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Exceptional collections on isotropic Grassmannians

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Abstract. We introduce a new construction of exceptional objects in the derived category of coherent sheaves on a compact homogeneous space of a semisimple algebraic group and show that it produces exceptional collections of the length equal to the rank of the Grothendieck group on homogeneous spaces of all classical groups.

Keywords. Exceptional collection, derived category of sheaves, isotropic Grassmannian

Contents

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1. Introduction

The study of derived categories of coherent sheaves on algebraic varieties has been an increasingly popular subject in algebraic geometry. One of important devices relevant for this study is the notion of an exceptional collection (see 1.1 below). In the present paper we give a new general construction of such collections in the derived categories of compact homogeneous spaces of semisimple algebraic groups and show that for classical groups it gives exceptional collections of maximal length.

1.1. An overview of exceptional collections on homogeneous varieties. Let k be a base field which we assume to be algebraically closed of characteristic 0. Recall that an object E of a k-linear triangulated category $\mathcal{T}$ is *exceptional* if

$$\mathsf{Ext}^\bullet(E, E) = \mathsf{k}$$

(that is, E is simple and has no higher self-Ext's). An ordered collection $E_1, \dots, E_m$ in $\mathcal{T}$ is an *exceptional collection* if each E_i is exceptional and

$$\mathsf{Ext}^\bullet(E_i, E_j) = 0$$

for all $i > j$. Finally, an exceptional collection $E_1, \dots, E_m$ is *full* if the smallest triangulated subcategory of $\mathcal{T}$ containing all the objects $E_1, \dots, E_m$ is $\mathcal{T}$ itself.

The simplest geometrical example of a full exceptional collection is the collection

$$\mathcal{O}, \mathcal{O}(1), \dots, \mathcal{O}(n-1), \mathcal{O}(n)$$

in the bounded derived category $\mathcal{D}(\mathbb{P}^n)$ of coherent sheaves on $\mathbb{P}^n$ constructed by Beilinson in his pioneering work [Bei]. Later a large number of exceptional collections were constructed by Kapranov [Kap]. In fact, he constructed full exceptional collections of vector bundles on all homogeneous spaces of simple algebraic groups of type A and on quadrics (which are special homogeneous spaces of types B and D). This naturally led to the following conjecture.

Conjecture 1.1. *If* $\mathbf{G}$ *is a semisimple algebraic group and* $\mathbf{P} \subset \mathbf{G}$ *is a parabolic subgroup of* $\mathbf{G}$ *then there is a full exceptional collection of vector bundles in* $\mathcal{D}(\mathbf{G}/\mathbf{P})$.

Up to now only partial results in this direction have been obtained. Below we list all minimal homogeneous varieties of simple groups (corresponding to maximal parabolic subgroups) for which a full exceptional collection was constructed. Recall that simple algebraic groups are classified by Dynkin diagrams that fall into types A, B, C, D, E, F and G. Maximal parabolic subgroups correspond to vertices of Dynkin diagrams for which we use the standard numbering (see [Bou]). Thus, we denote by $\mathbf{P}_i$ the maximal parabolic subgroup corresponding to the vertex i.

Type A_n: A full collection was constructed by Kapranov [Kap].

Type B_n: For $\mathbf{P} = \mathbf{P}_1$ (so that $\mathbf{G}/\mathbf{P} = Q^{2n-1}$, a quadric of dimension $2n-1$) a full exceptional collection was constructed by Kapranov [Kap]. For $\mathbf{P} = \mathbf{P}_2$ (so that $\mathbf{G}/\mathbf{P} = \mathsf{OGr}(2, 2n+1)$, the Grassmannian of lines on Q^{2n-1}) a full exceptional collection was constructed in [K08]. For $n = 4$ and $\mathbf{P} = \mathbf{P}_4$ (so that $\mathbf{G}/\mathbf{P} = \mathsf{OGr}(4, 9) = \mathsf{OGr}_+(5, 10)$) a full exceptional collection was constructed in [K06].

Type C_n: For $\mathbf{P} = \mathbf{P}_1$ (so that $\mathbf{G}/\mathbf{P} = \mathbb{P}^{2n-1}$) Beilinson's collection works. For $\mathbf{P} = \mathbf{P}_2$ (so that $\mathbf{G}/\mathbf{P} = \mathsf{SGr}(2, 2n)$, the Grassmannian of isotropic planes in a symplectic vector space) a full exceptional collection was constructed in [K08]. For $n = 3, 4, 5$ and $\mathbf{P} = \mathbf{P}_n$ (so that $\mathbf{G}/\mathbf{P} = \mathsf{SGr}(n, 2n)$, the Lagrangian Grassmannian) full exceptional collections were constructed in [S01] and [PS].

Type D_n: For $\mathbf{P} = \mathbf{P}_1$ (so that $\mathbf{G}/\mathbf{P} = Q^{2n-2}$, a quadric of dimension $2n-2$) a full exceptional collection was constructed by Kapranov [Kap]. For $\mathbf{P} = \mathbf{P}_2$ (so that $\mathbf{G}/\mathbf{P} = \mathsf{OGr}(2, 2n)$, the Grassmannian of isotropic lines on Q^{2n-2}) an almost full exceptional collection was constructed in [K08].

Type E_n: For $n = 6$ and $\mathbf{P} = \mathbf{P}_1$ (or $\mathbf{P} = \mathbf{P}_6$) an exceptional collection was constructed by Manivel [Man]. The collection was proved to be full in [FM].

Type F_4: For $\mathbf{P} = \mathbf{P}_4$ (so that $\mathbf{G}/\mathbf{P}$ is a hyperplane section of $E_6/\mathbf{P}_1$) an exceptional collection can be constructed by restricting Manivel's collection.

Type G_2: For $\mathbf{P} = \mathbf{P}_1$ (so that $\mathbf{G}/\mathbf{P} = Q^5$) Kapranov's collection works. For $\mathbf{P} = \mathbf{P}_2$ a full exceptional collection was constructed in [K06].

1.2. The statement of results. The main result of the present paper can be formulated as follows. Let us say that an exceptional collection in $\mathcal{D}(X)$, the bounded derived category of coherent sheaves on an algebraic variety X, is *of expected length* if its length is equal to the rank of the Grothendieck group, $\mathsf{rk}(K_0(X))$. Note that if $K_0(X)$ is a free abelian group then this implies that the corresponding classes generate $K_0(X)$.

Let us say that a simple group $\mathbf{G}$ is of *type BCD* if its type is B_n, C_n, or D_n, and is *classical* if its type is A_n, B_n, C_n, or D_n.

Theorem 1.2. *Let $\mathbf{G}$ be a simply connected simple group of type BCD. Then for each maximal parabolic subgroup $\mathbf{P} \subset \mathbf{G}$ there exists an exceptional collection of expected length in $\mathcal{D}(\mathbf{G}/\mathbf{P})$ consisting of objects that have a $\mathbf{G}$-equivariant structure.*

Note that the existence of a $\mathbf{G}$-equivariant structure here is a general result (see [P11, Lem. 2.2]) but also comes naturally from the construction. The $\mathbf{G}$-equivariant structure on objects of our collections allows one to construct a relative exceptional collection on any fibration with fiber $\mathbf{G}/\mathbf{P}$ (see [S07, Thm. 3.1]).

Corollary 1.3. *Let $\mathbf{G}$ and $\mathbf{P}$ be as in Theorem 1.2, and let $\mathcal{G} \to X$ be a principal $\mathbf{G}$-bundle over an arbitrary algebraic variety X. Consider the corresponding fibration $Y = \mathcal{G} \times_{\mathbf{G}} (\mathbf{G}/\mathbf{P}) \to X$. Then there exists a semiorthogonal decomposition of $\mathcal{D}^b(Y)$ consisting of $\mathsf{rk}(K_0(\mathbf{G}/\mathbf{P}))$ subcategories, each equivalent to $\mathcal{D}^b(X)$, and possibly an additional subcategory. In particular, if X has an exceptional collection of expected length then so does Y.*

Both Theorem 1.2 and Corollary 1.3 will be proved in Section 9.5.

Note that for an arbitrary (not maximal) parabolic subgroup $\mathbf{P} \subset \mathbf{G}$ the homogeneous space $\mathbf{G}/\mathbf{P}$ has a structure of an iterated fibration with fibers of the form $\mathbf{G}_i/\mathbf{P}_i$, where $\mathbf{G}_i$ are semisimple algebraic groups and $\mathbf{P}_i \subset \mathbf{G}_i$ are maximal parabolic subgroups. Moreover, if $\mathbf{G}$ is a classical group then all $\mathbf{G}_i$ are classical as well. So, applying Corollary 1.3 (or Kapranov's construction in type A) several times we conclude that

Corollary 1.4. *If $\mathbf{G}$ is a simple group of type BCD and $\mathbf{P} \subset \mathbf{G}$ is a (not necessarily maximal) parabolic subgroup then there exists an exceptional collection of expected length in $\mathcal{D}(\mathbf{G}/\mathbf{P})$.*

We conjecture that the exceptional collections we construct are full and possess further nice properties that we checked in some special cases (see Conjecture 1.9).

Finally, we would like to stress that our construction of an exceptional collection is quite general: we use special properties of types BCD only in some computations. So, we hope that the approach of this paper can be used to construct full exceptional collections for all the remaining homogeneous spaces (i.e., for the exceptional groups E_6, E_7, E_8 and F_4).

1.3. An overview of the construction. The main part of any construction of an exceptional collection is to find sufficiently many exceptional objects. For a homogeneous variety it is natural to try equivariant bundles.

Note that when we fix the type of a simple group we have several choices of the group itself, ranging from simply connected to adjoint cases. The simply connected group has the richest category of equivariant bundles. On the other hand, the variety $\mathbf{G}/\mathbf{P}$ does not change if we replace $\mathbf{G}$ by its simply connected covering. Because of this *from now on we will assume that $\mathbf{G}$ is simply connected.*

Recall that there is a natural equivalence of the category of $\mathbf{G}$-equivariant coherent sheaves on $\mathbf{G}/\mathbf{P}$ with the category of representations of $\mathbf{P}$:

$$\mathsf{Coh}^{\mathbf{G}}(\mathbf{G}/\mathbf{P}) \cong \mathsf{Rep}\,\mathbf{P}$$

(see [BK90]). In fact, it is an equivalence of tensor abelian categories. In particular, each representation of $\mathbf{P}$ can be considered as a vector bundle on $X = \mathbf{G}/\mathbf{P}$. The group $\mathbf{P}$ is not reductive, so its representation theory is rather complicated. Let us start by considering the semisimple part of the category, $\mathsf{Rep}^{\mathrm{ss}}\,\mathbf{P}$, i.e., the subcategory of representations on which the unipotent radical $\mathbf{U}$ of $\mathbf{P}$ acts trivially. Thus, if

$$\mathbf{L} = \mathbf{P}/\mathbf{U}$$

is the Levi quotient, then extending a representation of $\mathbf{L}$ to a representation of $\mathbf{P}$ via the projection $\mathbf{P} \to \mathbf{L}$ we get an equivalence $\mathsf{Rep}\,\mathbf{L} \cong \mathsf{Rep}^{\mathrm{ss}}\,\mathbf{P}$. The Levi group $\mathbf{L}$ is reductive, and its weight lattice $P_{\mathbf{L}}$ is canonically isomorphic to the weight lattice $P_{\mathbf{G}}$ of the group $\mathbf{G}$. Let us choose a maximal torus $\mathbf{T} \subset \mathbf{L}$ and a Borel subgroup $\mathbf{B}$ in $\mathbf{P}$ containing $\mathbf{T}$ such that $\mathbf{B} \cap \mathbf{L}$ is a Borel subgroup in $\mathbf{L}$. We denote the corresponding cones of $\mathbf{L}$-dominant and $\mathbf{G}$-dominant weights by $P^+_{\mathbf{L}} \subset P_{\mathbf{L}}$ and $P^+_{\mathbf{G}} \subset P_{\mathbf{G}}$, respectively. Irreducible representations of $\mathbf{L}$ are parameterized by their highest weights which are $\mathbf{L}$-dominant. For each $\mathbf{L}$-dominant weight $\lambda \in P^+_{\mathbf{L}}$ we denote by $V^\lambda_{\mathbf{L}}$ the corresponding irreducible representation of $\mathbf{L}$, as well as its extension to $\mathbf{P}$, and by $\mathcal{U}^\lambda$ the corresponding $\mathbf{G}$-equivariant bundle on $X = \mathbf{G}/\mathbf{P}$.

In type A there are sufficiently many exceptional bundles among the $\mathcal{U}^\lambda$'s, so one can construct an exceptional collection of expected length out of them. However, for other types the situation is not so nice. Although all the bundles $\mathcal{U}^\lambda$ are exceptional as objects of the derived category of equivariant sheaves $\mathcal{D}^{\mathbf{G}}(X)$, it turns out that only few of them are exceptional in $\mathcal{D}(X)$. For example, in the case when $\mathbf{G}$ is of type C_n and $\mathbf{P} = \mathbf{P}_n$, so that $X = \mathsf{SGr}(n, 2n)$ (the Lagrangian Grassmannian), one can check that $\mathcal{U}^\lambda$ is exceptional if and only if

$$\lambda = \omega_i + t\omega_n,$$

where ω_i is the fundamental weight of the vertex i of the Dynkin diagram and $t \in \mathbb{Z}$. Since the canonical bundle is $\omega_X = \mathcal{U}^{-(n+1)\omega_n}$, one can deduce easily that the maximal possible length of an exceptional collection in $\mathcal{D}(X)$ consisting of vector bundles of the form $\mathcal{U}^\lambda$ is $n(n+1)$ (we have n choices for i and $n+1$ choices for t in the above formula for λ), whereas $\mathsf{rk}(K_0(X)) = 2^n$. So, for $n \geq 5$ we have no chance to find an exceptional collection of expected length consisting only of $\mathcal{U}^\lambda$'s. In other words, we need to introduce another class of $\mathbf{P}$-modules. In fact, this is the most interesting problem discussed in this paper.

To explain how we do it let us return to the example of the group $\mathbf{G}$ of type C_n and of $\mathbf{P} = \mathbf{P}_n$. Recall that in this case the lattice of weights is

$$P_{\mathbf{L}} = P_{\mathbf{G}} = \mathbb{Z}^n = \{(\lambda_1, \dots, \lambda_n)\},$$

and the dominant cones can be described as

$$P^+_{\mathbf{G}} = \{\lambda_1 \geq \cdots \geq \lambda_n \geq 0\}, \qquad P^+_{\mathbf{L}} = \{\lambda_1 \geq \cdots \geq \lambda_n\}$$

(the Levi group $\mathbf{L}$ in this case is isomorphic to GL_n). Take any integer $0 \le a \le n$ and consider a subset (a *block*)

$$\mathsf{B}_a = \{n \ge \lambda_1 \ge \dots \ge \lambda_a \ge \lambda_{a+1} = \dots = \lambda_n = a\}.$$

Its elements can be viewed as Young diagrams inscribed in an $(n-a) \times a$ rectangle. In particular,

$$\#\mathsf{B}_a = \binom{n}{a}.$$

It turns out that for the weights λ, μ within such a block $\mathsf{B} = \mathsf{B}_a$ the following amusing property is satisfied: the canonical map

$$\bigoplus_{\nu \in \mathsf{B}} \mathsf{Ext}^\bullet_{\mathbf{G}}(\mathcal{U}^\lambda, \mathcal{U}^\nu) \otimes \mathsf{Hom}(\mathcal{U}^\nu, \mathcal{U}^\mu) \to \mathsf{Ext}^\bullet(\mathcal{U}^\lambda, \mathcal{U}^\mu) \qquad (\star)$$

is an isomorphism (here $\mathsf{Ext}_{\mathbf{G}}$ stands for the Ext groups in the derived category $\mathcal{D}^{\mathbf{G}}(X)$ of $\mathbf{G}$-equivariant coherent sheaves on X, and the map is given by the composition of equivariant Ext's with Hom's).

As already mentioned above, all the objects $\mathcal{U}^\lambda$ are exceptional when considered as objects of the derived category $\mathcal{D}^{\mathbf{G}}(X)$ of equivariant sheaves (and in fact form an exceptional collection), while when considered as objects of $\mathcal{D}(X)$ (by forgetting the equivariant structure), they are not exceptional in general. Now, having property $(\star)$ one can formally check that

- considering $\{\mathcal{U}^\lambda\}_{\lambda \in \mathsf{B}_a}$ as a (nonfull) exceptional collection in $\mathcal{D}^{\mathbf{G}}(X)$,
- passing to the *right dual exceptional collection* $\{\mathcal{E}^\lambda\}_{\lambda \in \mathsf{B}_a}$ in $\mathcal{D}^{\mathbf{G}}(X)$, and then
- forgetting the equivariant structure on all $\mathcal{E}^\lambda$,

one obtains an exceptional collection $\{\mathcal{E}^\lambda\}_{\lambda \in \mathsf{B}_a}$ in the nonequivariant category $\mathcal{D}(X)$ that generates the same subcategory as the original (nonexceptional) collection $\{\mathcal{U}^\lambda\}$. This strange procedure (see details in Section 3) can be considered as the central construction of the paper. To make it work in general we introduce the notion of an exceptional block. By definition, an *exceptional block* is a subset $\mathsf{B} \subset P^+_{\mathbf{L}}$ of $\mathbf{L}$-dominant weights such that the morphism $(\star)$ is an isomorphism. The procedure described above produces an exceptional collection $\{\mathcal{E}^\lambda\}_{\lambda \in \mathsf{B}}$ generating the subcategory

$$\mathcal{A}_{\mathsf{B}} := \langle \mathcal{U}^\lambda \rangle_{\lambda \in \mathsf{B}}.$$

However, in general one cannot find a single exceptional block of expected length. To obtain an exceptional collection of expected length we combine several exceptional blocks in a semiorthogonal sequence of blocks, i.e. with vanishing Ext's between blocks in the order-decreasing direction. For example, for $\mathbf{G}$ of type C_n and $\mathbf{P} = \mathbf{P}_n$ we take the blocks B_a described above for all a from 0 to n. Note that the total number of exceptional objects in the blocks B_a is $\sum_{a=0}^n \binom{n}{a} = 2^n$, which is the expected length in this case.

1.3.1. The choices and the restriction. Now let us describe the construction in the general case. The details can be found in Section 5. The construction depends on several choices (subject to one restriction) that we are going to explain now. Let $D = D_{\mathbf{G}}$ be the Dynkin diagram of $\mathbf{G}$. Denote by β the simple root (a vertex of D) corresponding to the maximal parabolic subgroup $\mathbf{P}$, and by ξ the corresponding fundamental weight of $\mathbf{G}$. The first choice is the following.

(C1) We choose a connected component of $D\backslash\beta$, called the *outer component* and denoted by $D_{\mathbf{out}}$. We also allow $D_{\mathbf{out}}$ to be empty.

The restriction is

(R) If $D_{\mathbf{out}}$ is nonempty then it is a Dynkin diagram of type A.

We denote the complement of β and $D_{\mathbf{out}}$ by $D_{\mathbf{inn}}$,

$$D_{\mathbf{inn}} = D_{\mathbf{G}} \setminus (D_{\mathbf{out}} \cup \beta),$$

and call it the *inner component* of $D_{\mathbf{G}}$. We consider the simply connected subgroups

$$\mathbf{L}_{\mathbf{out}}, \mathbf{L}_{\mathbf{inn}} \subset \mathbf{L}$$

corresponding to the subdiagrams $D_{\mathbf{out}}, D_{\mathbf{inn}} \subset D \setminus \beta = D_{\mathbf{L}}$ and denote by

$$i : \mathbf{L}_{\mathbf{inn}} \to \mathbf{L}, \qquad o : \mathbf{L}_{\mathbf{out}} \to \mathbf{L}$$

the embeddings. Abusing the notation we denote the embeddings of these subgroups into $\mathbf{G}$ by the same letters. Our restriction on $D_{\mathbf{out}}$ means that $\mathbf{L}_{\mathbf{out}} \simeq \mathsf{SL}_k$ for some $k \geq 1$.

The next choice is the following.

(C2) We choose a standard numbering of vertices in $D_{\mathbf{out}}$.

Since $D_{\mathbf{out}}$ is of type A, there are two possibilities for this choice (unless $D_{\mathbf{out}}$ is empty or consists of one vertex). Let b be the number corresponding to the vertex in $D_{\mathbf{out}}$ which is adjacent to β. The chain of vertices $1, 2, \dots, b$ of $D_{\mathbf{G}}$ will play an important role in the construction below.

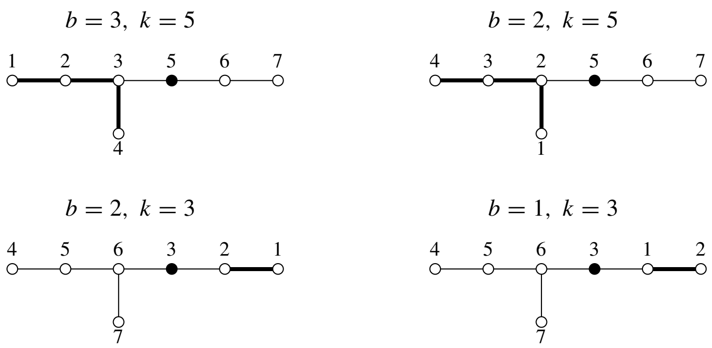

Fig. 1. Choices of the outer component and of the numbering

The possibilities of the choice of the outer component and of the numbering of its vertices are illustrated in Figure 1. We take the Dynkin diagram of type E_7; the black circle marks the vertex corresponding to the parabolic subgroup $\mathbf{P}$, and the thick lines mark the outer component of the diagram. So, there are four choices with nonempty outer component and the fifth choice (not illustrated in the picture) when $D_{\mathbf{out}}$ is empty.

We have the following decreasing chain of Dynkin subdiagrams in $\mathcal{D}_{\mathbf{G}}$:

$$D_a = D_{\mathbf{G}} \setminus \{1, \dots, a\}$$

for $a = 0, 1, \dots, b$ (so $D_0 = D_{\mathbf{G}}$). Let $h_a : \mathbf{H}_a \to \mathbf{G}$ be the embedding of the simply connected subgroup corresponding to the subdiagram D_a. Note that $\mathbf{L_{inn}} \subset \mathbf{H}_a$ since $D_{\mathbf{inn}} \subset D_a$, so the embedding $i : \mathbf{L_{inn}} \to \mathbf{G}$ factors through an embedding $\mathbf{L_{inn}} \to \mathbf{H}_a$ that we will also denote by i. If $\mathbf{K}$ is any of the groups $\mathbf{G}$, $\mathbf{L}$, $\mathbf{L_{inn}}$, $\mathbf{L_{out}}$, $\mathbf{H}_a$ then we denote by $P_{\mathbf{K}}$ (resp., $\mathbf{W_K}$) the corresponding weight lattice (resp., Weyl group).

The last choice is the following.

(C3) For each $a = 0, 1, \dots, b$ we choose a strictly dominant weight $\delta_a \in P^+_{\mathbf{H}_a}$.

For each $a = 0, 1, \dots, b$ we define a polyhedron in $P_{\mathbf{H}_a} \otimes \mathbb{R}$ by

$$\mathbf{R}_{\delta_a} = \{\lambda \in P_{\mathbf{H}_a} \otimes \mathbb{R} \mid \forall w \in \mathbf{W}_{\mathbf{H}_a}\ (\lambda, w\delta_a) \leq (\rho_{\mathbf{H}_a}, \delta_a)\},$$

where $\rho_{\mathbf{H}_a}$ is the sum of the fundamental weights of $\mathbf{H}_a$. We will refer to $\mathbf{R}_{\delta_a}$ as the *core* in $P_{\mathbf{H}_a} \otimes \mathbb{R}$.

1.3.2. The indexing set. The exceptional blocks that we construct are indexed by the set

$$\mathsf{J} = \{\mathsf{j} \in (\theta, P_{\mathbf{L}}) \mid 0 \leq \mathsf{j} < r\}.$$

Here θ is the unique element of $P_{\mathbf{L}} \otimes \mathbb{Q}$ such that

$$\theta \in \langle \omega_1, \dots, \omega_{k-1} \rangle^{\perp} \cap \operatorname{Ker} i^* \quad \text{and} \quad (\theta, \xi) = 1,$$

where ω_t is the fundamental weight of the vertex $t \in D_{\mathbf{G}}$, $i^* : P_{\mathbf{L}} \to P_{\mathbf{L_{inn}}}$ is the natural restriction map, and r is the index of the Grassmannian $\mathbf{G}/\mathbf{P}$ (the integer such that $\mathcal{U}^{-r\xi}$ is the canonical class of $\mathbf{G}/\mathbf{P}$). Note that the scalar product with θ defines a linear map $(\theta, -) : P_{\mathbf{L}} \to \mathbb{Q}$, its image $(\theta, P_{\mathbf{L}})$ is a finitely generated subgroup of $\mathbb{Q}$ containing $\mathbb{Z}$, so J is a finite totally ordered set.

1.3.3. The construction of the blocks. Recall that we have a chain of subgroups $\mathbf{H}_b \subset \dots \subset \mathbf{H}_1 \subset \mathbf{H}_0 = \mathbf{G}$. For each subgroup $\mathbf{H}_a$ denote by r_a the index of the Grassmannian $\mathbf{H}_a/(\mathbf{P} \cap \mathbf{H}_a)$. We prove that the sequence of integers r_a is strictly decreasing,

$$r = r_0 > r_1 > \dots > r_{b-1} > r_b > r_{b+1} := 0,$$

so it gives a subdivision of the indexing set $\mathsf{J} = \mathsf{J}_0 \sqcup \mathsf{J}_1 \sqcup \dots \sqcup \mathsf{J}_b$, where

$$\mathsf{J}_a = \{\mathsf{j} \in \mathsf{J} \mid r - r_a \leq \mathsf{j} < r - r_{a+1}\}.$$

We denote by a the function $\mathsf{J} \to \mathbb{Z}$ equal to a on J_a. In other words, it is defined by

$$r - r_{a(\mathsf{j})} \leq \mathsf{j} < r - r_{a(\mathsf{j})+1}.$$

For brevity we will write $\mathbf{H}_{\mathsf{j}} = \mathbf{H}_{a(\mathsf{j})}$, $h_{\mathsf{j}} = h_{a(\mathsf{j})}$ and $\mathbf{R}_{\mathsf{j}} = \mathbf{R}_{\delta_{a(\mathsf{j})}}$.

Now we are ready to describe the blocks. First, we construct for each $\mathsf{j} \in \mathsf{J}$ a block

$$\mathsf{B}_\mathsf{j} = \mathsf{B}^{\mathbf{out}}_\mathsf{j} + \mathsf{j}\xi + i_*(\mathsf{B}^{\mathbf{inn}}_\mathsf{j}) \subset P_\mathbf{L}$$

with $\mathsf{B}^{\mathbf{out}}_\mathsf{j} \subset \operatorname{Ker} h^*_\mathsf{j} = \langle \omega_1, \dots, \omega_{a(\mathsf{j})-1} \rangle$ (called the *outer part*), and $\mathsf{B}^{\mathbf{inn}}_\mathsf{j} \subset P_{\mathbf{L_{inn}}}$ (the *inner part*). The inner part is given by

$$\mathsf{B}^{\mathbf{inn}}_\mathsf{j} = \left\{ \nu \in P^+_{\mathbf{L_{inn}}} \;\middle|\; \begin{array}{l} (1)\ \rho_{\mathbf{H}_\mathsf{j}} \pm 2 i_*(w\nu) \in \mathbf{R}_\mathsf{j} \text{ for all } w \in \mathbf{W}_{\mathbf{L_{inn}}} \\ (2)\ \mathsf{j}\xi + i_*\nu \in P_\mathbf{L} \end{array} \right\},$$

and then the outer part is defined by

$$\mathsf{B}^{\mathbf{out}}_\mathsf{j} = \left\{ \mu \in \operatorname{Ker} h^*_\mathsf{j} \cap P^+_\mathbf{G} \;\middle|\; \begin{array}{l} \rho_{\mathbf{H}_\mathsf{j}} - h^*_\mathsf{j}(w_{\mathbf{L_{out}}}\mu) - i_*(w_{\mathbf{L_{inn}}}\nu) + i_*(w'_{\mathbf{L_{inn}}}\nu') \in \mathbf{R}_\mathsf{j} \\ \text{for all } \nu, \nu' \in \mathsf{B}^{\mathbf{inn}}_\mathsf{j},\ w_{\mathbf{L_{out}}} \in \mathbf{W}_{\mathbf{L_{out}}}, \text{ and } w_{\mathbf{L_{inn}}}, w'_{\mathbf{L_{inn}}} \in \mathbf{W}_{\mathbf{L_{inn}}} \end{array} \right\}.$$

Note that by definition of θ we have $(\theta, \mathsf{B}_\mathsf{j}) = \mathsf{j}$. So, the pairing with θ gives the ordering of the blocks.

We check that the blocks B_j constructed above are exceptional provided the group $\mathbf{G}$ is of type BCD (for other types one has to slightly modify the definition of the outer part; see details in Section 5.5). It follows that for each $\mathsf{j} \in \mathsf{J}$ the subcategory $\langle \mathcal{U}^\lambda \rangle_{\lambda \in \mathsf{B}_\mathsf{j}}$ is generated by an exceptional collection.

1.3.4. Modification of the blocks and the main result. It turns out that the subcategories $\langle \mathcal{U}^\lambda \rangle_{\lambda \in \mathsf{B}_\mathsf{j}}$ are not semiorthogonal, so we have to make our blocks slightly smaller. Let $\mathbf{R}^*_\mathsf{j}$ denote the interior of the core $\mathbf{R}_\mathsf{j}$. We define subsets $\bar{\mathsf{B}}^{\mathbf{inn}}_\mathsf{j} \subset \mathsf{B}^{\mathbf{inn}}_\mathsf{j}$ for $\mathsf{j} \in \mathsf{J}$ recursively (starting from $\mathsf{j} = 0$) by

$$\bar{\mathsf{B}}^{\mathbf{inn}}_\mathsf{j} = \left\{ \nu \in \mathsf{B}^{\mathbf{inn}}_\mathsf{j} \;\middle|\; \begin{array}{l} \text{for all } \mathsf{j}' < \mathsf{j},\ \nu' \in \bar{\mathsf{B}}^{\mathbf{inn}}_{\mathsf{j}'}, \text{ and } w_{\mathbf{L_{inn}}}, w'_{\mathbf{L_{inn}}} \in \mathbf{W}_{\mathbf{L_{inn}}} \\ \text{one has } \rho_{\mathbf{H}_{\mathsf{j}'}} - (\mathsf{j} - \mathsf{j}')\xi - w_{\mathbf{L_{inn}}} i_*\nu + w'_{\mathbf{L_{inn}}} i_*\nu' \in \mathbf{R}^*_{\mathsf{j}'} \end{array} \right\}.$$

Then we set

$$\bar{\mathsf{B}}^{\mathbf{out}}_\mathsf{j} = \left\{ \lambda_0 \in \mathsf{B}^{\mathbf{out}}_\mathsf{j} \;\middle|\; \begin{array}{l} \text{for all } \mathsf{j}' < \mathsf{j},\ \nu \in \bar{\mathsf{B}}^{\mathbf{inn}}_\mathsf{j},\ \nu' \in \bar{\mathsf{B}}^{\mathbf{inn}}_{\mathsf{j}'},\ w_{\mathbf{L_{inn}}}, w'_{\mathbf{L_{inn}}} \in \mathbf{W}_{\mathbf{L_{inn}}}, \text{ and } w_\mathbf{L} \in \mathbf{W}_\mathbf{L} \\ \text{one has } \rho_{\mathbf{H}_{\mathsf{j}'}} - h^*_{\mathsf{j}'}(w_\mathbf{L}\lambda_0 + (\mathsf{j} - \mathsf{j}')\xi) - w_{\mathbf{L_{inn}}} i_*\nu + w'_{\mathbf{L_{inn}}} i_*\nu' \in \mathbf{R}^*_{\mathsf{j}'} \end{array} \right\}$$

and, as before,

$$\bar{\mathsf{B}}_\mathsf{j} = \bar{\mathsf{B}}^{\mathbf{out}}_\mathsf{j} + \mathsf{j}\xi + i_*\bar{\mathsf{B}}^{\mathbf{inn}}_\mathsf{j}.$$

The subcategories $\mathcal{A}_\mathsf{j} = \langle \mathcal{U}^\lambda \rangle_{\lambda \in \bar{\mathsf{B}}_\mathsf{j}}$ generated by these smaller blocks are semiorthogonal.

This construction looks intimidating. However, we show in Section 8 that the definition of the blocks $\bar{\mathsf{B}}^{\mathbf{out}}_\mathsf{j}$ and $\bar{\mathsf{B}}^{\mathbf{inn}}_\mathsf{j}$ can be rewritten in terms of simple inequalities, and in Section 9 we describe these blocks for classical groups.

Here is a more precise version of our main result. Note that $\bar{\mathsf{B}}^{\mathbf{out}}_\mathsf{j}$ is a set of linear combinations of fundamental weights $\omega_1, \dots, \omega_{a(\mathsf{j})}$ with nonnegative coefficients. These can be considered as Young diagrams—a weight $x_1\omega_1 + \dots + x_a\omega_a$ corresponds to the Young diagram with x_i columns of length i. Let us say that the set $\bar{\mathsf{B}}^{\mathbf{out}}_\mathsf{j}$ is *closed under passing to Young subdiagrams* if the corresponding set of Young diagrams is.

Theorem 1.5. (i) *Let* **G** *be a simple simply connected group. For any choices* (C1)–(C3) *subject to the restriction* (R)*, the collection* $\{\mathcal{A}_{\mathsf{j}}\}_{\mathsf{j}\in\mathsf{J}}$ *of subcategories constructed above is semiorthogonal.*

(ii) *For* $\mathsf{j} \in \mathsf{J}$ *such that* $\bar{\mathsf{B}}^{\mathbf{out}}_{\mathsf{j}}$ *is closed under passing to Young subdiagrams, the block* $\bar{\mathsf{B}}_{\mathsf{j}}$ *is exceptional.*

(iii) *If* **G** *is a group of type* BCD *then the choices* (C1)–(C3) *can be made in such a way that the assumption of* (ii) *is satisfied for all* $\mathsf{j} \in \mathsf{J}$ *and the resulting exceptional collection*

$$\{\mathcal{E}^{\lambda}\}_{\lambda\in\bar{\mathsf{B}}_{\mathsf{j}},\ \mathsf{j}\in\mathsf{J}} \tag{1}$$

in $\mathcal{D}(X)$ *is of expected length.*

We will describe explicit choices in Theorem 1.5(iii) in Section 9 along with the explicit description of the blocks $\bar{\mathsf{B}}_{\mathsf{j}}$. The theorem is proved in Section 9.5. Note that Theorem 1.2 follows from this.

Conjecture 1.6. *The exceptional collections constructed in the Theorem* 1.5(iii) *are full.*

Remark 1.7. We conjecture that in fact every exceptional collection of expected length on $\mathbf{G}/\mathbf{P}$ is full. The more general Nonvanishing Conjecture of [K09] stating that every exceptional collection of expected length is full turned out to be false—counterexamples were constructed in [BBS], [AO], [GS], [BBKS]. Nevertheless, we believe that the conjecture is still true for homogeneous spaces.

1.3.5. Properties of the exceptional collection. Recall that an exceptional collection $E_1, \dots, E_m$ in a triangulated category $\mathcal{T}$ is *strong* if

$$\mathsf{Ext}^{\neq 0}(E_i, E_j) = 0$$

for all i, j. An advantage of a full strong exceptional collection is that it gives an equivalence of the category $\mathcal{T}$ with the derived category of modules over an algebra $\mathsf{End}(\bigoplus E_i)$ (for a nonstrong collection one has to deal with a DG-algebra). Let us say that an exceptional collection is *pure* if all E_i are vector bundles.

Theorem 1.8. *For the blocks of the collections constructed in Theorem* 1.5(i) *strongness and purity are equivalent.*

The proof will be given in Proposition 4.2. In fact, we conjecture the following.

Conjecture 1.9. *The collections constructed in Theorem* 1.5(iii) *are pure and their blocks are strong.*

We verify this conjecture for all maximal isotropic Grassmannians (symplectic and orthogonal).

1.4. Further questions.

There are several questions to be investigated.

Question 1.10. Is there a way to make choices (C1)–(C3) in a canonical way? Is restriction (R) really necessary for the construction?

This expectation is justified partly by a result of A. Fonarev [Fon], who proved that collections of objects, constructed in $\mathcal{D}^b(\mathsf{Gr}(k,n))$ by formally applying our procedure with choices violating restriction (R), are still exceptional. See Section 9.6 for more details.

Question 1.11. Assume that $\mathbf{G}$ is an exceptional group (types E_6, E_7, E_8 and F_4). Is it possible to make the choices (C1)–(C3) in a way analogous to Theorem 1.5(iii), so as to get an exceptional collection of expected length?

Note that in the case of groups of type BCD the equality of the length of the constructed collection with the rank of the Grothendieck group is a result of direct calculation without an a priori explanation. It would be nice to understand the combinatorics behind this coincidence. Recall that the rank of the Grothendieck group $K_0(\mathbf{G}/\mathbf{P})$ is equal to $|\mathbf{W_G}/\mathbf{W_L}|$, so the following question seems natural.

Question 1.12. Find a decomposition of the set $\mathbf{W_G}/\mathbf{W_L} = \bigsqcup_{\mathsf{j}\in\mathsf{J}} W_{\mathsf{j}}$ and a bijection between the sets W_{j} and the sets $\bar{\mathsf{B}}_{\mathsf{j}}$.

The above decomposition should depend on a chain of subgroups $\mathbf{H}_b \subset \cdots \subset \mathbf{H}_1 \subset \mathbf{H}_0 = \mathbf{G}$.

Question 1.13. What happens with our exceptional collections in positive characteristic?

For the case of Grassmannians of type A this was studied in [BLV].

