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Sets of β-expansions and the Hausdorff measure of slices through fractals

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Abstract. We study natural measures on sets of β-expansions and on slices through self-similar sets. In the setting of β-expansions, these allow us to better understand the measure of maximal entropy for the random β-transformation and to reinterpret a result of Lindenstrauss, Peres and Schlag in terms of equidistribution. Each of these applications is relevant to the study of Bernoulli convolutions. In the fractal setting this allows us to understand how to disintegrate Hausdorff measure by slicing, leading to conditions under which almost every slice through a self-similar set has positive Hausdorff measure, generalising long known results about almost everywhere values of the Hausdorff dimension.

Keywords. Bernoulli convolution, beta expansion, slicing fractals, conditional measures

1. Introduction

Given $\beta \in (1, 2)$, a β-*expansion* of a real number x is a sequence $\underline{a} \in \{0, 1\}^{\mathbb{N}}$ for which

$$\pi_\beta(\underline{a}) := \sum_{i=1}^{\infty} a_i \beta^{-i} = x.$$

We let $\mathcal{E}_\beta(x) := \pi_\beta^{-1}(x)$ denote the set of β-expansions of x.

The primary purpose of this article is to seek to understand measures on $\mathcal{E}_\beta(x)$. In particular, we study the family of measures $m_x := m|_{\mathcal{E}_\beta(x)}$ obtained by disintegrating the uniform $(1/2, 1/2)$ Bernoulli measure m on $\{0, 1\}^{\mathbb{N}}$. These measures appear as disintegrations of the measure of maximal entropy for the random β-transformation in [4], and are used to state an equidistribution result for β-expansions in [16].

We begin by assuming that the Bernoulli convolution ν_β (defined later) is absolutely continuous. In this setting we build a two-dimensional dynamical system which preserves Lebesgue measure and for which vertical fibres through the state space correspond to the sets $\mathcal{E}_\beta(x)$. By lifting one-dimensional Lebesgue measure on these fibres to the sets $\mathcal{E}_\beta(x)$ we obtain formulae for m_x in terms of the density of ν_β.

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We also consider Hausdorff measure on $\mathcal{E}_\beta(x)$. Results on the cardinality, branching rate and dimension of $\mathcal{E}_\beta(x)$ were given in a series of recent papers [2, 8, 14, 26]. We continue this line of research by showing that for certain β, including almost all $\beta \in (1, \sqrt{2})$, the set $\mathcal{E}_\beta(x)$ has positive finite Hausdorff measure, and in that case the normalised Hausdorff measure on $\mathcal{E}_\beta(x)$ coincides with m_x. Our necessary and sufficient condition for the positivity of Hausdorff measure is that the Bernoulli convolution ν_β is absolutely continuous with bounded density.

We then use the formulae for the measures m_x obtained by our natural extension to reinterpret the results of [16] as equidistribution results for the sets $\mathcal{E}_\beta(x)$. In particular, we show that for almost all $\beta \in (1, \sqrt{2})$ and almost all $x \in [0, 1/(\beta - 1)]$ the sets

$$\mathcal{O}^n(x) := \{\pi_\beta(\sigma^n(\underline{a})) : \underline{a} \in \mathcal{E}_\beta(x)\}$$

equidistribute with respect to Lebesgue measure as $n \to \infty$, where σ denotes the left shift. Hochman proved in [12] that if ν_β has dimension less than 1 then either there are 'exact overlaps'[1], or $\inf_x\{\inf\{|y - z| : y, z \in \mathcal{O}^n(x)\}\}$ tends to zero superexponentially. We conjecture that the sets $\mathcal{O}^n(x)$ equidistribute if and only if ν_β is absolutely continuous. We are also able to use our results to prove a finer result (Proposition 5.1) about the typical branching rate of sets of β-expansions, making progress towards Conjecture 1 of [14].

For each statement that we make about sets of β-expansions and Bernoulli convolutions, there is an analogous statement about slices through self-similar sets and projections of Hausdorff measure. We let $E \subset \mathbb{R}^n$ be a self-similar set of Hausdorff dimension s, where the similarities do not include rotations and satisfy another technical condition (Definition 6.1). We let E_θ be the orthogonal projection of E onto the line passing through the origin at angle $\theta = (\theta_1, \ldots, \theta_{n-1})$. We let $E_{\theta,x}$ be the intersection of E with the $(n-1)$-dimensional plane perpendicular to E_θ and passing through $x \in E_\theta$. We call the sets $E_{\theta,x}$ *slices* of E.

Our main theorem for fractals, Theorem 6.2, states that $\mathcal{H}^{s-1}(E_{\theta,x}) > 0$ for almost every $x \in E_\theta$ if and only if the orthogonal projection of Hausdorff measure on E to E_θ is absolutely continuous with bounded density. Theorem 6.2 could be seen as a measure-theoretic analogue of Furstenberg's 'dimension conservation theorem', Theorem 3.1 of [10]. The dimension conservation theorem relates the dimension of projections of a fractal to the dimension of typical slices perpendicular to this projection, whereas our theorem relates the density properties of projected Hausdorff measure to the Hausdorff measure of typical slices.

An example application is the following, we recall that the Menger sponge is the self-similar set defined recursively by subdividing $[0, 1]^3$ into 27 subcubes of side length 1/3, discarding the subcube at the centre of each face of our original cube and the subcube in the centre of our original cube, and then repeating the process for each of the 20 remaining subcubes.

Example 1. Let E be the Menger sponge. Then almost every plane slice through E has positive finite $(\log(20)/\log(3) - 1)$-dimensional Hausdorff measure.

[1] In our situation, exact overlaps in the coding IFS correspond to different sequences $\underline{a}, \underline{b}$ satisfying that $\pi_\beta(\underline{a}) = \pi_\beta(\underline{b})$ and that $\pi_\beta(\sigma^n(\underline{a})) = \pi_\beta(\sigma^n(\underline{b}))$ for some $n \in \mathbb{N}$.

Corresponding theorems due to Marstrand for the dimension of slices through fractals are well known, but the extension to the case of Hausdorff measure of slices through fractals is new.

In the final section we state a number of open questions related to our work.

2. Preliminaries

We define the left shift $\sigma : \{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$ by

$$\sigma(a_1 a_2 a_3 \cdots) = (a_2 a_3 \cdots).$$

Given a word $a_1 \cdots a_n \in \{0,1\}^n$ we let the cylinder $[a_1 \cdots a_n]$ be given by

$$[a_1 \cdots a_n] := \{\underline{b} \in \{0,1\}^{\mathbb{N}} : b_1 \cdots b_n = a_1 \cdots a_n\}.$$

We let m be the $(1/2, 1/2)$ Bernoulli measure on $\{0,1\}^{\mathbb{N}}$, which gives measure 2^{-n} to each cylinder $[a_1 \cdots a_n]$.

The Bernoulli convolution ν_β is the probability measure on $I_\beta := [0, 1/(\beta - 1)]$ defined by

$$\nu_\beta := m \circ \pi_\beta^{-1}.$$

An alternative definition of ν_β is that it is the unique probability measure satisfying the self-similarity relation

$$\nu_\beta = \tfrac{1}{2}(\nu_\beta \circ T_0 + \nu_\beta \circ T_1)$$

where the functions $T_i : \mathbb{R} \to \mathbb{R}$ are given by $T_i(x) := \beta x - i$.

There are a number of fascinating open questions relating to Bernoulli convolutions including the fundamental question of for which values of β the corresponding Bernoulli convolution is absolutely continuous. Solomyak [28] showed that ν_β is absolutely continuous for Lebesgue almost all $\beta \in (1,2)$, and has continuous density for almost all $\beta \in (1, \sqrt{2})$. Mauldin and Simon [19] showed that ν_β is actually equivalent to Lebesgue measure whenever it is absolutely continuous. Very recently, Shmerkin [25] has shown that the set of β for which ν_β is singular has Hausdorff dimension zero.

We let m_x be the disintegration of m by fibres $\mathcal{E}_\beta(x)$. This means that (m_x) is the ν_β-almost everywhere unique family of measures such that each m_x is a probability measure supported on the fibre $\mathcal{E}_\beta(x)$ and for every integrable function $f : \{0,1\}^{\mathbb{N}} \to \mathbb{R}$ we have

$$\int_{\{0,1\}^{\mathbb{N}}} f(\underline{a})\, dm(\underline{a}) = \int_{I_\beta} \int_{\mathcal{E}_\beta(x)} f(\underline{a})\, dm_x(\underline{a})\, d\nu_\beta(x). \tag{2.1}$$

The study of the measures m_x is the principal focus of this article.

Expansions of numbers in non-integer bases have been studied since the 1950s with the work of Rényi [24] and Parry [21] who were interested in the properties of the largest β-expansions of x with respect to the lexicographical ordering, known as the greedy β-expansion. The dynamics of the associated greedy β-transformation $x \to \beta x \pmod 1$ have been extensively studied over the last sixty years and are well understood.

