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Complexity of intersections of real quadrics and topology of symmetric determinantal varieties

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Abstract. Let W be a linear system of quadrics on the real projective space $\mathbb{R}\mathrm{P}^n$ and X be the base locus of that system (i.e. the common zero set of the quadrics in W). We prove a formula relating the topology of X to that of the discriminant locus Σ_W (i.e. the set of singular quadrics in W). The set Σ_W equals the intersection of W with the discriminant hypersurface for quadrics; its singularities are unavoidable (they might persist after a small perturbation of W) and we let $\{\Sigma_W^{(r)}\}_{r\geq 1}$ be its singular point filtration, i.e. $\Sigma_W^{(1)} = \Sigma_W$ and $\Sigma_W^{(r)} = \mathrm{Sing}(\Sigma_W^{(r-1)})$. With this notation, for a generic W the above mentioned formula reads

$$b(X) \leq b(\mathbb{R}\mathrm{P}^n) + \sum_{r\geq 1} b(\mathbb{P}\Sigma_W^{(r)}).$$

In the general case a similar formula holds, but we have to replace each $b(\mathbb{P}\Sigma_W^{(r)})$ with $\frac{1}{2}b(\Sigma_\epsilon^{(r)})$, where Σ_ϵ equals the intersection of the discriminant hypersurface with the unit sphere on the translation of W in the direction of a small negative definite form. Each $\Sigma_\epsilon^{(r)}$ is a determinantal variety on the sphere S^{k-1} defined by equations of degree at most $n+1$ (here k denotes the dimension of W); we refine Milnor's bound, proving that for such affine varieties, $b(\Sigma_\epsilon^{(r)}) \leq O(n)^{k-1}$.

Since the sum in the above formulas contains at most $O(k)^{1/2}$ terms, as a corollary we prove that if X is *any* intersection of k quadrics in $\mathbb{R}\mathrm{P}^n$ then the following *sharp* estimate holds:

$$b(X) \leq O(n)^{k-1}.$$

This bound refines Barvinok's style estimates (recall that the best previously known bound, due to Basu, is $O(n)^{2k+2}$).

Keywords. Real algebraic geometry, real quadrics, homological compexity, determinantal varieties

1. Introduction

This paper addresses the question of bounding the topology of the set

$$X = \text{intersection of } k \text{ quadrics in } \mathbb{R}\mathrm{P}^n.$$

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More specifically we are interested in finding a bound for its *homological complexity* $b(X)$, namely the sum of its Betti numbers.[1]

The problem of bounding the topology of semialgebraic sets belonging to some specified family dates back to the works of Milnor, Oleĭnik and Petrovskiĭ, Thom and Smith. Specifically, J. Milnor proved that a semialgebraic set S defined in $\mathbb{R}^n$ by a conjunction of s polynomial inequalities of degree at most d has complexity bounded by $b(S) \leq O(sd)^n$. What is special about sets defined by *quadratic* inequalities is that the role of s and n in the above can be "exchanged" to give A. Barvinok's bound $b(S) \leq n^{O(s)}$ (see [4]).

This kind of *duality* between the variables and the equations in the quadratic case is the leading theme of this paper.

Barvinok's bound, and its subsequent improvements by S. Basu, D. Pasechnik and M.-F. Roy [8] and S. Basu and M. Kettner [7], concern sets defined by s quadratic *inequalities*. The most refined estimate for the complexity of such sets is polynomial in n of degree s, but since we need two inequalities to produce an equality, this bound when applied to the set X of our interest (an intersection of k quadrics) produces (in its best form)

$$\text{Basu's bound:}\quad b(X) \leq O(n)^{2k+2} \qquad (s = 2k).$$

In this paper we focus on the *algebraic* case rather than the semialgebraic one. From the viewpoint of classical algebraic geometry our problem can be stated as follows. We are given a linear system W of real quadrics, i.e. the span of k quadratic forms in the space $\mathbb{R}[x_0, \ldots, x_n]_{(2)}$, and we consider the base locus $X = X_W$ of W, i.e. the set of points in $\mathbb{R}\mathrm{P}^n$ where all these forms vanish. What we consider to be the *dual* object to X_W is the set Σ_W of critical points of W, i.e. those *nonzero* elements in W that are degenerate. The set Σ_W is the intersection of W with the discriminant hypersurface Σ in $\mathbb{R}[x_0, \ldots, x_n]_{(2)}$.

Even if we allow W to be a generic subspace, the set Σ_W might have unavoidable singularities. Thus we consider $\Sigma_W^{(1)} = \Sigma_W$ and for $r \geq 2$,

$$\Sigma_W^{(r)} = \mathrm{Sing}(\Sigma_W^{(r-1)}).$$

In this notation one of the main results of this paper is the following formula, which holds for a generic[2] W:

$$b(X_W) \leq b(\mathbb{R}\mathrm{P}^n) + \sum_{r \geq 1} b(\mathbb{P}\Sigma_W^{(r)}). \tag{1}$$

The sum is finite, since for a generic W the set $\Sigma_W^{(r)}$ is empty for $\binom{r+1}{2} \geq k$. In fact we notice that for every natural r and a generic W the set $\Sigma_W^{(r+1)}$ coincides with the set of quadratic forms in W of *corank* at least r. The codimension of this singular locus is exactly $\binom{r+1}{2}$, thus it is empty for $r > \frac{1}{2}(-1 + \sqrt{8k-7})$.

[1] From now on, unless differently specified, all homology and cohomology groups are assumed to be with coefficients in $\mathbb{Z}_2$.

[2] We require W to be transversal to all strata of the singular point-stratification of Σ.

Example 1. In the case $k = 3$ the set X of interest is the intersection of *three* quadrics in $\mathbb{R}\mathrm{P}^n$. Let us assume for this example that W is generic. Thus $\mathbb{P}\Sigma_W$ is a real algebraic plane curve of degree $n + 1$. In this case statement (1) can be made even stronger:

$$|b(X) - b(\mathbb{P}\Sigma_W)| \leq O(n).$$

Thus for a generic choice of W we can replace the homological complexity of X with the one of $\mathbb{P}\Sigma_W$ and the error we are making in such replacement is bounded by $O(n)$; the generic choice produces a smooth X and also a smooth curve $\mathbb{P}\Sigma_W$ (such a curve is usually referred to as the *spectral variety* of X): the singularities of Σ have codimension three and in this case are avoidable (for a generic three-dimensional W the only singular point of Σ_W is the origin). Notice that essentially under this correspondence,

$$\text{number of variables in the equations for } X = \deg(\mathbb{P}\Sigma_W),$$
$$\text{number of equations for } X = \text{number of variables for } \mathbb{P}\Sigma_W.$$

Harnack's theorem on real plane curves states that

$$b(\mathbb{P}\Sigma_W) \leq n^2 - n + 2.$$

On the other hand, the complete intersection $X_{\mathbb{C}}$ of three real quadrics in $\mathbb{C}\mathrm{P}^n$ has $b(X_{\mathbb{C}}) = n^2 + O(1)$ and Smith's inequality implies

$$b(X) \leq n^2 + O(1).$$

A well known theorem of Vinnikov [19] (see also [12]) states that every real curve of degree $n + 1$ arises as the spectral variety of an intersection of three real quadrics (except for empty curves if $n + 1 \equiv 2 \bmod 4$). In particular (almost) maximal intersections of three real quadrics in $\mathbb{R}\mathrm{P}^n$ correspond to (almost) maximal curves of degree $n+1$ in $\mathbb{R}\mathrm{P}^2$.

Going back to the general case, if we remove the genericity assumption, a similar formula can be proved, but a *perturbation* of Σ_W is introduced. More specifically, we have to translate W in the direction of a small negative definite quadratic form $-\epsilon q$, getting in this way an *affine* space $W_\epsilon = W - \epsilon q$. We then consider a big enough sphere in W_ϵ and the set $\Sigma_\epsilon^{(r)}$ of quadratic forms on this sphere where the kernel has dimension at least r. The following formula holds (now for *any* X):

$$b(X) \leq b(\mathbb{R}\mathrm{P}^n) + \frac{1}{2}\sum_{r\geq 1} b(\Sigma_\epsilon^{(r)}). \tag{2}$$

The same remark on codimensions as above applies here and this sum is actually finite, containing no more than $O(k)^{1/2}$ summands. For a generic choice of W these two constructions coincide, since the set Σ_W deformation retracts on its intersection with any unit sphere in W; such an intersection double covers $\mathbb{P}\Sigma_W$, and the Gysin sequence produces (1) from (2). We will adopt this notation in what follows and think of Σ_W as its homotopy equivalent intersection with a sphere.

Remark 1 (A notion of geometric complexity for X). Notice that (1) becomes effective especially because we are in the framework of *real* algebraic geometry. Here the idea is to consider the number on the r.h.s. of (1) as a "geometric complexity" of X (other than the degree or the number of monomials appearing in the equations). As with fewnomial-type bounds, the structure of the equations (the arrangement in the space of quadratic forms of the linear system) gives finer predictions on the complexity of X.

The bounds (1) and (2) just reflected the above mentioned duality between equations and variables; the real tool encoding this duality is a spectral sequence introduced by A. A. Agrachev [1] and developed by him and the author [3].

The power of these bounds is that they are intrinsic, and different X might produce different ones: for example a set W whose nonzero forms have constant rank has base locus with complexity bounded by $b(\mathbb{R}\mathrm{P}^n)$. On the other hand, the bounds are sufficiently general to produce sharp numerical estimates. Indeed, using them we can get the following, which improves Basu's bound:

$$b(X) \leq O(n)^{k-1}. \tag{3}$$

This estimate is sharp in the following sense: if we let $B(k,n)$ be the maximum of $b(X)$ over *all* possible intersections X of k quadrics in $\mathbb{R}\mathrm{P}^n$, then

$$B(k,n) = O(n)^{k-1}. \tag{4}$$

The upper bound for $B(k,n)$ is provided by (3), and the lower bound by the existence of a maximal real complete intersection, i.e. a complete intersection M of k real quadrics in $\mathbb{C}\mathrm{P}^n$ satisfying $b(M_{\mathbb{R}}) = b(M)$. Such a complete intersection for $k \geq 2$ has the property

$$b(M_{\mathbb{R}}) = c_k n^{k-1} + O(n)^{k-2}.$$

(A smooth nonsingular quadric in $\mathbb{C}\mathrm{P}^n$ has total Betti number $n + \frac{1}{2}(1 + (-1)^{n+1})$.)

