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Jinhua Wang · Pin Yu

A large data regime for nonlinear wave equations

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Abstract. For semilinear wave equations with null form nonlinearities on $\mathbb{R}^{3+1}$, we exhibit an open set of initial data which are allowed to be large in energy spaces, yet we can still obtain global solutions in the future.

We also exhibit a set of localized data for which the corresponding solutions are strongly focused, which in geometric terms means that a wave travels along a specific incoming null geodesic in such a way that almost all of the energy is concentrated in a tubular neighborhood of the geodesic and almost no energy radiates out of this neighborhood.

Keywords. Large data problem, wave equation, null form

1. Introduction

In this paper, we study the Cauchy problem for the following semilinear wave equation on $\mathbb{R}^{3+1}$:

$$\Box\varphi = Q(\nabla\varphi, \nabla\varphi), \tag{1.1}$$

where Q is a null form (see Section 2.2 for definitions) and $\varphi : \mathbb{R}^{3+1} \to \mathbb{R}$ is a scalar function. The data that we will consider for (1.1) will be some specific large data. In fact, the size of the data is measured by a large parameter δ^{-1} (where δ is sufficiently small) in energy spaces. We remark that the results of the current work can be easily extended to higher dimensions and to a system of equations with null form nonlinearities in the obvious way.

1.1. Earlier results

We briefly summarize the progress on small data theory for nonlinear waves related to equations of type (1.1). Based on the decay mechanism of linear waves, we know much about the Cauchy problems for (1.1) on Minkowski space-times $\mathbb{R}^{n+1}$, especially for

J. Wang: Max Planck Institute for Gravitational Physics (Albert Einstein Institute), Potsdam-Golm, Germany; e-mail: jinhua.wang@aei.mpg.de

P. Yu: Mathematical Sciences Center, Tsinghua University, Beijing, China; e-mail: pin@math.tsinghua.edu.cn

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small initial data. In dimensions four and higher, since linear waves decay fast enough (at least at the rate $t^{-3/2}$), the small-data-global-existence type theorems for (1.1) hold for generic quadratic nonlinearities (which are not assumed to be *null*)—see the work of Klainerman [7]. However, in $\mathbb{R}^{3+1}$, linear waves decay more slowly and the quadratic nonlinearities control the dynamics of the system. In fact, there are quadratic forms Q for which a finite time blow-up phenomenon occurs even for arbitrarily small data. This has been shown by F. John [4].

The main breakthrough in understanding small-data-global-existence results for (1.1) was made by Klainerman [6] by introducing the *null condition* on the nonlinearities. Under this condition, Klainerman and Christodoulou [1] independently proved that small initial data lead to global in time classical solutions. Their proofs are different in nature. Klainerman's approach makes use of full conformal symmetries of $\mathbb{R}^{3+1}$ through vector fields, while Christodoulou's idea is to use the conformal compactification of $\mathbb{R}^{3+1}$. Nevertheless, their proofs rely essentially on special cancelations of null form nonlinearities, which are absent for generic quadratic nonlinearities.

The cancelation of null forms has far-reaching implications for other types of hyperbolic equations. Although many hyperbolic equations do not in general have a null quadratic form type nonlinearity, estimates for the nonlinear terms follow more or less the same philosophy: if one term behaves badly (i.e., is large in some norms in most cases) in the nonlinearities, it must be coupled with a good (i.e., with much better or smaller estimates) term. Thus, we hope that the good terms are strong enough to absorb the large contributions from the bad terms. One major application of this idea in general relativity appears in Christodoulou–Klainerman's proof of nonlinear stability of Minkowski spacetime [3]. They observed that a bad (worse decay) component of Weyl curvature is always coupled to either a good connection coefficient or a good curvature component, thus in most cases the bad components do not really affect the long time behavior of gravitational waves.

Although all of the aforementioned results require that the initial data be sufficiently small, the idea of using cancelations from null forms still can be used to handle certain large data problems. We shall briefly describe two very recent works on the dynamics of vacuum Einstein field equations in general relativity.

