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The logarithmic delay of KPP fronts in a periodic medium

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Abstract. We extend, to parabolic equations of the KPP type in periodic media, a result of Bramson which asserts that, in the case of a spatially homogeneous reaction rate, the time lag between the position of an initially compactly supported solution and that of a traveling wave grows logarithmically in time.

Keywords. Reaction-diffusion equations, periodic media, pulsating traveling fronts, Cauchy problem, asymptotic behavior, logarithmic shift

1. Introduction

1.1. Model and question

We study solutions $u(t, x)$ of the initial value problem

$$u_t = u_{xx} + f(x, u) \quad (t > 0,\ x \in \mathbb{R}), \qquad u(0, x) = u_0(x). \tag{1.1}$$

The initial datum u_0 is nonnegative, nonzero and compactly supported. The function f is of class $C^1[0, 1]$, 1-periodic in x, concave in u, and satisfies $f(x, 0) = 0$. We also assume that:

1. The first periodic eigenvalue of $-\partial_{xx} - g(x)$ is negative, where $g(x) = \partial_u f(x, 0)$.
2. We have $f(x, u) = g(x)u - q(x, u)$, with $q(x, u) \geq mu^2$ for large u.

Thus, $f(x, u) < 0$ as soon as u is larger than some $s_0 > 0$. We will sometimes say that f satisfies the *KPP assumptions*, in reference to the seminal paper of Kolmogorov,

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Petrovskiĭ and Piskunov [23]. Also note that they do not preclude g to be negative in some regions; this is important for models in ecology, where the nonlinearity $f(x,u) = g(x)u - u^2$ is of special interest.

Under those assumptions, there is (see [6]) a unique positive solution $\pi(x)$ to $-\pi'' = f(x,\pi)$ on $\mathbb{R}$, which is in addition 1-periodic. This function $\pi(x)$ attracts, locally uniformly, the solutions of (1.1). Thus, there is a moving transition between the values of $u(t,\cdot)$ that are close to $\pi(x)$, and those close to 0. We are going to study how this transition moves to the right, and to this end let us define

$$X(t) = \max\Big\{x \geq 0 : u(t,x) = \frac{1}{2}\inf_{\mathbb{R}} \pi\Big\}. \tag{1.2}$$

Then (Freidlin–Gärtner [17], Freidlin [15], Weinberger [32], Berestycki–Hamel–Nadin [5]) the function $X(t)/t$ tends, as $t \to \infty$, to a constant c^* which is the smallest speed of a pulsating front solution to (1.1) (we will come back to this definition later, in much more detail). The question we ask is the following: what can we say about $X(t) - c^*t$?

1.2. The case of a homogeneous medium

Here we just mean that f does not depend on x: $f(x,u) = f(u)$. Assumptions 1 and 2 translate into $f'(0) > 0$, $\pi(x) \equiv 1$ and $f'(1) < 0$. Then, given any $c \geq c^* = 2$, there exists a traveling wave solution $u(t,x) = U_c(x - ct)$ of (1.1), which satisfies $-cU_c' = U_c'' + f(U_c)$, $U_c(-\infty) = 1$, $U_c(\infty) = 0$ and $U_c > 0$. For $c > c^*$ the function $U_c(x)$ decays exponentially as $x \to \infty$: $U_c(x) \sim Ce^{-\lambda_c x}$, with the decay rate λ_c being the smallest positive solution of $\lambda^2 - c\lambda + 1 = 0$.

On the other hand, at $c = c^*$ the traveling wave asymptotics is $U_{c^*}(x) \sim Cxe^{-\lambda^* x}$, with $\lambda^* = 1$. It has been shown in the pioneering work of Bramson [8, 9] that

$$X(t) = c^*t - \frac{3}{2\lambda^*}\log t + O(1) \quad \text{as } t \to \infty.$$

There is even a little more: the region in $\mathbb{R}_+$ where $u(t,x)$ transitions from the value $u \approx 1$ to $u \approx 0$ has a width that is uniformly bounded in time, and is located at the distance $(3/(2\lambda^*))\log t$ behind the location of the traveling wave with minimal speed c^*. Bramson's proofs were based on probabilistic techniques, and were later extended by Gärtner to higher dimensions [16], and recently revisited by Roberts [29], while a PDE proof of this result was later given in [31]. It was extended in [24], with the additional assumption $f'(s) \leq f'(0)$ on $[0,1]$, to initial data that decay faster than the wave with minimal speed. These results were recently revisited in [20], which is actually a companion paper to the present one.

1.3. Main results

The goal of this paper is to understand whether we can generalize Bramson's results to the periodic case (1.1). While there is, as we just saw, a rather large literature concerning the homogeneous case, nothing of that sort exists in the case of coefficients that are not space-homogeneous. We are going to prove that there is still a logarithmic lag in the periodic case, and we will identify it precisely. As a by-product of our analysis, we will obtain the convergence of the solution to a family of traveling pulsating waves, in the correct reference frame.

So, let us recall the notion of a pulsating traveling wave that generalizes the notion of a traveling wave to periodic media. A *pulsating front* with speed $c > 0$ is a function $U_c(t, x)$ satisfying

$$U_t = U_{xx} + f(x, U), \qquad t, x \in \mathbb{R}, \tag{1.3}$$

and $U(t + 1/c, x) = U(t, x - 1)$, as well as the boundary conditions $U(t, -\infty) = 1$, $U(t, \infty) = 0$. Let us now recall some of the results about spreading speeds and pulsating traveling waves $U_c(t, x)$ [2, 4, 19, 21, 32, 33] under the given assumptions on $f(x, u)$. It is known that there is a minimal speed $c^* > 0$ such that for each $c \geq c^*$, there exists a unique (up to time shifts) pulsating traveling front $U_c(t, x)$, while no pulsating traveling front exists with a speed less than c^*. Furthermore, all pulsating traveling fronts are necessarily increasing in t. Lastly, the minimal speed c^* may be characterized as follows. Given $\lambda > 0$, let $\psi = \psi(x, \lambda) > 0$ be the principal eigenfunction of the 1-periodic eigenvalue problem

$$\begin{aligned} &\psi_{xx} - 2\lambda\psi_x + (\lambda^2 + g(x)f'(0))\psi = \gamma(\lambda)\psi, \\ &\psi(x + 1, \lambda) = \psi(x, \lambda), \qquad \psi(x, \lambda) > 0, \qquad x \in \mathbb{R}, \end{aligned} \tag{1.4}$$

and $\gamma(\lambda)$ the corresponding eigenvalue. The eigenfunction is normalized so that $\int_0^1 \psi(x, \lambda)\, dx = 1$ for all $\lambda > 0$. The minimal wave speed is given by $c^* = \min_{\lambda>0} \gamma(\lambda)/\lambda = c(\lambda^*)$. Here $\lambda^* > 0$ minimizes $\gamma(\lambda)/\lambda$. In particular, we have $\gamma'(\lambda^*) = \gamma(\lambda^*)/\lambda^* = c^*$. Our first main result is as follows.

Theorem 1.1. *Let $u(t, x)$ solve (1.1) with a nonnegative, nonzero, compactly supported initial datum $u_0(x)$. Then for any $\varepsilon > 0$ there exist $s(\varepsilon)$ and $L(\varepsilon)$ such that*

$$u(t, x) \geq \pi(x) - \varepsilon \quad \textit{for all } t > s(\varepsilon) \textit{ and } x \in \left[0, c^*t - \frac{3}{2\lambda^*}\log t - L(\varepsilon)\right]$$

and

$$u(t, x) < \varepsilon \quad \textit{for all } t > s(\varepsilon) \textit{ and } x \in \left[c^*t - \frac{3}{2\lambda^*}\log t + L(\varepsilon), \infty\right).$$

So, the front is located at distance $(3/(2\lambda^*)) \log t$ behind the pulsating front.

Let us explain informally, in PDE terms, how the logarithmic decay comes about. The main observation is that solutions of the nonlinear problem (1.1) behave very similar to those of the linearized problem $v_t = v_{xx} + g(x)v$, with the Dirichlet boundary condition $v(t, c^*t) = 0$ and any rapidly decaying initial datum. With $g(x) \equiv 1$, $c^* = 2$ and $\lambda^* = 1$, let us write $v(t, x) = p(t, x - 2t)e^{-(x-2t)}$. Then $p(t, x)$ satisfies the standard heat equation $p_t = p_{yy}$, $p(t, 0) = 0$. It follows that $p(t, y = 1) \sim t^{-3/2}$ as $t \to \infty$, or, in the original variables, $v(t, x = 2t + 1) \sim t^{-3/2}$. Assuming that the solution $u(t, x)$ of the nonlinear problem has the same behavior as $v(t, x)$, and has the exponential asymptotics $u(t, x) \sim e^{-(x-X(t))}$, we deduce that $X(t) \sim 2t - (3/2) \log t$. For the homogeneous case $g \equiv 1$, we have worked out this argument in detail in [20], producing quite a short proof of the Bramson shift. This is the idea that we will put to work here, at the unfortunate expense of much heavier technicalities.

In the proof of Theorem 1.1, one shows actually more precise exponential estimates on $u(t,x)$ for $x \geq c^*t - (3/(2\lambda^*))\log t$. These estimates imply that the solution u is asymptotically trapped between two finite space shifts of the minimal front U_{c^*} around the position $x = c^*t - (3/(2\lambda^*))\log t$. Equivalently, u is asymptotically trapped between two finite time shifts of the minimal front U_{c^*} around the time $t-(3/(2c^*\lambda^*))\log t$. Then, by passing to the limit along any level set, any limiting solution is necessarily equal to a shift of the minimal front: this follows from a new Liouville-type result which is similar to what had already been known in the homogeneous case. So, our result is:

Theorem 1.2. *There exist a constant $C \geq 0$ and a function $\xi : (0,\infty) \to \mathbb{R}$ such that $|\xi(t)| \leq C$ for all $t > 0$ and*

$$\lim_{t\to\infty} \left\| u(t,\cdot) - U_{c^*}\left(t - \frac{3}{2c^*\lambda^*}\log t + \xi(t), \cdot\right) \right\|_{L^\infty(0,\infty)} = 0. \tag{1.5}$$

Furthermore, for every $m \in (0, \inf \pi)$ and every sequence (t_n, x_n) such that $t_n \to \infty$ and $x_n - [x_n] \to x_\infty \in [0,1]$ as $n \to \infty$, and $u(t_n, x_n) = m$ for all $n \in \mathbb{N}$, we have

$$u(t+t_n, x+[x_n]) \xrightarrow[n\to\infty]{} U_{c^*}(t+T, x) \quad \text{locally uniformly in } (t,x) \in \mathbb{R}^2, \tag{1.6}$$

where $[x_n]$ denotes the integer part of x_n and $T \in \mathbb{R}$ denotes the unique real number such that $U_{c^*}(T, x_\infty) = m$.

This shows in particular the convergence to the family of minimal fronts along the level sets of u.

1.4. Discussion

To the best of our knowledge, Theorem 1.1 is the first of this type for models with periodic coefficients. As is well-known, most of the information is retrieved through the analysis of the linearized equation $v_t - v_{xx} = g(x)v$. The bulk of the proof is in getting the decay estimates for the heat kernel in a half-space for this equation, with a Dirichlet condition at a boundary moving with speed c^* to the right. Heat kernel estimates for second-order linear parabolic equations in the whole space are well-known, starting from the pioneering work of Nash [26] for operators in divergence form—a different viewpoint being provided by Fabes–Stroock [12]—and extended to general operators by Norris [28]. However, we are not aware of such results in a half-space for periodic coefficients. In fact, although the papers [12] and [28] were crucial to us, we had to introduce a new ingredient. Indeed, the Fabes–Stroock/Norris proofs need the conservation of the total mass—a trivial but indispensable property. Nothing of that sort is available here and, as a matter of fact, it should not be expected. What is true in the homogeneous case is the conservation of $\int_0^\infty xp(t,x)\,dx$ if $p(t,x)$ solves the heat equation on $\mathbb{R}_+$ with Dirichlet boundary conditions. However, we are dealing in the periodic case with an equation with variable coefficients, so trying to compute the integral of xp does not lead very far. One of our contributions in this paper is to have identified a family of multipliers which, integrated against a solution, yield a conserved—or controlled from above and below—quantity.

Turn to Theorem 1.2. This is a result of the type "convergence along level sets", i.e. it identifies a limiting profile for the solutions, in the (a priori unknown) reference frame of $X(t)$. The first—and most famous—one is the KPP theorem [23] for homogeneous equations. For equations with periodic coefficients, results of this type have been obtained recently in Ducrot–Giletti–Matano [11] for more general nonlinearities f and Heaviside initial conditions u_0, and in Giletti [18] for asymptotically periodic KPP functions f and compactly supported initial conditions u_0. See also Ducrot [10] that adapts our ideas in [20] to an equation that becomes asymptotically homogeneous in x. The proofs in [11, 18] are based on the time-decay property of the number of intersections of any two solutions and on the fact that the minimal fronts are the steepest ones. In particular, they do not identify whether or not the level sets of the solutions travel at the same speed as those of the traveling waves or, as opposed to that, if they travel with a time lag.

When $f(u) = u(1-u)$, there is a well-known connection between solutions of (1.1) and branching Brownian motion [8, 25]. Consider a branching Brownian motion with constant branching rate $g > 0$. Initially, there is one Brownian particle, $X_1(0) = 0$. At a random time T_1, which is an independent exponential random variable with rate g, this particle gives birth to two independent Brownian motions and then dies immediately itself. The two new particles start their motions from the final location of the parent particle. The process continues in this way, each living particle reproducing and dying at an independent random time, leaving two new Brownian particles as offspring. As shown by McKean [25], the function $u(t,x) = \mathbb{P}(\max_{k\in L(t)} X_k(t) > x \mid X_1(0) = 0)$ satisfies $u_t = \frac{1}{2}u_{xx} + gu(1-u)$ and $u(0,x)$ is the Heaviside function. The set $L(t)$ is the set of indices corresponding to particles that are alive at time t. The zero Dirichlet boundary condition corresponds to Gärtner's [16] strategy of killing the branching Brownian motion at a moving boundary. If $f(x,u)$ has the form $g(x)u(1-u)$, there is a similar interpretation of the solution u in terms of branching Brownian motion with spatially-variable branching rate $g(x)$. However, our general assumptions on f also include cases where the solution u seems not to have such a simple representation.

In a slightly different vein, let us mention the contribution of Fang–Zeitouni [13], where the medium is taken to be time-dependent, with the diffusion coefficient $\sigma(t)$ slowly and monotonically varying between two different values σ_1 and σ_2. The authors prove, by probabilistic techniques, that the lag behind $X(t)$ and the traveling front position depends strongly on the respective positions of σ_1 and σ_2. In particular, it is shown that it can be of the order $t^{1/3}$. In [27], we identify the lag to be $\sim -kt^{1/3} + O(\log t)$, the constant k being explicitly computed.

We end this section by a discussion of some issues that we are not treating here; they range from easy generalizations to truly difficult questions. The first one concerns equation (1.1) with a spatially periodic diffusion. We have chosen not to treat it, because it would only make the notation heavier. The results would be exactly the same. A more interesting question concerns what happens for equations of the type (1.1) in cylindrical geometries, or even in cylinders with oscillating boundaries. Dirichlet or Neumann conditions should be imposed. More than likely, the results would not change too much, but one might expect nontrivial technical issues in the study of the linearized equation. In the same (we believe) order of difficulty, one may ask about convergence to a single wave, rather than a family of waves. This is true in the homogeneous case. Moreover some of our intermediate results—the first moment conservation being one of them—would point towards this. We are not, however, in a position to be more conclusive. Finally, quite an interesting question is the multi-dimensional case, i.e. what is the shift in every direction if the initial datum is compactly supported? There is at the background a free boundary problem that is less than obvious, and so it is not a mere adaptation of Theorem 1.1. These last three questions are left for future research.

1.5. Organization of the paper

The proof of Theorem 1.1 is long and technical, so we will try to present it in a way that is the most reader-friendly as possible. As said before, the main effort is to be concentrated on the linearized equation $v_t - v_{xx} = g(x)v$; so, in Section 2, we state the main estimates that we would like to prove, and explain how these estimates entail the sought-for time shift: $X(t) - c^*t = (3/(2c^*\lambda^*)) \log t + O(1)$. Section 3 is an important part of the paper. We put the linearized equation in an almost self-adjoint form (as is done in Norris [28]), and we construct multipliers which, integrated against a solution, will produce conserved (or approximately conserved) quantities. In Section 4, we prove the estimate on the linearized equation that entail the lower bound on $X(t)$. In Section 5 we prove the estimates that imply the upper bound. Finally, in Section 6, we prove Theorem 1.2.

2. Computing the time shift from the linearized equation

This section is divided into two subsections, the first one dealing with the lower bound on the front location, the second with the upper bound. Both are organized in the same fashion: in the first paragraph, we state the linear estimates that we will prove later. In the subsequent paragraphs, we explain how these bounds turn into a lower estimate for the front location.

2.1. The lower bound

2.1.1. Estimates on the linearized Dirichlet problem. The proof of the lower bound in Theorem 1.1 is based on the analysis of the linearized equation

$$w_t = w_{xx} + g(x)w \ (x \geq c^*t), \qquad w(0,x) = u_0(x) \ (x \geq 0), \qquad w(t, c^*t) = 0, \qquad t \geq 0. \tag{2.1}$$

It is convenient to represent $w(t,x)$ in the form $w(t,x) = e^{-\lambda^*(x-c^*t)}\psi(x,\lambda^*)p(t,x)$, where $\psi(x,\lambda^*)$ is the normalized eigenfunction of (1.4) with $\lambda = \lambda^*$ satisfying $\gamma'(\lambda^*) = \gamma(\lambda^*)/\lambda^* = c^*$, and $p(t,x)$ satisfies

$$\begin{aligned} &p_t = p_{xx} + \frac{2\phi_x}{\phi} p_x, \qquad x \geq c^*t, \\ &p(t, c^*t) = 0, \qquad t > 0, \\ &p(0,x) = p_0(x) = u_0(x)e^{\lambda^* x}(\psi(x,\lambda^*))^{-1}, \qquad x > 0, \end{aligned} \tag{2.2}$$

with $\phi(t,x) = e^{-\lambda^*(x-c^*t)}\psi(x,\lambda^*)$. The initial datum $p_0(x)$ is nonnegative and compactly supported on $[0,\infty)$. For convenience, we define

$$\kappa(x) = \frac{2\phi_x}{\phi} = -2\lambda^* + 2\frac{\psi_x(x,\lambda^*)}{\psi(x,\lambda^*)}, \tag{2.3}$$

which is the drift term in (2.2). The function $\kappa(x)$ is 1-periodic in x and independent of t.

We will need two ingredients. The first one is an upper estimate on the solution $p(t,x)$ of (2.2).

Lemma 2.1. *There exists a constant $C > 0$ such that*

$$|p(t, x + c^*t)| \le \frac{Cx}{(t+1)^{3/2}} \int_0^\infty y p_0(y)\, dy \quad \textit{for all } t, x > 0. \tag{2.4}$$

For the homogeneous heat equation $p_t = p_{xx}$ on $\mathbb{R}_+$ (g is constant), with Dirichlet conditions, this is quite a classical result which can be seen by inspection of the solution $p(t,x) = (4\pi t)^{1/2} \int_0^t (e^{-(x-y)^2/4t} - e^{-(x+y)^2/4t}) p(0,y)\, dy$. The second ingredient is a lower bound when $x - c^*t$ is of order $\sqrt{t}$.