We list the first small values of c_k starting from $k = 2$ (the general problem is not trivial):

$$2, 1, \frac{2}{3}, \frac{1}{3}, \frac{2}{15}, \ldots$$

For small values of k the leading coefficient we get by expanding the r.h.s. of (1) in n is the same as the complete intersection one. This provides

$$B(1,n) = n, \quad B(2,n) = 2n, \quad B(3,n) = n^2 + O(n).$$

We conjecture that in general for $k \geq 2$ we have $B(k,n) = c_k n^{k-1} + O(n^{k-2})$. This conjecture can be tackled—and indeed this is the way we produce the numerical bound (3)—by studying the topology of symmetric determinantal varieties. In fact each set $\Sigma_\epsilon^{(r)}$ is defined by the vanishing of some minors of a symmetric matrix depending on parameters (in our case the parameter space is the unit sphere in W_ϵ).

The geometry of symmetric determinantal varieties over the complex numbers was studied in [13], where the degrees of such varieties were explicitly computed. Here we do not need this degree computation, though we use the fact that determinantal varieties are defined by (possibly many) polynomials of small degree. This property, combined with a refinement of Milnor's classical bound,[3] produces the general estimate

$$b(\Sigma_\epsilon^{(r)}) \leq (2n)^{k-1} + O(n)^{k-2}. \tag{5}$$

Notice that if we plug this into (1) we immediately get $B(k,n) \leq O(n)^{k-1}$ (this follows at once using the fact that there are less than $O(k)^{1/2}$ terms in the sum we consider).

On the other hand, such algebraic sets, among those defined by polynomials of degree less than n, are very special. For example they have unavoidable singularities—that is the reason for the appearance of higher order terms in (1). That is why we expect the leading coefficient of the bound (5) not to be optimal. In fact for $k = 1, 2, 3$ we have bounded the complexities of these varieties with a direct argument, getting the optimal coefficient.

As an example for these ideas ("determinantal varieties have small homological complexity") we compute the cohomology of the set Σ of (nonzero) symmetric matrices with zero determinant. This set coincides with the discriminant hypersurface of homogeneous polynomials of degree two (minus zero). The degree of this hypersurface is n and Milnor's bound gives $b(\Sigma) \leq O(n)^{\binom{n}{2}}$. On the other hand, Σ happens to be Spanier–Whitehead dual to a disjoint union of Grassmannians and

$$H^*(\Sigma) \simeq \bigoplus_{j=0}^{n} H_*(\mathrm{Gr}(j,n)). \tag{6}$$

In particular, the complexity of Σ is exactly 2^n, much smaller than Milnor's prediction.

The paper is organized as follows. Section 2 gives an account of the known numerical bounds. Section 3 introduces the spectral sequence approach, from which one can recover Barvinok's bound. Section 4 deals with symmetric determinantal varieties and contains the proof of (6). Section 5 is the technical bulk of the paper and deals with the transversality arguments needed in order to prove (2). Section 6 contains the proof of (1) and (2). Section 7 contains the proof of the numerical translation (3) of the previous bounds as well as a discussion of its sharpness. Section 8 brings some examples.

From now on all algebraic sets are assumed to be *real* (in particular projective spaces and Grassmannians are the real ones) unless otherwise specified. All homology and cohomology groups are with coefficients in $\mathbb{Z}_2$.

2. Complexity of intersections of real quadrics

The aim of this section is to review the numerical bounds that can be derived from the literature for the homological complexity of X.

We first mention the result due to J. Milnor, who proved that if Y is an algebraic set defined by homogeneous polynomials of degree at most d in $\mathbb{R}\mathrm{P}^n$, then $b(Y) \leq$

[3] Milnor's bound would give an estimate of the form $O(n)^k$, but the fact that we are on the sphere allows us to improve it, essentially using a combination of Alexander duality and a general position argument.

$nd(2d-1)^{n-1}$ [18, Corollary 3]. If we apply this bound to the set X we immediately get

$$\text{Milnor's bound:} \quad b(X) \leq 2n3^{n-1}.$$

What is interesting about this bound is that it does not depend on the *number* of equations defining Y (respectively X), but only on their degrees; it is thus natural to expect that this bound can be improved by confining oneself to a fixed number of equations.

A. Barvinok studied the complexity of basic semialgebraic subsets of $\mathbb{R}^n$ defined by a fixed number of inequalities of degree at most two. Using the main result from [4] we can derive another bound, whose shape is different from the previous one:[4]:

$$\text{Barvinok's bound:} \quad b(X) \leq n^{O(k)}.$$

Proof. Theorem (1.1) of [4] states that if Y is defined by k inequalities of degree at most two in $\mathbb{R}^n$ then $b(Y) \leq n^{O(k)}$.

At this point this is just a result for Y (a semialgebraic set in $\mathbb{R}^n$); we now explain how to use it for $X \subset \mathbb{R}\mathrm{P}^n$.

We decompose our X into its affine part A and its part at infinity B and we use a Mayer–Vietoris argument. More specifically, we let $A = X \cap \{x_0 \neq 0\}$ and $B = X \cap \{x_0^2 \leq \epsilon\}$. Now A is defined by $2k$ quadratic inequalities in $\mathbb{R}^n$ (each equation is equivalent to a pair of inequalities) and B by k quadratic equations in $\mathbb{R}\mathrm{P}^n$ (in fact this set for small ϵ deformation retracts to $X \cap \{x_0 = 0\}$). The intersection $A \cap B$ is defined in $\mathbb{R}^n$ by $2k+1$ quadratic inequalities: those defining A plus the one defining a big ball. We now apply Theorem (1.1) of [4] to A and $A \cap B$ to get a bound of the form $n^{O(k)}$ for their total Betti numbers. Induction on n and the Mayer–Vietoris long exact sequence of the semialgebraic pair (A, B) finally give

$$b(X) \leq b(A) + b(B) + b(A \cap B) \leq n^{O(k)}. \qquad \square$$

The subtlety of the previous bound is the implied constant in its definition: indeed in Theorem (1.1) of [4] this implied constant is at least two. This provides an implied constant of at least *four* in Barvinok's bound. The work [7] of S. Basu and M. Kettner provides a better estimate for this constant:

$$\text{Basu's bound:} \quad b(X) \leq O(n)^{2k+2}.$$

Proof. Corollary 1.7 of [7] states the following: Let S be a semialgebraic subset of $\mathbb{R}^n$ defined by k quadratic inequalities. Then

$$b(S) \leq \frac{n}{2} \sum_{j=0}^{k} \binom{k}{j} \binom{n+1}{j} 2^j = s(k, n).$$

[4] Following [9], in this context the notation $f(l) = O(l)$ means that there exists a natural number b such that $f(l) \leq bl$ for every $l \in \mathbb{N}$.

Let us show first that $s(k,n)$ behaves asymptotically as $O(n)^{k+1}$; indeed, let us prove that $\lim_n \frac{\log s(k,n)}{\log n} = k+1$. Notice first that for every k there exists $C_k > 0$ such that for every n,

$$\binom{n+1}{k} \leq \sum_{j=0}^{k} \binom{k}{j}\binom{n+1}{j} 2^j \leq C_k \binom{n+1}{k}.$$

The existence of C_k is due to the fact that the number of terms we are adding and the number $\binom{k}{j}$ do not depend on n but only on k. Using Stirling's asymptotic at infinity $n! \sim \sqrt{2\pi n}(n/e)^n$ we can write

$$\binom{n+1}{k} \sim \frac{1}{k!e^k} \frac{(n+1)^{n+1}}{(n+1-k)^{n+1-k}} \sqrt{\frac{n+1}{n+1-k}} \sim A_k n^k,$$

for some constant $A_k > 0$. The inequalities $\frac{n}{2}\binom{n+1}{k} \leq s(k,n) \leq \frac{C_k n}{2}\binom{n+1}{k}$ and the previous asymptotic immediately give the limit.

Proceeding now as in the proof of Barvinok's estimate, i.e. decomposing X into its affine and infinity part and using Mayer–Vietoris bounds, yields the result. □

The bound in [7] is the best known for semialgebraic sets defined by quadratic *inequalities*. Surprisingly enough, in the special case of our interest, i.e. algebraic sets, the exponent of Basu's bound can be lowered to $k-1$. This will be a straightforward consequence of a deeper approach to bounding the topology of X with the complexity of some determinantal varieties associated (and in a certain sense *dual*) to it. This is based on a spectral sequence argument and has strong consequences, besides the framework of bounding the topology of X.

3. The spectral sequence approach

In this section we will discuss a different approach to the study of intersections of real quadrics. This was first introduced by A. A. Agrachev [1], [2] for the nonsingular case and then extended in [3] to the general case.

Let $\mathcal{Q}_n$ denote the vector space of homogeneous polynomials of degree two in n variables, i.e. the space of quadratic forms over $\mathbb{R}^n$. Then X is the zero locus in the projective space of the elements $q_1, \dots, q_k \in \mathcal{Q}_{n+1}$ and we consider the *linear system*[5] defined by these elements:

$$W = \mathrm{span}\{q_1, \dots, q_k\} \subset \mathcal{Q}_{n+1}.$$

For a given quadratic form $p \in \mathcal{Q}_{n+1}$ we denote by $\mathrm{i}^+(p)$ its positive inertia index, the maximal dimension of a subspace of $\mathbb{R}^{n+1}$ such that the restriction of p to it is positive definite. The idea of the spectral sequence approach is to replace the geometry of X with the one of the restriction of the function i^+ to W. More precisely, let us consider the sets

$$W^j = \{q \in W \mid \mathrm{i}^+(q) \geq j\}, \qquad j \geq 1.$$

[5] In classical algebraic geometry, X is referred to as the base locus of W.

Notice that none of these sets contains zero and all of them are invariant under multiplication by positive numbers, hence they are deformation retracts of their intersections with a unit sphere in W (with respect to any scalar product):

$$\Omega^j = W^j \cap \{\text{any unit sphere in } W\}, \quad j \geq 1.$$

Even if not canonical (it depends on the choice of a scalar product on W) sometimes it is more convenient to use this family rather than the previous one; notice though that different scalar products will produce homeomorphic families. The spirit of this approach is to exploit the relation between X and the filtration

$$\Omega^{n+1} \subseteq \Omega^n \subseteq \cdots \subseteq \Omega^2 \subseteq \Omega^1.$$

This is made precise by a Leray spectral sequence argument, and produces the following

Theorem 1.

$$\text{Agrachev's bound:} \quad b(X) \leq n + 1 + \sum_{j \geq 0} b(\Omega^{j+1}).$$

Proof. Let S_W be any unit sphere in W and consider the topological space $B = \{(\omega, [x]) \in S_W \times \mathbb{R}\mathrm{P}^n \mid (\omega q)(x) > 0\}$ together with its two projections $p_1 : B \to S_W$ and $p_2 : B \to \mathbb{R}\mathrm{P}^n$. The image of p_2 is easily seen to be $\mathbb{R}\mathrm{P}^n \setminus X$ and the fibers of this map are contractible sets, hence p_2 gives a homotopy equivalence[6] $B \sim \mathbb{R}\mathrm{P}^n \setminus X$.