In his seminal work [2], Christodoulou discovered a remarkable mechanism responsible for the dynamical formation of black holes. For some carefully chosen initial data (which give an open set of the Sobolev space on an outgoing null hypersurface), called *short pulse* data in [2], he proved that a black hole (more precisely, a trapped surface) can form along the evolution due to the focusing of gravitational waves. Besides its significance in physics, this result is truly remarkable from a PDE perspective, because the result is for large data (roughly speaking, small data for Einstein equations in general relativity would lead to a space-time close to Minkowski space-time; so for small data, we do not expect black holes). One of the key observations used repeatedly in [2] is still related to the philosophy of null forms: we do have many bad (large) components in the estimates, but all of them must come with good (small) components to make the estimates work.

In [8], Klainerman and Rodnianski extended and significantly simplified Christodoulou's work. A key ingredient in their paper is the relaxed propagation estimates, namely, if one enlarges the admissible set of initial conditions, the corresponding propagation estimates are much easier to derive. They reduced the number of derivatives needed in the estimates from two derivatives on curvature (in Christodoulou's proof) to just one. We should note that the simpler proof of Klainerman–Rodnianski yields results weaker than those obtained by Christodoulou. In fact, within this more general initial data set, they can only show long time existence results for vacuum Einstein field equations; nevertheless, once such existence results are obtained, one can improve them by assuming more on the data, say, consistent with Christodoulou's assumptions, and then one can derive Christodoulou's results in a straightforward manner.

The results of this paper are strongly motivated by [2] and the proofs are very much inspired by [8]. In particular, the choice of initial data will be analogous to the short pulse ansatz in [2]; the proof will rely on a relaxed version of energy estimates similar to the relaxation of propagation estimates in [8]. We also have to mention another work [9] of Klainerman and Rodnianski where they managed to localize the data for Einstein equations to show the dynamical formation of locally trapped surfaces. One of our main results here concerning strongly focused waves is motivated by that work. Roughly speaking, it asserts that if the wave initially concentrates around a given point in a specific way, then it will be confined to a tubular neighborhood of an incoming null geodesic and there is only a negligible amount of energy dispersing out of this neighborhood. It is precisely in this sense that we say the wave is strongly focused. It seems to the authors that this result is new even for linear wave equations.

1.2. Main results

We study the following system of nonlinear wave equations:

$$\begin{aligned} \Box\varphi &= Q(\nabla\varphi, \nabla\varphi), \\ (\varphi, \partial_t\varphi)|_{t=0} &= (\varphi^{(0)}, \varphi^{(1)}), \end{aligned} \tag{1.2}$$

where Q is a null form. We emphasize that φ is a vector valued function, although we will not need to express φ in terms of its components. The initial data set $(\varphi^{(0)}, \varphi^{(1)})$ consists of two smooth functions. We define the initial energy as

$$\text{Energy}_{(1)}(\varphi^{(0)}, \varphi^{(1)}) = \frac{1}{2}\int_{\mathbb{R}^3} (|\nabla\varphi^{(0)}|^2 + |\varphi^{(1)}|^2)\, dx_1\, dx_2\, dx_3.$$

For $k \geq 2$, we can also define the higher order energies:

$$\text{Energy}_{(k)}(\varphi^{(0)}, \varphi^{(1)}) = \frac{1}{2}\int_{\mathbb{R}^2} (|\nabla^k\varphi^{(0)}|^2 + |\nabla^{k-1}\varphi^{(1)}|^2)\, dx_1\, dx_2\, dx_3.$$

Main Theorem 1. *For any given $E_0 > 0$, there exists a smooth initial data set $(\varphi^{(0)}, \varphi^{(1)})$ for (1.2) such that*

$$\text{Energy}_{(1)}(\varphi^{(0)}, \varphi^{(1)}) \geq E_0, \quad \text{Energy}_{(2)}(\varphi^{(0)}, \varphi^{(1)}) \geq E_0,$$

and this data set leads to a classical smooth solution with life-span $[0, \infty)$.