Given $\beta \in (1, 2)$, the β-expansion of $x \in I_\beta$ is typically not unique, indeed Lebesgue almost every $x \in I_\beta$ has uncountably many β-expansions [26]. There is a substantial amount of recent research trying to understand the properties of the sets $\mathcal{E}_\beta(x)$ for typical $x \in I_\beta$ (see for example [2, 3, 8, 14] and the references therein). Sets of β-expansions can be generated dynamically using the random β-transformation K_β of Dajani and Kraaikamp [6]. We define the random β-transformation $K_\beta : \{0,1\}^{\mathbb{N}} \times I_\beta \to \{0,1\}^{\mathbb{N}} \times I_\beta$ by

$$K_\beta(\omega, x) = \begin{cases} (\omega, T_0(x)), & x \in \left[0, \frac{1}{\beta}\right), \\ (\sigma(\omega), T_{\omega_1}(x)), & x \in \left[\frac{1}{\beta}, \frac{1}{\beta(\beta-1)}\right], \\ (\omega, T_1(x)), & x \in \left(\frac{1}{\beta(\beta-1)}, \frac{1}{\beta-1}\right]. \end{cases}$$

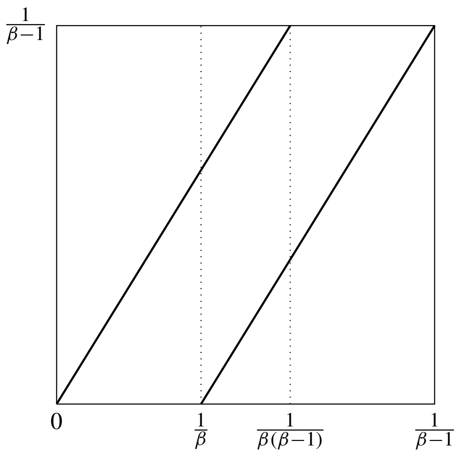

Fig. 1. The projection onto the second coordinate of K_β for $\beta = (1+\sqrt{5})/2$.

Given $x \in I_\beta$, β-expansions of x are generated by choosing some $\omega \in \{0,1\}^{\mathbb{N}}$ and iterating $K_\beta(\omega, x)$. If the ith iteration of $K_\beta(\omega, x)$ applies T_0 to the second coordinate we put $a_i = 0$, if it applies T_1 to the second coordinate we put $a_i = 1$. This generates a sequence (a_i) which is a β-expansion of x, and all β-expansions of x can be generated this way (see [6]).

Invariant measures for K_β were studied in [4, 5, 15]. In particular, the measure of maximal entropy of K_β was studied in [4] and was shown to project to the Bernoulli convolution on its second coordinate. The mapping which takes a pair (ω, x) to the β-expansion generated by (ω, x) is a bijection up to sets of measure zero with respect to the measure of maximal entropy, and thus the system K_β is a suitable dynamical system for studying both Bernoulli convolutions and sets of β-expansions.

A full description of the measure of maximal entropy for K_β was not given in [4]. The authors were able to show that it is not a product measure in general, but the behaviour of this measure on the first coordinate remains unknown in the general case. The measures on $\mathcal{E}_\beta(x)$ introduced in this article allow one to give a full description of the measure of maximal entropy for K_β in terms of the density of ν_β in the case that ν_β is absolutely continuous.

The method of coding β-expansions above gives a bijection (up to sets of measure zero) between $\{0,1\}^{\mathbb{N}}$ and $\{0,1\}^{\mathbb{N}} \times I_\beta$ by associating to a code $(a_i) \in \{0,1\}^{\mathbb{N}}$ the corresponding pair (ω, x). Then the space $\{0,1\}^{\mathbb{N}} \times I_\beta$ can be seen as a representation of $\{0,1\}^{\mathbb{N}}$ for which the complicated projection π_β becomes a simple projection onto the second coordinate, and horizontal fibres can be mapped onto the sets $\mathcal{E}_\beta(x)$. The dynamical system that we build in the next section uses effectively the same idea, except that the sets $\mathcal{E}_\beta(x)$ are represented in a different way which makes invariant measures much easier to study.

3. A dynamical system

We begin by building a dynamical system (X, ϕ, μ) which is measurably isomorphic to the full shift on two symbols (and hence also to the random β-transformation), but for which the invariant measure μ is Lebesgue measure. The sets $\mathcal{E}_\beta(x)$ correspond to vertical slices through the space X.

We assume that ν_β is absolutely continuous, and has $\mathcal{L}^1$ density function h_β. We define

$$X = \{(x, y) : x \in I_\beta,\ 0 \le y \le h_\beta(x)\}$$

and let λ^2 denote two-dimensional Lebesgue measure restricted to X.

Now since ν_β satisfies the self-similarity relation

$$\nu_\beta = \tfrac{1}{2}(\nu_\beta \circ T_0 + \nu_\beta \circ T_1),$$

h_β satisfies the relation

$$h_\beta(x) = \frac{\beta}{2}(h_\beta(T_0(x)) + h_\beta(T_1(x))). \tag{3.1}$$

Here we are considering h_β to be defined on the whole real line, although it takes value 0 outside of I_β. We partition X into two pieces with non-overlapping interior,

$$X_0 = \{(x, y) \in X : 0 \le y \le (\beta/2) h_\beta(\beta x)\}$$

and $X_1 = \overline{X \setminus X_0}$. Then X_1 and X_0 intersect in a set of Lebesgue measure zero. Note that X_0 is a scaled-down copy of X, and that X_1 is a scaled-down copy of X except that part of it is skewed to sit on top of X_0; this follows from equation (3.1).

We define a map $\phi : X \to X$ by

$$\phi(x, y) = \begin{cases} (\beta x, 2y/\beta) & (x, y) \in X_0, \\ \big(\beta x - 1, \frac{2}{\beta}\big(y - \frac{\beta}{2} h_\beta(\beta x)\big)\big) & (x, y) \in X_1. \end{cases}$$

The map ϕ is well defined except on the intersection of X_0 and X_1. Because of (3.1), ϕ maps each of X_0 and X_1 bijectively onto the whole space X and thus ϕ is conjugate to the full shift on two symbols. Furthermore, since ϕ stretches the first coordinate by a

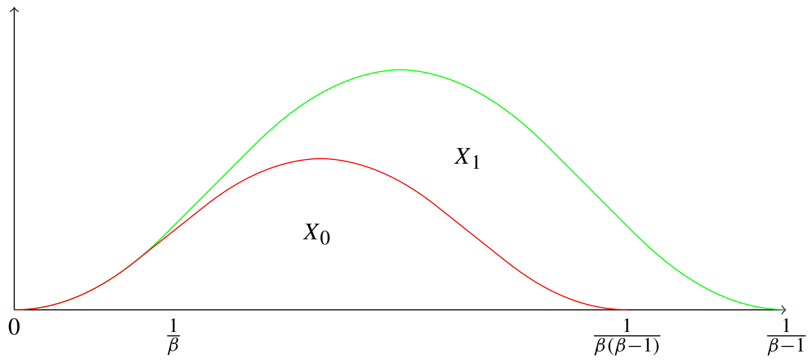

Fig. 2. A picture of X partitioned into X_0, X_1 for $\beta = 2^{1/3}$.

factor of β and the second coordinate by a factor of $2/\beta$, and each point has exactly two preimages under ϕ, we see that ϕ preserves Lebesgue measure λ^2.

The map ϕ allows us to assign a unique code $\underline{a}(x, y)$ to almost every point (x, y) in X by writing

$$a_n(x, y) = \begin{cases} 0, & \phi^{n-1}(x, y) \in X_0, \\ 1, & \phi^{n-1}(x, y) \in X_1. \end{cases}$$

There are problems only with boundaries of the partition X_0, X_1, as is typical for Markov partition constructions. The sequence $\underline{a}(x, y)$ satisfies $\pi_\beta(\underline{a}(x, y)) = x$ by the same arguments as given in [6] for codes arising from K_β.

We can describe this coding by a map $P\{0, 1\}^{\mathbb{N}} \to X$. Given a word $a_1 \cdots a_n \in \{0, 1\}^n$ we let $[a_1 \cdots a_n]$ denote the set of sequences $\{\underline{x} \in \{0, 1\}^{\mathbb{N}} : x_1 \cdots x_n = a_1 \cdots a_n\}$. We define

$$[a_1 \cdots a_n]_X := X_{a_1} \cap \phi^{-1}(X_{a_2}) \cap \cdots \cap \phi^{-(n-1)}(X_{a_n}).$$

For each $a_1 \cdots a_n \in \{0, 1\}^n$ we have $\lambda^2([a_1 \cdots a_n]|_X) = 2^{-n}$. Then we define $P : \{0, 1\}^{\mathbb{N}} \to X$ by

$$P(\underline{a}) := \bigcap_{n=1}^{\infty} [a_1 \cdots a_n]_X.$$

By construction, the coding map P is a measure isomorphism from $(\{0, 1\}^{\mathbb{N}}, \sigma, m)$ to (X, ϕ, λ^2).

3.1. Pulling back Lebesgue measure

This dynamical system gives rise to a natural measure on the sets $\mathcal{E}_\beta(x)$. Given a code $a_1 \cdots a_n \in \{0, 1\}^n$ we define

$$T_{a_1 \cdots a_n} := T_{a_n} \circ T_{a_{n-1}} \circ \cdots \circ T_{a_1}.$$

Then $T_{a_1\cdots a_n}(x) \in I_\beta$ if and only if $[a_1\cdots a_n] \cap \mathcal{E}_\beta(x)$ is non-empty (see [6] for a more detailed description of how to construct β-expansions).

Then for $x_0 \in I_\beta$ we define the fibre

$$X_{x_0} := \{(x, y) \in X : x = x_0\}$$

and see that $P^{-1}(X_x) = \mathcal{E}_\beta(x)$. So we can get a measure on the set $\mathcal{E}_\beta(x)$ by pulling back normalised one-dimensional Lebesgue measure on X_x.

