2^{\mathrm{nu}} \to \overline{\mathcal{M}} \setminus \{\omega\}$. Then, working near ω and invoking Luna type slice results, we obtain a neighborhood U of the unigonal divisor. The gluing of $\mathcal{P}_2^{\mathrm{nu}}$ and U gives $\widehat{\mathcal{P}}_2$ together with a forgetful morphism $\widehat{\mathcal{P}}_2 \to \widehat{\mathcal{M}}$.

The construction of $\mathcal{P}_2^{\mathrm{nu}}$ follows the example discussed in §4.1 (and [Laz09]). Simply, we consider the family of GIT quotients associated to pairs (C, L), where C is a plane sextic and L is a line:

$$\mathcal{P}(\alpha) := (\mathbb{P}H^0(\mathbb{P}^2, \mathcal{O}(6)) \times \check{\mathbb{P}}^2) /\!/_\alpha \mathrm{SL}(3).$$

As before, we have $\mathcal{P}(0) \cong \overline{\mathcal{M}}$ (the GIT quotient for plane sextics described in Thm. 1.3) and a forgetful map $\pi : \mathcal{P}(\epsilon) \to \overline{\mathcal{M}} \cong \mathcal{P}(0)$ (generically, a $\mathbb{P}^2$-fibration) for small ϵ. We

define

$$\mathcal{P}_2^{nu} = \pi^{-1}(\overline{\mathcal{M}} \setminus \{\omega\}) \subset \mathcal{P}(\epsilon), \tag{4.3}$$

i.e. we remove from $\mathcal{P}(\epsilon)$ the pairs (C, L) with C degenerating to the triple conic.

Notation 4.4. In what follows, we will denote by (C, L) a pair of a plane sextic and a line and by (X, H) the double cover associated to it (not necessarily normal). We will use (C, L) and (X, H) interchangeably to refer to points of $\mathcal{P}_2^{nu}$ (and $\widehat{\mathcal{P}}_2$).

From the general VGIT theory, it follows that if C is a stable/unstable sextic, then (C, L) is ϵ-stable/ϵ-unstable. We conclude that the pairs (X, H) consisting of a $K3$ surface X (possibly with ADE singularities) and an arbitrary degree 2 ample (Cartier) divisor H are stable in $\widehat{\mathcal{P}}_2$ (N.B. strictly speaking this applies to $\mathcal{P}_2^{nu}$ here, but the unigonal case is similar). Thus, $\widehat{\mathcal{P}}_2$ *is a compactification of the moduli space* $\mathcal{P}_2$ *of degree* 2 $K3$ *pairs*. Also from Proposition 1.5 (and Theorem 1.6) the semistable pairs (X, H) corresponding to points of $\mathcal{P}_2^{nu}$ (and then similarly for $\widehat{\mathcal{P}}_2$) have the property that X is a double cover of $\mathbb{P}^2$ (and later also Σ_4^0) with at worst slc singularities. What remains to be understood is how the strictly semistable orbits of sextics become separated when considered as pairs, and the connection between ϵ-GIT stability and ϵ-KSBA stability (i.e. stability in the sense of Definition 2.3). In fact, as already mentioned in a previous remark, ϵ-KSBA stability always implies ϵ-GIT stability.

We discuss the stability of pairs based on the stratification of $\overline{\mathcal{M}}$ given by Theorem 1.3. First note that the strata Z_4 (double cubic) and Z_3 (double conic plus another transversal conic) parameterize stable sextics. Thus the corresponding pairs in these cases are stable, and the forgetful map $\pi : \mathcal{P}^{nu} \to \overline{\mathcal{M}} \setminus \{\omega\}$ is a $\mathbb{P}^2$-fibration (up to finite stabilizers) in a neighborhood of $Z_3 \cup Z_4$. Furthermore, the pairs (X, H) parameterized by $\pi^{-1}(Z_3 \cup Z_4)$ are easily seen to be ϵ-KSBA stable (see §6.1 and §6.3 for a discussion of the geometry of (X, H) in these cases).

To handle the strictly semistable locus $(\overline{Z}_1 \cup \overline{Z}_2)$ we note the following: if C is semistable, there exists a 1-PS (one-parameter subgroup) λ adapted to C (i.e. $\mu(C, \lambda) = 0$), which then singles out a (possibly partial) flag $p_\lambda \in L_\lambda \subset \mathbb{P}^2$. This flag is always in a special position with respect to C; typically p_λ is a singular point of C, and L_λ is a line of highest multiplicity in the tangent cone at p_λ. The stability of the pair (C, L) (for C strictly semistable) is typically determined by the position of L with respect to the flag $p_\lambda \in L_\lambda$. In particular,

(a) if $p_\lambda \notin L$ for all λ with $\mu(C, \lambda) = 0$ (i.e. L is generic), then (C, L) is ϵ-stable;
(b) if $L = L_\lambda$ (i.e. L is very special), then (C, L) is ϵ-unstable.

While these two rules allow us to determine the stability/unstability of most pairs (C, L), there are additional possibilities that might lead to pairs (C, L) that are α-strictly semistable for α varying in an interval.

The behavior of stability conditions for pairs over the strictly semistable locus $\overline{Z}_1 \cup \overline{Z}_2$ is analyzed by the following two propositions:

Proposition 4.5. *Assume that C corresponds to a semistable orbit mapping to $\overline{Z}_2 \subset \overline{\mathcal{M}}$. Then either*

(i) *C has a singularity at a point p of multiplicity 4, in which case if $p \in L$, then (C, L) is ϵ-unstable; or*

(ii) *C contains a double line L_0, in which case if $L = L_0$ then (C, L) is ϵ-unstable. If C contains a double line L_0 which is also tangent to the residual quartic at a point p and $p \in L$, but $L \neq L_0$, then (C, L) is ϵ-semistable with associated minimal orbit*

$$(V(x_0^2 x_2^2 (x_0 x_2 - x_1^2)), V(x_1)).$$

If (C, L) is not one of the three degenerate cases above, then (C, L) is ϵ-stable. Furthermore, in this case, the associated double cover is ϵ-KSBA stable.

Proof. The Mumford numerical function for pairs is $\mu^\epsilon((C, L), \lambda) = \mu(C, \lambda) + \epsilon\mu(L, \lambda)$ (see also [Laz09, Sect. 2]). In the first case, by considering the 1-PS λ with weights $(2, -1, -1)$, we get $\mu(C, \lambda) = 0$ and then $\mu(L, \lambda) = -1$ if $p \in L$; thus, an unstable pair by the numerical criterion. Similarly, the second case follows by considering the 1-PS of weights $(1, 1, -2)$. The case of a double line tangent to a quartic is similar to Proposition 4.6 below. Finally, if none of these three degeneracy conditions is satisfied, we are in the situation (a) discussed above (i.e. generic line from the GIT point of view) and (C, L) will be stable. For the geometric analysis of these cases see §6.2. $\square$

Proposition 4.6. *Assume that C is strictly semistable (not necessarily with closed orbit) and such that it corresponds (in the GIT sense) to a point of $\overline{Z}_1 \setminus \{\zeta, \omega\}$. Then C has one or two singular points p of the following type: p is a triple point with tangent cone consisting of a triple line L_0, and the singularity at p is either $\widetilde{E}_8$ or $T_{2,3,r}$ ($r \geq 7$) or a degenerate cusp of type $x^2(x + y^2)$ (i.e. double conic tangent to the residual conic). The following hold for the GIT stability of the pair (C, L):*

(i) *if $p \notin L$ (for both p if two special singularities), then (C, L) is ϵ-stable;*

(ii) *$L = L_0$ (i.e. L is the special direction through p), then (C, L) is ϵ-unstable;*

(iii) *otherwise (i.e. L passes through p, but it is not special), (C, L) is ϵ-semistable. The minimal orbits in this case are given by:*

(Case Z_1) $\left(V((x_0 x_2 - a_1 x_1^2)(x_0 x_2 - a_2 x_1^2)(x_0 x_2 - a_3 x_1^2)), V(x_1)\right)$, *for a_i distinct;*
(Case τ) $\left(V((x_0 x_2 - x_1^2)^2 (x_0 x_2 - a x_1^2)), V(x_1)\right)$, *for $a \neq 1$.*

Furthermore, the double cover (X, H) is ϵ-KSBA stable iff (C, L) is ϵ-GIT stable.

Proof. In this situation, the adapted 1-PS λ has weights $(1, 0, -1)$. If L passes through p, but not in a special direction, then $\mu(L, \lambda) = 0$, and the configuration remains semistable for all $0 < \epsilon \ll 1$. Finally, to prove the equivalence of the two stability conditions it suffices to note that if L passes through p (regardless of whether $L = L_0$ or not), the associated pair $(X, \epsilon H)$ is not slc (see also §6.4). $\square$

The above discussion gives the non-unigonal case of Theorem 4.1:

Corollary 4.7. *Let (C, L) be a pair consisting of a plane sextic and a line. Let (X, H) be the associated double cover. Assume that*

(i) *(C, L) is ϵ-GIT stable;*

(ii) *the orbit closure of C does not contain the triple conic.*

Then (X, H) is ϵ-KSBA stable. Conversely, assume that (X, H) is ϵ-KSBA stable and that H is base point free. Then (X, H) is the double cover associated to a pair (C, L) satisfying the two conditions above.

4.3. Blow-up of the triple conic locus

Shah's construction of $\widehat{\mathcal{M}}$ replaces the degenerations to the triple conic ($\omega \in \overline{\mathcal{M}}$) by double covers of $\Sigma_4^0 \subset \mathbb{P}^5$, the cone over the rational normal curve of degree 4. Additionally, as noted in Theorem 1.6, all the semistable points in the GIT quotient $\widehat{\mathcal{M}}$ correspond to degenerations of $K3$ surfaces with slc singularities. We now consider pairs consisting of such a surface (semistable double cover of $\mathbb{P}^2$ or Σ_4^0) together with a hyperplane section. As explained, the goal here is to construct a neighborhood U of the unigonal divisor which can be glued to $\mathcal{P}_2^{\mathrm{nu}}$ to give $\widehat{\mathcal{P}}_2$.

To start, we note the following uniform description of the double covers of $\mathbb{P}^2$ and Σ_4^0: they are complete intersections of the form

$$\{z^2 - f_6(x_i, y) = f_2(x_i, y) = 0\} \subset \mathbb{P}(1, 1, 1, 2, 3). \tag{4.8}$$

The non-unigonal case corresponds to the situation $f_2(0, 0, 0, 1) \neq 0$. After a change of coordinates, one can normalize the equation in this case to

$$\{z^2 - f_6(x_i, y) = y = 0\} \subset \mathbb{P}(1, 1, 1, 2, 3),$$

which leads to the usual case of double covers of $\mathbb{P}^2$ branched along a sextic. Similarly, the double covers of Σ_4^0 correspond to the case when $f_2(0, 0, 0, 1) = 0$ and $f_2(x_1, x_2, x_3, 0)$ has maximal rank. Thus, one can choose the normal form

$$\{z^2 - f_6(x_i, y) = x_0 x_2 - x_1^2 = 0\} \subset \mathbb{P}(1, 1, 1, 2, 3).$$

Moreover, as $f_6(0, 0, 0, 1) \neq 0$ (i.e. the ramification curve does not pass through the vertex of Σ_4^0), we can further assume that $f_6(x_i, y) = y^3 + y g_4(x_i) + g_6(x_i)$.

Since the automorphism group of the weighted projective space $\mathbb{P}(1, 1, 1, 2, 3)$ is not reductive, a uniform GIT description would be difficult (see [Sha80, Sect. 4]). Instead, Shah [Sha80, Sect. 5] uses a local description of the GIT quotient $\overline{\mathcal{M}}$ near the orbit ω of the triple conic and a gluing construction to obtain the space $\widehat{\mathcal{M}}$ of Section 1. In more modern language, $\widehat{\mathcal{M}}$ is just the partial Kirwan blow-up of $\overline{\mathcal{M}}$ at the point ω (corresponding to the worst stabilizer for semistable plane sextics).

Remark 4.9. Here and elsewhere in this paper, locally means locally étale. Thus, the spaces we consider are at least algebraic spaces, and typically algebraic varieties (e.g. using the Kirwan point of view, we can see that $\overline{\mathcal{M}}$ is so). It might be even possible to prove that they are projective varieties, but this is of secondary concern for us (see Rem. 2.14). Furthermore, by the GIT construction, over the stable locus there exist (locally) flat proper families of surfaces or pairs. Thus, the spaces considered in this paper are coarse moduli spaces associated to certain algebraic stacks.

The following is a rephrasing of the main point of Shah's construction of $\widehat{\mathcal{M}}$.

Lemma 4.10. *Locally near ω, $\overline{\mathcal{M}}$ is identified with the (affine) quotient*

$$(\mathrm{Sym}^{12} V \times \mathrm{Sym}^8 V)/\mathrm{SL}(2),$$

where V is the standard $\mathrm{SL}(2)$-representation. The Kirwan blow-up $\widehat{\mathcal{M}} \to \overline{\mathcal{M}}$ is modeled on the weighted blow-up of the origin in the vector space $\mathrm{Sym}^{12} V \times \mathrm{Sym}^8 V$. In particular, the exceptional (unigonal) divisor is identified with the GIT quotient

$$(\mathbb{P}\,\mathrm{Sym}^{12} V \times \mathbb{P}\,\mathrm{Sym}^8 V) /\!\!/_{\mathcal{O}(3,2)} \mathrm{SL}(2).$$

Proof. By definition, $\overline{\mathcal{M}} \cong \mathbb{P}\,\mathrm{Sym}^6 W /\!\!/ \mathrm{SL}(3)$ where W is a standard representation of $\mathrm{SL}(3)$. Luna's slice theorem describes $\overline{\mathcal{M}}$ locally at ω as the quotient of a normal slice to the orbit of a triple conic by the stabilizer $\mathrm{SL}(2)$. Since the conic is $\mathbb{P}^1$ embedded by Veronese in $\mathbb{P}^2$, we get an identification of $W = \mathrm{Sym}^2 V$ as an $\mathrm{SL}(2)$-representation. Then the normal slice (as a $\mathrm{SL}(2)$-representation) is the summand $\mathrm{Sym}^{12} V \times \mathrm{Sym}^8 V$ in $\mathrm{Sym}^6 W \cong \mathrm{Sym}^6(\mathrm{Sym}^2 V)$. The lemma follows.