Remark (Largeness of the data). Recall that on $\mathbb{R}^{3+1}$, the critical H^s-exponent (with respect to scaling) of (1.2) is $3/2$. Therefore, $\text{Energy}_{(1)}$ is a subcritical quantity and $\text{Energy}_{(2)}$ is a supercritical quantity. This means that we cannot make both $\text{Energy}_{(1)}$ and $\text{Energy}_{(2)}$ small by the scaling invariance of the equation. It is in this sense (on the level of energy) that the data of (1.2) is large.

Moreover, for $k \geq 2$, we can show that

$$\text{Energy}_{(k)}(\varphi^{(0)}, \varphi^{(1)}) \geq \delta^{-(k-1)},$$

where δ is a small positive parameter. We note in passing that the higher order energies can be extremely large.

In the course of proving the above theorem, we will derive two other results which are of independent interest. To facilitate their statement, we introduce a bit of notation.

We review some geometric constructions on Minkowski space $\mathbb{R}^{3+1}$. Besides the standard coordinates (t, x_1, x_2, x_3), we shall mainly use the null-polar coordinates $(u, \underline{u}, \theta)$. We recall their definition. Let

$$r = \sqrt{x_1^2 + x_2^2 + x_3^2}$$

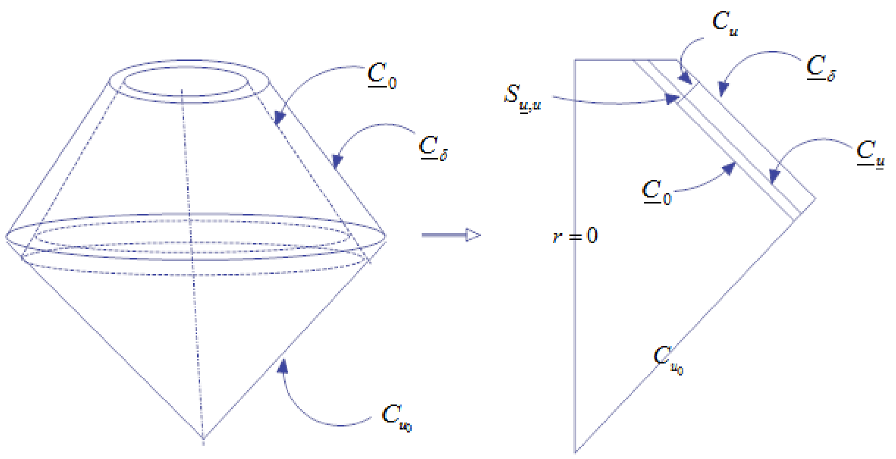
be the spatial radius function. Two optical functions u and $\underline{u}$ are defined by

$$u = \tfrac{1}{2}(t - r) \quad \text{and} \quad \underline{u} = \tfrac{1}{2}(t + r).$$

The angular argument θ denotes a point on the unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$.

The past null infinity $\mathcal{I}^-$ of $\mathbb{R}^{3+1}$ can be represented by the collection of past-pointing outgoing null lines. Therefore, $\mathcal{I}^-$ is parameterized by $(\underline{u}, \theta) \in \mathbb{R} \times \mathbb{S}^2$. We also use C_c to denote the level surface $u = c$, where c is a constant; similarly, $\underline{C}_{\underline{u}}$ denotes a level set of $\underline{u}$. Their intersection $C_u \cap \underline{C}_{\underline{u}}$ will be a two-sphere denoted by $S_{\underline{u},u}$.

We illustrate these definitions in the following picture:



We remark that, in Sections 3 and 4, which are the technical heart of the paper, the parameter u will be confined to the interval $[u_0, -1]$, where u_0 is a large negative number. The parameter $\underline{u}$ is confined to $[0, \delta]$, where δ is small positive parameter, which will be determined later. To simplify our presentation, we will ignore the θ directions in our pictures. Thus, instead of the left picture above, we will adopt the right picture; as such, the sphere $S_{\underline{u},u}$ is represented by a single point in the picture.