This measure can easily be described using h_β. We have

$$\phi^n(X_x \cap [a_1\cdots a_n]_X) = X_{T_{a_1\cdots a_n}(x)}.$$

Then since map ϕ expands vertical distances by $2/\beta$, we see that

$$\lambda(X_x \cap [a_1\cdots a_n]_X) = (\beta/2)^n \lambda(X_{T_{a_1\cdots a_n}(x)}) = (\beta/2)^n h_\beta(T_{a_1\cdots a_n}(x)),$$

where λ denotes one-dimensional Lebesgue measure. Summing over all words $a_1\cdots a_n \in \{0,1\}^n$ one recovers equation (3.1). Normalising λ to give the fibre total mass 1, and pulling back to the set $\mathcal{E}_\beta(x)$, we define the measure

$$m_x^1[a_1\cdots a_n] := \frac{1}{h_\beta(x)}\lambda(X_x \cap [a_1\cdots a_n]_X) = \left(\frac{\beta}{2}\right)^n \frac{h_\beta(T_{a_1\cdots a_n}(x))}{h_\beta(x)}.$$

The measure m_x^1 is a probability measure on $\mathcal{E}_\beta(x)$ defined whenever ν_β is absolutely continuous. We prove that it coincides with the measures m_x defined earlier.

Proposition 3.1. *The measure m_x^1 is equal to the measure m_x whenever m_x^1 is defined.*

Proof. We recall the measures $(m_x)_{x\in I_\beta}$ were defined as the ν_β-almost everywhere unique collection of probability measures supported on the sets $\mathcal{E}_\beta(x)$ satisfying (2.1). The measures m_x^1 are also probability measures supported on $\mathcal{E}_\beta(x)$, and so we need only show that they satisfy (2.1) in order to verify that $m_x = m_x^1$. But then, since the map P taking Σ to X is a bijection which maps m to two-dimensional Lebesgue measure on X and m_x^1 to one dimensional Lebesgue measure on X_x, it is enough to show that

$$\int_X f(x, y)\, d\lambda^2 = \int_{I_\beta}\int_{X_x} f(x, y)\, d\lambda(y)\, d\lambda(x)$$

for each integrable f. But this is just the classical Fubini theorem, and so we are done. □

3.2. *Comments on the map ϕ*

We briefly comment on the relationship between our map ϕ, the random β-transformation and the fat baker's transformation of [1]. Since these statements are rather outside of the main thrust of our arguments we make them without proof, but they can easily be deduced by looking at our construction.

Firstly we remark that the system (X, ϕ) is in fact rather similar to the random β-transformation K_β. In fact, if one studies the system $(\{0, 1\}^{\mathbb{N}} \times I_\beta, K_\beta, \hat{\nu}_\beta)$ where $\hat{\nu}_\beta$ is the measure of maximal entropy for K_β, then one sees that (X, ϕ, λ^2) and $(\Omega \times I_\beta, K_\beta, \hat{\nu}_\beta)$ are measurably isomorphic. One can prove this rather cheaply by observing that both systems are measurably isomorphic to the full shift on two symbols coupled with the $(1/2, 1/2)$ Bernoulli measure, but it is quite instructive to build the isomorphism directly. It was an open question stated in [4] to determine the behaviour of $\hat{\nu}_\beta$ on fibres; the above formula for the measure m_x^1 answers this question in the case that ν_β is absolutely continuous.

There is a simple invertible extension of (X, ϕ, μ) given by defining $\hat{X} = X \times [0, 1]$, $\hat{\mu} = \lambda^3|_{\hat{X}}$ where λ^3 denotes three-dimensional Lebesgue measure, and $\hat{\phi}((x, y), z) = (\phi(x, y), z/2 + i)$ whenever $(x, y) \in X_i$. The system $(\hat{X}, \hat{\phi}, \hat{\mu})$ is measurably isomorphic to $(\hat{\Sigma}, \hat{\sigma}, m)$ where $\hat{\Sigma}$ denotes the two-sided full shift on 2-symbols. $\hat{\phi}$ is invertible, and if one projects $\hat{\phi}^{-1}$ onto the first and third coordinates one recovers the fat baker's transformation. It was already known that the fat baker's transformation has the two-sided shift on two symbols as an invertible extension, but our map $\hat{\phi}^{-1}$ is perhaps a more interesting natural extension, since it preserves Lebesgue measure and maps down onto the factor system by orthogonal projection.

4. Hausdorff measure for sets of β-expansions

In this section we prove results about the Hausdorff measure of sets of β-expansions. For definitions of Hausdorff measure and Hausdorff dimension see [7]. We endow the space $\{0, 1\}^{\mathbb{N}}$ with the metric d defined by

$$d(\underline{a}, \underline{b}) = 2^{-\sup\{n : a_1 \cdots a_n = b_1 \cdots b_n\}}$$

if $a_1 = b_1$ and $d(\underline{a}, \underline{b}) = 1$ otherwise. We denote by $|A|$ the diameter of the set A, i.e. the supremum of the set of distances between pairs of points in A. The diameter of a cylinder set $[a_1 \cdots a_n]$ is 2^{-n}.

We recall that the density h_β of ν_β is an $\mathcal{L}^1$ function defined almost everywhere which satisfies (3.1). We have $h_\beta \geq 0$ and that $h_\beta(x) \to 0$ as x tends to 0 or $1/(\beta - 1)$. Since h_β is defined only almost everywhere, many of our statements about h_β will hold almost everywhere. In particular, we say that h_β is *bounded* if it is essentially bounded above, i.e. there exists a constant c such that $\lambda\{x \in I_\beta : h_\beta(x) > c\} = 0$. We have the following theorem.

Theorem 4.1. *The set $\mathcal{E}_\beta(x)$ of β-expansions of x has positive $\frac{\log(2/\beta)}{\log(2)}$-dimensional Hausdorff measure for Lebesgue almost every $x \in I_\beta$ if and only if the corresponding*

Bernoulli convolution ν_β is absolutely continuous with bounded density. In this case, normalised Hausdorff measure on the sets $\mathcal{E}_\beta(x)$ coincides with the measures m_x.

This theorem is proved using equation (3.1), which allows a rather simple method of studying the sets $\mathcal{E}_\beta(x)$. We split the theorem into three lemmas.

Lemma 4.2. *If the Bernoulli convolution ν_β is absolutely continuous with bounded density then the set $\mathcal{E}_\beta(x)$ of β-expansions of x has positive $\frac{\log(2/\beta)}{\log(2)}$-dimensional Hausdorff measure for Lebesgue almost every $x \in I_\beta$.*

Proof. Let $\tilde{\mathcal{U}}$ be a countable partition of $\{0,1\}^{\mathbb{N}}$ by cylinder sets $[a_1^i \cdots a_{n_i}^i]$ for $i \in \mathbb{N}$. We can iterate (3.1) to write

$$h_\beta(x) = \sum_{a_1^i \cdots a_{n_i}^i \in \tilde{\mathcal{U}}} (\beta/2)^{n_i} h_\beta(T_{a_1^i \cdots a_{n_i}^i}(x)).$$

Since $h_\beta(T_{a_1^i \cdots a_{n_i}^i}(x)) = 0$ whenever $T_{a_1^i \cdots a_{n_i}^i} \notin I_\beta$, we can remove those terms for which $T_{a_1^i \cdots a_{n_i}^i} \notin I_\beta$, or equivalently $[a_1^i \cdots a_{n_i}^i] \cap \mathcal{E}_\beta(x) = \emptyset$. Then on letting

$$\mathcal{U} = \{[a_1^i \cdots a_{n_i}^i] \in \tilde{\mathcal{U}} : [a_1^i \cdots a_{n_i}^i] \cap \mathcal{E}_\beta(x) \neq \emptyset\},$$

the previous equation becomes

$$h_\beta(x) = \sum_{a_1^i \cdots a_{n_i}^i \in \mathcal{U}} (\beta/2)^{n_i} h_\beta(T_{a_1^i \cdots a_{n_i}^i}(x)). \tag{4.1}$$

We stress that, since any open cover of $\mathcal{E}_\beta(x)$ can be obtained by taking a cover of $\{0,1\}^{\mathbb{N}}$ by cylinder sets and discarding those sets which do not intersect $\mathcal{E}_\beta(x)$, the above equation holds for all covers $\mathcal{U}$ of $\mathcal{E}_\beta(x)$ by cylinder sets.

Then for any disjoint cover $\mathcal{U}$ of $\mathcal{E}_\beta(x)$ by cylinder sets we have

$$\begin{aligned}\sum_{a_1^i \cdots a_{n_i}^i \in \mathcal{U}} |[a_1^i \cdots a_{n_i}^i]|^{\frac{\log(2/\beta)}{\log(2)}} &= \sum_{a_1^i \cdots a_{n_i}^i \in \mathcal{U}} 2^{-\frac{\log(2/\beta)}{\log(2)} n_i} = \sum_{a_1^i \cdots a_{n_i}^i \in \mathcal{U}} (\beta/2)^{n_i} \\ &= C(\mathcal{U}) \sum_{a_1^i \cdots a_{n_i}^i \in \mathcal{U}} (\beta/2)^{n_i} h_\beta(T_{a_1^i \cdots a_{n_i}^i}(x)) = C(\mathcal{U}) h_\beta(x),\end{aligned}$$

where

$$C(\mathcal{U}) := \frac{\sum_{a_1^i \cdots a_{n_i}^i \in \mathcal{U}} (\beta/2)^{n_i}}{\sum_{a_1^i \cdots a_{n_i}^i \in \mathcal{U}} (\beta/2)^{n_i} h_\beta(T_{a_1^i \cdots a_{n_i}^i}(x))}.$$

The final equality above followed from equation (4.1).