Proposition 2.2. *There exist constants $T_0, \sigma, C_0 > 0$ such that*

$$p(t, c^*t + \sigma\sqrt{t}) \ge C_0/t \quad \textit{for all } t \ge T_0.$$

Lemma 2.1 will be proved in Section 3, and Proposition 2.2 in Section 4. For the homogeneous heat equation, both are once again quite simple results.

2.1.2. From the linearized problem to a subsolution for the nonlinear problem. Given the lower bound of Proposition 2.2, the next step is to construct a subsolution for (1.1) using the solution of (2.1). If $\bar{w}(t,x) = a(t)w(t,x)$, then $\bar{w}(t,x)$ is a subsolution for (1.1), that is, $\bar{w}_t \le \bar{w}_{xx} + g(x)\bar{w} - q(x,\bar{w})$ with $q(x,\bar{w}) = g(x)\bar{w} - f(x,\bar{w}) = O(\bar{w}^2)$, provided that

$$a'(t)w(t,x) \le -q(x, a(t)w(t,x)). \tag{2.5}$$

So, (2.5) holds provided that

$$a'(t)w(t,x) \le -Ma(t)^2 w(t,x)^2 \tag{2.6}$$

with a large enough constant M. From Lemma 2.1, there exists $C_0 > 0$, depending on u_0, such that

$$w(t,x) \le \frac{C_0}{(t+1)^{3/2}} \quad \text{for all } t \ge 0,\ x \in \mathbb{R} \tag{2.7}$$

(we may define $w(t,x) = 0$ for $x < c^*t$). So, given (2.7), (2.6) holds provided that $a'(t) \le -M(t+1)^{-3/2}a(t)^2$, and we may take

$$a(t) = \frac{a(0)}{1 + 2Ma(0)(1 - (t+1)^{-1/2})}$$

with $a(0) > 0$, which satisfies $a(0)/1 + 2Ma(0) \le a(t) \le a(0)$ for all $t \ge 0$. If $a(0) < 1$, then $\bar{w}(0,x) \le u_0(x)$ for all $x \in \mathbb{R}$. Therefore, the comparison principle implies $u(t,x) \ge \bar{w}(t,x) = a(t)w(t,x) \ge Cw(t,x)$ for all $t \ge 0$ and $x \ge c^*t$. In particular, Proposition 2.2 yields

$$u(t, ct + \sigma\sqrt{t}) \ge Ct^{-1}e^{-\lambda^*\sigma\sqrt{t}} \quad \text{for } t \ge T_0. \tag{2.8}$$

2.1.3. From a lower bound on the far right to the bound at the front. Now we show that (2.8) implies the lower bound in Theorem 1.1. Let $\pi(x)$ be the positive steady solution of (1.1). Let $\varepsilon > 0$. We want to show that there is a constant $L(\varepsilon) \in \mathbb{R}$ such that, for large t,

$$u(t,x) \geq \pi(x) - 2\varepsilon, \qquad \forall x \in \left[0, c^*t - \frac{3}{2\lambda^*}\log t - L(\varepsilon)\right]. \tag{2.9}$$

The idea is to put a certain translate of the pulsating front U_{c^*} below u. However, $u(t,x)$ might be a little below $\pi(x)$ even in the areas where it should be close to π, so we have to slightly deform U_{c^*}. For every $\lambda \geq 1$, consider $f(\lambda, x, u) = g(x)u + \lambda q(x,u)$; we have $f(1,x,u) = f(x,u)$ and $\partial_\lambda f \leq 0$ due to the assumptions on f. The function $f(\lambda, x, \cdot)$ is still concave, and we still have $\partial_u f(\lambda, x, 0) = g(x)$. Let π_λ be the unique bounded positive solution of $-\pi'' = f(\lambda, x, \pi)$; it is still 1-periodic in x. From the linear stability [6] of π_λ with respect to (1.1), and the strong maximum principle, we have $\partial_\lambda \pi < 0$. Thus, if $\varepsilon > 0$ is small, let $\lambda_\varepsilon > 1$ be the largest λ such that $\pi_\lambda(x) \geq \pi(x) - \varepsilon$ for all x. By the Harnack inequality, there is $\delta_\varepsilon > 0$ such that $\pi(x) - \pi_\lambda(x) \geq \delta_\varepsilon$. We fix such a λ_ε, and denote it by λ for convenience. Because $\partial_u f(\lambda, x, 0) = g(x)$, c^* is still the minimal speed for the pulsating traveling front problem, with $f(x,u)$ replaced by $f(\lambda, x, u)$. Let $U^\lambda_{c^*}$ be such a traveling front; it connects monotonically π_λ on the left to 0 on the right.

To show (2.9), we will bound u from below by the function $\tilde{U}(t,x) = U^\lambda_{c^*}(t - r(t), x)$, with $r(t) = o(t)$ to be chosen. Since $\partial_t U^\lambda_{c^*}(t,x) > 0$ we have, provided that $r' \geq 0$,

$$\tilde{U}_t - \tilde{U}_{xx} - f(x, \tilde{U}) \leq \tilde{U}_t - \tilde{U}_{xx} - f(\lambda, x, \tilde{U}) = -r'(t)\partial_t U^\lambda_{c^*} \leq 0.$$

Since the first periodic eigenvalue of $-\partial_{xx} - g(x)$ is negative, it is known from [7] that $u(t,x) \to \pi(x)$ as $t \to \infty$ locally uniformly in $x \in \mathbb{R}$. Therefore, there exists $T_1 > 0$, depending on u_0 and ε, such that $u(t,0) \geq \pi(0) - \delta_\varepsilon/2$ for all $t \geq T_1$. Consequently, $\tilde{U}(t,0) < \pi(0) - \varepsilon \leq u(t,0)$ for all $t \geq T_1$. By taking T_1 larger if necessary, we may assume $T_1 > T_0$ so that (2.8) holds for all $t \geq T_1$. Therefore, the maximum principle and (2.8) imply that the bound

$$\tilde{U}(t,x) \leq u(t,x) \qquad \text{for all } t \geq T_1,\ x \in [0, c^*t + \sigma\sqrt{t}], \tag{2.10}$$

will hold if the following two conditions are satisfied:

$$\tilde{U}(T_1, x) \leq u(T_1, x), \qquad x \in [0, c^*T_1 + \sigma\sqrt{T_1}], \tag{2.11}$$

$$\tilde{U}(t, c^*t + \sigma\sqrt{t}) \leq \frac{C}{t} e^{-\lambda^*\sigma\sqrt{t}}, \qquad t > T_1. \tag{2.12}$$

We now claim that (2.11) and (2.12) hold with $r(t) = \frac{3}{2\lambda^* c^*}\log t + L_0$ if L_0 is sufficiently large. For (2.11), because of the monotonicity of $U^\lambda_{c^*}$, one just has to take L_0 large enough. As for (2.12), recall [19] that $U^\lambda_{c^*}(t,x) \leq C(x - c^*t)e^{-\lambda^*(x - c^*t)} \leq (C/t)e^{-\lambda^*\sigma\sqrt{t}}e^{-\lambda^*c^*L_0}$ for all $t > T_1$. So, if L_0 is sufficiently large, we have (2.12). With this choice of r, the lower bound of Theorem 1.1 follows from (2.10). □

2.2. *The upper bound*

As we have seen, the idea behind the $(3/(2\lambda^*))\log(t)$ delay is that the evolution is driven by the behavior of solutions to the Dirichlet problem (2.1), which is $z_t - z_{xx} - g(x)z = 0$, $x > c^*t$, with $z(t, c^*t) = 0$. The problem is that such solutions that are initially compactly supported will decay in time like $t^{-3/2}$, hence they cannot serve as supersolutions to the nonlinear problem. The correction to this inconvenience is to devise a reference frame in which the Dirichlet problem will have solutions that remain bounded both from above and below by positive constants for finite x, and this is exactly what the $(3/(2\lambda^*))\log t$ shift achieves.

2.2.1. The linearized problem in the logarithmically shifted reference frame. We expect the front to be at $x(t) = c^*t - r\log t$ with $r = 3/(2\lambda^*)$. For the moment, let us assume that the constant r is still general, and we will choose r appropriately later. Accordingly, we consider the Dirichlet problem

$$\begin{cases} z_t - z_{xx} - g(x)z = 0, & t > 0,\ x > c^*t - r\log(t+T) + r\log T, \\ z(t, c^*t - r\log(t+T) + r\log T) = 0, \end{cases}$$

with a given nonnegative continuous compactly supported initial condition $z(0,\cdot) \not\equiv 0$ in $(0,\infty)$.

Define the new time variable τ by $c^*\tau = c^*t - r\log(t+T) + r\log T$, and set $\tilde{z}(\tau, x) = z(t, x)$. Let us also denote $t = h(\tau)$, and choose $T > 0$ sufficiently large so that the function $h(\tau)$ is well-defined and monotonic. Then, set $\tilde{z}(\tau, x) = e^{-\lambda^*(x-c^*\tau)}\psi(x,\lambda^*)\alpha(\tau)\tilde{p}(\tau, x)$ with an increasing function $\alpha(\tau) > 0$ to be determined. Here, as before, $\psi(x,\lambda^*)$ is the eigenfunction of (1.4). The function $\tilde{p}(\tau, x)$ must satisfy

$$\frac{1}{h'(\tau)}\tilde{p}_\tau = \tilde{p}_{xx} + 2\frac{\phi_x}{\phi}\tilde{p}_x + \left(-\frac{1}{h'(\tau)}\frac{\alpha'(\tau)}{\alpha(\tau)} + \lambda^*c^*\left(1 - \frac{1}{h'(\tau)}\right)\right)\tilde{p} = 0, \quad \tau > 0,\ x > c^*\tau, \tag{2.13}$$

where $2\phi_x/\phi$ is as in (2.3). We first compute $h'(\tau)$:

$$\frac{1}{h'(\tau)} = 1 - \frac{r}{c^*(h(\tau)+T)} = 1 - \frac{r}{c^*(\tau+T) + r\log((t+T)/T)} = 1 - \frac{r}{c^*(\tau+T)} + \beta(\tau),$$

with

$$\begin{aligned} \beta(\tau) &= \frac{r}{c^*(\tau+T)} - \frac{r}{c^*(\tau+T) + r\log((t+T)/T)} \\ &= \frac{r^2\log((t+T)/T)}{c^*(\tau+T)\big(c^*(\tau+T) + r\log((t+T)/T)\big)}. \end{aligned} \tag{2.14}$$

Observe that $|\beta(\tau)| \le C\tau^{-3/2}$, and if $r > 0$, then $h'(\tau) > 1$ for all $\tau > 0$.

To eliminate the low-order term in (2.13), we now choose $\alpha(\tau)$ so that

$$\frac{\alpha'(\tau)}{\alpha(\tau)} = c^*\lambda^*(h'(\tau) - 1) = \frac{r\lambda^*}{\tau+T} + O\left(\frac{1}{(\tau+T)^{3/2}}\right),$$

hence

$$\alpha(\tau) = \exp[r\lambda^* \log(\tau + T) + O(\tau^{-1/2})] = (\tau + T)^{r\lambda^*}(1 + O(\tau^{-1/2})). \tag{2.15}$$

The function $\tilde{p}(\tau, x)$ then satisfies

$$\frac{1}{h'(\tau)}\tilde{p}_\tau = \tilde{p}_{xx} + 2\frac{\phi_x}{\phi}\tilde{p}_x, \quad \tau > 0,\ x > c^*\tau, \tag{2.16}$$

with the Dirichlet condition $\tilde{p}(\tau, c^*\tau) = 0$. Observe that if $r = 0$ (taking no logarithmic shift), and $h' \equiv 1$, this is identical to equation (2.2) which is satisfied by $p(t, x)$ that was used in the construction of a subsolution. However, we cannot take $r = 0$ and use $p(t, x)$ for a supersolution since $p(t, x)$ decays as $t^{-3/2}$ as $t \to \infty$, while for a supersolution we need $p(t, x)$ to stay bounded from above and below for finite values of x.

To bound the function $z(t, x) = \tilde{z}(\tau, x) = e^{-\lambda^*(x-c^*\tau)}\psi(x, \lambda^*)\alpha(\tau)\tilde{p}(\tau, x)$, we need an estimate on $\tilde{p}(\tau, x)$ from above and below. The main technical step in the proof of the upper bound in Theorem 1.1 is the following estimate on $\tilde{p}(\tau, x)$, which implies that $\tilde{p}$ has the same leading order behavior as p, even though $h'(\tau) \neq 1$ in (2.16). Let us set $\omega(\tau) = 1 - \frac{1}{h'(\tau)} = \frac{r}{c^*(\tau+T)} - \beta(\tau)$. Observe that $\omega(\tau) \sim r/c^*\tau$ as $\tau \to \infty$, and $|\omega(\tau)| \le C/\tau$, $|\omega'(\tau)| \le C/\tau^2$ for $\tau > \tau_0$. The linear estimate that we shall need is as follows.

Proposition 2.3. *Let $\tilde{p}(\tau, x)$ satisfy*

$$(1 - \omega(\tau))\tilde{p}_\tau = \tilde{p}_{xx} + 2\frac{\phi_x}{\phi}\tilde{p}_x, \quad x \ge c^*\tau, \tag{2.17}$$

with the Dirichlet boundary condition $\tilde{p}(\tau, c^*\tau) = 0$. Then there exist constants $k, K, \tau_0 > 0$ such that

$$\frac{k(x - c^*\tau)}{\tau^{3/2}} \le \tilde{p}(\tau, x) \le \frac{K(x - c^*\tau)}{\tau^{3/2}} \quad \textit{for all } \tau > \tau_0,\ x \in (c^*\tau, c^*\tau + k\sqrt{\tau}).$$

2.2.2. Proof of the upper bound in Theorem 1.1, knowing Proposition 2.3. In terms of the function $\tilde{z}(\tau, x)$, Proposition 2.3 says that

$$\frac{\alpha(\tau)}{\tau^{3/2}}k(x - c^*\tau)e^{-\lambda^*(x-c^*\tau)} \le \tilde{z}(\tau, x) \le \frac{\alpha(\tau)}{\tau^{3/2}}K(x - c^*\tau)e^{-\lambda^*(x-c^*\tau)}$$

for all $\tau > \tau_0$ and $x \in (c^*\tau, c^*\tau + k\sqrt{\tau})$, possibly after changing the positive constants k and K. Expression (2.15) for $\alpha(\tau)$ shows that the choice of $r = 3/(2\lambda^*)$ gives $K_1 \le \alpha(\tau)/\tau^{3/2} \le K_2$ for $\tau \ge \tau_0$, and therefore $k(x - c^*\tau)e^{-\lambda^*(x-c^*\tau)} \le \tilde{z}(\tau, x) \le K(x - c^*\tau)e^{-\lambda^*(x-c^*\tau)}$ for all $\tau > \tau_0$ and $x \in (c^*\tau, c^*\tau + k\sqrt{\tau})$.

Now, we go back to the t variable. Since $c^*\tau = c^*t - r\log(t + T) + r\log T$, we get the lower and upper bounds

$$\begin{aligned} z(t, x) &\ge k(x - c^*t + r\log(t + T) - r\log T)e^{-\lambda^*(x-c^*t+r\log(t+T)-r\log T)}, \\ z(t, x) &\le K(x - c^*t + r\log(t + T) - r\log T)e^{-\lambda^*(x-c^*t+r\log(t+T)-r\log T)}, \end{aligned} \tag{2.18}$$

for all $t \geq h(\tau_0)$, in the interval $c^*t - r\log(t+T) + r\log T \leq x \leq c^*t - r\log(t+T) + r\log T + kt^{1/2}$, possibly after decreasing the positive constant k.

Let $\pi(x)$ be the steady solution of (1.1). It follows from (2.18) that there exist $x_1, x_2 > 0$, both independent of $t \geq h(\tau_0)$, such that if we choose $M \geq \|\pi\|_\infty$ large enough, then (i) $Mz(t, c^*t - r\log(t+T) + r\log T + x_1) \geq 2\|p\|_\infty$ and (ii) $Mz(t, c^*t - r\log(t+T) + r\log T + x) \leq \frac{1}{2}\inf_{\mathbb{R}}\pi$ for all $x > c^*t - r\log(t+T) + r\log T + x_2$. Then we set

$$\bar{u}(t,x) = \begin{cases} \pi(x), & x \leq c^*t - r\log(t+T) + r\log T + x_1, \\ \min(\pi(x), Mz(t,x)), & x \geq c^*t - r\log(t+T) + r\log T + x_1, \end{cases} \tag{2.19}$$

for $t \geq h(\tau_0)$. Note that $\bar{u}(t,x) = Mz(t,x)$ for all $x > c^*t - r\log(t+T) + r\log T + x_2$. Moreover, $u(0,x) \leq \bar{u}(h(\tau_0), x)$ for all $x \in \mathbb{R}$, possibly after increasing the constant M. Therefore, since $\bar{u}(t,x)$ is a supersolution because of the KPP assumption, the maximum principle implies that $u(t,x) \leq \bar{u}(t + h(\tau_0), x)$ for all $t \geq 0$ and $x \in \mathbb{R}$. Therefore, for any $\gamma > 0$, we may choose $\bar{x}$ sufficiently large so that

$$u\left(t, x + c^*t - \frac{3}{2\lambda^*}\log(t)\right) \leq Mz\left(t + h(\tau_0), x + c^*t - \frac{3}{2\lambda^*}\log(t)\right) < \gamma$$

for all $t > 0$ and $x \geq \bar{x}$. □

3. Almost self-adjoint form and special solutions for the linearized equation

The first part of this section is standard, and simply consists in writting (2.2) in a form that is as close as possible to self-adjoint, as is done in [28]. This form is the best suited for studying moments of the solution. In the second part, we generalize to (2.2) the observation that, for the Dirichlet heat equation $p_t = p_{xx}$ on $\mathbb{R}_+$, there is first moment conservation: $\int_0^\infty xp(t,x)\,dx$ is time-constant. We are going to show that integrals of the form $I(t) = \int_{c^*t}^\infty \nu(x)f(t,x)p(t,x)\,dx$, where $f(t,x)$ solves an adjoint equation, are preserved. A more flexible version of this principle will also be presented, and will turn out to be useful in the more technical estimates of the solution of the linear equation. As an application, we will prove the $t^{-3/2}$ upper bound on the solutions $p(t,x)$.

3.1. *The almost self-adjoint form*

We summarize everything in

Lemma 3.1. *Let $\kappa(x) = 2\phi_x/\phi$ be defined by (2.3). There is a unique positive, periodic function $\nu(x)$ with mass 1 over a period, such that for any function $p(x)$,*

$$p_{xx} + \kappa(x)p_x = \frac{1}{\nu(x)}\frac{\partial}{\partial x}(\nu(x)p_x) - \frac{c^*}{\nu(x)}p_x. \tag{3.1}$$

Proof. The identity (3.1) means that

$$\nu'(x) = \kappa(x)\nu(x) - \bar{b} \tag{3.2}$$

with $\bar{b} = -c^*$, and hence $\nu_{xx} - (\kappa(x)\nu)_x = 0$. This equation has a positive periodic solution: indeed, $\tilde{\nu}(x) \equiv 1$ satisfies the adjoint problem $\tilde{\nu}_{xx} + \kappa(x)\tilde{\nu}_x = 0$, and the Krein–Rutman theorem applies.