We use L and $\underline{L}$ to denote the following future-pointing null vector fields:

$$L = \partial_t + \partial_r \quad \text{and} \quad \underline{L} = \partial_t - \partial_r.$$

We shall use $\nabla\!\!\!/$ to denote the intrinsic covariant derivative on $S_{\underline{u},u}$. It is the restriction of the usual covariant derivative of $(\mathbb{R}^{3+1}, g)$ to $S_{\underline{u},u}$, where g is the standard flat Lorentzian metric on $\mathbb{R}^{3+1}$.

As usual, we use $\mathfrak{so}(3)$ to denote the Lie algebra of the rotation group $SO(3)$ which acts on $\mathbb{R}^{3+1}$ in a canonical way. We choose generators $\Omega_1, \Omega_2, \Omega_3$ of $\mathfrak{so}(3)$ in the usual way, namely,

$$\begin{aligned}
\Omega_1 &= x_2 \frac{\partial}{\partial x_3} - x_3 \frac{\partial}{\partial x_2},\\
\Omega_2 &= x_3 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_3},\\
\Omega_3 &= x_1 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_1}.
\end{aligned}$$

In what follows, we shall use Ω to denote a generic Ω_i and use Ω^2 to denote a generic operator of the form $\Omega_i \Omega_j$, and so on. For a given function ϕ, we write $|\Omega\phi|$ for $\sum_{i=1}^3 |\Omega_i \phi|$ and write $|\Omega^2\phi|$ for $\sum_{1\le i,j\le 3} |\Omega_i\Omega_j\phi|$, and so on.

We observe that there are positive constants C_1 and C_2 such that on any $S_{\underline{u},u}$ we have[1]

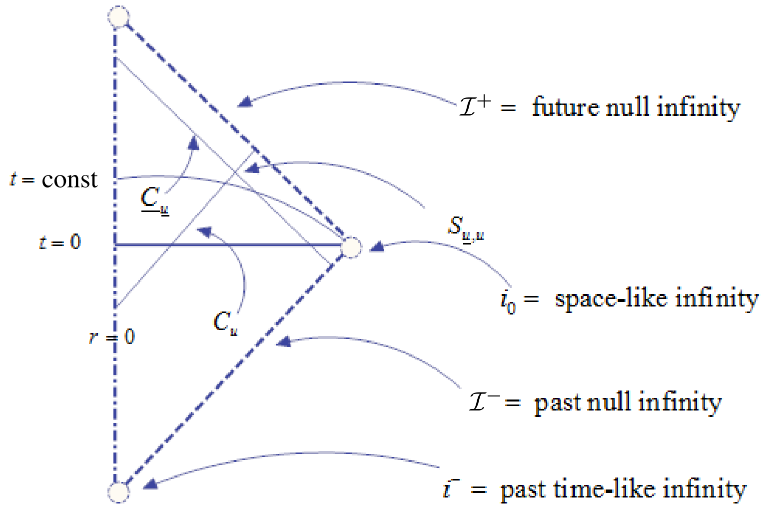
$$C_1 |u|\, |\nabla\!\!\!/ \phi| \le |\Omega\phi| \le C_2 |u|\, |\nabla\!\!\!/ \phi|.$$

For the sake of simplicity, we write this inequality as $|\Omega\phi| \sim |u|\, |\nabla\!\!\!/ \phi|$. The proof is straightforward: we first check it on the unit sphere and then use scaling to get the factor $|u|$. In general, for a given $k \in \mathbb{Z}_{\ge 0}$, we have

$$|\Omega^k \phi| \sim |u|^k |\nabla\!\!\!/^k \phi|. \tag{1.3}$$

The geometric picture all the way to null infinities is usually represented by the Penrose diagram of $(\mathbb{R}^{3+1}, g)$:

[1] Since $|\underline{u}| \le \delta$, for sufficiently small δ, $r = \underline{u} - u$ is comparable to $|u|$.



For many situations in the current work, one has to study a Goursat problem for (1.1), namely the characteristic problem, instead of the Cauchy problem. Therefore, the null hypersurfaces are the main geometric objects in what follows.