Alternatively, note that in the normal form described above, the group preserving it is $\mathrm{SL}(2)$. Then, by viewing x_i as sections of $\mathcal{O}_{\mathbb{P}^1}(2)$ we can identify $g_4(x_i)$ and $g_6(x_i)$ with binary forms $p_8(u, v)$ and $p_{12}(u, v)$ respectively.   □

From the perspective of the lemma, we can view a line in $\mathbb{P}^2$ (i.e. a section of the polarization) as an element of $\mathbb{P}\,\mathrm{Sym}^2 V$. We model the neighborhood U of the unigonal divisor as the quotient

$$\big(\mathrm{Bl}_0(\mathrm{Sym}^{12} V \times \mathrm{Sym}^8 V) \times \mathbb{P}\,\mathrm{Sym}^2 V\big) /\!\!/ \mathrm{SL}(2), \qquad (4.11)$$

where $\mathrm{Bl}_0(\mathrm{Sym}^{12} V \times \mathrm{Sym}^8 V)$ denotes the weighted blow-up of the origin from Lemma 4.10 (or equivalently a local model of the Kirwan blow-up). The map $\widehat{\mathcal{P}}_2 \to \widehat{\mathcal{M}}$ (locally near the unigonal divisor) is the induced map (at the level of quotients) by the first projection. By construction, U glues to the non-unigonal quotient $\mathcal{P}_2^{\mathrm{ns}}$ to give $\widehat{\mathcal{P}}_2$ together with a forgetful map $\widehat{\mathcal{P}}_2 \to \widehat{\mathcal{M}}$.

To complete the proof of Theorem 4.1, it remains to describe the stability conditions in the unigonal case. Obviously, it suffices to describe the stability condition for the points of the exceptional divisor of $\widehat{\mathcal{P}}_2 \to \mathcal{P}(\epsilon)$ (away from this locus, the stability was described in §4.2). Since the exceptional divisor of $\widehat{\mathcal{M}} \to \overline{\mathcal{M}}$ is $(\mathbb{P}\,\mathrm{Sym}^{12} V \times \mathbb{P}\,\mathrm{Sym}^8 V) /\!\!/_{\mathcal{O}(3,2)} \mathrm{SL}(2)$ (see Lem. 4.10) and the weight is small in the direction of the hyperplane section, we see that the exceptional divisor of $\widehat{\mathcal{P}}_2 \to \mathcal{P}(\epsilon)$ is modeled by

$$\big((\mathbb{P}\,\mathrm{Sym}^{12} V \times \mathbb{P}\,\mathrm{Sym}^8 V) \times \mathbb{P}\,\mathrm{Sym}^2 V\big) /\!\!/_{\mathcal{O}(3,2,\epsilon)} \mathrm{SL}(2).$$

Of course, the stability condition here is essentially determined by the stability on the first factor; this was analyzed by Shah. Specifically, we rephrase a result of Shah [Sha80, Thm. 4.3] as follows:

Theorem 4.12 (Shah). *With notation as above, let B be a curve of Σ_4^0 given by $f_6(x_i, y) = y^3 + y g_4(x_i) + g_6(x_i)$, and X be the associated double cover of Σ_4^0.*

(1) *If B is stable and reduced, then X has at most simple singularities.*

(2) *The minimal orbits of B which are strictly semistable and reduced give surfaces X with two $\widetilde{E}_8$ singularities. These minimal orbits are parameterized by an (affine) rational curve $U_1 \subset \widehat{Z}_1 \setminus \widehat{\tau}$ (see Fig. 2).*

(3) *If B is stable and non-reduced then $B = 2C_1 + C_2$ with C_i rational normal curves that intersect transversely. These orbits are parameterized by a rational curve $U_2 \subset \widehat{Z}_3 \setminus \widehat{\tau}$.*

(4) *In addition to the cases given by (1)–(3), there is a single additional minimal orbit corresponding to the case $B = 2C_1 + C_2$ with C_i rational normal curves that intersect tangentially at two points. This orbit maps to the point $\xi \in \widehat{\tau}$.*

Furthermore, the surfaces X degenerating to case (2), but not corresponding to minimal orbits, are rational with a unique $\widetilde{E}_8$ singularity. Similarly, the surfaces degenerating to case (4) have a singularity of type $T_{2,3,r}$ (for $r \geq 7$) or a degenerate cusp of type $z^2 + x^2(x + y^2)$.

Proof. As mentioned, this is precisely [Sha80, Thm. 4.3]. We only comment here on the singularities of the semistable objects. In case (2), the minimal orbits are given by $f_6(x_i, y) = y^3 + a_1 y u^4 v^4 + a_2 u^6 v^6$ (via the identification of x_i with binary quadrics as above). In affine coordinates X is given by $z^2 = y^3 + a_1 y u^4 + a_2 u^6$, which is an $\widetilde{E}_8$ singularity. Note that the discriminant condition to get an $\widetilde{E}_8$ singularity and not worse coincides with the non-degeneracy condition from [Sha80, Thm. 4.3]. By semicontinuity, one gets the same type of singularities for non-minimal orbits degenerating to case (2) (N.B. there has to be at least one non-ADE singularity, otherwise B would stable by (1); $\widetilde{E}_8$ deforms only to ADE singularities).

In cases (3) and (4), B has the form $f = (y + \theta)^2(y - 2\theta) = y^3 - 3y\theta^2 - 2\theta^3$, where $\theta = p_4(u, v)$. B is stable iff the two divisors defined by $y + \theta = 0$ and $y - 2\theta = 0$ intersect transversely in four distinct points. Similarly, the minimal orbit for the strictly semistable case corresponds to two double points. The singularity claim follows (see especially [AGZV85, §16.2.9]). □

We now conclude the analysis of stability in the unigonal case:

Corollary 4.13. *Let (B, L) be a pair defined as above and (X, H) be the associated double cover.*

(1) *(B, L) is ϵ-GIT stable iff L does not pass through the $\widetilde{E}_8$, $T_{2,3,r}$, or degenerate cusps singularities. If (B, L) is ϵ-GIT stable then (X, H) is ϵ-KSBA stable. Conversely, if (X, H) is ϵ-KSBA stable and X is a double cover of Σ_4^0, then (X, H) is induced by an ϵ-GIT stable (B, L) pair.*

(2) *The minimal orbits of ϵ-strictly semistable pairs (B, L) are given by B with minimal orbit (as in Thm. 4.12) and L such that it passes through the two $\widetilde{E}_8$ singularities if B is reduced or through the two tangent points otherwise.*

Proof. The stability analysis is similar to that of Proposition 4.6. For the converse, that ϵ-KSBA stable implies GIT stability, a closer look at the GIT stability conditions shows that unstable curves B have singularities worse than simple elliptic, cusp, or degenerate cusp (cf. Prop. 1.5). □

5. Classification of slc stable pairs via GIT

In Section 1 we have discussed Shah's compactification $\widehat{\mathcal{M}}$ which gives a compactification with a weak geometric meaning for the moduli space $\mathcal{F}_2$ of degree 2 $K3$ surfaces. In Section 2, we have shown that considering pairs (X, H) instead of polarized $K3$ surfaces $(X, \mathcal{O}_X(H))$ gives a proper and separated moduli stack $\overline{\mathcal{P}}_2$. Then, in Section 4, via GIT, we have constructed an approximation $\widehat{\mathcal{P}}_2$ of the space $\overline{\mathcal{P}}_2$. Namely, there is a birational map $\widehat{\mathcal{P}}_2 \dashrightarrow \overline{\mathcal{P}}_2$ which is an isomorphism over the stable locus $\mathcal{P}_2^s \subset \widehat{\mathcal{P}}_2$ (cf. Thm. 4.1). We also recall that the strictly semistable locus $\widehat{\mathcal{P}}_2 \setminus \mathcal{P}_2^s$ is a surface $\widetilde{Z}_1$ mapping one-to-one to the stratum $\widehat{Z}_1 \subset \widehat{\mathcal{M}}$, and that the pairs parameterized (in the sense of GIT) by $\widetilde{Z}_1$ are not KSBA stable.

   To complete the description of $\overline{\mathcal{P}}_2$, it remains to understand the KSBA replacement of the strictly semistable locus $\widetilde{Z}_1 \subset \widehat{\mathcal{P}}_2$. This is dealt with in the following theorem.

Theorem 5.1. *The birational map $\overline{\mathcal{P}}_2 \dashrightarrow \widehat{\mathcal{P}}_2$ replaces the strictly semistable locus $\widetilde{Z}_1$ in $\widehat{\mathcal{P}}_2$ by the locus of stable KSBA pairs (X, H) of type $X = V_1 \cup_E V_2$, where the V_i are degree 1 del Pezzo surfaces or allowable degenerations of them (in $\mathbb{P}(1, 1, 2, 3)$) glued along an anticanonical section E of V_i. At least set-theoretically, the birational transformation $\overline{\mathcal{P}}_2 \dashrightarrow \widehat{\mathcal{P}}_2$ is dominated by the Kirwan blow-up $\widetilde{\mathcal{P}}_2 \to \widehat{\mathcal{P}}_2$ (of the GIT model $\widehat{\mathcal{P}}_2$ along the strictly semistable locus $\widetilde{Z}_1$) as in diagram (5.3).*

Remark 5.2. We expect that the maps of diagram (5.3) are morphisms of algebraic varieties, and that the transformation $\overline{\mathcal{P}}_2 \dashrightarrow \widehat{\mathcal{P}}_2$ is a flip in the sense of VGIT. More precisely, we suspect that there is a common contraction $\mathcal{P}^\vee$ (with $\mathcal{P}^\vee \to (\mathcal{D}/\Gamma_2)^*$) of $\overline{\mathcal{P}}_2$ and $\widehat{\mathcal{P}}_2$ completing diagram (5.3), and that $\overline{\mathcal{P}}_2$, $\mathcal{P}^\vee$, and $\widehat{\mathcal{P}}_2$ are "+", "0", and "−" instances respectively in a VGIT set-up. This is plausible since the surfaces parameterized by $\overline{\mathcal{P}}_2$ and $\widehat{\mathcal{P}}_2$ are $(2, 6)$ complete intersections in the same space $W\mathbb{P}(1, 1, 1, 2, 3)$ (see Thm. 2.16). However, to make this work, there are two main issues: one needs to deal with GIT for weighted projective spaces and find a way around non-reductive stabilizers (see however [RT11]), and secondly the GIT analysis for complete intersections is already quite involved for $(2, 3)$ curves in $\mathbb{P}^3$ (see e.g. [CMJL14]). The analysis in this paper bypasses these issues, but the results are somewhat weaker.

Proof of Theorem 5.1. By Thompson's Theorem 2.16 (see also Sect. 3, especially Prop. 3.14) and the GIT analysis (see in particular Cor. 4.7 and Cor. 4.13), we see that the only KSBA stable pairs (X, H) that do not occur in the GIT quotient $\widehat{\mathcal{P}}_2$ are those for which X is a union of two del Pezzo surfaces of degree 1 glued along an anticanonical section E. More precisely, such an X is given as

$$X = V(z^2 - f_6(x_i, y), x_0 x_2) \subset \mathbb{P}(1, 1, 1, 2, 3)$$

and H is induced by a linear form in x_i. We denote by $\Delta_{2E_8} \subset \overline{\mathcal{P}}_2$ the closure of this locus. We have $\dim \Delta_{2E_8} = 19$ corresponding to one modulus for E, eight moduli for each of the del Pezzo surfaces V_i, and one modulus for each of the polarizing divisors $H_i \in |-K_{V_i}|$ on V_i. Note also that since X has at worst slc singularities, which is the same as cohomologically insignificant singularities, there is (at least set-theoretically) a

map $\overline{\mathcal{P}}_2 \to (\mathcal{D}/\Gamma_2)^*$ to the Baily–Borel compactification of $\mathcal{F}_2$ (see [Sha79]). From [Fri84, Thm. 5.4, §(5.2.2)] (see also Section 6 below), we find that $\Delta_{2E_8} \subset \overline{\mathcal{P}}_2$ maps to the closure $\overline{\mathrm{II}}_{2E_8+A_1} \cong \mathbb{P}^1$ of the Type II component labeled by $2E_8 + A_1$ (see Rem. 1.2 and Fig. 2). In other words, there is a fibration $\Delta_{2E_8} \to \mathbb{P}^1$ given by the j-invariant of the gluing curve E with 18-dimensional fibers.

On the other hand, for the strictly semistable locus $\widetilde{Z}_1 \subset \widehat{\mathcal{P}}_2$, we have the morphism

$$\widehat{\mathcal{P}}_2 \to \widehat{\mathcal{M}} \to (\mathcal{D}/\Gamma_2)^*, \quad \widetilde{Z}_1 \mapsto \widehat{Z}_1 \to \overline{\mathrm{II}}_{2E_8+A_1},$$

which realizes $\widetilde{Z}_1$ as a $\mathbb{P}^1$-fibration (up to finite stabilizer issues) over $\overline{\mathrm{II}}_{2E_8+A_1} \cong \mathbb{P}^1$; the fibration is given again by a j-invariant (see Rem. 1.10). Geometrically, the points of $\widetilde{Z}_1$ are in one-to-one correspondence with the pairs (X, H) where X is the double cover of $\mathbb{P}^2$ (or similarly for Σ_4^0) branched in the union of three conics pairwise tangent at two fixed points, and H is induced from the line passing through these two points. The surface X will have two $\widetilde{E}_8$ singularities. Since H passes through them, (X, H) is KSBA unstable. The KSBA replacement (obtained by applying Thm. 2.9) is analyzed in §5.1 below (see Figure 3 for a quick pictorial description). Essentially, the resolution V of X is a non-minimal elliptic ruled surface (over some elliptic curve E) with two disjoint (-1)-sections. Then the Kulikov semistable model associated to such a surface is $X_0 = V_1 \cup_E V \cup_E V_2$ with V_i degree 1 del Pezzo with a fixed anticanonical section E. The KSBA model contracts V resulting in $V_1 \cup_E V_2$ which corresponds to a point in Δ_{2E_8}. On the other hand, the GIT model contracts the surfaces V_i giving the surface X which corresponds to a point in $\widetilde{Z}_1$. Thus, the birational map $\overline{\mathcal{P}}_2 \dashrightarrow \widehat{\mathcal{P}}_2$ (defined over $(\mathcal{D}/\Gamma)^*$) replaces $\Delta_{2E_8} \subset \overline{\mathcal{P}}_2$ by $\widetilde{Z}_1 \subset \widehat{\mathcal{P}}_2$ by forgetting the modulus of V and $V_1 \cup V_2$ respectively.