To find $\bar{b}$, observe that the periodic function $\chi(x) = -\frac{1}{\psi(x,\lambda^*)}\frac{d\psi(x,\lambda^*)}{d\lambda}$ satisfies

$$\chi_{xx} + \kappa(x)\chi_x = -\kappa(x) - c^*. \tag{3.3}$$

Indeed, differentiating (1.4) in λ gives the following equation for $\psi_\lambda = d\psi/d\lambda$:

$$(\psi_\lambda)_{xx} - 2\lambda(\psi_\lambda)_x + \lambda^2\psi_\lambda - 2\psi_x + 2\lambda\psi + g(x)\psi_\lambda = \gamma'(\lambda)\psi + \gamma\psi_\lambda.$$

Then, using the identity $\gamma'(\lambda^*) = \gamma(\lambda^*)/\lambda^* = c^*$, we obtain at $\lambda = \lambda^*$, with $\psi_\lambda^*(x) = \psi_\lambda(x,\lambda^*)$,

$$(\psi_\lambda^*)_{xx} - 2\lambda^*(\psi_\lambda^*)_x + ((\lambda^*)^2 + g(x))\psi_\lambda^* - 2\psi_x^* + 2\lambda\psi^* = c^*\psi + c^*\lambda^*\psi_\lambda^*.$$

Writing now $\psi_\lambda^* = -\chi(x)\psi(x,\lambda^*)$ and using the definition of $\kappa(x)$ gives (3.3). Multiplying (3.3) by $\nu(x)$ and integrating over the period gives $\int_0^1(\kappa(x) + c^*)\nu\,dx = 0$. Therefore, since ν is of mass 1 we have $-c^* = \int_0^1 \kappa(x)\nu(x)\,dx$. It follows from (3.2) that $\bar{b} = \int_0^1 \kappa(x)\nu(x)\,dx = -c^*$. □

The periodic function $\chi(x)$ which satisfies (3.3) will be useful later. For this reason, let us remark that there is a unique periodic function $\chi^0(x)$ which satisfies both $\chi^0_{xx} + 2(\phi_x/\phi)\chi^0_x = -2\phi_x/\phi - c$ in $\mathbb{R}$, and $\int_0^1 \chi^0(x)\,dx = 0$, which is obtained by adding a suitable constant to χ. To end this paragraph, let us write the system satisfied by $p(t,x)$, the form which we shall work with from now on:

$$\begin{aligned} &\nu(x)p_t = (\nu(x)p_x)_x - c^*p_x, \qquad c^*t \le x, \\ &p(t,c^*t) = 0, \qquad t > 0, \\ &p(0,x) = p_0(x) = u_0(x)e^{\lambda^*x}(\psi(x,\lambda^*))^{-1}, \qquad x \ge 0. \end{aligned} \tag{3.4}$$

3.2. *Multipliers and approximate multipliers*

Let us consider the linear boundary value problem

$$\begin{cases} \nu(x)f_t + (\nu(x)f_x)_x + c^*f_x = 0, & t \in \mathbb{R},\ x > c^*t, \\ f(t,c^*t) = 0, & t \in \mathbb{R}, \end{cases} \tag{3.5}$$

and its adjoint form

$$\begin{cases} \nu(x)\zeta_t - (\nu(x)\zeta_x)_x - c^*\zeta_x = 0, & t \in \mathbb{R},\ x > c^*t, \\ \zeta(t,c^*t) = 0, & t \in \mathbb{R}. \end{cases} \tag{3.6}$$

Lemma 3.2. *There are functions $f(t,x)$ and $\zeta(t,x)$ and a constant $m > 0$ such that $f_t, \zeta_t < 0$ and*

$$m(x - c^*t) < f(t,x), \zeta(t,x) < m^{-1}(x - c^*t) \quad \textit{for all } t \in \mathbb{R},\ x > c^*t. \tag{3.7}$$

Proof. We only provide the construction for f. Observe that (3.5) has a solution of the form $Y(t,x) = (x - c^*t) + y(x)$, where $y(x)$ is periodic and satisfies $-c^*\nu(x) + (\nu(x)(1 + y_x))_x + c^*(1 + y_x) = 0$, or

$$(\nu(x)y_x)_x + c^*y_x = c^*(\nu(x) - 1) - \nu'(x). \tag{3.8}$$

Equation (3.8) has a periodic solution because the integral of the right side over the period vanishes, as ν is of mass 1. By subtracting a constant from y, we may assume $Y(t, c^*t) \le 0$. Although $Y(t,x)$ grows linearly in $x - c^*t$ and is a solution of (3.5) for all $t \in \mathbb{R}$ and $x \in \mathbb{R}$, it may not satisfy the desired Dirichlet boundary condition at $x = c^*t$. On the other hand, if $\beta(t)$ is the largest zero of Y then

$$|\beta(t) - c^*t| \le M, \tag{3.9}$$

with a constant M that does not depend on t.

A function $f(t,x)$ having the desired properties may be constructed as the limit of the sequence of functions $f^{(n)}(t,x)$ which satisfy

$$f_t^{(n)} + \frac{1}{\nu(x)}(\nu(x)f_x^{(n)})_x + \frac{c^*}{\nu(x)}f_x^{(n)} = 0, \quad t \le n,\ x > c^*t,$$
$$f^{(n)}(t, c^*t) = 0, \quad t \le n,$$
$$f^{(n)}(n,x) = \max(0, Y(n,x)), \quad x \ge c^*n.$$

It follows from the maximum principle and (3.9) that there exists a constant C, independent of n, such that

$$Y(t,x) - C \le f^{(n)}(t,x) \le Y(t,x) + C, \quad t \le n,\ x \ge c^*t. \tag{3.10}$$

Using (3.10), we can find positive constants L, M, m, independent of n, so that

$$f^{(n)}(t, ct + L) > M_1 \quad \text{for all } t \le n,$$

and, in addition, $m(x - c^*t) < f^{(n)}(t,x) < m^{-1}(x - c^*t)$ for $t < n/2$ and $x > c^*t + L$. Then the strong maximum principle and parabolic regularity imply that $f_x^{(n)}(t, c^*t) > c_0$ for all $t < n/2$, for some positive constant c_0 that does not depend on n or t. By parabolic regularity, we may then extract a subsequence converging to a limit $f(t,x)$ satisfying (3.5), (3.7) and the boundary condition $f(t, c^*t) = 0$ for all $t \in \mathbb{R}$. Note that $f_t^{(n)} \le 0$: this follows from the maximum principle since $f^{(n)}(t,x) \ge 0$ and $f^{(n)}(t,x) \ge Y(t,x)$ for all $t \le n$ and $x \ge c^*t$. It follows that in the limit we also have $f_t(t,x) \le 0$. □

Then, we need a more flexible quantity, which we call $\eta_\alpha(t,x)$, whose role will be to measure how much the solution $p(t,x)$ of (3.4) is concentrated in intervals of the form $[c^*t, c^*t + \sigma\sqrt{t}]$ (the parameter α will, as is often the case, play the role of $t^{-1/2}$).

Lemma 3.3. *There is a constant $C > 0$ such that for each α sufficiently small there is a constant $\mu(\alpha)$ and a function $\eta_\alpha(t, x)$ satisfying*

$$\nu(x)\frac{\partial \eta_\alpha}{\partial t} + \left(\nu(x)\frac{\partial \eta_\alpha}{\partial x}\right)_x + c^*\frac{\partial \eta_\alpha}{\partial x} = \mu(\alpha)\nu(x)\eta_\alpha, \qquad t \in \mathbb{R},\ x \geq c^* t, \tag{3.11}$$

$\eta_\alpha(t, c^* t) = 0$ for $t \in \mathbb{R}$, and

$$C\frac{e^{\alpha x} - e^{-\alpha x}}{\alpha} \leq \eta_\alpha(t, x + c^* t) \leq C^{-1}\frac{e^{\alpha x} - e^{-\alpha x}}{\alpha} \quad \text{for all } t \in \mathbb{R} \text{ and } x \geq 0.$$

In addition, there exists $\mu_0 > 0$ such that

$$\mu(\alpha) = \mu_0 \alpha^2 + O(\alpha^3) \quad \text{for all } \alpha > 0 \text{ sufficiently small.} \tag{3.12}$$

For the homogeneous medium, we have $\nu(x) \equiv 1$, and the function

$$\eta_\alpha(t, x) = \frac{e^{\alpha(x - c^* t)} - e^{-\alpha(x - c^* t)}}{\alpha}$$

satisfies (3.11) with $\mu(\alpha) = \alpha^2$. In the general case, η_α has exponential asymptotics as $x \to \infty$: $\eta_\alpha(t, x) \sim (1/\alpha)e^{\alpha(x - c^* t)}\bar{\eta}_\alpha(x)$as $x \to \infty$, where $\bar{\eta}_\alpha(x)$ is a positive periodic solution of

$$\left(\nu(x)\frac{\partial \bar{\eta}_\alpha}{\partial x}\right)_x + \alpha(\nu(x)\bar{\eta}_\alpha)_x + (c^* + \alpha\nu(x))\frac{\partial \bar{\eta}_\alpha}{\partial x} + c^*\alpha(1 - \nu(x))\bar{\eta}_\alpha = (\mu(\alpha) - \alpha^2)\nu(x)\bar{\eta}_\alpha,$$

and $\mu(\alpha)$ is the corresponding eigenvalue.

Proof of Lemma 3.3. The proof is divided into three steps, each corresponding to an item of the lemma.

1. The eigenvalue asymptotics for $\alpha \ll 1$. Consider the periodic eigenvalue problem

$$\begin{cases} \left(\nu(x)\frac{\partial \eta}{\partial x}\right)_x + \alpha(\nu(x)\eta)_x + (c^* + \alpha\nu(x))\frac{\partial \eta}{\partial x} + c^*\alpha(1 - \nu(x))\eta = \gamma(\alpha)\nu(x)\eta, \\ \eta(x+1) = \eta(x) > 0, \end{cases}$$

with $\gamma(\alpha) = \mu(\alpha) - \alpha^2$ and the normalization $\int_0^1 \nu(x)\eta(x)\,dx = 1$. Observe that $\gamma(0) = 0$ and $\eta(x, \alpha = 0) \equiv 1$. Moreover, as $\gamma(0) = 0$ is a simple eigenvalue, $\gamma(\alpha)$ is an analytic function of α for α sufficiently small. The function $\eta' = \partial\eta/\partial\alpha$ satisfies

$$\begin{aligned} \gamma\nu\eta' + \gamma'\nu\eta = {} & \left(\nu(x)\frac{\partial \eta'}{\partial x}\right)_x + \alpha(\nu(x)\eta')_x + (c^* + \alpha\nu(x))\frac{\partial \eta'}{\partial x} \\ & + c^*\alpha(1 - \nu(x))\eta' + (\nu\eta)_x + \nu\frac{\partial \eta}{\partial x} + c^*(1 - \nu)\eta. \end{aligned} \tag{3.13}$$

Setting $\alpha = 0$ we obtain

$$\gamma'\nu = \left(\nu(x)\frac{\partial \eta'}{\partial x}\right)_x + c^*\frac{\partial \eta'}{\partial x} + \nu_x + c^*(1 - \nu). \tag{3.14}$$

Integrating (3.14), we conclude that $\gamma'(0) = 0$. Next, η'' solves

$$\gamma\nu\eta'' + 2\gamma'\nu\eta' + \gamma''\nu\eta = \left(\nu(x)\frac{\partial\eta''}{\partial x}\right)_x + \alpha(\nu(x)\eta'')_x + (c^* + \alpha\nu(x))\frac{\partial\eta''}{\partial x} + c^*\alpha(1-\nu(x))\eta'' + 2(\nu\eta')_x + 2\nu\frac{\partial\eta'}{\partial x} + 2c^*(1-\nu)\eta'.$$

So, at $\alpha = 0$ we have

$$\left(\nu(x)\frac{\partial\eta''}{\partial x}\right)_x + c^*\frac{\partial\eta''}{\partial x} + 2(\nu\eta')_x + 2\nu\frac{\partial\eta'}{\partial x} + 2c^*(1-\nu)\eta' = \gamma''\nu.$$

Integrating this equation, we obtain

$$\gamma'' = 2\int_0^1 \left(\nu\frac{\partial\eta'}{\partial x} + c^*(1-\nu)\eta'\right)dx. \tag{3.15}$$

Since $\gamma'(0) = 0$, (3.14) implies that

$$c^*(1-\nu) = -\nu_x - c^*\frac{\partial\eta'}{\partial x} - \left(\nu\frac{\partial\eta'}{\partial x}\right)_x.$$

Plugging this into (3.15) yields

$$\gamma''(0) = 4\int_0^1 \nu\frac{\partial\eta'}{\partial x}\,dx + 2\int_0^1 \nu\left(\frac{\partial\eta'}{\partial x}\right)^2 dx.$$

Since $4y + 2y^2 \geq -2$ for all $y \in \mathbb{R}$, we conclude that $\gamma''(0) \geq -2\int_0^1 \nu(x)\,dx = -2$, with equality if and only if $\partial\eta'/\partial x \equiv -1$. Since η' is periodic, $\partial\eta'/\partial x = -1$ cannot hold at all x, so we must have $\gamma''(0) > -2$. Finally, since $\mu(\alpha) = \alpha^2 + \gamma(\alpha)$, we have $\mu''(0) = 2 + \gamma''(0) > 0$, proving (3.12).

Let us now denote the eigenfunction of (3.13) by $\bar{\eta}_\alpha$ to indicate its dependence on α.

2. Construction of the function $\eta_\alpha(t,x)$**.** We first claim that there is a constant C such that for all $\alpha > 0$ sufficiently small, there is $\beta(\alpha) > 0$ with $\mu(-\beta) = \mu(\alpha)$ and such that $|\beta/\alpha - 1| \leq C\alpha$ and $\sup_x |\bar{\eta}_\alpha(x) - 1| \leq C\alpha$, $\sup_x |\bar{\eta}_\beta(x) - 1| \leq C\alpha$. Indeed, the existence of such a β follows from the fact that $\mu(\alpha) \sim C\alpha^2$ for α small. The bounds on $\bar{\eta}_\alpha$ and $\bar{\eta}_\beta$ follow from elliptic regularity and the fact that for $\alpha = 0$, $\bar{\eta}_0(x) \equiv 1$.

So, choose $\beta = \beta(\alpha) > 0$ accordingly, and consider the terminal value problem

$$\nu(x)\frac{\partial\eta_{\alpha,T}}{\partial t} + \left(\nu(x)\frac{\partial\eta_{\alpha,T}}{\partial x}\right)_x + c^*\frac{\partial\eta_{\alpha,T}}{\partial x} = \mu(\alpha)\nu(x)\eta_{\alpha,T}, \quad t < T,\ x \geq c^*t, \tag{3.16}$$

with the terminal condition $\eta_{\alpha,T}(T,x) \geq 0$ to be determined. The function $\eta_\alpha(t,x)$ of Lemma 3.3 will be defined as $\lim_{T\to\infty}\eta_{\alpha,T}(t,x)$. Observe that for any constant C, the function

$$\alpha^{-1}e^{\alpha(x-c^*t)}\bar{\eta}_\alpha(x) - C\beta^{-1}e^{-\beta(x-c^*t)}\bar{\eta}_\beta(x)$$

satisfies (3.16), since $\mu(-\beta) = \mu(\alpha)$. If we choose the constant $C_u = \frac{\beta}{\alpha} \min_x \frac{\bar\eta_\alpha(x)}{\bar\eta_\beta(x)} > 0$, then the function

$$h_u(t,x) = \alpha^{-1} e^{\alpha(x-c^*t)} \bar\eta_\alpha(x) - C_u \beta^{-1} e^{-\beta(x-c^*t)} \bar\eta_\beta(x)$$

satisfies $h_u(t, c^*t) \geq 0$ for all $t \in \mathbb{R}$. Similarly, if we choose $C_l = \frac{\beta}{\alpha} \max_x \frac{\bar\eta_\alpha(x)}{\bar\eta_\beta(x)} > 0$, then the function

$$h_l(t,x) = \alpha^{-1} e^{\alpha(x-c^*t)} \bar\eta_\alpha(x) - C_l \beta^{-1} e^{-\beta(x-c^*t)} \bar\eta_\beta(x) \tag{3.17}$$

satisfies $h_l(t, c^*t) \leq 0$ for all $t \in \mathbb{R}$. Now, if we choose the terminal condition $\eta_{\alpha,T}(T,x) = \max(0, h_\ell(T,x))$, the maximum principle implies that

$$h_l(t,x) \leq \eta_\alpha(t,x) \leq h_u(t,x) \quad \text{for all } t \leq T,\ x \geq c^*t. \tag{3.18}$$

Although the constants C_u and C_l depend on α, we have $C_u = 1 + O(\alpha)$ and $C_l = 1 + O(\alpha)$ as $\alpha \to 0$.

Now, we claim there are constants $L, M > 0$, independent of T, such that

$$M \frac{e^{\alpha x} - e^{-\alpha x}}{\alpha} \leq \eta_{\alpha,T}(t, x + c^*t) \leq M^{-1} \frac{e^{\alpha x} - e^{-\alpha x}}{\alpha} \tag{3.19}$$

for all $x > L$ and $t \leq T$, and all α sufficiently small. Given this claim, parabolic regularity and the maximum principle imply that there is $b > 0$ universal such that $b < \frac{\partial \eta_{\alpha,T}}{\partial x}(t, c^*t) < b^{-1}$ for all $t \leq T - 1$ and $\alpha > 0$ sufficiently small. Since $\frac{d}{dx}\big(\frac{e^{\alpha x} - e^{-\alpha x}}{\alpha}\big)\big|_{x=0} = 2$, it follows by parabolic regularity that (3.19) also holds for all $x \geq 0$ and $t \leq T - 1$, with a constant C independent of T. Then letting $T \to \infty$ we may take a subsequence of functions $\eta_{\alpha,T_k}(x,t)$ such that $T_k \to \infty$ and η_{α,T_k} converges locally uniformly to a function $\eta_\alpha(t,x)$ satisfying all the criteria of Lemma 3.3.

3. The proof of (3.19). Let us derive the upper bound in (3.19). Because of (3.18), it suffices to show that

$$h_u(t, x + c^*t) \leq M^{-1} \frac{e^{\alpha x} - e^{-\alpha x}}{\alpha} \quad \text{for all } t \in \mathbb{R},\ x \geq L, \tag{3.20}$$

with $L > 0$ and M independent of α. Let us write

$$h_u(t, x + c^*t) = \alpha^{-1} \bar\eta_\alpha(x + c^*t) \left(e^{\alpha x} - C_u \frac{\alpha}{\beta} \frac{\bar\eta_\beta(x + c^*t)}{\bar\eta_\alpha(x + c^*t)} e^{-\beta x} \right).$$

Since $\bar\eta_\alpha$ is uniformly bounded in x, independently of $\alpha \in (0,1)$, the upper bound (3.20) holds if

$$e^{\alpha x} - C_u \frac{\alpha}{\beta} \frac{\bar\eta_\beta(x + c^*t)}{\bar\eta_\alpha(x + c^*t)} e^{-\beta x} \leq M_2 (e^{\alpha x} - e^{-\alpha x})$$

for some constant M_2, which is equivalent to

$$e^{-2\alpha x}\left(M_2 - C_u \frac{\alpha}{\beta}\frac{\bar{\eta}_\beta(x+c^*t)}{\bar{\eta}_\alpha(x+c^*t)}e^{-(\beta-\alpha)x}\right) \leq M_2 - 1. \tag{3.21}$$

Since C_u, $\bar{\eta}_\alpha$, $\bar{\eta}_\beta$ are positive, this inequality certainly holds if $e^{-2\alpha x}M_2 \leq M_2 - 1$. So, if we set $M_2 = 2$, then (3.21) holds for all $x \geq \log(2)/(2\alpha)$. Now consider (3.21) for $x \leq \log(2)/(2\alpha)$. By the beginning of step 2, $C_u \frac{\alpha}{\beta}\frac{\bar{\eta}_\beta(x+c^*t)}{\bar{\eta}_\alpha(x+c^*t)} = 1 + O(\alpha)$ as $\alpha \to 0$, uniformly in x and t. Moreover, $\beta - \alpha = O(\alpha^2)$, so that for $x \leq \log(2)/(2\alpha)$, we have $C_u \frac{\alpha}{\beta}\frac{\bar{\eta}_\beta(x+c^*t)}{\bar{\eta}_\alpha(x+c^*t)}e^{-(\beta-\alpha)x} = 1 + O(\alpha)$. Therefore, with $M_2 = 2$ and $x \leq \log(2)/(2\alpha)$, inequality (3.21) becomes

$$e^{-2\alpha x} \leq \frac{M_2 - 1}{M_2 - 1 + O(\alpha)} = 1 - O(\alpha).$$

Hence there is a constant L such that (3.21) holds for all $t \in \mathbb{R}$ and $x \geq L$, and all α sufficiently small. This establishes the upper bounds in (3.20) and (3.19).