1.5. The structure of the paper. We start by collecting in Section 2 the notation and basic facts about representation theory of algebraic groups.

In Section 3 we define exceptional blocks, prove that they produce exceptional collections, investigate their properties, and state a criterion of exceptionality of a block.

In Section 4 we discuss strongness and purity of the collection obtained from an exceptional block.

In Section 5 we define the blocks B_{j} and $\bar{\mathsf{B}}_{\mathsf{j}}$ and show that $(\mathcal{A}_{\mathsf{j}})$ is a semiorthogonal collection of subcategories.

In Section 6 we verify the first part of the exceptionality criterion from Section 3—the invariance condition—for the blocks B_{j} and $\bar{\mathsf{B}}_{\mathsf{j}}$.

In Section 7 we verify the second part of the criterion—the compatibility condition—modulo a technical assumption (that the outer part of each block is closed under passing to Young subdiagrams).

In Section 8 we rewrite the definition of the blocks in a more explicit form.

In Section 9 we write down the precise choices for classical groups and prove that they give exceptional collections of expected length.

Finally, in the Appendix (Section 10) we prove a certain property of representations of the general linear group which is used in the proof of the exceptionality of the blocks.

2. Preliminaries

2.1. Notation

(1) Groups:
- $\mathbf{G}$, a simple simply connected algebraic group;
- $\mathbf{P} \subset \mathbf{G}$, a maximal parabolic subgroup;
- $\mathbf{U} \subset \mathbf{P}$, the unipotent radical;
- $\mathbf{L} = \mathbf{P}/\mathbf{U}$, the Levi quotient, there is also an embedding $\mathbf{L} \subset \mathbf{P} \subset \mathbf{G}$;
- $\mathbf{L}_{\mathbf{inn}} \subset \mathbf{L}$, the inner part of $\mathbf{L}$, see Section 5.2;
- $\mathbf{L}_{\mathbf{out}} \subset \mathbf{L}$, the outer part of $\mathbf{L}$, see Section 5.2;
- $\mathbf{H}_a \subset \mathbf{G}$, $\mathbf{L}_{\mathbf{inn}} \subset \mathbf{H}_a$, a semisimple subgroup, see Section 5.2;
- $\mathbf{M}_a = \mathbf{L} \cap \mathbf{H}_a$, the Levi of $\mathbf{H}_a$;
- $\mathbf{M}_{a,\mathbf{inn}} = \mathbf{L}_{\mathbf{inn}} \cap \mathbf{H}_a = \mathbf{L}_{\mathbf{inn}}$, the inner part of the Levi of $\mathbf{H}_a$;
- $\mathbf{M}_{a,\mathbf{out}} = \mathbf{L}_{\mathbf{out}} \cap \mathbf{H}_a$, the outer part of the Levi of $\mathbf{H}_a$.

(2) Roots, weights:
- $D = D_{\mathbf{G}}$, $D_{\mathbf{L}_{\mathbf{inn}}} = D_{\mathbf{inn}} \subset D$, $D_{\mathbf{L}_{\mathbf{out}}} = D_{\mathbf{out}} \subset D$, $D_{\mathbf{H}_a} = D_a \subset D$, the Dynkin diagrams;
- $Q_{\mathbf{G}}$, $Q_{\mathbf{L}}$, $Q_{\mathbf{L}_{\mathbf{inn}}}$, $Q_{\mathbf{L}_{\mathbf{out}}}$, $Q_{\mathbf{H}_a}$, the root lattices;
- $Q^+_{\mathbf{G}}$, $Q^+_{\mathbf{L}}$, $Q^+_{\mathbf{L}_{\mathbf{inn}}}$, $Q^+_{\mathbf{L}_{\mathbf{out}}}$, $Q^+_{\mathbf{H}_a}$, the cones generated by simple roots;
- $P_{\mathbf{G}}$, $P_{\mathbf{L}} = P_{\mathbf{G}}$, $P_{\mathbf{L}_{\mathbf{inn}}}$, $P_{\mathbf{L}_{\mathbf{out}}}$, $P_{\mathbf{H}_a}$, the weight lattices;
- $P^+_{\mathbf{G}} \subset P_{\mathbf{G}}$, $P^+_{\mathbf{L}} \subset P_{\mathbf{L}}$, $P^+_{\mathbf{L}_{\mathbf{out}}} \subset P_{\mathbf{L}_{\mathbf{out}}}$, $P^+_{\mathbf{L}_{\mathbf{inn}}} \subset P_{\mathbf{L}_{\mathbf{inn}}}$, $P^+_{\mathbf{H}_a} \subset P_{\mathbf{H}_a}$, the dominant cones;
- α_i, the simple roots;
- ω_i, the fundamental weights;
- β, the simple root corresponding to the maximal parabolic $\mathbf{P}$;
- ξ, the fundamental weight corresponding to the maximal parabolic $\mathbf{P}$;
- $\rho = \rho_{\mathbf{G}} = \sum_{i \in D_{\mathbf{G}}} \omega_i \in P_{\mathbf{G}}$;
- $\rho_{\mathbf{H}_a} = \sum_{i \in D_a} \omega_i \in P_{\mathbf{H}_a}$;
- $(-,-)$, the scalar product on the root/weight lattices.

(3) Weyl groups:
- $\mathbf{W}_{\mathbf{G}}$, $\mathbf{W}_{\mathbf{L}}$, $\mathbf{W}_{\mathbf{L}_{\mathbf{inn}}}$, $\mathbf{W}_{\mathbf{L}_{\mathbf{out}}}$, $\mathbf{W}_{\mathbf{H}_a}$, the Weyl groups;
- s_α, $s_i = s_{\alpha_i}$, s_β, the simple reflections corresponding to simple roots;
- $\ell : \mathbf{W} \to \mathbb{Z}_{\geq 0}$, the length function on a Weyl group;
- $w_0^{\mathbf{G}}$, $w_0^{\mathbf{L}}$, $w_0^{\mathbf{L}_{\mathbf{inn}}}$, $w_0^{\mathbf{L}_{\mathbf{out}}}$, $w_0^{\mathbf{H}_a}$, the longest elements in the corresponding Weyl groups;
- $\mathsf{SR}^{\mathbf{L}}_{\mathbf{G}}$, the set of special representatives of left $\mathbf{W}_{\mathbf{L}}$-cosets in $\mathbf{W}_{\mathbf{G}}$, see Section 2.5;
- $\mathsf{SR}^{\mathbf{M}}_{\mathbf{H}}$, the set of special representatives of left $\mathbf{W}_{\mathbf{M}}$-cosets in $\mathbf{W}_{\mathbf{H}}$.

(4) Maps:
- $i : \mathbf{L}_{\mathbf{inn}} \to \mathbf{L}$, $\mathbf{L}_{\mathbf{inn}} \to \mathbf{H}_a$, $\mathbf{L}_{\mathbf{inn}} \to \mathbf{G}$, the natural embeddings;
- $o : \mathbf{L}_{\mathbf{out}} \to \mathbf{L}$, $\mathbf{L}_{\mathbf{out}} \to \mathbf{G}$, the natural embeddings;
- $h_a : \mathbf{H}_a \to \mathbf{G}$, the natural embedding;
- $i^* : P_{\mathbf{G}} \to P_{\mathbf{L}_{\mathbf{inn}}}$, $o^* : P_{\mathbf{G}} \to P_{\mathbf{L}_{\mathbf{out}}}$, $h_a^* : P_{\mathbf{G}} \to P_{\mathbf{H}_a}$, the restriction of weights;
- $i_* : Q_{\mathbf{L}_{\mathbf{inn}}} \to Q_{\mathbf{G}}$, $o_* : Q_{\mathbf{L}_{\mathbf{out}}} \to Q_{\mathbf{G}}$, $h_{a*} : Q_{\mathbf{H}_a} \to Q_{\mathbf{G}}$, the embedding of roots.

(5) Representations and bundles:
- $V_{\mathbf{G}}^{\lambda}$, the irreducible representation of **G** with the highest weight $\lambda \in P_{\mathbf{G}}^{+}$;
- $V_{\mathbf{L}}^{\lambda}$, the irreducible representation of **L** with the highest weight $\lambda \in P_{\mathbf{L}}^{+}$;
- $\mathcal{U}^{\lambda}$, the **G**-equivariant vector bundle on $\mathbf{G}/\mathbf{P}$ corresponding to $(V_{\mathbf{L}}^{\lambda})_{|\mathbf{P}}$.

(6) Other:
- $X = \mathbf{G}/\mathbf{P}$, the homogeneous space associated with a parabolic subgroup **P**;
- $\mathsf{SGr}(k, 2n)$ (resp., $\mathsf{OGr}(k, n)$), symplectic (resp., orthogonal) isotropic Grassmannian;
- $\mathcal{D}(X)$, the bounded derived category of coherent sheaves on X;
- $\mathcal{D}^{\mathbf{G}}(X)$, the bounded derived category of **G**-equivariant coherent sheaves on X.

2.2. Roots and weights. Let **G** be a simple algebraic group, **P** a maximal parabolic subgroup, and $\mathbf{G}/\mathbf{P} = X$. Let β be the corresponding simple root of **G**, and ξ the corresponding fundamental weight.

We denote by $\mathbf{U} \subset \mathbf{P}$ the unipotent radical of **P** and by $\mathbf{L} = \mathbf{P}/\mathbf{U}$ the Levi quotient. Recall that the projection $\mathbf{P} \to \mathbf{L}$ admits a splitting. We choose such a splitting and consider **L** as a subgroup of **P**, and hence of **G**. We also choose a maximal torus $\mathbf{T} \subset \mathbf{L}$ and a Borel subgroup **B** in **P** such that $\mathbf{T} \subset \mathbf{B}$ and $\mathbf{L} \cap \mathbf{B}$ is a Borel subgroup in **L**. Note that the set of simple roots of **L** is the complement of β in the set of simple roots of **G**.

The embedding of groups $\mathbf{L} \subset \mathbf{G}$ induces an isomorphism of weight lattices $P_{\mathbf{G}} \xrightarrow{\sim} P_{\mathbf{L}}$. We use this isomorphism to identify the lattices. Let $P_{\mathbf{L}}^{+}$ and $P_{\mathbf{G}}^{+}$ denote the dominant cones in P of **L** and **G** respectively.

We identify the simple roots of the group **G** with the vertices of the Dynkin diagram $D_{\mathbf{G}}$. In particular, we say that simple roots α and α' are *adjacent* if the corresponding vertices are connected by an edge, or equivalently if $\alpha \neq \alpha'$ but $(\alpha, \alpha') \neq 0$.

The fundamental weight of **G** corresponding to the vertex $i \in D_{\mathbf{G}}$ is denoted by ω_i. Also, we denote by $\rho = \rho_{\mathbf{G}}$ half the sum of the simple roots of **G**, or equivalently the sum of the fundamental weights.

We consider the root lattice $Q_{\mathbf{G}}$ of **G** as a sublattice of the weight lattice (roots are weights in the adjoint representation). We denote by $(-,-)$ the scalar product on the weight lattice. This scalar product is defined uniquely up to a multiplicative constant. We choose the standard scaling as in [Bou]. Note that with this choice all scalar products of roots are integers and scalar products of weights are rational.

2.3. Weyl group action. The simple reflection corresponding to a root $\alpha = \alpha_i$ is denoted by $s_\alpha = s_{\alpha_i} = s_i$. Note that

$$s_i(\omega_j) = \omega_j - \delta_{ij}\alpha_j, \tag{2}$$

which means that

$$(\omega_j, \alpha_i) = \delta_{ij}\alpha_i^2/2. \tag{3}$$

It follows that

$$s_i\rho = \rho - \alpha_i \tag{4}$$

for all i.

We identify $\mathbf{W_L}$ with the subgroup in $\mathbf{W_G}$ generated by all simple reflections s_{α_i} with $\alpha_i \neq \beta$. Together with (2) this immediately implies the following

Lemma 2.1. *The weight ξ is invariant under the action of $\mathbf{W_L}$.*

The length function on the Weyl group is denoted by ℓ (recall that $\ell(w)$ is the length of a minimal representation of w as a product of simple reflections). The following lemma is well-known (see [Hum, Lemma 10.3A and its proof]).

Lemma 2.2. *If $w \in \mathbf{W_G}$ and s_j is a simple reflection corresponding to the simple root α_j then one has $\ell(ws_j) > \ell(w)$ if and only if the root $w(\alpha_j)$ is positive.*

Recall that the dominant cone $P^+_{\mathbf{G}}$ is a fundamental domain for the action of $\mathbf{W_G}$ on $P_{\mathbf{G}}$. In particular, for each $\lambda \in P_{\mathbf{G}}$ there is an element $w \in \mathbf{W_G}$ such that $w\lambda \in P^+_{\mathbf{G}}$. Moreover, such a w is unique unless λ is orthogonal to a root of $\mathbf{G}$ (i.e., unless λ lies on a wall of a Weyl chamber).

Let us denote by $Q^+_{\mathbf{G}} \subset P_{\mathbf{G}}$ the cone of all linear combinations of simple roots with nonnegative integer coefficients. The following lemma is also well known but we provide a proof for completeness.

Lemma 2.3. *If λ is dominant then for any $w \in \mathbf{W_G}$ one has $\lambda - w\lambda \in Q^+_{\mathbf{G}}$.*

Proof. Since λ is a positive linear combination of fundamental weights, it is enough to check that for every ω_i and every w the weight $\omega_i - w\omega_i$ is a sum of positive roots. This can be checked by induction on the length of w. When w is a simple reflection s_j, this follows from (2). Let $s = s_j$ be a simple reflection, and assume $\ell(ws_j) = \ell(w) + 1$. Then

$$\omega_i - ws_j\omega_i = \omega_i - w\omega_i + w(\omega_i - s_j\omega_i) = \omega_i - w\omega_i + w(\delta_{ij}\alpha_j).$$

Now the assertion follows from the induction assumption and from Lemma 2.2. □

The following consequence of this lemma will be extremely important for us.

Corollary 2.4. *For a pair of weights λ and μ the maximum (resp., minimum) of the scalar product $(w\lambda, \mu)$, where w runs through the Weyl group $\mathbf{W}$, is achieved when $w\lambda$ and μ lie in the same Weyl chamber (resp., opposite Weyl chambers).*

Proof. Since the scalar product is $\mathbf{W}$-invariant, we can assume that μ is dominant. To prove the assertion about the maximum we have to check that if λ is also dominant then $(w\lambda, \mu) \leq (\lambda, \mu)$ for any $w \in \mathbf{W}$. But this follows easily from Lemma 2.3 since the scalar product of a positive root with a dominant weight is nonnegative by (3). The assertion about the minimum follows as well since $(w\lambda, \mu)$ is minimal exactly when $(w\lambda, -\mu)$ is maximal. □

Assume that $\mathbf{H}$ is a simply connected semisimple algebraic group and let $\mathbf{H} = \mathbf{H}_1 \times \cdots \times \mathbf{H}_k$ be its decomposition into the product of simple groups. Then $P_{\mathbf{H}} = P_{\mathbf{H}_1} \oplus \cdots \oplus P_{\mathbf{H}_k}$ and $P^+_{\mathbf{H}} = P^+_{\mathbf{H}_1} \times \cdots \times P^+_{\mathbf{H}_k}$. Denote by λ_i the component of a weight $\lambda \in P_{\mathbf{H}}$ in the summand $P_{\mathbf{H}_i}$.

Definition 2.5. A weight $\lambda \in P^+_{\mathbf{H}}$ is *strictly dominant* if all its components $\lambda_i \in P^+_{\mathbf{H}_i}$ are nonzero.

Lemma 2.6. *If $\lambda, \mu \in P^+_{\mathbf{H}}$ then $(\lambda, \mu) \geq 0$. Moreover, if λ is strictly dominant and $\mu \neq 0$ then $(\lambda, \mu) > 0$. In particular, if $\mathbf{H}$ is simple then the latter inequality holds for any pair of nonzero dominant weights.*

Proof. This follows immediately from the fact that all scalar products of fundamental weights of a simple group are strictly positive. ◻

Let β be the simple root corresponding to $\mathbf{P}$. The $\mathbf{W_L}$-orbit of β has the following nice description.

Lemma 2.7. *The $\mathbf{W_L}$-orbit of β consists of all roots of $\mathbf{G}$ that have the coefficient of β equal to 1, when expressed as a linear combination of simple roots, and have the same length as β.*

Proof. The coefficient of β in a root α is given by $(\xi, \alpha)/(\xi, \beta)$, where ξ is the fundamental weight corresponding to β. Since ξ is invariant under the action of $\mathbf{W_L}$, we have

$$(\xi, w_{\mathbf{L}}\beta) = (w_{\mathbf{L}}^{-1}\xi, \beta) = (\xi, \beta)$$

for all $w_{\mathbf{L}} \in \mathbf{W_L}$, so the coefficient of β is equal to 1 for all roots in the $\mathbf{W_L}$-orbit of β.

Conversely, let us check that if a positive root α has the coefficient of β equal to 1 and $(\alpha, \alpha) = (\beta, \beta)$ then α is in the $\mathbf{W_L}$-orbit of β. Let us write $\alpha = \sum c_i \alpha_i$, where α_i are simple roots. We will use induction on $\sum c_i$. If $\sum c_i = 1$ then $\alpha = \beta$, so the statement is true. Now assume that $\sum c_i > 1$. It is enough to prove that there exists a simple root $\alpha_i \neq \beta$ such that $(\alpha, \alpha_i) > 0$. Indeed, then $s_i\alpha$ will have a smaller sum of coefficients and by the induction assumption, we would deduce that $s_i\alpha$ is in the $\mathbf{W_L}$-orbit of β.

Suppose $(\alpha, \alpha_i) \leq 0$ for all $\alpha_i \neq \beta$. Then

$$(\alpha, \alpha) = \Big(\beta + \sum_{\alpha_i \neq \beta} c_i\alpha_i, \alpha\Big) = (\beta, \alpha) + \sum_{\alpha_i \neq \beta} c_i(\alpha_i, \alpha) \leq (\beta, \alpha).$$

Since $(\alpha, \alpha) = (\beta, \beta)$ by assumption, we get $(\beta, \beta) \leq (\beta, \alpha)$. But $s_\beta(\alpha) = \alpha - 2\frac{(\alpha,\beta)}{(\beta,\beta)}\beta$ should be a positive root (since $\alpha \neq \beta$). Looking at the coefficient of β in $s_\beta(\alpha)$ we obtain

$$2\frac{(\alpha, \beta)}{(\beta, \beta)} \leq 1,$$

which contradicts the previous inequality. ◻

We denote by $w_0^{\mathbf{G}}$ and $w_0^{\mathbf{L}}$ the longest elements of the Weyl groups $\mathbf{W_G}$ and $\mathbf{W_L}$ respectively. Note that

$$(w_0^{\mathbf{L}})^2 = (w_0^{\mathbf{G}})^2 = 1.$$

Note also that $w_0^{\mathbf{G}}$ takes any simple root of $\mathbf{G}$ to minus a simple root, and hence any fundamental weight to minus a fundamental weight. In particular,

$$w_0^{\mathbf{G}}\rho_{\mathbf{G}} = -\rho_{\mathbf{G}} \tag{5}$$

and $w_0^{\mathbf{G}}(P^+_{\mathbf{G}}) = -P^+_{\mathbf{G}}$, $w_0^{\mathbf{L}}(P^+_{\mathbf{L}}) = -P^+_{\mathbf{L}}$.

2.4. Representations. For each dominant weight $\lambda \in P_{\mathbf{G}}^+$ (resp., $\lambda \in P_{\mathbf{L}}^+$) we denote by $V_{\mathbf{G}}^\lambda$ (resp., $V_{\mathbf{L}}^\lambda$) the corresponding irreducible representation of $\mathbf{G}$ (resp., $\mathbf{L}$).

The dual of any irreducible representation is also irreducible. To be more precise,

$$(V_{\mathbf{L}}^\lambda)^\vee = V_{\mathbf{L}}^{-w_0^{\mathbf{L}}\cdot\lambda}. \tag{6}$$

Indeed, if λ is the highest weight of an irreducible representation of $\mathbf{L}$ then $w_0^{\mathbf{L}}\lambda$ is the lowest weight, so $-w_0^{\mathbf{L}}\lambda$ is the highest weight of the dual.

Since the group $\mathbf{L}$ is reductive, the tensor product of two irreducible representations of $\mathbf{L}$ is a direct sum of irreducibles. We denote by $\mathsf{mult}(V_{\mathbf{L}}^\nu, V_{\mathbf{L}}^\lambda \otimes V_{\mathbf{L}}^\mu)$ the multiplicity of $V_{\mathbf{L}}^\nu$ in the tensor product. The following simple result will be useful.

Lemma 2.8. *We have*

$$\mathsf{mult}(V_{\mathbf{L}}^\nu, V_{\mathbf{L}}^\lambda \otimes V_{\mathbf{L}}^\mu) = \dim \mathsf{Hom}_{\mathbf{L}}(V_{\mathbf{L}}^\nu, V_{\mathbf{L}}^\lambda \otimes V_{\mathbf{L}}^\mu) = \dim \mathsf{Hom}_{\mathbf{L}}(V_{\mathbf{L}}^\lambda \otimes V_{\mathbf{L}}^\mu, V_{\mathbf{L}}^\nu).$$

In particular, $\mathsf{mult}(V_{\mathbf{L}}^\nu, V_{\mathbf{L}}^\lambda \otimes V_{\mathbf{L}}^\mu) = \mathsf{mult}((V_{\mathbf{L}}^\mu)^\vee, V_{\mathbf{L}}^\lambda \otimes (V_{\mathbf{L}}^\nu)^\vee)$.

Proof. The first part follows from the fact that there are no maps between different irreducibles and a one-dimensional space of maps between isomorphic irreducibles.

The second part follows from the canonical isomorphism $\mathsf{Hom}_{\mathbf{L}}(V_{\mathbf{L}}^\nu, V_{\mathbf{L}}^\lambda \otimes V_{\mathbf{L}}^\mu) \cong \mathsf{Hom}_{\mathbf{L}}((V_{\mathbf{L}}^\mu)^\vee, V_{\mathbf{L}}^\lambda \otimes (V_{\mathbf{L}}^\nu)^\vee)$. □

We also need the following standard result that gives restrictions on the possible highest weights of irreducible summands of the tensor product of two irreducible representations (obtained e.g. by combining [FH, Thm. 14.18] with [Zhel, §131, Thm. 5]; see also [Hum, Exer. 24.12] and [FH, Exer. 25.33]).

Lemma 2.9. *If* $\mathsf{mult}(V_{\mathbf{L}}^\nu, V_{\mathbf{L}}^\lambda \otimes V_{\mathbf{L}}^\mu) > 0$ *then* $\nu \in \mathsf{Conv}(\lambda + w\mu)_{w\in\mathbf{W_L}}$*, where* Conv *stands for the convex hull and* $\mathbf{W_L}$ *is the Weyl group of* $\mathbf{L}$*. Similarly, if* $\mathsf{mult}(V_{\mathbf{L}}^\nu, V_{\mathbf{L}}^\lambda \otimes (V_{\mathbf{L}}^\mu)^\vee) > 0$ *then* $\nu \in \mathsf{Conv}(\lambda - w\mu)_{w\in\mathbf{W_L}}$.

2.5. Special representatives. For any $w \in \mathbf{W}$ the set $w(P_{\mathbf{G}}^+)$ belongs to a unique $\mathbf{W_L}$-chamber, so in the coset $\mathbf{W_L}w \subset \mathbf{W}$ of $\mathbf{W_L}$ there is a unique representative which takes the $\mathbf{G}$-dominant cone to the $\mathbf{L}$-dominant cone. We call it the $\mathbf{L}$-*special representative of the coset* and denote the set of all $\mathbf{L}$-special representatives in $\mathbf{W}$ by $\mathsf{SR}_{\mathbf{G}}^{\mathbf{L}}$. Note that the $\mathbf{W_L}$-chamber containing $w(P_{\mathbf{G}}^+)$ is determined by $w(\rho)$, hence the $\mathbf{L}$-special representative w_1 is determined by the condition $w_1(\rho) \in P_{\mathbf{L}}^+$.

The elements of $\mathsf{SR}_{\mathbf{G}}^{\mathbf{L}}$ can also be characterized as follows.

Lemma 2.10. *The set* $\mathsf{SR}_{\mathbf{G}}^{\mathbf{L}} \subset \mathbf{W}$ *consists of the elements that have minimal length in their left* $\mathbf{W_L}$*-cosets.*

Proof. Let $w \in \mathbf{W}$ be an element of minimal length in its left $\mathbf{W_L}$-coset. Then $\ell(w^{-1}s_j) = \ell(s_jw) > \ell(w) = \ell(w^{-1})$ for every simple reflection s_j in $\mathbf{W_L}$. Hence, by Lemma 2.2, the root $w^{-1}(\alpha_j)$ is positive for every simple root α_j that belongs to the root system of $\mathbf{L}$. Thus, $(w\rho, \alpha_j) = (\rho, w^{-1}\alpha_j) > 0$ for every such simple root, i.e., $w\rho$ is $\mathbf{L}$-dominant. Hence, w is a special representative. □

Recall that β is the simple root corresponding to $\mathbf{P}$.

Lemma 2.11. (0) *The only element of length* 0 *in* $\mathsf{SR}_{\mathbf{G}}^{\mathbf{L}}$ *is* 1.

(1) *The only element of length* 1 *in* $\mathsf{SR}_{\mathbf{G}}^{\mathbf{L}}$ *is* s_β.

(2) *All elements of length* 2 *in* $\mathsf{SR}_{\mathbf{G}}^{\mathbf{L}}$ *are of the form* $s_\beta s_\alpha$, *where* α *is a simple root of* $\mathbf{G}$ *adjacent to* β.

Proof. Part (0) is clear. For (1) we note that elements of length 1 in $\mathbf{W_G}$ are just simple reflections and for $\alpha \neq \beta$ the reflection s_α is in the same $\mathbf{W_L}$-coset as 1, which has smaller length. Similarly, all elements of length 2 are products $s_{\alpha_1} s_{\alpha_2}$ of simple reflections. If $\alpha_1 \neq \beta$ then s_{α_2} is in the same coset and has smaller length, hence $\alpha_1 = \beta$. And if $\alpha := \alpha_2$ is not adjacent to β then the reflections s_α and s_β commute, so $s_\beta s_\alpha = s_\alpha s_\beta$ is in the same coset as s_β, which has smaller length. □

Take any reductive subgroup $\mathbf{H} \subset \mathbf{G}$ compatible with the torus and the Borel subgroups $\mathbf{T} \subset \mathbf{B} \subset \mathbf{G}$, i.e. such that $\mathbf{H} \cap \mathbf{T} \subset \mathbf{H} \cap \mathbf{B}$ is a maximal torus and a Borel subgroup in $\mathbf{H}$, and such that $\mathbf{M} = \mathbf{H} \cap \mathbf{L}$ is the Levi subgroup in a parabolic subgroup of $\mathbf{H}$. Let $\mathbf{W_H}$ and $\mathbf{W_M}$ be the corresponding Weyl groups. Note that $\mathbf{W_M} = \mathbf{W_H} \cap \mathbf{W_L}$. It follows that $\mathbf{W_H}/\mathbf{W_M} \subset \mathbf{W_G}/\mathbf{W_L}$. Actually, the same inclusion holds for the sets of special representatives.

Lemma 2.12. *We have* $\mathsf{SR}_{\mathbf{G}}^{\mathbf{L}} \cap \mathbf{W_H} = \mathsf{SR}_{\mathbf{H}}^{\mathbf{M}}$.

Proof. The inclusion $\mathsf{SR}_{\mathbf{G}}^{\mathbf{L}} \cap \mathbf{W_H} \subset \mathsf{SR}_{\mathbf{H}}^{\mathbf{M}}$ is clear. Now let $w \in \mathsf{SR}_{\mathbf{H}}^{\mathbf{M}}$. We have to show that $(w\rho, \alpha_i) \geq 0$, where α_i is any simple root of $\mathbf{L}$. If α_i belongs to the root system of $\mathbf{H} \cap \mathbf{L} = \mathbf{M}$ then this follows from the definition of $\mathsf{SR}_{\mathbf{H}}^{\mathbf{M}}$. Otherwise, the simple reflection s_i associated with α_i is different from all simple reflections in $W_{\mathbf{H}}$, so $\ell(w^{-1}s_i) > \ell(w^{-1})$. Hence, by Lemma 2.2, $w^{-1}\alpha_i$ is a positive root, and so $(w\rho, \alpha_i) = (\rho, w^{-1}\alpha_i) \geq 0$. □

The following inequality is very important for us.

Lemma 2.13. *Assume that* $v \in \mathsf{SR}_{\mathbf{G}}^{\mathbf{L}}$. *Then*

$$(\xi, \rho - v\rho) \geq \ell(v)(\xi, \beta).$$

If $\ell(v) = 1$ *then this inequality becomes an equality.*

Proof. Let us prove this by induction on the length of v. In the case $v = 1$ both sides of our inequality are equal to zero. Now assume that $\ell(v) \geq 1$. Recall that v is the representative of minimal length in the coset $\mathbf{W_L}v$. Thus, we can write $v = us_i$, where s_i is a simple reflection, $\ell(u) = \ell(v) - 1$, and $u \in \mathsf{SR}_{\mathbf{G}}^{\mathbf{L}}$. We have

$$(\xi, \rho - v\rho) = (\xi, \rho - u\rho) + (\xi, u(\rho - s_i\rho)) = (\xi, \rho - u\rho) + (\xi, u(\alpha_i)).$$

The first summand on the right-hand side is $\geq \ell(u)(\xi, \beta)$ by the induction assumption. Thus, it suffices to check that $(\xi, u(\alpha_i)) \geq (\xi, \beta)$.

Since $\ell(us_i) = \ell(u) + 1$, the root $u(\alpha_i)$ is positive (by Lemma 2.2), so we only have to check that β appears in $u(\alpha_i)$ with nonzero coefficient, i.e., $(\xi, u(\alpha_i)) \neq 0$. Suppose $(\xi, u(\alpha_i)) = 0$. Then $u(\alpha_i) = \sum_{\alpha_j \neq \beta} n_j \alpha_j$ with $n_j \geq 0$. The fact that v has minimal length in its right $W_{\mathbf{L}}$-coset implies that $\ell(v^{-1}s_j) > \ell(v^{-1})$ for every j such that $\alpha_j \neq \beta$. Hence, all the roots $v^{-1}(\alpha_j)$ are positive, and therefore

$$-\alpha_i = s_i\alpha_i = v^{-1}u(\alpha_i) = \sum_{\alpha_j \neq \beta} n_j v^{-1}(\alpha_j)$$

should be positive, so we get a contradiction.

If $\ell(v) = 1$ then $v = s_\beta$ by Lemma 2.11, hence $\rho - v\rho = \beta$, and both sides are equal to (ξ, β). □

Remark 2.14. Note that if the root β is *cominuscule*, which means that the coefficient of β in any root of $\mathbf{G}$ does not exceed 1, then

$$(\xi, \rho - v\rho) = \ell(v)(\xi, \beta)$$

for all $v \in \mathsf{SR}_{\mathbf{G}}^{\mathbf{L}}$. Indeed, in the argument above we conclude that the coefficient of β in $u(\alpha_i)$ is precisely 1, hence we obtain an inductive proof of the equality.

2.6. Equivariant bundles $\mathcal{U}^\lambda$ and Borel–Bott–Weil Theorem. Since $X = \mathbf{G}/\mathbf{P}$ is a homogeneous variety, the category $\mathsf{Coh}^{\mathbf{G}}(X)$ of $\mathbf{G}$-equivariant coherent sheaves on X is equivalent to the category of representations of $\mathbf{P}$:

$$\mathsf{Coh}^{\mathbf{G}}(X) \cong \mathsf{Rep}\text{-}\mathbf{P} \tag{7}$$

(see [BK90], [Hille] and references therein). This equivalence is compatible with the structures of tensor abelian categories on both sides, i.e. it preserves tensor products and duals.

For each $\lambda \in P_{\mathbf{L}}^+$, a dominant weight of the Levi quotient $\mathbf{L} = \mathbf{P}/\mathbf{U}$, we consider $V_{\mathbf{L}}^\lambda$, the corresponding irreducible representation of $\mathbf{L}$. Extending $V_{\mathbf{L}}^\lambda$ to $\mathbf{P}$ (via the projection $\mathbf{P} \to \mathbf{L}$) we obtain a representation of $\mathbf{P}$, and hence a $\mathbf{G}$-equivariant vector bundle on X which we denote by $\mathcal{U}^\lambda$. Since the above equivalence preserves the tensor structure, we deduce from Lemma 2.8 and (6) that

$$\mathcal{U}^\lambda \otimes \mathcal{U}^{\lambda'} = \bigoplus_{\mu \in P_{\mathbf{L}}^+} \mathsf{Hom}(V_{\mathbf{L}}^\mu, V_{\mathbf{L}}^\lambda \otimes V_{\mathbf{L}}^{\lambda'}) \otimes \mathcal{U}^\mu, \qquad (\mathcal{U}^\lambda)^\vee \cong \mathcal{U}^{-w_0^{\mathbf{L}}\lambda}. \tag{8}$$

Note that $V_{\mathbf{L}}^\xi$ is a one-dimensional representation of $\mathbf{L}$, hence $\mathcal{U}^\xi$ is a line bundle on X. Moreover, it is the ample generator of $\mathsf{Pic}\, X = \mathbb{Z}$, so we will denote it by $\mathcal{O}_X(1)$. Thus,

$$\mathcal{O}_X(t) = \mathcal{U}^{t\xi}. \tag{9}$$

Similarly, we will denote the bundle $\mathcal{U}^{\lambda + t\xi}$ by $\mathcal{U}^\lambda(t)$.

The cohomology groups of the bundles $\mathcal{U}^\lambda$ can be computed via the Borel–Bott–Weil Theorem. Recall that a weight $\lambda \in P_{\mathbf{G}}$ is called $\mathbf{G}$-*singular* if it lies on a wall of a Weyl chamber of $\mathbf{G}$ (equivalently, if it is orthogonal to some root of $\mathbf{G}$). If a weight does not lie on a wall of a Weyl chamber, it is called $\mathbf{G}$-*regular*. If the group $\mathbf{G}$ is clear from the context, we will write just singular and regular. The sets of $\mathbf{G}$-singular and of $\mathbf{G}$-regular weights are invariant under the natural action of the Weyl group $\mathbf{W}_{\mathbf{G}}$ on $P_{\mathbf{G}}$.

Theorem 2.15 ([Bott, Thm. IV′]). *Take any* $\lambda \in P^+_{\mathbf{L}} \subset P_{\mathbf{L}} = P_{\mathbf{G}}$. *If* $\lambda + \rho_{\mathbf{G}}$ *is* $\mathbf{G}$-*singular then* $H^\bullet(X, \mathcal{U}^\lambda) = 0$. *If* $\lambda + \rho_{\mathbf{G}}$ *is* $\mathbf{G}$-*regular then there exists a unique* $w \in \mathbf{W_G}$ *such that* $w(\lambda + \rho_{\mathbf{G}})$ *is dominant. In this case*

$$H^{\ell(w)}(X, \mathcal{U}^\lambda) = V_{\mathbf{G}}^{w(\lambda+\rho_{\mathbf{G}})-\rho_{\mathbf{G}}}$$

and the other cohomology groups vanish. In particular, if λ *is* $\mathbf{G}$-*dominant then* $H^0(X, \mathcal{U}^\lambda) = V^\lambda_{\mathbf{G}}$.

Let $P_{\mathbf{G}}^{\mathsf{reg}}$ denote the set of all regular weights of $\mathbf{G}$, and $P_{\mathbf{G}}^{\mathsf{reg}} - \rho_{\mathbf{G}}$ denote the set of all weights $\mu \in P_{\mathbf{G}}$ such that $\mu + \rho_{\mathbf{G}} \in P_{\mathbf{G}}^{\mathsf{reg}}$. Further, for each $\mu \in P_{\mathbf{G}}^{\mathsf{reg}} - \rho_{\mathbf{G}}$ denote by w_μ the unique element of the Weyl group $\mathbf{W_G}$ such that $w_\mu(\mu + \rho_{\mathbf{G}})$ is $\mathbf{G}$-dominant. Combining the theorem above with (8) and Lemma 2.9 we deduce

Corollary 2.16. *We have*

$$\mathsf{Ext}^\bullet(\mathcal{U}^\lambda, \mathcal{U}^{\lambda'}) = \bigoplus_{\mu \in \mathsf{Conv}(\lambda' - w\lambda)_{w\in \mathbf{W_L}} \cap P^+_{\mathbf{L}} \cap (P^{\mathsf{reg}}_{\mathbf{G}} - \rho_{\mathbf{G}})} \mathsf{Hom}(V^\mu_{\mathbf{L}}, V^{\lambda'}_{\mathbf{L}} \otimes V_{\mathbf{L}}^{-w_0^{\mathbf{L}}\lambda}) \otimes V_{\mathbf{G}}^{w_\mu(\mu+\rho_{\mathbf{G}})-\rho_{\mathbf{G}}}[-\ell(w_\mu)],$$

where $[-\ell(w_\mu)]$ *stands for cohomological shift.*

We will also need a way to compute Ext-groups in the derived category $\mathcal{D}^{\mathbf{G}}(X)$ of $\mathbf{G}$-equivariant coherent sheaves on X. Let us denote these Ext groups between $F, F' \in \mathcal{D}^{\mathbf{G}}(X)$ by $\mathsf{Ext}^i_{\mathbf{G}}(F, F') = \mathsf{Hom}_{\mathcal{D}^{\mathbf{G}}(X)}(F, F'[i])$.

Proposition 2.17. *One has*

(i) $\mathsf{Ext}^i_{\mathbf{G}}(F, F') = (\mathsf{Ext}^i(F, F'))^{\mathbf{G}}$, *the space of* $\mathbf{G}$-*invariants in the* Ext-*group between* F *and* F' *in* $\mathcal{D}(X)$.