In other words, we obtain the following diagram:

$$\tag{5.3}$$

where $\widetilde{\mathcal{P}}$ is a blow-up of $\widehat{\mathcal{P}}_2$ along $\widetilde{Z}_1$ with exceptional divisor $\Delta_{2E_8+A_1}$ which parameterizes the pairs (X_0, H) with $X_0 = V_1 \cup V_0 \cup V_2$ and $H = H_1 + F_0 + H_2$ as discussed in §5.1. More precisely, we define $\widetilde{\mathcal{P}}$ as the Kirwan desingularization of $\widehat{\mathcal{P}}$ along the strictly semistable locus $\widetilde{Z}_1$ (N.B. there are only $\mathbb{C}^*$-stabilizers). It is clear that generically the points of the exceptional divisor of $\widetilde{\mathcal{P}} \to \widehat{\mathcal{P}}$ parameterize Kulikov models of type $X_0 = V_1 \cup V_0 \cup V_2$ as above. To see that the analysis extends also to the non-generic

case (N.B. the resulting surfaces might not be Kulikov, see the following remark for a discussion of what is allowed) we use (1) the analysis of the GIT quotient $\widehat{\mathcal{P}}$ along $\widetilde{Z}_1$ as discussed in §5.2, and (2) the explicit description of moduli of surfaces of type $V_1 \cup_E V_2$ or $V_1 \cup_E V \cup V_2$ based on some earlier work of Pinkham and Looijenga (see Lem. 5.13 and Cor. 5.15).

Note that $\widetilde{\mathcal{P}} \to \widehat{\mathcal{P}}_2$ is a morphism by construction, while $\widetilde{\mathcal{P}} \to \overline{\mathcal{P}}_2$ is a priori only set-theoretic (it maps $V_1 \cup V \cup V_2$ to $V_1 \cup V_2$, forgetting the modulus of the ruled surface V). $\qquad \square$

Remark 5.4. The following degenerations of $V_1 \cup_E V_2$ are allowed. First, E is either smooth elliptic (Type II case) or nodal irreducible with $p_a(E) = 1$ (Type III case). The del Pezzo surfaces V_i are allowed to have ADE singularities. As degenerate cases, we allow also cones, i.e. singularities of type $\widetilde{E}_8$ (the cone over a degree 1 elliptic curve) in the Type II case or the degenerate cusp which is obtained as the cone over a nodal curve. The normalization in this latter case is in fact $\mathbb{P}^2$ (see Rem. 7.5). A more detailed discussion of the possibilities for $V_1 \cup_E V_2$ (and how they fit together) is given in §6.4 (see especially Fig. 5) and §7.4.

Remark 5.5. The locus in $\widetilde{\mathcal{P}}$ of Kulikov models can be obtained by constructing a neighborhood of the Kulikov locus in $\Delta_{2E_8+A_1}$ by deformation theory as in [Fri84] (see Rem. 6.5) and then gluing it to the common open subset $\mathcal{P}_2^s$ of $\widehat{\mathcal{P}}_2$ and $\overline{\mathcal{P}}_2$.

### 5.1. The KSBA replacement of the strictly semistable locus

We are now interested in identifying the KSBA stable replacement for the strictly semistable locus $\widetilde{Z}_1 \subset \widehat{\mathcal{P}}_2$. As usual, we consider a family $(\mathcal{X}, \mathcal{H})/\Delta$ of semistable GIT pairs such that the central fiber (X, H) is strictly semistable (and thus $0 \in \Delta$ maps to $\widetilde{Z}_1 \subset \widehat{\mathcal{P}}_2$). In fact, without loss of generality we can assume that (X, H) corresponds to a minimal orbit. As usual, to understand the KSBA limit for $\mathcal{X}^*/\Delta^*$ one has to arrange $(\mathcal{X}, \mathcal{H})/\Delta$ in a semistable (or even Kulikov) form and then follow the arguments of Theorem 2.9 to obtain the KSBA limit. We sketch the computation below. Note that the semistable computations here are "generic" and their role is to give a geometric interpretation for Theorem 5.1.

5.1.1. The geometry of the minimal orbits of $\widehat{\mathcal{P}}_2$.
Consider the pairs (X, H) associated to a minimal orbit of a strictly semistable point (i.e. corresponding to a point in $\widetilde{Z}_1$). As discussed, X is the double cover of $\mathbb{P}^2$ branched along the sextic (the unigonal case is similar and left to the reader)

$$C = V((x_0 x_2 - \alpha_1 x_1^2)(x_0 x_2 - \alpha_2 x_1^2)(x_0 x_2 - \alpha_3 x_1^2)) \subset \mathbb{P}^2 \quad \text{for some } \alpha_1, \alpha_2, \alpha_3 \in \mathbb{C}.$$

The polarization H is the pull-back of the line $L = V(x_1)$. The surface X has two $\widetilde{E}_8$ singularities corresponding to the points $p_1 = (1 : 0 : 0)$ and $p_2 = (0 : 0 : 1)$. Consider the minimal resolution $V \to X$ (obtained via a single weighted blow-up of p_i). It is well known that the two exceptional divisors E_1 and E_2 are elliptic curves with $E_1^2 = E_2^2 = -1$.

We are interested here in understanding the geometry of the pair (V, H), where H is the polarizing divisor (i.e. $|H|$ defines the map $V \to X \to \mathbb{P}^2$, and H maps to $L \subset \mathbb{P}^2$). Using the standard procedure of resolving double covers, we obtain the following commutative diagram:
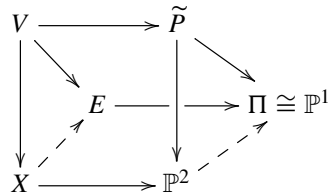
where

- $\Pi \cong \mathbb{P}^1$ is the pencil of conics passing through the points p_1, $p_2 \in \mathbb{P}^2$ and tangent to the lines $V(x_0)$ and $V(x_2)$;
- $E \to \Pi$ is the double cover branched at the points corresponding to the three special conics $V(x_0 x_2 - \alpha_i x_1^2)$ and to the double line $V(x_1^2)$ (in coordinates, E is the double cover of $\mathbb{P}^1$ branched at $\alpha_1, \alpha_2, \alpha_3, \infty$);
- $\widetilde{P}$ is the blow-up of $\mathbb{P}^2$ twice at each of the points p_1 and p_2, followed by the contraction of the resulting two (-2)-curves (alternatively, $\widetilde{P}$ is a weighted blow-up of $\mathbb{P}^2$ at the two special points; $\widetilde{P}$ has two A_1 singularities);
- the horizontal arrows are double covers;
- $\mathbb{P}^2 \dashrightarrow \Pi$ is the conic bundle fibration given by mapping a point $x (\neq p_i) \in \mathbb{P}^2$ to the unique member of the pencil Π that passes through x.

Lemma 5.6. *With notation as above, $V \to E$ is an elliptic ruled surface, and the two exceptional divisors E_1 and E_2 are two disjoint sections of self-intersection -1. Furthermore:*

 (i) *The strict transform of the line $L = V(x_1) \subset \mathbb{P}^2$ gives a special fiber F_0, which can be taken as the origin of E (and of the sections E_i).*
 (ii) *There are two reducible fibers for $V \to E$ corresponding to the reducible conic $V(x_0 x_2)$ in the pencil Π. In particular, V is the blow-up at two points of a geometrically ruled surface.*
(iii) *The pull-back of the line $L \subset \mathbb{P}^2$ to V is*

$$H = F_0 + E_1 + E_2,$$

and thus the linear system $|F_0 + E_1 + E_2|$ gives the map $V \to X \xrightarrow{2:1} \mathbb{P}^2$.

Proof. The claims follow easily from the above discussion. $\square$

Remark 5.7 (cf. Rem. 1.10). The cross-ratio associated to the elliptic curve E is $\lambda = \frac{\alpha_1 - \alpha_3}{\alpha_2 - \alpha_3}$ and then the j-invariant is $j(E) = 2^8 \cdot \frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2 (\lambda - 1)^2}$. Alternatively, the affine equation of X near a singular point, $z^2 = (x - \alpha_1 y^2)(x - \alpha_2 y^2)(x - \alpha_3 y^2)$, can be put into the Weierstrass form $z^2 = x^3 + Axy^4 + By^6$, where

$$A = \sigma_2 - (\sigma_1)^2/3, \qquad B = \sigma_3 + \sigma_1 \sigma_2 / 3 - 2(\sigma_1)^3/27,$$

and σ_i are the elementary symmetric functions in α_i. In this form, the discriminant and the j-invariant have the following expressions:

$$\Delta = 4A^3 + 27B^2, \quad j = 1728 \cdot \frac{4A^3}{27B^2}.$$

Note that $\Delta = -(\alpha_1 - \alpha_2)^2(\alpha_2 - \alpha_3)^2(\alpha_3 - \alpha_1)^2$, and thus the elliptic curve E is singular iff two of the α_i coincide, which corresponds to the case τ (i.e. a double conic together with another conic bitangent to it). The analysis of the lemma can be extended to the case of E singular. In that situation, V is a non-normal surface whose normalization is a (non-minimal) rational ruled surface (see Rem. 5.8 for further discussion).

5.1.2. The semistable reduction. We are now interested in understanding the Kulikov model associated to a one-parameter family with central fiber (X, H) as above. In the generic case, the weighted blow-up of the two $\widetilde{E}_8$ singularities of X (as above, but this time keeping track of the ambient threefold $\mathfrak{X}$) gives a semistable model with central fiber $X_0 = V_1 \cup V \cup V_2$, where

- V is the resolution of X as in §5.1.1, and V_1 and V_2 are degree 1 del Pezzo surfaces with a marked anticanonical section $D_i \cong E$;
- V_1 is glued to V along the elliptic curve $D_0 \cong E \cong E_1$ (and similarly for V_2); the gluing is such that the unique base point of $|-K_{V_1}|$ matches with the point $E_1 \cap F_0$.
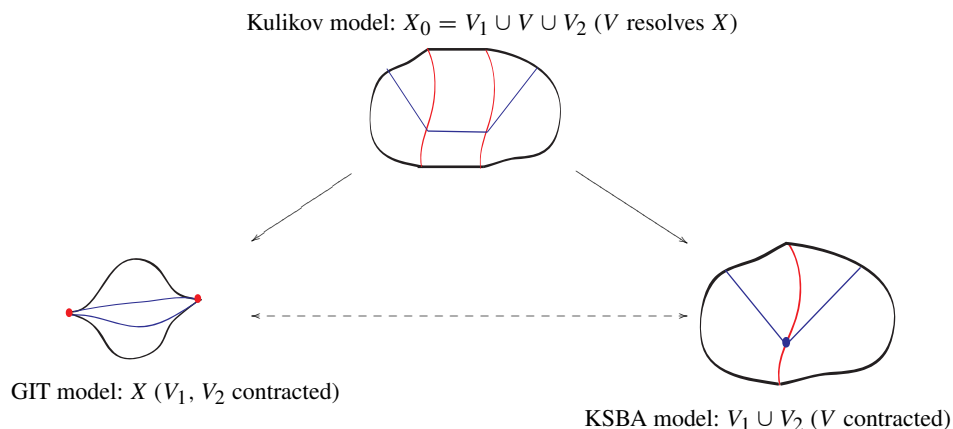
Kulikov model: $X_0 = V_1 \cup V \cup V_2$ (V resolves X)

GIT model: X (V_1, V_2 contracted)

KSBA model: $V_1 \cup V_2$ (V contracted)

Fig. 3. Semistable replacement of the two $\widetilde{E}_8$ surfaces.

Note that the triple point formulas (e.g. $D_1^2 + E_1^2 = 1 + (-1) = 0$) are satisfied, and we have indeed a Kulikov model.

Keeping track of the polarization, we obtain for the GIT model the polarized components

$$(V_1, D_1; 0), \quad (V, E_1 + E_2; F_0 + E_1 + E_2), \quad (V_2, D_2, 0).$$

This leads to a KSBA unstable limit since $(V, E_1 + E_2; \epsilon(F_0 + E_1 + E_2))$ is not slc (the polarizing divisor contains a double curve). Then the KSBA replacement is obtained

by twisting the polarization by $\mathcal{O}(-V)$ (on the total threefold space), resulting in the polarized components

$$(V_1, D_1; H_1), \quad (V, E_1 + E_2; F_0), \quad (V_2, D_2; H_2),$$

with $H_i \in |-K_{V_i}|$. In this model, V becomes a 1-surface and thus will be contracted to the elliptic curve E. We conclude that the KSBA limit in this situation is $\overline{X}_0 = V_1 \cup_E V_2$, two degree 1 del Pezzo's glued along an anticanonical section (see Fig. 3).

Remark 5.8. Note that the same analysis can be easily extended to cover the Type III case (i.e. E nodal) as well. For example in this case, the central fiber $X_0 = V_1 \cup V \cup V_2$ of the corresponding semistable degeneration will contain again two del Pezzo surfaces V_i, but the double curves will be nodal anticanonical sections. Additionally, V is a rational surface which is a degeneration of an elliptic ruled surface. Namely, V is a non-normal surface whose normalization $\widetilde{V}$ is a rational ruled surface blown up at two points. For instance, V can be obtained by blowing up two points on a section of self-intersection 1 on the Hirzebruch surface $\mathbb{F}_1$, and then gluing two irreducible fibers of the resulting surface to get V.