Consider now the projection p_1; for a point $\omega \in S_W$ the fiber $p_1^{-1}(\omega)$ has the homotopy type of a projective space of dimension $\mathrm{i}^+(\omega q) - 1$, thus the Leray spectral sequence[7] for p_1 converges to $H^*(\mathbb{R}\mathrm{P}^n \setminus X)$ and has the second term $E_2^{i,j}$ isomorphic to $H^i(\Omega^{j+1})$. A detailed proof of these statements can be found in [3]. Since $\mathrm{rk}(E_\infty) \leq \mathrm{rk}(E_2)$, we have $b(\mathbb{R}\mathrm{P}^n \setminus X) \leq \sum_{j \geq 0} b(\Omega^{j+1})$. Recalling that by Alexander–Pontryagin duality $H_{n-*}(X) \simeq H^*(\mathbb{R}\mathrm{P}^n, \mathbb{R}\mathrm{P}^n \setminus X)$, then the exactness of the long cohomology exact sequence of the pair $(\mathbb{R}\mathrm{P}^n, \mathbb{R}\mathrm{P}^n \setminus X)$ gives the desired inequality. □

It is interesting to notice that Agrachev's bound implies Barvinok's. Indeed, let us fix a scalar product on $\mathbb{R}^{n+1}$; then the formula $\langle x, Qx \rangle = q(x)$ for $x \in \mathbb{R}^{n+1}$ defines a symmetric matrix Q whose number of positive eigenvalues equals $\mathrm{i}^+(q)$. Consider the polynomial

$$f(t, Q) = \det(Q - tI) = a_0(Q) + \cdots + a_n(Q)t^n \pm t^{n+1}$$

defined over $\mathbb{R} \times W = \mathbb{R} \times \mathrm{span}\{q_1, \ldots, q_k\}$. Then by Descartes' rule of signs the positive inertia index of Q is given by the sign variation in the sequence $(a_0(Q), \ldots, a_n(Q))$.

6 Strictly speaking, if we want to use the Vietoris–Begle theorem, we should check that p_2 closed. This can be avoided if we replace B with $B_\epsilon = \{(\omega, [x]) \in S_W \times \mathbb{R}\mathrm{P}^n \mid (\omega q)(x) \geq \epsilon\}$: this set is now compact, $p_2|_{B_\epsilon}$ is closed and its fibers are still contractible. By semialgebraic triviality for small ϵ the inclusion $i : B_\epsilon \to B$ is a homotopy equivalence; since the projection $p_2|_{B(\epsilon)}$ factors as $p_2 \circ i$, p itself is a homotopy equivalence.

7 Notice that this is not the spectral sequence of the filtration $\{p_2^{-1}(\Omega^{j+1})\}_{j=1}^n$: the latter would converge in $n + 1$ steps, whereas our Leray spectral sequence converges in $k + 1$ steps.

Thus the sets Ω^{j+1} are defined on the unit sphere in W by sign conditions (quantifier-free formulas) whose atoms belong to a set of $n+1$ polynomials in k variables and of degree less than $n+1$. For such sets we have the estimate, proved in [9]: $b(\Omega^{j+1}) \leq n^{O(k)}$. Putting all this together we get

$$b(X) \leq n+1+\sum_{j\geq 0} b(\Omega^{j+1}) \leq n^{O(k)}.$$

Example 2. Before going on we give an idea of which direction this spectral sequence approach will lead us in; this will be just a motivation for the next sections, the detailed theory being developed in the final part of the paper. Let S_W be the unit sphere in W and assume that the set

$$\Sigma_W = \{q \in S_W \mid \ker(q) \neq 0\}$$

is a *smooth* manifold and each time we cross it the index function changes exactly by ± 1. Then the components of this manifold are exactly the boundaries of the sets Ω^j and $b(\Sigma_W) = \sum_{\Omega^j \neq S_W} b(\partial\Omega^j)$. On the other hand, each $\partial\Omega^j$ is a submanifold of the sphere and it is not difficult to show that $b(\partial\Omega^j) = 2b(\Omega^j)$ (we will give an argument for this in Lemma 9). Inserting all this into Agrachev's bound we get

$$b(X) \leq n+1+\tfrac{1}{2}b(\Sigma_W), \tag{7}$$

which relates the topology of X (the base locus of W) to the topology of Σ_W (the singular locus of W). In the general case Σ_W will not be smooth, nor the index function well behaving, and a more refined approach is needed. This approach is based on the study of the topology of Σ_W and its singularities. These two objects are very particular algebraic sets: they are defined by the set of points where a family of matrices has some rank degeneracy, i.e. they are determinantal varieties.

4. Symmetric determinantal varieties

The aim of this section is to bound the topology and describe some geometry of symmetric determinantal varieties. In a broad sense these will be defined by rank degeneracy conditions of (algebraic) families of symmetric matrices. Recall that our interest is in families of quadratic forms; we switch to symmetric matrices simply by establishing a linear isomorphism between $\mathcal{Q}_n$ and $\mathrm{Sym}_n(\mathbb{R})$. This can be done once a scalar product on $\mathbb{R}^n$ has been fixed, by associating to each quadratic form q the matrix Q defined by

$$q(x) = \langle x, Qx\rangle \quad \forall x \in \mathbb{R}^n.$$

Notice that the dimension of the vector space $\mathrm{Sym}_n(\mathbb{R})$ is $\binom{n}{2}$.

Suppose now that Y is an algebraic subset of $\mathrm{Sym}_n(\mathbb{R})$; for every natural number r we define the rank degeneracy locus

$$Y^{(r)} = \{Q \in Y \mid \dim\ker(Q) \geq r\}.$$

Using the bound of [18] we can immediately prove the following proposition, which exploits the idea that symmetric determinantal varieties have relatively simple topology.

Proposition 2. *Let Y be defined by polynomials of degrees less than d in* $\mathrm{Sym}_n(\mathbb{R})$ *and $\mathbb{R}^k$ be a subspace; let also $\delta = \max\{d, n-r+1\}$. Then*

$$b(Y^{(r)} \cap \mathbb{R}^k) \leq \delta(2\delta - 1)^{k-1}.$$

Proof. The set $Y^{(r)}$ is defined in $\mathrm{Sym}_n(\mathbb{R})$ be the same equations defining Y plus all the equations for the vanishing of minors of order $r+1$; these last equations have degree $r+1$. Once we intersect $Y^{(r)}$ with a linear space of dimension k, we get a set defined by equations of degree at most δ in k variables, and Milnor's estimate applies. □

Let us now fix a scalar product also on the space $\mathrm{Sym}_n(\mathbb{R})$, e.g. we can take $\langle A, B\rangle = \frac{1}{2}\mathrm{tr}(AB)$. We consider the set of singular matrices of norm one:

$$\Sigma = \{\|Q\|^2 = 1,\ \det(Q) = 0\}.$$

This set is a deformation retract of the set of *nonzero* matrices with determinant zero and for $n > 1$ it is defined by equations of degree at most n in $\mathrm{Sym}_n(\mathbb{R})$. The previous proposition would produce a bound of the form $an^{\binom{n}{2}}$ for its topological complexity; indeed the bound is much better, as shown in the next theorem.

Theorem 3.

$$H^*(\Sigma) \simeq \bigoplus_{j=0}^{n} H^*(\mathrm{Gr}(j,n)) \quad \textit{and} \quad b(\Sigma) = 2^n.$$

Proof. In the space of all symmetric matrices let us consider the open set A where the determinant does not vanish; this set deformation retracts to $S^N \setminus \Sigma$ and by Alexander–Pontryagin duality it follows that

$$H^*(\Sigma) \simeq H_*(A). \tag{8}$$

On the other hand A is the *disjoint union* of the open sets

$$G_{j,n} = \{\det(Q) \neq 0,\ \mathrm{i}^+(Q) = j\}, \qquad j = 0, \ldots, n.$$

We prove that each of these sets is homotopy equivalent to a Grassmannian; this, together with equation (8), will give the desired result. More specifically, we show that the semi-algebraic map

$$p_k : G_{j,n} \to \mathrm{Gr}(j,n)$$

which sends each matrix Q to its positive eigenspace, is a homotopy equivalence. In fact, let $\{e_1, \ldots, e_n\}$ be the standard basis of $\mathbb{R}^n$ and E_j be the span of the first j basis elements. The preimage of E_j under p_j equals the set of all symmetric block matrices of the form

$$Q = \begin{pmatrix} D^2 & 0 \\ 0 & Q' \end{pmatrix}$$

with D diagonal invertible and Q' invertible and negative definite, i.e. $Q' \in G_{0,n-j}$. In particular, since the set $G_{0,n-j}$ is an open cone, it is contractible and

$$p_j^{-1}(E_j) \simeq (\mathbb{R}^+)^j \times G_{0,n-j} \quad \text{is contractible.}$$

For $W \in \mathrm{Gr}(j,n)$ let M be any orthogonal matrix such that $MW = E_j$; then clearly $p_j^{-1}(W) = M^{-1}p_j^{-1}(E_j)M$ and all the fibers of p_j are homeomorphic. Notice that the matrix M can be chosen to depend continuously (and indeed semialgebraically) on W; hence p_j is a semialgebraic fibration with contractible fibers, hence a homotopy equivalence.