(ii) $\mathsf{Ext}^\bullet_{\mathbf{G}}(\mathcal{U}^\lambda, \mathcal{U}^{\lambda'}) = \bigoplus_{v \in \mathsf{SR}^{\mathbf{L}}_{\mathbf{G}}} \mathsf{Hom}(V_{\mathbf{L}}^{v\rho-\rho}, V^{\lambda'}_{\mathbf{L}} \otimes V_{\mathbf{L}}^{-w_0^{\mathbf{L}}\lambda})[-\ell(v)]$.

(iii) $\mathsf{Ext}^1_{\mathbf{G}}(\mathcal{U}^\lambda, \mathcal{U}^{\lambda'}) = \mathsf{Hom}(V_{\mathbf{L}}^{-\beta}, V^{\lambda'}_{\mathbf{L}} \otimes V_{\mathbf{L}}^{-w_0^{\mathbf{L}}\lambda})$.

Proof. (i) This follows from $\mathsf{Hom}_{\mathbf{G}}(F, F') = \mathsf{Hom}(F, F')^{\mathbf{G}}$ because the functor of invariants is exact (since the group $\mathbf{G}$ is reductive).

(ii) Note that $(V^\nu_{\mathbf{G}})^{\mathbf{G}}$ is zero for $\nu \neq 0$ and is equal to k for $\nu = 0$, hence μ from the formula of Corollary 2.16 contributes to $\mathsf{Ext}_{\mathbf{G}}$ if and only if $w_\mu(\mu + \rho) - \rho = 0$, that is, if $\mu = v\rho - \rho$ for some $v \in \mathbf{W_G}$. Since μ should be $\mathbf{L}$-dominant, the element v should be a special representative, that is, $v \in \mathsf{SR}^{\mathbf{L}}_{\mathbf{G}}$. Of course, if $v\rho - \rho \notin \mathsf{Conv}(\lambda' - w\lambda)$ then Hom is zero, so we can forget this restriction.

(iii) This follows from (ii) using the fact that by Lemma 2.11(1) the only special representative of length 1 is s_β, and $s_\beta\rho = \rho - \beta$. □

2.7. The canonical class. Let $\mathbf{G}$ be a semisimple algebraic group (not necessarily simple). Recall that by [Hille, Sec. 1.5], the canonical class of $X = \mathbf{G}/\mathbf{P}$ is the line bundle corresponding to the weight equal to minus the sum of all positive roots of $\mathbf{G}$ which are not roots of $\mathbf{L}$. The following formula is also well known but we give a proof for completeness.

Lemma 2.18. *The canonical class ω_X of $X = \mathbf{G}/\mathbf{P}$ is isomorphic to the line bundle $\mathcal{U}^{w_0^{\mathbf{L}} w_0^{\mathbf{G}}\rho-\rho}$.*

Proof. Recall that ρ is half the sum of all positive roots of $\mathbf{G}$. As $w_0^{\mathbf{G}}$ takes all positive roots of $\mathbf{G}$ to negative roots and $w_0^{\mathbf{L}}$ takes all negative roots of $\mathbf{L}$ to positive roots of $\mathbf{L}$, it follows that $w_0^{\mathbf{L}} w_0^{\mathbf{G}}\rho$ is half the sum of all positive roots of $\mathbf{L}$ minus half the sum of all positive roots of $\mathbf{G}$ which are not roots of $\mathbf{L}$. So, subtracting ρ we obtain minus the sum of all positive roots of $\mathbf{G}$ which are not the roots of $\mathbf{L}$. □

We will also need the following more explicit formula.

Lemma 2.19. *Let β be the simple root corresponding to $\mathbf{P}$, and ξ the corresponding fundamental weight. There exists a maximal root in the $\mathbf{W_L}$-orbit of β, i.e., a positive root $\bar\beta \in \mathbf{W_L}\beta$ satisfying $\bar\beta - w\bar\beta \in Q^+_{\mathbf{G}}$ for any $w \in \mathbf{W_L}$. Then $\omega_X = \mathcal{O}_X(-r) = \mathcal{U}^{-r\xi}$, where*

$$r = (\rho, \bar\beta + \beta)/(\xi, \beta).$$

Proof. The Picard group of $\mathbf{G}/\mathbf{P}$ is generated by $\mathcal{U}^{\xi}$, hence $\omega_X \cong \mathcal{U}^{w_0^{\mathbf{L}} w_0^{\mathbf{G}}\rho-\rho} \cong \mathcal{U}^{-k\xi}$ for some $k \in \mathbb{Z}$. To find k we compute the scalar product with β. We get

$$k = (\rho - w_0^{\mathbf{L}} w_0^{\mathbf{G}}\rho, \beta)/(\xi, \beta).$$

Further $(-w_0^{\mathbf{L}} w_0^{\mathbf{G}}\rho, \beta) = (w_0^{\mathbf{L}}\rho, \beta) = (\rho, w_0^{\mathbf{L}}\beta)$ by (5). Note that β considered as a weight of $\mathbf{L}$ is antidominant (its scalar products with simple roots of $\mathbf{L}$ are nonpositive), hence $w_0^{\mathbf{L}}\beta$ is $\mathbf{L}$-dominant. By Lemma 2.3 we conclude that $\bar\beta := w_0^{\mathbf{L}}\beta$ is the maximal root in the $\mathbf{W_L}$-orbit of β. Finally, it is a positive root since $(\xi, w_0^{\mathbf{L}}\beta) = (w_0^{\mathbf{L}}\xi, \beta) = (\xi, \beta) > 0$, because ξ is $\mathbf{W_L}$-invariant. □

Remark 2.20. By Lemma 2.7, $\bar\beta$ is in fact the maximal root of the same length as β and with the coefficient of β equal to 1. This gives an easy way to find $\bar\beta$ just by looking at the table of roots.

Remark 2.21. The integer r is called the *index* of the Grassmannian $\mathbf{G}/\mathbf{P}$.

The following consequence of the above formula is useful.

Corollary 2.22. *Let $\mathbf{P}$ be a maximal parabolic subgroup in $\mathbf{G}$, and β the corresponding simple root. Let $\mathbf{H} \subset \mathbf{H}' \subset \mathbf{G}$ be a pair of semisimple subgroups corresponding to a pair of Dynkin subdiagrams $D_{\mathbf{H}} \subset D_{\mathbf{H}'} \subset D_{\mathbf{G}}$ such that $\beta \in D_{\mathbf{H}}$ and there is a simple root $\alpha \in D_{\mathbf{H}'} \setminus D_{\mathbf{H}}$ adjacent to the connected component of β in $D_{\mathbf{H}}$. Let r and r' be the indices of the Grassmannians $\mathbf{H}/(\mathbf{H} \cap \mathbf{P})$ and $\mathbf{H}'/(\mathbf{H}' \cap \mathbf{P})$ respectively. Then $r' > r$.*

Proof. Let $\mathbf{M} = \mathbf{L} \cap \mathbf{H}$ and $\mathbf{M}' = \mathbf{L} \cap \mathbf{H}'$. Let $\bar\beta$ be the maximal root in the $\mathbf{W_M}$-orbit of β, and $\bar\beta'$ the maximal root in the $\mathbf{W_{M'}}$-orbit of β. Let $C \subset D_{\mathbf{H}}$ denote the connected component of β in $D_{\mathbf{H}}$, and let α be a simple root of $\mathbf{H}'$ adjacent to C. Note that since $\bar\beta$ is maximal, the coefficient of any simple root of C in $\bar\beta$ is strictly positive. In particular, the coefficients of simple roots in C adjacent to α are positive, hence the scalar product $(\alpha, \bar{\beta})$ is strictly negative. Therefore,

$$s_\alpha(\bar{\beta}) = \bar{\beta} - 2\frac{(\alpha, \bar{\beta})}{\alpha^2}\alpha$$

has a strictly positive coefficient of α. Therefore,

$$(\rho, \bar{\beta}') \geq (\rho, s_\alpha(\bar{\beta})) \geq (\rho, \bar{\beta}) + (\rho, \alpha) > (\rho, \bar{\beta})$$

since $(\rho, \alpha) = \alpha^2/2 > 0$. Now the assertion follows from Lemma 2.19. □

3. Exceptional blocks

Let $\mathbf{G}$ be a simple simply connected algebraic group and $\mathbf{P} \subset \mathbf{G}$ a maximal parabolic subgroup. We take $X = \mathbf{G}/\mathbf{P}$ and denote by $\mathcal{D}(X)$ the bounded derived category of coherent sheaves on X, and by $\mathcal{D}^{\mathbf{G}}(X)$ the bounded derived category of $\mathbf{G}$-equivariant coherent sheaves. We denote by $\mathsf{Fg} : \mathcal{D}^{\mathbf{G}}(X) \to \mathcal{D}(X)$ the forgetful functor.

We denote as usual $\mathsf{Ext}^i(F, F') = \mathsf{Hom}(F, F'[i])$, the Ext-groups in the category $\mathcal{D}(X)$. Similarly, Ext-groups in the equivariant category $\mathcal{D}^{\mathbf{G}}(X)$ are denoted by $\mathsf{Ext}^i_{\mathbf{G}}(F, F')$. Recall that $\mathsf{Ext}^i_{\mathbf{G}}(F, F') = \mathsf{Ext}^i(F, F')^{\mathbf{G}}$ by Proposition 2.17(i). Note that the forgetful functor induces a linear map

$$\mathsf{Fg} : \mathsf{Ext}^i_{\mathbf{G}}(F, F') \to \mathsf{Ext}^i(F, F').$$

For each triple of $\mathbf{L}$-dominant weights $\lambda, \mu, \nu \in P^+_{\mathbf{L}}$ consider the map

$$\mathsf{Ext}^\bullet_{\mathbf{G}}(\mathcal{U}^\lambda, \mathcal{U}^\nu) \otimes \mathsf{Hom}(\mathcal{U}^\nu, \mathcal{U}^\mu) \to \mathsf{Ext}^\bullet(\mathcal{U}^\lambda, \mathcal{U}^\mu),$$

the composition of the action of the forgetful functor with the Yoneda multiplication.

Now we can introduce the main notion of this section.

Definition 3.1. A set of $\mathbf{L}$-dominant weights $\mathsf{B} \subset P^+_{\mathbf{L}}$ is called an *exceptional block* if for all $\lambda, \mu \in \mathsf{B}$ the canonical map

$$\bigoplus_{\nu \in \mathsf{B}} \mathsf{Ext}^\bullet_{\mathbf{G}}(\mathcal{U}^\lambda, \mathcal{U}^\nu) \otimes \mathsf{Hom}(\mathcal{U}^\nu, \mathcal{U}^\mu) \to \mathsf{Ext}^\bullet(\mathcal{U}^\lambda, \mathcal{U}^\mu) \tag{10}$$

is an isomorphism.

The goal of this section is to show that for any exceptional block $\mathsf{B} \subset P^+_{\mathbf{L}}$ the category

$$\mathcal{D}_{\mathsf{B}}(X) = \langle \mathcal{U}^\lambda \rangle_{\lambda \in \mathsf{B}} \subset \mathcal{D}(X)$$

generated in $\mathcal{D}(X)$ by the bundles $\mathcal{U}^\lambda$ with $\lambda \in \mathsf{B}$ has a full exceptional collection.

3.1. The ξ-ordering.

Recall that β is the simple root of $\mathbf{G}$ corresponding to the maximal parabolic $\mathbf{P}$, and ξ is the corresponding fundamental weight. By Lemma 2.1 it is invariant under the action of $\mathbf{W_L}$.

Consider the partial ordering on the weight lattice $P_{\mathbf{L}}$ defined by:

$$\begin{aligned} \lambda \prec \mu \quad & \text{if } (\xi, \lambda) < (\xi, \mu), \\ \lambda \preceq \mu \quad & \text{if either } \lambda \prec \mu \text{ or } \lambda = \mu. \end{aligned} \tag{11}$$

We will call it the *ξ-ordering*.

Lemma 3.2. *If* $\mathsf{Hom}(\mathcal{U}^\lambda, \mathcal{U}^\mu) \neq 0$ *then* $\lambda \preceq \mu$.

Proof. By Corollary 2.16 if $\mathsf{Hom}(\mathcal{U}^\lambda, \mathcal{U}^\mu) \neq 0$ then there is a nontrivial **L**-map $V_{\mathbf{L}}^\kappa \subset (V_{\mathbf{L}}^\lambda)^\vee \otimes V_{\mathbf{L}}^\mu$ for some **G**-dominant weight κ. This means that there is a nontrivial **L**-map $V_{\mathbf{L}}^\kappa \otimes V_{\mathbf{L}}^\lambda \to V_{\mathbf{L}}^\mu$, hence $\mu \in \mathsf{Conv}(\lambda + w\kappa)_{w \in \mathbf{W_L}}$ by Lemma 2.9. But for any $w \in \mathbf{W_L}$,

$$(\xi, \lambda + w\kappa) - (\xi, \lambda) = (\xi, w\kappa) = (w^{-1}\xi, \kappa) = (\xi, \kappa) \geq 0,$$

where the last inequality follows from Lemma 2.6 since both ξ and κ are **G**-dominant. Moreover, since **G** is simple, the inequality is strict unless $\kappa = 0$. Thus, $\lambda \prec \mu$ unless $\kappa = 0$. But if $\kappa = 0$ then $V_{\mathbf{L}}^\kappa \otimes V_{\mathbf{L}}^\lambda = V_{\mathbf{L}}^\lambda$, hence $\mu = \lambda$. □

Thus, we see that nonzero Hom groups between $\mathcal{U}^\lambda$ in $\mathcal{D}(X)$ are compatible with the ξ-ordering (they cannot go from a bigger weight to a smaller weight). It turns out that Ext groups in the equivariant category go in the opposite direction!

Lemma 3.3. *If* $\mathsf{Ext}^\bullet_{\mathbf{G}}(\mathcal{U}^\lambda, \mathcal{U}^\mu) \neq 0$ *then* $\mu \preceq \lambda$. *More precisely, if* $\mathsf{Ext}^i_{\mathbf{G}}(\mathcal{U}^\lambda, \mathcal{U}^\mu) \neq 0$ *then*

$$(\xi, \lambda) - (\xi, \mu) \geq i(\xi, \beta),$$

and for $i = 1$ this inequality becomes an equality. Also, each bundle $\mathcal{U}^\lambda$ is exceptional in $\mathcal{D}^{\mathbf{G}}(X)$.

Proof. By Proposition 2.17, if $\mathsf{Ext}^i_{\mathbf{G}}(\mathcal{U}^\lambda, \mathcal{U}^\mu) \neq 0$ then there is a nontrivial **L**-map $V_{\mathbf{L}}^{v\rho-\rho} \to (V_{\mathbf{L}}^\lambda)^\vee \otimes V_{\mathbf{L}}^\mu$ for some $v \in \mathsf{SR}^{\mathbf{L}}_{\mathbf{G}}$ with $\ell(v) = i$. This means that there is a nontrivial **L**-map $V_{\mathbf{L}}^{v\rho-\rho} \otimes V_{\mathbf{L}}^\lambda \to V_{\mathbf{L}}^\mu$, hence $\mu \in \mathsf{Conv}(\lambda + w(v\rho - \rho))_{w \in \mathbf{W_L}}$ by Lemma 2.9. Now by Lemma 2.13, for any $w \in \mathbf{W_L}$ we have

$$(\xi, \lambda + w(v\rho - \rho)) - (\xi, \lambda) = (\xi, w(v\rho - \rho)) = (w^{-1}\xi, v\rho - \rho) = (\xi, v\rho - \rho) \leq -i(\xi, \beta),$$

where the last inequality becomes an equality for $i = 1$. This implies that

$$(\xi, \mu) - (\xi, \lambda) \leq -i(\xi, \beta)$$

with equality for $i = 1$, as required. Thus, we see that $\mu \prec \lambda$ unless $v = 1$. But if $v = 1$ then $V_{\mathbf{L}}^{v\rho-\rho} \otimes V_{\mathbf{L}}^\lambda = V_{\mathbf{L}}^\lambda$, hence $\mu = \lambda$. Also, if $v = 1$ then $i = \ell(v) = 0$, so $\mathsf{Ext}^{>0}_{\mathbf{G}}(\mathcal{U}^\lambda, \mathcal{U}^\lambda) = 0$ and by Proposition 2.17 we have $\mathsf{Hom}_{\mathbf{G}}(\mathcal{U}^\lambda, \mathcal{U}^\lambda) = \mathsf{Hom}_{\mathbf{L}}(V_{\mathbf{L}}^\lambda, V_{\mathbf{L}}^\lambda) = \mathsf{k}$, hence $\mathcal{U}^\lambda$ is exceptional in $\mathcal{D}^{\mathbf{G}}(X)$. □

Lemma 3.3 has the following important consequence.

Theorem 3.4. *The bundles $\{\mathcal{U}^\lambda\}_{\lambda \in P_{\mathbf{L}}^+}$, ordered with respect to any total ordering refining the opposite of the ξ-ordering, constitute a full exceptional collection in the derived category of equivariant sheaves $\mathcal{D}^{\mathbf{G}}(X)$.*

Proof. The fact that we get an exceptional collection follows from Lemma 3.3. It remains to check that it is full.

Indeed, let us show that every object belongs to the triangulated subcategory generated by this collection. It suffices to check this for pure objects, that is, for **G**-equivariant coherent sheaves. As we know, the category of **G**-equivariant coherent sheaves is equivalent to the category of **P**-representations. But each such representation has a filtration (a refinement of the radical filtration) with the quotients that are simple **L**-representations, i.e. correspond to bundles $\mathcal{U}^\lambda$ with appropriate $\lambda \in P_{\mathbf{L}}^+$. Thus, it is contained in the subcategory generated by the $\mathcal{U}^\lambda$. □

Remark 3.5. The fact that the orderings of Hom's in $\mathcal{D}(X)$ and Ext's in $\mathcal{D}^{\mathbf{G}}(X)$ are opposite is the reason for the fact that an object $\mathcal{U}^\lambda$ is typically not exceptional in $\mathcal{D}(X)$—one can construct a nontrivial element of $\mathsf{Ext}^\bullet(\mathcal{U}^\lambda, \mathcal{U}^\lambda)$ by composing Hom's and equivariant Ext's. As we will see in Section 3.3 below, the cure is, in a sense, to reverse one of the orderings.

3.2. The forgetful functor and its adjoint.

Let $\mathsf{B} \subset P_{\mathbf{L}}^+$ be an exceptional block. Let

$$\mathcal{D}_{\mathsf{B}}^{\mathbf{G}}(X) = \langle \mathcal{U}^\lambda \rangle_{\lambda \in \mathsf{B}}$$

denote the subcategory of $\mathcal{D}^{\mathbf{G}}(X)$ generated by $\mathcal{U}^\lambda$ with λ in B. Since the collection $\{\mathcal{U}^\lambda\}_{\lambda \in \mathsf{B}}$ is exceptional, the category $\mathcal{D}_{\mathsf{B}}^{\mathbf{G}}$ is saturated (see [BK89]), hence the forgetful functor $\mathsf{Fg} : \mathcal{D}_{\mathsf{B}}^{\mathbf{G}}(X) \to \mathcal{D}_{\mathsf{B}}(X)$ has a right adjoint functor $\mathsf{Fg}^! : \mathcal{D}_{\mathsf{B}}(X) \to \mathcal{D}_{\mathsf{B}}^{\mathbf{G}}(X)$ (cf. [BK89, Prop. 2.7]).

The crucial observation is the following

Proposition 3.6. *If* B *is an exceptional block then*

$$\mathsf{Fg}^!(\mathsf{Fg}(\mathcal{U}^\mu)) = \bigoplus_{\nu \in \mathsf{B}} \mathsf{Hom}(\mathcal{U}^\nu, \mathcal{U}^\mu) \otimes \mathcal{U}^\nu,$$

where $\mathsf{Hom}(\mathcal{U}^\lambda, \mathcal{U}^\mu)$ *are considered just as vector spaces, not as representations of* **G**.

Proof. Let

$$\tilde{\mathcal{U}}^\mu := \bigoplus_{\nu \in \mathsf{B}} \mathsf{Hom}(\mathcal{U}^\nu, \mathcal{U}^\mu) \otimes \mathcal{U}^\nu \in \mathcal{D}_{\mathsf{B}}^{\mathbf{G}}(X).$$

We have a canonical evaluation map $\mathsf{ev} : \mathsf{Fg}(\tilde{\mathcal{U}}^\mu) \to \mathsf{Fg}(\mathcal{U}^\mu)$ in $\mathcal{D}(X)$. By adjunction it gives a map $\tilde{\mathcal{U}}^\mu \to \mathsf{Fg}^!\mathsf{Fg}(\mathcal{U}^\mu)$. Let us show it is an isomorphism. For this let us check that the induced map

$$f : \mathsf{Ext}^\bullet_{\mathbf{G}}(\mathcal{U}^\lambda, \tilde{\mathcal{U}}^\mu) \to \mathsf{Ext}^\bullet_{\mathbf{G}}(\mathcal{U}^\lambda, \mathsf{Fg}^!\mathsf{Fg}(\mathcal{U}^\mu))$$

is an isomorphism for all $\lambda \in \mathsf{B}$. Indeed, we have a commutative diagram

$$\begin{array}{ccccc}
\bigoplus_{\nu \in \mathsf{B}} \mathsf{Ext}^\bullet_{\mathbf{G}}(\mathcal{U}^\lambda, \mathcal{U}^\nu) \otimes \mathsf{Hom}(\mathcal{U}^\nu, \mathcal{U}^\mu) & = & \mathsf{Ext}^\bullet_{\mathbf{G}}(\mathcal{U}^\lambda, \tilde{\mathcal{U}}^\mu) & \xrightarrow{f} & \mathsf{Ext}^\bullet_{\mathbf{G}}(\mathcal{U}^\lambda, \mathsf{Fg}^!\mathsf{Fg}(\mathcal{U}^\mu)) \\
\downarrow{\scriptstyle \mathsf{Fg} \otimes 1} & & \downarrow{\scriptstyle \mathsf{Fg}} & & \| \\
\bigoplus_{\nu \in \mathsf{B}} \mathsf{Ext}^\bullet(\mathcal{U}^\lambda, \mathcal{U}^\nu) \otimes \mathsf{Hom}(\mathcal{U}^\nu, \mathcal{U}^\mu) & = & \mathsf{Ext}^\bullet(\mathsf{Fg}(\mathcal{U}^\lambda), \mathsf{Fg}(\tilde{\mathcal{U}}^\mu)) & \xrightarrow{\mathsf{ev}} & \mathsf{Ext}^\bullet(\mathsf{Fg}(\mathcal{U}^\lambda), \mathsf{Fg}(\mathcal{U}^\mu))
\end{array}$$

The composition of the left vertical map with the maps in the bottom row is the map (10), which is an isomorphism since B is an exceptional block. Hence, the map f in the top row is an isomorphism as well.

It follows that the cone of the map $\tilde{\mathcal{U}}^\mu \to \mathsf{Fg}^!\mathsf{Fg}(\mathcal{U}^\mu)$ is orthogonal to all $\mathcal{U}^\lambda$ in $\mathcal{D}^{\mathbf{G}}_{\mathsf{B}}(X)$. But $\mathcal{U}^\lambda$ generate this category, hence the cone is zero. □

Question 3.7. It would be of interest to find a general formula for $\mathsf{Fg}^!$ (or maybe for the composition $\mathsf{Fg}^! \circ \mathsf{Fg}$).

3.3. Exceptional bundles $\mathcal{E}^\lambda$. The crucial step is to replace the exceptional collection $\mathcal{U}^\lambda$ in $\mathcal{D}^{\mathbf{G}}_{\mathsf{B}}(X)$ by its *right dual exceptional collection* (see [B]).

Recall that if (E, F) is an exceptional pair in a triangulated category $\mathcal{T}$ then the *right mutation* $\mathbf{R}_F(E)$ is defined as the (shifted) cone

$$\mathbf{R}_F(E) := \mathsf{Cone}(E \xrightarrow{\mathsf{coev}} \mathsf{Hom}^\bullet(E, F)^\vee \otimes F)[-1],$$

It is well known that $(F, \mathbf{R}_F(E))$ is also an exceptional pair which generates the same subcategory in $\mathcal{T}$ as the initial pair (E, F).

Now assume that $E_1, \dots, E_n$ is an exceptional collection. Its right dual collection is defined as the collection obtained by a sequence of right mutations

$$(E_n, \mathbf{R}_{E_n} E_{n-1}, \mathbf{R}_{E_n}\mathbf{R}_{E_{n-1}} E_{n-2}, \dots, \mathbf{R}_{E_n} \cdots \mathbf{R}_{E_2} E_1).$$

This collection is exceptional and generates the same subcategory as the initial collection. Note that the composition of mutations $\mathbf{R}_{E_n} \cdots \mathbf{R}_{E_{n-i}}$ depends only on the subcategory generated by $E_n, \dots, E_{n-i}$, so we denote it by $\mathbf{R}_{\langle E_n, \dots, E_{n-i}\rangle}$.

Now we apply this construction to the exceptional collection $(\mathcal{U}^\lambda)_{\lambda \in \mathsf{B}}$ (with respect to some total ordering refining the opposite of the ξ-ordering) in the derived category of equivariant sheaves $\mathcal{D}^{\mathbf{G}}(X)$ and denote by

$$\mathcal{E}^\lambda_{\mathsf{B}} := \mathbf{R}_{\langle \mathcal{U}^\mu \rangle_{\{\mu \in \mathsf{B} \mid \mu \prec \lambda\}}} \mathcal{U}^\lambda \tag{12}$$

the objects of the right dual collection (as this formula indicates, $\mathcal{E}^\lambda_{\mathsf{B}}$ does not depend on the choice of the total ordering). Further on we will frequently drop the index B in the notation $\mathcal{E}^\lambda_{\mathsf{B}}$ if it is clear which block B is considered.

By definition, the objects $\mathcal{E}^\lambda$ are exceptional in the derived category of equivariant sheaves. Our goal now is to show that the objects $\mathsf{Fg}(\mathcal{E}^\lambda)$ in the usual derived category $\mathcal{D}(X)$ are also exceptional and moreover form a full exceptional collection in $\mathcal{D}_{\mathsf{B}}(X)$.

First of all, recall that the standard property of the right dual exceptional collections gives

$$\mathsf{Ext}^\bullet_{\mathbf{G}}(\mathcal{E}^\lambda, \mathcal{U}^\mu) = \begin{cases} \mathsf{k} & \text{for } \lambda = \mu, \\ 0 & \text{otherwise} \end{cases} \tag{13}$$

(see e.g. [B]). Also, it follows from the construction of the dual collection that the subcategories both in $\mathcal{D}^{\mathbf{G}}(X)$ and $\mathcal{D}(X)$ generated by the objects $\mathcal{E}^\mu$ and $\mathcal{U}^\mu$ coincide:

$$\langle \mathcal{E}^\mu \rangle_{\mu \preceq \lambda} = \langle \mathcal{U}^\mu \rangle_{\mu \preceq \lambda}, \tag{14}$$

and moreover for each λ there is a morphism $\mathcal{E}^\lambda \to \mathcal{U}^\lambda$ such that

$$\mathsf{Cone}(\mathcal{E}^\lambda \to \mathcal{U}^\lambda) \in \langle \mathcal{U}^\mu \rangle_{\mu \prec \lambda}. \tag{15}$$

Corollary 3.8. *For all $\lambda, \mu \in \mathsf{B}$ we have*

$$\mathsf{Ext}^\bullet(\mathsf{Fg}(\mathcal{E}^\lambda), \mathcal{U}^\mu) = \mathsf{Hom}(\mathcal{U}^\lambda, \mathcal{U}^\mu). \tag{16}$$

Proof. Indeed, by Proposition 3.6 we have

$$\mathsf{Ext}^\bullet(\mathsf{Fg}(\mathcal{E}^\lambda), \mathcal{U}^\mu) \cong \mathsf{Ext}^\bullet_{\mathbf{G}}(\mathcal{E}^\lambda, \mathsf{Fg}^!(\mathcal{U}^\mu)) \cong \mathsf{Ext}^\bullet_{\mathbf{G}}\Big(\mathcal{E}^\lambda, \bigoplus_{\nu \in \mathsf{B}} \mathsf{Hom}(\mathcal{U}^\nu, \mathcal{U}^\mu) \otimes \mathcal{U}^\nu\Big).$$

Now note that by (13) we have $\mathsf{Ext}^\bullet_{\mathbf{G}}(\mathcal{E}^\lambda, \mathcal{U}^\nu) = 0$ unless $\lambda = \nu$. Thus, the RHS equals $\mathsf{Hom}(\mathcal{U}^\lambda, \mathcal{U}^\mu)$. □

Proposition 3.9. *For an exceptional block B the objects $\mathsf{Fg}(\mathcal{E}^\lambda)$ form a full exceptional collection in $\mathcal{D}_{\mathsf{B}}(X)$ with respect to any total ordering refining the ξ-ordering.*

Proof. First, take $\mu \prec \lambda$. By (16) and Lemma 3.2 we have $\mathsf{Ext}^\bullet(\mathsf{Fg}(\mathcal{E}^\lambda), \mathcal{U}^\mu) = 0$. Then (14) implies that $\mathsf{Ext}^\bullet(\mathsf{Fg}(\mathcal{E}^\lambda), \mathsf{Fg}(\mathcal{E}^\mu)) = 0$ as well. On the other hand, using this semiorthogonality and (15) we deduce that $\mathsf{Ext}^\bullet(\mathsf{Fg}(\mathcal{E}^\lambda), \mathsf{Fg}(\mathcal{E}^\lambda)) \cong \mathsf{Ext}^\bullet(\mathsf{Fg}(\mathcal{E}^\lambda), \mathcal{U}^\lambda) = \mathsf{Hom}(\mathcal{U}^\lambda, \mathcal{U}^\lambda) = \mathsf{k}$, so each $\mathsf{Fg}(\mathcal{E}^\lambda)$ is exceptional. Finally, the fullness of the collection $\{\mathsf{Fg}(\mathcal{E}^\lambda)\}_{\lambda \in \mathsf{B}}$ in $\mathcal{D}_{\mathsf{B}}(X)$ follows from (14). □

From now on to unburden the notation we will denote $\mathsf{Fg}(\mathcal{E}^\lambda)$ simply by $\mathcal{E}^\lambda$.

3.4. Properties of exceptional blocks.

Let B be any subset of $P^+_{\mathbf{L}}$ and $\mu \in P^+_{\mathbf{L}}$. Denote

$$\mathsf{B} + \mu = \{\lambda + \mu \mid \lambda \in \mathsf{B}\}.$$

Lemma 3.10. *If B is an exceptional block then for each $t \in \mathbb{Z}$ the block $\mathsf{B} + t\xi$ is exceptional. Moreover, $\mathcal{E}^{\lambda+t\xi}_{\mathsf{B}+t\xi} = \mathcal{E}^\lambda_{\mathsf{B}}(t)$.*

Proof. Recall that $\mathcal{U}^{t\xi} = \mathcal{O}_X(t)$ and twisting by this bundle takes $\mathcal{U}^\lambda$ to $\mathcal{U}^{\lambda+t\xi}$. Since such a twisting is an autoequivalence, it follows that it preserves the exceptionality of a block. □

Let us say that a subset $\mathsf{B}' \subset \mathsf{B}$ is *downward closed with respect to the ξ-ordering* if for any $\lambda, \mu \in \mathsf{B}$, if $\lambda \in \mathsf{B}'$ and $\mu \preceq \lambda$ then $\mu \in \mathsf{B}'$.

Lemma 3.11. *Let B be an exceptional block, and let $\mathsf{B}' \subset \mathsf{B}$ be a subset downward closed with respect to the ξ-ordering. Then B' is an exceptional block. Moreover, $\mathcal{E}^\lambda_{\mathsf{B}'} = \mathcal{E}^\lambda_{\mathsf{B}}$ for all $\lambda \in \mathsf{B}'$.*

Proof. Take $\lambda, \mu \in \mathsf{B}'$ and consider the map (10). It is an isomorphism since B is exceptional. On the other hand, $\nu \in \mathsf{B}$ contributes to the LHS only if $\mathsf{Ext}^\bullet_{\mathbf{G}}(\mathcal{U}^\lambda, \mathcal{U}^\nu) \neq 0$ which by Lemma 3.3 implies that $\nu \prec \lambda$. But then $\nu \in \mathsf{B}'$ since B' is downward closed with respect to the ξ-ordering. Thus, the LHS of (10) coincides with the LHS of the analogous map written for the block B', hence B' is exceptional.

Isomorphism of $\mathcal{E}^\lambda_{\mathsf{B}'}$ and $\mathcal{E}^\lambda_{\mathsf{B}}$ follows immediately from the definition (12). □

3.5. The output set and the criterion of exceptionality. In this section we give a criterion for a block B to be exceptional in terms of the Weyl group action on weights and the representation theory of **L**. We start with some preparations.

Lemma 3.12. *Let* $\mu \in P^+_{\mathbf{L}} \cap (P^{\mathsf{reg}}_{\mathbf{G}} - \rho)$*. Then there exists a unique pair* (κ, v)*, where* $\kappa \in P^+_{\mathbf{G}}$ *and* $v \in \mathbf{W_G}$ *such that*

$$\mu = v(\kappa + \rho) - \rho.$$

Moreover, $v \in \mathsf{SR}^{\mathbf{L}}_{\mathbf{G}}$.

Proof. The existence and uniqueness of the pair (κ, v) follow from the regularity of $\mu+\rho$. And since $\mu \in P^+_{\mathbf{L}}$ we conclude that $v \in \mathsf{SR}^{\mathbf{L}}_{\mathbf{G}}$. □

Using this simple observation we can rewrite the formula of Corollary 2.16 as follows:

$$\mathsf{Ext}^\bullet(\mathcal{U}^\lambda, \mathcal{U}^{\lambda'}) = \bigoplus_{\kappa\in P^+_{\mathbf{G}},\, v\in\mathsf{SR}^{\mathbf{L}}_{\mathbf{G}} \,|\, v(\kappa+\rho)-\rho\in\mathsf{Conv}(\lambda'-w\lambda)_{w\in\mathbf{W_L}}} \mathsf{Hom}(V_{\mathbf{L}}^{v(\kappa+\rho)-\rho}, V_{\mathbf{L}}^{\lambda'}\otimes V_{\mathbf{L}}^{-w_0^{\mathbf{L}}\lambda})\otimes V_{\mathbf{G}}^{\kappa}[-\ell(v)].$$

It is clear from this formula that it is convenient to have control over the set of all pairs (κ, v) which can appear on the RHS. So, we define the *output set* for the pair of weights λ, λ' of **L** as

$$\mathsf{OP}(\lambda, \lambda') = \{(\kappa, v) \in P^+_{\mathbf{G}} \times \mathsf{SR}^{\mathbf{L}}_{\mathbf{G}} \mid v(\kappa + \rho) - \rho \in \mathsf{Conv}(\lambda' - w\lambda)_{w\in\mathbf{W_L}}\}.$$

Consequently, we define the *output set* of a block B to be

$$\mathsf{OP}(\mathsf{B}) = \bigcup_{\lambda,\lambda'\in\mathsf{B}} \mathsf{OP}(\lambda, \lambda') \subset P^+_{\mathbf{G}} \times \mathsf{SR}^{\mathbf{L}}_{\mathbf{G}},$$

and we denote by $\mathsf{OP}_1(\mathsf{B}) \subset P^+_{\mathbf{G}}$ and $\mathsf{OP}_2(\mathsf{B}) \subset \mathsf{SR}^{\mathbf{L}}_{\mathbf{G}}$ the projections of $\mathsf{OP}(\mathsf{B})$ to $P^+_{\mathbf{G}}$ and $\mathsf{SR}^{\mathbf{L}}_{\mathbf{G}}$ respectively, so that

$$\mathsf{OP}(\mathsf{B}) \subset \mathsf{OP}_1(\mathsf{B}) \times \mathsf{OP}_2(\mathsf{B}).$$

Using these definitions we can rewrite the formula of Corollary 2.16 as follows:

$$\mathsf{Ext}^\bullet(\mathcal{U}^\lambda, \mathcal{U}^{\lambda'}) = \bigoplus_{(\kappa,v)\in\mathsf{OP}(\lambda,\lambda')} \mathsf{Hom}(V_{\mathbf{L}}^{v(\kappa+\rho)-\rho}, V_{\mathbf{L}}^{\lambda'} \otimes V_{\mathbf{L}}^{-w_0^{\mathbf{L}}\lambda}) \otimes V_{\mathbf{G}}^{\kappa}[-\ell(v)]. \quad (17)$$

Note that we can extend the range of summation in the above formula. Indeed, if for a pair (κ, v) one has $v(\kappa+\rho)-\rho \notin \mathsf{Conv}(\lambda'-w\lambda)_{w\in\mathbf{W_L}}$ then $\mathsf{Hom}(V_{\mathbf{L}}^{v(\kappa+\rho)-\rho}, V_{\mathbf{L}}^{\lambda'}\otimes V_{\mathbf{L}}^{-w_0^{\mathbf{L}}\lambda})=0$ by Lemma 2.9, and we have no contribution. So, we can replace $\mathsf{OP}(\lambda, \lambda')$ by $\mathsf{OP}(\mathsf{B})$, or even by $\mathsf{OP}_1(\mathsf{B}) \times \mathsf{OP}(\mathsf{B}_2)$.