If h_β is bounded then $1/h_\beta(T_{a_1^i\cdots a_{n_i}^i}(x)) \geq C > 0$ where $C := 1/\text{ess}\sup\{h(x)\ :\ x \in I_\beta\}$ is independent of $a_1^i\cdots a_{n_i}^i$ and x. Then $C(\mathcal{U}) \geq C$ and thus

$$\sum_{a_1^i\cdots a_{n_i}^i \in \mathcal{U}} |[a_1^i\cdots a_{n_i}^i]|^{\frac{\log(2/\beta)}{\log(2)}} \geq Ch(x)$$

for any cover $\mathcal{U}$ of $\mathcal{E}_\beta(x)$. We conclude that

$$\mathcal{H}^{\frac{\log(2/\beta)}{\log(2)}}(\mathcal{E}_\beta(x)) \geq Ch_\beta(x) > 0$$

for all $x \in I_\beta$ such that $h(x) > 0$, and in particular for almost all $x \in I_\beta$. □

We define the measure m_x^2 on $\mathcal{E}_\beta(x)$ by

$$m_x^2(A) = \frac{\mathcal{H}^{\frac{\log(2/\beta)}{\log(2)}}(A)}{\mathcal{H}^{\frac{\log(2/\beta)}{\log(2)}}(\mathcal{E}_\beta(x))}.$$

This is well defined for almost every $x \in I_\beta$ whenever $h_\beta(x)$ is bounded. The second step of the proof of Theorem 4.1 is the following.

Lemma 4.3. *The measures m_x^2 and m_x^1 are equal whenever they are both defined, i.e. whenever the Bernoulli convolution ν_β is absolutely continuous with bounded density.*

Proof. We first observe that one has the bound

$$\mathcal{H}^{\frac{\log(2/\beta)}{\log(2)}}(\mathcal{E}_\beta(x)) \leq 2h_\beta(x)$$

for $x \in I_\beta$. To prove this one takes the cover of $\mathcal{E}_\beta(x)$ by all cylinders of depth n which intersect $\mathcal{E}_\beta(x)$. It was proved in [14, Lemma 3.4], following a similar argument in [23, Appendix C], that

$$\limsup_{n\to\infty}(\beta/2)^n|\{a_1\cdots a_n \in \{0,1\}^n : [a_1\cdots a_n] \cap \mathcal{E}_\beta(x) \neq \emptyset\}| \leq 2h_\beta(x).$$

Then for all $\epsilon > 0$ we can, by taking n large enough, find a cover $\mathcal{U}$ of $\mathcal{E}_\beta(x)$ by cylinder sets of depth n for which

$$\sum_{a_1\cdots a_n \in \mathcal{U}} |[a_1^i\cdots a_{n_i}^i]|^{\frac{\log(2/\beta)}{\log(2)}} = |\mathcal{U}|(\beta/2)^n \leq 2h_\beta(x) + \epsilon.$$

In particular, we see that

$$0 < \int_{I_\beta} \mathcal{H}^{\frac{\log(2/\beta)}{\log(2)}}(\mathcal{E}_\beta(x))dx \leq 2.$$

We define

$$g(x) := \mathcal{H}^{\frac{\log(2/\beta)}{\log(2)}}(\mathcal{E}_\beta(x)).$$

Now given a cylinder $[a_1 \cdots a_n]$ we have

$$|[a_1 \cdots a_n]| = 2^{-1}|[a_2 \cdots a_n]| = 2^{-1}|\sigma[a_1 \cdots a_n]|.$$

Then given any set A which is contained in either [0] or [1] we have

$$\mathcal{H}^{\frac{\log(2/\beta)}{\log(2)}}(A) = 2^{-\frac{\log(2/\beta)}{\log(2)}}\mathcal{H}^{\frac{\log(2/\beta)}{\log(2)}}(\sigma(A)) = (\beta/2)\mathcal{H}^{\frac{\log(2/\beta)}{\log(2)}}(\sigma(A)).$$

The tree structure of the set of β-expansions means that

$$\begin{aligned} g(x) &= \mathcal{H}^{\frac{\log(2/\beta)}{\log(2)}}(\mathcal{E}_\beta(x)) = \mathcal{H}^{\frac{\log(2/\beta)}{\log(2)}}(\mathcal{E}_\beta(x) \cap [0]) + \mathcal{H}^{\frac{\log(2/\beta)}{\log(2)}}(\mathcal{E}_\beta(x) \cap [1]) \\ &= (\beta/2)\mathcal{H}^{\frac{\log(2/\beta)}{\log(2)}}(\sigma(\mathcal{E}_\beta(x) \cap [0])) + (\beta/2)\mathcal{H}^{\frac{\log(2/\beta)}{\log(2)}}(\sigma(\mathcal{E}_\beta(x) \cap [1])) \\ &= (\beta/2)(g(T_0(x)) + g(T_1(x))). \end{aligned}$$

Thus g is an $\mathcal{L}^1$ function with positive integral satisfying equation (3.1), and since $\mathcal{L}^1$ solutions to (3.1) are unique up to multiplication by constants, we see that $g(x) = Kh_\beta(x)$ for some constant K. In particular, m_x^2 on $\mathcal{E}_\beta(x)$ assigns mass

$$\frac{(\beta/2)^n g(T_{a_1 \cdots a_n}(x))}{g(x)} = \left(\frac{\beta}{2}\right)^n \frac{h_\beta(T_{a_1 \cdots a_n}(x))}{h_\beta(x)}$$

to the cylinder $[a_1 \cdots a_n]$ for any choice of $a_1 \cdots a_n$, and thus the measures m_x^2 and m_x^1 coincide. By Proposition 3.1 we conclude that all three measures m_x, m_x^1 and m_x^2 coincide when they are defined. □

The following lemma deals with the case of h_β unbounded.

Lemma 4.4. *If h_β is unbounded then $\mathcal{H}^{\frac{\log(2/\beta)}{\log(2)}}(\mathcal{E}_\beta(x)) = 0$ for almost every $x \in I_\beta$.*

Proof. We stress that, since h_β is defined only almost everywhere, we take the statement 'h_β is unbounded' to mean that for each $C \in \mathbb{R}$ the set $A_C := \{x \in I_\beta : h_\beta(x) > C\}$ has positive Lebesgue measure.

We begin by supposing that

$$g(x) := \mathcal{H}^{\frac{\log(2/\beta)}{\log(2)}}(\mathcal{E}_\beta(x))$$

is positive for a positive Lebesgue measure set of $x \in I_\beta$. Then g is an $\mathcal{L}^1$ function of positive integral and the conclusion of Lemma 4.3 holds.

Now we define

$$B_C := \{\underline{a} \in \Sigma : \pi_\beta(\underline{a}) \in A_C\}$$

and see that $m(B_C) > 0$. We let $B \subset \Sigma$ be the set of sequences $\underline{a} \in \Sigma$ such that $\sigma^n(\underline{a}) \in B_C$ infinitely often. Since the system (Σ, σ, m) is ergodic, we see that $m(B) = 1$. In particular, $m_x(B^c) = 0$ for almost every x, giving

$$\mathcal{H}^{\frac{\log(2/\beta)}{\log(2)}}(\mathcal{E}_\beta(x) \cap B^c) = 0$$

for almost every x. Here we have used the assumption that the conclusion of Lemma 4.3 holds, allowing us to replace m_x with normalised Hausdorff measure. We now find efficient covers for $\mathcal{E}_{\beta,x} \cap B$.

Let $\delta > 0$ and let $N \in \mathbb{N}$ satisfy $2^{-N} < \delta$. For $n \geq N$ we define

$$A_{n,x} := \{\underline{a} \in \mathcal{E}_\beta(x) : \sigma^n(\underline{a}) \in B_C,\ \underline{a} \notin A_{N,x}, \ldots, A_{n-1,x}\}.$$

Each $A_{n,x}$ consists of a finite number of cylinder sets, and the union of these collections of cylinder sets over $n \geq N$ forms a δ-cover of $\mathcal{E}_\beta(x) \cap B$. Furthermore, on each of these cylinder sets forming $A_{n,x}$ one has $h_\beta(\pi_\beta(\sigma^n(\underline{a})) > C$. Then letting $\mathcal{U}$ be the δ-cover of $\mathcal{E}_\beta(x) \cap B$ using the cylinder sets in $A_{n,x}$ for $n \geq N$, we have

$$\mathcal{C}(\mathcal{U}) < 1/C.$$

Using the final lines of the proof of Lemma 4.2, we see that this gives

$$\mathcal{H}^{\frac{\log(2/\beta)}{\log(2)}}(\mathcal{E}_\beta(x)) = \mathcal{H}^{\frac{\log(2/\beta)}{\log(2)}}(\mathcal{E}_\beta(x) \cap B^c) + \mathcal{H}^{\frac{\log(2/\beta)}{\log(2)}}(\mathcal{E}_\beta(x) \cap B) \leq 0 + h_\beta(x)/C,$$

and since C was arbitrary we are done. □

All that is left to complete the proof of Theorem 4.1 is to prove that if ν_β is singular then slices have zero Hausdorff measure.

Lemma 4.5. *If ν_β is singular then $\mathcal{H}^{\frac{\log(2/\beta)}{\log(2)}}(\mathcal{E}_\beta(x)) = 0$ for Lebesgue a.e. x.*

Proof. This follows directly from [27]. We cover $\mathcal{E}_\beta(x)$ by cylinders of length n, and let

$$\mathcal{N}_n(x;\beta) := |\{a_1 \cdots a_n \in \{0,1\}^n : [a_1 \cdots a_n] \cap \mathcal{E}_\beta(x) \neq \emptyset\}|,$$

be the number of sets in this covering. Each of the sets in this cover is of diameter 2^{-n}, and so letting n tend to infinity we have

$$\mathcal{H}^{\frac{\log(2/\beta)}{\log(2)}}(\mathcal{E}_\beta(x)) \leq \lim_{n\to\infty} \mathcal{N}_n(x;\beta) 2^{-n\frac{\log(2/\beta)}{\log(2)}} = \lim_{n\to\infty} (\beta/2)^n \mathcal{N}_n(x;\beta) = 0$$

for Lebesgue almost every x. The final equality was proved in [27], answering the first part of Conjecture 1 of [14]. This completes the proof of the lemma and of Theorem 4.1. □

5. Equidistribution results

In this section we use our understanding gained in the last section of the disintegration of m by the sets $\mathcal{E}_\beta(x)$ to turn some results of [16] into equidistribution results for sets of β-expansions. It is likely that, by suitably adapting the results of [16] to the case of projecting and slicing self-similar sets, one could prove similar results for equidistribution of slices of fractals. Our main question is the following.