The last part of the theorem follows from the well known fact that $b(\mathrm{Gr}(j,n)) = \binom{n}{j}$ and the formula $\sum_{j=0}^{n}\binom{n}{j} = 2^n$. □

Let Z be the algebraic set of all singular matrices in $\mathrm{Sym}_n(\mathbb{R})$; we will be interested in greater generality in the filtration

$$\{0\} = Z^{(n)} \subset Z^{(n-1)} \subset \cdots \subset Z^{(2)} \subset Z^{(1)} = Z. \tag{9}$$

We recall that each $Z^{(r)}$ is a real algebraic subset of $\mathrm{Sym}_n(\mathbb{R})$ of codimension $\binom{r+1}{2}$ and that the singular loci of these varieties are related by

$$\mathrm{Sing}(Z^{(j)}) = Z^{(j+1)}. \tag{10}$$

References for this statements are [1] and [3]; in particular Proposition 9 of [3] shows that Z is Nash stratified by the smooth semialgebraic sets $N_r = Z^{(r)} \setminus Z^{(r+1)}$ (see [10] for the definition and properties of Nash stratifications). Notice also that using the above notation we have the equalities $Y^{(r)} = Y \cap Z^{(r)}$ and $\Sigma = \{\|Q\|^2 = 1\} \cap Z^{(1)}$.

The degrees of the complexifications $Z_{\mathbb{C}}^{(r)}$ of these varieties are computed in [13]:

$$\deg Z_{\mathbb{C}}^{(r)} = \prod_{\alpha=0}^{r-1} \frac{\binom{n+\alpha}{r-\alpha}}{\binom{2\alpha+1}{\alpha}} = O(n)^{r(r+1)/2}.$$

Notice that they have big degree but small topological complexity and the same holds for their hyperplane sections. High degree is essentially due to their unavoidable singularities, small complexity (via Milnor's bound) to the fact that they can be defined by many equations of low degree.

Let us denote by $\mathrm{i}^-(Q)$ the number of *negative* eigenvalues of a symmetric matrix Q and recall that

$$P_j = \{Q \in \mathrm{Sym}_n(\mathbb{R}) \mid \mathrm{i}^-(Q) \le j\}, \quad 0 \le j \le n-1,$$

is a (noncompact) topological submanifold of $\mathrm{Sym}_n(\mathbb{R}) \simeq \mathcal{Q}_n$ with boundary (see [1]). Let us also set

$$A_j = \partial P_j, \quad 0 \le j \le n-1.$$

The following proposition describes in more detail the structure of the sets $Z^{(r)}$, using the combinatorics of the A_j's.

Proposition 4. *For every $r \ge 0$ let I_r be the set of all the subsets α of $\{0,\ldots,n-1\}$ consisting of r consecutive integers. Then*

$$Z^{(r)} = \bigcup_{\alpha \in I_r} \bigcap_{j \in \alpha} A_j.$$

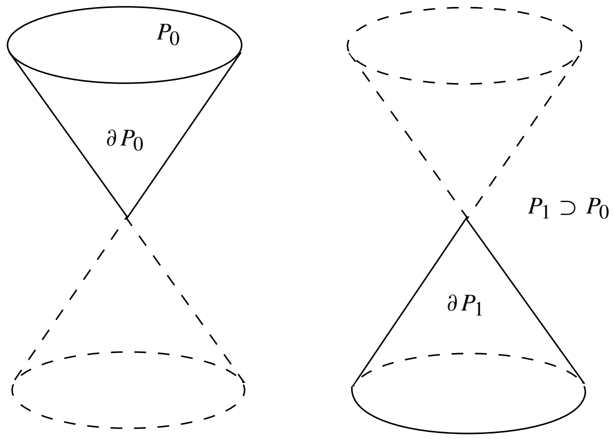

Fig. 1. A picture of the filtration $P_0 \subset P_1 \subset \mathrm{Sym}_2(\mathbb{R})$. The set P_0 is the closed cone of positive semidefinite matrices, its boundary is a topological manifold; P_1 is the set of sign-indefinite matrices, it contains P_0 and its boundary is again a topological manifold. The union $\partial P_0 \cup \partial P_1$ is the set of singular matrices and this set is not a manifold: its singular locus (the zero matrix) is given by $\partial P_0 \cap \partial P_1$.

Proof. For $l \geq 0$ let us say that a matrix Q has the *property* $s(l)$ if there exists a sequence $\{Q_n\}_{n\geq 0}$ converging to Q such that $\mathrm{i}^-(Q_n) \geq l$. Using this notation we have $A_j = \{\mathrm{i}^-(Q) \leq j$ and Q has the property $s(j+1)\}$. From this it follows that for every $r \geq 0$,

$$A_i \cap A_{i+r-1} = \{\mathrm{i}^-(Q) \leq i \text{ and } Q \text{ has the property } s(i+r)\},$$

which also says that $A_i \cap A_{i+1} \cap \cdots \cap A_{i+r-1} = A_i \cap A_{i+r-1}$. Let now $Q \in Z^{(r)}$ and M be an orthogonal matrix such that

$$M^{-1}QM = \mathrm{diag}(-\lambda_1^2, \ldots, -\lambda_{\mathrm{i}^-(Q)}^2, \mu_{\mathrm{i}^-(Q)+1}, \ldots, \mu_{n-r}, 0, \ldots, 0)$$

with the λ_i's greater than zero. Let now D_n be defined by changing each zero on the diagonal of the previous matrix to $-1/n$. Then if we set $Q_n = MD_nM^{-1}$ we find that Q satisfies the property $s(\mathrm{i}^-(Q)+r)$ and thus belongs to $\bigcup_{\alpha\in I_r}\bigcap_{j\in\alpha} A_j$. Conversely, let Q be in $\bigcup_{\alpha\in I_r}\bigcap_{j\in\alpha} A_j$; then Q satisfies $s(\mathrm{i}^-(Q)+r)$ and there exists $\{Q_n\}_{n\geq 0}$ such that

$$\dim\ker Q = n - \mathrm{i}^+(Q) - \mathrm{i}^-(Q) \geq n - \mathrm{i}^+(Q_n) - \mathrm{i}^-(Q_n) + r \geq r$$

(for the inequality $\mathrm{i}^+(Q_n) \geq \mathrm{i}^+(Q)$ we have used the fact that $\{\mathrm{i}^+(Q) \geq j\}$ is an open set), i.e. Q is in $Z^{(r)}$. □

5. Transversality arguments

In this section we will discuss the following idea. Suppose we are given X by the vanishing of some quadratic polynomials in $\mathbb{R}\mathrm{P}^n$ and let W be the span of these polynomials as in the previous sections. The homological complexity of X (up to an $n+1$ term) can be bounded, using Agrachev's bound, by the sum of the complexities of the sets Ω^j. To have an alternative description of these sets let us introduce the following notation. Let $q_1, \ldots, q_k \in \mathcal{Q}_{n+1}$ be the quadratic forms defining X and $q : S^{k-1} \to \mathcal{Q}_{n+1}$ the map defined by

$$\omega = (\omega_1, \ldots, \omega_k) \mapsto \omega q = \omega_1 q_1 + \cdots + \omega_k q_k.$$

The map q is the restriction to the sphere of the linear map sending the standard basis of $\mathbb{R}^k$ to $\{q_1, \ldots, q_k\}$. We redefine now

$$\Omega^j = \{\omega \in S^{k-1} \mid \mathrm{i}^+(\omega q) \leq j\}, \qquad j \geq 1.$$

If $q_1, \ldots, q_k$ are linearly independent, then this definition agrees with the previous one; if they are not linearly independent, the map q is no longer an embedding, though a look at the proof of Agrachev's bound shows that it still holds:

$$b(X) \leq n + 1 + \sum_{j \geq 0} b(\Omega^{j+1})$$

(it is sufficient to use the set $B' = \{(\omega, [x]) \in S^{k-1} \times \mathbb{R}\mathrm{P}^n \mid (\omega q)(x) \geq 0\}$ instead of B and the proof works the same; actually it can also be proved that these new sets deformation retract onto the previously defined ones). The question we address is now the following: what happens if we perturb the map q?

The perturbations we will be interested in are of the form

$$q_\epsilon : \omega \mapsto \omega q - \epsilon p,$$

where p is a positive definite quadratic form; in other words we will be interested in small *affine* translations $q - \epsilon p$ of the map q. It turns out that if p is a positive definite quadratic form and $\epsilon > 0$ is small enough then each set Ω^j is homotopy equivalent to

$$\Omega_{n-j}(\epsilon) = \{\omega \in S^{k-1} \mid \mathrm{i}^-(\omega q - \epsilon p) \leq n - j\},$$

where i^- denotes the negative inertia index, i.e. $\mathrm{i}^-(\omega q - \epsilon p) = \mathrm{i}^+(\epsilon p - \omega q)$. In particular the Betti numbers of Ω^{j+1} and of its perturbation $\Omega_{n-j}(\epsilon)$ are the same, as proved in the following lemma from [17].

Lemma 5. *For every positive definite form $p \in \mathcal{Q}_{n+1}$ and for every $\epsilon > 0$ sufficiently small,*

$$b(\Omega^{j+1}) = b(\Omega_{n-j}(\epsilon)).$$

Proof. Let us first prove that $\Omega^{j+1} = \bigcup_{\epsilon > 0} \Omega_{n-j}(\epsilon)$. Let $\omega \in \bigcup_{\epsilon > 0} \Omega_{n-j}(\epsilon)$; then there exists $\overline{\epsilon}$ such that $\omega \in \Omega_{n-j}(\epsilon)$ for every $\epsilon < \overline{\epsilon}$. Since for ϵ small enough,

$$\mathrm{i}^-(\omega q - \epsilon p) = \mathrm{i}^-(\omega q) + \dim(\ker(\omega q)),$$

it follows that

$$\mathrm{i}^+(\omega q) = n + 1 - \mathrm{i}^-(\omega q) - \dim(\ker \omega q) \geq j + 1.$$

Conversely, if $\omega \in \Omega^{j+1}$ the previous inequality proves $\omega \in \Omega_{n-j}(\epsilon)$ for ϵ small enough, i.e. $\omega \in \bigcup_{\epsilon>0} \Omega_{n-j}(\epsilon)$.