Also, for each set of **L**-dominant weights $S \subset P^+_{\mathbf{L}}$ denote by $\Pi_S : \mathsf{Rep}\,\mathbf{L} \to \mathsf{Rep}\,\mathbf{L}$ the projector onto the subcategory formed by all $V_{\mathbf{L}}^{\nu}$ with $\nu \in S$. In other words, Π_S is a functor such that

$$\Pi_S(V_{\mathbf{L}}^{\lambda}) = \begin{cases} V_{\mathbf{L}}^{\lambda} & \text{if } \lambda \in S, \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 3.13. *Assume that a subset* $\mathsf{B} \subset P_{\mathbf{L}}^{+}$ *has the following two properties:*

(a) *for all* $\kappa \in \mathsf{OP}_1(\mathsf{B})$ *and* $\upsilon \in \mathsf{OP}_2(\mathsf{B})$ *we have* $\upsilon\kappa = \kappa$*;*

(b) *for all* $\kappa \in \mathsf{OP}_1(\mathsf{B})$*,* $\upsilon \in \mathsf{OP}_2(\mathsf{B})$ *and* $\lambda \in \mathsf{B}$ *the canonical map*

$$\Pi_{\mathsf{B}}(V_{\mathbf{L}}^{\kappa+\upsilon\rho-\rho} \otimes V_{\mathbf{L}}^{\lambda}) \to \Pi_{\mathsf{B}}(V_{\mathbf{L}}^{\kappa} \otimes \Pi_{\mathsf{B}}(V_{\mathbf{L}}^{\upsilon\rho-\rho} \otimes V_{\mathbf{L}}^{\lambda})) \tag{18}$$

is an isomorphism.

Then the block B *is exceptional.*

In what follows we will refer to part (a) of this criterion as the *invariance condition*, and to part (b) as the *compatibility condition*.

Proof. Fix a pair of weights $\lambda, \lambda' \in \mathsf{B}$. We have to check that the map (10) (with $\mu = \lambda'$) is an isomorphism.

We start by rewriting (17) in a more convenient form. First of all, we extend the summation to $\mathsf{OP}_1(\mathsf{B}) \times \mathsf{OP}_2(\mathsf{B})$ (as mentioned above, this does not spoil the equality). Next, we use the isomorphism

$$\begin{aligned}\mathsf{Hom}(V_{\mathbf{L}}^{\upsilon(\kappa+\rho)-\rho}, V_{\mathbf{L}}^{\lambda'} \otimes V_{\mathbf{L}}^{-w_0^{\mathbf{L}}\lambda}) &\simeq \mathsf{Hom}(V_{\mathbf{L}}^{\lambda'}, V_{\mathbf{L}}^{\upsilon(\kappa+\rho)-\rho} \otimes V_{\mathbf{L}}^{\lambda})^{\vee} \\ &\simeq \mathsf{Hom}(V_{\mathbf{L}}^{\lambda'}, \Pi_{\mathsf{B}}(V_{\mathbf{L}}^{\upsilon(\kappa+\rho)-\rho} \otimes V_{\mathbf{L}}^{\lambda}))^{\vee},\end{aligned}$$

where the second isomorphism follows from the condition $\lambda' \in \mathsf{B}$. Finally, by the invariance condition we have $\upsilon(\kappa+\rho)-\rho = \kappa+\upsilon\rho-\rho$. Thus,

$$\begin{aligned}&\mathsf{Ext}^{\bullet}(\mathcal{U}^{\lambda}, \mathcal{U}^{\lambda'}) \\ &\quad = \bigoplus_{\kappa\in\mathsf{OP}_1(\mathsf{B}),\, \upsilon\in\mathsf{OP}_2(\mathsf{B})} \mathsf{Hom}(V_{\mathbf{L}}^{\lambda'}, \Pi_{\mathsf{B}}(V_{\mathbf{L}}^{\kappa+\upsilon\rho-\rho} \otimes V_{\mathbf{L}}^{\lambda}))^{\vee} \otimes V_{\mathbf{G}}^{\kappa}[-\ell(\upsilon)].\end{aligned} \tag{19}$$

Now specializing (19) we can obtain an expression for $\mathsf{Ext}_{\mathbf{G}}$ and Hom on the LHS of (10). To obtain an expression for $\mathsf{Ext}_{\mathbf{G}}$ we should restrict to the case $\kappa = 0$. Replacing also λ' by $\nu \in \mathsf{B}$ gives

$$\mathsf{Ext}_{\mathbf{G}}^{\bullet}(\mathcal{U}^{\lambda}, \mathcal{U}^{\nu}) = \bigoplus_{\upsilon\in\mathsf{OP}_2(\mathsf{B})} \mathsf{Hom}(V_{\mathbf{L}}^{\nu}, \Pi_{\mathsf{B}}(V_{\mathbf{L}}^{\upsilon\rho-\rho} \otimes V_{\mathbf{L}}^{\lambda}))^{\vee}[-\ell(\upsilon)]. \tag{20}$$

On the other hand, to obtain an expression for Hom we should restrict to $\upsilon = 1$. Replacing also λ by ν we obtain

$$\mathsf{Hom}(\mathcal{U}^{\nu}, \mathcal{U}^{\lambda'}) = \bigoplus_{\kappa\in\mathsf{OP}_1(\mathsf{B})} \mathsf{Hom}(V_{\mathbf{L}}^{\lambda'}, \Pi_{\mathsf{B}}(V_{\mathbf{L}}^{\kappa} \otimes V_{\mathbf{L}}^{\nu}))^{\vee} \otimes V_{\mathbf{G}}^{\kappa}. \tag{21}$$

Combining (20) with (21) we rewrite the LHS of (10) as

$$\begin{aligned}&\bigoplus_{\nu\in\mathsf{B}} \mathsf{Ext}_{\mathbf{G}}^{\bullet}(\mathcal{U}^{\lambda}, \mathcal{U}^{\nu}) \otimes \mathsf{Hom}(\mathcal{U}^{\nu}, \mathcal{U}^{\lambda'}) = \\ &\bigoplus_{\nu\in\mathsf{B},\, \kappa\in\mathsf{OP}_1(\mathsf{B}),\, \upsilon\in\mathsf{OP}_2(\mathsf{B})} \mathsf{Hom}(V_{\mathbf{L}}^{\nu}, \Pi_{\mathsf{B}}(V_{\mathbf{L}}^{\upsilon\rho-\rho} \otimes V_{\mathbf{L}}^{\lambda}))^{\vee} \otimes \mathsf{Hom}(V_{\mathbf{L}}^{\lambda'}, \Pi_{\mathsf{B}}(V_{\mathbf{L}}^{\kappa} \otimes V_{\mathbf{L}}^{\nu}))^{\vee} \otimes V_{\mathbf{G}}^{\kappa}[-\ell(\upsilon)] \\ &\quad = \bigoplus_{\kappa\in\mathsf{OP}_1(\mathsf{B}),\, \upsilon\in\mathsf{OP}_2(\mathsf{B})} \mathsf{Hom}(V_{\mathbf{L}}^{\lambda'}, \Pi_{\mathsf{B}}(V_{\mathbf{L}}^{\kappa} \otimes \Pi_{\mathsf{B}}(V_{\mathbf{L}}^{\upsilon\rho-\rho} \otimes V_{\mathbf{L}}^{\lambda})))^{\vee} \otimes V_{\mathbf{G}}^{\kappa}[-\ell(\upsilon)],\end{aligned}$$

where the second equality follows from the formula

$$\Pi_{\mathsf{B}}(V_{\mathbf{L}}^{\nu\rho-\rho}\otimes V_{\mathbf{L}}^{\lambda})=\bigoplus_{\nu\in\mathsf{B}}\mathsf{Hom}(V_{\mathbf{L}}^{\nu},\Pi_{\mathsf{B}}(V_{\mathbf{L}}^{\nu\rho-\rho}\otimes V_{\mathbf{L}}^{\lambda}))^{\vee}\otimes V_{\mathbf{L}}^{\nu}.$$

To conclude we compare the expression for the LHS of (10) with the expression (19) for the RHS and note that the map from the LHS of (10) to the RHS is induced by the map (18). Thus, if the compatibility property (b) holds then this map is an isomorphism, hence the block B is exceptional. □

4. On strongness and purity

Note that a priori the exceptional objects $\mathcal{E}^{\lambda}$ constructed above are complexes. However, we have the following

Conjecture 4.1. *For any exceptional block $\mathsf{B}\subset P_{\mathbf{L}}^{+}$ and any $\lambda\in\mathsf{B}$ the object $\mathcal{E}^{\lambda}$ is a vector bundle.*

Note that the standard t-structure on $\mathcal{D}^{\mathbf{G}}(X)$ restricts to a t-structure on the category $\mathcal{D}^{\mathbf{G}}_{\mathsf{B}}(X)$ whose heart $\mathcal{C}_{\mathsf{B}}$ consists of $\mathbf{G}$-equivariant coherent sheaves that are obtained by successive extensions from $\mathcal{U}^{\lambda}$ with $\lambda\in\mathsf{B}$. As already mentioned, the category of $\mathbf{G}$-equivariant coherent sheaves on X is equivalent to the category of finite-dimensional representations of $\mathbf{P}$, which in turn is equivalent to the category of finite-dimensional representations of a certain infinite quiver with relations $(\mathcal{Q},\mathcal{I})$ (see [Hille]). Recall that the vertices of $\mathcal{Q}$ are in bijection with the set $P_{\mathbf{L}}^{+}$ of dominant weights of $\mathbf{L}$, and there is an arrow $\lambda\to\mu$ if and only if $V_{\mathbf{L}}^{\mu}$ appears in $V_{\mathbf{L}}^{-\beta}\otimes V_{\mathbf{L}}^{\lambda}$ (i.e., when there is a nontrivial $\mathsf{Ext}^1_{\mathbf{G}}(\mathcal{U}^{\lambda},\mathcal{U}^{\mu})$). Note that by Lemma 3.3, this quiver is leveled by the function

$$\boldsymbol{w}(\lambda)=-(\xi,\lambda)/(\xi,\beta),\tag{22}$$

which means that for every arrow $\lambda\to\mu$ one has $\boldsymbol{w}(\mu)=\boldsymbol{w}(\lambda)+1$. The subcategory $\mathcal{C}_{\mathsf{B}}$ corresponds to the subcategory of representations supported at the vertices $\mathsf{B}\subset P_{\mathbf{L}}^{+}$. Hence, it is equivalent to the category of finite-dimensional representations $\mathsf{k}[\mathcal{Q}_{\mathsf{B}}]/\mathcal{I}_{\mathsf{B}}$, where $\mathsf{k}[\mathcal{Q}_{\mathsf{B}}]$ is the path algebra of the full subquiver $\mathcal{Q}_{\mathsf{B}}\subset\mathcal{Q}$ corresponding to the set of vertices B, and $\mathcal{I}_{\mathsf{B}}$ is an ideal of relations.

We refer to [ARS] for an introduction to quivers and representation theory of finite-dimensional algebras. In particular, we use the notion of a projective cover P of a simple object S corresponding to a vertex (such a P is an indecomposable projective object with a surjective map $P\to S$).

Proposition 4.2. *The following conditions are equivalent:*

(i) *Each $\mathcal{E}^{\lambda}$ for $\lambda\in\mathsf{B}$ is a vector bundle.*
(ii) *For each $\lambda\in\mathsf{B}$, $\mathcal{E}^{\lambda}$ is a projective cover of $\mathcal{U}^{\lambda}$ in the category $\mathcal{C}_{\mathsf{B}}$.*
(iii) *The natural map $\mathsf{Ext}^{\bullet}_{\mathcal{C}_{\mathsf{B}}}(\mathcal{U}^{\lambda},\mathcal{U}^{\mu})\to\mathsf{Ext}^{\bullet}_{\mathbf{G}}(\mathcal{U}^{\lambda},\mathcal{U}^{\mu})$ is an isomorphism for any $\lambda,\mu\in\mathsf{B}$.*
(iv) *The exceptional collection $(\mathcal{E}^{\lambda})_{\lambda\in\mathsf{B}}$ in $\mathcal{D}(X)$ is strong.*

Furthermore, under these conditions the canonical map $\mathcal{E}^\lambda \to \mathcal{U}^\lambda$ *induces an isomorphism*

$$\mathsf{Hom}(\mathcal{U}^\lambda, \mathcal{E}^\mu) \xrightarrow{\sim} \mathsf{Ext}^\bullet(\mathcal{E}^\lambda, \mathcal{E}^\mu), \tag{23}$$

where $\lambda, \mu \in \mathsf{B}$.

Proof. (i)⇒(ii). If the objects $\mathcal{E}^\lambda$ are vector bundles then they belong to $\mathcal{C}_\mathsf{B}$. Furthermore, since $\mathcal{C}_\mathsf{B}$ is the heart of a t-structure of a full subcategory $\mathcal{D}^{\mathbf{G}}_\mathsf{B}(X)$ of $\mathcal{D}^{\mathbf{G}}(X)$ we have $\mathsf{Ext}^1_{\mathcal{C}_\mathsf{B}}(\mathcal{E}^\lambda, \mathcal{U}^\mu) \simeq \mathsf{Ext}^1_{\mathbf{G}}(\mathcal{E}^\lambda, \mathcal{U}^\mu) = 0$ for $\lambda, \mu \in \mathsf{B}$. This implies that $\mathsf{Ext}^1_{\mathcal{C}_\mathsf{B}}(\mathcal{E}^\lambda, \mathcal{F}) = 0$ for any $\mathcal{F}$ in $\mathcal{C}_\mathsf{B}$, i.e., $\mathcal{E}^\lambda$ is projective.

(ii)⇒(i). If $\mathcal{E}^\lambda$ is a projective cover of $\mathcal{U}^\lambda$ in $\mathcal{C}_\mathsf{B}$ then $\mathcal{E}^\lambda$ itself is an object of $\mathcal{C}_\mathsf{B}$, hence a successive extension of $\mathcal{U}^\mu$ with $\mu \in \mathsf{B}$. In particular, it is a vector bundle on X.

(ii)⇒(iii). Using (ii) we can construct for any object $\mathcal{F}$ in $\mathcal{C}_\mathsf{B}$ a projective resolution consisting of direct sums of objects $\mathcal{E}^\lambda$. Computing $\mathsf{Ext}^\bullet_{\mathcal{C}_\mathsf{B}}(\mathcal{F}, \mathcal{U}^\mu)$ using such a resolution and using the isomorphisms

$$\mathsf{Hom}_{\mathcal{C}_\mathsf{B}}(\mathcal{E}^\lambda, \mathcal{U}^\mu) \simeq \mathsf{Ext}^\bullet_{\mathbf{G}}(\mathcal{E}^\lambda, \mathcal{U}^\mu)$$

(coming from the assumption that $\mathcal{E}^\lambda$ is a projective cover of $\mathcal{U}^\lambda$ and from the orthogonality relations (13)), we derive that the map $\mathsf{Ext}^\bullet_{\mathcal{C}_\mathsf{B}}(\mathcal{F}, \mathcal{U}^\mu) \to \mathsf{Ext}^\bullet_{\mathbf{G}}(\mathcal{F}, \mathcal{U}^\mu)$ is an isomorphism.

(iii)⇒(ii). For $\lambda \in \mathsf{B}$ let $\mathcal{P}^\lambda \to \mathcal{U}^\lambda$ be the projective cover of $\mathcal{U}^\lambda$ in $\mathcal{C}_\mathsf{B}$. Condition (iii) implies that the natural map

$$\mathsf{Ext}^\bullet_{\mathcal{C}_\mathsf{B}}(\mathcal{P}^\lambda, \mathcal{U}^\mu) \to \mathsf{Ext}^\bullet_{\mathbf{G}}(\mathcal{P}^\lambda, \mathcal{U}^\mu)$$

is an isomorphism. It follows that $(\mathcal{P}^\lambda)$ satisfies the orthogonality relations (13), characterizing the right dual exceptional sequence to $(\mathcal{U}^\lambda)_{\lambda \in \mathsf{B}}$, so we get $\mathcal{P}^\lambda \simeq \mathcal{E}^\lambda$ for $\lambda \in \mathsf{B}$.

(i)⇒(iv). Condition (10) implies that the natural map

$$\bigoplus_{\nu \in \mathsf{B}} \mathsf{Ext}^\bullet_{\mathbf{G}}(A, \mathcal{U}^\nu) \otimes \mathsf{Hom}(\mathcal{U}^\nu, B) \to \mathsf{Ext}^\bullet(A, B)$$

is an isomorphism for any $A, B \in \mathcal{C}_\mathsf{B}$. Applying this to $A = \mathcal{E}^\lambda$, $B = \mathcal{E}^\mu$, where $\lambda, \mu \in \mathsf{B}$, and using (13), we derive the isomorphism (23), which implies that the exceptional collection $(\mathcal{E}^\lambda)$ is strong.

(iv)⇒(i). Choose any ordering of $\mathcal{U}^\lambda$ compatible with the partial ordering $\prec$. Let $\mathcal{U}_p$ denote the p-th object for this ordering. Let $\mathcal{E}_p$ be the objects of the dual collection. Then for any $\mathcal{F} \in \mathcal{D}^{\mathbf{G}}_\mathsf{B}(X)$ there is a spectral sequence $\mathsf{Ext}^q_{\mathbf{G}}(\mathcal{E}_p[p], \mathcal{F}) \otimes \mathcal{U}_p \Rightarrow H^{q-p}\mathcal{F}$. For $\mathcal{F} = \mathcal{E}^\lambda$ this spectral sequence implies (i). □

Now we are going to prove two criteria for the equivalent conditions of Proposition 4.2 to hold.

Proposition 4.3. *Assume that the subquiver* $\mathcal{Q}_\mathsf{B} \subset \mathcal{Q}$ *contains entirely any path in* $\mathcal{Q}$ *that starts and ends in* $\mathcal{Q}_\mathsf{B}$. *Then the equivalent conditions of Proposition 4.2 hold.*

Proof. Recall that the projective cover of a simple object of a vertex λ is the representation of $(\mathcal{Q}_{\mathsf{B}}, \mathcal{I}_{\mathsf{B}})$ associating with a vertex $\mu \in \mathsf{B}$ the vector space generated by all paths in the quiver from λ to μ (modulo the relations). The assumption of the proposition ensures that this representation is isomorphic to the restriction to $\mathcal{Q}_{\mathsf{B}}$ of the projective cover of the simple object of the vertex λ in the category of representations of $\mathcal{Q}$. It follows that Hom's between projective objects in $\mathcal{Q}_{\mathsf{B}}$ are the same as in $\mathcal{Q}$, and moreover, minimal projective resolutions of simple objects in $\mathcal{Q}_{\mathsf{B}}$ are the restrictions of their minimal projective resolutions in $\mathcal{Q}$. Combining all this we deduce that Ext's between simple objects in $\mathcal{C}_{\mathsf{B}}$ are isomorphic to those in $\mathcal{C} = \mathsf{Coh}^{\mathbf{G}}(X)$, i.e. condition (iii) of Proposition 4.2 holds. □

Also, the properties of purity and strongness of the collection $\mathcal{E}^{\lambda}$ are related to the Koszulity of a certain algebra. We refer to [BGS], [PP] for basic facts about Koszul algebras.

Proposition 4.4. (i) *Assume that the graded algebra*

$$A_{\mathsf{B}} = \bigoplus_{\lambda,\mu \in \mathsf{B}} \mathsf{Ext}^{\bullet}_{\mathbf{G}}(\mathcal{U}^{\lambda}, \mathcal{U}^{\mu})$$

is Koszul (with respect to the cohomological grading). Then the equivalent conditions of Proposition 4.2 hold.

(ii) *If the algebra A_{B} is one-generated then Koszulity of A_{B} is equivalent to the conditions of Proposition 4.2.*

Proof. (i) This follows from the main result of [Pos, Cor. 8] (see also [P97, proofs of Theorems 4.1 and 4.2]).

(ii) If condition (iii) of Proposition 4.2 is satisfied then A_{B} is isomorphic (as a graded algebra) to the Ext-algebra between simple objects in the abelian category $\mathcal{C}_{\mathsf{B}}$. Thus, the assumption that A_{B} is one-generated implies that (22) is a Koszul weight function on the set of simple objects of $\mathcal{C}_{\mathsf{B}}$, i.e., $\mathsf{Ext}^i_{\mathcal{C}_{\mathsf{B}}}(\mathcal{U}^{\lambda}, \mathcal{U}^{\mu}) \neq 0$ only for $\boldsymbol{w}(\mu) - \boldsymbol{w}(\lambda) = i$. Thus, by [BGS, Prop. 2.1.3], the algebra $\mathsf{k}[\mathcal{Q}_{\mathsf{B}}]/\mathcal{I}_{\mathsf{B}}$ is Koszul, hence its Yoneda algebra A_{B} is also Koszul. □

Remark 4.5. In the case when the unipotent radical of $\mathbf{P}$ is abelian (in this case the Grassmannian $X = \mathbf{G}/\mathbf{P}$ is called *cominuscule*) and the subquiver $\mathcal{Q}_{\mathsf{B}} \subset \mathcal{Q}$ contains entirely any path that starts and ends in $\mathcal{Q}_{\mathsf{B}}$, the algebra A_{B} is Koszul, as follows from the main result of [Hille] and from Proposition 4.3.

Remark 4.6. In Section 9.3 we will give an example (Example 9.5) of an exceptional block for which Proposition 4.3 does not apply, and at the same time the inequality of Lemma 3.3 becomes strict in some cases (and so the algebra A_{B} is not one-generated) and so Proposition 4.4 does not apply either, but the equivalent conditions of Proposition 4.2 still hold. See Conjecture 9.6 for a possible explanation of this.

5. Constructing exceptional blocks

In this section we suggest a construction of a semiorthogonal collection of blocks, which will be proved to be exceptional in Sections 6 and 7.

5.1. Cores. Let $\mathbf{H}$ be a semisimple group. Let $\delta \in P_{\mathbf{H}}^{+}$ be a strictly dominant weight (see Definition 2.5).

Definition 5.1. The polyhedron

$$\mathbf{R}_{\delta} = \{\lambda \in P_{\mathbf{H}} \otimes \mathbb{R} \mid \forall w \in \mathbf{W_H}\ (w\delta, \lambda) \le (\delta, \rho_{\mathbf{H}})\} \tag{24}$$

is called the *core* of shape δ.

We will denote by

$$\mathbf{R}_{\delta}^{*} := \{\lambda \in P_{\mathbf{H}} \otimes \mathbb{R} \mid \forall w \in \mathbf{W_H}\ (w\delta, \lambda) < (\delta, \rho_{\mathbf{H}})\} \tag{25}$$

the interior of the core $\mathbf{R}_{\delta}$. Note that both $\mathbf{R}_{\delta}$ and $\mathbf{R}_{\delta}^{*}$ are $\mathbf{W_H}$-invariant and convex.

Lemma 5.2. *The intersection of a core with the set of dominant weights is given by*

$$\mathbf{R}_{\delta} \cap P_{\mathbf{H}}^{+} = \{\lambda \in P_{\mathbf{H}}^{+} \mid (\delta, \lambda) \le (\delta, \rho_{\mathbf{H}})\}.$$

Similarly,

$$\mathbf{R}_{\delta}^{*} \cap P_{\mathbf{H}}^{+} = \{\lambda \in P_{\mathbf{H}}^{+} \mid (\delta, \lambda) < (\delta, \rho_{\mathbf{H}})\}.$$

Proof. Let us check the first equality (the second is proved analogously). By definition, the LHS is contained in the RHS. On the other hand, since both λ and δ are $\mathbf{H}$-dominant, by Corollary 2.4 we have $(w\delta, \lambda) \le (\delta, \lambda)$ for all $w \in \mathbf{W_H}$, hence the RHS is contained in the LHS. □

We will say that a point of $P_{\mathbf{H}} \otimes \mathbb{R}$ is *integral* if it lies in the weight lattice $P_{\mathbf{H}} \subset P_{\mathbf{H}} \otimes \mathbb{R}$.

Lemma 5.3. *All integral points of* $\mathbf{R}_{\delta}^{*}$ *are* $\mathbf{H}$*-singular. All* $\mathbf{H}$*-regular integral points of the core* $\mathbf{R}_{\delta}$ *are contained in the* $\mathbf{W_H}$*-orbit of* $\rho_{\mathbf{H}}$.

Proof. Assume that $\lambda \in P_{\mathbf{H}} \cap \mathbf{R}_{\delta}$ is $\mathbf{H}$-regular. Take $w \in \mathbf{W_H}$ such that $w\lambda$ is $\mathbf{H}$-dominant. Then $w\lambda \in \mathbf{R}_{\delta} \cap P_{\mathbf{H}}^{+}$, and since $w\lambda$ is $\mathbf{H}$-regular, we can write $w\lambda = \rho_{\mathbf{H}} + \mu$ with $\mu \in P_{\mathbf{H}}^{+}$. Therefore,

$$(\delta, \rho_{\mathbf{H}} + \mu) = (\delta, w\lambda) = (w^{-1}\delta, \lambda) \le (\delta, \rho_{\mathbf{H}}),$$

hence $(\delta, \mu) \le 0$. Since δ is strictly dominant, this implies that $\mu = 0$ by Lemma 2.6, hence $\lambda = w^{-1}\rho_{\mathbf{H}}$. □

5.2. The setup. Consider the complement $D_{\mathbf{G}} \setminus \beta$ of the vertex β of the Dynkin diagram $D_{\mathbf{G}}$ of $\mathbf{G}$. In general it consists of several (up to three) connected components of different types. We choose one component of type A (possibly empty) to be called the *outer component* and denote it by $D_{\mathbf{out}}$. The union of the other components will be called the *inner component* and denoted by $D_{\mathbf{inn}}$. We denote the corresponding connected semisimple groups by $\mathbf{L_{out}}$ and $\mathbf{L_{inn}}$, and by

$$o : \mathbf{L_{out}} \to \mathbf{L}, \qquad i : \mathbf{L_{inn}} \to \mathbf{L}$$

the canonical embeddings. Abusing the notation we will also denote by o (resp., i) the embedding of $\mathbf{L_{out}}$ (resp., $\mathbf{L_{inn}}$) into $\mathbf{G}$. Note that the groups $\mathbf{L_{out}}$ and $\mathbf{L_{inn}}$ are simply connected (this follows from the fact that an embedding of Dynkin diagrams induces a surjection of the weight lattices). In particular, we have

$$\mathbf{L_{out}} \cong \mathsf{SL}_k \tag{26}$$

for some $k \geq 1$. We fix a numbering of the vertices of $D = D_{\mathbf{G}}$ as follows. First, we number the vertices of the outer part $D_{\mathbf{out}} = A_{k-1}$ by integers from 1 to $k-1$ in a standard way (if $k \geq 3$ there are two ways to number the vertices of $D_{\mathbf{out}}$; see Section 1.3.1 for an illustration). Then we number the vertex β by k and the remaining vertices in an arbitrary way. We denote by b the number of the vertex in $D_{\mathbf{out}}$ which is adjacent to β (note that such a vertex is unique).

Note that we have the following decomposition of the Weyl group of $\mathbf{L}$:

$$\mathbf{W_L} = \mathbf{W_{L_{out}}} \times \mathbf{W_{L_{inn}}}$$

(since $D_{\mathbf{out}}$ and $D_{\mathbf{inn}}$ are not adjacent, the corresponding simple reflections commute).

Now consider the chain of subdiagrams

$$D_b \subset D_{b-1} \subset \cdots \subset D_1 \subset D_0 = D_{\mathbf{G}}, \qquad D_a = D_{\mathbf{G}} \setminus \{1, \ldots, a\}.$$

Let

$$\mathbf{H}_b \subset \mathbf{H}_{b-1} \subset \cdots \subset \mathbf{H}_1 \subset \mathbf{H}_0 = \mathbf{G}$$

be the corresponding chain of semisimple subgroups of $\mathbf{G}$. For $a = 0, \ldots, b$ we denote by $h_a : \mathbf{H}_a \to \mathbf{G}$ the embedding. Note that any $\mathbf{H}_a$ contains $\mathbf{L_{inn}}$. Abusing the notation we will denote the corresponding embedding by $i : \mathbf{L_{inn}} \to \mathbf{H}_a$.

For each $a = 0, \ldots, b$ we choose a strictly dominant weight $\delta_a \in P^+_{\mathbf{H}_a}$ (in the sense of Definition 2.5—note that the Dynkin diagram D_a may be disconnected, so the group $\mathbf{H}_a$ may be nonsimple) and consider the corresponding core $\mathbf{R}_{\delta_a} \subset P_{\mathbf{H}_a} \otimes \mathbb{R}$. To ease the notation we denote this core by $\mathbf{R}_a$. The interior of this core will be denoted by $\mathbf{R}^*_a$.

Let r be the index of $\mathbf{G}/\mathbf{P}$ and let r_a be the index of $\mathbf{H}_a/(\mathbf{H}_a \cap \mathbf{P})$. Note that by Corollary 2.22 we have

$$0 < r_b < r_{b-1} < \cdots < r_1 < r_0 = r.$$

5.3. The indexing set. Let us denote by θ an element of $P_{\mathbf{L}} \otimes \mathbb{Q}$ such that

$$\theta \in \langle \omega_1, \ldots, \omega_{k-1} \rangle^{\perp} \cap \operatorname{Ker} i^* \quad \text{and} \quad (\theta, \xi) = 1. \tag{27}$$

Since $\omega_1, \ldots, \omega_{k-1}, \xi$ form a basis of $\operatorname{Ker} i^* \subset P_{\mathbf{L}} \otimes \mathbb{Q}$, such a θ exists and unique. Note that the set $(\theta, P_{\mathbf{L}})$ of all scalar products of θ with weights of $\mathbf{L}$ is a cyclic subgroup of $\mathbb{Q}$ containing $\mathbb{Z}$. We consider the intersection of this subgroup with the interval $[0, r) \subset \mathbb{Q}$:

$$\mathsf{J} = \{\mathsf{j} \in (\theta, P_{\mathbf{L}}) \mid 0 \leq \mathsf{j} < r\}.$$

This set will number the blocks in the collection. Note that it is naturally linearly ordered. The blocks will be shown to be semiorthogonal with respect to this order.

For each $\mathsf{j} \in \mathsf{J}$ there is a unique integer $a(\mathsf{j})$ in the interval $0 \le a(\mathsf{j}) \le b$ such that

$$r - r_{a(\mathsf{j})} \le \mathsf{j} < r - r_{a(\mathsf{j})+1}, \tag{28}$$

where we set $r_{b+1} = 0$. To unburden the notation we will write $\mathbf{H}_\mathsf{j} = \mathbf{H}_{a(\mathsf{j})}$, $h_\mathsf{j} = h_{a(\mathsf{j})}$ and $\mathbf{R}_\mathsf{j} = \mathbf{R}_{\delta_{a(\mathsf{j})}}$.

Below we will need the following simple observation.

Lemma 5.4. *For any $\nu \in P_{\mathbf{L_{inn}}}$ there is a rational number $p \in (\theta, P_\mathbf{L})$ such that $p\xi + i_*\nu \in P_\mathbf{L}$.*

Proof. Since any ν is a linear combination of fundamental weights, it suffices to consider the case of $\nu = i^*\omega_t$ for some $t \in D_\mathbf{inn}$. Then it is clear that $i_*\nu = i_*i^*\omega_t$ is just the orthogonal projection of ω_t onto the subspace $i_*(P_{\mathbf{L_{inn}}} \otimes \mathbb{Q}) \subset P_\mathbf{L} \otimes \mathbb{Q}$. Its orthogonal complement is generated by the lattice $Q_{\mathbf{L_{out}}}$ and by the weight ξ. Moreover, ω_t is orthogonal to the lattice $Q_{\mathbf{L_{out}}}$ since $t \in D_\mathbf{inn}$. Hence,

$$i_*i^*\omega_t = \omega_t - \frac{(\omega_t, \xi)}{\xi^2}\xi.$$

It remains to check that $(\omega_t, \xi)/\xi^2 \in (\theta, P_\mathbf{L})$. For this we apply the linear function $(\theta, -)$ to the above equality. Since θ is orthogonal to the image of i_*, we conclude that $(\omega_t, \xi)/\xi^2 = (\theta, \omega_t) \in (\theta, P_\mathbf{L})$. □

5.4. The first approximation. For each element $\mathsf{j} \in \mathsf{J}$ of the indexing set we will define a subset $\hat{\mathsf{B}}_\mathsf{j} \subset P_\mathbf{L}^+$. We will show that this is an exceptional block if $\mathbf{G}$ is of type BCD. In other cases we will have to replace $\hat{\mathsf{B}}_\mathsf{j}$ by an appropriate smaller subset B_j.

First, we define the inner part as

$$\hat{\mathsf{B}}_\mathsf{j}^\mathbf{inn} = \left\{ \nu \in P_{\mathbf{L_{inn}}}^+ \;\middle|\; \begin{array}{l} (1)\ \rho_{\mathbf{H}_\mathsf{j}} \pm 2i_*(w\nu) \in \mathbf{R}_\mathsf{j} \text{ for all } w \in \mathbf{W}_{\mathbf{L_{inn}}} \\ (2)\ \mathsf{j}\xi + i_*\nu \in P_\mathbf{L} \end{array} \right\}. \tag{29}$$

After that we define the outer part as

$$\hat{\mathsf{B}}_\mathsf{j}^\mathbf{out} = \left\{ \mu \in \operatorname{Ker} h_\mathsf{j}^* \cap P_\mathbf{G}^+ \;\middle|\; \begin{array}{l} \rho_{\mathbf{H}_\mathsf{j}} - h_\mathsf{j}^*(w_{\mathbf{L_{out}}}\mu) - i_*(w_{\mathbf{L_{inn}}}\nu) + i_*(w'_{\mathbf{L_{inn}}}\nu') \in \mathbf{R}_\mathsf{j} \\ \text{for all } \nu, \nu' \in \hat{\mathsf{B}}_\mathsf{j}^\mathbf{inn},\ w_{\mathbf{L_{out}}} \in \mathbf{W}_{\mathbf{L_{out}}}, \text{ and } w_{\mathbf{L_{inn}}}, w'_{\mathbf{L_{inn}}} \in \mathbf{W}_{\mathbf{L_{inn}}} \end{array} \right\}. \tag{30}$$

And finally, we consider the set

$$\hat{\mathsf{B}}_\mathsf{j} = \hat{\mathsf{B}}_\mathsf{j}^\mathbf{out} + \mathsf{j}\xi + i_*(\hat{\mathsf{B}}_\mathsf{j}^\mathbf{inn}). \tag{31}$$

Remark 5.5. In both definitions (29) and (30) we can replace all terms of the form $i_*(w\nu)$ with $w \in \mathbf{W}_{\mathbf{L_{inn}}}$ by $wi_*(\nu)$ and allow w to run through the entire group $\mathbf{W_L}$. Indeed, this follows from the decomposition $\mathbf{W_L} = \mathbf{W}_{\mathbf{L_{out}}} \times \mathbf{W}_{\mathbf{L_{inn}}}$ together with the fact that $\mathbf{W}_{\mathbf{L_{out}}}$ acts trivially on the image of i_*.

5.5. Very special representatives. In this section we will define a certain class of elements of the set $\mathsf{SR}_\mathbf{H}^\mathbf{M}$ and use them to define a subblock $\mathsf{B}_\mathsf{j} \subset \hat{\mathsf{B}}_\mathsf{j}$. In fact, $\mathsf{B}_\mathsf{j} = \hat{\mathsf{B}}_\mathsf{j}$ if $\mathbf{G}$ is of type BCD.

Recall that $\mathbf{L_{out}} = \mathsf{SL}_k$ (see (26)). We will use the following representation of the weight lattice of SL_k:

$$P_{\mathsf{SL}_k} = \left\{(\lambda_1, \dots, \lambda_k) \in \mathbb{Q}^k \;\middle|\; \lambda_i - \lambda_{i+1} \in \mathbb{Z} \text{ for all } 1 \le i \le k-1 \text{ and } \sum_{i=1}^{k} \lambda_i = 0\right\},$$

where the simple roots and the fundamental weights are given by

$$\alpha_t = (\underbrace{0, \dots, 0}_{t-1}, 1, -1, \underbrace{0, \dots, 0}_{k-t-1}), \qquad \omega_t = \Bigg(\underbrace{\frac{k-t}{k}, \dots, \frac{k-t}{k}}_{t}, \underbrace{-\frac{t}{k}, \dots, -\frac{t}{k}}_{k-t}\Bigg).$$

Remark 5.6. Note that this representation fixes the scaling of the scalar product as $\alpha_t^2 = 2$ for all $1 \le t \le k-1$. From now on we fix this scaling.

Let $\mathbf{H} = \mathbf{H}_a$ for some a with $1 \le a \le b$. For each $v \in \mathsf{SR}^{\mathbf{M}}_{\mathbf{H}}$ define a rational number

$$\phi(v) := \frac{(\xi, \rho - v\rho)}{k(\xi, \omega_1)} \left(1 - k\frac{(\xi, \omega_1)^2}{\xi^2}\right). \tag{32}$$

Definition 5.7. An element $v \in \mathsf{SR}^{\mathbf{M}}_{\mathbf{H}}$ is *very special* if $\phi(v)$ is a positive integer.

Lemma 5.8. *If* $\mathbf{G}$ *is a group of type* B*,* C *or* D*, then there are no very special elements.*

Proof. Consider the standard numbering of vertices. Let $\beta = \alpha_k$. Note that if we take $D_{\mathbf{out}}$ to be empty then we have nothing to check (since we assumed $a \ge 1$). This means that we only have to consider the case when $D_{\mathbf{out}}$ consists of vertices from 1 to $k-1$.

First, assume that either $k \le n-1$ for type B and $k \le n-2$ for type D or any k for type C. Then $(\xi, \omega_1) = 1$, $\xi^2 = k$ and we see that the second factor in (32) vanishes, hence $\phi(v) = 0$. In the remaining cases ($k = n$ for type B and $k = n$ for type D) we have $(\xi, \omega_1) = 1/2$, $\xi^2 = n/4$, and $k = n$, so the second factor vanishes as well. □

Remark 5.9. It seems plausible that for types E, F and G there are no very special elements either, although we have not checked this. On the contrary, for type A,

$$\phi(v) = (\xi, \rho - v\rho)/(n+1-k),$$

so very special elements correspond to permutations $v \in \mathfrak{S}_{n+1}$ such that $v(n+1) = k$.