In a similar manner, we now prove the lower bound in (3.19). It suffices to show that

$$h_l(t,x) \geq M\frac{e^{\alpha x} - e^{-\alpha x}}{\alpha} \tag{3.22}$$

for all $t \in \mathbb{R}$ and $x \geq L$. Let us write

$$h_l(t, x+c^*t) = \alpha^{-1}\bar{\eta}_\alpha(x+c^*t)\left(e^{\alpha x} - C_l \frac{\alpha}{\beta}\frac{\bar{\eta}_\beta(x+c^*t)}{\bar{\eta}_\alpha(x+c^*t)}e^{-\beta x}\right).$$

Since $\bar{\eta}_\alpha(x)$ is uniformly bounded away from zero, independently of $\alpha \in (0,1)$, the lower bound (3.22) holds if

$$M_3\left(e^{\alpha x} - C_l \frac{\alpha}{\beta}\frac{\bar{\eta}_\beta(x+c^*t)}{\bar{\eta}_\alpha(x+c^*t)}e^{-\beta x}\right) \geq e^{\alpha x} - e^{-\alpha x}$$

for some constant M_3, which is equivalent to

$$M_3 - 1 \geq M_3 C_l \frac{\alpha}{\beta}\frac{\bar{\eta}_\beta(x+c^*t)}{\bar{\eta}_\alpha(x+c^*t)}e^{-(\beta+\alpha)x} - e^{-2\alpha x}. \tag{3.23}$$

This bound certainly holds if

$$M_3 - 1 \geq M_3 C_l \frac{\alpha}{\beta}\frac{\bar{\eta}_\beta(x+c^*t)}{\bar{\eta}_\alpha(x+c^*t)}e^{-(\beta+\alpha)x}.$$

By construction of η_α we know that $C_l \frac{\alpha}{\beta}\frac{\bar{\eta}_\beta(x+c^*t)}{\bar{\eta}_\alpha(x+c^*t)} = 1 + O(\alpha) \leq 2$ uniformly in x and t if α is sufficiently small. So, if we set $M_3 = 2$, then (3.23) holds for all $x \geq \log(2)/\alpha$.

Now consider (3.23) for $x \leq \log(2)/\alpha$. Recall that $\beta + \alpha = 2\alpha + O(\alpha^2)$, so that for $x \leq \log(2)/\alpha$, we have

$$C_l \frac{\alpha}{\beta}\frac{\bar{\eta}_\beta(x+c^*t)}{\bar{\eta}_\alpha(x+c^*t)}e^{-(\beta-\alpha)x} = e^{-2\alpha x}(1 + O(\alpha)).$$

Therefore, with $M_3 = 2$ and $x \le \log(2)/\alpha$, inequality (3.23) becomes

$$M_3 - 1 \ge (M_3(1 + O(\alpha)) - 1)e^{-2\alpha x},$$

which is $e^{-2\alpha x} \le \frac{1}{2(1+O(\alpha))-1} = 1 - O(\alpha)$. Hence there is a constant L such that (3.23) holds for all $t \in \mathbb{R}$ and $x \ge L$, and all α sufficiently small. This proves the lower bounds in (3.22) and (3.19), completing the proof of Lemma 3.3. □

Lemma 3.4. (i) *There is a constant $C > 0$ such that $|\partial_\tau \zeta(\tau, x)| \le C$ for all $x > c^*\tau$.*
(ii) *There is a constant C such that $|\partial_\tau \eta_\alpha(\tau, x)| \le C$ for all $x \in (c^*\tau, c^*\tau + \alpha^{-1})$.*
(iii) *There is a constant C such that $|\partial_\tau \eta_\alpha(\tau, x)| \le C\alpha\eta_\alpha(\tau, x)$ for all $x > c^*\tau$.*

Proof. Part (i) just comes from parabolic regularity. As for (ii), we come back to the notation of Lemma 3.3. Let $T > 0$; at $\tau = T$ we have, just using the equation for η_α,

$$\partial_\tau \eta_\alpha(T, x) = O(e^{\alpha(x-c^*T)} + e^{-\alpha(x-c^*T)}) + d\mu_\alpha(x)$$

where μ_α is a measure carried by the (compact) zero set of the function h_l, which was defined at (3.17), and whose mass is uniformly bounded with respect to α. So, the equation for $\partial_\tau \eta_\alpha$ (recall that it solves the same equation as η_α) yields

$$\partial_\tau \eta_\alpha(T - 1, x) = O(e^{\alpha(x-c^*T)} + e^{-\alpha(x-c^*T)}) + O(1) = O(e^{\alpha(x-c^*T)}).$$

Running the equation for $\tau \le T - 1$ yields

$$|\partial_\tau \eta_\alpha(\tau, x)| \le Ce^{\alpha(x-c^*\tau)}\bar{\eta}_\alpha(x),$$

and so $\partial_\tau \eta_\alpha(\tau, x) = O(e^{\alpha(x-c^*\tau)})$, which is sufficient to prove the claim. □

3.3. Application: the $t^{-3/2}$ bound

Proof of Lemma 2.1. We are working with the almost self-adjoint form of (2.2), which is (3.4). We use a duality argument, and the main step is to derive the L^2 bound

$$\left(\int_{c^*t}^{\infty} p(t, x)^2\, dx\right)^{1/2} \le \frac{C}{t^{3/4}} \int_0^{\infty} x p_0(x)\, dx, \qquad \forall t > 0. \tag{3.24}$$

It follows from (3.4) that

$$\frac{1}{2}\frac{d}{dt}\int_{c^*t}^{\infty} \nu(x) p(t, x)^2\, dx = -\int_{c^*t}^{\infty} \nu(x) p_x(t, x)^2\, dx. \tag{3.25}$$

The right side of (3.25) may be bounded from above by using a Nash-type inequality: there is a constant C such that

$$\int_0^{\infty} |\beta(x)|^2\, dx \le C\left(\int_0^{\infty} \beta_x^2\, dx\right)^{3/5}\left(\int_0^{\infty} x\beta(x)\, dx\right)^{4/5} \tag{3.26}$$

for all $\beta \in L^1([0,\infty)) \cap H^1([0,\infty))$ satisfying $\beta(0) = 0$ and $\beta(x) \geq 0$ for $x \geq 0$. This inequality can be verified in the usual manner: if $\xi(x)$ is an odd extension of $\beta(x)$ to all of $\mathbb{R}$, then

$$\int_{-\infty}^{\infty} |\xi(x)|^2\, dx = C \int_{-\infty}^{\infty} |\hat{\xi}(k)|^2\, dk, \tag{3.27}$$

where $\hat{\xi}(k)$ is the Fourier transform of $\xi(x)$. Note that $\hat{\xi}(0) = 0$ and $\left|\frac{d}{dk}\hat{\xi}(k)\right| \leq C\int_0^\infty x\beta(x)\, dx$, whence $|\hat{\xi}(k)| \leq C|k|\, \|x\beta\|_1$. It follows from (3.27) that for any $R > 0$,

$$\begin{aligned}\int_{-\infty}^{\infty} |\xi(x)|^2\, dx &\leq C \int_{|k|\leq R} |\hat{\xi}(k)|^2\, dk + C \int_{|k|\geq R} \frac{|k|^2}{R^2} |\hat{\xi}(k)|^2\, dk \\ &\leq CR^3 \|x\beta\|_1^2 + \frac{C}{R^2}\|\beta_x\|_2^2.\end{aligned}$$

Choosing $R = (\|\beta_x\|_2^2 / \|x\beta\|_1^2)^{1/5}$ gives (3.26).

Going back to (3.25), since $\nu(x)^{-1} > 0$ is bounded, we conclude that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_{c^*t}^{\infty} \nu(x) p(t,x)^2\, dx \\ &\qquad \leq -C\left(\int_{c^*t}^{\infty} p(t,x)^2\, dx\right)^{5/3} \left(\int_{c^*t}^{\infty} (x - c^*t) p(t,x)\, dx\right)^{-4/3}.\end{aligned} \tag{3.28}$$

Next, we work toward an estimate of the right side of (3.28). Let us multiply (3.4) by a function $\nu(x) f(t,x)$ with $f(t, c^*t) = 0$ and integrate:

$$\begin{aligned}\frac{d}{dt}\int_{c^*t}^{\infty} \nu(x) f(t,x) p(t,x)\, dx &= \int_{c^*t}^{\infty} \nu(x) f_t(t,x) p(t,x)\, dx \\ &\quad - \int_{c^*t}^{\infty} \nu(x) f_x(t,x) p_x(t,x)\, dx - c^* \int_{c^*t}^{\infty} f p_x\, dx.\end{aligned}$$

We choose f to be a solution of the backward equation, as in Lemma 3.2. Recall that the integral $I(t) = \int_{c^*t}^{\infty} \nu(x) f(t,x) p(t,x)\, dx$ is preserved: $I(t) = I(0)$ for all $t \geq 0$. Moreover, (3.7) implies that

$$\left(\int_{c^*t}^{\infty} (x - c^*t) p(t,x)\, dx\right)^{-4/5} \geq C\left(\int_{c^*t}^{\infty} \nu(x) f(t,x) p(t,x)\, dx\right)^{-4/5} = CI(0)^{-4/5}$$

for all $t > 0$. So, if $I_2(t) = \int_{c^*t}^{\infty} \nu(x) p(t,x)^2\, dx$, we conclude from (3.28) that

$$\frac{dI_2(t)}{dt} \leq -C\frac{I_2(t)^{5/3}}{I(0)^{4/3}}.$$

It follows that $I_2(t)^{-2/3} \geq CtI(0)^{-4/3}$ for all $t > 0$, which implies the L^2 bound (3.24).

The standard duality argument can now be applied. If S_t is the solution operator mapping $p_0(\cdot)$ to $p(t,\cdot)$, then the adjoint operator S_t^* is of the same form as S_t except that

c^* is replaced by $-c^*$ and the direction of time is changed. Hence, the $L^1 \to L^2$ bound (3.24) for S_t also implies the dual $L^2 \to L^\infty$ bound:

$$|p(t,x)| \le \frac{C(x-c^*t)}{t^{3/4}}\|p_0\|_{L^2}, \qquad t>0,\ x>c^*t.$$

Finally, writing $S_t = S_{t/2} \circ S_{t/2}$ we obtain the conclusion of Lemma 2.1. □

4. Estimate from below for the linearized equation

Recall that we are dealing with the almost self-adjoint form (3.4), and that we wish to prove Proposition 2.2, namely: $p(t,x)$ is larger than $O(t^{-1})$ if $x - c^*t$ is of order $\sqrt{t}$. We will take three steps: in the first one, we will show that the estimate is true for a lot of points in the range $x - c^*t \sim \sqrt{t}$; this will be an integral estimate. This is not good enough to propagate the estimate inside, and so another step will be to prove a Harnack-type inequality (Section 5.2), which will retrieve all the points of the real line. The last item is proved in Section 5.3.

4.1. Proposition 2.2 is true in the integral sense

Proposition 4.1. *There exist a time $T_0>0$ and constants $c_0, \beta, N>0$, depending only on the initial data, such that for any $t>T_0$ there exists a set $I_t \subset [c^*t+\sqrt{t}/N, c^*t+N\sqrt{t}]$ with $|I_t| \ge \beta\sqrt{t}$ and*

$$p(t,x) \ge c_0/t \quad \textit{for all } x \in I_t. \tag{4.1}$$

Proof. We define the second exponential moment by

$$V_\alpha(t) = \int_{c^*t}^\infty \nu(x)\eta_{2\alpha}(t,x)p(t,x)q(t,x)\,dx = \int_{c^*t}^\infty \nu(x)\eta_{2\alpha}(t,x)\zeta(t,x)q(t,x)^2\,dx.$$

Then

$$\begin{aligned}\frac{dV_\alpha(t)}{dt} &= \int_{c^*t}^\infty \nu(\partial_t\eta_{2\alpha})pq\,dx + \int_{c^*t}^\infty \nu\eta_{2\alpha}p_tq\,dx + \int_{c^*t}^\infty \nu\eta_{2\alpha}pq_t\,dx\\ &= \mu(2\alpha)V_\alpha(t) - \int_{c^*t}^\infty \nu(\mathcal{L}^*\eta_{2\alpha})pq\,dx + \int_{c^*t}^\infty \nu\eta_{2\alpha}p_tq\,dx + \int_{c^*t}^\infty \nu\eta_{2\alpha}pq_t\,dx,\end{aligned}$$

where $\mathcal{L}^*\eta = \nu^{-1}(\nu\eta_x)_x + \nu^{-1}c^*\eta_x$. Since $p_t = \mathcal{L}p$ and $q_t = \mathcal{L}q + 2\frac{\zeta_x}{\zeta}q_x$ we have

$$V_\alpha'(t) = \mu(2\alpha)V_\alpha(t) - 2\int_{c^*t}^\infty \nu\eta_{2\alpha}p_xq_x\,dx + 2\int_{c^*t}^\infty \nu\eta_{2\alpha}p\frac{\zeta_x}{\zeta}q_x\,dx. \tag{4.2}$$

As $p = \zeta q$, we have $p_x = \zeta_x q + \zeta q_x$ and so $p\frac{\zeta_x}{\zeta}q_x = q\zeta_xq_x = p_xq_x - \zeta q_x^2$. Therefore, the last two terms in (4.2) reduce to

$$V_\alpha'(t) = \mu(2\alpha)V_\alpha(t) - 2\int_{c^*t}^\infty \nu\eta_{2\alpha}\zeta q_x^2\,dx = \mu(2\alpha)V_\alpha(t) - 2D_\alpha(t), \tag{4.3}$$

where $D_\alpha(t) = \int_{c^*t}^\infty \nu\eta_{2\alpha}\zeta q_x^2\,dx$.

The quantity $V_\alpha(t)$ is the one we need to estimate; we do this by bounding the right side of (4.3). We claim that there is a constant $C > 0$ such that $D_\alpha(t) \geq C V_\alpha(t)^{5/3}/I_\alpha(t)^{4/3}$ for all $t > 1$ and $\alpha > 0$ sufficiently small. Since $\nu > 0$ is periodic, this is equivalent to the statement that for any $\alpha > 0$,

$$\left(\int_{c^*t}^{\infty} \eta_{2\alpha}\zeta q^2\, dx\right)^{5/3} \leq C\left(\int_{c^*t}^{\infty} \eta_\alpha \zeta q\, dx\right)^{4/3}\left(\int_{c^*t}^{\infty} \eta_{2\alpha}\zeta q_x^2\, dx\right). \tag{4.4}$$

By Lemma 3.3, $\zeta(t,x)$ is comparable with the linear function $x - c^*t$, and $\eta_\alpha(t,x)$ with $(e^{\alpha x} - e^{-\alpha x})/\alpha$. That is, for $\alpha > 0$ sufficiently small,

$$\int_{c^*t}^{\infty} \eta_{2\alpha}\zeta q^2\, dx \leq C_1 \int_0^\infty \frac{e^{2\alpha x} - e^{-2\alpha x}}{2\alpha} x q(t, x + c^*t)^2\, dx =: C_1 \hat{V}_\alpha,$$

$$\int_{c^*t}^{\infty} \eta_\alpha \zeta q\, dx \geq C_2 \int_0^\infty \frac{e^{\alpha x} - e^{-\alpha x}}{\alpha} x q(t, x + c^*t)\, dx =: C_2 \hat{I}_\alpha,$$

and

$$\int_{c^*t}^{\infty} \eta_{2\alpha}\zeta q_x^2\, dx \geq C_3 \int_0^\infty \frac{e^{2\alpha x} - e^{-2\alpha x}}{2\alpha} x\big(q_x(t, x + c^*t)^2 - \alpha^2 q(t, x + c^*t)^2\big)\, dx =: C_3 \hat{D}_\alpha.$$

The Nash inequality in $\mathbb{R}^3$ [30, Lemma I.1.1] gives $\hat{V}_\alpha^{5/3} \leq C \hat{I}_\alpha^{4/3} \hat{D}_\alpha$, and (4.4) follows for all $t > 1$.