Notice now that if $\omega \in \Omega_{n-j}(\epsilon)$ then, possibly choosing a smaller ϵ, we may assume ϵ properly separates the spectrum of ω and thus, by continuity of q, there exists an open neighborhood U of ω such that ϵ also properly separates the spectrum of ηq for every $\eta \in U$. Hence every $\eta \in U$ also belongs to $\Omega_{n-j}(\epsilon)$. From this consideration it easily follows that each compact set in Ω^{j+1} is contained in some $\Omega_{n-j}(\epsilon)$ and thus

$$\varinjlim_{\epsilon} H_*(\Omega_{n-j}(\epsilon)) = H_*(\Omega^{j+1}).$$

It remains to prove that the topology of $\Omega_{n-j}(\epsilon)$ is definitely stable as ϵ goes to zero. Consider the semialgebraic compact set $S_{n-j} = \{(\omega, \epsilon) \in S^{k-1} \times [0, \infty) \mid \mathrm{i}^-(\omega q - \epsilon p) \leq n - j\}$. By Hardt's triviality theorem (see [10]) the projection $(\omega, \epsilon) \mapsto \omega$ is a locally trivial fibration over $(0, \epsilon)$ for ϵ small enough; from this the conclusion follows. □

The following is a variation of [17, Lemma 4] and describes the structure of the sets of degenerate quadratic forms on the 'perturbed sphere'. We recall that the space Z of all degenerate forms in $\mathcal{Q}_{n+1}$ admits the semialgebraic Nash stratification[8] $Z = \coprod N_r$ where $N_r = Z^{(r)} \setminus Z^{(r+1)}$ (as above we use the linear identification between quadratic forms and symmetric matrices).

Lemma 6. *There exists a positive definite form $p \in \mathcal{Q}_{n+1}$ such that for every $\epsilon > 0$ small enough the map $q_\epsilon : S^{k-1} \to \mathcal{Q}_{n+1}$ defined by*

$$\omega \mapsto \omega q - \epsilon p$$

is transversal to all strata of $Z = \coprod N_r$. In particular $q_\epsilon^{-1}(Z) = \coprod q_\epsilon^{-1}(N_r)$ is a Nash stratification, the closure of $q_\epsilon^{-1}(N_r)$ equals $q_\epsilon^{-1}(Z^{(r)})$ and

$$\mathrm{Sing}(q_\epsilon^{-1}(Z^{(r)})) = q_\epsilon^{-1}(Z^{(r+1)}).$$

Proof. Let $\mathcal{Q}^+$ be the set of positive definite quadratic forms in $\mathcal{Q}_{n+1}$ and consider the map $F : S^{k-1} \times \mathcal{Q}^+$ defined by

$$(\omega, p) \mapsto \omega q - p.$$

Since $\mathcal{Q}^+$ is open in $\mathcal{Q}$, the map F is a submersion and $F^{-1}(Z)$ is Nash-stratified by $\coprod F^{-1}(N_i)$. Then for $p \in \mathcal{Q}^+$ the evaluation map $\omega \mapsto f(\omega) - p$ is transversal to all strata of Z if and only if p is a regular value for the restriction of the second factor projection $\pi : S^{k-1} \times \mathcal{Q}^+ \to \mathcal{Q}^+$ to each stratum of $F^{-1}(Z) = \coprod F^{-1}(N_i)$. Thus let $\pi_i = \pi|_{F^{-1}(N_i)} : F^{-1}(N_i) \to \mathcal{Q}^+$; since all data are smooth semialgebraic, by semialgebraic Sard's Lemma (see [10]) the set $\Sigma_i = \{\hat{q} \in \mathcal{Q}^+ \mid \hat{q}$ is a critical value of $\pi_i\}$ is a semialgebraic subset of $\mathcal{Q}^+$ of dimension strictly less than $\dim(\mathcal{Q}^+)$. Hence $\Sigma = \bigcup_i \Sigma_i$ is also a semialgebraic subset of $\mathcal{Q}^+$ with $\dim(\Sigma) < \dim(\mathcal{Q}^+)$ and for every $p \in \mathcal{Q}^+ \setminus \Sigma$ the map $\omega \mapsto f(\omega) - p$ is transversal to each N_i. Since Σ is semialgebraic of codimension at least one, there exists $p \in \mathcal{Q}^+ \setminus \Sigma$ such that $\{tp\}_{t>0}$ intersects Σ in a finite number of points, i.e. for every $\epsilon > 0$ sufficiently small, $\epsilon p \in \mathcal{Q}^+ \setminus \Sigma$. This concludes the proof. □

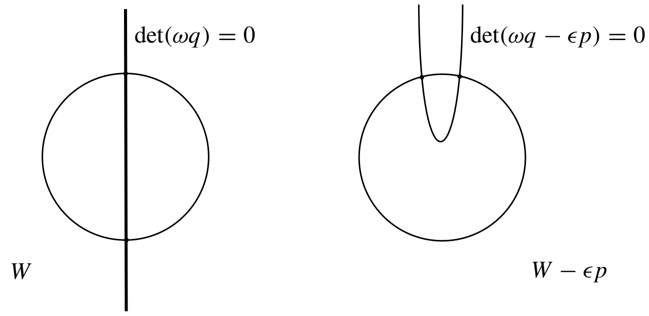

Fig. 2. Effect on the spectral variety of the perturbation from Lemma 6.

Since the codimension of $Z^{(r)}$ is $\binom{r+1}{2}$, we can immediately derive the following.

Corollary 7. *Assume* $r > \frac{1}{2}(-1+\sqrt{8k-7})$. *Then there exists a positive definite form* p *such that for* $\epsilon > 0$ *small enough,*

$$\{\omega \in S^{k-1} \mid \dim\ker(\omega q - \epsilon p) \geq r\} = \emptyset.$$

We recall the following result describing the local topology of the space of quadratic forms (see [3, Proposition 9]).

Proposition 8. *Let* $q_0 \in \mathcal{Q}$ *be a quadratic form and let* V *be its kernel. Then there exists a neighborhood* U_{q_0} *of* q_0 *and a smooth semialgebraic map* $\phi : U_{q_0} \to \mathcal{Q}(V)$ *such that:*

1) $\phi(q_0) = 0$;
2) $\mathrm{i}^-(q) = \mathrm{i}^-(q_0) + \mathrm{i}^-(\phi(q))$;
3) $\dim\ker(q) = \dim\ker(\phi(q))$;
4) $d\phi_{q_0}(p) = p|_V$ *for every* $p \in \mathcal{Q}$.

Combining Lemma 6 and the previous proposition we can prove the following corollary, which shows that after the perturbation the sets $\Omega_{n-j}(\epsilon)$ have a very nice structure.

Corollary 9. *Let* p *be the positive definite form provided by Lemma 6. Then for every* $\epsilon > 0$ *small enough,*

$$\Omega_{n-j}(\epsilon) \textit{ is a } \text{topological } \textit{submanifold of } S^{k-1} \textit{ with boundary}.$$

Proof. Let p be the quadratic form given by Lemma 6 and $f = q_\epsilon : S^{k-1} \to \mathcal{Q}_{n+1}$ the map defined there. Let us consider a point ω in $\Omega_{n-j}(\epsilon)$ and the map $\phi : U_{f(\omega)} \to \mathcal{Q}(\ker f(\omega))$ given by Proposition 8. Since $d\phi_{f(\omega)}p = p|_{\ker f(\omega)}$, the map $d\phi_{f(\omega)}$ is surjective. On the other hand, by transversality of f to each stratum N_r we have

$$\mathrm{im}(df_\omega) + T_{f(\omega)}N_r = \mathcal{Q}_{n+1}.$$

[8] The symbol $\coprod$ means disjoint union.

Since $\phi(N_r) = \{0\}$ (notice that this condition implies $(d\phi_{f(\omega)})|_{T_{f(\omega)}N_r} = 0$), we have

$$\mathcal{Q}(\ker f(\omega)) = \mathrm{im}(d\phi_{f(\omega)}) = \mathrm{im}(d(\phi \circ f)_\omega),$$

which tells us that $\phi \circ f$ is a submersion at ω. Thus by the Rank Theorem there exist an open neighborhood U_ω of ω and an open diffeomorphism ψ onto its image such that the following diagram is commutative:

$$\begin{array}{ccc} U_\omega & \xrightarrow{\ \psi\ } & \mathcal{Q}(\ker f(\omega)) \times \mathbb{R}^l \\ & {\scriptstyle \phi\circ f} \searrow \quad \swarrow {\scriptstyle p_1} & \\ & \mathcal{Q}(\ker f(\omega)) & \end{array}$$

(in particular $\psi(U_\omega)$ is an open subset of $\mathcal{Q}(\ker f(\omega)) \times \mathbb{R}^l$). Let us pick an open neighborhood of $\psi(\omega)$ of the form $A \times B$, with $A \subset \mathcal{Q}(\ker f(\omega))$ and $B \subset \mathbb{R}^l$ contractible, and consider the open set $U' = U_\omega \cap \psi^{-1}(A \times B)$ and the commutative diagram

$$\begin{array}{ccc} U' & \xrightarrow{\ \psi\ } & A \times B \\ & {\scriptstyle \phi\circ f} \searrow \quad \swarrow {\scriptstyle p_1} & \\ & A & \end{array}$$

Notice now that for every η in U' the second point of Proposition 8 implies that $\mathrm{i}^-(f(\eta)) = \mathrm{i}^-(f(\omega)) + \mathrm{i}^-(\phi(f(\eta)))$. In particular, $U' \cap \Omega_{n-j}(\epsilon)$ is homeomorphic, through ψ, to the set

$$\big(A \cap \{q \in \mathcal{Q}(\ker f(\omega)) \mid \mathrm{i}^-(q) \le n - j - \mathrm{i}^-(f(\omega))\}\big) \times B.$$

The first factor is the intersection of A with the set of quadratic forms in $\mathcal{Q}(\ker f(\omega))$ with *negative* inertia index $\le n - j - \mathrm{i}^-(f(\omega))$; since this set is a topological submanifold with boundary in $\mathcal{Q}(\ker f(\omega))$, it follows that $U' \cap \Omega_{n-j}$ is homeomorphic to an open neighborhood of a topological manifold with boundary. This proves that every point ω in $\Omega_{n-j}(\epsilon)$ has an open neighborhood U'_ω such that $U'_\omega \cap \Omega_{n-j}(\epsilon)$ is homeomorphic to an open set of a topological manifold with boundary; thus $\Omega_{n-j}(\epsilon)$ itself is a topological manifold with boundary (the boundary being possibly empty). □

6. A topological bound

The aim of this section is to provide a formula which generalizes (7) from Example 2. The idea is to use Lemma 5 and Corollary 9 in Agrachev's bound: the first says that we can perturb each set Ω^j to a set $\Omega_{n-j}(\epsilon)$ without changing its Betti numbers, the second says that we can do that *and* make the new sets topological manifolds with boundary. As we will see, we can use the topological manifold structure of these sets to get more information out of Agrachev's bound.

We start by proving the following lemma from algebraic topology (see also the proof of [18, Theorem 2]).