5.2. The structure of the Kirwan blow-up $\widetilde{\mathcal{P}}$ of $\widehat{\mathcal{P}}_2$ along the strictly semistable locus

We now analyze the structure of the Kirwan blow-up $\widetilde{\mathcal{P}}$ of $\widehat{\mathcal{P}}_2$ along the strictly semistable locus $\widetilde{Z}_1$. Namely, we show that $\widetilde{\mathcal{P}} \to \widehat{\mathcal{P}}_2$ is a fibration over $\widetilde{Z}_1$ with fiber isomorphic to $W\mathbb{P}^9 \times W\mathbb{P}^9$, which can be identified with the moduli space of polarized surfaces of type $V_1 \cup_E V_2$ with V_i degree 1 del Pezzo (and E fixed). On the other hand $\widetilde{Z}_1$ is a $\mathbb{P}^1$-fibration over $\overline{\mathrm{II}}_{2E_8 + A_1}$. Combining this with the geometric analysis of §5.1, we get diagram (5.3), completing the proof of Theorem 5.1.

5.2.1. Preliminaries on degree 1 del Pezzo surfaces.
We recall that a degree 1 del Pezzo surface has the following anticanonical model:

$$\{z^2 = y^3 + yg_4(x_0, x_1) + g_6(x_0, x_1)\} \subset \mathbb{P}(1, 1, 2, 3), \tag{5.9}$$

and the point $(0 : 0 : 1 : 1)$ is the base locus of the anticanonical linear system. We are interested in surfaces of type $V_1 \cup_E V_2$, where both V_i are degree 1 del Pezzo surfaces, E is an anticanonical section, and the gluing is such that the base points of the anticanonical systems on V_i match (a necessary condition for $V_1 \cup_E V_2$ to occur as a central fiber in a degree 2 $K3$ degeneration). As already noted, $V_1 \cup_E V_2$ has the following description:

$$V_1 \cup_E V_2 = \{z^2 = y^3 + yg_4(x_0, x_1, x_2) + g_6(x_0, x_1, x_2), x_0 x_2 = 0\} \subset \mathbb{P}(1, 1, 1, 2, 3),$$

which is compatible with (4.8) and Theorem 2.16. Note that the gluing curve is given by intersecting with $V(x_0, x_2)$:

$$E = \{z^2 = y^3 + Byx_1^4 + Cx_1^6\} \subset \mathbb{P}(1, 2, 3),$$

an elliptic curve in Weierstrass form. The polarizing divisor in this situation is given by a linear form $l(x_0, x_1, x_2)$. Finally, the cone over E is given by

$$\{z^2 = y^3 + Byx_1^4 + Cx_1^6\} \subset \mathbb{P}(1, 1, 2, 3),$$

with vertex at $(1 : 0 : 0 : 0) \in \mathbb{P}(1, 1, 2, 3)$; it appears when x_0 (or x_2) does not occur in g_4 and g_6.

5.2.2. The moduli of degree 1 del Pezzo's with a marked anticanonical section. A theorem of Pinkham and Looijenga (see e.g. [Pin77] and [Loo77, Loo78]) identifies the moduli space of pairs (V, D) consisting of a degree 1 del Pezzo surface V with a marked hyperplane section D to the weighted projective space $\mathbb{P}(1, 2, 2, 3, 3, 4, 4, 5, 6)$ (N.B. $2, 2, \ldots, 5, 6$ are the coefficients of the simple roots α_i of E_8 in the highest root $\widetilde{\alpha}$). One way of seeing this is to consider the versal deformation of an $\widetilde{E}_8$ singularity $\{z^2 = y^3 + Byx^4 + Cx^6\} \subset (\mathbb{C}^3, 0)$, which is given by

$$\{z^2 = y^3 + Byx^4 + Cx^6 + t_1 yx^3 + \cdots + t_5 x^5 + \cdots + t_{10}\} \subset (\mathbb{C}^3, 0) \times (\mathbb{C}^{10}, 0). \quad (5.10)$$

Since this is a singularity with $\mathbb{G}_m$-action in the sense of Pinkham, one gets a $\mathbb{C}^*$-action on the germ $(t_i) \in (\mathbb{C}^{10}, 0)$ with weights $0, -1, -2, -2, -3, -3, -4, -4, -5, -6$. The deformations in the 0-direction correspond to equisingular deformations (i.e. keep the $\widetilde{E}_8$ singularity, but modify the j-invariant). The deformations in the negative weight correspond to smoothing deformations, and modulo $\mathbb{C}^*$ the resulting quotient $\mathbb{P}(1, 2, 2, 3, 3, 4, 4, 5, 6)$ is the moduli space of del Pezzo pairs (V, D) with D isomorphic to the fixed elliptic curve $E = V(z^2 - y^3 + Byx^4 + Cx^6)$. Simply, this corresponds to homogenizing the equation of the versal deformation of $\widetilde{E}_8$; the result is the del Pezzo equation (5.9) (here $x = x_1/x_0$ and the section D corresponds to the hyperplane at infinity $V(x_0)$).

Remark 5.11. The homogenized version of (5.10) can be understood as a normal form for pairs (V, D) (where, as above, D is the hyperplane at infinity). For fixed D, such a normal form is unique up to the $\mathbb{C}^*$ scaling of the parameters t_i, and possibly some finite group action. Furthermore, we note that this normal form is still valid in case the curve D becomes nodal.

Remark 5.12. Alternatively, the moduli space of pairs (V, D) with $D \cong E$ (a fixed elliptic curve) can be obtained by considering the mixed Hodge structure (MHS) on $V \setminus D$. Since V is the blow-up of $\mathbb{P}^2$ at eight points lying on E, the classifying space for these MHS is $E \otimes_{\mathbb{Z}} E_8$. Thus, the moduli space of pairs (V, D) is

$$(E \otimes_{\mathbb{Z}} E_8)/W(E_8) \cong \mathbb{P}(1, 2, 2, 3, 3, 4, 4, 5, 6),$$

the isomorphism to the weighted projective space being the content (in more general circumstances) of the above mentioned theorem of Looijenga [Loo77]. This description allows us to see the moduli space of semistable models $X_0 = V_1 \cup_E V \cup_E V_2$ (for E fixed) as the product $\mathbb{P}(1, 2, 2, 3, 3, 4, 4, 5, 6) \times \mathbb{P}^1 \times \mathbb{P}(1, 2, 2, 3, 3, 4, 4, 5, 6)$ (by applying Looijenga's theorem to the root lattice $R = 2E_8 + A_1$; see also Rem. 6.5).

Here, we are interested in the moduli space of triples $(V, D; H)$ where V is a degree 1 del Pezzo, $D \cong E$ is a fixed anticanonical section, and H is another anticanonical section. By a simple modification of the arguments above for (V, D), we get:

Lemma 5.13. *The moduli space of triples $(V, D; H)$, where V is a degree 1 del Pezzo, $D, H \in |-K_V|$, $D \neq H$, and $D \cong E$ is fixed, is the quasi-projective variety*

$$\mathbb{P}(1, 1, 2, 2, 3, 3, 4, 4, 5, 6) \setminus \{(1 : 0 : \cdots : 0)\}.$$

Furthermore, it has a 1-point compactification to the weighted projective space $\mathbb{P}^9(1, \ldots, 6)$. The extra point naturally corresponds to the cone over E (with an $\widetilde{E}_8$ singularity at the vertex v) together with a hyperplane section away from v.

Proof. We obtain this by considering as before the negative weight deformations of $\widetilde{E}_8$. We define H to be the hyperplane $\{x = a\}$ in affine coordinates for $a \in \mathbb{C}$ (or equivalently $\{x_1 = ax_0\}$ after homogenization). As before we obtain the affine quotient $(\mathbb{C} \times \mathbb{C}^9)/\mathbb{C}^*$, which gives the weighted projective space from the theorem. Note that if the $\mathbb{C}^9$ component is non-zero we obtain a unique pair (V, D) with V a degree 1 del Pezzo (with at worst ADE singularities) and $D \cong E$. From the $\mathbb{C}$ component of the parameter space, we also get a hyperplane section $H \in |-K_V| \setminus \{D\}$ (N.B. $H = D$ corresponds to $a = \infty$). Finally, if the $\mathbb{C}^9$ component vanishes, we must have $a \neq 0$ and then we get the cone over E together with a hyperplane section not passing through the vertex $v = (1 : 0 : 0 : 0)$ of the cone. $\square$

Remark 5.14. The weighted blow-up of $(1 : 0 : \cdots : 0) \in \mathbb{P}^9(1, 1, 2, \ldots, 6)$ will give a $\mathbb{P}^1$-fibration over $\mathbb{P}^8(1, 2, \ldots, 6)$. This corresponds geometrically to the triples $(V, D; H)$ with $D \cong E$ fixed and H moving in the linear system $H \in |-K_V| \cong \mathbb{P}^1$ with no restriction on H. Thus, the difference from the moduli space of the lemma is that all triples $(V, D; H)$ with $D = H$ are replaced in Lemma 5.13 by the cone over E (plus a general hyperplane section). This is the correct moduli space from the KSBA perspective (see also §6.4, especially case $2E_8 + A_1$ (C) and Fig. 5).

We conclude

Corollary 5.15. *The moduli space of pairs (X, H) with $X = V_1 \cup_E V_2$, where the V_i are degree 1 del Pezzo or degenerations (in $\mathbb{P}(1, 1, 2, 3)$), $H_{|V_i} \in |-K_{V_i}|$, and such that E is fixed and $(X, \epsilon H)$ is slc, is*

$$\mathbb{P}(1, 1, 2, 2, 3, 3, 4, 4, 5, 6) \times \mathbb{P}(1, 1, 2, 2, 3, 3, 4, 4, 5, 6).$$

Proof. This follows from the previous lemma by noting that (X, H) is uniquely determined by $(V_i, D_i; H_i)$ (where $H_i = H_{|V_i}$ and $D_i \cong E$). $\square$

5.2.3. The structure of $\widehat{\mathcal{P}}_2$ near $\widetilde{Z}_1$. Let $x = (c, l) \in \mathbb{P}^N \times \mathbb{P}^2$ (where $\mathbb{P}^N$ is the Hilbert scheme of sextics) be a point corresponding to the minimal orbit (C, L) given by $C = V((x_0x_2 - \alpha_1 x_1^2)(x_0x_2 - \alpha_2 x_1^2)(x_0x_2 - \alpha_3 x_1^2))$, and $L = V(x_1)$. We are interested in the structure of the quotient $\widehat{\mathcal{P}}_2$ near the projection $\bar{x} \in \widetilde{Z}_1$ of x.

As before, by Luna's slice theorem a local model is given by the normal slice $\mathcal{N}_x$ to the orbit $G \cdot x$. It is immediate to see that the stabilizer group G_x acts on the space of sextics $T_x \mathbb{P}^N$ with weights

Weight	± 6	± 5	± 4	± 3	± 2	± 1	0
Multiplicity	1	1	2	2	3	3	3

and on the space of linear forms $T_l \mathbb{P}^2$ by weights ± 1. Since $G \cdot x = G/G_x$, the group G_x acts on the tangent space $T_x(G \cdot x)$ to the orbit by weights $\pm 2, \pm 1$, and 0 with multiplicities 1, 2, and 1 respectively. We conclude that $G_x \cong \mathbb{C}^*$ acts on $N_x \cong \mathbb{C}^{22}$ by

Weight	± 6	± 5	± 4	± 3	± 2	± 1	0
Multiplicity	1	1	2	2	2	2	2

Thus locally (in etale topology) near $\bar{x}$, $\widehat{\mathcal{P}}_2$ is the quotient of $\mathbb{C}^{22}$ by $\mathbb{C}^*$ with weights given as above. The two-dimensional 0-weight direction corresponds to deformations preserving the strictly minimal orbit. We conclude:

Lemma 5.16. *The fiber of the Kirwan blow-up $\widetilde{\mathcal{P}} \to \widehat{\mathcal{P}}_2$ over $\bar{x} \in \widetilde{Z}_1$ is*

$$\mathbb{P}(1, 1, 2, 2, 3, 3, 4, 4, 5, 6) \times \mathbb{P}(1, 1, 2, 2, 3, 3, 4, 4, 5, 6). \qquad \square$$

Using this lemma and the $\mathbb{P}^1$-fibration of $\widetilde{Z}_1$ given by the j-invariant (see Rem. 1.10), we find that the exceptional divisor $\Delta_{2E_8+A_1}$ of $\widetilde{\mathcal{P}}$ has a fibration over $\mathbb{P}^1$ (the compactified j-line) with fiber $W\mathbb{P}^9 \times W\mathbb{P}^9 \times \mathbb{P}^1$ (cf. Rem. 5.12). Geometrically, these fibers parameterize surfaces of type $V_1 \cup_E V \cup_E V_2$ as described in §5.1. The projection $\widetilde{\mathcal{P}} \to \overline{\mathcal{P}}_2$ is then given by the contraction of the $\mathbb{P}^1$ direction. For a further discussion of the geometry in this case see §6.4 (especially Fig. 5).

6. Classification of type II degenerations

As established above, the moduli space $\overline{\mathcal{P}}_2$ of stable pairs maps to the Baily–Borel $(\mathcal{D}/\Gamma_2)^*$, which is generically a $\mathbb{P}^2$-bundle. In this section we discuss the structure of the boundary of $\overline{\mathcal{P}}_2$ over the Type II boundary in $(\mathcal{D}/\Gamma_2)^*$.

We recall that the semistable model in a Type II degeneration is a chain of surfaces $X_0 = V_0 \cup \cdots \cup V_r$ (with V_i meeting V_{i+1}) such that

- V_0 and V_r are rational surfaces;
- the V_i are elliptic ruled surfaces;
- the double curves are smooth elliptic and isomorphic to a fixed curve E;
- the double curves are anticanonical sections for V_0 and V_r and sections for V_i for $i \neq 0, r$.