Question. What can one say about the distribution of the multisets

$$\mathcal{O}^n(x) := \{T_{a_1\cdots a_n}(x) : [a_1 \cdots a_n] \cap \mathcal{E}_\beta(x) \neq \emptyset\} = \pi_\beta(\sigma^n(\mathcal{E}_\beta(x))),$$

where the multiplicity of $y \in \mathcal{O}^n(x)$ is defined as being equal to the number of words $a_1 \cdots a_n$ for which $T_{a_1\cdots a_n}(x) = y$? In particular, what is the relationship between the limiting distribution of $\mathcal{O}^n(x)$ for typical x and the absolute continuity of ν_β?

If β is non-algebraic then there do not exist words $a_1 \cdots a_n \neq b_1 \cdots b_n \in \{0, 1\}^n$ such that $T_{a_1\cdots a_n}(x) = T_{b_1\cdots b_n}(x)$, and thus the multiplicity of elements of $\mathcal{O}^n(x)$ is always equal to 1.

We have

$$\mathcal{N}_n(x; \beta) = |\mathcal{O}^n(x)| = |\{a_1 \cdots a_n \in \{0, 1\}^n : T_{a_1\cdots a_n}(x) \in I_\beta\}|.$$

In [14] we were able to link the growth rate of $\mathcal{N}_n(x; \beta)$ for typical $x \in I_\beta$ with the question of the absolute continuity of ν_β. In particular we defined

$$\bar{f}(x) := \limsup_{n\to\infty} (\beta/2)^n \mathcal{N}_n(x; \beta),$$

and $\underline{f}(x)$ as above but with the lim sup replaced by a lim inf. We proved that if either $\bar{f}$ or $\underline{f}$ are $\mathcal{L}^1$ functions with positive integral then ν_β is absolutely continuous. We conjectured that for absolutely continuous ν_β one has $\bar{f} = \underline{f}$.

We are interested in the extent to which equidistribution of $\mathcal{O}^n(x)$ is implied by the absolute continuity of ν_β. We are not able to answer this question, but we can at least show that equidistribution is typical for $\beta \in (1, \sqrt{2})$. This in turn leads to some results on the typical growth of $|\mathcal{O}^n(x)|$ (see Proposition 5.1). The following theorem is a restatement in our language of [16, Theorem 1.2].

Theorem 5.1 (Lindenstrauss, Peres and Schlag). *For almost every $\beta \in (1, 2)$, for each $a_1 \cdots a_m \in \{0, 1\}^m$ and for almost every $x \in I_\beta$ we have*

$$m_x\{\underline{w} \in \mathcal{E}_\beta(x) : \sigma^n(\underline{w}) \in [a_1 \cdots a_m]\} \xrightarrow[n\to\infty]{} 2^{-m}.$$

Given $\epsilon > 0$ and an interval $A \subset I_\beta$, we can approximate A from below by a finite collection $\mathcal{U}_1$ of disjoint cylinder sets such that

$$\sum_{[a_1\cdots a_m]\in\mathcal{U}_1} m[a_1 \cdots a_m] > \nu_\beta(A) - \epsilon$$

and $\pi_\beta[a_1 \cdots a_m] \subset A$ for each $[a_1 \cdots a_m] \in \mathcal{U}_1$. Similarly we can approximate A from above with a collection $\mathcal{U}_2$ of cylinder sets such that

$$\sum_{[a_1\cdots a_m]\in\mathcal{U}_2} m[a_1 \cdots a_n] < \nu_\beta(A) + \epsilon$$

and

$$\pi_\beta(\underline{a}) \in A \ \Rightarrow \ \underline{a} \in \bigcup_{[a_1\cdots a_m]\in\mathcal{U}_2} [a_1\cdots a_m].$$

Then an immediate corollary to Theorem 5.1 is that for almost every $\beta \in (1,2)$, almost every $x \in I_\beta$ and each interval $A \subset I_\beta$ we have

$$m_x\{\underline{w} \in \mathcal{E}_\beta(x) : \pi_\beta(\sigma^n(\underline{w})) \in A\} \xrightarrow[n\to\infty]{} \nu_\beta(A).$$

Equivalently,

Corollary 5.2. *For almost every $\beta \in (1,2)$ and for almost every $x \in I_\beta$ the probability measures*

$$\nu_{n,x} := \sum_{a_1\cdots a_n\in\{0,1\}^n} \delta_{T_{a_1\cdots a_n}(x)} m_x[a_1\cdots a_n]$$

converge weak to ν_β as $n \to \infty$.*

Note that we could restrict the sum to those $a_1\cdots a_n \in \{0,1\}^n$ such that $[a_1\cdots a_n] \cap \mathcal{E}_\beta(x) \neq \emptyset$, since m_x gives zero measure to words $a_1\cdots a_n$ for which $T_{a_1\cdots a_n}(x) \notin I_\beta$. This corollary is an equidistribution result stated in terms of conditional measures, and was well suited to the purposes of [16] as it allowed answering an old question of Sinai and Rokhlin about conditional entropy. However, if one is interested in the distribution of the sets $\mathcal{O}^n(x)$ it would be more natural to seek equidistribution results that did not depend on the conditional measures m_x. We define probability measures

$$\mu_{n,x} := \frac{1}{\mathcal{N}_n(x;\beta)} \sum_{y\in\mathcal{O}^n(x)} \delta_y$$

and have the following theorem.

Theorem 5.3. *For almost every $\beta \in (1,\sqrt{2})$, we have*

$$\mu_{n,x} \to \lambda|_{I_\beta}$$

weakly as $n \to \infty$ for almost every $x \in I_\beta$.

Here $\lambda|_{I_\beta}$ is Lebesgue measure on the interval I_β, normalised by multiplying by $\beta - 1$ to give $\lambda|_{I_\beta} = 1$. We conjecture that the conclusion of this theorem holds whenever ν_β is absolutely continuous.[2]

Given the description of the measures m_x in the earlier sections, it seems natural that Theorem 5.3 follows from Corollary 5.2. In some sense, all we are doing is dividing by the density $h_\beta(x)$ to turn ν_β into $\lambda|_{I_\beta}$ on the right hand side and $\nu_{n,x}$ into $\mu_{n,x}$ on the left. However, we have to do this formally, and also to be careful to ensure that too much of $\mu_{n,x}$ is not concentrated at the edges of I_β (where $h_\beta \to 0$ and so dividing by h_β is problematic).

[2] Some progress in this direction was announced by C. Bandt at a recent conference in Hong Kong; at the time of writing no preprint was available.

Proof of Theorem 5.3. We assume that β is such that ν_β is absolutely continuous with continuous density h_β which is strictly positive on $(0, 1/(\beta - 1))$ and that β satisfies the conclusion of Corollary 5.2. This holds for almost every $\beta \in (1, \sqrt{2})$; the fact that h_β is strictly positive on $(0, 1/(\beta - 1))$ for almost every $\beta \in (1, \sqrt{2})$ was proved in [13].

Now let $A \subset I_\beta$ be such that there exists a constant $h_\beta(A)$ for which

$$h_\beta(A)(1 - \epsilon) < h_\beta(x) < h_\beta(A)(1 + \epsilon) \tag{5.1}$$

for each $x \in A$. Then

$$\begin{aligned}&\sum_{a_1\cdots a_n \in \{0,1\}^n} m_x[a_1 \cdots a_n]\chi_A(T_{a_1\cdots a_n}(x))\\ &= \sum_{a_1\cdots a_n \in \{0,1\}^n} \left(\frac{\beta}{2}\right)^n \frac{h_\beta(T_{a_1\cdots a_n}(x))}{h_\beta(x)} \chi_A(T_{a_1\cdots a_n}(x)) \le \left(\frac{\beta}{2}\right)^n \frac{h_\beta(A)(1+\epsilon)}{h_\beta(x)} |\mathcal{O}^n(x) \cap A|.\end{aligned}$$

Now Corollary 5.2 says that

$$\sum_{a_1\cdots a_n \in \{0,1\}^n} m_x[a_1 \cdots a_n]\chi_A(T_{a_1\cdots a_n}(x)) \xrightarrow[n\to\infty]{} \nu_\beta(A) \ge \lambda(A)h_\beta(A)(1 - \epsilon).$$

Then using the fact that $m_x = m_x^1$, and approximating $h_\beta(T_{a_1\cdots a_n}(x))$ by $h_\beta(A)$, we find for sufficiently large n that

$$\lambda(A)h_\beta(A)(1 - \epsilon)^2 \le \left(\frac{\beta}{2}\right)^n \frac{h_\beta(A)(1+\epsilon)}{h_\beta(x)} |\mathcal{O}^n(x) \cap A|,$$

giving

$$|\mathcal{O}^n(x) \cap A| \ge \lambda(A)h_\beta(x)\left(\frac{2}{\beta}\right)^n \frac{(1-\epsilon)^2}{1+\epsilon}.$$

Similarly,

$$|\mathcal{O}^n(x) \cap A| \le \lambda(A)h_\beta(x)\left(\frac{2}{\beta}\right)^n \frac{(1+\epsilon)^2}{1-\epsilon}. \tag{5.2}$$

The following proposition will end the proof of the theorem.