Lemma 10. *Let M be a semialgebraic topological submanifold of the sphere S^n with nonempty boundary and nonempty interior. Then*

$$b(M) = \frac{1}{2} b(\partial M).$$

Proof. By assumption also $N = S^n \setminus \mathrm{int}(M)$ is a semialgebraic topological manifold with boundary $\partial N = \partial M$. Let us consider collar neighborhoods A of M and B of N such that $A \cap B$ deformation retracts onto ∂M (such collar neighborhoods certainly exist by semialgebraicity and the Collaring Theorem). From the reduced Mayer–Vietoris sequence for the pair (A, B) we get $\tilde{b}_i(A) + \tilde{b}_i(B) = \tilde{b}_i(A \cap B)$ for $i \neq n-1$, and $\tilde{b}_{n-1}(A) + \tilde{b}_{n-1}(B) = \tilde{b}_{n-1}(A \cap B) - 1$. Adding all these equalities we obtain

$$\tilde{b}(A) + \tilde{b}(B) = \tilde{b}(A \cap B) - 1$$

(here we are using the notation $\tilde{b}(Y)$ for the sum of the *reduced* Betti numbers of a semialgebraic set Y). Alexander–Pontryagin duality implies that $\tilde{b}(N) = \tilde{b}(M)$; on the other hand, A and B deformation retract respectively onto M and N, which means $\tilde{b}(A) = \tilde{b}(M) = \tilde{b}(N) = \tilde{b}(B)$. Plugging this equality in the previous formula immediately gives the statement. □

We now prove the main technical theorem of the paper. A toy model proof in the case when Σ_ϵ is smooth was provided in Example 2; another proof for the case Σ_ϵ has only isolated singularities is given in Example 5; the reader uncomfortable with technical details is advised to take a look at them first.

Theorem 11. *Let X be defined by the vanishing of the quadratic forms $q_1, \ldots, q_k$ in $\mathbb{R}\mathrm{P}^n$. For every quadratic form p and real number ϵ define*

$$\Sigma_\epsilon = \{\omega \in S^{k-1} \mid \det(\omega q - \epsilon p) = 0\},$$

where $q = (q_1, \ldots, q_k)$. Then there exists a positive definite form $p \in \mathcal{Q}_{n+1}$ such that for every $\epsilon > 0$ small enough:

(i) *the map $q_\epsilon : S^{k-1} \to \mathcal{Q}_{n+1}$ given by $\omega \mapsto \omega q - \epsilon p$ is transversal to all strata of Z, stratified as in (9); in particular*

$$\Sigma_\epsilon^{(r)} = \{\omega \in S^{k-1} \mid \dim\ker(\omega q - \epsilon p) \geq r\}$$

is an algebraic subset of S^{k-1} of codimension $\binom{r+1}{2}$;

(ii) *if we let μ and ν be respectively the maximum and the minimum of the negative inertia index on the image of q_ϵ, then*

$$b(X) \leq n + 1 - 2(\mu - \nu) + \frac{1}{2} \sum_{r \geq 1} b(\Sigma_\epsilon^{(r)}). \tag{11}$$

The last sum is indeed finite since for $\binom{r+1}{2} \geq k$ part (i) *implies $\Sigma_\epsilon^{(r)} = \emptyset$.*

Proof. The first statement follows directly from Lemma 6. For the second, in order to get the $-2(\mu-\nu)$ term in (11), we will use a refined version of Agrachev's bound. The refined bound follows by considering, in the proof of Agrachev's bound, a spectral sequence converging directly to $H_*(X)$ and whose second term is isomorphic to $E_2 \simeq \bigoplus_j H^*(B^k, \Omega^{j+1})$, where B^k is the unit ball in $\mathbb{R}^k$ such that $\partial B^k = S^{k-1}$. The existence of such a spectral sequence is the content of [3, Theorem A]; repeating verbatim the above argument we get $b(X) \leq \sum_{j\geq 0} b(B, \Omega^{j+1})$. Let now p be given by Lemma 6; since p is positive definite, Lemma 5 implies $b(\Omega^{j+1}) = b(\Omega_{n-j}(\epsilon))$ for $\epsilon > 0$ sufficiently small and for every $j \geq 0$. In particular we can rewrite the refined Agrachev bound as

$$b(X) \leq n+1-2(\mu-\nu) + \sum_{\nu\leq j\leq \mu-1} b(\Omega_j(\epsilon)). \tag{12}$$

The rest of the proof is devoted to bounding $\sum b(\Omega_j(\epsilon))$. First notice that Corollary 9 says that each nonempty $\Omega_j(\epsilon)$ is a topological submanifold of S^{k-1} with boundary and nonempty interior; thus applying Lemma 10 we get

$$b(\Omega_j(\epsilon)) = \tfrac{1}{2} b(\partial\Omega_j(\epsilon)), \qquad \nu \leq j \leq \mu - 1.$$

For convenience of notation let us rename these boundaries as follows:

$$C_j = \partial\Omega_{\nu+j-1}(\epsilon), \qquad j = 1, \ldots, l = \mu - \nu.$$

Thus (12) can be rewritten as

$$b(X) \leq n+1-2(\mu-\nu) + \frac{1}{2}\sum_{j=1}^{l} b(C_j). \tag{13}$$

Let us now analyze the structure of $\Sigma_\epsilon = \Sigma_\epsilon^{(1)}$. By construction this set equals the union of all the C_j's, but the union is not disjoint since Σ_ϵ might have singularities, which occur precisely when two sets C_j and C_{j+1} intersect (this immediately follows from the fact that q_ϵ is transversal to all the strata of Z, and Σ_ϵ is stratified by the preimages of the strata of Z as described in Lemma 6). For convenience of notation, let $S(\omega, j)$ denote the assertion that there exists a sequence $\{\omega_n\}_{n\geq 0}$ converging to ω such that $\mathrm{i}^-(q_\epsilon(\omega_n)) \geq j$. Corollary 9 implies now that $C_j = \Omega_{\nu+j-1}(\epsilon) \cap \mathrm{Cl}(\Omega_{\nu+j-1}(\epsilon)^c)$, i.e.

$$C_j = \{\omega \mid \mathrm{i}^-(q_\epsilon(\omega)) \leq j \text{ and } S(\omega, j+1)\}. \tag{14}$$

Let I_r be the set of all subsets α of $\{1, \ldots, l\}$ consisting of r consecutive integers; if $\alpha = \{\alpha_1, \ldots, \alpha_r\} \in I_r$, we assume its elements are arranged in increasing order, $\alpha_1 < \cdots < \alpha_r$. Let now $r \in \{1, \ldots, l\}$, $\alpha \in I_r$ and for $i \in \{1, \ldots, l-r\}$ consider the sets

$$E_{i,r} = \bigcup_{\alpha_1 \leq i} \bigcap_{j\in\alpha} C_j, \qquad F_{i+1,r} = \bigcap_{j=i+1}^{i+r} C_j.$$

For example if $r = 1$ we have $E_{i,1} = C_1 \cup \cdots \cup C_i$ and $F_{i+1,1} = C_{i+1}$; if $r = 2$ then $E_{i,2} = (C_1 \cap C_2) \cup \cdots \cup (C_{i-1} \cap C_i)$ and $F_{i+1,2} = C_{i+1} \cap C_{i+2}$. We have the following combinatorial properties:

$$E_{i,r} \cup F_{i+1,r} = E_{i+1,r} \quad \text{and} \quad E_{i,r} \cap F_{i+1,r} = \bigcap_{j=i}^{i+r} C_j.$$

The first equality is clear from the definition; for the second, notice that (14) implies

$$\begin{aligned} E_{i,r} \cap F_{i+1,r} &= \bigcup_{\alpha_1 \le i} \{\omega \mid \mathrm{i}^-(q_\epsilon(\omega)) \le \alpha_1 \text{ and } S(\omega, l+r+1)\} \\ &= \{\omega \mid \mathrm{i}^-(q_\epsilon(\omega)) \le i \text{ and } S(\omega, l+r+1)\} = \bigcap_{j=i}^{i+r} C_j. \end{aligned}$$

Plugging these equalities in the semialgebraic Mayer–Vietoris exact sequence of the pair $(E_{i,r}, F_{i+1,r})$ we get

$$b\Big(\bigcap_{j=i+1}^{i+r} C_j\Big) \le b\Big(\bigcup_{\alpha_1 \le j+1} \bigcap_{j\in\alpha} C_j\Big) + b\Big(\bigcap_{j=i}^{i+r} C_j\Big) - b\Big(\bigcup_{\alpha_1 \le j} \bigcap_{j\in\alpha} C_j\Big). \tag{15}$$

If we now add all these inequalities, we obtain

$$\sum_{i=0}^{l-r} b\Big(\bigcap_{j=i+1}^{i+r} C_j\Big) \le b\Big(\bigcup_{\alpha_1 \le l-r+1} \bigcap_{j\in\alpha} C_j\Big) + \sum_{i=0}^{l-r-1} b\Big(\bigcap_{j=i}^{i+r} C_j\Big). \tag{16}$$

In fact, in the sum all the first and the last terms of the r.h.s. of (15) cancel (since they appear with opposite signs), except for the last inequality which gives the contribution $b(\bigcup_{\alpha_1 \le l-r+1} \bigcap_{j\in\alpha} C_j)$. Moreover since q_ϵ is transversal to all strata of Z, Proposition 4 implies

$$\bigcup_{\alpha_1 \le l-r+1} \bigcap_{j\in\alpha} C_j = \bigcup_{\alpha \in I_r} \bigcap_{j\in\alpha} C_j = \Sigma_\epsilon^{(r)}.$$

Substituting this formula into (16) we finally get

$$\sum_{i=0}^{l-r} b\Big(\bigcap_{j=i+1}^{i+r} C_j\Big) \le b(\Sigma_\epsilon^{(r)}) + \sum_{i=0}^{l-r-1} b\Big(\bigcap_{j=i}^{i+r} C_j\Big). \tag{17}$$

In particular we have the following chain of inequalities (we keep on substituting at each step what we get from (17)):

$$\begin{aligned} \sum_{i=1}^{l} b(C_i) &= \sum_{i=0}^{l-1} b\Big(\bigcap_{j=i+1}^{i+1} C_j\Big) \le b(\Sigma_\epsilon^{(1)}) + \sum_{i=0}^{l-2} b\Big(\bigcap_{j=i}^{i+1} C_j\Big) \\ &\le b(\Sigma_\epsilon^{(1)}) + b(\Sigma_\epsilon^{(2)}) + \sum_{i=0}^{l-3} b\Big(\bigcap_{j=i}^{i+2} C_j\Big) \le \cdots \le \sum_{r\ge 1} b(\Sigma_\epsilon^{(r)}). \end{aligned}$$

Substituting this into (13) and recalling that $\Sigma_\epsilon^{(r)} = \emptyset$ for $\binom{r+1}{2} \ge k$ yields the result. □

As a corollary we immediately get the following theorem.