Returning to (4.3) we now have

$$V_\alpha'(t) \leq \mu(2\alpha) V_\alpha(t) - C\frac{V_\alpha(t)^{5/3}}{I_\alpha(t)^{4/3}}$$

where $I_\alpha'(t) = \mu(\alpha) I_\alpha(t)$. For $V_\alpha(t) = e^{\mu(2\alpha)t} Z_\alpha(t)$, this implies the bound

$$Z_\alpha'(t) \leq -C\frac{e^{-t\mu(2\alpha)} e^{t5\mu(2\alpha)/3} Z_\alpha(t)^{5/3}}{e^{t4\mu(\alpha)/3} I_\alpha(0)^{4/3}} = -C\frac{Z_\alpha(t)^{5/3}}{I_\alpha(0)^{4/3}} e^{tR_\alpha} \tag{4.5}$$

for $t \geq 1$, where $R_\alpha = \frac{2}{3}\mu(2\alpha) - \frac{4}{3}\mu(\alpha) = \frac{1}{3}\mu(2\alpha) + O(\alpha^3)$. We used (3.12) in the last step above. We deduce from (4.5) that

$$Z_\alpha(t) \leq C\left(\frac{I_\alpha(0)^{4/3} R_\alpha}{e^{tR_\alpha} - e^{R_\alpha}}\right)^{3/2} = C\frac{I_\alpha(0)^2}{(t-1)^{3/2}}\left(\frac{tR_\alpha - R_\alpha}{e^{tR_\alpha} - e^{R_\alpha}}\right)^{3/2}. \tag{4.6}$$

Note that since e^x is a convex function, we have $\frac{b-a}{e^b - e^a} \leq e^{-a}$ for all $b > a$. Moreover, $R_\alpha > 0$ for α sufficiently small, so $R_\alpha t > R_\alpha$ for $t > 1$. Hence, (4.6) implies

$$Z_\alpha(t) \leq C\frac{I_\alpha(0)^2}{(t-1)^{3/2}} e^{-3R_\alpha/2} \leq \frac{C I_\alpha(0)^2}{(t-1)^{3/2}}.$$

Therefore, we have $V_\alpha(t) \le Ce^{\mu(2\alpha)t} I_\alpha(0)^2/(t-1)^{3/2}$, which is

$$\left(\int_{c^*t}^\infty \eta_{2\alpha}(t,x)\nu(x)\zeta(t,x)q^2\,dx\right)^{1/2} \le C\frac{e^{\mu(2\alpha)t}}{(t-1)^{3/4}}\int_0^\infty \eta_\alpha(0,x)\nu(x)\zeta(0,x)p_0(x)\,dx.$$

By Lemma 3.3 and the definition of $q(t,x)$, this implies

$$\left(\int_0^\infty \frac{e^{2\alpha x}-e^{2\alpha x}}{2\alpha x} p(t,c^*t+x)^2\,dx\right)^{1/2} \le C\frac{e^{\mu(2\alpha)t}}{(t-1)^{3/4}}\int_0^\infty \frac{e^{\alpha x}-e^{\alpha x}}{\alpha} x p_0(x)\,dx.$$

From now on, we take $\alpha = 1/\sqrt{t}$. If T_0 is sufficiently large, and $t > T_0$, then for any $x \in \operatorname{supp} p_0$ we have $\frac{e^{\alpha x}-e^{-\alpha x}}{\alpha} \le 4x$. So, for all $t > T_0$ we have

$$\left(\int_0^\infty \frac{e^{2x/\sqrt{t}}-e^{-2x/\sqrt{t}}}{2x/\sqrt{t}} p(t,x)^2\,dx\right)^{1/2} \le \frac{C}{t^{3/4}}\int_0^\infty x p_0(x)\,dx, \tag{4.7}$$

or

$$\left(\int_0^\infty \frac{e^{2x/\sqrt{t}}-e^{-2x/\sqrt{t}}}{x} p(t,x)^2\,dx\right)^{1/2} \le \frac{C}{t}\int_0^\infty x p_0(x)\,dx. \tag{4.8}$$

Let us take $N > 1$ sufficiently large (but independent of t); then for $x > N\sqrt{t}$ we have $e^{2x/\sqrt{t}} > 2e^{-2x/\sqrt{t}}$, thus (4.8) implies

$$\left(\int_{N\sqrt{t}}^\infty \frac{e^{2x/\sqrt{t}}}{x} p(t,x)^2\,dx\right)^{1/2} \le \frac{C}{t}\int_0^\infty x p_0(x)\,dx.$$

Moreover, we have

$$\begin{aligned}\int_{N\sqrt{t}}^\infty x p(t,x)\,dx &\le \int_{N\sqrt{t}}^\infty \frac{e^{x/\sqrt{t}}}{\sqrt{x}} p(t,x) e^{-x/\sqrt{t}} x^{3/2}\,dx\\ &\le \left(\int_{N\sqrt{t}}^\infty \frac{e^{2x/\sqrt{t}}}{x} p(t,x)^2\,dx\right)^{1/2}\left(\int_{N\sqrt{t}}^\infty e^{-2x/\sqrt{t}} x^3\,dx\right)^{1/2}\\ &\le C\left(\int_0^\infty x p_0(x)\,dx\right)\left(\int_N^\infty y^3 e^{-y}\,dy\right)^{1/2}\\ &\le 2N^3 e^{-N/2}\int_0^\infty x p_0(x)\,dx = I(0)/4\end{aligned}$$

as long as $N > N_0$ is large enough (but independent of t). Recall now the conservation of $I(t) = \int_{c^*t}^\infty \nu(x) f(t,x) p(t,x)\,dx$, together with the fact that $m(x-c^*t) \le f(t,x) \le m^{-1}(x-c^*t)$ for some $m > 0$ and all $x \ge c^*t$. It follows that $\int_0^{N\sqrt{t}} x p(t,x)\,dx \ge 3I_0/4$. From Lemma 2.1 have $\int_0^{\sqrt{t}/N} x p(t,x)\,dx \le CN^{-3} I_0$. Therefore, by taking N larger if necessary, we have $\int_{\sqrt{t}/N}^{N\sqrt{t}} x p(t,x)\,dx \ge I_0/2$. For $c_0 > 0$ to be chosen, let $H_t^+ =$

$\{x \in [\sqrt{t}/N, N\sqrt{t}] : p(t,x) \geq c_0/t\}$ and $H_t^- = \{x \in [\sqrt{t}/N, N\sqrt{t}] : p(t,x) < c_0/t\}$. We have

$$\frac{I_0}{2} \leq \int_{H_t^+} xp(t,x)\,dx + \int_{H_t^-} xp(t,x)\,dx \leq \int_{H_t^+} xp(t,x)\,dx + \frac{c_0}{2}N^2.$$

so that by choosing $c_0 \leq I_0/(2N^2)$, we have $I_0/4 \leq \int_{H_t^+} xp(t,x)\,dx$. Now, Lemma 2.1 again yields

$$\frac{I_0}{4} \leq \int_{H_t^+} xp(t,x)\,dx \leq \frac{CI_0}{t^{3/2}} \int_{H_t^+} x^2\,dx \leq \frac{CI_0}{t^{3/2}} |H_t^+| N^2 t.$$

It follows that $|H_t^+| \geq \sqrt{t}/(4N^2C)$. This proves Proposition 4.1. □

4.2. A Harnack type estimate

For $R > 0$ and $\xi \in \mathbb{R}$ fixed, let $\bar{\Gamma}(t,x,s,y) = \bar{\Gamma}(t,x,s,y;R,\xi)$ denote the heat kernel for $\nu\rho_t - (\nu\rho_x)_x + c^*\rho_x = 0$ in the tilted cylinder

$$T(\xi,R,s) = \{(t,x) \in \mathbb{R}^2 : t \geq s,\ |x - \xi - c^*t| < R\}$$

with the Dirichlet boundary conditions on the lateral boundary of the cylinder. That is, if $s \in \mathbb{R}$ and $|y - \xi - cs| < R$, $\bar{\Gamma}(t,x,s,y)$ satisfies the PDE for $(t,x) \in T(\xi,R,s)$, with the boundary condition $\bar{\Gamma}(t,x,s,y) = 0$ if $|x - \xi - c^*t| = R$, and the initial condition $\lim_{t\searrow s} \bar{\Gamma}(t,x,s,y) = \nu(y)^{-1}\delta_y(x)$. The following lemma gives a lower bound on $\bar{\Gamma}(t,x,s,y)$, provided that x and y are sufficiently far from the boundary of $T(\xi,R,s)$. It is directly inspired by Fabes–Stroock [12, Lemma 5.1].

Lemma 4.2. *For all $\delta \in (0,1)$, there are constants $\alpha, K > 0$ such that*

$$\bar{\Gamma}(t,x,s,y - c^*(t-s);R,\xi) \geq \frac{\alpha}{2K(t-s)^{1/2}} e^{-K|y-x|^2/(t-s)}$$

for all $R > 0$, $t \in (s, s+R^2]$, and $x, y \in (c^*t + \xi - \delta R, c^*t + \xi + \delta R)$.

Proof. Let $\rho(t,x) = \int_{\mathbb{R}} \Gamma(t,x,s,y)\rho(s,y)\nu(y)\,dy$ where $\Gamma(t,x,s,y)$ denotes the free-space heat kernel for $t \geq s$. We have the following estimates of Norris [28, Theorem 1.1]: there is a constant $K > 0$ such that

$$\frac{e^{-K|x-y|^2/(t-s)}}{K|t-s|^{1/2}} \leq \Gamma(t,x,s,y - c^*(t-s)) \leq \frac{Ke^{-|x-y|^2/(K(t-s))}}{|t-s|^{1/2}} \tag{4.9}$$

for all $x, z \in \mathbb{R}$ and $t > s$. Obviously, (4.9) implies the upper bound

$$\bar{\Gamma}(t,x,s,y - c^*(t-s);R,\xi) \leq \Gamma(t,x,s,y - c^*(t-s)) \leq K|t-s|^{-1/2}e^{-|x-y|^2/(K(t-s))}.$$

It suffices to assume $s = 0$ and $\xi = 0$. The first step is the identity

$$\bar{\Gamma}(t,x,0,y) = \Gamma(t,x,0,y) - \int_0^t \left(\Gamma(t,x,r,c^*r+R)h^+(r) + \Gamma(t,x,r,c^*r-R)h^-(r)\right)dr \tag{4.10}$$

where $h^{\pm}(r) \geq 0$ depends on y and R, but $\int_0^t (h^+(r) + h^-(r))\,dr \leq 1$ always holds. This is analogous to a statement in [12, p. 335], obtained by integrating the equation for ρ against a test function. By combining (4.10) with the estimate (4.9) for Γ, we obtain a lower bound on $\bar{\Gamma}$:

$$\bar{\Gamma}(t, x, 0, y - c^* t) \geq \frac{e^{-K|y-x|^2/t}}{K t^{1/2}} - K \sup_{0<\tau\leq t} \frac{e^{-R^2(1-\delta)^2/(K\tau)}}{\tau^{1/2}} \tag{4.11}$$

for all $x \in [-\delta R, \delta R]$, $y \in [-R, R]$ and $t > 0$. The unique maximum of the function $0 < \tau \mapsto \beta(\tau) = e^{-R^2(1-\delta)^2/(K\tau)}/\tau^{1/2}$ occurs at the point $\tau^* = 2R^2(1-\delta)^2/K$. So, if $\varepsilon^2 < 2(1-\delta)^2/K$ and $t \leq \varepsilon^2 R^2$, we have $t \leq \tau^*$. In this case, (4.11) gives us the bound

$$\begin{aligned}\bar{\Gamma}(t, x, 0, y - c^* t) &\geq \frac{e^{-K|y-x|^2/t}}{K t^{1/2}} - K \sup_{0<\tau\leq t} \frac{e^{-R^2(1-\delta)^2/(K\tau)}}{\tau^{1/2}} \\ &= \frac{e^{-K|y-x|^2/t}}{K t^{1/2}} \big(1 - K^2 e^{-R^2(1-\delta)^2/(Kt) + K|x-y|^2/t}\big).\end{aligned}$$

If also $x \in [-\delta R, \delta R]$ and $|x - y| \leq \varepsilon R$, and $\varepsilon^2 < (1-\delta)^2/(2K^2)$ is small enough, then

$$\begin{aligned}1 - K^2 e^{-R^2(1-\delta)^2/(Kt) + K|x-y|^2/t} &\geq 1 - K^2 e^{-t^{-1} R^2 (1-\delta)^2/(2K)} \\ &\geq 1 - K^2 e^{-\varepsilon^{-2}(1-\delta)^2/(2K)} > 1/2.\end{aligned}$$

This implies that for any $\delta \in (0, 1)$ and $R > 0$,

$$\bar{\Gamma}(t, x, 0, y - c^* t) \geq \frac{1}{2K t^{1/2}} e^{-K|y-x|^2/t}$$

if $x \in [-\delta R, \delta R]$ and $|x - y| \leq \varepsilon R$, $t \leq \varepsilon^2 R^2$, and ε is sufficiently small, depending only on δ and K. A chaining argument, as in [12], now shows that for any $\delta \in (0, 1)$, there must be a constant α, depending only on δ and K, such that

$$\bar{\Gamma}(t, x, 0, y - c^* t) \geq \frac{\alpha}{2K t^{1/2}} e^{-K|y-x|^2/t}$$

for all $x, y \in [-\delta R, \delta R]$ and $t \leq R^2$ (i.e. rather than just $t \leq \varepsilon^2 R^2$). Although $\bar{\Gamma}$ depends on R, α and K are independent of R. This finishes the proof of Lemma 4.2. □

4.3. *Proof of Proposition 2.2*

By Proposition 4.1 we have $p(s, x) \geq c_0/s$ for all $s \geq T_0$ and $x \in I_s$, where $I_s \subset [c^* s + N^{-1}\sqrt{s}, c^* s + N\sqrt{s}]$ and $|I_s| \geq \beta\sqrt{s}$. Let $s \geq T_0$, $R = \sqrt{s}(N^{-1} + N)/2$, $\xi = c^* s + R$, and $\bar{\Gamma} = \bar{\Gamma}(t, x, s, y; R, \xi)$ be the heat kernel in the tilted cylinder $T(\xi, R, s)$ with Dirichlet boundary conditions. For $t > s$ and $x \in [c^* t, c^* t + 2R]$, we have

$$p(t, x) \geq \int_{cs}^{cs+2R} \bar{\Gamma}(t, x, s, y) p(s, y) v(y)\, dy. \tag{4.12}$$

Set $\delta = \frac{N-N^{-1}}{N+N^{-1}} \in (0,1)$ and $t = s+R^2$. Observe that $I_s \subset [c^*s+N^{-1}\sqrt{s}, c^*s+N\sqrt{s}] = [c^*s+(1-\delta)R, c^*s+(1+\delta)R]$. By Lemma 4.2, we have

$$\bar{\Gamma}(t,x,s,y) \geq \frac{\alpha}{2(t-s)^{1/2}} e^{-K|x-y|^2/(t-s)} = \frac{\alpha}{2KR} e^{-K|x-y|^2/R^2}$$

for all

$$x \in [c^*t+(1-\delta)R, c^*t+(1+\delta)R] = [c^*t+N^{-1}\sqrt{s}, c^*t+N\sqrt{s}],$$
$$y \in [c^*s+(1-\delta)R, c^*s+(1+\delta)R] = [c^*s+N^{-1}\sqrt{s}, c^*s+N\sqrt{s}].$$

Therefore, by combining $p(s,x) \geq c_0/s$ and (4.12) we obtain

$$\begin{aligned} p(t,x) &\geq \int_{I_s} \bar{\Gamma}(t,x,s,y)p(s,y)\nu(y)\,dy \\ &\geq |I_s| \min_{y\in I_s} \bar{\Gamma}(t,x,s,y)p(s,y)\nu(y) \geq |I_s| \frac{C}{\sqrt{s}} \min_{y\in I_s} p(s,y) \geq \frac{C}{s} \end{aligned}$$

for all $x \in [c^*t+(1-\delta)R, c^*t+(1+\delta)R]$. Since $R = \sqrt{s}(N^{-1}+N)/2$ and $t = s+R^2$ we have shown that for $\sigma = 1+(N^{-1}+N)^2/4$, there is $C>0$ such that $p(t, c^*t+\sigma\sqrt{t}) \geq C/s = C\sigma/t$ for $t \geq \sigma T_0$. □

5. The perturbed linearized equation in the diffusive range

Recall that the upper bound in Theorem 1.1 was reduced in Section 2.2 to the proof of Proposition 2.3, which we present in this section. Let $\tilde{p}(\tau,x)$ be as in this proposition, that is,

$$(1-\omega(\tau))\,\tilde{p}_\tau = \tilde{p}_{xx} + 2\frac{\phi_x}{\phi}\tilde{p}_x, \qquad x \geq c^*\tau, \tag{5.1}$$

with the Dirichlet boundary condition $\tilde{p}(\tau, c^*\tau) = 0$. The coefficient $\omega(\tau)$ satisfies $\omega(\tau) \sim 3/(2c^*\tau)$ as $\tau \to \infty$, and $|\omega(\tau)| \leq C/\tau$, $|\omega'(\tau)| \leq C/\tau^2$ for $\tau > \tau_0$. The general philosophy is that the correction $\omega(\tau)$ does not play a role in most of the decay estimates, and the function $\tilde{p}(t,x)$ behaves essentially as $p(t,x)$, which is the solution of (5.1) with $\omega(\tau) = 0$, and which we have studied in detail in the preceding sections. We could think of re-using the arguments already displayed in the preceding section and, in particular, trying to adapt the proof of Lemma 2.1. However, as far as the perturbed equation is concerned, we do not have exact linear solutions anymore. As a consequence, the computations of Proposition 2.2 and Lemma 2.1 would yield big errors, which would in the end yield not sufficiently precise estimates. So, we have chosen a different way, which in turn allows us to gain a little more insight in the heat kernel.

Proposition 5.1. *For any $L_0, \varepsilon > 0$, there is $C_\varepsilon > 0$ such that*

$$\frac{1}{C_\varepsilon \tau} \leq \tilde{p}(\tau, c^*\tau + L_0 + \varepsilon\sqrt{\tau}) \leq \frac{C_\varepsilon}{\tau} \quad \text{for all } \tau \geq 1.$$

This is a direct generalization of Proposition 2.2 and Lemma 2.1 to the case $\omega(\tau) \neq 0$. We will also need a more or less explicit solution of the approximate equation that we will need to compare with $\tilde{p}(t, x)$. It is described in the next proposition.

Proposition 5.2. *Let $\bar{\chi} \in \mathbb{R}$ and let*

$$\chi(x) = -\frac{1}{\psi(x, \lambda^*)} \frac{d\psi(x, \lambda^*)}{d\lambda}.$$

There is a function $\theta^{\mathrm{app}}(\tau, x)$ such that for any $\sigma > 0$,

$$(1 - \omega(\tau)) \frac{\partial \theta^{\mathrm{app}}}{\partial \tau} - \theta^{\mathrm{app}}_{xx} - 2\frac{\phi_x}{\phi} \theta^{\mathrm{app}}_x = O(\tau^{-3}), \qquad \tau \geq 1,\ c^*\tau < x < c^*\tau + \sigma\sqrt{\tau},$$

and there is a constant C (depending on σ and m) such that

$$\left| \theta^{\mathrm{app}}(\tau, x) - \frac{x - c^*\tau + \chi(x) + \bar{\chi}}{\tau^{3/2}}\, e^{-\frac{(x - c^*\tau)^2}{4(1+\kappa)\tau}} \right| \leq C\tau^{-3/2} \left(\frac{x - c^*\tau}{\sqrt{\tau}} \right)^2 + O(\tau^{-2}) \tag{5.2}$$

for all $\tau \geq 1$ and $x \in [c^*\tau, c^*\tau + \sigma\sqrt{\tau}]$. The constant κ in the exponential factor is defined by formula (5.11) below and satisfies $1 + \kappa > 0$.

Observe that the approximate solution θ^{app} satisfies the conclusion of Proposition 2.3. So, the last step is to transfer these estimates to the true solution.

Proposition 5.3. *Let $\sigma > 0$ be fixed, and let $\theta^{\mathrm{app}}(\tau, x)$ be defined as in Proposition 5.2 for some $\bar{\chi} \in \mathbb{R}$. Let $\xi(\tau, x)$ solve*

$$(1 - \omega(\tau)) \frac{\partial \xi}{\partial \tau} = \xi_{xx} + 2\frac{\phi_x}{\phi} \xi_x, \qquad \tau > 1,\ x \in (c^*\tau, c^*\tau + \sigma\sqrt{\tau}), \tag{5.3}$$

with the boundary conditions

$$\begin{aligned} \xi(\tau, c^*\tau) &= \theta^{\mathrm{app}}(\tau, c^*\tau), \\ \xi(\tau, c^*\tau + \sigma\sqrt{\tau}) &= \theta^{\mathrm{app}}(\tau, c^*\tau + \sigma\sqrt{\tau}). \end{aligned} \tag{5.4}$$

There is $\tau_0 > 0$ such that $|\xi(\tau, x) - \theta^{\mathrm{app}}(\tau, x)| \leq C\tau^{-3/2}$ for $\tau \geq \tau_0$ and $c^*\tau < x < c^*\tau + \sigma\sqrt{\tau}$.