Proposition 5.1. *For almost all $\beta \in (1, \sqrt{2})$ and almost all $x \in I_\beta$ we have*

$$\lim_{n\to\infty} (\beta - 1)\left(\frac{\beta}{2}\right)^n |\mathcal{O}^n(x)| = h_\beta(x).$$

Then we see that, since ϵ was arbitrary in (5.2),

$$\mu_{n,x}(A) = \frac{(\beta/2)^n|\mathcal{O}^n(x) \cap A|}{(\beta/2)^n|\mathcal{O}^n(x)|} \to \lambda(A)(\beta - 1)\frac{h_\beta(x)}{h_\beta(x)} = \lambda|_{I_\beta}(A)$$

as $n \to \infty$. Now any interval $B \subset (\delta, 1/(\beta - 1) - \delta)$ can be written as a union of intervals A_i for which there is a constant $h_\beta(A)$ such that (5.1) holds, and so the proof of

Theorem 5.3 will be complete once we have proved that not too much mass is concentrated in sets $[0, \delta)$. This is included in the proof of Proposition 5.1. $\square$

Proposition 5.1 is interesting in its own right, showing that Conjecture 1 of [14] holds at least for almost every $\beta \in (1, \sqrt{2})$. It was conjectured in [14] that this proposition holds for almost all $x \in I_\beta$ for all β such that ν_β is absolutely continuous; this conjecture remains open. A similar question was asked in [11] relating to solutions of the Schilling equation, which share many similarities with Bernoulli convolutions.

Proof of Proposition 5.1. We assume that β is non-algebraic and that the conditions of the previous theorem hold, i.e. ν_β is absolutely continuous with continuous density h_β which is strictly positive on $(0, 1/(\beta-1))$ and β satisfies the conclusion of Corollary 5.2. This holds for almost every $\beta \in (1, \sqrt{2})$. Since $h_\beta(x) > 0$ on the interior of I_β and h_β is uniformly continuous, for any $\delta > 0$ we can cover $(\delta, 1/(\beta-1)-\delta)$ with intervals $A_1, \ldots, A_k$ such that there exist constants $h_\beta(A_i)$ satisfying (5.1).

We first suppose that for some $n_k \to \infty$, too much of $|\mathcal{O}^{n_k}(x)|$ is concentrated in $[0, \delta)$ for some $\delta > 0$. To be concrete, we suppose that there exists some $K > 2/(2-\beta)$ for which

$$|\mathcal{O}^{n_k}(x) \cap [0, \delta)| > h_\beta(x)\left(\frac{2}{\beta}\right)^{n_k} K\delta.$$

But

$$|\mathcal{O}^{n_k}(x) \cap [0, \delta)| = \left|\mathcal{O}^{n_k-1}(x) \cap \left[0, \frac{\delta}{\beta}\right)\right| + \left|\mathcal{O}^{n_k-1}(x) \cap \left[\frac{1}{\beta}, \frac{1+\delta}{\beta}\right)\right|,$$

where the first and second summand correspond to applying T_0^{-1} and T_1^{-1} respectively to the sets $\mathcal{O}^{n_k}(x) \cap [0, \delta)$. For any $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$\left|\mathcal{O}^{n}(x) \cap \left[\frac{1}{\beta}, \frac{1+\delta}{\beta}\right)\right| < (1+\epsilon)h_\beta(x)\left(\frac{2}{\beta}\right)^{n} \frac{\delta}{\beta} \tag{5.3}$$

for all $n > N$ by (5.2). Combining the last two inequalities, we must have

$$\begin{aligned}\left|\mathcal{O}^{n_k-1}(x) \cap \left[0, \frac{\delta}{\beta}\right)\right| &> h(x)\left(\frac{2}{\beta}\right)^{n_k} K\delta - (1+\epsilon)h_\beta(x)\left(\frac{2}{\beta}\right)^{n_k-1} \frac{\delta}{\beta} \\ &= h_\beta(x)\left(\frac{2}{\beta}\right)^{n_k-1} K\delta\left(\frac{2}{\beta} - \frac{1+\epsilon}{\beta K}\right) > h_\beta(x)\left(\frac{2}{\beta}\right)^{n_k-1} K\delta,\end{aligned}$$

since $\frac{2}{\beta} - \frac{1+\epsilon}{\beta K} > 1$ for sufficiently small ϵ by our choice of K. We iterate this inequality to get

$$\left|\mathcal{O}^{n_k-m}(x) \cap \left[0, \frac{\delta}{\beta^m}\right)\right| \le \left|\mathcal{O}^{n_k-m-1}(x) \cap \left[0, \frac{\delta}{\beta^{m+1}}\right)\right| + \left|\mathcal{O}^{n_k-1}(x) \cap \left[\frac{1}{\beta}, \frac{1+\delta}{\beta}\right)\right|,$$

where we are using the interval $[1/\beta, (1+\delta)/\beta)$ rather than $[1/\beta, (1+\delta)/\beta^m)$ on the right because it allows us to use (5.3). Iterating this equation to stage $n_k - N$ gives

$$|\mathcal{O}^{N}(x) \cap [0, \delta/\beta^{n_k-N})| > h_\beta(x)(2/\beta)^N K\delta.$$

Taking $n_k \to \infty$ we see that the multiset $\mathcal{O}^N(x)$ must contain the value 0 multiple times. Since we have assumed that β is non-algebraic, this is a contradiction. By symmetry, the same arguments show that not too much of $\mathcal{O}^n(x)$ can be concentrated in $[1/(\beta-1)-\delta, 1/(\beta-1)]$.

Then building on the proof of the previous theorem, we can cover $(\delta, 1/(\beta-1)-\delta)$ with intervals A_i upon which $h_\beta(x)$ is constant up to multiplicative error ϵ. Summing over A_i and using the bounds in the proof of the previous theorem gives

$$\begin{aligned}\left|\mathcal{O}^n(x) \cap \left(\delta, \frac{1}{\beta-1}-\delta\right)\right| &= \sum_{i=1}^{k} |\mathcal{O}^n(x) \cap A_i| \\ &\geq (h_\beta(x)\left(\frac{2}{\beta}\right)^n \frac{1}{\beta-1} \frac{(1-\epsilon)^2}{1+\epsilon} \sum_{i=1}^{k} \lambda(A_i) \\ &\geq h_\beta(x)\left(\frac{2}{\beta}\right)^n \frac{1}{\beta-1} \frac{(1-\epsilon)^2}{1+\epsilon}(1-2\delta).\end{aligned}$$

Then

$$\begin{aligned}|\mathcal{O}^n(x)| &= |\mathcal{O}^n(x) \cap [0,\delta)| + \left|\mathcal{O}^n(x) \cap \left(\frac{1}{\beta-1}-\delta, \frac{1}{\beta-1}\right]\right| + \sum_{i=1}^{k} |\mathcal{O}^n(x) \cap A_i| \\ &\leq h_\beta(x)\left(\frac{2}{\beta}\right)^n \left(2\delta\frac{2}{2-\beta} + \frac{(1-\epsilon)^2}{1+\epsilon}\,\frac{1-2\delta}{\beta-1}\right).\end{aligned}$$

Since δ and ϵ were arbitrary, we see that

$$\lim_{n\to\infty} (\beta-1)(\beta/2)^n |\mathcal{O}^n(x)| = h_\beta(x)$$

as required. □

5.1. *Absolute continuity from strong equidistribution*

For a partial converse, we show that if the sets $\mathcal{O}^n(x)$ equidistribute in a strong sense for almost every x then ν_β is absolutely continuous. We note that the normalised Lebesgue measure $\lambda|_{I_\beta}$ of the switch region $S := [1/\beta, 1/(\beta(\beta-1))]$ is

$$\left(\frac{1}{\beta(\beta-1)} - \frac{1}{\beta}\right)(\beta-1) = \frac{2}{\beta} - 1.$$

Thus if the measures $\mu_{n,x}$ converge weak* to λ_{I_β} we would expect

$$k_n(x) := \frac{\beta}{2}(\mu_{n,x}(S) + 1)$$

to converge to 1. The following proposition shows that fast equidistribution of $\mathcal{O}^n(x)$ implies the absolute continuity of ν_β.

Proposition 5.2. *Suppose that $\prod_{n=1}^{\infty}(k_n(x))$ is an $\mathcal{L}^1$ function of x. Then ν_β is absolutely continuous.*

In particular, if the sets $\mathcal{O}^n(x)$ equidistribute fast enough and uniformly across x then $k_n(x)$ will tend to 1 quickly and so the condition of the proposition will be satisfied and ν_β will be absolutely continuous.

Proof of Proposition 5.2. We see that we have a choice of the value of a_{n+1} if and only if $T_{a_1\cdots a_n}(x) \in S$, otherwise there is a unique a_{n+1} such that $T_{a_1\cdots a_{n+1}}(x) \in I_\beta$. Then

$$\begin{aligned}(\beta/2)^{n+1}\mathcal{N}_{n+1}(x) &= (\beta/2)^{n+1}|\mathcal{O}^{n+1}(x)| = (\beta/2)^{n+1}(|\mathcal{O}^n(x)| + |\mathcal{O}^n(x) \cap S|) \\ &= (\beta/2)^{n+1}|\mathcal{O}^n(x)|(1 + \mu_{n,x}(S)) = (\beta/2)^{n+1}\prod_{i=1}^{n}(1 + \mu_{i,x}(S)) \\ &= (\beta/2)^{n+1}(2/\beta)^n \prod_{i=1}^{n} k_i(x) = (\beta/2)\prod_{i=1}^{n} k_i(x),\end{aligned}$$

which converges to an $\mathcal{L}^1$ function by assumption. But the main theorem of [14] states that if $f_n(x) := (\beta/2)^n\mathcal{N}_n(x)$ converges to an $\mathcal{L}^1$ function then ν_β is absolutely continuous.