Our second main theorem is a semiglobal existence result for a Goursat problem for (1.1) where the data is described on a part of a virtual null hypersurface, i.e., the past null infinity $\mathcal{I}^-$. More precisely, the initial data of (1.1) will be a radiation field given on the subset

$$\mathcal{I}_\delta^- = \{(\underline{u}, \theta) \in \mathcal{I}^- \mid \underline{u} \le \delta\},$$

in the asymptotic sense, where δ is a positive small parameter to be determined later. Explicitly, the data is given by a smooth function

$$\varphi_{-\infty} : \mathcal{I}_\delta^- \to \mathbb{R}, \qquad (\underline{u}, \theta) \mapsto \varphi_{-\infty}(\underline{u}, \theta),$$

and we require the solution of (1.1) to obey the asymptotic condition

$$\varphi(u, \underline{u}, \theta) \sim \frac{1}{|u|}\varphi_{-\infty}(\underline{u}, \theta) + o(1/|u|).$$

We remark that, for linear waves, $1/|u|$ is the expected decay rate towards past null infinity. We also remark that, when we speak about the smallness or largeness of the data, we always mean the smallness or largeness of the radiation field $\varphi_{-\infty}$ instead of φ itself (which vanishes on $\mathcal{I}^-$).

In this work, we require the initial datum $\varphi_{-\infty}$ to have the following form:

$$\varphi_{-\infty}(\underline{u}, \theta) = \begin{cases} 0 & \text{if } \underline{u} \le 0, \\ \delta^{1/2}\psi_0(\underline{u}/\delta, \theta) & \text{if } 0 \le \underline{u} \le \delta, \end{cases} \tag{1.4}$$

where $\psi_0 : (0, 1) \times \mathbb{S}^2 \to \mathbb{R}$ is a fixed compactly supported smooth function. More generally, we can take $\varphi_{-\infty}$ from an open set of certain Sobolev spaces defined on $\mathcal{I}^-$.

We do not pursue this point at the moment and we shall revisit it in the last section of the paper.

The datum given in the above form is called a *short pulse*, a name coined by Christodoulou [2]. In his work, he prescribes the shear (more precisely, the conformal geometry) of the initial null hypersurface in a similar form. The shear in the situation of [2] is exactly the initial data for the Einstein vacuum equation.

One may argue that the datum (1.4) is small when δ is small, at least pointwise. In fact, the L^∞ norm is irrelevant to equation (1.1) since we may always add a constant to get a new solution. The size of the datum should be measured at the level of derivatives. The $\partial_{\underline{u}}$ derivative of the datum can be extremely *large* if δ is small. In what follows, we shall see that the energy of φ will be comparable to 1 and the higher order energy of φ will be comparable to some δ^{-a} with $a > 0$. Therefore, the datum is no longer small in energy spaces.

Before the statement of the second main theorem, we recall that the domain of dependence $\mathcal{D}^+(\mathcal{I}_\delta^-)$ of $\mathcal{I}_\delta^-$ is the backward light-cone in $\mathbb{R}^{3+1}$ with vertex at $(\delta, 0, 0, 0)$.

Main Theorem 2. *Consider the nonlinear wave equation*

$$\Box\varphi = Q(\nabla\varphi, \nabla\varphi),$$

where Q is a null form, and prescribe the following asymptotic characteristic initial datum on the future null infinity $\mathcal{I}^-$:

$$\lim_{u\to-\infty} |u|\varphi(u, \underline{u}, \theta) = \varphi_{-\infty}(\underline{u}, \theta) \quad \textit{for all } (\underline{u}, \theta), \tag{1.5}$$

where $\varphi_{-\infty} \in C^\infty(\mathcal{I}_\delta^-)$ is given by

$$\varphi_{-\infty}(\underline{u}, \theta) = \begin{cases} 0 & \textit{if } \underline{u} \le 0, \\ \delta^{1/2}\psi_0(\underline{u}/\delta, \theta) & \textit{if } 0 \le \underline{u} \le \delta. \end{cases}$$

Here $\psi_0 : (0, 1) \times \mathbb{S}^2 \to \mathbb{R}$ is a fixed compactly supported smooth function. If δ is sufficiently small, there exists a unique classical solution φ on $\mathcal{D}^+(\mathcal{I}_\delta^-) \cap \{t \le -1\}$ such that the radiation field of φ is exactly $\varphi_{-\infty}$, i.e. as described in (1.5).