5.1. Proof of Proposition 2.3, granting Propositions 5.1–5.3

Observe that by choosing $\bar{\chi} > \|\chi\|_\infty$ in Proposition 5.2, we may arrange that $\theta^{\mathrm{app}}(\tau, c^*\tau) > 0$ for τ sufficiently large. Similarly, with $\bar{\chi} < -\|\chi\|_\infty$, we have $\theta^{\mathrm{app}}(\tau, c^*\tau) < 0$ for τ sufficiently large. Let us define θ^{app}_+ to be a solution with $\bar{\chi} = 2\|\chi\|_\infty$ and $\theta^{\mathrm{app}}_+(\tau, c^*\tau) > 0$; let θ^{app}_- be a solution with $m = -2\|\chi\|_\infty$ and $\theta^{\mathrm{app}}_-(\tau, c^*\tau) < 0$. To prove Proposition 2.3, we wish to compare $\tilde{p}(\tau, x)$ with the functions $\theta^{\mathrm{app}}_\pm$. We know from Proposition

5.2 that $|\theta_\pm^{\rm app}(\tau, c^*\tau + \sigma\sqrt{\tau}) - C\sigma/\tau| \le C/\tau^{3/2}$. Combining this with Proposition 5.1, we see that there must be $C_1 > 0$ such that

$$\tilde p(\tau, c^*\tau + \sigma\sqrt{\tau}) \le C_1\theta_+^{\rm app}(\tau, c^*\tau + \sigma\sqrt{\tau}),$$
$$\tilde p(\tau, c^*\tau + \sigma\sqrt{\tau}) \ge C_1^{-1}\theta_-^{\rm app}(\tau, c^*\tau + \sigma\sqrt{\tau}),$$

for all $\tau \ge 1$. Now if $\xi_\pm(\tau, x)$ solve (5.3) for $\tau \ge 1$ with the boundary conditions (5.4) using $\theta^{\rm app} = \theta_\pm^{\rm app}$, we have

$$\xi_+(\tau, c^*\tau) = \theta_+^{\rm app}(\tau, c^*\tau) > 0 = C_1^{-1}\tilde p(\tau, c^*\tau),$$
$$\xi_+(\tau, c^*\tau + \sigma\sqrt{\tau}) = \theta_+^{\rm app}(\tau, c^*\tau + \sigma\sqrt{\tau}) \ge C_1^{-1}\tilde p(\tau, c^*\tau),$$
$$\xi_-(\tau, c^*\tau) = \theta_-^{\rm app}(\tau, c^*\tau) < 0 = C_1\tilde p(\tau, c^*\tau),$$
$$\xi_-(\tau, c^*\tau + \sigma\sqrt{\tau}) = \theta_-^{\rm app}(\tau, c^*\tau + \sigma\sqrt{\tau}) \le C_1\tilde p(\tau, c^*\tau).$$

The maximum principle implies $C_1^{-1}\xi_-(\tau, x) \le \tilde p(\tau, x) \le C_1\xi_+(\tau, x)$ for all τ sufficiently large and $x \in [c^*\tau, c^*\tau + \sigma\sqrt{\tau}]$. Proposition 5.3 implies that for any $\delta > 0$ there exists x_δ such that $|\xi_\pm(\tau, x) - \theta_\pm^{\rm app}(\tau, x)| \le \delta\theta_\pm^{\rm app}(\tau, x)$ for $c^*\tau + x_\delta < x < c^*\tau + \varepsilon\sqrt{\tau}$, if $\tau \ge \tau_0$. It follows that $(C_1^{-1}/2)\theta_-^{\rm app}(\tau, x) \le \tilde p(\tau, x) \le 2C_1\theta_+^{\rm app}(\tau, x)$ for all $\tau \ge \tau_0$ and $c^*\tau + x_\delta < x < c^*\tau + \varepsilon\sqrt{\tau}$. Proposition 2.3 follows from (5.2) and parabolic regularity. □

5.2. *The proof of Proposition 5.1*

The proof of Proposition 5.1 is as in the case $\omega(\tau) = 0$ (i.e. Proposition 2.2 and Lemma 2.1) but a little more technical—we focus only on the differences. The first ingredient needed is a quantity that is bounded from above and below.

Lemma 5.4. *Let $\tilde p(\tau, x)$ be as in Proposition 2.3. There is $C > 0$ such that*

$$C^{-1} \le \int_{c^*\tau}^{\infty} (x - c^*\tau)\tilde p(\tau, x)\,dx \le C, \qquad \forall\tau \ge 0.$$

Proof. It suffices to bound the integral $I(\tau) = \int_{c^*\tau}^{\infty} \nu(x)(1 - \omega(\tau))f(\tau, x)\tilde p(\tau, x)\,dx$, where $f(\tau, x)$ is defined in Lemma 3.2 with $m(x - c^*\tau) \le f(\tau, x) \le m^{-1}(x - c^*\tau)$. In the case $\omega \equiv 0$, $I(\tau)$ is conserved. We compute

$$\frac{dI}{d\tau} = -\omega' \int_{c^*\tau}^{\infty} \nu f\tilde p\,dx - \omega\int_{c^*\tau}^{\infty} \nu f_\tau\tilde p\,dx = O(\tau^{-2})I(\tau) - \omega\int_{c^*\tau}^{\infty} \nu f_\tau\tilde p\,dx. \tag{5.5}$$

For an upper bound on $I(\tau)$, we treat the spurious term $\int_{c^*\tau}^{\infty} \nu f_\tau\tilde p\,dx$ as follows:

$$\int_{c^*\tau}^{\infty} \nu f_\tau\tilde p\,dx = \int_{c^*\tau}^{c^*\tau+\tau^{1/4}} \nu f_\tau\tilde p\,dx + \int_{c^*\tau+\tau^{1/4}}^{\infty} \nu f_\tau\tilde p\,dx =: II + III.$$

By parabolic regularity, there is a constant $C > 0$ such that $|\partial_\tau f(\tau, x)| \le C$, hence

$$|III| \le C\tau^{-1/4} \int_{c^*\tau}^{\infty} x\tilde{p}\,dx \le C\tau^{-1/4} \int_{c^*\tau}^{\infty} \nu(x) f(\tau, x)\tilde{p}(\tau, x)\,dx.$$

Recall that equation (2.17) for $\tilde{p}$ is equivalent to

$$(1 - \omega(\tau))\tilde{p}_\tau = \frac{1}{\nu(x)}(\nu(x)\tilde{p}_x)_x - \frac{c^*}{\nu(x)}\tilde{p}_x, \qquad x > c^*\tau, \tag{5.6}$$

with $\tilde{p}(\tau, c^*\tau) = 0$. The time change $d\tau' = (1 - \omega(\tau))^{-1}d\tau$ shows that the heat kernel bounds of [28] in the whole space hold (with the time change) for the perturbed equation

$$(1 - \omega(\tau))P_\tau = \frac{1}{\nu(x)}(\nu(x)P_x)_x - \frac{c^*}{\nu(x)}P_x, \qquad x \in \mathbb{R}.$$

In particular, we have $|P(\tau, x)| \le C\tau^{-1/2}\int_{\mathbb{R}} |P(0, y)|\,dy$. So, because $\tilde{p}(\tau, x)$ is less than the solution of (5.6) in the whole space with the same initial data $\tilde{p}(0, \cdot)$, we have

$$|II| \le C\tau^{-1/2} \int_{c^*\tau}^{c^*\tau+\tau^{1/4}} \int_{\mathbb{R}} |\tilde{p}(0, y)|\,dy\,dx = C\tau^{-1/4} \int_0^{\infty} \tilde{p}(0, x)\,dx.$$

Gathering these estimates we conclude $I'(\tau) \le O(\tau^{-2})I + O(\tau^{-5/4})I + O(\tau^{-5/4})$, which implies the existence of $C > 0$ such that $I(\tau) \le C(1 + I(0))$.

For a lower bound, note that $f_\tau \le 0$, while $\nu, \tilde{p} \ge 0$. Therefore, the term $-\omega\int_{c^*\tau}^{\infty} \nu f_\tau \tilde{p}\,dx$ in (5.5) is nonnegative. This implies $I'(\tau) \ge O(\tau^{-2})I$, so that $I(t) \ge CI(0) > 0$, with some universal constant $C > 0$. □

We are going to estimate

$$V_\alpha(\tau) = (1 - \omega(\tau)) \int_{c^*\tau}^{\infty} \nu(x)\eta_{2\alpha}(\tau, x)\tilde{p}(\tau, x)q(\tau, x)\,dx,$$

which is the main step in the proof of Proposition 5.1. Here $q(\tau, x) = \tilde{p}(\tau, x)/\zeta(\tau, x)$ and $\zeta(\tau, x)$ is defined by Lemma 3.2. The function $\eta_\alpha(\tau, x)$ is defined by Lemma 3.3.

Proof of Proposition 5.1. A straightforward computation shows that

$$\frac{dV_\alpha}{d\tau} = (\mu(2\alpha) - \omega')V_\alpha + \omega \int_{c^*\tau}^{\infty} \left(\nu\eta_{2\alpha}\zeta_\tau q^2 - \nu(\partial_\tau\eta_{2\alpha})pq\right)dx - 2D_\alpha.$$

Here, as in the case $\omega = 0$, we have defined $D_\alpha(\tau) = \int_{c^*\tau} \nu\eta_{2\alpha}\zeta q_x^2\,dx$. We now use the following fact: for all $M > 0$, there is a constant $\kappa_M > 0$ such that for all nonnegative functions $u(x) \in C^1([0, 1])$ such that $|u'(x)| \le M\int_0^1 u(x)\,dx$ we have $\int_0^1 u(x)\,dx \le \kappa_M \int_0^1 xu(x)\,dx$. If not, there is a sequence u_n of such functions with unit mass and uniformly bounded derivatives whose first moments tend to 0, an impossibility. Now, from this remark $\omega\int_{c^*\tau}^{\infty} \nu|\zeta_\tau\eta_{2\alpha}|q^2\,dx \le C\tau^{-1}V_\alpha$, and from Lemma 3.4,

$$\omega \int_{c^*\tau}^{\infty} \nu|\partial_\tau\eta_{2\alpha}|\tilde{p}q\,dx \le C\omega \int_{c^*\tau}^{\infty} \nu\eta_{2\alpha}\tilde{p}q\,dx \le C\tau^{-1}V_\alpha.$$

Because of Lemma 5.4, we have (following the lines of the proof of Proposition 4.1)

$$\frac{dV_\alpha}{d\tau} \le (\mu(2\alpha)+O(\tau^{-1}))V_\alpha(\tau) - C\frac{V_\alpha^{5/3}}{I_\alpha^{4/3}} = -C\frac{V_\alpha^{5/3}}{I_\alpha(0)^{4/3}}e^{\tau R_\alpha} + (\mu(2\alpha)+O(\tau^{-1}))V_\alpha(\tau).$$

Let us choose $T > 0$ and examine the above differential inequality with $\alpha = T^{-1}$ and $\tau \le T$. For $\Lambda > 0$ large enough, the function $\Lambda\tau^{-3/2}$ is a supersolution for $\tau \le T$, showing that $V_\alpha(T) = O(T^{-3/2})$. So, for all $\tau > 0$, we have $V_\alpha(\tau) \le C\tau^{-3/2}$, and the rest of the proof follows as in Proposition 4.1. □

5.3. *The proof of Proposition 5.2*

The proof is by a multiple-scale expansion. We will construct a function θ^{app} having the form $\theta^{\text{app}}(\tau, x) = a(\tau)v(\tau, (x - c^*\tau)/R(\tau), x)$ which satisfies $\theta^{\text{app}}(\tau, c^*\tau) = 0$ with $R(\tau) = \tau^{1/2}$. Plugging this ansatz into $(1-\omega(\tau))\theta_\tau = \theta_{xx} + 2\frac{\phi_x}{\phi}\theta_x$, we see that $v(\tau, z, x)$ should satisfy

$$(1-\omega)\left[\frac{a'}{a}v + v_\tau - \frac{zR'}{R}v_z - \frac{c^*}{R}v_z\right] = \frac{1}{R^2}v_{zz} + \frac{2}{R}v_{zx} + v_{xx} + 2\frac{\phi_x}{\phi}v_x + \frac{2}{R}\frac{\phi_x}{\phi}v_z.$$

We will construct an approximate solution given by the expansion

$$v = v(\tau, z, x) = v^0(z) + \frac{1}{R}v^1(z, x) + \frac{1}{R^2}v^2(z, x) + \frac{1}{R^3}v^3(z, x),$$

where $v^1(z, x)$ and $v^2(z, x)$ are uniformly bounded in each compact set in z and x, and are both periodic in x. Therefore, the desired equality is

$$\begin{aligned}
&(1-\omega)\frac{a'}{a}\left(v^0 + \frac{1}{R}v^1 + \frac{1}{R^2}v^2 + \frac{1}{R^3}v^3\right) - (1-\omega)\frac{R'}{R^2}\left(v^1 + \frac{2}{R}v^2 + \frac{3}{R^2}v^3\right)\\
&- (1-\omega)\frac{zR'}{R}\left(v_z^0 + \frac{1}{R}v_z^1 + \frac{1}{R^2}v_z^2 + \frac{1}{R^3}v_z^3\right) - (1-\omega)\frac{c^*}{R}\left(v_z^0 + \frac{1}{R}v_z^1 + \frac{1}{R^2}v_z^2 + \frac{1}{R^3}v_z^3\right)\\
&\qquad = \frac{1}{R^2}v_{zz}^0 + \frac{1}{R^3}v_{zz}^1 + \frac{1}{R^4}v_{zz}^2 + \frac{2}{R^2}v_{zx}^1 + \frac{2}{R^3}v_{zx}^2 + \frac{2}{R^4}v_{zx}^3\\
&\qquad\quad + \frac{1}{R}v_{xx}^1 + \frac{1}{R^2}v_{xx}^2 + \frac{1}{R^3}v_{xx}^3 + \frac{2}{R}\frac{\phi_x}{\phi}v_x^1 + \frac{2}{R^2}\frac{\phi_x}{\phi}v_x^2 + \frac{2}{R^3}\frac{\phi_x}{\phi}v_x^3\\
&\qquad\quad + \frac{2}{R}\frac{\phi_x}{\phi}v_z^0 + \frac{2}{R^2}\frac{\phi_x}{\phi}v_z^1 + \frac{2}{R^3}\frac{\phi_x}{\phi}v_z^2 + \frac{2}{R^4}\frac{\phi_x}{\phi}v_z^3. \qquad (5.7)
\end{aligned}$$

Set $a(\tau) = \tau^{-m}$, so that $a'/a = -m\tau^{-1} = O(R^{-2})$. Now we choose v_i, $i \in \{0, \ldots, 3\}$, so that terms of order $O(R^{-1})$, $O(R^{-2})$ and $O(R^{-3})$ will cancel. Recall that $\omega(\tau) \sim 3/(2c^*\lambda^*\tau)$, so ω will not play a role until we equate terms of order $O(R^{-3})$, and even then the only term to contribute is $\omega c^* v_z^0/R$. All other terms involving $\omega(\tau)$ are smaller than $O(\tau^{-3/2})$.

If we equate the leading order terms (of order $O(R^{-1})$), we obtain an equation for v^1 in terms of v^0:

$$v^1_{xx} + 2\frac{\phi_x}{\phi}v^1_x = -\left(2\frac{\phi_x(x)}{\phi(x)} + c\right)v^0_z(z). \tag{5.8}$$

Let us re-introduce the solution $\chi(x)$ of $\chi_{xx} + 2\frac{\phi_x}{\phi}\chi_x = -2\frac{\phi_x}{\phi} - c$; we see that (5.8) has a solution of the form $v^1(z,x) = v^0_z(z)\chi^0(x) - p^0(z)$ with $\chi^0(x) = \chi(x) + \bar{\chi}$ being periodic in x, and $\bar{\chi}$ any constant. For any choice of $\bar{\chi}$ and $p^0(z)$, (5.8) holds and the $O(R^{-1})$ terms in (5.7) cancel.

Let us now equate the terms of $O(R^{-2})$ in (5.7) to obtain

$$v^2_{xx} + 2\frac{\phi_x}{\phi}v^2_x + mv^0 + \frac{z}{2}v^0_z + v^0_{zz} + cv^1_z + 2v^1_{zx} + 2\frac{\phi_x}{\phi}v^1_z = 0. \tag{5.9}$$

Consider the operator $\rho_{xx} + 2\frac{\phi_x(x)}{\phi(x)}\rho_x = \hat{\phi}^{-2}(\hat{\phi}^2\rho_x)_x$ acting on 1-periodic functions, where $\hat{\phi} = e^{-\mu x}\psi(x)$. We claim that the adjoint operator has one-dimensional kernel. A function η is in the kernel of the adjoint operator if and only if $(\hat{\phi}^2(\hat{\phi}^{-2}\eta)_x)_x = 0$, which holds if and only if $\eta(x) = k_1\hat{\phi}(x)^2\int_0^x \hat{\phi}(s)^{-2}\,ds + k_2\hat{\phi}(x)^2$ for some constants k_1 and k_2. If $k_1 = 0$, the function η cannot be periodic, since $\hat{\phi}(x)^2 = e^{-2\mu x}\psi(x)^2$ is not periodic. So, we may assume $k_1 = 1$. However, the function $\eta(x) = \hat{\phi}(x)^2\int_0^x \hat{\phi}(s)^{-2}\,ds + k_2\hat{\phi}(x)^2$ will be periodic only for $k_2 = \frac{\hat{\phi}(1)^2}{\hat{\phi}(0)^2 - \hat{\phi}(1)^2}\int_0^1 \hat{\phi}(s)^{-2}\,ds > 0$. Any other solution of the equation for $\hat{\phi}$ must be a multiple of this function η. Observe that $\eta(x) > 0$ for all x.

If $\eta(x)$ is 1-periodic and spans the kernel of $(\hat{\phi}^2(\hat{\phi}^{-2}\eta)_x)_x$, then equation (5.9) is solvable if and only if the sum $mv^0 + \frac{z}{2}v^0_z + v^0_{zz} + cv^1_z + 2v^1_{zx} + 2\frac{\phi_x}{\phi}v^1_z$ is orthogonal to η for each $z \in \mathbb{R}$. Using $v^1 = v^0_z(z)\chi(x) - p^0(z)$, we write the sum as

$$mv^0 + \frac{z}{2}v^0_z + v^0_{zz} + cv^0_{zz}\chi^0 + 2v^0_{zz}\chi^0_x + 2\frac{\phi_x}{\phi}v^0_{zz}\chi^0 - \left(c + 2\frac{\phi_x}{\phi}\right)p^0_z. \tag{5.10}$$

So, the solvability condition is

$$\begin{aligned}\left(mv^0 + \frac{z}{2}v^0_z + v^0_{zz}\right)\int_0^1 \eta(x)\,dx &= -\int_0^1\left(cv^1_z + 2v^1_{zx} + 2\frac{\phi_x}{\phi}v^1_z\right)\eta(x)\,dx \\ &= -\int_0^1\left(cv^0_{zz}\chi^0 + 2v^0_{zz}\chi^0_x + 2\frac{\phi_x}{\phi}v^0_{zz}\chi^0\right)\eta(x)\,dx.\end{aligned}$$

Here we have used the fact that $\int_0^1(c+2\phi_x/\phi)\eta(x)\,dx = 0$, so that the terms involving p^0_z cancel after integration against η. Hence, $v^0(z)$ should solve $mv^0 + (z/2)v^0_z + (1+\kappa)v^0_{zz} = 0$ where

$$\kappa = \left(\int_0^1 \eta(x)\,dx\right)^{-1}\int_0^1\left(c\chi^0(x) + 2\chi^0_x(x) + 2\frac{\phi_x}{\phi}\chi^0(x)\right)\eta(x)\,dx. \tag{5.11}$$

It is not difficult to show that $1 + \kappa = \int \eta(1+\chi^0_x)^2\,dx / \int \eta\,dx > 0$. In particular, κ is independent of the normalization of $\chi^0(x)$ (the choice of $\bar{\chi}$). Thus, we choose $v^0(z) > 0$

to be the principal eigenfunction of $mv^0 + (z/2)v_z^0 + (1+\kappa)v_{zz}^0 = 0$, $z > 0$, $v^0(0) = 0$, which forces $m = 1$, and $v^0(z) = ze^{-z^2/(4(1+\kappa))}$.