Now we are ready to define the block—we just set

$$\begin{aligned} \mathsf{B}^{\mathbf{out}}_{\mathsf{j}} &= \{\lambda \in \hat{\mathsf{B}}^{\mathbf{out}}_{\mathsf{j}} \mid (\lambda + \rho - v\rho, \alpha_1 + \dots + \alpha_{k-1}) < \phi(v) \text{ for all very special } v\}, \\ \mathsf{B}^{\mathbf{inn}}_{\mathsf{j}} &= \hat{\mathsf{B}}^{\mathbf{inn}}_{\mathsf{j}}, \\ \mathsf{B}_{\mathsf{j}} &= \mathsf{B}^{\mathbf{out}}_{\mathsf{j}} + \mathsf{j}\xi + i_*(\mathsf{B}^{\mathbf{inn}}_{\mathsf{j}}), \end{aligned} \tag{33}$$

Further we will show that the block B_{j} defined by (33) is exceptional if its outer part $\mathsf{B}^{\mathbf{out}}_{\mathsf{j}}$, viewed as a set of Young diagrams, is closed under passing to a subdiagram. In fact, we will prove part (a) of the criterion 3.13 for the block B_{j} in Section 6 (without additional conditions). Part (b) of this criterion will be proved in Section 7 assuming that $\mathsf{B}^{\mathbf{out}}_{\mathsf{j}}$ is closed under passing to a subdiagram. Finally, we will verify the latter condition for groups of type BCD by a direct computation in Section 9.

5.6. Exceptional collections. Before we proceed to the proof that the blocks constructed are exceptional, we will explain how one can make these blocks smaller in order to achieve semiorthogonality of the subcategories of $\mathcal{D}^b(X)$ generated by the corresponding equivariant bundles.

First, we define

$$\bar{\mathsf{B}}^{\mathbf{inn}}_{\mathsf{j}} = \left\{ \nu \in \mathsf{B}^{\mathbf{inn}}_{\mathsf{j}} \;\middle|\; \begin{array}{l} \text{for all } \mathsf{j}' < \mathsf{j},\ \nu' \in \bar{\mathsf{B}}^{\mathbf{inn}}_{\mathsf{j}'}, \text{ and } w_{\mathbf{L}_{\mathbf{inn}}}, w'_{\mathbf{L}_{\mathbf{inn}}} \in \mathbf{W}_{\mathbf{L}_{\mathbf{inn}}} \\ \text{one has } \rho_{\mathbf{H}_{\mathsf{j}'}} - (\mathsf{j}-\mathsf{j}')\xi - w_{\mathbf{L}_{\mathbf{inn}}} i_*\nu + w'_{\mathbf{L}_{\mathbf{inn}}} i_*\nu' \in \mathbf{R}^*_{\mathsf{j}'} \end{array} \right\}. \tag{34}$$

Note that the above formula is recursive—it describes $\bar{\mathsf{B}}^{\mathbf{inn}}_{\mathsf{j}}$ in terms of all $\bar{\mathsf{B}}^{\mathbf{inn}}_{\mathsf{j}'}$ with $\mathsf{j}' < \mathsf{j}$. We also set

$$\bar{\mathsf{B}}^{\mathbf{out}}_{\mathsf{j}} = \left\{ \lambda_0 \in \mathsf{B}^{\mathbf{out}}_{\mathsf{j}} \;\middle|\; \begin{array}{l} \text{for all } \mathsf{j}' < \mathsf{j},\ \nu \in \bar{\mathsf{B}}^{\mathbf{inn}}_{\mathsf{j}},\ \nu' \in \bar{\mathsf{B}}^{\mathbf{inn}}_{\mathsf{j}'},\ w_{\mathbf{L}_{\mathbf{inn}}}, w'_{\mathbf{L}_{\mathbf{inn}}} \in \mathbf{W}_{\mathbf{L}_{\mathbf{inn}}}, \text{ and } w_{\mathbf{L}} \in \mathbf{W}_{\mathbf{L}} \\ \text{one has } \rho_{\mathbf{H}_{\mathsf{j}'}} - h^*_{\mathsf{j}'}(w_{\mathbf{L}}\lambda_0 + (\mathsf{j}-\mathsf{j}')\xi) - w_{\mathbf{L}_{\mathbf{inn}}} i_*\nu + w'_{\mathbf{L}_{\mathbf{inn}}} i_*\nu' \in \mathbf{R}^*_{\mathsf{j}'} \end{array} \right\}. \tag{35}$$

Note that by Remark 5.5, we can let the elements $w_{\mathbf{L}_{\mathbf{inn}}}$ and $w'_{\mathbf{L}_{\mathbf{inn}}}$ run through the entire group $\mathbf{W}_{\mathbf{L}}$ in the definitions (34) and (35). Finally, we set

$$\bar{\mathsf{B}}_{\mathsf{j}} = \bar{\mathsf{B}}^{\mathbf{out}}_{\mathsf{j}} + \mathsf{j}\xi + i_*\bar{\mathsf{B}}^{\mathbf{inn}}_{\mathsf{j}}, \tag{36}$$

and define the subcategory

$$\mathcal{A}_{\mathsf{j}} := \langle \mathcal{U}^{\lambda} \rangle_{\lambda \in \bar{\mathsf{B}}_{\mathsf{j}}}.$$

Theorem 5.10. *The collection $\{\mathcal{A}_{\mathsf{j}}\}_{\mathsf{j} \in \mathsf{J}}$ of subcategories ordered by increasing j is semi-orthogonal.*

Proof. Assume that $\mathsf{j}' < \mathsf{j}$. Let $\lambda_0 \in \bar{\mathsf{B}}^{\mathbf{out}}_{\mathsf{j}}$, $\lambda'_0 \in \bar{\mathsf{B}}^{\mathbf{out}}_{\mathsf{j}'}$, $\nu \in \bar{\mathsf{B}}^{\mathbf{inn}}_{\mathsf{j}}$, $\nu' \in \bar{\mathsf{B}}^{\mathbf{inn}}_{\mathsf{j}'}$. We have to check that

$$\mathsf{Ext}^\bullet(\mathcal{U}^{\lambda_0 + \mathsf{j}\xi + i_*\nu}, \mathcal{U}^{\lambda'_0 + \mathsf{j}'\xi + i_*\nu'}) = 0.$$

By Corollary 2.16 we have to check that for any $\mathbf{L}$-dominant weight

$$\mu \in \mathsf{Conv}(\lambda'_0 - w_{\mathbf{L}}\lambda_0 + (\mathsf{j}' - \mathsf{j})\xi + i_*\nu' - w_{\mathbf{L}} i_*\nu)_{w_{\mathbf{L}} \in \mathbf{W}_{\mathbf{L}}}$$

the sum $\mu + \rho_{\mathbf{G}}$ is $\mathbf{G}$-singular. Note that $h^*_{\mathsf{j}'}(\lambda'_0) = 0$ since $\lambda'_0 \in \bar{\mathsf{B}}^{\mathbf{out}}_{\mathsf{j}'} \subset \mathsf{Ker}\, h^*_{\mathsf{j}'}$, hence

$$h^*_{\mathsf{j}'}(\rho + \lambda'_0 - w_{\mathbf{L}}\lambda_0 + (\mathsf{j}' - \mathsf{j})\xi + i_*\nu' - w_{\mathbf{L}} i_*\nu) = \rho_{\mathbf{H}_{\mathsf{j}'}} - h^*_{\mathsf{j}'}(w_{\mathbf{L}}\lambda_0 - (\mathsf{j} - \mathsf{j}')\xi) + i_*\nu' - w_{\mathbf{L}} i_*\nu.$$

By definition of $\bar{\mathsf{B}}^{\mathbf{out}}_{\mathsf{j}}$, all these weights for $w_{\mathbf{L}} \in \mathbf{W}_{\mathbf{L}}$ lie in the interior of the core $\mathbf{R}_{\mathsf{j}'}$, hence we have $h^*_{\mathsf{j}'}(\mu + \rho) \in \mathbf{R}^*_{\mathsf{j}'}$, and so by Lemma 5.3, $h^*_{\mathsf{j}'}(\mu + \rho)$ is $\mathbf{H}_{\mathsf{j}'}$-singular. But the map $h^*_{\mathsf{j}'}$ preserves regularity, hence $\mu + \rho$ is $\mathbf{G}$-singular as well. □

6. Verification of the invariance condition

In this section we prove that the blocks B_j and $\bar{\mathsf{B}}_\mathsf{j}$ constructed in Section 5 satisfy the invariance condition (part (a) of the criterion 3.13).

First, we will need the following simple fact. Assume that $\mathbf{H} \subset \mathbf{H}'$ is an embedding of semisimple groups corresponding to the embedding of the Dynkin diagrams $D_\mathbf{H} \subset D_{\mathbf{H}'}$ such that $D_{\mathbf{H}'} \setminus D_\mathbf{H}$ consists only of one vertex. Let α be the corresponding simple root and η the corresponding fundamental weight of $\mathbf{H}'$.

Lemma 6.1. *There is a positive integer $k = k_{\mathbf{H}',\mathbf{H}}$ such that*

$$\rho_{\mathbf{H}'} - k\eta = w_0^{\mathbf{H}} w_0^{\mathbf{H}'} \rho_{\mathbf{H}'}.$$

Moreover, for all $0 < c < k$ the weight $\rho_{\mathbf{H}'} - c\eta$ is $\mathbf{H}'$-singular.

Proof. Let us denote the embedding $\mathbf{H} \to \mathbf{H}'$ by h. Then $h^*\rho_{\mathbf{H}'} = \rho_\mathbf{H}$ and $\mathsf{Ker}\, h^* = \mathbb{Z}\eta$. Since

$$h^* w_0^{\mathbf{H}} w_0^{\mathbf{H}'} \rho_{\mathbf{H}'} = -h^* w_0^{\mathbf{H}} \rho_{\mathbf{H}'} = -w_0^{\mathbf{H}} \rho_\mathbf{H} = \rho_\mathbf{H},$$

we get

$$\rho_{\mathbf{H}'} - w_0^{\mathbf{H}} w_0^{\mathbf{H}'} \rho_{\mathbf{H}'} = k\eta$$

for some $k \in \mathbb{Z}$. Moreover, the LHS is a sum of positive roots by Lemma 2.3, hence $(k\eta, \eta) > 0$, hence k is positive. This proves the first statement.

For the second, by Lemma 5.3 it is enough to show that $\rho_{\mathbf{H}'} - c\eta$ with $0 < c < k$ is in the interior of a core $\mathbf{R}^*_\delta$ for some strictly dominant δ. In fact, we will show that one can take any strictly dominant δ. Indeed, since $\mathbf{R}^*_\delta$ is convex and $\rho_{\mathbf{H}'} - c\eta$ lies in the convex hull of $\rho_{\mathbf{H}'} - \eta$ and $\rho_{\mathbf{H}'} - (k-1)\eta$, it is enough to check that the last two weights are in $\mathbf{R}^*_\delta$. Fix some strictly dominant δ.

First, we have $(\delta, \rho_{\mathbf{H}'} - \eta) = (\delta, \rho_{\mathbf{H}'}) - (\delta, \eta) < (\delta, \rho_{\mathbf{H}'})$, so since $\rho_{\mathbf{H}'} - \eta$ is dominant we have $\rho_{\mathbf{H}'} - \eta \in \mathbf{R}^*_\delta$ by Lemma 5.2. On the other hand,

$$\rho_{\mathbf{H}'} - (k-1)\eta = w_0^{\mathbf{H}} w_0^{\mathbf{H}'} \rho_{\mathbf{H}'} + \eta = w_0^{\mathbf{H}} w_0^{\mathbf{H}'} (\rho_{\mathbf{H}'} + w_0^{\mathbf{H}'} w_0^{\mathbf{H}} \eta) = w_0^{\mathbf{H}} w_0^{\mathbf{H}'} (\rho_{\mathbf{H}'} + w_0^{\mathbf{H}'} \eta).$$

Since $-w_0^{\mathbf{H}'}\eta$ is a fundamental weight of $\mathbf{H}'$, the same argument as above shows that

$$\rho_{\mathbf{H}'} + w_0^{\mathbf{H}'} \eta = \rho_{\mathbf{H}'} - (-w_0^{\mathbf{H}'} \eta) \in \mathbf{R}^*_\delta.$$

Hence, $\rho_{\mathbf{H}'} - (k-1)\eta$ is also in $\mathbf{R}^*_\delta$. This finishes the proof. □

Remark 6.2. One can also deduce the claim geometrically. Consider the Grassmannian of $\mathbf{H}'$ corresponding to the root α. Then its Picard group is $\mathbb{Z}$ and its generator is the line bundle corresponding to the weight η. By Lemma 2.18 the canonical class of the Grassmannian is given by the weight $w_0^{\mathbf{H}} w_0^{\mathbf{H}'} \rho_{\mathbf{H}'} - \rho_{\mathbf{H}'}$. On the other hand, it is equal to the line bundle corresponding to the weight $-k\eta$ for some $k \in \mathbb{Z}$. This gives the equality. Having all this, it is clear that the weights the weights $\rho_{\mathbf{H}'} - c\eta$ with $0 < c < k$ are singular. Indeed, by the Borel–Bott–Weil Theorem the fact that $\rho - c\eta$ is singular is equivalent to the vanishing of the cohomology of the line bundle corresponding to the weight $-c\eta$, which indeed vanishes by the Kodaira vanishing theorem.

Now we can verify the invariance condition formulated in Proposition 3.13(a).

Proposition 6.3. *Let* $\kappa \in \mathsf{OP}_1(\mathsf{B_j})$ *and* $v \in \mathsf{OP}_2(\mathsf{B_j})$. *Then* $\kappa \in \operatorname{Ker} h_{\mathsf{j}}^*$ *and* $v \in \mathbf{W}_{\mathbf{H}_{\mathsf{j}}}$. *In particular,* $v\kappa = \kappa$.

Proof. Take arbitrary $\lambda, \lambda' \in \mathsf{B_j}$. Then $\lambda = \lambda_0 + \mathsf{j}\xi + i_*\nu$ and $\lambda' = \lambda_0' + \mathsf{j}\xi + i_*\nu'$ with $\lambda_0, \lambda_0' \in \mathsf{B}_{\mathsf{j}}^{\mathbf{out}}$ and $\nu, \nu' \in \mathsf{B}_{\mathsf{j}}^{\mathbf{inn}}$. Note that for any $w_{\mathbf{L}} \in \mathbf{W_L}$ we have

$$h_{\mathsf{j}}^*(\rho + \lambda' - w_{\mathbf{L}}\lambda) = h_{\mathsf{j}}^*(\rho + \lambda_0' + i_*\nu' - w_{\mathbf{L}}\lambda_0 - w_{\mathbf{L}} i_*\nu)$$
$$= h_{\mathsf{j}}^*(\rho - w_{\mathbf{L}}\lambda_0) + i_*\nu' - w_{\mathbf{L}} i_*\nu \qquad (37)$$

since $\lambda_0' \in \operatorname{Ker} i^*$ and $h_{\mathsf{j}} \circ i = i$. So, by definition of $\mathsf{B_j}$ (using Remark 5.5) we conclude that the weight (37) is in $\mathbf{R}_\delta$.

Let $(\kappa, v) \in \mathsf{OP}(\mathsf{B_j})$, that is, $(\kappa, v) \in \mathsf{OP}(\lambda, \lambda')$ for some $\lambda, \lambda' \in \mathsf{B_j}$. By definition of the output set the weight

$$\mu := v(\kappa + \rho) - \rho \in \mathsf{Conv}(\lambda' - w_{\mathbf{L}}\lambda)_{w_{\mathbf{L}} \in \mathbf{W_L}}$$

is $\mathbf{L}$-dominant and $\mu + \rho$ is $\mathbf{G}$-regular. Moreover, $h_{\mathsf{j}}^*(\mu + \rho)$ is in the convex hull of the weights (37) (where $w_{\mathbf{L}}$ runs through $\mathbf{W_L}$), hence it is in the core $\mathbf{R}_\delta$. So, Proposition 6.4 below applies and we conclude that $\kappa \in \operatorname{Ker} h_{\mathsf{j}}^*$ and $v \in \mathbf{W}_{\mathbf{H}_{\mathsf{j}}}$. □

Proposition 6.4. *Assume that a weight* $\mu \in P_{\mathbf{L}}$ *satisfies*

$$\mu \in P_{\mathbf{L}}^+, \qquad \mu + \rho \in P_{\mathbf{G}}^{\mathsf{reg}}, \qquad h_a^*(\mu + \rho) \in \mathbf{R}_\delta, \qquad (38)$$

for some a with $0 \le a \le b$. *Let also* $\mu = v(\kappa + \rho) - \rho$ *be the unique presentation of* μ *with* $\kappa \in P_{\mathbf{G}}^+$ *and* $v \in \mathsf{SR}_{\mathbf{G}}^{\mathbf{L}}$. *Then*

$$v \in \mathsf{SR}_{\mathbf{G}}^{\mathbf{L}} \cap \mathbf{W}_{\mathbf{H}_a} \quad \textit{and} \quad \kappa \in P_{\mathbf{G}}^+ \cap \operatorname{Ker} h_a^*.$$

In particular, $v\kappa = \kappa$.

Proof. To simplify the notation we write $\mathbf{H}$ instead of $\mathbf{H}_a$ and h instead of h_a. Set $\mathbf{M} = \mathbf{L} \cap \mathbf{H}$. Note that h^* takes $\mathbf{G}$-regular $\mathbf{L}$-dominant weights of $P_{\mathbf{G}}$ to $\mathbf{H}$-regular $\mathbf{M}$-dominant weights of $P_{\mathbf{H}}$, hence $h^*(\mu + \rho)$ is $\mathbf{H}$-regular and $\mathbf{M}$-dominant. On the other hand, $h^*(\mu + \rho) \in \mathbf{R}_\delta$, so Lemma 5.3 implies that $h^*(\mu + \rho) = v\rho_{\mathbf{H}}$ with $v \in \mathbf{W_H}$. Thus, $v\rho_{\mathbf{H}}$ is $\mathbf{M}$-dominant, so we have $v \in \mathsf{SR}_{\mathbf{H}}^{\mathbf{M}}$. Further, $v\rho_{\mathbf{H}} = h^*(v\rho)$, hence $h^*(\mu + \rho - v\rho) = 0$. Denoting

$$\kappa = \mu + \rho - v\rho$$

we see that $\kappa \in \operatorname{Ker} h^*$ and $\mu = v\rho - \rho + \kappa$. Since $\kappa \in \operatorname{Ker} h^*$ and $v \in \mathbf{W_H}$, we have $v\kappa = \kappa$, so μ can be written as $v(\kappa + \rho) - \rho$. So it remains to check that κ is $\mathbf{G}$-dominant.

To check the dominance of a weight we should check that its inner products with all simple roots are nonnegative. We divide the simple roots into three groups.

Case 1: the simple roots of $\mathbf{H}$. If $\alpha \in D_{\mathbf{H}}$ then $(\kappa, \alpha) = 0$ since $\kappa \in \operatorname{Ker} h^*$.

Case 2: the simple roots of $\mathbf{G}$ not adjacent to $D_{\mathbf{H}}$. If α is such a root then $v^{-1}\alpha = \alpha$ since $v \in \mathbf{W_H}$, hence $(v\rho, \alpha) = (\rho, v^{-1}\alpha) = (\rho, \alpha)$, therefore $(\kappa, \alpha) = (\mu, \alpha) \geq 0$. Here the last inequality follows from the $\mathbf{L}$-dominance of μ since simple roots not adjacent to $D_{\mathbf{H}}$ are roots of $\mathbf{L}$.

Case 3: the simple root, adjacent to $D_{\mathbf{H}}$. Let α be such a root, and let $\mathbf{H}'$ be the reductive subgroup of $\mathbf{G}$ such that $D_{\mathbf{H}'} = D_{\mathbf{H}} \cup \{\alpha\}$. Let $\eta \in P_{\mathbf{H}'}$ be the fundamental weight of $\mathbf{H}'$ corresponding to the root α. Let $h' : \mathbf{H}' \to \mathbf{G}$ be the embedding, and let h denote the embeddings $\mathbf{H} \to \mathbf{H}'$ and $\mathbf{H} \to \mathbf{G}$. Note that $\mathsf{Ker}(h^* : P_{\mathbf{H}'} \to P_{\mathbf{H}}) = \mathbb{Z}\eta$.

Since $h^*(h')^*(\mu+\rho) = h^*(\mu+\rho) = v\rho_{\mathbf{H}} = h^*(v\rho_{\mathbf{H}'})$, it follows that $(h')^*(\mu+\rho) = v\rho_{\mathbf{H}'} + c\eta = v(\rho_{\mathbf{H}'} + c\eta)$. It is enough to show that $c \geq 0$. Indeed, since α is a root of $\mathbf{H}'$ we have $\alpha = h'_*\alpha$, so

$$\begin{aligned}(\kappa, \alpha) &= (\kappa, h'_*\alpha) = ((h')^*\kappa, \alpha) = ((h')^*(\mu + \rho - v\rho), \alpha) \\ &= (v\rho_{\mathbf{H}'} + c\eta - v\rho_{\mathbf{H}'}, \alpha) = c(\eta, \alpha) = c\alpha^2/2 \geq 0,\end{aligned}$$

and we are done. So, assume that $c < 0$. Since $v^{-1}(h')^*(\mu+\rho)$ is $\mathbf{H}'$-regular, Lemma 6.1 implies that $v^{-1}(h')^*(\mu + \rho) = \rho_{\mathbf{H}'} + c\eta = -w_0^{\mathbf{H}}\rho_{\mathbf{H}'} - c'\eta$ with $c' \geq 0$. Then

$$(h')^*\mu = v(-w_0^{\mathbf{H}}\rho_{\mathbf{H}'} - c'\eta) - (h')^*\rho = -vw_0^{\mathbf{H}}\rho_{\mathbf{H}'} - \rho_{\mathbf{H}'} - c'\eta.$$

Let us check that the scalar product of this weight with α is always negative. Indeed, $(\rho_{\mathbf{H}'}, \alpha) > 0$ since α is a simple root of $\mathbf{H}'$. Further, the root $w_0^{\mathbf{H}}v^{-1}\alpha$ is positive since $(\eta, w_0^{\mathbf{H}}v^{-1}\alpha) = (vw_0^{\mathbf{H}}\eta, \alpha) = (\eta, \alpha) > 0$. Therefore, $(vw_0^{\mathbf{H}}\rho_{\mathbf{H}'}, \alpha) = (\rho_{\mathbf{H}'}, w_0^{\mathbf{H}}v^{-1}\alpha) > 0$. Finally, $(c'\eta, \alpha) \geq 0$ since $c' \geq 0$. Thus, we see that

$$((h')^*\mu, \alpha) < 0.$$

But this is equal to (μ, α), which is nonnegative since μ is $\mathbf{L}$-dominant. This contradiction shows that we actually have $c \geq 0$, which completes the proof. □

7. Adapted weights and compatibility condition

Let L be a reductive algebraic group. For any subset $S \subset P_L^+$ of the set of dominant weights of L we denote by $\mathsf{Rep}_S(L)$ the subcategory of $\mathsf{Rep}(L)$ consisting of direct sums of irreducible representations with highest weights in S. We also denote by $\Pi_S : \mathsf{Rep}(L) \to \mathsf{Rep}(L)$ the corresponding projector (that leaves unchanged only representations in $\mathsf{Rep}_S(L)$).

A morphism $f : V_1 \to V_2$ in $\mathsf{Rep}(L)$ is called an *S-isomorphism* if $\Pi_S(f) : \Pi_S(V_1) \to \Pi_S(V_2)$ is an isomorphism. In other words, f is an S-isomorphism if it induces an isomorphism on λ-isotypical components for any $\lambda \in S$.

We say that a pair of L-dominant weights (κ, λ) is *adapted to S* (or *S-adapted*) if the natural map

$$V_L^{\kappa+\lambda} \otimes V_L^{\mu} \to V_L^{\kappa} \otimes V_L^{\lambda} \otimes V_L^{\mu} \to V_L^{\kappa} \otimes \Pi_S(V_L^{\lambda} \otimes V_L^{\mu}) \tag{39}$$

is an S-isomorphism for any $\mu \in S$.

The goal of this section is to show that for all $(\kappa, \upsilon) \in \mathsf{OP}_1(\mathsf{B}) \times \mathsf{OP}_2(\mathsf{B})$ the pair $(\kappa, \upsilon\rho - \rho)$ (considered as a pair of weights of the Levi subgroup $\mathbf{L}$) is B-adapted for either $\mathsf{B} = \mathsf{B}_\mathsf{j}$ or $\mathsf{B} = \bar{\mathsf{B}}_\mathsf{j}$, which will give the compatibility condition of Proposition 3.13. In fact, we will prove a more general statement.

Let us return to the setup of Section 5.2, i.e., fix a choice of the outer component $D_{\mathbf{out}}$ of $D_{\mathbf{G}} \setminus \beta$ of type A_{k-1}, a standard numbering of its vertices, and a subdiagram $D_a = D_{\mathbf{G}} \setminus \{1, \dots, a\}$. We will write $\mathbf{H}$ for the corresponding semisimple subgroup $\mathbf{H}_a \subset \mathbf{G}$ and h for its embedding into $\mathbf{G}$ and set $\mathbf{M} = \mathbf{L} \cap \mathbf{H}$. Recall also that the subgroups $\mathbf{L_{out}} \subset \mathbf{L}$ and $\mathbf{L_{inn}} \subset \mathbf{L}$ correspond to the outer and the inner parts of $D_{\mathbf{G}}$.

Assume that subsets $\mathsf{B}^{\mathbf{inn}} \subset P^+_{\mathbf{L_{inn}}}$, $\mathsf{B}^{\mathbf{out}} \subset P^+_{\mathbf{G}} \cap \operatorname{Ker} h^*$ and $\mathsf{j} \in \mathbb{Q}$ are given such that $\mathsf{j}\xi + i_*\mathsf{B}^{\mathbf{inn}} \subset P_{\mathbf{L}}$. Set

$$\mathsf{B} = \mathsf{B}^{\mathbf{out}} + \mathsf{j}\xi + i_*\mathsf{B}^{\mathbf{inn}}.$$

Note that elements of $\mathsf{B}^{\mathbf{out}}$, being linear combinations of the fundamental weights $\omega_1, \dots, \omega_a$ with nonnegative coefficients can be viewed as Young diagrams: a weight $x_1\omega_1 + \dots + x_a\omega_a$ corresponds to the Young diagram with x_i columns of length i. Recall the notion of a very special element of $\mathsf{SR}^{\mathbf{M}}_{\mathbf{H}}$ (Definition 5.7) and the function $\phi(\upsilon)$ given by (32).

Theorem 7.1. *Assume that the set* $\mathsf{B}^{\mathbf{out}}$ *has the following two properties:*

(1) *for all* $\lambda \in \mathsf{B}^{\mathbf{out}}$ *and all very special* $\upsilon \in \mathsf{SR}^{\mathbf{M}}_{\mathbf{H}}$ *we have* $(\lambda + \rho - \upsilon\rho, \alpha_1 + \dots + \alpha_{k-1}) < \phi(\upsilon)$*;*
(2) *the set* $\mathsf{B}^{\mathbf{out}}$ *is closed under passing to Young subdiagrams.*

Then for any $\kappa \in P^+_{\mathbf{G}} \cap \operatorname{Ker} h^*$ *and any* $\upsilon \in \mathsf{SR}^{\mathbf{M}}_{\mathbf{H}}$ *the pair* $(\kappa, \upsilon\rho - \rho)$ *is* B*-adapted.*

This result applies to the blocks B_j and $\bar{\mathsf{B}}_\mathsf{j}$ defined by (33) and (36).

Corollary 7.2. *Assume for some* $\mathsf{j} \in \mathsf{J}$ *the set* $\mathsf{B}^{\mathbf{out}}_\mathsf{j}$ *(resp.,* $\bar{\mathsf{B}}^{\mathbf{out}}_\mathsf{j}$*) is closed under passing to Young subdiagrams. Then the block* B_j *(resp.,* $\bar{\mathsf{B}}_\mathsf{j}$*) is exceptional.*

Proof. Set $\mathsf{B} = \mathsf{B}_\mathsf{j}$ (resp., $\bar{\mathsf{B}}_\mathsf{j}$). It is enough to check the two conditions of Proposition 3.13 for B. The invariance condition holds for this block by Proposition 6.3. To check the compatibility condition we can apply Theorem 7.1. The first condition of that theorem holds by the definition (33) of the block B_j, while the second holds by assumption. It remains to observe that for any pair $\kappa \in \mathsf{OP}_1(\mathsf{B})$, $\upsilon \in \mathsf{OP}_2(\mathsf{B})$ we have $\kappa \in P^+_{\mathbf{G}} \cap \operatorname{Ker} h^*$ and $\upsilon \in \mathsf{SR}^{\mathbf{M}}_{\mathbf{H}}$. Hence, Theorem 7.1, applied to B and a pair $\kappa \in \mathsf{OP}_1(\mathsf{B})$, $\upsilon \in \mathsf{OP}_2(\mathsf{B})$, implies that the compatibility condition is satisfied for B. □

Unfortunately, we have not been able to find an abstract way of checking that $\mathsf{B}^{\mathbf{out}}_\mathsf{j}$ or $\bar{\mathsf{B}}^{\mathbf{out}}_\mathsf{j}$ is closed under passing to Young subdiagrams. So, we will check it for classical groups in Section 9 as a result of an explicit description of the blocks.

7.1. Preparations. We start with a description of the connected component of the center of $\mathbf{L}$.

Lemma 7.3. *Let $\mathbf{Z} \subset \mathbf{L}$ be the connected component of the center of $\mathbf{L}$. Then $\mathbf{Z} \cong \mathbb{G}_{\mathrm{m}}$ and the map $P_{\mathbf{L}} \to P_{\mathbf{Z}} = \mathbb{Z}$, induced by the embedding $\mathbf{Z} \to \mathbf{L}$, is given by the scalar product with the minimal rational multiple $c\xi$ of ξ such that $(c\xi, -)$ is an integer valued function on $P_{\mathbf{L}}$.*

Proof. First, note that $\mathbf{Z} \cong \mathbb{G}_{\mathrm{m}}$ since it is a 1-dimensional (since $\mathbf{P}$ is maximal) connected commutative reductive group. As a consequence, $P_{\mathbf{Z}} \cong \mathbb{Z}$. Since the map $P_{\mathbf{L}} \to P_{\mathbf{Z}}$ is dual to the embedding of $\mathbf{Z}$ into a maximal torus of $\mathbf{L}$, it is surjective. Note also that the adjoint representation of the semisimple part of $\mathbf{L}$ is a trivial representation of $\mathbf{Z}$, hence all simple roots of $\mathbf{L}$ are mapped to zero. This implies that the map is given by the scalar product with a multiple $c\xi$ of ξ. Moreover, since the scalar product should be a map to $\mathbb{Z}$, it follows that $(c\xi, -)$ should be an integral function on $P_{\mathbf{L}}$ (and in particular, c should be rational since the scalar product has rational values on the weight lattice), and the surjectivity of the map implies that c is minimal with this property. □

Consider the diagram of groups

$$\mathsf{GL}_k \times \mathbb{G}_{\mathrm{m}} \times \mathbf{L_{inn}} \xleftarrow{\ \varpi\ } \mathbf{L_{out}} \times \mathbb{G}_{\mathrm{m}} \times \mathbf{L_{inn}} \xrightarrow{\ \pi\ } \mathbf{L}$$

where π and ϖ are defined as follows. The morphism π is induced by the embeddings $o : \mathbf{L_{out}} \to \mathbf{L}$, $i : \mathbf{L_{inn}} \to \mathbf{L}$ and by the isomorphism $\mathbb{G}_{\mathrm{m}} \cong \mathbf{Z}$. The restriction of ϖ to $\mathbf{L_{out}} = \mathsf{SL}_k$ (resp., $\mathbf{L_{inn}}$) is given by the natural embedding $\mathsf{SL}_k \subset \mathsf{GL}_k$ (resp., the identity map to $\mathbf{L_{inn}}$). Finally, the restriction of ϖ to $\mathbb{G}_{\mathrm{m}}$ is given by $z \mapsto (z^{(c\xi,\omega_1)} \times 1, z, 1)$. Note that the map π is an isogeny and the map ϖ is an embedding.

Now take any $\kappa \in \operatorname{Ker} h^* \cap P_{\mathbf{G}}^+$, $v \in \mathsf{SR}_{\mathbf{H}}^{\mathbf{M}}$ and $\mu \in \mathsf{B}$, and consider the morphisms

$$V_{\mathbf{L}}^{\kappa+v\rho-\rho} \otimes V_{\mathbf{L}}^{\mu} \to V_{\mathbf{L}}^{\kappa} \otimes V_{\mathbf{L}}^{v\rho-\rho} \otimes V_{\mathbf{L}}^{\mu} \to V_{\mathbf{L}}^{\kappa} \otimes \Pi_{\mathsf{B}}(V_{\mathbf{L}}^{v\rho-\rho} \otimes V_{\mathbf{L}}^{\mu}). \tag{40}$$

Our goal is to show that after application of Π_{B} this map becomes an isomorphism. For this we pull back the map (40) via π to a map of representations of the group $\mathbf{L_{out}} \times \mathbb{G}_{\mathrm{m}} \times \mathbf{L_{inn}}$ and check that the same map can be realized as a pullback via ϖ of a map of representations of $\mathsf{GL}_k \times \mathbb{G}_{\mathrm{m}} \times \mathbf{L_{inn}}$. We also express the action of the projector Π_{B} in terms of group $\mathsf{GL}_k \times \mathbb{G}_{\mathrm{m}} \times \mathbf{L_{inn}}$ and thus reduce the verification to the latter group. It turns out that the components $\mathbb{G}_{\mathrm{m}}$ and $\mathbf{L_{inn}}$ play no role, and the statement essentially reduces to a similar statement for representations of the group GL_k. The latter statement is proved in the Appendix.

Recall that irreducible representations of GL_k are numbered by nonincreasing sequences of integers of length k. For a sequence $\kappa_\bullet = (\kappa_1 \ge \cdots \ge \kappa_k)$ we will denote by $V_{\mathsf{GL}_k}^{\kappa_\bullet}$ the corresponding GL_k-representation.

Lemma 7.4. *For any* $\lambda \in P^+_{\mathbf{L}}$ *we have*

$$\pi^* V^{\lambda}_{\mathbf{L}} = V^{o^*\lambda}_{\mathbf{L_{out}}} \otimes V^{(c\xi,\lambda)}_{\mathbb{G}_m} \otimes V^{i^*\lambda}_{\mathbf{L_{inn}}}.$$

On the other hand, for any nonincreasing sequence $\kappa_\bullet = (\kappa_1 \geq \cdots \geq \kappa_k)$ *of integers, any* $z \in \mathbb{Z}$ *and any* $\nu \in P^+_{\mathbf{L_{inn}}}$ *we have*

$$\varpi^*(V^{\kappa_\bullet}_{\mathsf{GL}_k} \otimes V^{z}_{\mathbb{G}_m} \otimes V^{\nu}_{\mathbf{L_{inn}}}) = V^{\kappa}_{\mathbf{L_{out}}} \otimes V^{(c\xi,\omega_1)\sum_{i=1}^k \kappa_i + z}_{\mathbb{G}_m} \otimes V^{\nu}_{\mathbf{L_{inn}}},$$

where $\kappa = \sum_{i=1}^{k-1}(\kappa_i - \kappa_{i+1})\omega_i$ *is the weight of* $\mathbf{L_{out}}$ *corresponding to* $\kappa_\bullet$.

Proof. Straightforward (to compute the $\mathbb{G}_m$-component of $\pi^* V^{\lambda}_{\mathbf{L}}$ use Lemma 7.3). □

Now we give a description of the pullbacks via π of the representations $V^{\kappa}_{\mathbf{L}}$, $V^{\upsilon\rho-\rho}_{\mathbf{L}}$ and $V^{\mu}_{\mathbf{L}}$ entering into (40) as the pullbacks via ϖ of appropriate representations of $\mathsf{GL}_k \times \mathbb{G}_m \times \mathbf{L_{inn}}$. In fact, such a description is not unique (which is clear from Lemma 7.4), so we choose a description which is most convenient for our purposes.

Lemma 7.5. *Let* $\kappa \in \operatorname{Ker} h^* \cap P^+_{\mathbf{G}}$. *Then there exists a unique nonincreasing sequence of integers* $\kappa_\bullet = (\kappa_1 \geq \cdots \geq \kappa_a \geq \kappa_{a+1} = \cdots = \kappa_k = 0)$ *such that*

$$\pi^* V^{\kappa}_{\mathbf{L}} = \varpi^* V^{\kappa_\bullet}_{\mathsf{GL}_k}.$$

Proof. By definition, κ is a nonnegative linear combination of $\omega_1, \ldots, \omega_a$. Let $\kappa_1 - \kappa_2$, $\kappa_2 - \kappa_3, \ldots, \kappa_{a-1} - \kappa_a$ and κ_a be the coefficients. Then $\kappa_1 \geq \cdots \geq \kappa_a \geq 0$. Extending this sequence by $\kappa_{a+1} = \cdots = \kappa_k = 0$ we obtain a sequence $\kappa_\bullet$. To prove the required isomorphism we use Lemma 7.4. By that lemma, we only have to check that $(c\xi, \kappa) = (c\xi, \omega_1)\sum_{i=1}^k \kappa_i$. For this we note that for $i < b$ we have $\alpha_i = 2\omega_i - \omega_{i-1} - \omega_{i+1}$, hence $(c\xi, \omega_i) = i(c\xi, \omega_1)$, so

$$(c\xi, \kappa) = (c\xi, \omega_1)\sum_{i=1}^{a} i(\kappa_i - \kappa_{i-1}) = (c\xi, \omega_1)\sum_{i=1}^{a} \kappa_i = (c\xi, \omega_1)\sum_{i=1}^{k} \kappa_i,$$

as required. □

Lemma 7.6. *Let* $\upsilon \in \mathsf{SR}^{\mathbf{M}}_{\mathbf{H}}$. *Set* $\nu_\upsilon = i^*(\upsilon\rho - \rho)$. *Then there exists a unique sequence of integers* $\tau_\bullet = (0 = \tau_1 = \cdots = \tau_a \geq \tau_{a+1} \geq \cdots \geq \tau_k)$ *such that*

$$\pi^* V^{\upsilon\rho-\rho}_{\mathbf{L}} = \varpi^*(V^{\tau_\bullet}_{\mathsf{GL}_k} \otimes V^{z(\upsilon)}_{\mathbb{G}_m} \otimes V^{\nu_\upsilon}_{\mathbf{L_{inn}}}),$$

where

$$z(\upsilon) = (\upsilon\rho - \rho, c\xi)(1 - k(\omega_1, \xi)^2/\xi^2). \tag{41}$$

Proof. Consider the restriction $o^*(\upsilon\rho - \rho)$. It is a weight of SL_k. Every weight of SL_k can be thought of as a weight of GL_k up to adding a central character. In other words, it is given by a nonincreasing sequence of integers up to a simultaneous translation. Consider the sequence $\tau_1 \geq \cdots \geq \tau_k$ representing $o^*(\upsilon\rho - \rho)$ such that $\tau_1 = 0$. Note that $\upsilon\rho - \rho$ is orthogonal to $\alpha_1, \ldots, \alpha_{a-1}$ (because these roots are orthogonal to the roots of $\mathbf{H}$ and hence are υ-invariant), hence $\tau_1 = \cdots = \tau_a$.