Theorem 12. *Let* $\sigma_k = \lfloor \frac{1}{2}(-1+\sqrt{8k-7})\rfloor$. *Then we have*

$$\text{Topological bound:} \quad b(X) \leq b(\mathbb{R}\mathrm{P}^n) + \frac{1}{2}\sum_{r=1}^{\sigma_k} b(\Sigma_\epsilon^{(r)}).$$

Proof. This is simply a reformulation of the previous theorem in a nicer form. In fact $\mu - \nu \geq 0$, $n+1 = b(\mathbb{R}\mathrm{P}^n)$ and σ_k is given by Corollary 7. □

If we also assume nondegeneracy of the linear system W we get the following theorem.

Theorem 13. *For a generic choice of* $W = \mathrm{span}\{q_1, \ldots, q_k\}$ *and* $r \geq 1$,

$$\Sigma_W^{(r)} = \{q \in W \setminus \{0\} \mid \dim\ker(q) \geq r\} = \mathrm{Sing}(\Sigma_W^{(r-1)})$$

and

$$b(X) \leq b(\mathbb{R}\mathrm{P}^n) + \frac{1}{2}\sum_{r\geq 1} b(\Sigma_W^{(r)}).$$

Proof. Let us fix a scalar product on $\mathcal{Q}_{n+1}$; then for a generic choice of $q_1, \ldots, q_k$ the unit sphere S^{k-1} in W is transversal to all strata of $Z = \coprod N_r$ and the first part of the statement follows from (10).

Notice that the set of linear affine embeddings $f : \mathbb{R}^k \to \mathcal{Q}_{n+1}$ whose restriction to S^{k-1} is transversal to all the strata of Z is an open dense set; moreover if two such embeddings f_0 and f_1 are joined by a nondegenerate homotopy, then by the Thom Isotopy Lemma the sets $f_0^{-1}(Z^{(r)})$ and $f_1^{-1}(Z^{(r)})$ are homotopy equivalent. In particular for $\epsilon > 0$ small enough the map q_ϵ given by Theorem 11 is nondegenerate homotopic to $S^{k-1} \hookrightarrow \mathcal{Q}_{n+1}$ and thus for every $r \geq 0$ we can substitute

$$b(\Sigma_\epsilon^{(r)}) = b(\Sigma_W^{(r)})$$

in (11), which gives the result. □

Remark 2. From the point of view of classical agebraic geometry, it is natural to consider the projectivization $\mathbb{P}W$ rather than W itself; similarly we can consider $\mathbb{P}\Sigma$ and by the Gysin exact sequence we get, for a generic W,

$$b(X) \leq b(\mathbb{R}\mathrm{P}^n) + \sum_{r\geq 1} b(\mathbb{P}\Sigma_W^{(r)}).$$

Unfortunately there is no such formula for the general case; this is due to the fact that the perturbation Σ_ϵ is not invariant by the antipodal map.

7. A numerical bound

From the previous discussion we can derive quantitative bounds on the homological complexity of the intersection of real quadrics. We start by proving the following proposition, which essentially refines [6, Corollary 2.3].

Proposition 14. *Let $Y \subset S^{k-1}$ be defined by polynomial equations of degree $\leq d$. Then*

$$b(Y) \leq (2d)^{k-1} + \frac{1}{8}\binom{k+1}{3}(6d)^{k-2}.$$

Proof. By Alexander–Pontryagin duality our problem is equivalent to that of bounding $b(S^{k-1} \setminus Y) = b(Y)$. Let Y be defined on the sphere by the polynomials $f_1, \dots, f_R$ and consider the new polynomial $F = f_1^2 + \cdots + f_R^2$; then clearly Y is also defined by $\{F = 0\}$ on the sphere, and since $F \geq 0$ we have $S^{k-1} \setminus Y = \{F|_{S^{k-1}} > 0\}$; notice that the degree of F is $\delta = 2d$. By semialgebraic triviality, for $\epsilon > 0$ small enough we have the homotopy equivalences

$$S^{k-1} \setminus Y \sim \{F|_{S^{k-1}} > \epsilon\} \sim \{F|_{S^{k-1}} \geq \epsilon\}.$$

Let now $\epsilon > 0$ be a small enough regular value of $F|_{S^{k-1}}$; then $\{F|_{S^{k-1}} \geq \epsilon\}$ is a submanifold of the sphere with smooth boundary $\{F|_{S^{k-1}} = \epsilon\}$, and by Lemma 10 we obtain

$$b(S^{k-1} \setminus Y) = \tfrac{1}{2} b(\{F|_{S^{k-1}} = \epsilon\}).$$

Thus we have reduced the problem to the study of the topology of $\{F|_{S^{k-1}} = \epsilon\}$, a set given in $\mathbb{R}^k$ by the two equations $F - \epsilon = 0$ and $\|\omega\|^2 - 1 = 0$. Equivalently we can consider their homogenization $g_1 = {}^hF - \epsilon\omega_0^{2d} = 0$ and $g_2 = \|\omega\|^2 - \omega_0^2 = 0$ and their common zero locus in $\mathbb{R}\mathrm{P}^k$; since there are no common solutions on $\{\omega_0 = 0\}$ (the hyperplane at infinity), these two equations still define $\{F|_{S^{k-1}} = \epsilon\}$. By [17, Fact 1] we can *real* perturb the coefficients of g_1 and g_2 and make their common zero set in $\mathbb{C}\mathrm{P}^k$ a smooth complete intersection. This perturbation of the coefficients will not change the topology of the zero locus set in $\mathbb{R}\mathrm{P}^k$ since before the perturbation it was a smooth manifold; the fact that the perturbation is *real* allows us to use Smith's theory. Thus let $\tilde{g}_1$ and $\tilde{g}_2$ be the perturbed polynomials. Then

$$b(\{F|_{S^{k-1}} = \epsilon\}) = b(Z_{\mathbb{R}\mathrm{P}^k}(\tilde{g}_1, \tilde{g}_2)) \leq b(Z_{\mathbb{C}\mathrm{P}^k}(\tilde{g}_1, \tilde{g}_2)),$$

where in the last step we have used Smith's inequalities. Eventually we end up with the problem of bounding the homological complexity of the complete intersection C of multidegree $(2, \delta)$ in $\mathbb{C}\mathrm{P}^k$. Let us first compute the Euler characteristic of C. By Hirzebruch's formula this is given by the $(k-2)$th coefficient in the series expansion around zero of the function

$$H(x) = \frac{2\delta(1+x)^{k+1}}{(1+2x)(1+\delta x)}.$$

In other words,

$$\chi(C) = \frac{H^{(k-2)}(0)}{(k-2)!}.$$

To compute this number write $H(x) = F(x)G(x)$ with $F(x) = \frac{2\delta(1+x)^{k+1}}{1+2x}$ and $G(x) = \frac{1}{1+\delta x}$. Then

$$H^{(k-2)}(0) = \sum_{j=0}^{k-2} \binom{k-2}{j} F^{(j)}(0) G^{(k-2-j)}(0).$$

To compute the derivatives of F we do the same trick as for H: we write $F(x) = A(x)B(x)$ where $A(x) = 2\delta(1+x)^{k+1}$ and $B(x) = \frac{1}{1+2x}$. In this way, using the series expansion $B(x) = \sum_{i=0}^{\infty}(-1)^i 2^i x^i$, we get

$$F^{(j)}(0) = j!2(-2)^j \delta \sum_{i=0}^{j} \binom{k+1}{i}\left(-\frac{1}{2}\right)^i.$$

Moreover $G^{(i)}(0) = (-1)^i \delta^i i!$ (from the series expansion $G(x) = \sum_{i=0}^{\infty}(-1)^i\delta^i x^i$ around zero). Plugging these equalities into the above one we get:

$$\chi(C) = (-1)^k \sum_{j=0}^{k-2}\left(\sum_{i=0}^{j}\binom{k+1}{i}\left(-\frac{1}{2}\right)^i\right)2^{j+1}\delta^{k-j-1}.$$

Recall now that the formula $b(C) = (k-1)(1+(-1)^{k+1}) + (-1)^k\chi(C)$ gives

$$\begin{aligned} b(C) &= (k-1)(1+(-1)^{k+1}) + \sum_{j=0}^{k-2}\left(\sum_{i=0}^{j}\binom{k+1}{i}\left(-\frac{1}{2}\right)^i\right)2^{j+1}\delta^{k-j-1} \\ &= (k-1)(1+(-1)^{k+1}) + 2\delta^{k-1} + \sum_{j=1}^{k-2}\left(\sum_{i=0}^{j}\binom{k+1}{i}\left(-\frac{1}{2}\right)^i\right)2^{j+1}\delta^{k-j-1} \end{aligned}$$

Since $|\sum_{i=0}^{j}\binom{k+1}{i}\left(-\frac{1}{2}\right)^i| \leq \binom{k+1}{3}\left(\frac{3}{2}\right)^{k-2}$, from the above equality we can deduce

$$b(C) \leq 2(k-1) + 2\delta^{k-1} + 2\delta^{k-1}\left(\frac{3}{2}\right)^{k-2}\binom{k+1}{3}\sum_{j=1}^{k-2}\left(\frac{2}{\delta}\right)^j.$$

Since now $\delta^{k-1}\sum_{j=1}^{k-2}(\frac{2}{\delta})^j = 2(\delta^{k-2} + 2\delta^{k-3} + \cdots + 2^{k-3}\delta + 2^{k-2})$ and $2^{k-2} \geq k-1$, we can bound $2(k-1) + 2\delta^{k-1}\sum_{j=1}^{k-2}(2/\delta)^j$ by $2^k\delta^{k-2}$ and finally write

$$b(C) \leq 2\delta^{k-1} + \frac{1}{4}\binom{k+1}{3}(3\delta)^{k-2}. \tag{18}$$

This inequality, together with $b(Y) \leq \frac{1}{2}b(C)$ and $\delta = 2d$, gives the result. □

Remark 3. We notice that as long as d is large enough with respect to k, the previous bound improves Milnor's, which gives $b(Y) \leq d(2d-1)^{k-1}$; here it is essential that Y is on the sphere, as we have used Alexander–Pontryagin duality.