The function $p^0(z)$ is undetermined so far. With $v^0(z)$ chosen in this way, there exists a function $v^2(z,x)$ which is periodic in x and satisfies (5.9). Thus, the $O(R^{-2})$ terms cancel. In view of (5.10) and the definition of v^0, we see that (5.9) is equivalent to

$$v_{xx}^2 + 2\frac{\phi_x}{\phi}v_x^2 = -v_{zz}^0(z)\left(c\chi^0 + 2\chi_x^0 + 2\frac{\phi_x}{\phi}\chi^0 - \kappa\right) - \left(c + 2\frac{\phi_x}{\phi}\right)p_z^0.$$

Therefore, $v^2(z,x)$ must have the form $v^2(z,x) = v_{zz}^0(z)\hat{v}^2(x) - p_z^0(z)\chi^0(x) + p^1(z)$, where $\hat{v}^2(x)$ is a periodic solution of $\hat{v}_{xx}^2 + 2\frac{\phi_x}{\phi}\hat{v}_x^2 = -\left(c\chi^0 + 2\chi_x^0 + 2\frac{\phi_x}{\phi}\chi^0 - \kappa\right)$. Finally, equating the R^{-3} terms suggests choosing $v^3(x,z)$ to satisfy

$$v_{xx}^3 + 2\frac{\phi_x}{\phi}v_x^3 = \frac{3}{2\lambda^*}v_z^0 - (m+1)v^1 - \frac{z}{2}v_z^1 - v_{zz}^1 - \left(c^* + 2\frac{\phi_x}{\phi}\right)v_z^2 - 2v_{zx}^2. \quad (5.12)$$

The right hand side is

$$\frac{3}{2\lambda^*}v_z^0 - 2v_z^0\chi^0 + 2p^0 - \frac{z}{2}v_{zz}^0\chi^0 + \frac{z}{2}p_z^0 - v_{zzz}^0\chi^0 + p_{zz}^0$$
$$- \left(c^* + 2\frac{\phi_x}{\phi}\right)(v_{zzz}^0\hat{v}^2 - p_{zz}^0\chi^0 + p_z^1) - 2(v_{zzz}^0\hat{v}_x^2 - p_{zz}^0\chi_x^0).$$

Therefore, the solvability condition implies that $p^0(z)$ should satisfy

$$2p^0 + \frac{z}{2}p_z^0 + (1+\kappa)p_{zz}^0 = \beta_1 v_{zzz}^0 + \beta_2\frac{z}{2}v_{zz}^0 + \left(\frac{3}{2\lambda^*} - 2\beta_2\right)v_z^0$$

where

$$\beta_1 = \left(\int_0^1 \eta(x)\,dx\right)^{-1}\int_0^1\left(\chi^0 + \left(c^* + 2\frac{\phi_x}{\phi}\right)\hat{v}^2 + 2\hat{v}_x^2\right)\eta(x)\,dx,$$
$$\beta_2 = \left(\int_0^1 \eta(x)\,dx\right)^{-1}\int_0^1 \chi^0\eta\,dx,$$

and we would like to have $p^0(0) = 0$. The p^1 term does not appear in the solvability condition. Therefore, we may take $p^1(z) \equiv 0$. We let $p^0(z)$ be the unique solution of the initial value problem

$$2p^0 + \frac{z}{2}p_z^0 + (1+\kappa)p_{zz}^0 = \beta_1 v_{zzz}^0 + \beta_2\frac{z}{2}v_{zz}^0 + \left(\frac{3}{2\lambda^*} - 2\beta_2\right)v_z^0, \qquad z > 0,$$

with the initial data $p^0(z) = 0$ and $p_z^0(0) = 0$.

Having chosen p^0 in this way, we take v^3 to be a solution of (5.12), which is unique up to addition of a function $p^3(z)$. So, the $O(R^{-3}) = O(\tau^{-3/2})$ terms have canceled. Our approximate solution is

$$\theta^{\text{app}}(t,x) = \tau^{-1}v^0(z) + \tau^{-3/2}v^1(z,x) + \tau^{-2}v^2(z,x) + \tau^{-5/2}v^3(z,x),$$

with

$$v^0(z) = ze^{-\frac{z^2}{4(1+\kappa)}}, \qquad v^1(z,x) = \chi^0(x)e^{-\frac{z^2}{4(1+\kappa)}} - \frac{z^2\chi^0(x)}{2(1+\kappa)}e^{-\frac{z^2}{4(1+\kappa)}} - p^0(z).$$

Now, fix a constant $\sigma > 0$. Having chosen $p^0(0) = 0$ and $p_z^0(0) = 0$, we may choose $C_1 > 0$ so that $|p^0(z)| \le C_1 z^2$ for all $z \in [0,\sigma]$. Consequently, there is a constant $C_2 > 0$ such that for all $\tau > 1$ and $x \in [c^*\tau, c^*\tau + \sigma\sqrt{\tau}]$ we have

$$\left|\theta^{\text{app}}(t,x) - \frac{x - c^*\tau + \chi^0(x)}{\tau^{3/2}}e^{-\frac{(x-c^*\tau)^2}{4(1+\kappa)\tau}}\right| \le C_2\tau^{-3/2}\left(\frac{x-c^*\tau}{\sqrt{\tau}}\right)^2 + O(\tau^{-2})$$

The last term $O(\tau^{-2})$ comes from v^2 and v^3 and the fact that v^2 and v^3 are uniformly bounded over $[0,\sigma] \times \mathbb{R}$.

Since the periodic function $\chi^0(x) = \chi(x) + \bar{\chi}$ is unique up to addition of a constant, we may choose $\bar{\chi} < 0$ so that $\max_x \chi^0(x) < -1$. Then at the point $x = c^*\tau$ we have

$$\theta^{\text{app}}(t, c^*\tau) \le \tau^{-3/2}\chi^0(c^*\tau) + O(\tau^{-2}) \le -\tau^{-3/2} + O(\tau^{-2}),$$

which is negative for all $\tau > 1$ sufficiently large. Alternatively, we could choose $\bar{\chi} > 0$ so that $\min_x \chi^0(x) > 0$. Then we would have $\theta^{\text{app}}(\tau, c^*\tau) > 0$ for all τ sufficiently large. □

5.4. *The proof of Proposition 5.3*

Using Lemma 3.1 we bring this problem into the form

$$(1-\omega(\tau))\xi_\tau = \frac{1}{\nu(x)}\frac{\partial}{\partial x}(\nu(x)\xi_x) - \frac{c^*}{\nu(x)}\xi_x. \tag{5.13}$$

Let $\Phi(\tau,x) = \xi(\tau,x) - \theta^{\text{app}}(\tau,x)$ so that $\Phi(\tau, c^*\tau) = 0$ and $\Phi(\tau, c^*\tau + L_0 + \varepsilon\sqrt{\tau}) = 0$. We have

$$(1-\omega(\tau))\nu(x)\Phi_\tau = (\nu(x)\Phi_x)_x - c^*\Phi_x + O(\tau^{-3}).$$

Multiplying by $\Phi(\tau,x)$ and integrating by parts over $I = [c^*\tau, c^*\tau + L_0 + \varepsilon\sqrt{\tau}]$, we obtain

$$\frac{1}{2}\frac{d}{d\tau}\int_I \nu(x)(1-\omega(\tau))\Phi^2\,dx + \frac{\omega'(\tau)}{2}\int_I \nu(x)\Phi^2\,dx$$
$$= -\int_I \nu(x)\Phi_x^2\,dx + O(\tau^{-3})\int_I \Phi\,dx.$$

Note that since $\Phi(\tau, c^*\tau) = 0$, we have $|\omega'(\tau)|\int_I \nu\Phi^2\,dx \le \frac{C}{\tau^2}\varepsilon^2\tau\int_I \nu\Phi_x^2\,dx$ and

$$\left|O(\tau^{-3})\int_I \Phi\,dx\right| \le \frac{C\varepsilon}{\tau^{9/2}} + \frac{1}{\tau}\int_I \Phi^2\,dx \le \frac{C}{\tau^4} + C\varepsilon^2\int_I \nu\Phi_x^2\,dx.$$

If now ε is small enough so that the constant $C\varepsilon^2$ is less than $1/4$, it follows that, for $\tau > \tau_0$ large enough,

$$\begin{aligned}\frac{1}{2}\frac{d}{d\tau}\int_I \nu(x)(1-\omega(\tau))\Phi^2\,dx &\le -\frac{1}{2}\int_I \nu(x)\Phi_x^2\,dx + \frac{C}{\tau^4}\\ &\le -\frac{1}{C(L_0+\varepsilon\sqrt{\tau})^2}\int_I \nu(x)(1-\omega(\tau))\Phi^2\,dx + \frac{C}{\tau^4}.\end{aligned}$$

We conclude that, for ε sufficiently small,

$$\int_I \nu(x)\Phi^2\,dx \le \frac{C_\varepsilon}{(1+\tau)^{1/\varepsilon^2}} + \frac{C_\varepsilon}{(1+\tau)^3}.$$

Now, parabolic regularity implies that $|\Phi(\tau,x)| \le C/(1+\tau)^{3/2}$ for $\tau > \tau_0$ sufficiently large. This completes the proof of Proposition 5.3. □

6. Convergence to a family of waves

This section is devoted to the proof of the convergence of the solution u to the family of shifted minimal fronts U_{c^*}. We first recall that u is bounded away from 0 or $\pi(x)$ around the position $c^*t - (3/(2\lambda^*))\log t$ for large t. To the right of this position, the solution u has the same type of decay as the critical front U_{c^*}, as follows from the estimates of Sections 2 and 3. Therefore, u is almost trapped between two finite shifts of the profile of the front U_{c^*}. From a Liouville-type result, similar to that in [3] and based on the sliding method, the convergence to the shifted approximate minimal fronts will follow. First, we derive from Sections 2 and 3 some exponential bounds of u to the right of the position $c^*t - (3/(2\lambda^*))\log t$.

Lemma 6.1. *Let $\sigma > 0$ be as in Proposition 2.2. There are constants $0 < \kappa \le \rho$ such that*

$$\kappa y e^{-\lambda^* y} \le u\Big(t, c^*t - \frac{3}{2\lambda^*}\log t + y\Big) \quad \text{for all } t \ge 1 \text{ and } 0 \le y \le \sigma\sqrt{t} \tag{6.1}$$

and

$$u\Big(t, c^*t - \frac{3}{2\lambda^*}\log t + y\Big) \le \rho y e^{-\lambda^* y} \quad \text{for all } t \ge 1 \text{ and } y \ge 1. \tag{6.2}$$

Proof. The lower bound (6.1) is a simple consequence of (2.10). On the other hand, it follows from (2.18), (2.19) and the fact that $u(t,x)$ is below one of its translates in time that there exist positive constants $\overline{T}$, $\overline{y}$ and ρ such that $u(t, c^*t - (3/(2\lambda^*))\log t + y) \le \rho y e^{-\lambda^* y}$ for all $t \ge \overline{T}$ and $y \ge \overline{y}$, hence the inequality (6.2) for a possibly different ρ. □

The main ingredient in the proof of Theorem 1.2 is a Liouville-type lemma, whose proof is postponed to the end of the section.

Lemma 6.2. *For any solution* $0 \le u_\infty(t,x) \le \pi(x)$ *of* (6.7) *in* $\mathbb{R}^2$ *satisfying* (6.8) *and* (6.9) *for some positive constants* κ *and* ρ*, there is* $\xi_0 \in \mathbb{R}$ *such that* $u_\infty(t,x) = U_{c^*}(t+\xi_0, x)$ *for all* $(t,x) \in \mathbb{R}^2$.

Proof of Theorem 1.2. First, let $\sigma > 0$ and $0 < \kappa \le \rho$. Write the pulsating front U_{c^*} as

$$U_{c^*}(t,x) = \phi_{c^*}(x - c^* t, x), \tag{6.3}$$

where $0 < \phi_{c^*}(s,x) < \pi(x)$ is continuous in $\mathbb{R}\times\mathbb{R}$, 1-periodic in x, and $\phi_{c^*}(-\infty,\cdot) = \pi$, $\phi_{c^*}(\infty,\cdot) = 0$. From [19], there is a constant $B > 0$ such that

$$\phi_{c^*}(s,x) \sim B\psi(x,\lambda^*) s e^{-\lambda^* s} \quad \text{as } s \to \infty, \text{ uniformly in } x \in \mathbb{R}. \tag{6.4}$$

Choose now any real number $\widetilde{C} \ge 0$ so that

$$B \max \psi(\cdot,\lambda^*) e^{-c^*\lambda^*\widetilde{C}} \le \kappa \le \rho e^{\lambda^*} \le B \min \psi(\cdot,\lambda^*) e^{c^*\lambda^*\widetilde{C}}. \tag{6.5}$$

Let us prove that (1.5) holds with $C = \widetilde{C} + 1/c^*$. Assume not. There are then $\varepsilon > 0$ and a sequence $(t_n)_{n\in\mathbb{N}}$ of positive times such that $t_n \to \infty$ as $n \to \infty$ and

$$\min_{|\xi| \le \widetilde{C} + 1/c^*} \left\| u(t_n,\cdot) - U_{c^*}\left(t_n - \frac{3}{2c^*\lambda^*}\log t_n + \xi, \cdot\right) \right\|_{L^\infty(0,\infty)} \ge \varepsilon$$

for all $n \in \mathbb{N}$. Since $\phi_{c^*}(-\infty,\cdot) = \pi$ and $\phi_{c^*}(\infty,\cdot) = 0$ uniformly in $\mathbb{R}$, and $\phi(s,x)$ is 1-periodic in x, it follows from (6.3) and Theorem 1.1 that there exists a constant $\theta \ge 0$ such that

$$\min_{|\xi| \le \widetilde{C}} \left(\max_{|y| \le \theta} \left| u\left(t_n, y + \left[c^* t_n - \frac{3}{2\lambda^*}\log t_n\right]\right) - U_{c^*}(\xi, y) \right| \right) \ge \varepsilon \tag{6.6}$$

for all $n \in \mathbb{N}$, where $[c^* t_n - (3/(2\lambda^*))\log t_n]$ denotes the integer part of $c^* t_n - (3/(2\lambda^*))\log t_n$.

For each $n \in \mathbb{N}$, set $u_n(t,x) = uf(t + t_n, x + [c^* t_n - (3/(2\lambda^*))\log t_n])$. Up to extraction of a subsequence, the functions u_n converge locally uniformly in $\mathbb{R}^2$ to a solution u_∞ of

$$(u_\infty)_t = (u_\infty)_{xx} + f(x, u_\infty) \quad \text{in } \mathbb{R}^2 \tag{6.7}$$

such that $0 \le u_\infty(t,x) \le \pi(x)$ in $\mathbb{R}^2$. Furthermore, Theorem 1.1 implies that

$$\begin{aligned} &\lim_{A\to\infty} \Big(\sup_{(t,x)\in\mathbb{R}^2,\, x \ge c^* t + A} u_\infty(t,x) \Big) = 0, \\ &\lim_{A\to-\infty} \Big(\sup_{(t,x)\in\mathbb{R}^2,\, x \le c^* t + A} (\pi(x) - u_\infty(t,x)) \Big) = 0. \end{aligned} \tag{6.8}$$

On the other hand, for each fixed $t \in \mathbb{R}$ and $y > 2$, and n large enough, write

$$u_n(t, c^* t + y) = u\left(t + t_n, c^*(t + t_n) - \frac{3}{2\lambda^*}\log(t + t_n) + y + \gamma_n\right),$$

where $\gamma_n = [c^* t_n - (3/(2\lambda^*)) \log t_n] - (c^* t_n - (3/(2\lambda^*)) \log(t + t_n))$. We have $t + t_n \geq 1$ and $1 \leq y + \gamma_n \leq \sigma\sqrt{t + t_n}$ for n large enough, whence $\kappa(y + \gamma_n)e^{-\lambda^*(y+\gamma_n)} \leq u_n(t, c^*t + y) \leq \rho(y + \gamma_n)e^{-\lambda^*(y+\gamma_n)}$ for n large enough, from Lemma 6.1. Since $-1 \leq \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n \leq 0$, it follows that

$$\kappa(y - 1)e^{-\lambda^* y} \leq u_\infty(t, c^*t + y) \leq \rho y e^{-\lambda^*(y-1)} \quad \text{for all } t \in \mathbb{R} \text{ and } y \geq 2. \tag{6.9}$$

Now, it follows from Lemma 6.2, (6.3), (6.9) and the exponential decay (6.4) of ϕ_{c^*} that $\kappa \leq B \max \psi(\cdot, \lambda^*)e^{c^*\lambda^*\xi_0}$ and $B \min \psi(\cdot, \lambda^*)e^{c^*\lambda^*\xi_0} \leq \rho e^{\lambda^*}$, whence $|\xi_0| \leq \widetilde{C}$ from (6.5). But since (at least for a subsequence) $u_n \to u_\infty$ locally uniformly in $\mathbb{R}^2$, it follows in particular that $u_n(0, \cdot) - U_{c^*}(\xi_0, \cdot) \to 0$ uniformly in $[-\theta, \theta]$, that is,

$$\max_{|y|\leq\theta} \left| u\left(t_n, y + \left[c^* t_n - \frac{3}{2\lambda^*} \log t_n\right]\right) - U_{c^*}(\xi_0, y)\right| \to 0 \quad \text{as } n \to \infty.$$

Since $|\xi_0| \leq \widetilde{C}$, one gets a contradiction with (6.6). Therefore, (1.5) is proved.

Let us now turn to the proof of (1.6). Let $m \in (0, \min_{\mathbb{R}} \pi)$ be fixed and let $(t_n)_{n\in\mathbb{N}}$ and $(x_n)_{n\in\mathbb{N}}$ be sequences of positive real numbers such that $t_n \to \infty$ as $n \to \infty$ and $u(t_n, x_n) = m$ for all $n \in \mathbb{N}$. Set $X_n = [x_n] - [c^* t_n - (3/(2\lambda^*)) \log t_n]$. Theorem 1.1 implies that the sequence $(X_n)_{n\in\mathbb{N}}$ of integers is bounded, and may then be assumed to be equal to a constant integer X_∞, up to extraction of a subsequence. In the notation of the previous paragraphs, the functions

$$v_n(t, x) = u(t+t_n, x+[x_n]) = u\left(t+t_n, x+X_\infty+\left[c^* t_n-\frac{3}{2\lambda^*} \log t_n\right]\right) = u_n(t, x+X_\infty)$$

converge locally uniformly in $\mathbb{R}^2$, up to extraction of another subsequence, to the function

$$v_\infty(t, x) = u_\infty(t, x + X_\infty) = U_{c^*}(t + \xi, x + X_\infty) = U_{c^*}\left(t + \xi - \frac{X_\infty}{c^*}, x\right)$$

for some real number ξ. Since $v_n(0, x_n - [x_n]) = m$ for all $n \in \mathbb{N}$ and $x_n - [x_n] \to x_\infty$ as $n \to \infty$, one gets $U_{c^*}(\xi - X_\infty/c^*, x_\infty) = m$, that is, $\xi - X_\infty/c^* = T$, where T is the unique real number such that $U_{c^*}(T, x_\infty) = m$. Finally, the limit v_∞ is uniquely determined and the whole sequence $(v_n)_{n\in\mathbb{N}}$ therefore converges to the pulsating front $U_{c^*}(t + T, x)$. The proof of Theorem 1.2 is thereby complete. □

Proof of Lemma 6.2. In the homogeneous case, the function u_∞ is assumed to be trapped between two shifts of the minimal traveling front, so the conclusion follows directly from [3, Theorem 3.5]. In our periodic case, the comparisons (6.9) and the exponential behavior (6.4) of the minimal front U_{c^*} imply that u_∞ is actually trapped between two finite time shifts of U_{c^*} in the region $\{x - c^*t \geq 0\}$. In the region where $x - c^*t$ is very negative, $u_\infty(t, x)$ is close to $\pi(x)$ and the maximum principle can be applied since $f(x, s)/s$ is decreasing with respect to $s > 0$, at least when s is close to $\pi(x)$. The solution u_∞ can then be compared with some of its shifts in this region. We finally complete the proof of the lemma by using a sliding method: we shift the function $u_\infty(t, x + 1)$ in time, we compare it with the function u_∞, and we show that $u_\infty(t + 1/c^*, x + 1) = u_\infty(t, x)$

in $\mathbb{R}^2$. Together with (6.8) and (6.9), this will mean that u_∞ is a pulsating front. From the uniqueness of the pulsating fronts up to time shifts [21], the conclusion of the lemma will follow. More precisely, for all $\xi \in \mathbb{R}$ and $(t,x) \in \mathbb{R}^2$, we set $v^\xi(t,x) = u_\infty(t+\xi, x+1)$. We shall prove that $v^\xi \geq u_\infty$ in $\mathbb{R}^2$ for all ξ large enough. We will then prove that $v^\xi \equiv u_\infty$ in $\mathbb{R}^2$ for the smallest such ξ, and finally that this critical shift is equal to $1/c^*$.