Recall that we denote by ν_v the weight $i^*(v\rho-\rho)$. Then the representations $\pi^*V_{\mathbf{L}}^{v\rho-\rho}$ and $\varpi^*(V_{\mathsf{GL}_k}^{\tau_\bullet}\otimes V_{\mathbf{L_{inn}}}^{\nu_v})$ have the same restrictions to $\mathbf{L_{out}}$ and $\mathbf{L_{inn}}$, so it remains to compare the central characters. First, the central character of $V_{\mathbf{L}}^{v\rho-\rho}$ is $(c\xi, v\rho-\rho)$. Further, the central character of $V_{\mathsf{GL}_k}^{\tau_\bullet}$ is $(c\xi,\omega_1)\sum\tau_i$, while the central character of $V_{\mathbf{L_{inn}}}^{\nu_v}$ is 0. Note that since $\tau_1=0$ and $k\omega_1=(k-1,-1,\dots,-1)$, we have

$$\begin{aligned}\sum\tau_i &= -(k\omega_1, o^*(v\rho-\rho)) = (v\rho-\rho, -ko_*\omega_1)\\ &= (v\rho-\rho, -k\omega_1+k((\omega_1,\xi)/\xi^2)\xi) = k(v\rho-\rho,\xi)(\omega_1,\xi)/\xi^2.\end{aligned}$$

In the third equality above we use the formula $o_*\omega_1=\omega_1-\frac{(\omega_1,\xi)}{\xi^2}\xi$ analogous to one in the proof of Lemma 5.4. So, we see that the difference of characters is

$$(v\rho-\rho,c\xi)-(c\xi,\omega_1)k(v\rho-\rho,\xi)(\omega_1,\xi)/\xi^2 = (v\rho-\rho,c\xi)(1-k(\omega_1,\xi)^2/\xi^2)=z(v).$$

Thus, twisting $V_{\mathsf{GL}_k}^{\tau_\bullet}\otimes V_{\mathbf{L_{inn}}}^{\nu_v}$ by $V_{\mathbb{G}_m}^{z(v)}$ we obtain an isomorphism. □

Lemma 7.7. *Let $\mu=\mu_{\mathbf{out}}+\mathsf{j}\xi+i_*\mu_{\mathbf{inn}}\in\mathsf{B}$. Then there exists a unique nonincreasing sequence of integers $\mu_\bullet=(\mu_1\ge\dots\ge\mu_a\ge\mu_{a+1}=\dots=\mu_k=0)$ such that*

$$\pi^*V_{\mathbf{L}}^{\mu}=\varpi^*(V_{\mathsf{GL}_k}^{\mu_\bullet}\otimes V_{\mathbb{G}_m}^{c\mathsf{j}\xi^2}\otimes V_{\mathbf{L_{inn}}}^{\mu_{\mathbf{inn}}}).$$

Proof. Note that $V_{\mathbf{L}}^{\mu}=V_{\mathbf{L}}^{\mu_{\mathbf{out}}}\otimes V_{\mathbf{L}}^{\mathsf{j}\xi+i_*\mu_{\mathbf{inn}}}$. Since $\mu_{\mathbf{out}}\in\operatorname{Ker}h^*\cap P_{\mathbf{G}}^+$, we already know from Lemma 7.5 that $\pi^*V_{\mathbf{L}}^{\mu_{\mathbf{out}}}=\varpi^*V_{\mathsf{GL}_k}^{\mu_\bullet}$ for a uniquely determined sequence $\mu_\bullet=(\mu_1\ge\dots\ge\mu_a\ge\mu_{a+1}=\dots=\mu_k=0)$. So, it remains to express $\pi^*V_{\mathbf{L}}^{\mathsf{j}\xi+i_*\mu_{\mathbf{inn}}}$ as a product of representations of $\mathbb{G}_m$ and $\mathbf{L_{inn}}$. Since $i^*(\mathsf{j}\xi+i_*\mu_{\mathbf{inn}})=\mu_{\mathbf{inn}}$, the $\mathbf{L_{inn}}$-component is $V_{\mathbf{L_{inn}}}^{\mu_{\mathbf{inn}}}$. On the other hand, the $\mathbb{G}_m$-component has weight $(c\xi,\mathsf{j}\xi+i_*\mu_{\mathbf{inn}})=c\mathsf{j}\xi^2$. □

Proposition 7.8. *A representation $\varpi^*(V_{\mathsf{GL}_k}^{\lambda_\bullet}\otimes V_{\mathbb{G}_m}^{z}\otimes V_{\mathbf{L_{inn}}}^{\nu})$ is isomorphic to the pullback via π of a representation in B if and only if $\nu\in\mathsf{B}^{\mathbf{inn}}$,*

$$\sum_{i=1}^{a}(\lambda_i-\lambda_{i+1})\omega_i\in\mathsf{B}^{\mathbf{out}}, \tag{42}$$

and

$$\lambda_{a+1}=\dots=\lambda_k=\frac{c\mathsf{j}\xi^2-z}{k(c\xi,\omega_1)}. \tag{43}$$

Proof. Note that by Lemma 7.4 for any $s\in\mathbb{Z}$ we have

$$\varpi^*(V_{\mathsf{GL}_k}^{(\lambda_1,\dots,\lambda_k)}\otimes V_{\mathbb{G}_m}^{z}\otimes V_{\mathbf{L_{inn}}}^{\nu})\cong\varpi^*(V_{\mathsf{GL}_k}^{(\lambda_1-s,\dots,\lambda_k-s)}\otimes V_{\mathbb{G}_m}^{z+sk(c\xi,\omega_1)}\otimes V_{\mathbf{L_{inn}}}^{\nu}).$$

So, taking $s=\lambda_k$ and using Lemma 7.7 we deduce $z+k\lambda_k(c\xi,\omega_1)=c\mathsf{j}\xi^2$, and the proposition follows. □

Denote by $\pi^*\mathsf{B}$ the set of all representations of $\mathbf{L_{out}}\times\mathbb{G}_m\times\mathbf{L_{inn}}$ which are pullbacks via π of representations of $\mathbf{L}$ from the block B. Now we can rewrite the action of the projector $\Pi_{\pi^*\mathsf{B}}$ on the subcategory of representations of $\mathbf{L_{out}}\times\mathbb{G}_m\times\mathbf{L_{inn}}$ with a given $\mathbb{G}_m$-component.

Corollary 7.9. *We have*

$$\Pi_{\pi^*\mathsf{B}}(\varpi^*(V^{\lambda_\bullet}_{\mathsf{GL}_k} \otimes V^{z(v)+c\mathsf{j}\xi^2}_{\mathbb{G}_\mathrm{m}} \otimes V^{\nu}_{\mathbf{L_{inn}}})) = \varpi^*(\Pi_{S^{\mathbf{out}}}(V^{\lambda_\bullet}_{\mathsf{GL}_k}) \otimes V^{z(v)+c\mathsf{j}\xi^2}_{\mathbb{G}_\mathrm{m}} \otimes \Pi_{\mathsf{B}^{\mathbf{inn}}}(V^{\nu}_{\mathbf{L_{inn}}})),$$

where $S^{\mathbf{out}}$ is the set of all $\lambda_\bullet$ such that (42) *holds and*

$$\lambda_{a+1} = \cdots = \lambda_k = \phi(v). \tag{44}$$

Proof. Substituting $z = z(v) + c\mathsf{j}\xi^2$ into (43) and comparing (41) with (32) yields (44). ◻

7.2. Proof of the compatibility. The goal of this section is to prove Theorem 7.1. So we take $\mu = \mu_{\mathbf{out}} + \mathsf{j}\xi + i_*\mu_{\mathbf{inn}} \in \mathsf{B}$ and consider the tensor product $V^{\kappa}_{\mathbf{L}} \otimes V^{v\rho-\rho}_{\mathbf{L}} \otimes V^{\mu}_{\mathbf{L}}$. We have

$$\pi^*(V^{\kappa+v\rho-\rho}_{\mathbf{L}} \otimes V^{\mu}_{\mathbf{L}}) = \varpi^*\big((V^{\kappa_\bullet+\tau_\bullet}_{\mathsf{GL}_k} \otimes V^{\mu_\bullet}_{\mathsf{GL}_k}) \otimes V^{z(v)+c\mathsf{j}\xi^2}_{\mathbb{G}_\mathrm{m}} \otimes (V^{\nu_v}_{\mathbf{L_{inn}}} \otimes V^{\mu_{\mathbf{inn}}}_{\mathbf{L_{inn}}})\big),$$

$$\pi^*(V^{\kappa}_{\mathbf{L}} \otimes V^{v\rho-\rho}_{\mathbf{L}} \otimes V^{\mu}_{\mathbf{L}}) = \varpi^*\big((V^{\kappa_\bullet}_{\mathsf{GL}_k} \otimes V^{\tau_\bullet}_{\mathsf{GL}_k} \otimes V^{\mu_\bullet}_{\mathsf{GL}_k}) \otimes V^{z(v)+c\mathsf{j}\xi^2}_{\mathbb{G}_\mathrm{m}} \otimes (V^{\nu_v}_{\mathbf{L_{inn}}} \otimes V^{\mu_{\mathbf{inn}}}_{\mathbf{L_{inn}}})\big),$$

$$\begin{aligned}\pi^*(V^{\kappa}_{\mathbf{L}} &\otimes \Pi_{\mathsf{B}}(V^{v\rho-\rho}_{\mathbf{L}} \otimes V^{\mu}_{\mathbf{L}})) \\ &= \varpi^*\big((V^{\kappa_\bullet}_{\mathsf{GL}_k} \otimes \Pi_{S^{\mathbf{out}}}(V^{\tau_\bullet}_{\mathsf{GL}_k} \otimes V^{\mu_\bullet}_{\mathsf{GL}_k})) \otimes V^{z(v)+c\mathsf{j}\xi^2}_{\mathbb{G}_\mathrm{m}} \otimes \Pi_{\mathsf{B}^{\mathbf{inn}}}(V^{\nu_v}_{\mathbf{L_{inn}}} \otimes V^{\mu_{\mathbf{inn}}}_{\mathbf{L_{inn}}})\big).\end{aligned}$$

So, the π-pullback of (40) is equal to the ϖ-pullback of the tensor product of

$$V^{\kappa_\bullet+\tau_\bullet}_{\mathsf{GL}_k} \otimes V^{\mu_\bullet}_{\mathsf{GL}_k} \to V^{\kappa_\bullet}_{\mathsf{GL}_k} \otimes V^{\tau_\bullet}_{\mathsf{GL}_k} \otimes V^{\mu_\bullet}_{\mathsf{GL}_k} \to V^{\kappa_\bullet}_{\mathsf{GL}_k} \otimes \Pi_{S^{\mathbf{out}}}(V^{\tau_\bullet}_{\mathsf{GL}_k} \otimes V^{\mu_\bullet}_{\mathsf{GL}_k}) \tag{45}$$

with

$$V^{z(v)+c\mathsf{j}\xi^2}_{\mathbb{G}_\mathrm{m}} \xrightarrow{\mathsf{id}} V^{z(v)+c\mathsf{j}\xi^2}_{\mathbb{G}_\mathrm{m}} \quad \text{and} \quad V^{\nu_v}_{\mathbf{L_{inn}}} \otimes V^{\mu_{\mathbf{inn}}}_{\mathbf{L_{inn}}} \to \Pi_{\mathsf{B}^{\mathbf{inn}}}(V^{\nu_v}_{\mathbf{L_{inn}}} \otimes V^{\mu_{\mathbf{inn}}}_{\mathbf{L_{inn}}}).$$

Since the last map is a $\mathsf{B}^{\mathbf{inn}}$-isomorphism, we only have to check that (45) is an $S^{\mathbf{out}}$-isomorphism.

Let $\widetilde{S}^{\mathbf{out}}$ be the set of all $\lambda_\bullet$ satisfying only (44). We claim that if we replace in (45) the projector $\Pi_{S^{\mathbf{out}}}$ by $\Pi_{\widetilde{S}^{\mathbf{out}}}$, then the resulting map

$$V^{\kappa_\bullet+\tau_\bullet}_{\mathsf{GL}_k} \otimes V^{\mu_\bullet}_{\mathsf{GL}_k} \to V^{\kappa_\bullet}_{\mathsf{GL}_k} \otimes V^{\tau_\bullet}_{\mathsf{GL}_k} \otimes V^{\mu_\bullet}_{\mathsf{GL}_k} \to V^{\kappa_\bullet}_{\mathsf{GL}_k} \otimes \Pi_{\widetilde{S}^{\mathbf{out}}}(V^{\tau_\bullet}_{\mathsf{GL}_k} \otimes V^{\mu_\bullet}_{\mathsf{GL}_k}) \tag{46}$$

is an $\widetilde{S}^{\mathbf{out}}$-isomorphism. Indeed, if $\phi(v)$ is a nonpositive integer then this is Corollary 10.2 from Appendix. If $\phi(v)$ is not an integer, then $\widetilde{S}^{\mathbf{out}} = \emptyset$, so any map is an $\widetilde{S}^{\mathbf{out}}$-isomorphism. Finally, if $\phi(v)$ is a positive integer then v is very special, hence $(\mu + \rho - v\rho, \alpha_1 + \cdots + \alpha_{k-1}) < \phi(v)$. This means that $\mu_1 + \tau_k < \phi(v)$, so by the Littlewood–Richardson rule the tensor products $V^{\kappa_\bullet+\tau_\bullet}_{\mathsf{GL}_k} \otimes V^{\mu_\bullet}_{\mathsf{GL}_k}$ and $V^{\tau_\bullet}_{\mathsf{GL}_k} \otimes V^{\mu_\bullet}_{\mathsf{GL}_k}$ contain no terms $V^{\lambda_\bullet}_{\mathsf{GL}_k}$ with $\lambda_k = \phi(v)$, and a fortiori no terms in $\widetilde{S}^{\mathbf{out}}$. Thus, both the source and the target of (45) become zero after applying $\Pi_{\widetilde{S}^{\mathbf{out}}}$, hence the map becomes an isomorphism. This finishes the proof that (46) is an $\widetilde{S}^{\mathbf{out}}$-isomorphism.

Since $S^{\mathbf{out}} \subset \widetilde{S}^{\mathbf{out}}$, it remains to check that

$$\Pi_{S^{\mathbf{out}}}(V^{\kappa_\bullet}_{\mathsf{GL}_k} \otimes \Pi_{\widetilde{S}^{\mathbf{out}}}(V^{\tau_\bullet}_{\mathsf{GL}_k} \otimes V^{\mu_\bullet}_{\mathsf{GL}_k})) = \Pi_{S^{\mathbf{out}}}(V^{\kappa_\bullet}_{\mathsf{GL}_k} \otimes \Pi_{S^{\mathbf{out}}}(V^{\tau_\bullet}_{\mathsf{GL}_k} \otimes V^{\mu_\bullet}_{\mathsf{GL}_k})). \tag{47}$$

Indeed, if (47) is true then the result of applying $\Pi_{\mathcal{S}^{\mathrm{out}}}$ to (45) coincides with that of applying $\Pi_{\mathcal{S}^{\mathrm{out}}}$ to (46). Since (46) becomes an isomorphism already after applying $\Pi_{\widetilde{\mathcal{S}}^{\mathrm{out}}}$, the assertion would follow.

Now to verify (47) we have to check that for any $\lambda_\bullet \in \widetilde{\mathcal{S}}^{\mathrm{out}}$ such that $V^{\lambda_\bullet}_{\mathsf{GL}_k}$ appears as a summand in $V^{\tau_\bullet}_{\mathsf{GL}_k} \otimes V^{\mu_\bullet}_{\mathsf{GL}_k}$ and $\Pi_{\mathcal{S}^{\mathrm{out}}}(V^{\kappa_\bullet}_{\mathsf{GL}_k} \otimes V^{\lambda_\bullet}_{\mathsf{GL}_k}) \neq 0$, one has $\lambda_\bullet \in \mathcal{S}^{\mathrm{out}}$. Let $\lambda'_\bullet \in \mathcal{S}^{\mathrm{out}}$ be such that $V^{\lambda'_\bullet}_{\mathsf{GL}_k}$ is a summand in $V^{\kappa_\bullet}_{\mathsf{GL}_k} \otimes V^{\lambda_\bullet}_{\mathsf{GL}_k}$. Note that both $\lambda_\bullet$ and $\lambda'_\bullet$ satisfy (44). Since $\kappa_\bullet$ is nonnegative and $V^{\lambda'_\bullet}_{\mathsf{GL}_k}$ is a summand in $V^{\kappa_\bullet}_{\mathsf{GL}_k} \otimes V^{\lambda_\bullet}_{\mathsf{GL}_k}$, the Young diagram corresponding to the weight $\sum_{i=1}^{a}(\lambda_i - \lambda_{i+1})\omega_i$ is a subdiagram in the Young diagram corresponding to $\sum_{i=1}^{a}(\lambda'_i - \lambda'_{i+1})\omega_i$. The latter is in $\mathsf{B}^{\mathrm{out}}$ since $\lambda'_\bullet \in \mathcal{S}^{\mathrm{out}}$. Hence, the former is also in $\mathsf{B}^{\mathrm{out}}$, since $\mathsf{B}^{\mathrm{out}}$ is closed with respect to passing to Young subdiagrams. Thus, $\lambda_\bullet$ is in $\mathcal{S}^{\mathrm{out}}$ and we are done.

8. Explicit description of the exceptional blocks

In this section we will pass from the abstract description of the blocks B_{j} given in Subsections 5.4 and 5.5 to a more explicit description which will be used later to deal with concrete examples. We show in fact that both the inner and the outer parts of the blocks are described by several simple inequalities, numbered by $\mathbf{W}_{\mathbf{M}_{\mathsf{j}}}$-orbits in the $\mathbf{W}_{\mathbf{H}_{\mathsf{j}}}$-orbit of the weight $\delta_{a(\mathsf{j})}$ (the shape of the core $\mathbf{R}_{\mathsf{j}}$).

Let us fix $\mathsf{j} \in \mathsf{J}$. It will be convenient to write the shape $\delta = \delta_{a(\mathsf{j})} \in P^+_{\mathbf{H}_{\mathsf{j}}}$ of the core $\mathbf{R}_{\mathsf{j}} = \mathbf{R}_\delta$ in the form

$$\delta = -h^*_{\mathsf{j}}\gamma, \tag{48}$$

where $\gamma \in P_{\mathbf{G}}$.

Remark 8.1. Since the action of the Weyl group on roots is much better understood than on arbitrary weights (for example, one can use tables of roots), the most convenient choice of γ is the simple root of the vertex of $D_{\mathbf{G}}$ adjacent to $D_{\mathbf{H}_{\mathsf{j}}}$. In this case the $\mathbf{W}_{\mathbf{H}_{\mathsf{j}}}$-orbit of γ is described in Lemma 2.7.

8.1. The big blocks.

First, we give a description of the block B_{j}.

Assume that $\gamma \in P_{\mathbf{G}}$ and that δ defined by (48) is $\mathbf{H}_{\mathsf{j}}$-dominant. To ease the notation we will write $\mathbf{H}$ for $\mathbf{H}_{\mathsf{j}}$, h for h_{j}, and $\mathbf{M}$ for $\mathbf{M}_{\mathsf{j}} = \mathbf{L} \cap \mathbf{H}_{\mathsf{j}}$. Since $\mathbf{W}_{\mathbf{M}} \subset \mathbf{W}_{\mathbf{H}}$, the $\mathbf{W}_{\mathbf{H}}$-orbit of γ splits into several $\mathbf{W}_{\mathbf{M}}$-orbits. We number the orbits by integers $0, \dots, m$ in such a way that the 0-th orbit is the $\mathbf{W}_{\mathbf{M}}$-orbit of γ itself.

In each $\mathbf{W}_{\mathbf{M}}$-orbit we have two special elements: the unique $\mathbf{M}$-dominant representative γ_{t+} and the unique $\mathbf{M}$-antidominant representative γ_{t-} (where $0 \le t \le m$). Note that $\gamma_{0-} = \gamma$, since we have assumed that $h^*\gamma = -\delta$ is $\mathbf{H}$-antidominant. Using these data we can describe the block B_{j} more explicitly. We start with the inner part of the block.

Proposition 8.2. *We have*

$$\mathsf{B}^{\mathbf{inn}}_{\mathsf{j}} = \left\{ \nu \in P^+_{\mathbf{L}_{\mathbf{inn}}} \;\middle|\; \begin{array}{l} \max\{(i^*\gamma_{t+}, \nu), -(i^*\gamma_{t-}, \nu)\} \le \frac{1}{2}(h^*(\gamma_{t-} - \gamma), \rho_{\mathbf{H}}) \\ \text{for all } 0 \le t \le m \text{ and } \mathsf{j}\xi + i_*\nu \in P_{\mathbf{L}} \end{array} \right\}. \tag{49}$$

Proof. By definition, $\mathsf{B}_{\mathsf{j}}^{\mathbf{inn}}$ is the set of all ν with $\mathsf{j}\xi + i_*\nu \in P_{\mathbf{L}}$ and $\rho_{\mathbf{H}} \pm 2w_{\mathbf{L_{inn}}} i_*\nu \in \mathbf{R}_\delta$ for all $w_{\mathbf{L_{inn}}} \in \mathbf{W}_{\mathbf{L_{inn}}}$. We only need to rework the second condition. Substituting the Definition 5.1 of the core $\mathbf{R}_\delta$, it can be rewritten as

$$-(w_{\mathbf{H}} h^*\gamma, \rho_{\mathbf{H}} \pm 2w_{\mathbf{L_{inn}}} i_*\nu) \le -(h^*\gamma, \rho_{\mathbf{H}}).$$

Since h^* is $\mathbf{W_H}$-equivariant, this inequality can be rewritten as

$$\pm(h^* w_{\mathbf{H}}\gamma, 2w_{\mathbf{L_{inn}}} i_*\nu) \le (h^*(w_{\mathbf{H}}\gamma - \gamma), \rho_{\mathbf{H}}).$$

Note that $w_{\mathbf{H}}\gamma = w_{\mathbf{M}}\gamma_{t+}$ for appropriate $w_{\mathbf{M}} \in \mathbf{W_M}$ and $t \in \{0, 1, \dots, m\}$. After such a substitution the inequality takes the form

$$\pm(h^* w_{\mathbf{M}}\gamma_{t+}, 2w_{\mathbf{L_{inn}}} i_*\nu) \le (h^*(w_{\mathbf{M}}\gamma_{t+} - \gamma), \rho_{\mathbf{H}}).$$

Let $\mathbf{M_{out}} = \mathbf{L_{out}} \cap \mathbf{H}$ and $\mathbf{M_{inn}} = \mathbf{L_{inn}} \cap \mathbf{H} = \mathbf{L_{inn}}$. Then $\mathbf{W_M} = \mathbf{W_{M_{out}}} \times \mathbf{W_{M_{inn}}}$. In particular, we can write $w_{\mathbf{M}} = w_{\mathbf{M_{out}}} w_{\mathbf{M_{inn}}}$ with $w_{\mathbf{M_{out}}} \in \mathbf{W_{M_{out}}}$ and $w_{\mathbf{M_{inn}}} \in \mathbf{W_{M_{inn}}}$. Moreover, $i_*\nu$ is fixed by $\mathbf{W_{M_{out}}}$, hence the LHS is equal to

$$\pm 2(h^*\gamma_{t+}, w_{\mathbf{M}}^{-1} w_{\mathbf{L_{inn}}} i_*\nu) = \pm 2(h^*\gamma_{t+}, w_{\mathbf{M_{out}}}^{-1} w_{\mathbf{M_{inn}}}^{-1} w_{\mathbf{L_{inn}}} i_*\nu) = \pm 2(h^*\gamma_{t+}, w'_{\mathbf{L_{inn}}} i_*\nu),$$

where $w'_{\mathbf{L_{inn}}} = w_{\mathbf{M_{inn}}}^{-1} w_{\mathbf{L_{inn}}}$. Note that $w'_{\mathbf{L_{inn}}}$ on the LHS runs through $\mathbf{W_{L_{inn}}}$ independently of $w_{\mathbf{M}}$ on the RHS running through $\mathbf{W_M}$. Hence, the inequality for all $w'_{\mathbf{L_{inn}}} \in \mathbf{W_{L_{inn}}}$ and $w_{\mathbf{M}} \in \mathbf{W_M}$ is equivalent to

$$\max_{w'_{\mathbf{L_{inn}}} \in \mathbf{W_{L_{inn}}}} \{\pm 2(h^*\gamma_{t+}, w'_{\mathbf{L_{inn}}} i_*\nu)\} \le \min_{w_{\mathbf{M}} \in \mathbf{W_M}} \{(h^*(w_{\mathbf{M}}\gamma_{t+} - \gamma), \rho_{\mathbf{H}})\}.$$

The expression under the maximum can be rewritten as $\pm 2((w'_{\mathbf{L_{inn}}})^{-1} i^*\gamma_{t+}, \nu)$. Since both ν and $i^*\gamma_{t+}$ are $\mathbf{L_{inn}}$-dominant, the expression with the "+" sign is maximal when $w'_{\mathbf{L_{inn}}} = 1$, and the one with the "−" sign is maximal when $(w'_{\mathbf{L_{inn}}})^{-1} i^*\gamma_{t+} = i^*\gamma_{t-}$. Thus, the LHS is

$$\max\{2(i^*\gamma_{t+}, \nu), -2(i^*\gamma_{t-}, \nu)\}.$$

Similarly, since $\rho_{\mathbf{H}}$ is $\mathbf{M}$-dominant, the expression on the RHS is minimal when $w_{\mathbf{M}}\gamma_{t+} = \gamma_{t-}$. The claim follows. □

Now let us rewrite more explicitly the definition of the outer part of the block $\mathsf{B}_{\mathsf{j}}^{\mathbf{out}}$. Denote by $\hat{\gamma}_t$ the $\mathbf{L_{out}}$-dominant representative in the $\mathbf{W_{L_{out}}}$-orbit of $h_* h^*\gamma_{t+}$. Also, set

$$d_{\mathsf{j}}^{t,+} := \max\{(i^*\gamma_{t+}, \nu) \mid \nu \in \mathsf{B}_{\mathsf{j}}^{\mathbf{inn}}\}, \qquad d_{\mathsf{j}}^{t,-} := -\min\{(i^*\gamma_{t-}, \nu) \mid \nu \in \mathsf{B}_{\mathsf{j}}^{\mathbf{inn}}\}. \tag{50}$$

Proposition 8.3. *We have*

$$\hat{\mathsf{B}}_{\mathsf{j}}^{\mathbf{out}} = \{\lambda \in \mathsf{Ker}\, h^* \cap P_{\mathbf{G}}^+ \mid (\lambda, \hat{\gamma}_t) + d_{\mathsf{j}}^{t,+} + d_{\mathsf{j}}^{t,-} \le (\rho_{\mathbf{H}}, \gamma_{t-} - \gamma) \text{ for all } 0 \le t \le m\}. \tag{51}$$

Proof. Take $\lambda \in \operatorname{Ker} h^* \cap P_{\mathbf{G}}^+$. By definition $\lambda \in \hat{\mathsf{B}}_{\mathsf{j}}^{\mathbf{out}}$ if and only if $h^*(\rho - w_{\mathbf{L}}\lambda) - w_{\mathbf{L_{inn}}} i_* \nu + w'_{\mathbf{L_{inn}}} i_* \nu' \in \mathbf{R}_\delta$. By definition of $\mathbf{R}_\delta$ this is equivalent to

$$(h^*(\rho - w_{\mathbf{L}}\lambda) - w_{\mathbf{L_{inn}}} i_* \nu + w'_{\mathbf{L_{inn}}} i_* \nu', -v_{\mathbf{H}} h^* \gamma) \leq (\rho_{\mathbf{H}}, -h^*\gamma)$$

for all $\nu, \nu' \in \mathsf{B}_{\mathsf{j}}^{\mathbf{inn}}$, $w_{\mathbf{L}} \in \mathbf{W_L}$, $w_{\mathbf{L_{inn}}}, w'_{\mathbf{L_{inn}}} \in \mathbf{W_{L_{inn}}}$, and $v_{\mathbf{H}} \in \mathbf{W_H}$. Note that $\mathbf{W_L} = \mathbf{W_{L_{out}}} \times \mathbf{W_{L_{inn}}}$ and that λ is $\mathbf{W_{L_{inn}}}$-invariant. So we can rewrite the above condition as

$$(h^*(\rho - w_{\mathbf{L_{out}}}\lambda) - w_{\mathbf{L_{inn}}} i_* \nu + w'_{\mathbf{L_{inn}}} i_* \nu', -v_{\mathbf{H}} h^* \gamma) \leq (\rho_{\mathbf{H}}, -h^*\gamma).$$

Since h^* is $\mathbf{W_H}$-equivariant, we have $v_{\mathbf{H}} h^*\gamma = h^*(v_{\mathbf{H}}\gamma)$. Further, each weight $v_{\mathbf{H}}\gamma$ can be written as $v_{\mathbf{M}}\gamma_{t+}$ for some $0 \leq t \leq m$ and $v_{\mathbf{M}} \in \mathbf{W_M}$. This allows us to rewrite the condition as

$$(h^*(\rho - w_{\mathbf{L_{out}}}\lambda) - w_{\mathbf{L_{inn}}} i_* \nu + w'_{\mathbf{L_{inn}}} i_* \nu', -v_{\mathbf{M}} h^* \gamma_{t+}) \leq (\rho_{\mathbf{H}}, -h^*\gamma)$$

for all $\nu, \nu' \in \mathsf{B}_{\mathsf{j}}^{\mathbf{inn}}$, $w_{\mathbf{L_{out}}} \in \mathbf{W_{L_{out}}}$, $w_{\mathbf{L_{inn}}}, w'_{\mathbf{L_{inn}}} \in \mathbf{W_{L_{inn}}}$, $v_{\mathbf{M}} \in \mathbf{W_M}$, and $0 \leq t \leq m$. Now recall that $h^*\rho = \rho_{\mathbf{H}}$ and move it from the LHS to the RHS:

$$(h^*(-w_{\mathbf{L_{out}}}\lambda) - w_{\mathbf{L_{inn}}} i_* \nu + w'_{\mathbf{L_{inn}}} i_* \nu', -v_{\mathbf{M}} h^* \gamma_{t+}) \leq (\rho_{\mathbf{H}}, h^*(v_{\mathbf{M}}\gamma_{t+} - \gamma)).$$

Writing $v_{\mathbf{M}} = v_{\mathbf{M_{out}}} v_{\mathbf{L_{inn}}}$ on the LHS, taking into account that h^* is $\mathbf{W_M}$-equivariant, and replacing $v_{\mathbf{M_{out}}}^{-1} w_{\mathbf{L_{out}}}$ with $w_{\mathbf{L_{out}}}$, $v_{\mathbf{L_{inn}}}^{-1} w_{\mathbf{L_{inn}}}$ with $w_{\mathbf{L_{inn}}}$, and $v_{\mathbf{L_{inn}}}^{-1} w'_{\mathbf{L_{inn}}}$ with $w'_{\mathbf{L_{inn}}}$ we rewrite the condition as

$$(h^*(-w_{\mathbf{L_{out}}}\lambda) - w_{\mathbf{L_{inn}}} i_* \nu + w'_{\mathbf{L_{inn}}} i_* \nu', -h^* \gamma_{t+}) \leq (\rho_{\mathbf{H}}, h^*(v_{\mathbf{M}}\gamma_{t+} - \gamma)).$$

Finally, using the adjunction of h^* and h_* and of i^* and i_* we rewrite this as

$$(w_{\mathbf{L_{out}}}\lambda, h_* h^* \gamma_{t+}) + (w_{\mathbf{L_{inn}}}\nu, i^*\gamma_{t+}) + (-w'_{\mathbf{L_{inn}}}\nu', i^*\gamma_{t+}) \leq (\rho_{\mathbf{H}}, h^*(v_{\mathbf{M}}\gamma_{t+} - \gamma)).$$

Note that each term on both sides contains an action of a Weyl group element, and these elements run through the corresponding Weyl groups independently. Therefore, one can replace each summand by its maximum (on the LHS) or minimum (on the RHS) to obtain an equivalent inequality.

The maxima of the second and the third summands on the LHS are given by $d_{\mathsf{j}}^{t,\pm}$ by definition. The first summand can be rewritten as $(\lambda, w_{\mathbf{L_{out}}}^{-1} h_* h^* \gamma_{t+})$, and since λ is $\mathbf{L_{out}}$-dominant, to achieve the maximum one should choose $w_{\mathbf{L_{out}}}^{-1}$ in such a way that the corresponding weight is also $\mathbf{L_{out}}$-dominant. By definition, it is $\hat{\gamma}_t$, hence the maximum of the first summand is $(\lambda, \hat{\gamma}_t)$. Finally, as in Proposition 8.2, the minimum on the RHS is equal to $(\rho_{\mathbf{H}}, \gamma_{t-} - \gamma)$. Combining all this, we obtain the result. □

8.2. The small blocks. Now we will give a description of the blocks $\bar{\mathsf{B}}_{\mathsf{j}}$.

Take $\mathsf{j}, \mathsf{j}' \in \mathsf{J}$ and assume that $\mathsf{j}' < \mathsf{j}$. As before we write $\mathbf{H}$ for $\mathbf{H}_{\mathsf{j}}$, h for h_{j}, and $\mathbf{M}$ for $\mathbf{M}_{\mathsf{j}} = \mathbf{L} \cap \mathbf{H}_{\mathsf{j}}$. In addition, we will write $\mathbf{H}'$ for $\mathbf{H}_{\mathsf{j}'}$, h' for $h_{\mathsf{j}'}$, and $\mathbf{M}'$ for $\mathbf{M}_{\mathsf{j}'} = \mathbf{L} \cap \mathbf{H}_{\mathsf{j}'}$. Similarly we denote by γ and γ' the weights such that $\delta = -h^*\gamma$ and $\delta' = -(h')^*\gamma'$ are the shapes of the corresponding cores. We number the orbits of $\mathbf{W}_{\mathbf{M}'}$ on $\mathbf{W}_{\mathbf{H}'}\gamma'$ from 0 to m', and we denote by $\gamma'_{t\pm}$ the $\mathbf{M}'$-dominant and antidominant representatives of these orbits.

The proof of the next two results is analogous to that of Propositions 8.2 and 8.3.

Proposition 8.4. *The inner part of the block* $\bar{\mathsf{B}}_{\mathsf{j}}$ *can be described as*

$$\bar{\mathsf{B}}_{\mathsf{j}}^{\mathbf{inn}} = \left\{ \nu \in \mathsf{B}_{\mathsf{j}}^{\mathbf{inn}} \;\middle|\; \begin{array}{l} (\mathsf{j} - \mathsf{j}')((h')^*\xi, (h')^*\gamma'_{t+}) + (i^*\gamma'_{t+}, \nu) + \bar{d}_{\mathsf{j}'}^{t,-} < (\rho_{\mathbf{H}'}, (h')^*(\gamma'_{t-} - \gamma')) \\ \textit{for all } \mathsf{j}' < \mathsf{j} \textit{ and for all } 0 \le t \le m' \end{array} \right\}, \tag{52}$$

where for $\mathsf{j}' < \mathsf{j}$,

$$\bar{d}_{\mathsf{j}'}^{t,-} := -\min\{(i^*\gamma'_{t-}, \nu') \mid \nu' \in \bar{\mathsf{B}}_{\mathsf{j}'}^{\mathbf{inn}}\}. \tag{53}$$

Proposition 8.5. *The outer part of the block* $\bar{\mathsf{B}}_{\mathsf{j}}$ *can be described as*

$$\bar{\mathsf{B}}_{\mathsf{j}}^{\mathbf{out}} = \left\{ \lambda \in \mathsf{B}_{\mathsf{j}}^{\mathbf{out}} \;\middle|\; \begin{array}{l} (\lambda, \hat{\gamma}'_t) + (\mathsf{j} - \mathsf{j}')((h')^*\xi, (h')^*\gamma'_{t+}) + \bar{d}_{\mathsf{j}',\mathsf{j}}^{t+} + \bar{d}_{\mathsf{j}'}^{t-} < (\rho_{\mathbf{H}'}, (h')^*(\gamma'_{t-} - \gamma')) \\ \textit{for all } \mathsf{j}' < \mathsf{j} \textit{ and for all } 0 \le t \le m' \end{array} \right\}, \tag{54}$$

where

$$\bar{d}_{\mathsf{j}',\mathsf{j}}^{t,+} := \max\{(i^*\gamma'_{t+}, \nu) \mid \nu \in \bar{\mathsf{B}}_{\mathsf{j}}^{\mathbf{inn}}\} \tag{55}$$

and $\hat{\gamma}'_t$ *is the* $\mathbf{L_{out}}$*-dominant representative in* $\mathbf{W}_{\mathbf{L_{out}}} h'_*(h')^*\gamma'_{t+}$.