The normalization X^ν of the central fiber $X = \overline{X}_0$ of a relative log canonical model will have at most two simple elliptic singularities (besides ADE singularities), and the double curve of the normalization will be either empty (if X is normal with simple elliptic singularities) or give disjoint elliptic curves, all isomorphic to E.

Associated to every Type II degeneration of $K3$ surfaces there are two basic invariants: a continuous invariant, the modulus of E (possibly with a level structure), and a discrete invariant, the isometry class of the lattice $\mathrm{Gr}_2^W H^2(X_0)_{\mathrm{prim.}}$ (N.B. it is defined over $\mathbb{Z}$, see [Fri84, §3] for details). The discrete invariant determines the Type II boundary component to which a one-parameter semistable degeneration with central fiber X_0 would map. The continuous invariant determines the actual point in the Type II component where the degeneration maps (recall that the Type II components are quotients of $\mathfrak{H}$ by modular groups).

For degree two, there are four Type II Baily–Borel boundary components, labeled by the root lattices A_{17}, $E_7 + D_{10}$, $D_{16} + A_1$, and $2E_8 + A_1$. The geometric meaning of these components (via GIT) was explained in Section 1 (see Figure 2). Furthermore, Friedman [Fri84, Thm. 5.4] has classified the semistable models corresponding to these four cases, subject to the following normalization assumptions: there are only two components for X_0 (i.e. a union of two rational surfaces), and the polarization meets the double curve (i.e. $L_i.D_i > 0$) (N.B. any Type II degeneration can be arranged to satisfy these conditions, cf. [Fri84, Thm. 2.2]).

As we will see below, the Friedman semistable models can be used to understand all the Type II boundary pairs in $\overline{\mathcal{P}}_2$. However, as the proof of Theorem 2.9 shows, one needs to allow two operations on Friedman's models: base changes (introducing elliptic ruled surfaces in the middle) and twists by components V_i of X_0 (these have the effect of modifying the polarization on (V_i, D_i) from L_i to $L_i - D_i$). Combining the list of polarized semistable models of Friedman with the GIT analysis of Section 4, one gets a clear picture of $\overline{\mathcal{P}}_2 \to (\mathcal{D}/\Gamma_2)^*$ over Type II strata. We discuss this below. The discussion can be summarized as follows:

Theorem 6.1. *The preimage in $\overline{\mathcal{P}}_2$ of the Type II boundary in $(\mathcal{D}/\Gamma_2)^*$ consists of six irreducible components II_i as summarized in Table 1 (the index i corresponds to the case in the table). Furthermore, via $\overline{\mathcal{P}}_2 \to (\mathcal{D}/\Gamma_2)^*$,*

(i) II_1 *maps to* $\mathrm{II}_{A_{17}}$ *(see Prop. 6.3);*
(ii) II_2 *and* II_5 *map to* $\mathrm{II}_{2E_8+A_1}$ *(see Prop. 6.8);*
(iii) II_3 *maps to* $\mathrm{II}_{D_{16}+A_1}$ *(see Prop. 6.7);*
(iv) II_4 *and* II_6 *map to* $\mathrm{II}_{E_7+D_{10}}$ *(see Prop. 6.4).*

Proof. As discussed, the stable limits of degenerations of $K3$ surfaces are essentially determined by the components that are 0-surfaces. The polarized 0-surfaces in degree 2 are classified by Proposition 3.14; this gives the six cases of Table 1. The fact that these cases occur and that there is exactly one boundary component associated to each case follows from the GIT analysis. A detailed discussion of the GIT models and of the connection to the abstract point of view is done in Propositions 6.3, 6.4, 6.7, and 6.8 below. $\square$

Remark 6.2. The analysis of the Type II boundary of $\overline{\mathcal{P}}_2$ does not depend (after ignoring finite quotient issues) on the j-invariant associated to the corresponding geometric object (cf. Rem. 1.10). Thus, the preimages of Type II components in $\overline{\mathcal{P}}_2$ will be certain fibrations over the affine j-line. The limits as $j \to \infty$ give Type III pairs; the classification of those will be discussed in Section 7.

6.1. Case A_{17} ([Fri84, (5.2.1)], [Sha80, Thm. 2.4 (II.4)], [Tho10, Table 1 (II.3)])

The Type II Baily–Borel boundary component $\mathrm{II}_{A_{17}}$ corresponds to the stratum Z_4 in the GIT quotient $\widehat{\mathcal{M}}$, and in fact $\widehat{\mathcal{M}} \to (\mathcal{D}/\Gamma_2)^*$ is an isomorphism along this stratum. Since the stratum Z_4 corresponds to stable GIT points, it follows that $\widehat{\mathcal{P}}_2 \to \widehat{\mathcal{M}}$ is a $\mathbb{P}^2$-fibration (up to finite stabilizers) over $Z_4 \cong \mathbb{A}^1$. Finally, $\overline{\mathcal{P}}_2$ and $\widehat{\mathcal{P}}_2$ agree over this stratum. Thus, we conclude (see §4.1 for the last statement):

Proposition 6.3. *The moduli space $\overline{\mathcal{P}}$ of pairs is* (*up to finite stabilizers*) *a $\mathbb{P}^2$-fibration over the Type II boundary component $\mathrm{II}_{A_{17}}$. In fact, the closure of this locus remains a $\mathbb{P}^2$-bundle over $\overline{\mathrm{II}}_{A_{17}} \cong \mathbb{P}^1$.*

6.1.1. GIT model. The underlying surface $\overline{X}_0$ is

$$z^2 = f_3(x_i)^2$$

for a smooth plane cubic f_3. The normalization of $\overline{X}_0$ is two copies of $\mathbb{P}^2$ with the double curve being the elliptic curve $E = V(f_3)$. Since the plane sextic f_3^2 is stable and we are working with $0 < \epsilon \ll 1$ linearization, the choice of polarizing divisor is irrelevant here. The same is true from the KSBA perspective, as the polarizing divisor (a line in each copy of $\mathbb{P}^2$) cannot have a common component with the double curve.

6.1.2. Friedman's model. The semistable surface $X_0 = V_1 \cup V_2$ is obtained by gluing two copies of $\mathbb{P}^2$ along an elliptic curve, and then blowing up 18 times along the elliptic curve. The polarization is the pull-back to V_i of a line in $\mathbb{P}^2$. Thus, the relative log canonical model $\overline{X}_0$ is obtained by contracting all these (-1)-curves, and coincides with the GIT model. Note also that in this situation the polarizations L_i on the components (V_i, D_i) satisfy $L_i^2 = 1$ and $L_i.D_i = 3$. Thus, no twist is possible (cf. Lem. 3.5).

6.2. Case $E_7 + D_{10}$ ([Fri84, (5.2.3)], [Sha80, Thm. 2.4 (II.2)], [Tho10, Table 1 (II.0h), (II.1)])

In this case, the corresponding GIT stratum is Z_2. Again $\widehat{\mathcal{M}}$ and $(\mathcal{D}/\Gamma_2)^*$ agree over this stratum. Also, $\widehat{\mathcal{P}}_2$ and $\overline{\mathcal{P}}_2$ agree over the preimage of this stratum. However, in contrast to the previous case, $\widehat{\mathcal{P}}_2 \to \widehat{\mathcal{M}}$ is not a $\mathbb{P}^2$-bundle over Z_2. Here the choice of polarizing divisor instead of line bundle is essential: without a divisor one only gets strictly semistable points; in contrast when the divisor is considered, all the pairs are either stable or unstable (in the GIT sense, but this coincides with the KSBA stability here). The analysis of the models associated to this stratum gives the following result:

Proposition 6.4. *The fiber over a point of the boundary component* $\mathrm{II}_{E_7+D_{10}}$ *consists of the closure of the following two strata:*

(A) *a 9-dimensional stratum parameterizing the triples* $(V, D; H)$ *where V is a degree 2 del Pezzo surface, $D, H \in |-K_V|$, $D \cong E$ is a fixed elliptic curve, and $H \neq D$;*

(B) *a 12-dimensional component parameterizing the surfaces that are double covers of $\mathbb{P}^2$ branched in a reduced sextic with an $\widetilde{E}_7$ singularity and a hyperplane section not passing through the singularity.*

The closure of each of these components is obtained by adding a rational curve, which is common to both. The gluing curve parameterizes the pairs (X, H) with X a minimal elliptic ruled surface with a section of self-intersection 2 and another section collapsed to an $\widetilde{E}_7$ singularity, and a divisor $H \in |\sigma + 2f|$ (where σ is the class of the (-2)-section, and f the class of a fiber).

6.2.1. GIT model. The minimal orbits corresponding to points of Z_2 are given by $x_0^2 f_4(x_1, x_2)$ for a binary quartic with distinct roots. One distinguishes three distinct geometric possibilities:

(A) a sextic containing a double line: the normalized double cover will be a degree 2 del Pezzo, and the line will give (as the double curve of the normalization) the anticanonical section D;

(B) a reduced sextic with a unique $\widetilde{E}_7$ singularity;

(C) a sextic with both an $\widetilde{E}_7$ and a double line.

When considering additionally a hyperplane section (i.e. passing from $\widehat{\mathcal{M}}$ to $\widehat{\mathcal{P}}$), the orbits become separated as in the toy example of §4.1. As discussed in Proposition 4.5, the restrictions for the hyperplane section are not to pass through the $\widetilde{E}_7$ singularity or to coincide with the double line. A simple dimension count (for a fixed j-invariant) gives the dimensions of the proposition. For example, the moduli space of degree 2 del Pezzo surfaces containing a fixed elliptic curve E is 7-dimensional and isomorphic to the weighted projective space $(E \otimes_{\mathbb{Z}} E_7) / W(E_7) \cong \mathbb{P}^7(1, 1, 2, 2, 2, 3, 3, 4)$. The choice of polarizing divisor gives two additional dimensions.

Finally, case (C) is a specialization of both (A) and (B). The double cover associated to the sextic of case (C) is, after normalization and resolution of the $\widetilde{E}_7$ singularity, a minimal elliptic ruled surface with marked sections of self-intersection 2 and -2. Because of the $\mathbb{C}^*$ stabilizer in case (C), there is (up to isomorphism) only a one-dimensional choice for the hyperplane section H. Explicitly, X is the normalization of the double cover $z^2 = x_0^2 f_4(x_1, x_2)$ and H is the pull-back of the line $L = V(x_0 + bx_1 + cx_2)$ in $\mathbb{P}^2$ and the modulus is given by $(b : c) \in \mathbb{P}^1$.

6.2.2. Friedman's model. The semistable model in this case, $X_0 = V_1 \cup V_2$, gives, after the contraction of the (-1)-curves orthogonal to the polarization, the following two relatively minimal polarized anticanonical surfaces: a degree 2 del Pezzo $(\overline{V}_1, \overline{D}_1; \overline{L}_1)$ with $\overline{D}_1, \overline{L}_1 \in |-K_{\overline{V}_1}|$, and $(\overline{V}_2, \overline{D}_2; \overline{L}_2) \cong (\mathbb{F}_1, 2\sigma + 4f; f)$. Note that to get the semistable model X_0 starting from $\overline{V}_1 \cup \overline{V}_2$ one needs ten additional blow-ups, but these are irrele-

vant from the perspective of the relative log canonical model. Also note that in this model V_1 is 0-surface, and V_2 is a 1-surface. Thus, the central fiber of the relative log canonical model will be just $\overline{V}_1$ with $\overline{D}_1$ marked. This is precisely case (A) above.
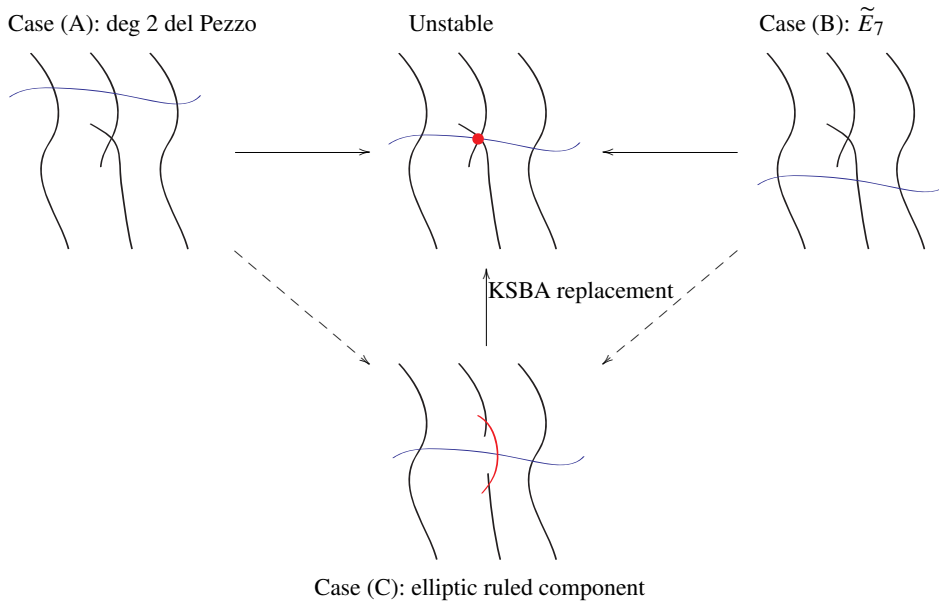
Case (A): deg 2 del Pezzo   Unstable   Case (B): $\widetilde{E}_7$

Case (C): elliptic ruled component

Fig. 4. The degenerations in the $E_7 + D_{10}$ case.

It is easy to understand how the other models occur: when trying to compactify the moduli space of pairs, one has to replace the case $\overline{H}_1 = \overline{D}_1$ (the polarizing divisor coincides with the double curve) by a slc model. This is achieved by replacing the semistable model $V_1 \cup V_2$ by a model $V_1 \cup V \cup V_2$ where V is a minimal elliptic ruled surface glued along two sections: D_1 of self-intersection -2 and D_2 of self-intersection 2. When $H_1 = D_1$ occurs, one applies the twist, and moves the polarization from V_1 to V (see Figure 4).