As a corollary we get the following theorem.

Theorem 15. *Let X be the intersection of k quadrics in $\mathbb{R}\mathrm{P}^n$. Then*

$$b(X) \leq O(n)^{k-1}.$$

Proof. We use the bound given in Theorem 12; the proof is essentially collecting the estimates given by the previous proposition for each summand $b(\Sigma_\epsilon^{(r)})$. By construction, $\Sigma_\epsilon^{(r)}$ is a determinantal variety and it is defined by polynomials $f_1, \ldots, f_r$ of degree $d = n - r + 2$ on the sphere S^{k-1}. The result now follows by plugging the bounds given in Proposition 14 in the summands of Theorem 12 (there are only σ_k such summands). □

Remark 4. As suggested to the author by S. Basu, there are two other possible ways to get such numerical estimates. The first one is using a general position argument similar to [6] and combinatorial Mayer–Vietoris bounds as in [9]; the second one is using again a general position argument and stratified Morse theory (which in the semialgebraic case is very well controlled, as noticed in [5]). Both these approaches work also in the *affine* case producing a bound of the same shape; in the projective case it seems that also the leading *coefficient* is the same. As for numerical uniform bounds, the advantage of the first one is that it is applicable to more general cases, i.e. besides the quadratic one. To the author's knowledge nothing has been published on the subject; together with S. Basu the author plans to give an account of these different techniques in a forthcoming paper.

We introduce the following notation:

$$B(k, n) = \max\{b(X) \mid X \text{ is the intersection of } k \text{ quadrics in } \mathbb{R}\mathrm{P}^n\}.$$

We now discuss the sharpness of the previous bound, showing that[9]

$$B(k, n) = \Theta(n)^{k-1}.$$

Theorem 15 gives the inequality $B(k, n) \leq O(n)^{k-1}$; for the opposite inequality we need to produce for every k and n an intersection $M_\mathbb{R}$ of k quadrics in $\mathbb{R}\mathrm{P}^n$ with $b(M_\mathbb{R}) \geq Cn^{k-1}$, $C > 0$. Let us first notice that repeating the argument of Proposition 14, we can deduce that a complete intersection M of k quadrics in $\mathbb{C}\mathrm{P}^n$ has

$$b(M) = b(M; \mathbb{Z}) = \Theta(n)^{k-1}$$

(this computation is already performed in [6]). It is known that there exists a real maximal M, i.e. a complete intersection of k *real* quadrics in $\mathbb{C}\mathrm{P}^n$ whose real part $M_\mathbb{R}$ satisfies

$$b(M_\mathbb{R}) = b(M).$$

Such an existence result holds in general for any complete intersection of multidegree $(d_1, \ldots, d_k)$. An asymptotic construction is provided in [14]; the proof for the general case has not been published yet but the author has been informed that it will be the subject of a forthcoming paper of the authors of [14].

9 Here $f(n) = \Theta(g(n))$ means that there exist constants a, b such that $ag(n) \leq f(n) \leq bg(n)$ for all $n \in \mathbb{N}$.

8. Examples

Example 3 ($k = 2$). If X is the intersection of *two* quadrics in $\mathbb{R}\mathrm{P}^n$, the previous ideas produce the sharp bound $b(X) \leq 2n$: in fact by inequality (13) we have

$$b(X) \leq n + 1 - 2(\mu - \nu) + \frac{1}{2}b(\Sigma_\epsilon)$$

(every $\Sigma_\epsilon^{(r)}$ with $r > 1$ is empty). On the other hand, Σ_ϵ is defined by an equation of degree $n + 1$ on the circle S^1 and thus it consists of at most $2(n + 1)$ points. This gives

$$b(X) \leq n + 1 - 2(\mu - \nu) + n + 1 \leq 2n$$

(in the case $\mu = \nu$ we have $b(X) \leq n+1$). Moreover for every n there exist two quadrics in $\mathbb{R}\mathrm{P}^n$ whose intersection X satisfies $b(X) = 2n$ (see [16, Example 2]). Notice that the example provided there in the case of n *odd* gives a *singular* X. Using the notation introduced above, this reads

$$B(2, n) = 2n.$$

What is interesting now is that for *odd* n the number $B(2, n)$ is attained only by a singular intersection of quadrics: the nonsingular one has at most $b(X) \leq 2n - 2$ (this follows from Smith's inequality and Hirzebruch's formula for the complete intersection of two quadrics in $\mathbb{C}\mathrm{P}^n$). For a more detailed discussion the reader is referred to [16, Section 6].

Example 4 ($k = 3$). If X is the intersection of three quadrics, then inequality (13) gives

$$b(X) \leq n + 1 - 2(\mu - \nu) + \frac{1}{2}b(\Sigma_\epsilon),$$

Again, since the codimension of $\Sigma_\epsilon^{(r)}$ is greater than 3 for $r \geq 2$, in this case all these sets except Σ_ϵ are empty (since $k = 3$ these sets are subsets of the sphere S^2). This also says that Σ_ϵ is a smooth curve on S^2; let $f = \|\omega\|^2 - 1$ and $g = \det(\omega q - \epsilon p)$ be the polynomials defining this curve and F, G their homogenizations. Then there exists a *real* perturbation $\tilde{G}$ of G that makes the common zero locus C of $\tilde{G}$ and F a smooth complete intersection in $\mathbb{C}\mathrm{P}^3$. Since Σ_ϵ is nondegenerate, the real part $C_\mathbb{R}$ of this complete intersection has the same topology of C, and by Theorem 12,

$$b(X) \leq \frac{1}{2}b(C) + O(n).$$

Recall that equation (18) was for a complete intersection of multidegree $(2, \delta)$ in $\mathbb{C}\mathrm{P}^3$; specifying it to this case $\delta = n + 1$ we get $b(C) \leq 2n^2 + O(n)$, which when plugged into the previous inequality gives

$$b(X) \leq n^2 + O(n).$$

(Indeed Theorem 1 of [17] gives the refined bound $b(X) \leq n^2 + n$.) We notice now that in this case

$$B(3, n) = n^2 + O(n).$$

In fact the previous inequality provides the upper bound, and the lower bound is given by the existence of almost maximal real complete intersections of three quadrics (see [15] for an explicit construction of such maximal varieties).

For X smooth, using the spectral sequence approach the authors of [11] have proved that the maximum value $B_0(3,n)$ of $b_0(X)$ satisfies

$$\tfrac{1}{4}(n-1)(n+5)-2 < B_0(3,n) \le \tfrac{3}{2}l(l-1)+2$$

where $l=\lfloor n/2\rfloor+1$. Notice in particular that $1/4 \le \liminf B_0(n,3)/n^2 \le 3/8$ as n goes to infinity.

Example 5 ($k=4$). This is the first case where we need to take into account the complexity of the singular points of Σ_ϵ. As promised, we give a simplified proof of part (ii) of Theorem 11 for this case, aiming to acquaint the reader with the idea of that proof. Let $p \in \mathcal{Q}_{n+1}$ be the positive definite form given by Lemma 6. Then by Lemma 5 and Agrachev's bound we get

$$b(X) \le \sum_{\nu \le j \le \mu-1} b(\Omega_j(\epsilon)) + O(n)$$

where the $\Omega_j(\epsilon)$ are now subsets of S^3. Corollary 9 says that each of them is a manifold with boundary; let us rename the boundaries as $C_j=\partial\Omega_{\nu+j-1}(\epsilon)$ for $j=1,\dots,l=\mu-\nu$. Lemma 10 allows us now to use $\frac{1}{2}b(C_j)$ instead of $b(\Omega_{\nu+j-1}(\epsilon))$ in the previous bound to get

$$b(X) \le \frac{1}{2}\sum_{j=1}^{l} b(C_j) + O(n).$$

Now Σ_ϵ is a surface on S^3 given by $C_1\cup\cdots\cup C_l$, but this union is not disjoint since singular points may occur. They are isolated, since their union (if nonempty) has codimension 3 on the sphere S^3. The set $\Sigma_\epsilon^{(2)}=\mathrm{Sing}(\Sigma_\epsilon)$ equals exactly the set of points where two different C_j intersect. On the other hand, if $|i-j|\ge 2$ then $C_j\cap C_i=\emptyset$, since any point on this intersection would have kernel at least of dimension three. Thus Σ_ϵ is made by taking the abstract disjoint union of the sets $C_1,\dots,C_l$ and identifying the points on $C_j\cap C_{j+1}$ for $j=1,\dots,l-1$. This identification procedure can increase the number of generators of the fundamental group; the number of connected components instead can decrease at most by $b(\mathrm{Sing}(\Sigma_\epsilon))$, that is,

$$b(\Sigma_\epsilon) \ge b\Big(\coprod_{j=1}^{l} C_j\Big) - b(\mathrm{Sing}(\Sigma_\epsilon)).$$

Plugging this into the above inequality for $b(X)$ we get

$$b(X) \le \tfrac{1}{2}\{b(\Sigma_\epsilon)+b(\mathrm{Sing}(\Sigma_\epsilon))\}+O(n).$$

Proposition 14 says that both Σ_ϵ and $\mathrm{Sing}(\Sigma_\epsilon)$ have complexity bounded by $16n^3$ and thus

$$b(X) \le 16n^3 + O(n). \tag{19}$$

On the other hand, we notice that if X is the real part of a smooth complete intersection of four real quadrics in $\mathbb{C}\mathrm{P}^n$, then $b(X)\le \frac{2}{3}n^3+O(n)$; thus the above bound is sharp only in its shape. By the above discussion on the topology of determinantal varieties, we expect that (19) can be improved.

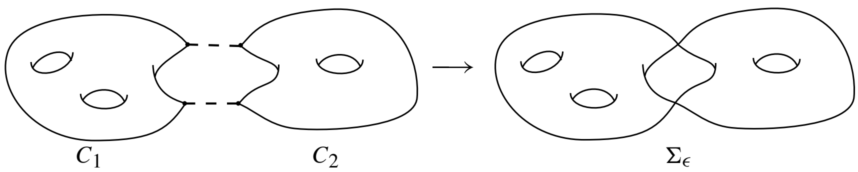

Fig. 3. An example of the way the identification procedure works: Σ_ϵ has two singular points and is obtained by glueing the disjoint union of C_1 and C_2 along two copies of these singular points (one copy is on C_1 and one on C_2).

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