9. Explicit collections for classical groups

Now we will show that the construction of the previous section leads to (conjecturally full) exceptional collections for isotropic Grassmannians of types B, C and D, and to many interesting collections in type A.

So, assume that $\mathbf{G}$ is of type B, C or D and consider the standard numbering of the vertices of its Dynkin diagram.

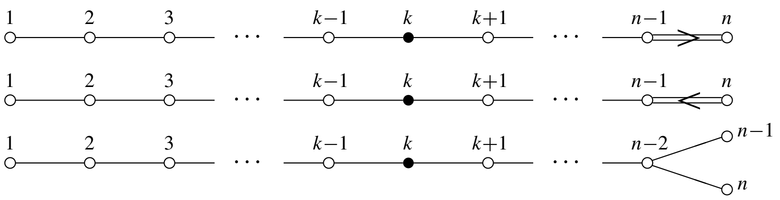

Diagram B_n

Diagram C_n

Diagram D_n

To treat these cases simultaneously it is convenient to denote

$$e = \begin{cases} 1/2 & \text{if } \mathbf{G} \text{ is of type } B, \\ 1 & \text{if } \mathbf{G} \text{ is of type } C, \\ 0 & \text{if } \mathbf{G} \text{ is of type } D. \end{cases} \tag{56}$$

Then the weight lattice $P_{\mathbf{G}}$ can be identified with the sublattice of $\mathbb{Q}^n$ spanned by ω_i $(1 \le i \le n)$ with

$$\begin{aligned} \omega_i^{B,C,D} &= (\underbrace{1, \dots, 1}_{i}, \underbrace{0, \dots, 0}_{n-i}), \quad 1 \le i \le n-2+2e, \\ \omega_n^{B,D} &= (1/2, \dots, 1/2), \\ \omega_{n-1}^{D} &= (1/2, \dots, 1/2, -1/2). \end{aligned}$$

Let k be the number of the vertex of the Dynkin diagram of $\mathbf{G}$ corresponding to the maximal parabolic subgroup $\mathbf{P}$, so that $\xi = \omega_k$.

9.1. Isotropic Grassmannians. First, we assume that

$$k \le n + 2e - 2.$$

In other words, $k \le n-1$ for type B, $k \le n$ for type C, and $k \le n-2$ for type D. Then

$$X := \mathbf{G}/\mathbf{P} = \begin{cases} \mathsf{OGr}(k, 2n+1), & k \le n-1 \ (\mathbf{G} \text{ is of type } B_n), \\ \mathsf{SGr}(k, 2n), & k \le n \qquad (\mathbf{G} \text{ is of type } C_n), \\ \mathsf{OGr}(k, 2n), & k \le n-2 \ (\mathbf{G} \text{ is of type } D_n), \end{cases}$$

where OGr (resp., SGr) denotes the orthogonal (resp., symplectic) isotropic Grassmannian.

Let $D_{\mathbf{out}}$ be the component of $D \setminus \beta$ containing the vertices from 1 to $k-1$. Then $b = k-1$ and $D_{\mathbf{inn}}$ is the component containing the vertices from $k+1$ to n. Note that i^* is the projection onto the last $n-k$ coordinates with respect to the standard basis $\varepsilon_1, \dots, \varepsilon_n$ in $P_{\mathbf{G}} = \mathbb{Q}^n$. The simple roots are

$$\begin{aligned} \alpha_i^{B,C,D} &= \varepsilon_i - \varepsilon_{i+1}, \quad 1 \le i \le n-1, \\ \alpha_n^{B} &= \varepsilon_n, \quad \alpha_n^{C} = 2\varepsilon_n, \quad \alpha_n^{D} = \varepsilon_{n-1} + \varepsilon_n. \end{aligned}$$

Note also that

$$\rho = (n+e-1, n+e-2, \dots, e),$$

thus $(\rho, \varepsilon_i) = n + e - i$.

Now take any $a \le k-1$. Then h_a^* is the projection onto the last $n-a$ coordinates (it kills all ε_i with $i \le a$). The simple root corresponding to $\mathbf{P}$ is $\beta = \varepsilon_k - \varepsilon_{k+1}$, so the maximal root of $\mathbf{H}_a$ with the coefficient of β equal to 1 is $\bar{\beta}_a = \varepsilon_{a+1} + \varepsilon_{k+1}$, so by Lemma 2.19 the index of the Grassmannian $\mathbf{H}_a/(\mathbf{H}_a \cap \mathbf{P})$ is

$$r_a = (\rho, \beta + \bar{\beta}_a)/(\xi, \beta) = 2n + 2e - a - k - 1.$$

In particular, when a decreases by 1, r_a increases by 1. Also, $r_{k-1} = 2n + 2e - 2k$, while

$$r = r_0 = 2n + 2e - k - 1.$$

Further, the weight θ defined by (27) in this case is

$$\theta = (\underbrace{0, \dots, 0}_{k-1}, 1, \underbrace{0, \dots, 0}_{n-k}).$$

It follows that $(\theta, P_{\mathbf{L}}) = \frac{1}{2}\mathbb{Z}$ if $\mathbf{G}$ is of type B or D and $(\theta, P_{\mathbf{L}}) = \mathbb{Z}$ if $\mathbf{G}$ is of type C, and

$$\mathsf{J} = \begin{cases} \frac{1}{2}\mathbb{Z} \cap [0, 2n - k - 1/2] & \text{if } \mathbf{G} \text{ is of type } B, \\ \mathbb{Z} \cap [0, 2n - k] & \text{if } \mathbf{G} \text{ is of type } C, \\ \frac{1}{2}\mathbb{Z} \cap [0, 2n - k - 3/2] & \text{if } \mathbf{G} \text{ is of type } D. \end{cases}$$

Applying (28) we conclude that

$$a(\mathsf{j}) = \begin{cases} \lfloor \mathsf{j} \rfloor & \text{if } \mathsf{j} < k, \\ k - 1 & \text{if } \mathsf{j} \ge k. \end{cases}$$

Now we are going to apply Propositions 8.2–8.5. We take

$$\gamma_a = \alpha_a = \varepsilon_a - \varepsilon_{a+1}.$$

Note that $\mathbf{W}_{\mathbf{H}_a}$ acts by permutations of the last $n - a$ coordinates and changes of signs of the coordinates (in the case of type D by pairwise changes of signs), while $\mathbf{W}_{\mathbf{M}_a}$ acts by permuting coordinates from $a + 1$ to k and from $k + 1$ to n separately and (pairwise) changes of signs only of the last $n - k$ coordinates. Thus, the $\mathbf{W}_{\mathbf{H}_a}$-orbit of γ_a consists of all vectors $\varepsilon_a \pm \varepsilon_i$, $a + 1 \le i \le n$, and it splits into three $\mathbf{W}_{\mathbf{M}_a}$-orbits:

$$\{\varepsilon_a - \varepsilon_i\}_{a+1 \le i \le k}, \quad \{\varepsilon_a \pm \varepsilon_i\}_{k+1 \le i \le n}, \quad \text{and} \quad \{\varepsilon_a + \varepsilon_i\}_{a+1 \le i \le k}.$$

Thus, using the notation of Section 8 we have $m = 2$ (unless $\mathbf{G}$ has type C and $k = n$, in which case the second orbit is empty and so $m = 1$), and the characteristic weights and quantities from Section 8 are given by the following table:

t	γ_{t-}	$(\rho_{\mathbf{H}}, \gamma_{t-} - \gamma)$	γ_{t+}	$h_a^*\gamma_{t+}$	$\hat{\gamma}_t$	$(h_a^*\xi, h_a^*\gamma_{t+})$	$i^*\gamma_{t+}$	$i^*\gamma_{t-}$
0	$\varepsilon_a - \varepsilon_{a+1}$	0	$\varepsilon_a - \varepsilon_k$	$-\varepsilon_k$	$-\varepsilon_k$	-1	0	0
1	$\varepsilon_a - \varepsilon_{k+1}$	$k - a$	$\varepsilon_a + \varepsilon_{k+1}$	ε_{k+1}	ε_{k+1}	0	ε_{k+1}	$-\varepsilon_{k+1}$
2	$\varepsilon_a + \varepsilon_k$	$2n + 2e - a - k - 1$	$\varepsilon_a + \varepsilon_{a+1}$	ε_{a+1}	ε_1	1	0	0

(if $\mathbf{G}$ has type C and $k = n$ then the line $t = 1$ should be omitted).

Applying Proposition 8.2 we obtain the following description of $\mathsf{B}_{\mathsf{j}}^{\mathbf{inn}}$:

$$\mathsf{B}_{\mathsf{j}}^{\mathbf{inn}} = \{(\nu_{k+1}, \dots, \nu_n) \in P_{\mathbf{L_{inn}}}^+ \mid 2\nu_{k+1} \le k - a(\mathsf{j}) \text{ and } \nu_i \equiv \mathsf{j} \ (\mathrm{mod}\, \mathbb{Z})\}.$$

Further, we apply (50) and compute

$$d_{\mathrm{j}}^{1,\pm} = \{\mathrm{j}\} + \lfloor (k - a(\mathrm{j}))/2 - \{\mathrm{j}\}\rfloor$$

(where $\{-\}$ stands for the fractional part), and for other values of t we have $d_{\mathrm{j}}^{t,\pm} = 0$.

Now we can describe $\mathsf{B}_{\mathrm{j}}^{\mathbf{out}}$. Note that $(\hat{\gamma}_t, \operatorname{Ker} h_a^*) = 0$ unless $t = 2$. So, for $t = 0$, Proposition 8.3 gives an empty condition, and for $t = 1$ we obtain $d_{\mathrm{j}}^{1,+} + d_{\mathrm{j}}^{1,-} \le k - a$, which holds by the definition of $d_{\mathrm{j}}^{1,\pm}$. Finally, the condition for $t = 2$ gives

$$\mathsf{B}_{\mathrm{j}}^{\mathbf{out}} = \{(\lambda_1, \dots, \lambda_{a(\mathrm{j})}, 0, \dots, 0) \mid 2n + 2e - a(\mathrm{j}) - k - 1 \ge \lambda_1 \ge \dots \ge \lambda_{a(\mathrm{j})} \ge 0\}.$$

Note that $\mathsf{B}_{\mathrm{j}}^{\mathbf{out}}$ is the set of Young diagrams inscribed in the rectangle $a(\mathrm{j}) \times (2n + 2e - a(\mathrm{j}) - k - 1)$, hence it is closed under taking subdiagrams. Thus, the second condition of Theorem 7.1 is satisfied. Since there are no very special elements by Lemma 5.8, the first condition is satisfied as well, so the theorem applies, and we conclude that the block B_{j} consisting of all $(\lambda_1, \dots, \lambda_n) \in P_{\mathbf{L}}^+$ such that

$$\begin{aligned}
&2n + 2e + \mathrm{j} - a(\mathrm{j}) - k - 1 \ge \lambda_1 \ge \dots \ge \lambda_{a(\mathrm{j})} \ge \mathrm{j} = \lambda_{a(\mathrm{j})+1} = \dots = \lambda_k,\\
&(k - a(\mathrm{j}))/2 \ge \lambda_{k+1} \ge \dots \ge \lambda_n,\\
&\lambda_1, \dots, \lambda_n \equiv \mathrm{j} \pmod{\mathbb{Z}}
\end{aligned}$$

is exceptional.

Now we are going to apply Proposition 8.4. First, let us show that

$$\bar{\mathsf{B}}_{\mathrm{j}}^{\mathbf{inn}} = \mathsf{B}_{\mathrm{j}}^{\mathbf{inn}} \quad \text{for } \mathrm{j} < k \tag{57}$$

and $\bar{d}_{\mathrm{j}}^{1-} = d_{\mathrm{j}}^{1-} = \{\mathrm{j}\} + \lfloor (k - a(\mathrm{j}))/2 - \{\mathrm{j}\}\rfloor$. We use induction on j. The base of induction, $\mathrm{j} = 0$, is clear. Assume that the statement is proved for all $\mathrm{j}' < \mathrm{j}$. Then by Proposition 8.4, the additional condition defining $\bar{\mathsf{B}}_{\mathrm{j}}^{\mathbf{inn}}$ is

$$\nu_{k+1} + \{\mathrm{j}'\} + \lfloor (k - a(\mathrm{j}'))/2 - \{\mathrm{j}'\}\rfloor < k - a(\mathrm{j}').$$

We claim that this condition is always satisfied for $\nu \in \mathsf{B}_{\mathrm{j}}^{\mathbf{inn}}$. Indeed, we have

$$\begin{aligned}
&\{\mathrm{j}\} + \lfloor (k - a(\mathrm{j}))/2 - \{\mathrm{j}\}\rfloor + \{\mathrm{j}'\} + \lfloor (k - a(\mathrm{j}'))/2 - \{\mathrm{j}'\}\rfloor\\
&\qquad \le (k - a(\mathrm{j}))/2 + (k - a(\mathrm{j}'))/2 = k - (a(\mathrm{j}) + a(\mathrm{j}'))/2 \le k - a(\mathrm{j}'),
\end{aligned} \tag{58}$$

and equality is possible only if $a(\mathrm{j}) = a(\mathrm{j}')$ and both $(k - a(\mathrm{j}))/2 - \{\mathrm{j}\}$ and $(k - a(\mathrm{j}))/2 - \{\mathrm{j}'\}$ are integers. But for $\mathrm{j}', \mathrm{j} < k$ one has $a(\mathrm{j}) = \lfloor \mathrm{j} \rfloor$, so the first condition shows that the integer parts of j and j′ are equal, while the second shows that the difference $\mathrm{j} - \mathrm{j}'$ is an integer. This is possible only if $\mathrm{j} = \mathrm{j}'$, which is a contradiction. Hence, one of the inequalities in (58) is strict as claimed. This finishes the proof of (57).

Now let us check that

$$\bar{\mathsf{B}}^{\mathbf{inn}}_{\mathsf{j}} = \{0\} \qquad \text{for integer } \mathsf{j} \geq k,$$
$$\bar{\mathsf{B}}^{\mathbf{inn}}_{\mathsf{j}} = \emptyset \qquad \text{for half-integer } \mathsf{j} \geq k.$$

Indeed, if j is half-integer take $\mathsf{j}' = k - 1/2$. Then $\bar{d}^{1-}_{\mathsf{j}'} = \{\mathsf{j}'\} + \lfloor (k - a(\mathsf{j}')/2 - \{\mathsf{j}'\} \rfloor = 1/2 + \lfloor 1/2 - 1/2 \rfloor = 1/2$, so the inequality defining $\bar{\mathsf{B}}^{\mathbf{inn}}_{\mathsf{j}} \subset \mathsf{B}^{\mathbf{inn}}_{\mathsf{j}}$ is

$$\nu_{k+1} + 1/2 < 1.$$

On the other hand, ν_{k+1} should be a nonnegative half-integer, so $\bar{\mathsf{B}}^{\mathbf{inn}}_{\mathsf{j}} = \emptyset$. For an integer $\mathsf{j} \geq k$ we note that already $\mathsf{B}^{\mathbf{inn}}_{\mathsf{j}} = \{0\}$, so we only have to check that the inequality (52) is satisfied for $\nu = 0$. Indeed, if $\mathsf{j}' < k$ then $a(\mathsf{j}') \leq k - 1$, hence

$$0 + \bar{d}^{1-}_{\mathsf{j}'} = \{\mathsf{j}'\} + \lfloor (k - a(\mathsf{j}'))/2 - \{\mathsf{j}'\} \rfloor \leq \{\mathsf{j}'\} + (k - a(\mathsf{j}'))/2 - \{\mathsf{j}'\} = (k - a(\mathsf{j}'))/2 < k - a(\mathsf{j}').$$

Further, if $\mathsf{j}' \geq k$ is a half-integer then we already know that $\bar{\mathsf{B}}^{\mathbf{inn}}_{\mathsf{j}'}$ is empty, so $\bar{d}^{1,-}_{\mathsf{j}'} = -\infty$ and so we have no restriction on ν. Finally, if $\mathsf{j}' \geq k$ is an integer then by the induction hypothesis we have $\bar{d}^{1-}_{\mathsf{j}'} = 0$ while $a(\mathsf{j}') = k - 1$, so $\nu_{k+1} + \bar{d}^{1-}_{\mathsf{j}'} < k - a(\mathsf{j}')$ in this case.

Now let us describe the outer parts of the blocks, $\bar{\mathsf{B}}^{\mathbf{out}}_{\mathsf{j}}$. The inequality (54) gives

$$\lambda_1 + \mathsf{j} - \mathsf{j}' < 2n + 2e - a(\mathsf{j}') - k - 1.$$

It can be rewritten as

$$\lambda_1 < 2n + 2e - \mathsf{j} - k - 1 + (\mathsf{j}' - a(\mathsf{j}')).$$

Since this should hold for all $\mathsf{j}' < \mathsf{j}$, we can replace the last summand by its minimum, which is 0. So, the defining inequality of $\bar{\mathsf{B}}^{\mathbf{out}}_{\mathsf{j}}$ is $\lambda_1 < 2n + 2e - \mathsf{j} - k - 1$. Since λ_1 should be an integer, this is equivalent to $\lambda_1 \leq 2n + 2e - \lfloor \mathsf{j} \rfloor - k - 2$.

Now we can write down the answer obtained. We denote by $\mathcal{A}_{\mathsf{j}}$ the subcategory in $\mathcal{D}(X)$ corresponding to the block $\bar{\mathsf{B}}_{\mathsf{j}} = \bar{\mathsf{B}}^{\mathbf{out}}_{\mathsf{j}} + \mathsf{j}\xi + \bar{\mathsf{B}}^{\mathbf{inn}}_{\mathsf{j}}$.

Theorem 9.1. *Let* $\mathbf{G}$ *be of type B or D. Assume that* $k \leq n-1$ *for type B and* $k \leq n-2$ *for type D. For each integer* t *with* $0 \leq t \leq k-1$, *consider the subcategories* $\mathcal{A}_t$ *and* $\mathcal{A}_{t+1/2}$ *in* $\mathcal{D}(X)$ *defined by*

$$\mathcal{A}_t = \left\langle \mathcal{E}^\lambda \;\middle|\; \begin{array}{l} 2n + 2e - k - 2 \geq \lambda_1 \geq \cdots \geq \lambda_t \geq t = \lambda_{t+1} = \cdots = \lambda_k, \\ (k-t)/2 \geq \lambda_{k+1} \geq \cdots \geq \lambda_n \geq (2e-1)\lambda_{n-1},\ \lambda_i \in \mathbb{Z} \end{array} \right\rangle,$$
$$\mathcal{A}_{t+1/2} = \left\langle \mathcal{E}^\lambda \;\middle|\; \begin{array}{l} 2n + 2e - k - 3/2 \geq \lambda_1 \geq \cdots \geq \lambda_t \geq t + 1/2 = \lambda_{t+1} = \cdots = \lambda_k, \\ (k-t)/2 \geq \lambda_{k+1} \geq \cdots \geq \lambda_n \geq (2e-1)\lambda_{n-1},\ \lambda_i \in 1/2 + \mathbb{Z} \end{array} \right\rangle,$$

where e *is defined by* (56). *Also, for each integer* t *with* $k \leq t \leq 2n + 2e - k - 2$, *consider the subcategory*

$$\mathcal{A}_t = \left\langle \mathcal{E}^\lambda \;\middle|\; \begin{array}{l} 2n + 2e - k - 2 \geq \lambda_1 \geq \cdots \geq \lambda_{k-1} \geq \lambda_k = t, \\ \lambda_{k+1} = \cdots = \lambda_n = 0,\ \lambda_i \in \mathbb{Z} \end{array} \right\rangle.$$

Then the collection of subcategories

$$\mathcal{A}_0, \mathcal{A}_{1/2}, \mathcal{A}_1, \mathcal{A}_{3/2}, \dots, \mathcal{A}_{k-1}, \mathcal{A}_{k-1/2}, \mathcal{A}_k, \mathcal{A}_{k+1}, \dots, \mathcal{A}_{2n+2e-k-2}$$

is semiorthogonal, and each subcategory is generated by an exceptional collection.

Theorem 9.2. *Assume* **G** *is of type C and $k \le n$. Consider the following subcategories in $\mathcal{D}(X)$ indexed by integers $t = 0, \dots, 2n-k$:*

$$\mathcal{A}_t = \begin{cases} \left\langle \mathcal{E}^\lambda \;\middle|\; \begin{array}{l} 2n-k \ge \lambda_1 \ge \dots \ge \lambda_t \ge t = \lambda_{t+1} = \dots = \lambda_k, \\ \lfloor (k-t)/2 \rfloor \ge \lambda_{k+1} \ge \dots \ge \lambda_n \ge 0 \end{array} \right\rangle & \text{for } t \le k-1, \\ \left\langle \mathcal{E}^\lambda \;\middle|\; \begin{array}{l} 2n-k \ge \lambda_1 \ge \dots \ge \lambda_{k-1} \ge \lambda_k = t, \\ \lambda_{k+1} = \dots = \lambda_n = 0 \end{array} \right\rangle & \text{for } t \ge k. \end{cases}$$

Then the collection of subcategories $\mathcal{A}_0, \mathcal{A}_1, \dots, \mathcal{A}_{2n-k}$ is semiorthogonal, and each subcategory is generated by an exceptional collection.

9.2. Orthogonal maximal isotropic Grassmannians. Note that if **G** is of type D and $k = n-1$ or $k = n$ then the Grassmannian **G**/**P** is isomorphic to the Grassmannian of type B_{n-1} with $k = n-1$. Thus, the only remaining case with **G** classical is when **G** is of type B_n and $k = n$, which we will now consider. Note that in this case

$$X = \mathbf{G}/\mathbf{P} = \mathsf{OGr}(n, 2n+1).$$

As before we take $D_{\mathbf{out}}$ to be the component containing the vertices from 1 to $n-1$, and thus $D_{\mathbf{inn}} = \emptyset$. Further, $\beta = \varepsilon_n$, so $\bar{\beta}_a = \varepsilon_{a+1}$ and

$$r_a = (\rho, \beta + \bar{\beta}_a)/(\xi, \beta) = 2n - 2a.$$

Hence, when a increases by 1, the index decreases by 2. In particular, $r = r_0 = 2n$. The weight θ defined by (27) is $\theta = (0, \dots, 0, 2)$, hence $(\theta, P_{\mathbf{L}}) = \mathbb{Z}$ and $\mathsf{J} = \mathbb{Z} \cap [0, 2n-1]$. Applying (28) we deduce that $a(\mathsf{j}) = \lfloor \mathsf{j}/2 \rfloor$.

As before we take $\gamma_a = \alpha_a = \varepsilon_a - \varepsilon_{a+1}$. Note that $\mathbf{W}_{\mathbf{H}_a}$ acts by permutations of the last $n-a$ coordinates and by changes of signs of the coordinates, while $\mathbf{W}_{\mathbf{M}_a}$ acts just by permutations. Thus, the $\mathbf{W}_{\mathbf{H}_a}$-orbit of γ_a consists of all vectors $\varepsilon_a \pm \varepsilon_i$, $a+1 \le i \le n$, and it splits into two $\mathbf{W}_{\mathbf{M}_a}$-orbits:

$$\{\varepsilon_a - \varepsilon_i\}_{a+1 \le i \le n} \quad \text{and} \quad \{\varepsilon_a + \varepsilon_i\}_{a+1 \le i \le n}.$$

Thus, using the notation of Section 8 we have $m = 1$ and

t	γ_{t-}	$(\rho_{\mathbf{H}}, \gamma_{t-} - \gamma)$	γ_{t+}	$h_a^*\gamma_{t+}$	$\hat{\gamma}_t$	$(h_a^*\xi, h_a^*\gamma_{t+})$
0	$\varepsilon_a - \varepsilon_{a+1}$	0	$\varepsilon_a - \varepsilon_n$	$-\varepsilon_n$	$-\varepsilon_n$	$-1/2$
1	$\varepsilon_a + \varepsilon_n$	$n-a$	$\varepsilon_a + \varepsilon_{a+1}$	ε_{a+1}	ε_1	$1/2$

Since $P_{\mathbf{L_{inn}}} = 0$ and $a(\mathsf{j}) < k = n$ for all $\mathsf{j} \in \mathsf{J}$, we have

$$\mathsf{B}_{\mathsf{j}}^{\mathbf{inn}} = 0 \quad \text{for all } \mathsf{j} \in \mathsf{J}.$$

In particular, $d_{\mathsf{j}}^{t,\pm} = 0$ and thus

$$\mathsf{B}_{\mathsf{j}}^{\mathbf{out}} = \{n - a(\mathsf{j}) \geq \lambda_1 \geq \cdots \geq \lambda_{a(\mathsf{j})} \geq 0\}.$$

Note that this is the set of Young diagrams inscribed into the rectangle $a(\mathsf{j}) \times (n - a(\mathsf{j}))$, hence it is closed under taking subdiagrams. Thus, the second condition of Theorem 7.1 is satisfied. Since there are no very special elements by Lemma 5.8, the first condition is satisfied as well, so the Theorem applies, and we conclude that the block $\mathsf{B}_{\mathsf{j}} = \mathsf{B}_{\mathsf{j}}^{\mathbf{out}} + \mathsf{j}\xi$ consisting of all $(\lambda_1, \ldots, \lambda_n)$ such that

$$\begin{aligned} &n + \mathsf{j}/2 - a(\mathsf{j}) \geq \lambda_1 \geq \cdots \geq \lambda_{a(\mathsf{j})} \geq \mathsf{j}/2 = \lambda_{a(\mathsf{j})+1} = \cdots = \lambda_n, \\ &\lambda_i \equiv \mathsf{j}/2 \ (\mathrm{mod}\, \mathbb{Z}) \end{aligned}$$

is exceptional.

On the other hand, the condition (54) gives $\lambda_1 + (\mathsf{j} - \mathsf{j}')/2 < n - a(\mathsf{j}') = n - \lfloor \mathsf{j}'/2 \rfloor$. It can be rewritten as

$$\lambda_1 < n - \mathsf{j}/2 + \{\mathsf{j}'/2\}.$$

Since this should be satisfied for all $\mathsf{j}' < \mathsf{j}$, we conclude that $\lambda_1 < n - \mathsf{j}/2$. On the other hand, λ_1 should be an integer, so we obtain $\lambda_1 \leq n - 1 - \lfloor \mathsf{j}/2 \rfloor$.

Now we can write down the answer obtained. Recall that $\mathcal{A}_{\mathsf{j}}$ is the subcategory of $\mathcal{D}(X)$ corresponding to the block $\bar{\mathsf{B}}_{\mathsf{j}} = \bar{\mathsf{B}}_{\mathsf{j}}^{\mathbf{out}} + \mathsf{j}\xi + \bar{\mathsf{B}}_{\mathsf{j}}^{\mathbf{inn}}$.

Theorem 9.3. *Assume* $\mathbf{G}$ *is of type* B_n *and* $k = n$. *Consider the following subcategories in* $\mathcal{D}(X)$ (*where* t *is an integer,* $0 \leq t \leq n-1$)*:*

$$\begin{aligned} \mathcal{A}_{2t} &= \langle \mathcal{E}^\lambda \mid n - 1 \geq \lambda_1 \geq \cdots \geq \lambda_t \geq t = \lambda_{t+1} = \cdots = \lambda_n,\ \lambda_i \in \mathbb{Z} \rangle, \\ \mathcal{A}_{2t+1} &= \langle \mathcal{E}^\lambda \mid n - 1/2 \geq \lambda_1 \geq \cdots \geq \lambda_t \geq t + 1/2 = \lambda_{t+1} = \cdots = \lambda_n,\ \lambda_i \in 1/2 + \mathbb{Z} \rangle. \end{aligned}$$

Then the collection of subcategories $\mathcal{A}_0, \mathcal{A}_1, \ldots, \mathcal{A}_{2n-1}$ *is semiorthogonal, and each subcategory is generated by an exceptional collection.*

9.3. Purity for maximal isotropic Grassmannians.

Recall that for an exceptional block B the exceptional collection $(\mathcal{E}^\lambda)_{\lambda \in \mathsf{B}}$ is strong if and only if it consists of vector bundles (see Proposition 4.2). Using the explicit form of the blocks we can check that this is true in the case of maximal isotropic Grassmannians (symplectic or orthogonal).

Theorem 9.4. *The exceptional collections of Theorem 9.2 for* $k = n$ *and of Theorem 9.3 consist of vector bundles.*

Proof. By Proposition 4.3, it is enough to check that for each of the blocks B appearing in the collection the subquiver $\mathcal{Q}_{\mathsf{B}} \subset \mathcal{Q}$ contains any path that starts and ends in $\mathcal{Q}_{\mathsf{B}}$.

First, let us consider the case when $\mathbf{G}$ is of type C_n and $k = n$ (so $\mathbf{G}/\mathbf{P}$ is the Lagrangian Grassmannian $\mathsf{SGr}(n, 2n)$). In this case $\mathbf{L} = \mathsf{GL}_n$, so the quiver $\mathcal{Q}$ has vertices numbered by dominant weights of GL_n and there is an arrow $\lambda \to \mu$ if and only if

$$\mathsf{Hom}_{\mathsf{GL}_n}(V^\mu, V^\lambda \otimes (V^{2\omega_1})^\vee) = \mathsf{Hom}_{\mathsf{GL}_n}(V^\mu \otimes V^{2\omega_1}, V^\lambda) \neq 0.$$

Thus, if μ corresponds to a Young diagram then so does λ, and μ is contained in λ as a subdiagram. Since all the blocks consist of Young diagrams and are closed under passing to subdiagrams, this implies that they satisfy our condition on paths.

In the case when $\mathbf{G}$ is of type B_n and $k = n$, the Levi group $\mathbf{L}$ is a twofold covering of GL_n. If j is integer then all λ and μ from this block are restricted from GL_n and the arrow $\lambda \to \mu$ in $\mathcal{Q}$ exists if and only if

$$\mathsf{Hom}_{\mathsf{GL}_n}(V^\mu \otimes V^{\omega_1}, V^\lambda) \neq 0,$$

so the above argument shows that the block B_j satisfies the condition on paths. If j is half-integer then $\mathsf{B}_\mathsf{j} = \mathsf{B}_{\mathsf{j}-1/2} + \xi$, and since the twist by ξ is an autoequivalence, we conclude that the block B_j satisfies the condition on paths as well. □

Example 9.5. Assume that $\mathbf{G}$ is of type C_4 and $k = 3$, i.e. $X = \mathbf{G}/\mathbf{P} = \mathsf{SGr}(3, 8)$, and take the block

$$\mathsf{B}_1 = \{5 \geq \lambda_1 \geq 1 = \lambda_2 = \lambda_3,\ 1 \geq \lambda_4 \geq 0\}.$$

Note also that $\mathbf{L} = \mathsf{GL}_3 \times \mathsf{SL}_2$ and $V_\mathbf{L}^{-\beta} = V_\mathbf{L}^{0,0,-1;1}$. In particular, we have a path

$$(3, 1, 1; 1) \to (2, 1, 1; 2) \to (1, 1, 1; 1)$$

in the quiver $\mathcal{Q}$ that starts and ends in the block B_1, while its second vertex is not in the block. So, the assumption of Proposition 4.3 does not hold. On the other hand, the assumption of Proposition 4.4(i) is not satisfied either. Indeed, if $\lambda = (4, 1, 1; 0)$ and $\mu = (1, 1, 1; 1)$ and $v = s_3 s_4 \in \mathsf{SR}_\mathbf{G}^\mathbf{L}$ then $v\rho - \rho = (0, 0, -3; 1)$, hence $V_\mathbf{L}^\mu \subset V_\mathbf{L}^\lambda \otimes V_\mathbf{L}^{v\rho-\rho}$, so by Proposition 2.17(ii) we have $\mathsf{Ext}^2(V_\mathbf{L}^\lambda, V_\mathbf{L}^\mu) \neq 0$. On the other hand, $\xi = (1, 1, 1, 0)$, so $(\xi, \lambda) - (\xi, \mu) = 6 - 3 = 3$. So, in the algebra A_{B_1} its bigrading is $(2, 3)$, while the first (in the cohomological grading) component of the algebra has bigrading $(1, 1)$ by Lemma 3.3. Thus, the algebra cannot be one-generated, and in particular it is not Koszul.

On the other hand, one can check that the objects $\mathcal{E}^\lambda$ with $\lambda \in \mathsf{B}_1$ are still vector bundles. To illustrate what goes on let us consider the case $\lambda = (4, 1, 1; 0)$. By definition, $\mathcal{E}^{(4,1,1;0)}$ is the right mutation of $\mathcal{U}^{(4,1,1;0)}$ through the subcategory generated by $\mathcal{U}^\mu$ with smaller μ. This mutation is a composition of several simple mutations. The first simple mutation is the right mutation through $\mathcal{U}^{(3,1,1;1)}$. It is easy to see that $\mathsf{Ext}^\bullet(\mathcal{U}^{(4,1,1;0)}, \mathcal{U}^{(3,1,1;1)}) = \mathsf{k}[-1]$, i.e. Ext^1 is one-dimensional and $\mathsf{Ext}^i = 0$ for $i \neq 1$. This means that the result R_1 of the first mutation fits into an exact sequence

$$0 \to \mathcal{U}^{(3,1,1;1)} \to R_1 \to \mathcal{U}^{(4,1,1;0)} \to 0.$$

The second simple mutation is the right mutation of R_1 through $\mathcal{U}^{(2,1,1;0)}$. It is easy to see that $\mathsf{Ext}^\bullet(\mathcal{U}^{(4,1,1;0)}, \mathcal{U}^{(2,1,1;0)}) = 0$ and $\mathsf{Ext}^\bullet(\mathcal{U}^{(3,1,1;1)}, \mathcal{U}^{(2,1,1;0)}) = \mathsf{k}[-1]$, hence $\mathsf{Ext}^\bullet(R_1, \mathcal{U}^{(2,1,1;0)}) = \mathsf{k}[-1]$, so the second mutation is again given by the extension

$$0 \to \mathcal{U}^{(2,1,1;0)} \to R_2 \to R_1 \to 0,$$

where R_2 is the result of the mutation. The last simple mutation is the right mutation of R_2 through $\mathcal{U}^{(1,1,1;1)}$. It is easy to see that we have $\mathsf{Ext}^\bullet(\mathcal{U}^{(4,1,1;0)}, \mathcal{U}^{(1,1,1;1)}) = \mathsf{k}[-2]$, $\mathsf{Ext}^\bullet(\mathcal{U}^{(3,1,1;1)}, \mathcal{U}^{(1,1,1;1)}) = 0$, and $\mathsf{Ext}^\bullet(\mathcal{U}^{(2,1,1;0)}, \mathcal{U}^{(1,1,1;1)}) = \mathsf{k}[-1]$. It follows that there is an exact sequence

$$\begin{aligned} 0 \to \mathsf{Ext}^1(R_2, \mathcal{U}^{(1,1,1;1)}) &\to \mathsf{Ext}^1(\mathcal{U}^{(2,1,1;1)}, \mathcal{U}^{(1,1,1;1)}) \\ &\to \mathsf{Ext}^2(\mathcal{U}^{(4,1,1;0)}, \mathcal{U}^{(1,1,1;1)}) \to \mathsf{Ext}^2(R_2, \mathcal{U}^{(1,1,1;1)}) \to 0, \end{aligned} \tag{59}$$

and that all other Ext spaces from R_2 to $\mathcal{U}^{(1,1,1;1)}$ vanish. Indeed, the map in the middle is a map $\mathsf{k} \to \mathsf{k}$, and a direct computation shows that it is an isomorphism. Thus, $\mathsf{Ext}^\bullet(R_2, \mathcal{U}^{(1,1,1;1)}) = 0$, so the last mutation changes nothing and $\mathcal{E}^{(4,1,1;0)} = R_2$ has a filtration of length 3 with factors being $\mathcal{U}^{(2,1,1;0)}$, $\mathcal{U}^{(3,1,1;1)}$, and $\mathcal{U}^{(4,1,1;0)}$. In particular, it is a vector bundle.

It is clear from the above argument that the key point is the surjectivity of the middle morphism in the 4-term exact sequence (59). In fact, it is equivalent to the surjectivity of the Massey triple product

$$\begin{aligned} \mathsf{Ext}^1(\mathcal{U}^{(4,1,1;0)}, \mathcal{U}^{(3,1,1;1)}) \otimes \mathsf{Ext}^1(\mathcal{U}^{(3,1,1;1)}, \mathcal{U}^{(2,1,1;0)}) \otimes \mathsf{Ext}^1(\mathcal{U}^{(2,1,1;0)}, \mathcal{U}^{(1,1,1;1)}) \\ \to \mathsf{Ext}^2(\mathcal{U}^{(4,1,1;0)}, \mathcal{U}^{(1,1,1;1)}). \end{aligned}$$

Since the Massey products are induced by the higher products in the natural A_∞-structure of the algebra A_{B_1}, this surjectivity can be reinterpreted as the fact that the algebra A_{B_1} is one-generated as an A_∞-algebra. This leads to the following conjecture.

Conjecture 9.6. *The algebra A_{B} is one-generated as an A_∞-algebra. Its Koszul dual is a usual algebra.*

This conjecture implies the purity and strongness of the collections $\mathcal{E}^\lambda$.

9.4. Numbers of objects. It turns out that the collections constructed in Sections 9.1 and 9.2 contain the maximal possible number of objects. It is well known that the rank of the Grothendieck group of $\mathbf{G}/\mathbf{P}$ is equal to the cardinality of $\mathbf{W_G}/\mathbf{W_L}$ (this rank is equal to the rank of the homology group of X due to the Bruhat cell decomposition, and the homology of X was computed in [BGG, Prop. 5.2]). For the series B, C and D these ranks are given by

$$r(n,k) = \binom{n}{k} 2^k,$$

where in the case of type D we assume that $k \le n-2$ (as was explained before, for type D we do not need to consider the case $k = n-1$ or n).