Remark. We clarify the meaning of a *solution* in the above theorem: we always assume that the asymptotic behavior of φ (as $t \to -\infty$) resembles linear waves, i.e.

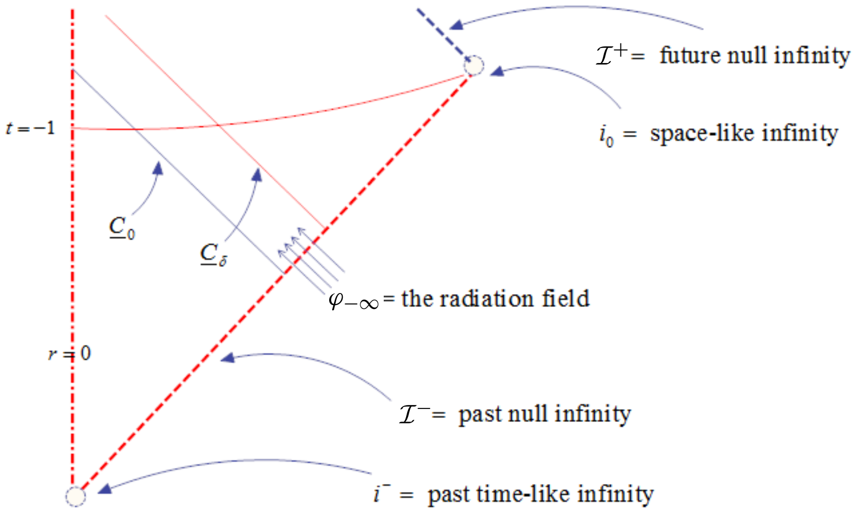
$$|\varphi| = O(1/t), \quad |L\varphi| = O(1/t), \quad |\underline{L}\varphi| + |\nabla\!\!\!/\varphi| = O(1/t^{3/2}).$$

In other words, we only search for solutions with the above conditions. The uniqueness statement is also understood in this class: if φ and ϕ are two solutions that have the same linear asymptotics, i.e.

$$\lim_{u\to-\infty}\big[|u|\big(\varphi(u, \underline{u}, \theta) - \phi(u, \underline{u}, \theta)\big) + |u|\big(L\varphi(u, \underline{u}, \theta) - L\phi(u, \underline{u}, \theta)\big)\big] = \varphi_{-\infty}(\underline{u}, \theta) \tag{1.6}$$

for all $(\underline{u}, \theta)$, then $\varphi \equiv \phi$.

The theorem can also be depicted as follows (notice that the region $\mathcal{D}^+(\mathcal{I}_\delta^-) \cap \{t \le -1\}$ is enclosed by the four red lines):



In the course of the proof of Main Theorem 2, we shall see that the energy flux through $\underline{C}_\delta$ is bounded above by δ^a for some $a > 0$. Thus, almost all of the energy is confined to $\mathcal{D}^+(\mathcal{I}_\delta^-) \cap \{t \le -1\}$ and very little energy is radiated to future null infinity. Intuitively, the waves travel from past null infinity in the incoming direction all the way up to the finite time $t = -1$ with almost no loss of energy.

We now turn to the last main theorem of the paper where the data are prescribed on a fixed finite null hypersurface C_{u_0} instead of past null infinity. Only for this third main theorem, can we fix a finite $u_0 \le -2$, say $u_0 = -10$.