To do so, we first notice that, for all $a \leq b \in \mathbb{R}$,

$$\inf_{(t,x)\in\mathbb{R}^2,\, a\leq x-c^*t\leq b} u_\infty(t,x) > 0, \qquad \inf_{(t,x)\in\mathbb{R}^2,\, a\leq x-c^*t\leq b} (\pi(x) - u_\infty(t,x)) > 0. \tag{6.10}$$

This a consequence of the strong maximum principle, parabolic regularity, and the fact that the solution $0 < u_\infty(t,x) < \pi(x)$ converges to two different limits (0 and $\pi(x)$) as $x - c^*t \to \pm\infty$. Now, if $f(x,s)$ is of the type $f(x,s) = g(x)\tilde{f}(s)$ with, for instance, $\tilde{f}$ concave and $\tilde{f}(1) = 0$, there is $\delta \in (0,1)$ such that $\tilde{f}(s)/s$ is decreasing in $[1-\delta, 1]$; by defining $f(x,s) = 0$ for all $(x,s) \in \mathbb{R} \times (1,\infty)$, it follows that $s \mapsto f(x,s)/s$ is nonincreasing on $[1-\delta,\infty)$ for every $x \in \mathbb{R}$. Whether $f(x,s)$ is of the type $g(x)\tilde{f}(s)$ or not, one then deduces from the general assumptions of Section 1 and from the definition of $\pi(x)$ that $s \mapsto f(x,s)/s$ is nonincreasing on $[(1-\delta)\pi(x),\infty)$ for every $x \in \mathbb{R}$. From (6.8) and the fact that $\min_{\mathbb{R}} \pi > 0$, there is $A > 0$ such that

$$u_\infty(t,x) \geq (1-\delta)\pi(x) \quad \text{for all } (t,x) \in \mathbb{R}^2 \text{ such that } x - c^*t \leq -A. \tag{6.11}$$

As far as the region $\{x - c^*t \geq -A\}$ is concerned, we claim that there is $\overline{\xi} \in \mathbb{R}$ such that

$$v^\xi(t,x) = u_\infty(t+\xi, x+1) \geq u_\infty(t,x) \quad \text{for } x - c^*t \geq -A \text{ and } \xi \geq \overline{\xi}. \tag{6.12}$$

Assume not. Then there exist some sequences $(\xi_n)_{n\in\mathbb{N}}$ in $[0,\infty)$ and $(t_n,x_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^2$ such that $\lim_{n\to\infty} \xi_n = \infty$, while $x_n - c^*t_n \geq -A$ and $u_\infty(t_n+\xi_n, x_n+1) = v^{\xi_n}(t_n,x_n) < u_\infty(t_n,x_n)$ for all $n \in \mathbb{N}$. By (6.8)–(6.10), the sequence $(x_n - c^*t_n - c^*\xi_n)_{n\in\mathbb{N}}$ is bounded from below by a constant M. Thus, (6.9) and (6.10) provide the existence of some positive constants $\widetilde{\kappa}$ and $\widetilde{\rho}$ such that

$$\begin{aligned} &\widetilde{\kappa}(x_n - c^*t_n - c^*\xi_n - M + 1)e^{-\lambda^*(x_n - c^*t_n - c^*\xi_n)} \\ &\quad \leq u_\infty(t_n+\xi_n, x_n+1) < u_\infty(t_n,x_n) \leq \widetilde{\rho}(x_n - c^*t_n + A + 1)e^{-\lambda^*(x_n - c^*t_n)} \end{aligned} \tag{6.13}$$

for all $n \in \mathbb{N}$. On the other hand,

$$\begin{aligned} x_n - c^*t_n + A + 1 &= (x_n - c^*t_n - c^*\xi_n - M + 1) + (c^*\xi_n + M + A) \\ &\leq 2(x_n - c^*t_n - c^*\xi_n - M + 1)(c^*\xi_n + M + A) \end{aligned}$$

for n large enough. Putting this into (6.13) and letting $n \to \infty$ (with $\xi_n \to \infty$ as $n \to \infty$) leads to a contradiction. Thus, the claim (6.12) is proved.

Without loss of generality, one can assume that $\overline{\xi} \geq 1/c^*$. In this paragraph, we fix ξ in $[\overline{\xi},\infty)$. Set

$$\varepsilon^* = \min\{\varepsilon \geq 0 : (1+\varepsilon)v^\xi(t,x) \geq u_\infty(t,x) \text{ for all } (t,x) \in \mathbb{R}^2 \text{ such that } x - c^*t \leq -A\}.$$

Notice first that v^ξ is bounded from below by a positive constant in the region $\{x - c^*t \le -A\}$ by (6.8) and (6.10), while u_∞ is bounded from above, whence ε^* is a nonnegative real number. Let us prove that $\varepsilon^* = 0$. Assume that $\varepsilon^* > 0$. Since u_∞ is globally Lipschitz continuous and since $v^\xi \ge u_\infty$ on $\{x - c^*t = -A\}$ by (6.12), and both functions $v^\xi(t,x)$ and $u_\infty(t,x)$ converge to $\pi(x)$ as $x - c^*t \to -\infty$, there are a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ of positive real numbers, a sequence $(t_n, x_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^2$ and a real number $y_\infty < -A$ such that

$$\varepsilon_n \to \varepsilon^*,\ x_n - c^*t_n \to y_\infty \text{ as } n \to \infty \text{ and } (1+\varepsilon_n)v^\xi(t_n, x_n) < u_\infty(t_n, x_n) \text{ for all } n \in \mathbb{N}.$$

Without loss of generality, one can also assume that $x_n - [x_n] \to x_\infty$ and $t_n - [x_n]/c^* \to \tau$ as $n \to \infty$, with $y_\infty = x_\infty - c^*\tau$. Up to extraction of a subsequence, the functions $U_n(t,x) = u_\infty(t + [x_n]/c^*, x + [x_n])$ converge locally uniformly in $\mathbb{R}^2$ to a solution $0 \le U_\infty \le \pi$ of (6.7) satisfying (6.8) and (6.9). Set $V^\xi(t,x) = U_\infty(t+\xi, x+1)$ for $(t,x) \in \mathbb{R}^2$. Then $(1+\varepsilon^*)V^\xi(t,x) \ge U_\infty(t,x)$ for all $(t,x) \in \mathbb{R}^2$ with $x - c^*t \le -A$, with equality at the point (τ, x_∞) such that $x_\infty - c^*\tau = y_\infty < -A$. On the other hand, for all $(t,x) \in \mathbb{R}^2$ with $x - c^*t \le -A$, we have $(1+\varepsilon^*)V^\xi(t,x) \ge V^\xi(t,x) \ge (1-\delta)\pi(x)$ from (6.11), the definition of the functions V^ξ and U_n, and the assumption $\xi \ge 1/c^*$. Consequently,

$$(1+\varepsilon^*)V^\xi_t(t,x) - (1+\varepsilon^*)V^\xi_{xx}(t,x) = (1+\varepsilon^*)f(x, V^\xi(t,x)) \ge f(x, (1+\varepsilon^*)V^\xi(t,x))$$

for all $(t,x) \in \mathbb{R}^2$ such that $x - c^*t \le -A$, since $s \mapsto f(x,s)/s$ is nonincreasing on $[(1-\delta)\pi(x), \infty)$ for every $x \in \mathbb{R}$. Since U_∞ solves (6.7), it follows from the strong parabolic maximum principle that $(1+\varepsilon^*)V^\xi(t,x) = U_\infty(t,x)$ for all $(t,x) \in \mathbb{R}^2$ such that $t \le \tau$ and $x - c^*t \le -A$. The positivity of ε^* is in contradiction with the fact that $V^\xi(t,x)$ and $U_\infty(t,x)$ converge to $\pi(x) > 0$ uniformly as $x - c^*t \to -\infty$. Therefore, $\varepsilon^* = 0$, whence

$$v^\xi(t,x) \ge u_\infty(t,x) \quad \text{for all } (t,x) \in \mathbb{R}^2 \text{ such that } x - c^*t \le -A. \tag{6.14}$$

Together with (6.12), one finally gets $v^\xi \ge u_\infty$ in $\mathbb{R}^2$ for all $\xi \ge \overline{\xi}$.

Set now $\xi_* = \min\{\xi \in \mathbb{R} : v^{\xi'} \ge u_\infty \text{ in } \mathbb{R}^2 \text{ for all } \xi' \ge \xi\}$, which is a well-defined real number such that $\xi_* \le \overline{\xi}$ (notice that $v^\xi(t,x) \to 0$ as $\xi \to -\infty$ for each fixed $(t,x) \in \mathbb{R}^2$, while $u_\infty > 0$ in $\mathbb{R}^2$). Our goal is to prove that $\xi_* \le 1/c^*$, which will then yield $v^{1/c^*} \ge u_\infty$, and a symmetric argument will then give the desired conclusion. Assume then by way of contradiction that $\xi_* > 1/c^*$. Remember that $v^{\xi_*} \ge u_\infty$ in $\mathbb{R}^2$ by definition of ξ_*. We first claim that, for any $a \le b$ in $\mathbb{R}$,

$$\inf_{(t,x)\in\mathbb{R}^2,\, a \le x - c^*t \le b} (v^{\xi_*}(t,x) - u_\infty(t,x)) > 0. \tag{6.15}$$

Otherwise, by a usual limiting argument, there would exist a solution $0 \le U_\infty \le \pi$ of (6.7) satisfying (6.8) and (6.9), and such that $U_\infty(t + \xi_*, x+1) \ge U_\infty(t,x)$ for all $(t,x) \in \mathbb{R}^2$, with equality somewhere. From the strong maximum principle and the uniqueness of the solutions of the Cauchy problem associated with (6.7), it would then follow that $U_\infty(t+\xi_*, x+1) = U_\infty(t,x)$ for all $(t,x) \in \mathbb{R}^2$ and then $U_\infty(t+k\xi_*, x+k) =$

$U_\infty(t,x)$ in $\mathbb{R}^2$ for all $k \in \mathbb{N}$. Since we have assumed that $\xi_* > 1/c^*$ and since U_∞ satisfies (6.8), the limit as $k \to \infty$ implies that $U_\infty(t,x) = \pi(x)$ for all $(t,x) \in \mathbb{R}^2$, which is clearly impossible, because of property (6.9) satisfied by U_∞.

Therefore, (6.15) holds. In particular, since u_∞ is Lipschitz, there is $\underline{\xi} \in (1/c^*, \xi_*)$ such that

$$v^\xi(t,x) \geq u_\infty(t,x) \quad \text{for all } (t,x) \in \mathbb{R}^2 \text{ with } x - c^*t = -A \text{ and all } \xi \in [\underline{\xi}, \xi_*].$$

Furthermore, $v^\xi(t,x) \geq (1-\delta)\pi(x)$ for all $(t,x) \in \mathbb{R}^2$ with $x - c^*t \leq -A$ and all $\xi \in [\underline{\xi}, \xi_*] \subset [1/c^*, \infty)$, from (6.11) and the definition of v^ξ. As in the proof of (6.14), it then follows that

$$v^\xi(t,x) \geq u_\infty(t,x) \quad \text{for all } (t,x) \in \mathbb{R}^2 \text{ with } x - c^*t \leq -A \text{ and all } \xi \in [\underline{\xi}, \xi_*]. \tag{6.16}$$

On the other hand, the definition of ξ_* implies that there exist a sequence $(\xi_n)_{n\in\mathbb{N}}$ in $(\xi_* - 1, \xi_*)$ and a sequence $(t_n, x_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^2$ such that

$$\xi_n \to \xi_* \text{ as } n \to \infty \quad \text{and} \quad v^{\xi_n}(t_n, x_n) < u_\infty(t_n, x_n) \text{ for all } n \in \mathbb{N}. \tag{6.17}$$

Property (6.16) yields $x_n - c^*t_n > -A$ for all n large enough, and (6.15) and (6.17) imply then that $x_n - c^*t_n \to \infty$ as $n \to \infty$. Up to extraction of a subsequence, one can assume that $x_n - [x_n] \to x_\infty \in [0,1]$ as $n \to \infty$.

Define now

$$U_n(t,x) = \frac{u_\infty(t+t_n, x+[x_n])}{u_\infty(t_n, [x_n])} \quad \text{and} \quad V_n(t,x) = \frac{v^{\xi_*}(t+t_n, x+[x_n])}{u_\infty(t+t_n, x+[x_n])}$$

for all $(t,x) \in \mathbb{R}^2$ and $n \in \mathbb{N}$. From (6.9) and $\lim_{n\to\infty}(x_n - c^*t_n) = \infty$, it follows that the sequences $(U_n)_{n\in\mathbb{N}}$ and $(V_n)_{n\in\mathbb{N}}$ are bounded in $L^\infty_{loc}(\mathbb{R}^2)$. From standard parabolic estimates and the fact that $u_\infty(t_n, [x_n]) \to 0$ as $n \to \infty$, the functions U_n converge locally uniformly in $\mathbb{R}^2$, up to extraction of a subsequence, to a nonnegative classical solution U_∞ of $(U_\infty)_t = (U_\infty)_{xx} + g(x)U_\infty$ in $\mathbb{R}^2$. Furthermore, $(U_n)_x \to (U_\infty)_x$ locally uniformly in $\mathbb{R}^2$ as $n \to \infty$ and $U_\infty(0,0) = 1$, whence $U_\infty > 0$ in $\mathbb{R}^2$ from the maximum principle. In particular, the functions

$$\frac{(u_\infty)_x(t+t_n, x+[x_n])}{u_\infty(t+t_n, x+[x_n])} = \frac{(U_n)_x(t,x)}{U_n(t,x)}$$

are locally bounded. As far as the functions V_n are concerned, they obey

$$\begin{aligned}(V_n)_t(t,x) = (V_n)_{xx}(t,x) + 2\frac{(U_n)_x(t,x)}{U_n(t,x)}(V_n)_x(t,x) \\ + \frac{f(x, u_\infty(t+t_n, x+[x_n])V_n(t,x))}{u_\infty(t+t_n, x+[x_n])} - \frac{f(x, u_\infty(t+t_n, x+[x_n]))}{u_\infty(t+t_n, x+[x_n])}V_n(t,x)\end{aligned}$$

in $\mathbb{R}^2$. Since $(U_n)_x/U_n \to (U_\infty)_x/U_\infty$ and $u_\infty(t+t_n, x+[x_n]) \to 0$ locally uniformly in $\mathbb{R}^2$ as $n \to \infty$, and since the functions V_n are locally bounded, it follows from standard parabolic estimates that, up to extraction of a subsequence, the functions V_n converge locally uniformly in $\mathbb{R}^2$ to a classical solution V_∞ of

$$(V_\infty)_t = (V_\infty)_{xx} + 2\frac{(U_\infty)_x}{U_\infty}(V_\infty)_x \quad \text{in } \mathbb{R}^2. \tag{6.18}$$

Owing to the definitions of V_n and ξ_*, one has $V_n \geq 1$, whence $V_\infty \geq 1$ in $\mathbb{R}^2$. On the other hand,

$$V_n(\xi_n-\xi_*, x_n-[x_n]) = \frac{v^{\xi_n}(t_n,x_n)}{u_\infty(t_n,x_n)} \times \frac{U_n(0,x_n-[x_n])}{U_n(\xi_n-\xi_*,x_n-[x_n])} \leq \frac{U_n(0,x_n-[x_n])}{U_n(\xi_n-\xi_*,x_n-[x_n])}$$

from (6.17). By passing to the limit as $n \to \infty$, one infers that $V_\infty(0,x_\infty) \leq 1$. Finally, $V_\infty(0,x_\infty) = 1$. Therefore, $V_\infty = 1$ in $\mathbb{R}^2$ from the strong parabolic maximum principle and the uniqueness of the Cauchy problem associated with (6.18).

Thus

$$\frac{u_\infty(t+t_n+\xi_*, x+[x_n]+1)}{u_\infty(t+t_n,x+[x_n])} = \frac{v^{\xi_*}(t+t_n,x+[x_n])}{u_\infty(t+t_n,x+[x_n])} \to 1 \quad \text{locally uniformly in } \mathbb{R}^2$$

as $n \to \infty$. It follows by immediate induction that, for each $p \in \mathbb{N}$,

$$\frac{u_\infty(t+t_n+p\xi_*, x+[x_n]+p)}{u_\infty(t+t_n,x+[x_n])} \to 1 \quad \text{locally uniformly in } \mathbb{R}^2 \text{ as } n \to \infty.$$

Fix $p \in \mathbb{N}$. Property (6.9) and the limit $\lim_{n\to\infty}(x_n - c^* t_n) = \infty$ imply that, for n large enough,

$$\frac{u_\infty(t_n+p\xi_*, [x_n]+p)}{u_\infty(t_n,[x_n])} \geq \frac{\kappa([x_n]+p-c^*t_n-pc^*\xi_*-1)e^{-\lambda^*([x_n]+p-c^*t_n-pc^*\xi_*)}}{\rho([x_n]-c^*t_n)e^{-\lambda^*([x_n]-c^*t_n-1)}}.$$

By letting $n \to \infty$, one gets $1 \geq (\kappa/\rho)e^{p\lambda^*(c^*\xi_*-1)-\lambda^*}$. Since this inequality holds for all $p \in \mathbb{N}$ and since it was assumed that $\xi_* > 1/c^*$, this leads to a contradiction. We conclude that $\xi_* \leq 1/c^*$, whence $v^{1/c^*} \geq u_\infty$ in $\mathbb{R}^2$.

By sliding $u_\infty(t,x+1)$ in the other t-direction, one can prove similarly that $v^\xi \leq u_\infty$ in $\mathbb{R}^2$ for all $\xi \leq \xi_-$ for some real number ξ_-, and that the largest such ξ cannot be smaller than $1/c^*$. Therefore, $v^{1/c^*} \leq u_\infty$ in $\mathbb{R}^2$.

Finally, $v^{1/c^*} = u_\infty$ in $\mathbb{R}^2$, that is, $u_\infty(t+1/c^*, x+1) = u_\infty(t,x)$ for all $(t,x) \in \mathbb{R}^2$. In other words, u_∞ is a pulsating front with speed c^*, connecting 0 and $\pi(x)$. The conclusion follows from the uniqueness up to time shifts of the pulsating fronts, for a given speed (see [21]). The proof of Lemma 6.2 is thereby complete. □

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