Proposition 9.7. *The total number of objects in the collections of Theorems 9.1–9.3 equals the rank of the Grothendieck group of the corresponding Grassmannian.*

Proof. Let us denote

$$c_k(n) = |\{n \geq \lambda_1 \geq \cdots \geq \lambda_k \geq 0,\ \lambda_i \in \mathbb{Z}\}| = \binom{n+k}{k}.$$

We will consider the types B, C and D separately.

1. Type B_n, $k \leq n-1$. In this case we have

$$\begin{aligned} |\bar{\mathsf{B}}_t| &= c_t(2n-k-1-t)c_{n-k}(\lfloor (k-t)/2 \rfloor), && \text{for integer } 0 \leq t \leq k-1,\\ |\bar{\mathsf{B}}_{t+1/2}| &= c_t(2n-k-1-t)c_{n-k}(\lfloor (k-t-1)/2 \rfloor), && \text{for integer } 0 \leq t \leq k-1,\\ |\bar{\mathsf{B}}_t| &= c_{k-1}(2n-k-1-t), && \text{for integer } k \leq t \leq 2n-k-1. \end{aligned}$$

Hence, the total number of objects in the collection of Theorem 9.1 in this case is

$$\begin{aligned} N^B(n,k) = &\sum_{t=0}^{k-1} c_t(2n-k-1-t)\cdot (c_{n-k}(\lfloor (k-t)/2 \rfloor) + c_{n-k}(\lfloor (k-t-1)/2 \rfloor))\\ &+ \sum_{t=k}^{2n-k-1} c_{k-1}(2n-k-1-t). \end{aligned}$$

But

$$\begin{aligned} \sum_{t=k}^{2n-k-1} c_{k-1}(2n-k-1-t) = \sum_{i=0}^{2n-2k-1} c_{k-1}(i) = \sum_{i=0}^{2n-2k-1} \binom{k-1+i}{k-1} = \binom{2n-k-1}{k}\\ = c_k(2n-2k-1). \end{aligned}$$

Thus,

$$\begin{aligned} &N^B(n,k)\\ &= \sum_{t=0}^{k-1} c_t(2n-k-1-t)\cdot \big(c_{n-k}(\lfloor (k-t)/2 \rfloor) + c_{n-k}(\lfloor (k-t-1)/2 \rfloor)\big) + c_k(2n-2k-1)\\ &\qquad = \sum_{t=0}^{k} \binom{2n-k-1}{t} \cdot \big(c_{n-k}(\lfloor (k-t)/2 \rfloor) + c_{n-k}(\lfloor (k-t-1)/2 \rfloor)\big). \end{aligned}$$

Hence, $N^B(n,k)$ is the coefficient of x^k in $(1+x)^{2n-k-1} f^B_{n-k}(x)$, where

$$\begin{aligned} f^B_{n-k}(x) &= \sum_{i \geq 0} \big(c_{n-k}(\lfloor i/2 \rfloor) + c_{n-k}(\lfloor (i-1)/2 \rfloor)\big) x^i\\ &= (1+2x+x^2) \sum_{j \geq 0} c_{n-k}(j) x^{2j} = \frac{(1+x)^2}{(1-x^2)^{n-k+1}}. \end{aligned}$$

Therefore, $N^B(n,k)$ is the coefficient of x^k in

$$\frac{(1+x)^{2n-k+1}}{(1-x^2)^{n-k+1}} = \frac{(1+x)^n}{(1-x)^{n-k+1}} = (1+x)^n \sum_{i \geq 0} \binom{n-k+i}{i} x^i.$$

Finally, this gives

$$N^B(n,k) = \sum_{i=0}^{k} \binom{n}{k-i}\binom{n-k+i}{i} = \sum_{i=0}^{k} \frac{n!}{(k-i)!i!(n-k)!} = \binom{n}{k}\sum_{i=0}^{k}\binom{k}{i}$$
$$= \binom{n}{k}2^k.$$

1′. Type $B_n,\ k = n$. In this case

$$|\bar{\mathsf{B}}_{2t}| = |\bar{\mathsf{B}}_{2t+1}| = c_t(n-t-1) = \binom{n-1}{t},$$

and the total number of objects is

$$N^B(n,n) = 2\sum_{t=0}^{n-1}\binom{n-1}{t} = 2\cdot 2^{n-1} = 2^n.$$

2. Type C_n. We have

$$|\bar{\mathsf{B}}_t| = c_t(2n-k-t)c_{n-k}(\lfloor (k-t)/2\rfloor) \quad \text{for integer } 0 \le t \le k-1,$$
$$|\bar{\mathsf{B}}_t| = c_{k-1}(2n-k-t) \quad \text{for integer } k \le t \le 2n-k.$$

Thus, the total number of objects is

$$N^C(n,k) = \sum_{t=0}^{k-1} c_t(2n-k-t)c_{n-k}(\lfloor (k-t)/2\rfloor) + \sum_{t=k}^{2n-k} c_{k-1}(2n-k-t)$$
$$= \sum_{t=0}^{k-1} c_t(2n-k-t)c_{n-k}(\lfloor (k-t)/2\rfloor) + c_k(2n-2k) = \sum_{t=0}^{k} c_t(2n-k-t)c_{n-k}(\lfloor (k-t)/2\rfloor).$$

In other words, $N^C(n,k)$ is the coefficient of x^k in $(1+x)^{2n-k} f^C_{n-k}(x)$, where

$$f^C_{n-k}(x) = \sum_{i\ge 0} c_{n-k}(\lfloor i/2\rfloor)x^i = (1+x)\sum_{j\ge 0} c_{n-k}(j)x^{2j} = \frac{1+x}{(1-x^2)^{n-k+1}}.$$

Therefore, $N^C(n,k)$ is the coefficient of x^k in $(1+x)^{2n-k+1}(1-x^2)^{-(n-k+1)}$, so we get

$$N^C(n,k) = N^B(n,k) = \binom{n}{k}2^k.$$

3. Type $D_n,\ k \le n-2$. First, we observe that

$$s_k(n) := |\{n \ge \lambda_1 \ge \cdots \ge \lambda_k \ge -\lambda_{k-1},\ \lambda_i \in \mathbb{Z}\}|$$
$$= \sum_{p\ge 0} |\{n \ge \lambda_1 \ge \cdots \ge \lambda_{k-1} = p, \lambda_i \in \mathbb{Z}\}|(2p+1) = \sum_{p\ge 0}(2p+1)c_{k-2}(n-p),$$

and so

$$\sum_{n\ge 0} s_k(n)x^n = \Big(\sum_{p\ge 0}(2p+1)x^p\Big)\frac{1}{(1-x)^{k-1}} = \frac{1+x}{(1-x)^{k+1}}.$$

Similarly,

$$\begin{aligned} t_k(n) &:= |\{n+1/2 \ge \lambda_1 \ge \cdots \ge \lambda_k \ge -\lambda_{k-1},\ \lambda_i \in 1/2+\mathbb{Z}\}| \\ &= \sum_{p\ge 0} |\{n+1/2 \ge \lambda_1 \ge \cdots \ge \lambda_{k-1} = p+1/2,\ \lambda_i \in 1/2+\mathbb{Z}\}|(2p+2) \\ &= \sum_{p\ge 0}(2p+2)c_{k-2}(n-p), \end{aligned}$$

and so

$$\sum_{n\ge 0} t_k(n)x^n = \Big(\sum_{p\ge 0}(2p+2)x^p\Big)\frac{1}{(1-x)^{k-1}} = \frac{2}{(1-x)^{k+1}}.$$

Now

$$\begin{aligned} |\bar{\mathsf{B}}_t| &= c_t(2n-k-2-t)s_{n-k}(\lfloor (k-t)/2\rfloor) && \text{for integer } 0 \le t \le k-1, \\ |\bar{\mathsf{B}}_{t+1/2}| &= c_t(2n-k-2-t)t_{n-k}(\lfloor (k-t-1)/2\rfloor) && \text{for integer } 0 \le t \le k-1, \\ |\bar{\mathsf{B}}_t| &= c_{k-1}(2n-k-2-t) && \text{for integer } k \le t \le 2n-k-2. \end{aligned}$$

Hence, the total number is

$$\begin{aligned} &N^D(n,k) = \\ &\sum_{t=0}^{k-1} c_t(2n-k-2-t)\big(s_{n-k}(\lfloor (k-t)/2\rfloor)+t_{n-k}(\lfloor (k-t-1)/2\rfloor)\big) + \sum_{t=k}^{2n-k-2} c_{k-1}(2n-k-2-t) \\ &\qquad = \sum_{t=0}^{k} c_t(2n-k-2-t)\big(s_{n-k}(\lfloor (k-t)/2\rfloor)+t_{n-k}(\lfloor (k-t-1)/2\rfloor)\big). \end{aligned}$$

Thus, $N^D(n,k)$ is the coefficient of x^k in $(1+x)^{2n-k-2} f^D_{n-k}(x)$, where

$$\begin{aligned} f^D_{n-k}(x) &= \sum_{i\ge 0}\big(s_{n-k}(\lfloor i/2\rfloor)+t_{n-k}(\lfloor (i-1)/2\rfloor)\big)x^i \\ &= (1+x)\sum_{j\ge 0} s_{n-k}(j)x^{2j} + x(1+x)\sum_{j\ge 0} t_{n-k}(j)x^{2j} \\ &= \frac{(1+x)(1+x^2)}{(1-x^2)^{n-k+1}} + \frac{2(1+x)x}{(1-x^2)^{n-k+1}} = \frac{(1+x)^3}{(1-x^2)^{n-k+1}}. \end{aligned}$$

Therefore, $N^D(n,k)$ is the coefficient of x^k in $(1+x)^{2n-k+1}(1-x^2)^{-(n-k+1)}$, which gives

$$N^D(n,k) = N^B(n,k) = \binom{n}{k}2^k.$$

This completes the proof. □

9.5. Proofs. Here we explain how the results of the paper imply the theorems from the Introduction.

Proof of Theorem 1.2. The exceptional collections are constructed in Theorems 9.1–9.3. They have equivariant structure by construction. The number of objects equals the rank of the Grothendieck group by Proposition 9.7. □

Proof of Corollary 1.3. Recall that $Y = \mathcal{G} \times_{\mathbf{G}} (\mathbf{G}/\mathbf{P}) = (\mathcal{G} \times (\mathbf{G}/\mathbf{P}))/\mathbf{G}$, with respect to the natural right action of $\mathbf{G}$ on $\mathcal{G}$ and the left action on $\mathbf{G}/\mathbf{P}$. By [El, Theorem 9.6], the derived category $\mathcal{D}(Y)$ is equivalent to $\mathcal{D}(\mathcal{G} \times (\mathbf{G}/\mathbf{P}))^{\mathbf{G}}$, the category of $\mathbf{G}$-equivariant objects in $\mathcal{D}(\mathcal{G} \times (\mathbf{G}/\mathbf{P}))$. Consider the object $\mathcal{O}_{\mathcal{G}} \boxtimes \mathcal{E}^{\lambda} \in \mathcal{D}(\mathcal{G} \times (\mathbf{G}/\mathbf{P}))$ with its natural $\mathbf{G}$-equivariant structure. By the above observation it gives an object $\mathcal{E}^{\lambda}_{Y} \in \mathcal{D}(Y)$ such that for any point $x \in X$ we have

$$(\mathcal{E}^{\lambda}_{Y})_{|p^{-1}(x)} \cong \mathcal{E}^{\lambda}.$$

Thus we can apply Theorem 3.1 from [S07] to conclude that the functors

$$\Phi^{\lambda} : \mathcal{D}(X) \to \mathcal{D}(Y), \qquad F \mapsto p^{*}F \otimes \mathcal{E}^{\lambda}_{Y},$$

are fully faithful and the subcategories $\Phi_{\lambda}(\mathcal{D}(X)) \subset \mathcal{D}(Y)$ are semiorthogonal. This means that we have a semiorthogonal decomposition

$$\mathcal{D}^{b}(Y) = \langle \{\Phi^{\lambda}(\mathcal{D}(X))\}_{\lambda \in \mathsf{B}}, \mathcal{A} \rangle,$$

where $\mathcal{A} = \bigcap_{\lambda \in \mathsf{B}} {}^{\perp}\Phi^{\lambda}(\mathcal{D}(X))$. Now if X has an exceptional collection F_i of length $N = \operatorname{rk} K_0(X)$ then the objects $p^{*}F_i \otimes \mathcal{E}^{\lambda}_{Y}$ form an exceptional collection of length $N \cdot \#\mathsf{B}$ in $\mathcal{D}(Y)$, so if $\#\mathsf{B} = \operatorname{rk} K_0(\mathbf{G}/\mathbf{P})$ then this number equals $\operatorname{rk} K_0(X) \cdot \operatorname{rk} K_0(\mathbf{G}/\mathbf{P}) = \operatorname{rk} K_0(Y)$, so we have an exceptional collection of expected length on Y. □

Proof of Theorem 1.5. Part (i) is given by Theorem 5.10. Part (ii) follows from Proposition 3.13 combined with Proposition 6.3 and Theorem 7.1. Part (iii) is a combination of Theorems 9.1–9.3 with Proposition 9.7. □

Proof of Theorem 1.8. This is just Proposition 4.2. □

9.6. Usual Grassmannians. In this section we speculate that our construction might still work with a certain weakening of the assumption (26) (so that $D_{\mathbf{out}}$ is not necessarily connected). Namely, we consider the case $X = \mathsf{Gr}(k, n)$, the usual Grassmannian, and formally apply the procedure of Section 5 to the data for which (26) does not hold to construct collections of expected length in $\mathcal{D}^{b}(X)$. Of course, our proof of part (b) of the criterion of exceptionality (see Proposition 3.13) does not work in this situation, so we do not have a proof of the exceptionality of this collection.

Let $\mathbf{G} = \mathsf{SL}_n$ and $\mathbf{L} = (\mathsf{GL}_k \times \mathsf{GL}_l) \cap \mathsf{SL}_n$ $(n = k + l)$. In the framework of the paper we could take $D_{\mathbf{out}}$ to be either of the two connected components of $D_{\mathbf{G}} \setminus \beta$. Let us take instead $D_{\mathbf{out}}$ to be *the union* of both, that is, $D_{\mathbf{out}} = D_{\mathbf{G}} \setminus \beta$. Of course we violate here the assumption (26).

Moreover, we arbitrarily renumber the vertices of $D_{\mathbf{G}}$ in such a way that $D_a = D_{\mathbf{G}} \setminus \{1, \dots, a\}$ is always connected and contains $\beta = \alpha_{n-1}$. In other words, to obtain from $D_{\mathbf{G}}$ the chain of Dynkin diagrams D_a we keep chopping off one of the end-points of the diagram until only β is left.

It is clear that such renumberings are in a bijection with isotopy classes of monotone curves C in a $k \times l$ rectangle on an integer grid going from the point (k, l) to the point $(0, 0)$ and not passing through integer points. We will describe a conjectural exceptional collection corresponding to an isotopy class of such a curve.

Moreover, in fact we will allow the curve to pass through integer points (this corresponds to allowing both end-points to be chopped off simultaneously).

So, assume we are given such a curve C. Consider the sequence of points $Q_0, Q_1, \dots, Q_m$ of intersection of C with the edges of the grid squares (some of the points Q_i can lie at the vertices of the squares) and let (x_i, y_i) be the coordinates of Q_i. Set

$$a_i = \lfloor x_i \rfloor, \quad b_i = \lfloor y_i \rfloor, \quad c_i = k - \lceil x_i \rceil, \quad d_i = l - \lceil y_i \rceil.$$

Then consider the blocks

$$\mathsf{B}_i = \left\{ \begin{matrix} d_i + i \ge \lambda_1 \ge \dots \ge \lambda_{a_i} \ge i = \lambda_{a_i+1} = \dots = \lambda_k, \\ \lambda_{k+1} = \dots = \lambda_{n-b_i} = 0 \ge \lambda_{n-b_i+1} \ge \dots \ge \lambda_n \ge -c_i \end{matrix} \right\} \tag{60}$$

(in particular, $\mathsf{B}_0 = \{0\}$). Note that the total number of weights in those blocks is

$$\#(\mathsf{B}_0 \sqcup \mathsf{B}_1 \sqcup \dots \sqcup \mathsf{B}_m) = \sum_{i=0}^{m} \binom{a_i + d_i}{a_i} \binom{b_i + c_i}{b_i} = \binom{k+l}{k},$$

which is the rank of the Grothendieck group of $X = \mathsf{Gr}(k, n)$. The equality above has a simple combinatorial proof—the RHS is the number of Young diagrams inscribed in the rectangle, we divide the set of all such diagrams into subsets numbered by the point of intersection of the border of the diagram with the curve C, and the summands on the LHS correspond to the parts of this decomposition.

In the first version of this paper we suggested the following conjecture.

Conjecture 9.8. *The blocks B_i given by (60) are exceptional and the collection $\langle \mathcal{A}_0, \mathcal{A}_1, \dots, \mathcal{A}_m \rangle$ with subcategories $\mathcal{A}_i = \langle \mathcal{U}^\lambda \rangle_{\lambda \in \mathsf{B}_i}$ is a semiorthogonal decomposition of $\mathcal{D}^b(\mathsf{Gr}(k, n))$, each component of which is generated by an exceptional collection.*

This conjecture was recently proved by A. Fonarev [Fon].

Remark 9.9. One special case is interesting. Assume $l = k$, and take for C the segment of the straight line from (k, k) to $(0, 0)$. Then $m = k$ and $Q_i = (i, i)$ so that $a_i = b_i = i$, $c_i = d_i = k - i$. The corresponding exceptional collection is invariant with respect to the outer automorphism of $\mathsf{Gr}(k, 2k)$ (passing to the orthogonal complement with respect to a nondegenerate bilinear form).

10. Appendix. Key technical proposition

In this Appendix we prove a certain auxiliary result on GL_n-representations.

For a dominant weight $\lambda = (\lambda_1 \ge \dots \ge \lambda_n)$ of GL_n we denote by V^λ the corresponding irreducible GL_n-representation. We write $\lambda \ge 0$ (and say that λ is *nonnegative*) if $\lambda_n \geq 0$. Such weights correspond to partitions with at most n parts. Let w_0 denote the longest element of the symmetric group $\mathfrak{S}_n$, i.e. the permutation which takes i to $n+1-i$ for all i.

For an integer a with $0 \leq a \leq n$, and an integer $l \geq 0$, let Π^a_{-l} be the projector on the category of GL_n-representations which acts identically on V^λ, where $\lambda_{a+1} = \cdots = \lambda_n = -l$, and sends all other irreducible representations to zero. We say that a map of GL_n-representations is a Π^a_{-l}*-isomorphism* (resp. Π^a_{-l}*-injection*) if applying Π^a_{-l} to this map we get an isomorphism (resp. injection).

The main result of this Appendix is the following

Proposition 10.1. *Fix an integer a with $0 \leq a \leq n$. Let κ be a partition with at most a parts, and let τ be a partition with at most $n - a$ parts (both viewed as weights of GL_n). Finally, let W be a representation which is a direct summand of $V^{\otimes N}$, where V is the standard n-dimensional representation of GL_n. Then the natural map*

$$V^{\kappa - w_0\tau} \otimes W \to V^\kappa \otimes V^{-w_0\tau} \otimes W \to V^\kappa \otimes \Pi^a_0(V^{-w_0\tau} \otimes W) \tag{61}$$

is a Π^a_0-isomorphism.

The following corollary of this proposition is used in Section 7.2.

Corollary 10.2. *Fix a with $0 \leq a \leq n-1$. Let κ be a partition with at most a parts, τ a partition with at most $n-a$ parts, and μ a partition with at most n parts. Then for any $l \geq 0$ the natural map*

$$V^{\kappa - w_0\tau} \otimes V^\mu \to V^\kappa \otimes V^{-w_0\tau} \otimes V^\mu \to V^\kappa \otimes \Pi^a_{-l}(V^{-w_0\tau} \otimes V^\mu)$$

induces an isomorphism after applying Π^a_{-l}.

Proof. Denote by (l) the autoequivalence of the category of representations of GL_n that takes a representation with a highest weight $\lambda = (\lambda_1, \dots, \lambda_n)$ to the representation with the highest weight $(\lambda_1 + l, \dots, \lambda_n + l)$. In other words, it is the twist by $(\det V)^{\otimes l}$. Then for $W = V^\mu(l)$ we have

$$\begin{aligned}
\Pi^a_{-l}(V^{\kappa - w_0\tau} \otimes V^\mu)(l) &= \Pi^a_0(V^{\kappa - w_0\tau} \otimes W), \\
\Pi^a_{-l}(V^\kappa \otimes V^{-w_0\tau} \otimes V^\mu)(l) &= \Pi^a_0(V^\kappa \otimes V^{-w_0\tau} \otimes W), \\
\Pi^a_{-l}(V^\kappa \otimes \Pi^a_{-l}(V^{-w_0\tau} \otimes V^\mu))(l) &= \Pi^a_0(V^\kappa \otimes \Pi^a_{-l}(V^{-w_0\tau} \otimes V^\mu)(l)) \\
&= \Pi^a_0(V^\kappa \otimes \Pi^a_0(V^{-w_0\tau} \otimes W)),
\end{aligned}$$

so applying Π^a_{-l} to the map in the corollary and twisting by (l) we obtain the map (61) acted upon by Π^a_0. The latter is an isomorphism by Proposition 10.1, hence so is the former. □

We start the proof of Proposition 10.1 with the following numerical observation.

Lemma 10.3. *Under the assumptions of Proposition 10.1 one has*

$$\dim \Pi^a_0(V^{\kappa - w_0\tau} \otimes W) = \dim \Pi^a_0(V^\kappa \otimes \Pi^a_0(V^{-w_0\tau} \otimes W)).$$

Proof. It is enough to check that the multiplicities of V^μ, where μ is a partition with at most a parts, in $V^{\kappa - w_0\tau} \otimes W$ and in $V^\kappa \otimes \Pi_0^a(V^{-w_0\tau} \otimes W)$ are equal. To this end we replace W with any of its irreducible summand of the form V^λ, where $\lambda \geq 0$, and apply the Littlewood–Richardson rule. The dimension of the space $\mathsf{Hom}(V^\mu, V^{\kappa - w_0\tau} \otimes V^\lambda)$ is given by the number of semistandard skew tableaux S of shape $(\mu) \setminus (\kappa - w_0\tau)$ with the content of weight λ, satisfying the lattice permutation condition. Every such skew tableau contains a skew subtableau S' of shape $\mu \setminus \kappa$ that still satisfies the lattice permutation condition. Let ν be the weight of the content of S'. Then to give S is the same as giving $\nu \subset \lambda$ together with a pair:

(i) a semistandard skew tableau of shape $\mu \setminus \kappa$ with content of weight ν,
(ii) a semistandard skew tableau of shape $\nu \setminus (-w_0\tau)$ with content λ.

Let N_1 (resp., N_2) be the number of choices in (i) (resp., in (ii)). We have

$$N_1 = \dim \mathsf{Hom}(V^\mu, V^\kappa \otimes V^\nu).$$

On the other hand,

$$N_2 = \dim \mathsf{Hom}(V^\nu, V^{-w_0\tau} \otimes V^\lambda).$$

Thus, the above argument gives the equality

$$\begin{aligned} &\dim \mathsf{Hom}(V^\mu, V^{\kappa - w_0\tau} \otimes V^\lambda) \\ &\quad = \sum_{\nu \geq 0,\, \nu \subset \mu,\, \nu \subset \lambda} \dim(\mathsf{Hom}(V^\mu, V^\kappa \otimes V^\nu)) \cdot \dim(\mathsf{Hom}(V^\nu, V^{-w_0\tau} \otimes V^\lambda)). \end{aligned} \tag{62}$$

Note that the condition $\nu \subset \mu$ here is automatic since otherwise $\mathsf{Hom}(V^\mu, V^\kappa \otimes V^\nu)$ is zero. On the other hand, we have a decomposition

$$\begin{aligned} &\mathsf{Hom}(V^\mu, V^\kappa \otimes \Pi_0^a(V^{-w_0\tau} \otimes V^\lambda)) \\ &\qquad = \bigoplus_{\nu \geq 0,\, \nu \subset \mu,\, \nu \subset \lambda} \mathsf{Hom}(V^\mu, V^\kappa \otimes V^\nu) \otimes \mathsf{Hom}(V^\nu, V^{-w_0\tau} \otimes V^\lambda). \end{aligned} \tag{63}$$

Indeed, the summation over $\nu \geq 0$ on the right-hand side comes from decomposing $\Pi_0^a(V^{-w_0\tau} \otimes V^\lambda)$ into irreducibles. The condition $\nu \subset \mu$ can be added for the same reason as before, and the condition $\nu \subset \lambda$ is added because otherwise $\mathsf{Hom}(V^\nu, V^{-w_0\tau} \otimes V^\lambda)$ vanishes. According to the definition of Π_0^a we also have to require ν to have at most a parts, but this follows from the inclusion $\nu \subset \mu$. Comparing the dimensions in (63) with (62), we get the required equality. □

The above lemma reduces the proof of Proposition 10.1 to showing that the map (61) is Π_0^a-injective. We will deduce this from a more general Proposition 10.4 below. To state it we need more notation.

Let us define the *depth* of a dominant weight $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n)$ of GL_n as the sum of the absolute values of all its negative entires. In other words, we take $1 \leq i \leq n$ such that $\lambda_i \geq 0 \geq \lambda_{i+1}$, and set

$$\mathsf{depth}(\lambda) = -\lambda_{i+1} - \cdots - \lambda_n.$$

Note that the depth is always nonnegative, and it is zero if and only if $\lambda \geq 0$.

Let Π_d be the the projector on the category of representations of GL_n which acts identically on all V^λ with $\mathsf{depth}(\lambda) = d$, and sends all other irreducible representations to zero. Also, set $\Pi_{\geq d_0} := \sum_{d \geq d_0} \Pi_d$.

Consider the GL_n-representations

$$V_p := V^{\otimes p} \quad \text{and} \quad V_{p,q} := V^{\otimes p} \otimes (V^*)^{\otimes q}.$$

We will derive the Π_0^a-injectivity of (61) from the following result.

Proposition 10.4. *Fix integers $k, t, N \geq 0$. The natural map*

$$\Pi_t(V_{k,t}) \otimes V_N \to V_{k+N,t} \to V_k \otimes \Pi_0(V_{N,t}) \tag{64}$$

is Π_0-injective, i.e. it becomes injective after applying Π_0.

To prove Proposition 10.4 we will use some simple facts about partial contraction maps between the GL_n-representations $V_{p,q}$. First, for each $i \leq p$ and $j \leq q$ consider the partial trace map $\mathsf{Tr}_{i,j} : V_{p,q} \to V_{p-1,q-1}$ given by

$$\begin{aligned} &\mathsf{Tr}_{i,j}((v_1 \otimes \cdots \otimes v_p) \otimes (f_1 \otimes \cdots \otimes f_q)) \\ &\qquad = f_j(v_i) v_1 \otimes \cdots \otimes \widehat{v_i} \otimes \cdots \otimes v_p \otimes f_1 \otimes \cdots \otimes \widehat{f_j} \otimes \cdots \otimes f_q. \end{aligned} \tag{65}$$

Clearly it is GL_n-equivariant.

Lemma 10.5. *The maximal depth of an irreducible representation occurring in $V_{p,q}$ is equal to q. The intersection of the kernels of all maps $\mathsf{Tr}_{i,j}$ for $1 \leq i \leq p$ and $1 \leq j \leq q$ contains the direct sum of all irreducibles of depth q in $V_{p,q}$:*

$$\Pi_q(V_{p,q}) \subset \bigcap_{1 \leq i \leq p,\ 1 \leq j \leq q} \mathsf{Ker}\, \mathsf{Tr}_{i,j}\ .$$

Proof. The first assertion follows easily from the Littlewood–Richardson rule. The second follows immediately from the first, as $V_{p-1,q-1}$ does not contain irreducible representations of depth q. □

Next, for $p \geq q$ and a permutation $\sigma \in \mathfrak{S}_p$ let us define the corresponding contraction map

$$\begin{aligned} &\mathsf{Tr}_\sigma : V_{p,q} \to V_{p-q}, \\ &(v_1 \otimes \cdots \otimes v_p) \otimes (f_1 \otimes \cdots \otimes f_q) \mapsto f_1(v_{\sigma_p}) \cdots f_q(v_{\sigma_{p-q+1}}) v_{\sigma_1} \otimes \cdots \otimes v_{\sigma_{p-q}}. \end{aligned} \tag{66}$$

In other words, Tr_σ is the composition of the action of $\sigma \otimes \mathsf{id}_{V_{0,q}}$ followed by q consecutive contractions of the factors $V \otimes V^*$.

Lemma 10.6. *The intersection of the kernels of all maps Tr_σ for $\sigma \in \mathfrak{S}_p$ contains the direct sum of all irreducibles of positive depth in $V_{p,q}$:*

$$\Pi_{\geq 1}(V_{p,q}) \subset \bigcap_{\sigma \in \mathfrak{S}_p} \mathsf{Ker}\, \mathsf{Tr}_\sigma = \mathsf{Ker}\Big(\sum_{\sigma \in \mathfrak{S}_p} \mathsf{Tr}_\sigma \Big).$$

Proof. This follows from the fact that irreducible representations of positive depth do not occur in V_{p-q}. $\square$

Lemma 10.7. *Any GL_n-map $V_{p,q} \to V_{p-q}$, where $p \geq q$, is a linear combination $\sum_{\sigma \in \mathfrak{S}_p} a_\sigma \operatorname{Tr}_\sigma$ of the contraction maps (66). Moreover, the kernel of the map*

$$\sum \operatorname{Tr}_\sigma : V_{p,q} \to \bigoplus_{\sigma \in \mathfrak{S}_p} V_{p-q}$$

is $\Pi_{\geq 1}(V_{p,q})$. In particular, the restriction of this map to $\Pi_0(V_{p,q})$ is injective.

Proof. The first part follows immediately from the first fundamental theorem on invariants of GL_n (see e.g. [CB, Sec. 12]). For the second part, since we already know that $\Pi_{\geq 1}(V_{p,q})$ is in the kernel, we have to check that for each irreducible summand $V^\mu \subset V_{p,q}$ with $\mu \geq 0$ the map $\sum \operatorname{Tr}_\sigma$ is injective on V^μ.

So, let $V^\mu \subset V_{p,q}$ be an irreducible summand with $\mu \geq 0$. Note that μ is a partition of $p-q$, in particular, V^μ is a direct summand of V_{p-q}. Choose a splitting $V_{p,q} \to V^\mu$ of the given embedding and an embedding $V^\mu \to V_{p-q}$. Then the composition

$$V^\mu \to V_{p,q} \to V^\mu \to V_{p-q}$$

is an embedding. On the other hand, the composition of the second and third arrows is a linear combination of the maps Tr_σ. It follows that for some σ the map Tr_σ restricted to V^μ is nonzero, hence injective. Therefore $\sum \operatorname{Tr}_\sigma$ is also injective on V^μ. $\square$

Proof of Proposition 10.4. If $N < t$ then by the Littlewood–Richardson rule, $\Pi_0(V_{N,t}) = 0$, hence the third term in (64) is zero. Similarly, in this case $\Pi_0(V^\lambda \otimes V_N) = 0$ for any λ of depth t, hence the first term in (64) becomes zero after applying Π_0. Thus, the composition (64) is Π_0-injective.

From now on assume that $N \geq t$. By Lemma 10.7, we have a left exact sequence

$$0 \to \Pi_{\geq 1}(V_{N,t}) \to V_{N,t} \xrightarrow{\sum \operatorname{Tr}_\sigma} \bigoplus_{\sigma \in \mathfrak{S}_N} V_{N-t}.$$

Since the complement of $\Pi_{\geq 1}(V_{N,t})$ in $V_{N,t}$ is $\Pi_0(V_{N,t})$, this means that the projection $V_{N,t} \to \Pi_0(V_{N,t})$ fits into a commutative diagram

$$\begin{array}{ccc} V_{N,t} & \longrightarrow & \Pi_0(V_{N,t}) \\ & {\scriptstyle \sum_{\sigma \in \mathfrak{S}_N} \operatorname{Tr}_\sigma} \searrow \quad \swarrow\!\hookleftarrow & \\ & \bigoplus_{\sigma \in \mathfrak{S}_N} V_{N-t} & \end{array}$$

with an injective right bottom arrow. Tensoring it with V_k we obtain the commutative diagram

$$\begin{array}{ccc} V_{k+N,t} = V_k \otimes V_{N,t} & \longrightarrow & V_k \otimes \Pi_0(V_{N,t}) \\ & {\scriptstyle \sum_{\sigma \in \mathfrak{S}_N} \operatorname{id}_{V_k} \otimes \operatorname{Tr}_\sigma} \searrow \quad \swarrow\!\hookleftarrow & \\ & \bigoplus_{\sigma \in \mathfrak{S}_N} V_{k+N-t} & \end{array}$$

Now consider the composition (64) and assume that $V^\mu \subset \Pi_t(V_{k,t}) \otimes V_N$ is an irreducible summand with $\mu \geq 0$ whose image in $V_{k+N,t}$ is mapped to zero by the projection $V_{k+N,t} \to V_k \otimes \Pi_0(V_{N,t})$. By the above commutative diagram this means that

$$(\mathsf{id}_{V_k} \otimes \mathsf{Tr}_\sigma)(V^\mu) = 0 \tag{67}$$

for all $\sigma \in \mathfrak{S}_N$. On the other hand, since $V^\mu \subset \Pi_t(V_{k,t}) \otimes V_N \subset V_{k+N,t}$, by Lemma 10.5 we have

$$\mathsf{Tr}_{i,j}(V^\mu) = 0 \tag{68}$$

for all $1 \leq i \leq k$ and $1 \leq j \leq t$. Let us show that (67) and (68) lead to a contradiction. Indeed, since $\mu \geq 0$, we know by Lemma 10.7 that for some $\sigma \in \mathfrak{S}_{k+N}$ the trace map $\mathsf{Tr}_\sigma : V_{k+N,t} \to V_{k+N-t}$ is injective on V^μ. Fix such a σ. There are two possibilities: either

(1) for each $1 \leq i \leq k$ we have $\sigma_i \leq k + N - t$, or
(2) for some $1 \leq i \leq k$ we have $\sigma_i > k + N - t$.

In the first case the map Tr_σ can be rewritten as the composition of $\mathsf{id}_{V_k} \otimes \mathsf{Tr}_{\sigma'}$ with some $\sigma' \in \mathfrak{S}_N$, followed by an appropriate permutation acting on V_{k+N-t}. In particular, by (67) it vanishes on V^μ. In the second case the map Tr_σ factors through $\mathsf{Tr}_{i,j} : V_{k+N,t} \to V_{k+N-1,t-1}$ for $j = N + k + 1 - \sigma_i$, and so it vanishes on V^μ by (68). This contradiction finishes the proof. □

Now we can finish the proof of our key technical proposition.

Proof of Proposition 10.1. By Lemma 10.3, it is enough to prove that the map (61) is Π_0^a-injective. Let k be the sum of the parts of κ, and let t be the sum of the parts of τ. Note that the representation $V^{\kappa - w_0\tau}$ has depth t. Let us choose some embeddings $V^\kappa \subset V_k$, $V^{-w_0\tau} \subset V_{0,t}$ and $W \subset V_N$. Their tensor product gives an embedding of $V^\kappa \otimes V^{-w_0\tau} \otimes W$ into $V_{k+N,t}$ that fits into a commutative diagram

$$\begin{array}{ccccc}
V^{\kappa - w_0\tau} \otimes W & \longrightarrow & V^\kappa \otimes V^{-w_0\tau} \otimes W & \longrightarrow & V^\kappa \otimes \Pi_0^a(V^{-w_0\tau} \otimes W) \\
\vdots & & \downarrow & & \vdots \\
\Pi_t(V_{k,t}) \otimes V_N & \longrightarrow & V_{k+N,t} & \longrightarrow & V_k \otimes \Pi_0(V_{N,t})
\end{array}$$

(the left dotted arrow comes from the embedding $V^{\kappa - w_0\tau} \subset \Pi_t(V_{k,t})$ and the right dotted arrow is obtained by the functoriality of the projector Π_0^a). Note that all vertical arrows are injective. Applying the projector Π_0 (and dropping the middle terms) we obtain a commutative square

$$\begin{array}{ccc}
\Pi_0(V^{\kappa - w_0\tau} \otimes W) & \longrightarrow & \Pi_0(V^\kappa \otimes \Pi_0(V^{-w_0\tau} \otimes W)) \\
\downarrow & & \downarrow \\
\Pi_0(\Pi_t(V_{k,t}) \otimes V_N) & \longrightarrow & \Pi_0(V_k \otimes \Pi_0(V_{N,t}))
\end{array}$$

with injective vertical arrows. The bottom line is injective by Proposition 10.4, hence so is the top line. Applying additionally the projector Π_0^a we conclude that the map

$$\Pi_0^a(V^{\kappa - w_0\tau} \otimes W) \to \Pi_0^a(V^\kappa \otimes \Pi_0(V^{-w_0\tau} \otimes W))$$

is also injective. But

$$\Pi_0^a(V^\kappa \otimes \Pi_0(V^{-w_0\tau} \otimes W)) = \Pi_0^a(V^\kappa \otimes \Pi_0^a(V^{-w_0\tau} \otimes W)).$$

Indeed, by the Littlewood–Richardson rule, the tensor product of V^κ with V^μ for non-negative μ has a summand V^λ with $\lambda_{a+1} = \cdots = \lambda_n = 0$ only if $\mu_{a+1} = \cdots = \mu_n = 0$. We conclude that the map

$$\Pi_0^a(V^{\kappa - w_0\tau} \otimes W) \to \Pi_0^a(V^\kappa \otimes \Pi_0^a(V^{-w_0\tau} \otimes W))$$

is injective. □

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