Finally, note that the semistable model $X_0 = V_1 \cup V \cup V_2$ dominates all the GIT models (A), (B), and (C). The difference between the three cases is given by the lift of the polarization $\mathcal{L}^*$ in a one-parameter degeneration $\mathcal{X}/\Delta$ from the generic fiber to the family. Explicitly, encoding the lift $\mathcal{L}$ by the degrees on the components of X_0, we get:

- $(2, 0, 0)$ corresponds to (A) (degree 2 del Pezzo);
- $(0, 2, 0)$ corresponds to (C) (elliptic ruled with a section collapsed to an $\widetilde{E}_7$ singularity);
- $(0, 0, 2)$ corresponds to (B) (rational with an $\widetilde{E}_7$ singularity).

Note that the lifts of $\mathcal{L}$ are related by the twist operation.

Remark 6.5 ($\overline{\mathcal{P}}_d$ vs. partial toroidal compactifications). The deformation theory of a semistable model X_0 is well understood: one can obtain a partial compactification of the moduli space of $K3$ surfaces by adding a divisor parameterizing semistable models of

a fixed combinatorial type and satisfying the *d-semistability condition* of Friedman (see [Fri84, §4], [Fri83a], [KN94], and [Ols04]). On the other hand, for Type II boundary components (for Type III see [FS85]), it is known that there exists a unique toroidal compactification over a Type II boundary stratum in $(\mathcal{D}/\Gamma_d)^*$. One can show that the geometric divisor obtained by local trivial deformations of semistable models and the toroidal divisor can be identified via the theory of degenerations of Hodge structures (this is essentially the content of [Fri84]). However, the issue is that, in the polarized case, there are several possible liftings of the polarization. Thus one obtains several boundary divisors (three in the example above) which are not distinguishable at the level of Hodge theory. The choice of polarizing divisor (giving $\mathbb{P}^g$-bundles over these boundary divisors) allows one to glue the various boundary divisors. In the $\mathbb{P}^g$-bundle $\overline{\mathcal{P}}_d$ over $\mathcal{F}_d$ these divisors are contracted to smaller-dimensional components.

For instance, in the $E_7 + D_{10}$ case discussed above, the deformation theory for $X_0 = V_1 \cup V \cup V_2$ will give a boundary divisor which is essentially a $W\mathbb{P}^7 \times W\mathbb{P}^{10}$-fibration (coming from $(E_7 \otimes E)/W(E_7)$ and $(D_{10} \otimes E)/W(D_{10})$ respectively) over the j-line. The choice of the lifting polarization gives three copies of this divisor, say $\Delta_{(2,0,0)}$, $\Delta_{(0,2,0)}$, and $\Delta_{(0,0,2)}$ (N.B. these can be viewed as divisors in a partial non-separated compactification of $\mathcal{F}_2$, cf. [Ols04]). The choice of a polarizing divisor H gives $\mathbb{P}^2$-bundles, say $\widetilde{\Delta}_{(2,0,0)} \to \Delta_{(2,0,0)}$, over each of these copies (N.B. $\widetilde{\Delta}_{(2,0,0)}$ can be viewed as divisors in a partial compactification of $\mathcal{P}_2$). Then the case $H_1 = D_1$ can be viewed as giving a gluing of the copy $\widetilde{\Delta}_{(2,0,0)}$ with the $\widetilde{\Delta}_{(0,2,0)}$ copy; and similar gluing for the divisor corresponding to $(0, 2, 0)$ and $(0, 0, 2)$. Thus at the level of pairs, it is possible to give a partial toroidal-like compactification for $\mathcal{P}_2$ by adding the divisors $\widetilde{\Delta}_{(*,*,*)}$. Finally, in $\overline{\mathcal{P}}_d$ these divisors will be collapsed to three smaller-dimensional strata (e.g. in the $(2, 0, 0)$ case the $\mathbb{P}^2$-bundle $\widetilde{\Delta}_{(0,2,0)}$ over $W\mathbb{P}^7 \times W\mathbb{P}^{10}$ will be collapsed to a $\mathbb{P}^2$-bundle over $W\mathbb{P}^7$, giving the 9-dimensional stratum (A)).

6.3. Case $D_{16} + A_1$ ([Fri84, (5.2.4)], [Sha80, Thm. 2.4 (II.2)], [Tho10, Table 1 (II.2)])

This case is quite similar to the A_{17} case: both Z_3 (this case) and Z_4 correspond to stable GIT loci. We note first that $\widehat{\mathcal{M}} \to (\mathcal{D}/\Gamma)^*$ is a $\mathbb{P}^1$-bundle over the component $\mathrm{II}_{D_{16}+A_1}$. Specifically, $\widehat{Z}_3 \setminus \hat{\tau} \to \mathrm{II}_{D_{16}+A_1} \cong \mathbb{A}^1$ is a $\mathbb{P}^1$-bundle, the map being given by the j-invariant (cf. Rem. 1.10). This corresponds to the following geometric fact:

Lemma 6.6. *The choice of a point in $\widehat{Z}_3 \setminus \hat{\tau}$ corresponds to the choice of an elliptic normal curve of degree 4 in $\mathbb{P}^3$ together with a quadric containing it.*

The points of $\widehat{Z}_3 \setminus \hat{\tau}$ are stable GIT points, and by construction it follows that $\widehat{\mathcal{P}}_2 \to \widehat{\mathcal{M}}$ is a $\mathbb{P}^2$-bundle over this locus. Finally, $\overline{\mathcal{P}}_2$ and $\widehat{\mathcal{P}}_2$ agree here (the preimage of $\widehat{Z}_3 \setminus \hat{\tau}$ is away from the flip locus). We conclude:

Proposition 6.7. *Over the component $\mathrm{II}_{D_{16}+A_1}$, $\overline{\mathcal{P}}_2 \to (\mathcal{D}/\Gamma)^*$ is a $\mathbb{P}^2 \times \mathbb{P}^1$-bundle.*

6.3.1. The GIT model. The equation of the sextic corresponding to this case is $q_0^2 q$, with the conditions that q_0 is smooth, q is reduced, and q_0 and q intersect transversely. The

normalization V_1 of the double cover $z^2 = q_0^2 q$ is the quadric surface $z^2 = q$ in $\mathbb{P}^3$. The double curve of the normalization will be an elliptic curve D_1 which is the double cover of the conic $V(q_0)$ branched at the four intersection points. A similar picture holds also in the unigonal case (i.e. $U_3 \subset \widehat{Z}_3$). This concludes the proof of Lemma 6.6. Note that in this case all the points are stable, thus this stratum is modular even without the choice of a divisor.

Finally, the hyperplane section is the pull-back of a line in $\mathbb{P}^2$. It is a hyperplane section of the quadric V_1 but it is not an arbitrary section: in fact, it lies in a two-dimensional linear subsystem characterized by the property that it cuts the elliptic curve D_1 in two conjugate points. This somewhat surprising fact is explained by the analysis of the semistable model below.

6.3.2. Friedman's model. The relatively minimal models of the two components in this case are $(\overline{V}_1, \overline{D}_1; \overline{H}_1) = (\mathbb{F}_0, 2f_1 + 2f_2; f_1 + f_2)$, $(\overline{V}_2, \overline{D}_2; \overline{H}_2) = (\mathbb{F}_1, 2\sigma + 4f; 2f)$. Note that $H_1^2 = 2$, $H_2^2 = 0$, $H_1.D_1 = H_2.D_2 = 4$. Since $H_1.D_1 > H_1^2$, it follows that no twisting is possible. This means that in contrast to the $E_7 + D_{10}$ case there is only one model. Note that the condition on the hyperplane section noted in the previous paragraph (i.e. H_1 cuts the elliptic double curve D_1 in two conjugate points) is imposed by the requirement of extending the polarization to the second component (even though this component is a 1-surface, which is contracted to the double curve in the log canonical model).

Abstractly, this case corresponds to the case of polarized anticanonical pairs $(V, D; L)$ with $L^2 = 2$ and $L.D = 4$. From Proposition 3.8, we know that V has to be a scroll and in fact a quadric surface in $\mathbb{P}^3$. The results of Harbourne (e.g. Thm. 3.11) say that the linear system $|L|$ is base point free. Thus, the occurrence of the unigonal case might seem contradictory. The resolution of this apparent contradiction was given above: the allowed polarizing divisors are members of a linear subsystem of $|L|$.

6.4. Case $2E_8 + A_1$ ([Fri84, (5.2.2)], [Sha80, Thm. 2.4 (II.1)], [Tho10, Table 1 (II.0h)], [Tho10, Table 2 (II.0u)], [Tho10, Table 2 (II.4)])

This case is the most involved one. Specifically, on the GIT side this corresponds to the stratum $\widehat{Z}_1$, parameterizing curves with $\widetilde{E}_8$ singularities; these are strictly semistable points. When we consider the polarizing divisor, the orbits become stable if the divisor does not pass through the $\widetilde{E}_8$ singularity. If it passes through the singularity, we obtain a strictly semistable object, which will be replaced via the flip discussed in Section 5 by the case of two components (which both have to be del Pezzo's of degree 1). We conclude:

Proposition 6.8. *The fiber in $\overline{\mathcal{P}}_2$ over a point in $\mathrm{II}_{2E_8+A_1} \subset (\mathcal{D}/\Gamma)^*$ consists of two components:*

(A) *A component of dimension 18 parameterizing (X, H) with $X = V_1 \cup_E V_2$, with both V_i being degree 1 del Pezzo surfaces glued along an elliptic curve E (such that the base points $p_i \in E$ of the anticanonical systems are matched). This case can further degenerate to cases (C) and (D) (where one or both V_i degenerate to elliptic ruled surfaces with $\widetilde{E}_8$ singularities).*

(B) *A component of dimension* 11 *parameterizing rational surfaces with* $\widetilde{E}_8$ *singularities together with hyperplane sections not passing through the singularities. This can further degenerate to case* (E) (*i.e. elliptic ruled surfaces with two* $\widetilde{E}_8$ *singularities*).

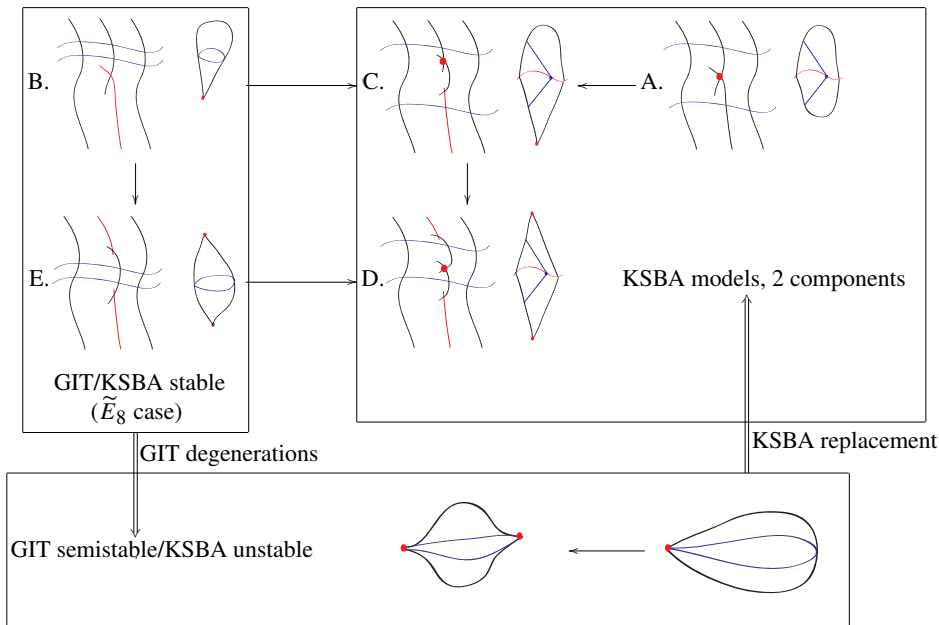*The two components are glued along the* 9-*dimensional* (*closure of the*) *stratum* (C) (*see Fig.* 5).

Fig. 5. The degenerations in the $2E_8 + A_1$ case.

Remark 6.9. As already discussed in Section 5, one can be very precise about the structure of the various strata occurring in the proposition. For instance, the closure of the stratum (A) is the product of two weighted projective spaces $\mathbb{P}^9(1, 1, \ldots, 5) \times \mathbb{P}^9(1, 1, \ldots, 5)$ (see Cor. 5.15).

6.4.1. GIT model. Here we have several models. First, we have plane sextic curves with a unique $\widetilde{E}_8$ (depending on ten moduli, one of which is the j-invariant) and sextics with two $\widetilde{E}_8$ (depending on two moduli) which are obtained from $q_1 q_2 q_3$ with a common axis. In the classification above, these cases correspond to (B) and (E). In case (E), via a partial smoothing one obtains case (B) (see Lem. 3.16).

Next, we consider additionally the hyperplane section. If the hyperplane does not pass through the $\widetilde{E}_8$ singularity, then the resulting pair is both GIT and KSBA stable. If the hyperplane passes through the singularity, the pair is GIT semistable and slc unstable. By applying a semistable reduction as discussed in Section 5, one obtains the case of two components which (unless they are cones) have to be degree 1 del Pezzo surfaces.