□

6. Slicing fractal sets

We now turn to the question of disintegrating Hausdorff measure for self-similar sets. The techniques that we used in the symbolic case can be combined with a few technical lemmas to show that slices through certain fractals have positive Hausdorff measure if and only if the corresponding projected measures are absolutely continuous with bounded density. We begin with some background on fractals.

Let $E \subset \mathbb{R}^n$ be a self-similar set without rotations, that is, a set satisfying

$$E = \bigcup_{i=1}^{l} S_i(E)$$

where the maps $S_i : \mathbb{R}^n \to \mathbb{R}^n$ are of the form $S_i(x) = \lambda_i x + d_i$ for some $\lambda_i \in (0, 1)$ and $d_i \in \mathbb{R}^n$. We further suppose that our iterated function system satisfies the *open set condition*, i.e. there is a non-empty open set $V \subset \mathbb{R}^n$ such that $V \supset \bigcup_{i=1}^{l} S_i(V)$ where the union is disjoint. Then E has Hausdorff dimension s satisfying

$$\sum_{i=1}^{l} \lambda_i^s = 1.$$

Furthermore, the s-dimensional Hausdorff measure ν on E is positive and finite and satisfies the self-similarity relation

$$\nu(A) = \sum_{i=1}^{l} \lambda_i^s \nu(\tilde{T}_i(A)),$$

where $\tilde{T}_i(x) := S_i^{-1}(x)$. The open set condition implies that for almost every $x \in E$ there is a unique code $\underline{a} \in \Sigma := \{0, \ldots, l\}^{\mathbb{N}}$ such that

$$x \in [a_1 \cdots a_n]_E := S_{a_n} \circ S_{a_{n-1}} \circ \cdots \circ S_{a_1}(E)$$

for each $n \in \mathbb{N}$. We call $\underline{a}$ the *address* of x.

We let π_θ denote orthogonal projection of $\mathbb{R}^n$ down a line l_θ through the origin at angle $\theta = (\theta_1, \ldots, \theta_{n-1})$. We let $\nu_\theta = \nu \circ \pi_\theta^{-1}$. Then ν_θ satisfies the relation

$$\nu_\theta(A) = \sum_{i=1}^{l} \lambda_i^s \nu_\theta(T_i(A))$$

where $T_i(x) = \lambda_i^{-1} x - \pi_\theta(a_i)$ is the projection of the map $\tilde{T}_i$ under π_θ.

Now if $s > 1$ then the Marstrand projection theorem says that for almost every value of θ the projection ν_θ is absolutely continuous. The Marstrand slicing theorem says that for almost every θ and almost every $x \in E_\theta$ the slice $E_{\theta,x}$ has Hausdorff dimension $s-1$ and has finite $(s-1)$-dimensional Hausdorff measure. We refer the reader to [7, 22] for proofs and discussions of the Marstrand slicing and projection theorems. The Marstrand slicing theorem does not say anything about the Hausdorff measure of slices; indeed, an example was given in [18] of a set whose slices are $(s-1)$-dimensional but for which almost every slice has zero $(s-1)$-dimensional Hausdorff measure.

We let $h_\theta : \mathbb{R} \to \mathbb{R}^+$ be the density of ν_θ if it exists; h_θ takes value 0 outside of E_θ. Differentiating the self-similarity equation for ν_θ we see that

$$h_\theta(x) = \sum_{i=1}^{l} \lambda_i^{s-1} h_\theta(T_i(x)), \tag{6.1}$$

where we have used the fact that the derivative of each T_i is λ_i. For $a_1 \cdots a_n \in \{0, \ldots l\}^n$ we define

$$[a_1 \cdots a_n]_{E_{\theta,x}} := E_{\theta,x} \cap (S_{a_n} \circ S_{a_{n-1}} \circ \cdots \circ S_{a_1}(E)).$$

Equation (6.1) is our main tool in the proof of our theorem about the positivity of Hausdorff measure of slices through self-similar sets. The proof of Theorem 6.2 is similar to that of Theorem 4.1, but we require some extra lemmas to estimate the diameter of sets $[a_1 \cdots a_n]_{E_{\theta,x}}$, because, unlike in the symbolic case, this diameter is not purely determined by the length of the word $a_1 \cdots a_n$. We let $|A|$ denote the Euclidean diameter of a set A. This issue with diameters also means that we need the following condition:

Definition 6.1. We say that a self-similar set E satisfies the *slice coding condition* if for all θ there exists a constant δ such that for all $x \in \pi_\theta(E)$, either $|E_{\theta,x}| > \delta$ or $E_{\theta,x} \subset [a]_E$ for some $a \in \{1, \ldots, l\}$.

We suspect that all self-similar sets where the self-similarities do not contain rotations satisfy this condition, but we are unable to prove this. We assume for the rest of the article that the slice coding condition is satisfied.

Lemma 6.1. *Suppose that h_θ is bounded. Then there exists a constant C such that $h_\theta(x)/|E_{\theta,x}|^{s-1} < C$ for all $x \in \pi_\theta(E)$.*

Proof. First let $C := \sup\{h_\theta(x)/|E_{\theta,x}|^{s-1} : |E_{\theta,x}| \geq \delta\}$ where δ was defined in the Definition 6.1. The fact that h_θ is bounded implies that C is finite.

Now suppose that $0 < |E_{\theta,x}| < \delta$. Then since $E_{\theta,x}$ satisfies the slice coding condition, there exists a unique $n \in \mathbb{N}$ and word $a_1 \cdots a_n$ such that $E_{\theta,x} \subset [a_1 \cdots a_n]_{E_{\theta,x}}$ but $E_{\theta,x} \not\subset [a_1 \cdots a_{n+1}]_{E_{\theta,x}}$ for any choice of $a_{n+1} \in \{1, \ldots, n\}$. In particular, we have that $E_{\theta,T_{a_1\cdots a_n}(x)} \not\subset [a_{n+1}]_{E_{\theta,T_{a_1\cdots a_n}(x)}}$ for any choice of a_{n+1}, and so $|E_{\theta,T_{a_1\cdots a_n}(x)}| > \delta$.

Then using equation (6.1) we have

$$h_\theta(x) = (\lambda_{a_n}\lambda_{a_{n-1}} \cdots \lambda_{a_1})^{s-1} h_\theta(T_{a_1\cdots a_n}(x)).$$

By the self-similarity of E we have

$$|E_{\theta,x}| = \lambda_{a_n}\lambda_{a_{n-1}} \cdots \lambda_{a_1} |E_{\theta,T_{a_1\cdots a_n}(x)}|.$$

Then

$$\frac{h_\theta(x)}{|E_{\theta,x}|^{s-1}} = \frac{h_\theta(T_{a_1\cdots a_n}(x))}{|E_{\theta,T_{a_1\cdots a_n}(x)}|^{s-1}} \leq C$$

where the final inequality follows from the definition of C because $|E_{\theta,T_{a_1\cdots a_n}(x)}| \geq \delta$. $\square$

Then following the proof of Theorem 4.1, we have the following theorem.

Theorem 6.2. *Suppose that E is the attractor of an IFS without rotations satisfying the open set condition and Definition 6.1. Further assume that the projection of Hausdorff measure on E onto the line at angle θ through the origin is absolutely continuous with bounded density. Then $\mathcal{H}^{s-1}(E_{\theta,x}) > 0$ for Lebesgue almost every $x \in \pi_\theta(E)$.*

Proof. We prove the statement for ν_θ-a.e. x, and use the fact that Lebesgue measure on $\pi_\theta(E)$ and ν_θ are equivalent whenever ν_θ is absolutely continuous This was proved for Bernoulli convolutions in [19] but an identical proof works for the present case.

We recall that Hausdorff measure is defined as the limit as $\delta \to 0$ of the infimum over all δ-coverings $\mathcal{U} = \{\mathcal{U}_i\}$ of the quantity $\sum_{i=1}^{\infty} |\mathcal{U}_i|^s$. It is enough to consider coverings which are unions of cylinder sets $[a_1 \cdots a_n]_{E_{\theta,x}}$. Then we have

$$|[a_1 \cdots a_n]_{E_{\theta,x}}| = \lambda_{a_1} \cdots \lambda_{a_n} |E_{\theta,T_{a_1\cdots a_n}(x)}|.$$

Following our proof of Theorem 4.1, we have

$$\begin{aligned}
h_\theta(x) &= \sum_{[a_1\cdots a_n]_{E_{\theta,x}} \in \mathcal{U}} (\lambda_{a_1} \cdots \lambda_{a_n})^{s-1} h_\theta(T_{a_1\cdots a_n}(x)) \\
&= \sum_{[a_1\cdots a_n]_{E_{\theta,x}} \in \mathcal{U}} |[a_1 \cdots a_n]_{E_{\theta,x}}|^{s-1} \left(\frac{\lambda_{a_1} \cdots \lambda_{a_n}}{|[a_1 \cdots a_n]_{E_{\theta,x}}|} \right)^{s-1} h_\theta(T_{a_1\cdots a_n}(x)) \\
&= \sum_{[a_1\cdots a_n]_{E_{\theta,x}} \in \mathcal{U}} |[a_1 \cdots a_n]_{E_{\theta,x}}|^{s-1} \frac{h_\theta(T_{a_1\cdots a_n}(x))}{|E_{\theta,T_{a_1\cdots a_n}(x)}|^{s-1}} \\
&= C(\mathcal{U}) \sum_{[a_1\cdots a_n]_{E_{\theta,x}} \in \mathcal{U}} |[a_1 \cdots a_n]_{E_{\theta,x}}|^{s-1},
\end{aligned}$$

where $C(\mathcal{U})$ is a weighted average of the values of $h_\theta(T_{a_1\cdots a_n}(x))/|E_{\theta,T_{a_1\cdots a_n}(x)}|^{s-1}$ over different $a_1\cdots a_n \in \mathcal{U}$. In particular, since $C(\mathcal{U}) < C$ for all covers $\mathcal{U}$, where C is the constant defined in Lemma 6.1, we see that

$$\sum_{[a_1\cdots a_n]_{E_{\theta,x}}\in\mathcal{U}} |[a_1\cdots a_n]_{E_{\theta,x}}|^{s-1} > h_\theta(x)/C$$

for each cover $\mathcal{U}$ of $E_{\theta,x}$, finally yielding $\mathcal{H}^{s-1}(E_{\theta,x}) > h_\theta(x)/C$, which is positive for ν_θ-a.e. x. □

Subsequent work has shown that the packing measure of almost every slice $E_{\theta,x}$ is infinite under the conditions of Theorem 6.2 (see [20]).