6.4.2. Friedman's model. Each of the two relatively minimal models $(\overline{V}_i, \overline{D}_i; \overline{H}_i)$ of the components of [Fri84, (5.2.2)] is a degree 1 del Pezzo surface with $\overline{D}_i, \overline{H}_i \in |-K_{\overline{V}_i}|$. This gives case (A) discussed above. As in the $E_7 + D_{10}$ case, additional models can be obtained by base change and twisting. For instance, starting with case (A), one needs to blow up two more times to get a semistable model. Then applying a twist gives case (B): a single rational component V_1, which is the blow-up of ten points on a cubic in $\mathbb{P}^2$. Here, $H_1^2 = 2$ (and $H_2^2 = 0$), $H_1.D_1 = 0$, $D_1^2 = -1$. By the results discussed in Section 3, such a case either leads to a double cover of $\mathbb{P}^2$ branched along a sextic with an $\widetilde{E}_8$ singularity if there is no fixed component, or to a unigonal type case (which corresponds to the $U_1 \subset \widehat{Z}_1$ stratum in the GIT model). Finally, applying base changes to Friedman's model followed by twists leads to cases (C), (D), and (E). It is interesting to note that all cases discussed in [Tho10, p. 21] occur in the $2E_8 + A_1$ situation.

7. Classification of Type III degenerations

We now discuss the case of Type III degenerations. According to the classification given by Proposition 3.14, every 0-surface $(V, D; L)$ that occurs in a Type III degeneration has a partial smoothing to a Type II case $(V', D'; L')$, i.e. as polarized surfaces (V, L) and (V', L') are deformation equivalent, D' is a smoothing of the cycle of rational curves D. Thus, we obtain

Theorem 7.1. *The Type III locus in $\overline{\mathcal{P}}_2$ is the closure of the Type II locus, in the sense of taking the closure (as $j \to \infty$) of the fibrations over the Type II Baily–Borel boundary components in $(\mathcal{D}/\Gamma_2)^*$. In particular, there are six Type III boundary components III_i in $\overline{\mathcal{P}}_2$ of dimensions 2, 18, 3, 9, 11, and 12 as described in Table 1. Each of these components is irreducible except III_3 which splits into two irreducible components III_γ and III_δ. The incidence relations of Type III components are summarized in Table 2.*

Remark 7.2. We note that the statement that all 0-surfaces of Type III have a deformation to a Type II polarized anticanonical pair is only true for low degrees. For instance, the surfaces $\mathbb{F}_n$ (for $n \geq 3$) carry an effective anticanonical divisor of Type III, but none of Type II. Thus, for large degrees, at least a priori, there might be degenerations of Type III that are not limits of Type II degenerations.

We will denote the six Type III boundary components by III_i for $i \in \{1, \dots, 6\}$ according to Table 1 and Proposition 3.14. The generic point of each of these components was already described. Also, their basic structure is similar to that of their Type II counterparts (see Props. 6.3, 6.4, 6.7, and 6.8). The only significant difference is that the gluing of the Type III strata is more involved, reflecting the fact that it is easier for the polarizing divisor to pass through a log canonical center (i.e. H might pass through a triple point, or contain a component of the anticanonical cycle vs. H has to contain the anticanonical curve in the Type II case). A summary of the strata resulting from incidence of several Type III boundary components III_i is given in Table 2 below. Note that the components III_i for $i \neq 3$ are irreducible, but for $i = 3$ we have a decomposition into irreducible compo-

Table 2. The incidence of Type III boundary components in $\overline{\mathcal{P}}_2$.

	Description (generic point)	dim	Contained in
α	$X = V_0 \cup_D V_1$, $V_i \cong \mathbb{P}^2$, D a nodal cubic	1	$\mathrm{III}_1, \mathrm{III}_\beta, \mathrm{III}_\delta$
β	X^ν a deg 2 del Pezzo with an A_1 at $p \in D$	8	$\mathrm{III}_4, \mathrm{III}_6$
γ	X^ν a quadric in $\mathbb{P}^3$, D a union of two conics	3	comp. of $\mathrm{III}_3, \mathrm{III}_\beta$
δ	X^ν a quadric in $\mathbb{P}^3$, D a nodal quartic	3	comp. of $\mathrm{III}_3, \mathrm{III}_5$
ϵ	X^ν a quadric, $D = C \cup L_1 \cup L_2$, C a conic, L_i a line	2	$\mathrm{III}_\gamma, \mathrm{III}_\delta$
ϕ	$X = V_0 \cup_D V_1$, V_0 a deg 1 del Pezzo, $V_1^\nu \cong \mathbb{P}^2$, D nodal	9	$\mathrm{III}_2, \mathrm{III}_5$
ζ'	$X = V_0 \cup_D V_1$, $V_i^\nu \cong \mathbb{P}^2$, D nodal	0	III_ϕ
ζ	$X = V_0 \cup_D V_1$, $V_i \cong \mathbb{P}^2$, D a triangle	0	$\mathrm{III}_\alpha, \mathrm{III}_\epsilon$

nents $\mathrm{III}_3 = \mathrm{III}_\gamma \cup \mathrm{III}_\delta$. We include III_γ and III_δ in the table as they are included in some other components III_i.

Remark 7.3. As for Table 1, when describing the stable (X, H) corresponding to the generic point of a boundary stratum we ignore the polarizing divisor H. The dimension is the dimension of the stratum in $\overline{\mathcal{P}}_2$ and thus takes H into account. Note that sometimes X has positive-dimensional stabilizer, leading to strata of dimension less than 2 (cf. Rem. 2.2). Finally, D refers to a cycle of rational curves which is an anticanonical divisor on the normalized components of X. In some cases D passes through a canonical singularity of (the normalization of) X; a resolution of the singularity will bring X and D in a standard form.

To begin, we note that the gluing of Type III strata will reflect the structure of the boundary in the GIT quotient $\widehat{\mathcal{M}}$ (see Figure 2) and the gluing of the type II components in $\overline{\mathcal{P}}_2$. Namely, we recall that in the GIT quotient $\widehat{\mathcal{M}}$, the Type III stratum is mapped to the rational curve $\hat{\tau} \cup \zeta$. The point ζ corresponds to the gluing of all strata. Similarly, the affine curve $\hat{\tau}$ corresponds to the gluing of the strata corresponding to $E_8^2 + A_1$ and $D_{16} + A_1$. At the level of Type II pairs, the only gluing occurs for $E_7 + D_{10}$ cases (A) and (B) (corresponding to III_4 and III_6) and for $E_8^2 + A_1$ cases (A) and (B) (corresponding to III_2 and III_5). Using this information, we now identify for each Type III component III_i some substrata along which the given component glues to other Type III components.

Remark 7.4. In general, given a boundary pair (X, H) in $\mathcal{P}_d$, further degenerations of it will have no fewer components. It follows that the two boundary components III_1 and III_2 that parameterize the degenerations $X = V_1 \cup_E V_2$ with two components are disjoint. These two boundary components will meet the other boundary components along III_α and $\mathrm{III}_\zeta \in \mathrm{III}_\alpha$ for III_1 and along III_ϕ and $\mathrm{III}_{\zeta'} \in \mathrm{III}_\phi$ for III_2. The points III_ζ and $\mathrm{III}_{\zeta'}$ are in a certain sense the deepest degenerations for degree 2 $K3$ pairs. They correspond to the so called *pillow degenerations* of $K3$ surfaces (see e.g. [CMT01]), i.e. unions of $\mathbb{P}^2$'s (polarized by $\mathcal{O}(1)$) glued according to the combinatorics of a triangulation of S^2 with d triangles (see also Rem. 7.5). An ongoing project of Gross–Hacking–Keel (see e.g. [GHK15]) investigates the deformations of such pillow surfaces (in all degrees) and thus (in particular) describes neighborhoods of III_ζ and $\mathrm{III}_{\zeta'}$ in $\overline{\mathcal{P}}_2$.

7.1. A_{17} case

As already mentioned in Proposition 6.3, the closure of the Type II locus in this case is still a $\mathbb{P}^2$-bundle over $\overline{\mathrm{II}}_{A_{17}} \cong \mathbb{P}^1$. The corresponding Type III stratum is III_1 and is two-dimensional. It parameterizes the stable pairs (X, H) where X is a union of two $\mathbb{P}^2$'s glued along a nodal cubic C, and H corresponds to the choice of a line not passing through the nodes of C (cf. §4.1). The generic case is when C is irreducible nodal; the stable pairs in this case belong only to the component III_1. The component III_1 will be glued to other components along the locus where C becomes reducible. We denote by $\mathrm{III}_\alpha \subset \mathrm{III}_1$ the rational curve corresponding to C reducible and by $\mathrm{III}_\zeta \in \mathrm{III}_\alpha$ the point corresponding to C becoming a triangle.

Note that the entire component $\mathrm{III}_1 \subset \overline{\mathcal{P}}_2$ maps to the point ζ in $\widehat{\mathcal{M}}$ (via $\overline{\mathcal{P}}_2 \dashrightarrow \widehat{\mathcal{P}}_2 \to \widehat{\mathcal{M}}$; N.B. the III_1 locus is not affected by the flip of Section 5). Note also that the point $\mathrm{III}_\zeta \in \overline{\mathcal{P}}_2$ corresponds to the minimal orbit associated to $\zeta \in \widehat{\mathcal{M}}$. We reiterate here that the main point is that at the level of pairs the moduli functor is separated and thus III_ζ corresponds to a single geometric object, in contrast to the point ζ which hides several orbits.

7.2. $E_7 + D_{10}$ case

As already discussed, in this case there are two geometric possibilities:

(III$_4$) Degenerations of $E_7 + D_{10}$ (A): the elliptic section of the degree 2 del Pezzo becomes nodal, or, in terms of sextics, a quartic plus a line tangent to it (the line being counted with multiplicity 2).

(III$_6$) Degenerations of $E_7 + D_{10}$ (B): the $\widetilde{E}_7$ singularity degenerates to a cusp singularity $T_{2,4,5}$ and then further to other cusps of type $T_{2,q,r}$ with $q \geq 4$, $r \geq 5$.

For a fixed invariant j, the Type II components $E_7 + D_{10}$ (A) and (B) are glued along a curve (see Prop. 6.4). The Type III limit (i.e. $j \to \infty$) of this curve in $\overline{\mathcal{P}}_2$ is the point III_ζ. Thus, we have $\mathrm{III}_\zeta \subseteq \mathrm{III}_4 \cap \mathrm{III}_6$. However, it is immediate to see that the intersection of these two Type III components is larger. Namely, the maximal stratum which is a common degeneration of both the degree 2 del Pezzo and $T_{2,4,5}$ cases corresponds to the double cover of $\mathbb{P}^2$ branched in a nodal quartic with a double line passing through it. From the del Pezzo perspective, this would be a nodal degree 2 del Pezzo with a hyperplane section through it (as an anticanonical section). From the perspective of cusp singularities, this is a degenerate cusp singularity, which has a partial smoothing to cusp singularities of type $T_{2,4,r}$. We call this stratum III_β. Note that $\mathrm{III}_\alpha \subset \mathrm{III}_\beta$ (and then $\mathrm{III}_\zeta \subset \mathrm{III}_\alpha \subset \mathrm{III}_\beta$).

We note that there is another special stratum (which is contained in III_β) in this case: the case of the double cover X of $\mathbb{P}^2$ branched along two double lines plus a generic quadric. The intersection of the two double lines leads to a degenerate cusp singularity, which has a partial smoothing to $T_{2,q,r}$ with $q, r \geq 5$. The normalization of X will be a quadric in $\mathbb{P}^3$ and the double curves of the normalization will be the union of two hyperplane sections (i.e. $(1, 1)$ curves in the case $X^\nu \cong \mathbb{P}^1 \times \mathbb{P}^1$). We call this stratum III_γ.

7.3. $D_{16} + A_1$ case

We recall that the type II degenerations corresponding to this case are surfaces X such that their normalizations are quadrics in $\mathbb{P}^3$, and the double curve E is an elliptic quartic curve in $\mathbb{P}^3$. These are obtained by considering double covers of type $z^2 = q_0^2 q$ of $\mathbb{P}^2$ (where q_0 and q are two conics). There are two distinct Type III degenerations (in codimension 1) in this situation: either $V(q_0)$ becomes singular (i.e. union of two lines) or the two conics become tangent. The first case was labeled as III_γ above. We call the second case III_δ. In other words, the Type III component in this case is reducible:

$$\mathrm{III}_3 = \mathrm{III}_\gamma \cup \mathrm{III}_\delta.$$

The two components intersect in the stratum corresponding to the double cover of $\mathbb{P}^2$ branched in two double lines together with a conic which is tangent to one of the lines. Equivalently, at the level of quadrics in $\mathbb{P}^3$, this corresponds to the case where the anti-canonical divisor D splits as $L_1 \cup L_2 \cup C$ with L_1 of type $(1, 0)$, L_2 of type $(0, 1)$ and C of type $(1, 1)$. We call this case III_ϵ.

7.4. $2E_8 + A_1$ case

Here again we have two possibilities:

(III_2) Degenerations of $2E_8 + A_1$ (A): the two degree 1 del Pezzo surfaces are glued along a nodal section. This can degenerate to the case when one or both of the del Pezzo surfaces become cones over this nodal curve. We denote this case by III_ϕ (a degeneration of the Type II case $2E_8 + A_1$ (C)).

(III_5) Degenerations of $2E_8 + A_1$ (B): the $\widetilde{E}_8$ becomes a $T_{2,3,7}$ or worse singularity. In the closure, we can obtain case III_ϕ as a replacement of the case when the polarizing divisor passes through the $T_{2,3,7}$ singularity (cf. Prop. 6.8), or case III_δ (a double conic plus another conic tangent to it), which is a degenerate cusp singularity that has a partial smoothing to $T_{2,3,7}$.

Remark 7.5. As already noted, a degree 1 del Pezzo can degenerate to the cone over an elliptic curve of degree 1 giving an $\widetilde{E}_8$ singularity (the Type II case $2E_8 + A_1$ (C)). This can further degenerate to the cone $\overline{V}$ over an irreducible nodal curve C of arithmetic genus 1 (case III_ϕ above). We note that the normalization V of $\overline{V}$ is $\mathbb{P}^2$. In fact, $\overline{V}$ is obtained from $V \cong \mathbb{P}^2$ by gluing together two lines. The nodal curve C is the image of a third line (forming a triangle D) in $\mathbb{P}^2$. In other words, $(\overline{V}, C)$ (or the normalized pair (V, D)) regarded as a 0-component of a degree 2 degeneration fits into the classification given by Proposition 3.14. However, the gluing in the limit surface X is somewhat surprising and hints at the difficulty of an analogous classification for larger degrees. Finally, we note that the two surfaces corresponding to III_ζ and $\mathrm{III}_{\zeta'}$ are obtained by gluing two copies of $\mathbb{P}^2$ with a marked triangle in each according to the two possible triangulations of S^2 with two triangles (see also [Thu98], [Laz08]).

References

[Ale94]   Alexeev, V.: Boundedness and K^2 for log surfaces. Int. J. Math. **5**, 779–810 (1994) Zbl 0838.14028 MR 1298994

[Ale96]   Alexeev, V.: Moduli spaces $M_{g,n}(W)$ for surfaces. In: Higher-Dimensional Complex Varieties (Trento, 1994), de Gruyter, Berlin, 1–22 (1996) Zbl 0896.14014 MR 1463171

[Ale02]   Alexeev, V.: Complete moduli in the presence of semiabelian group action. Ann. of Math. (2) **155**, 611–708 (2002) Zbl 1052.14017 MR 1923963

[Ale13]   Alexeev, V.: Moduli of weighted hyperplane arrangements with applications. Notes for the advanced course "Compactifying moduli spaces" (Barcelona, May 2013)

[Alp13]   Alper, J.: Good moduli spaces for Artin stacks. Ann. Inst. Fourier (Grenoble) **63**, 2349–2402 (2013) Zbl 314.14095 MR 3237451

[AJ89]   Arapura, D., Jaffe, D. B.: On Kodaira vanishing for singular varieties. Proc. Amer. Math. Soc. **105**, 911–916 (1989) Zbl 0685.32016 MR 0952313

[AGZV85] Arnol'd, V. I., Guseĭn-Zade, S. M., Varchenko, A. N.: Singularities of Differentiable Maps. Vol. I. Monogr. Math. 82, Birkhäuser Boston, Boston, MA (1985) Zbl 0554.58001 MR 0777682

[Bea04]   Beauville, A.: Fano threefolds and $K3$ surfaces. In: The Fano Conference, Univ. Torino, Turin, 175–184 (2004) Zbl 1096.14034 MR 2112574

[CMJL14] Casalaina-Martin, S., Jensen, D., Laza, R.: Log canonical models and variation of GIT for genus 4 canonical curves. J. Algebraic Geom. **23**, 727–764 (2014). Zbl 06353750 MR 3263667

[Che08]   Cheltsov, I.: Log canonical thresholds of del Pezzo surfaces. Geom. Funct. Anal. **18**, 1118–1144 (2008) Zbl 1161.14030 MR 2465686

[CMT01]   Ciliberto, C., Miranda, R., Teicher, M.: Pillow degenerations of $K3$ surfaces. In: Applications of Algebraic Geometry to Coding Theory, Physics and Computation (Eilat, 2001), NATO Sci. Ser. II Math. Phys. Chem. 36, Kluwer, Dordrecht, 53–63 (2001) Zbl 1006.14014 MR 1866895

[Fri83a]   Friedman, R.: Global smoothings of varieties with normal crossings. Ann. Math. (2) **118**, 75–114 (1983) Zbl 0569.14002 MR 0707162

[Fri83b]   Friedman, R.: Linear systems on anticanonical pairs. In: The Birational Geometry of Degenerations (Cambridge, MA, 1981), Progr. Math. 29, Birkhäuser Boston, Boston, MA, 162–171 (1983)

[Fri84]   Friedman, R.: A new proof of the global Torelli theorem for $K3$ surfaces. Ann. of Math. (2) **120**, 237–269 (1984) Zbl 0559.14004 MR 0763907

[FM83a]   Friedman, R., Miranda, R.: Smoothing cusp singularities of small length. Math. Ann. **263**, 185–212 (1983) Zbl 0488.14006 MR 0698002

[FM83b]   Friedman, R., Morrison, D. R. (eds.): The Birational Geometry of Degenerations. Progr. Math. 29, Birkhäuser Boston, MA (1983) Zbl 0493.00005 MR 0690262

[FS85] Friedman, R., Scattone, F.: Type III degenerations of $K3$ surfaces. Invent. Math. **83**, 1–39 (1985) Zbl 0613.14030 MR 0813580

[Fuj12] Fujino, O.: Semipositivity theorems for moduli problems. arXiv:1210.5784 (2012)

[GH94] Griffiths, P., Harris, J.: Principles of Algebraic Geometry. Wiley Classics Library, Wiley, New York (1994); reprint of the 1978 original Zbl 0836.14001 MR 1288523

[GHK15] Gross, M., Hacking, P., Keel, S.: Mirror symmetry for log Calabi–Yau surfaces I. Publ. Math. Inst. Hautes Études Sci. **122**, 65–168 (2015) MR 3415066

[Hac04] Hacking, P.: Compact moduli of plane curves. Duke Math. J. **124**, 213–257 (2004) Zbl 1060.14034 MR 2078368

[Hac12] Hacking, P.: Compact moduli spaces of surfaces of general type. In: Compact Moduli Spaces and Vector Bundles, Contemp. Math. 564, Amer. Math. Soc., Providence, RI, 1–18 (2012) Zbl 1254.14043 MR 2895182

[HKT09] Hacking, P., Keel, S., Tevelev, J.: Stable pair, tropical, and log canonical compactifications of moduli spaces of del Pezzo surfaces. Invent. Math. **178**, 173–227 (2009) Zbl 1205.14012 MR 2534095

[Har97a] Harbourne, B.: Anticanonical rational surfaces. Trans. Amer. Math. Soc. **349**, 1191–1208 (1997) Zbl 0860.14006 MR 1373636

[Har97b] Harbourne, B.: Birational morphisms of rational surfaces. J. Algebra **190**, 145–162 (1997) Zbl 0883.14017 MR 1442149

[HM98] Harris, J., Morrison, I.: Moduli of Curves. Grad. Texts Math. 187, Springer, New York (1998) Zbl 0913.14005 MR 1631825

[Has03] Hassett, B.: Moduli spaces of weighted pointed stable curves. Adv. Math. **173**, 316–352 (2003) Zbl 1072.14014 MR 1957831

[HK04] Hassett, B., J. Kovács, S.: Reflexive pull-backs and base extension. J. Algebraic Geom. **13**, 233–247 (2004) Zbl 1081.14017 MR 2047697

[Kaw88] Kawamata, Y.: Crepant blowing-up of 3-dimensional canonical singularities and its application to degenerations of surfaces. Ann. of Math. (2) **127**, 93–163 (1988) Zbl 0651.14005 MR 0924674

[KN94] Kawamata, Y., Namikawa, Y.: Logarithmic deformations of normal crossing varieties and smoothing of degenerate Calabi–Yau varieties. Invent. Math. **118**, 395–409 (1994) Zbl 0848.14004 MR 1296351

[KM97] Keel, S., Mori, S.: Quotients by groupoids. Ann. of Math. (2) **145**, 193–213 (1997) Zbl 0881.14018 MR 1432041

[KL04] Kim, H., Lee, Y.: Log canonical thresholds of semistable plane curves. Math. Proc. Cambridge Philos. Soc. **137**, 273–280 (2004) Zbl 1055.14032 MR 2090618

[KL89] Kirwan, F. C., Lee, R.: The cohomology of moduli spaces of $K3$ surfaces of degree 2. I. Topology **28**, 495–516 (1989) Zbl 0671.14018 MR 1030990

[Kol] Kollár, J.: Moduli for Varieties of General Type. Book in preparation.

[Kol90] Kollár, J.: Projectivity of complete moduli. J. Differential Geom. **32**, 235–268 (1990) Zbl 0684.14002 MR 1064874

[Kol08] Kollár, J.: Hulls and husks. arXiv:0805.0576 (2008)

[Kol10] Kollár, J.: Moduli of varieties of general type. In: Handbook of Moduli, Vol. II, Adv. Lectures Math. 25, Int. Press, Somerville, MA, 131–157 (2013) Zbl 1322.14006 MR 3184176

[KK10] Kollár, J., J. Kovács, S.: Log canonical singularities are Du Bois. J. Amer. Math. Soc. **23**, 791–813 (2010) Zbl 1202.14003 MR 2629988

[KM98] Kollár, J., Mori, S.: Birational Geometry of Algebraic Varieties. Cambridge Tracts in Math. 134, Cambridge Univ. Press, Cambridge (1998) Zbl 0926.14003 MR 1658959

[KSB88]   Kollár, J., I. Shepherd-Barron, N.: Threefolds and deformations of surface singularities. Invent. Math. **91**, 299–338 (1988) Zbl 0642.14008 MR 0922803

[KSS10]   Kovács, S. J., Schwede, K., Smith, K. E.: The canonical sheaf of Du Bois singularities. Adv. Math. **224**, 1618–1640 (2010) Zbl 1198.14003 MR 2646306

[Laz08]   Laza, R.: Triangulations of the sphere and degenerations of $K3$ surfaces. arXiv:0809.0937 (2008)

[Laz09]   Laza, R.: Deformations of singularities and variation of GIT quotients. Trans. Amer. Math. Soc. **361**, 2109–2161 (2009) Zbl 1174.14004 MR 2465831

[Loo78]   Looijenga, E.: On the semi-universal deformation of a simple-elliptic hypersurface singularity. II. The discriminant. Topology **17**, 23–40 (1978) Zbl 0392.57013 MR 0492380

[Loo81]   Looijenga, E.: Rational surfaces with an anticanonical cycle. Ann. of Math. (2) **114**, 267–322 (1981) Zbl 0509.14035 MR 0632841

[Loo86]   Looijenga, E.: New compactifications of locally symmetric varieties. In: Proceedings of the 1984 Vancouver Conference in Algebraic Geometry, CMS Conf. Proc. 6, Amer. Math. Soc., Providence, RI, 341–364 (1986) Zbl 0624.14008 MR 0846027

[Loo03]   Looijenga, E.: Compactifications defined by arrangements. II. Locally symmetric varieties of type IV. Duke Math. J. **119**, 527–588 (2003) Zbl 1079.14045 MR 2003125

[Loo77]   Looijenga, E.: Root systems and elliptic curves. Invent. Math. **38**, 17–32 (1976/77) Zbl 0358.17016 MR 0466134

[Nam80]   Namikawa, Y.: Toroidal Compactification of Siegel Spaces. Lecture Notes in Math. 812, Springer, Berlin (1980) Zbl 0466.14011 MR 0584625

[Ols04]   Olsson, M. C.: Semistable degenerations and period spaces for polarized $K3$ surfaces. Duke Math. J. **125**, 121–203 (2004) Zbl 1073.14054 MR 2097359

[Pin77]   Pinkham, H. C., Simple elliptic singularities, del Pezzo surfaces and Cremona transformations. In: Several Complex Variables, Proc. Sympos. Pure Math. 30, Part 1, Amer. Math. Soc., Providence, RI, 69–71 (1977) Zbl 0391.14006 MR 0441969

[RT11]   Ross, J., Thomas, R.: Weighted projective embeddings, stability of orbifolds, and constant scalar curvature Kähler metrics. J. Differential Geom. **88**, 109–159 (2011) Zbl 1244.32013 MR 2819757

[Sca87]   Scattone, F.: On the compactification of moduli spaces for algebraic $K3$ surfaces. Mem. Amer. Math. Soc. **70**, no. 374, x+86 (1987) Zbl 0633.14019 MR 0912636

[Ser06]   Sernesi, E.: Deformations of Algebraic Schemes. Grundlehren Math. Wiss. 334, Springer, Berlin (2006) Zbl 1102.14001 MR 2247603

[Sha79]   Shah, J., Insignificant limit singularities of surfaces and their mixed Hodge structure. Ann. of Math. (2) **109**, 497–536 (1979) Zbl 0414.14022 MR 0534760

[Sha80]   Shah, J.: A complete moduli space for $K3$ surfaces of degree 2. Ann. of Math. (2) **112**, 485–510 (1980) Zbl 0412.14016 MR 0595204

[SB83a]   Shepherd-Barron, N. I., Degenerations with numerically effective canonical divisor. In: The Birational Geometry of Degenerations (Cambridge, MA, 1981), Progr. Math. 29, Birkhäuser Boston, Boston, MA, 33–84 (1983) Zbl 0506.14028 MR 0690263

[SB83b]   Shepherd-Barron, N. I.: Extending polarizations on families of $K3$ surfaces. In: The Birational Geometry of Degenerations (Cambridge, MA, 1981), Progr. Math. 29, Birkhäuser Boston, Boston, MA, 135–171 (1983) Zbl 0509.14040 MR 0690265

[Ste81]   Steenbrink, J. H. M.: Cohomologically insignificant degenerations. Compos. Math. **42**, 315–320 (1980/81) Zbl 0428.32017 MR 0607373

[Tho10]   Thompson, A.: Degenerations of $K3$ surfaces of degree two. Trans. Amer. Math. Soc. **366**, 219–243 (2014) Zbl 1303.14023 MR 3118396

[Thu98]   Thurston, W. P.: Shapes of polyhedra and triangulations of the sphere. In: The Epstein Birthday Schrift, Geom. Topol. Monogr. 1, Geom. Topol. Publ., Coventry, 511–549 (1998)   Zbl 0931.57010   MR 1668340

[Usu06]   Usui, S.: Images of extended period maps. J. Algebraic Geom. **15**, 603–621 (2006)   Zbl 1119.14009   MR 2237263

[Vie95]   Viehweg, E.: Quasi-Projective Moduli for Polarized Manifolds. Ergeb. Math. Grenzgeb. (3) 30, Springer, Berlin (1995)   Zbl 0844.14004   MR 1368632

[Vie10]   Viehweg, E.: Compactifications of smooth families and of moduli spaces of polarized manifolds. Ann. of Math. (2) **172**, 809–910 (2010)   Zbl 1238.14009   MR 2680483

[Wal99]   Wall, C. T. C.: Sextic curves and quartic surfaces with higher singularities. www.liv.ac.uk/~ctcw/hsscqs.ps (1999)