6.1. Further fractal results

In this section we outline how the remaining results of sections 3 and 4 transfer over to the fractal case. We have done the difficult part (turning Lemma 4.2 into Theorem 6.2); the remaining results are extremely straightforward and we do not cover them in detail.

First we remark that one can build a dynamical system analogous to that of Section 3 related to the set E. We define

$$X_\theta := \{(x, y) \in \mathbb{R}^2 : x \in E_\theta,\ 0 \le y \le h_\theta(x)\}.$$

The self-similarity equation (3.1) for h_β is directly analogous to the self-similarity equation (6.1) for h_θ, and using the transformations $T_1, \ldots, T_l$ one can partition X_θ into subsets $X_\theta^1, \ldots, X_\theta^l$ in the same way that X was partitioned into X_1, X_2. We define a dynamical system on X_θ using the transformations $T_1, \ldots, T_l$ in the same way as was done in the construction of ϕ in Section 3, and this induces a coding of elements of X_θ. By mapping elements of E to the elements of X_θ which have the same code, one has an isomorphism (up to sets of measure zero) between $(E, \mathcal{H}^s|_E)$ and $(X_\theta, \lambda^2|_{X_\theta})$ where λ^2 is two-dimensional Lebesgue measure.

Now one can define a measure μ_x^1 on the slice $E_{\theta,x}$ by pulling back normalised Lebesgue measure from the fibres $\{(x, y) : 0 \le y \le h_\theta(x)\}$. This gives

$$\mu_x^1([a_1\cdots a_n]_{E_{\theta,x}}) := \frac{(\lambda_{a_1}\cdots\lambda_{a_n})^{s-1} h_\theta(T_{a_1\cdots a_n}(x))}{h_\theta(x)}$$

for $a_1\cdots a_n \in \{0, \ldots, l\}^n$.

By the same Fubini argument given in the proof of Proposition 3.1 we see that the probability measures μ_x^1 disintegrate Hausdorff measure $\mathcal{H}^s$ on E.

We now wish to show that this disintegration coincides with normalised Hausdorff measure on slices. We define μ_x^2 on the sets $E_{\theta,x}$ by

$$\mu_2(A) = \mathcal{H}^{s-1}(A)/\mathcal{H}^{s-1}(E_{\theta,x})$$

for $A \subset E_{\theta,x}$, which is well defined ν_θ-almost everywhere by Theorem 6.2. In [17], Marstrand proved that

$$\mathcal{H}^s(E) \geq \int_{\pi_\theta(E)} \mathcal{H}^{s-1}(E_{\theta,x})\,dx.$$

Combined with our previous theorem this shows that, under the conditions of Theorem 6.2,

$$g(x) := \mathcal{H}^{s-1}(E_{\theta,x})$$

is an $\mathcal{L}^1$ function with positive integral. But then following the proof of Corollary 4.3, we see that g satisfies equation (6.1), and therefore there is a constant $K(\theta)$ such that

$$g(x) = K(\theta)h_\theta(x).$$

Finally, we note that $[a_1 \cdots a_n]_{E_{\theta,x}}$ is a copy of $E_{\theta,T_{a_1\cdots a_n}(x)}$ scaled down by a factor of $\lambda_{a_1}\cdots\lambda_{a_n}$, and so

$$\mathcal{H}^{s-1}([a_1\cdots a_n]_{E_{\theta,x}}) = (\lambda_{a_1}\cdots\lambda_{a_n})^{s-1}\mathcal{H}^{s-1}(E_{\theta,T_{a_1\cdots a_n}(x)}).$$

Plugging this into our definition of μ_x^2 we see that the measures μ^2 and μ^1 coincide whenever they are both defined. Since μ^1 was a disintegration of Hausdorff measure on E, we have the following theorem.

Theorem 6.3. *Suppose that the conditions of Theorem 6.2 are satsified. Then the probability measures μ_x^2, which are the normalised $(s-1)$-dimensional Hausdorff measure on slices through E, disintegrate the measure $\mathcal{H}^s$ on E.*

The fact that the typical Hausdorff measure of slices $E_{\theta,x}$ is 0 whenever the projected measure ν_θ is singular or absolutely continuous with unbounded density also follows directly using the methods of the proofs of Lemmas 4.4 and 4.5.

7. Further comments, examples and questions

We begin by demonstrating that the example given in the introduction is really a special case of Theorem 6.2. First we need a strengthening of Marstrand's projection theorem for self-similar sets with uniform contraction.

Proposition 7.1. *Let E be a self-similar set Hausdorff dimension $s > 2$ for which the generating IFS does not contain rotations and for which each contraction has the same contraction ratio. Then for almost every $\theta = (\theta_1, \theta_2) \in [0, \pi)^2$ the orthogonal projection of s-dimensional Hausdorff measure on E down to the line l_θ is an absolutely continuous measure with continuous density.*

Proof. This proposition, which is probably classical, is proved by a simple convolution argument analogous to one given by Solomyak [28] to prove that Bernoulli convolutions associated to a.e. parameter $\beta \in (1, \sqrt{2})$ are absolutely continuous with continuous density. If the set E is generated by contractions $S_1, \ldots, S_l$ where $S_i(\underline{x}) = \lambda_i(\underline{x}) + \underline{a}_i$ then we can write the measure ν_θ as the distribution of the sums

$$\sum_{n=1}^{\infty} \lambda^i (\pi_\theta \underline{a}_{i_n}),$$

where the i_n are picked uniformly at random from the set $\{1, \ldots, l\}$. But these sums can be decomposed into odd and even terms, so we see that $\nu_\theta = \nu_\theta^{\text{odd}} * \nu_\theta^{\text{even}}$ where these are the measures which give the distribution of the above sums restricted to odd and even terms respectively. Now the Hausdorff dimension $s > 2$ is the unique solution of

$$\sum_{i=1}^{l} \lambda^s = 1,$$

and so if λ were to be replaced with λ^2 then the Hausdorff dimension of the corresponding set would be $s/2 > 1$. In particular, ν_θ^{odd} and ν_θ^{even} are both absolutely continuous for almost all θ, since they correspond to projections of Hausdorff measure on sets of dimension $s/2 > 1$. Hence the convolution $\nu_\theta = \nu_\theta^{\text{odd}} * \nu_\theta^{\text{even}}$ is almost surely absolutely continuous with continuous density, since the convolution of two absolutely continuous measures is absolutely continuous with continuous density. □

The Menger sponge has Hausdorff dimension $\log(20)/\log(3) > 2$; it is a self-similar set without rotations and satisfies the condition of Definition 6.1, thus projections onto lines in $\mathbb{R}^3$ are almost surely absolutely continuous with bounded density. Hence by Theorem 6.2, almost every plane slice through them has positive finite $(s-1)$-dimensional Hausdorff measure.

Question 1. In loose terms, the above proposition show that for self-similar sets E with uniform contraction ratios and without rotations one can expect more regularity of the measures $\nu_\theta(E)$ (in terms of n-fold differentiability of the density) when the Hausdorff dimension of E is larger. Does one have such a principle if the condition that the contraction ratios are uniform is removed? What about general sets without any self-similarity?

Question 2. Does a self-similar set E for which the generating contractions do not contain rotations automatically satisfy the conditions of Definition 6.1?

Question 3. Is the statement '$\mu_{n,x} \to \lambda|_{I_\beta}$ in the weak* topology for Lebesgue almost every $x \in I_\beta$' equivalent to 'ν_β is absolutely continuous'? What about the measures $\nu_{\beta,x}$? Or about the analogous questions on slices and projections of fractals?

Question 4. Suppose that ν_β is singular. Can one describe the measures m_x? Do the quantities $m_x[0]/m_x[1]$ mean anything? When h_β is well defined they relate in a natural way to h_β through the formulation of m_x^1.

Question 5. Do there exist values of β for which ν_β is absolutely continuous with unbounded density? We note that Feng and Wang [9] found some non-Pisot values of β for which ν_β is either singular, or absolutely continuous with unbounded density. One might hope that geometric analytic methods may forbid the possibility that $\mathcal{E}_\beta(x)$ has zero Hausdorff measure for each value of x, and hence rule out the possibility that ν_β is absolutely continuous with unbounded density, this would be very interesting as it would provide non-Pisot examples of singular Bernoulli convolutions.

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