The initial outgoing null hypersurface is

$$C_{u_0} = \{(u, \underline{u}, \theta) \mid u_0 \le \underline{u} \le \delta\},$$

and we also require the initial datum φ_{u_0} to behave like a pulse:

$$\varphi_{u_0}(\underline{u}, \theta) = \begin{cases} 0 & \text{if } u_0 \le \underline{u} \le 0, \\ \dfrac{\delta^{1/2}}{|u_0|}\psi_0(\underline{u}/\delta, \theta) & \text{if } 0 \le \underline{u} \le \delta. \end{cases} \tag{1.7}$$

Besides the above requirement, we impose the condition that ψ_0 is localized in the angular argument θ, i.e., there is a fixed angle $\theta_0 \in \mathbb{S}^2$ such that the support of ψ_0 is contained in the geodesic ball $B_{\delta^{1/2}}(\theta_0)$ centered at θ_0 of radius $\delta^{1/2}$ on $\mathbb{S}^2$. Moreover, we can require that ψ_0 satisfy the following estimate on $\mathbb{S}^2$:

$$\sum_{k=1}^{4} \delta^{(k-1)/2} \|\nabla\!\!\!/^k \psi_0\|_{L^\infty(\mathbb{S}^2)} \lesssim 1,$$

where $\nabla\!\!\!/$ is the covariant derivative for the standard metric on $\mathbb{S}^2$. We call such an initial datum $\varphi_{u_0}(\underline{u}, \theta)$ a *short pulse localized in* $B_{\delta^{1/2}}(\theta_0)$.

Main Theorem 3. *Consider the nonlinear wave equation*

$$\Box\varphi = Q(\nabla\varphi, \nabla\varphi)$$

with characteristic initial data given by a short pulse datum localized in $B_{\delta^{1/2}}(\theta_0)$. *If* δ *is small enough, there exists a unique classical solution* φ *on* $\mathcal{D}^+(C_{u_0}) \cap \{u \leq -1\}$.

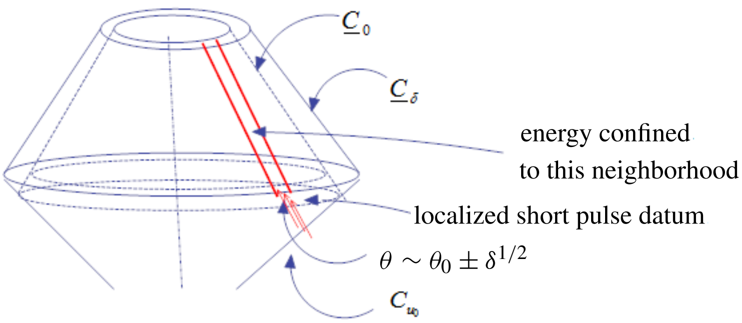
Moreover, if $C_u^o \triangleq \{u\} \times [0, \delta) \times B_{\delta^{1/2}}(\theta_0)$ *denotes a small set on the outgoing null hypersurface* C_u, *the energy is almost localized in* $\bigcup_{u\in[u_0,-1]} C_u^o$, *which is a tubular neighborhood of some incoming null geodesic parameterized by* $u \in [u_0, -1]$ *with fixed* $\underline{u}$ *and* $\theta = \theta_0$. *More precisely, for all* $u \in [u_0, -1]$, *the incoming energy outside the small neighborhood* C_u^o *is bounded as follows:*

$$\int_{C_u - C_u^o} (|L\varphi|^2 + |\nabla\!\!\!\!/\,\varphi|^2) \lesssim \delta^2.$$

And the energy inside C_u^o *is almost conserved,*

$$\left| \int_{C_u^o} (|L\varphi|^2 + |\nabla\!\!\!\!/\,\varphi|^2) - \int_{C_{u_0}^o} (|L\varphi|^2 + |\nabla\!\!\!\!/\,\varphi|^2) \right| \lesssim \delta.$$

The above theorem can also be depicted as follows:



We can also show that there is almost no energy radiating out through the incoming null hypersurface $\underline{C}_\delta$. Quantitatively, we have

$$\int_{\underline{C}_\delta} (|\underline{L}\varphi|^2 + |\nabla\!\!\!\!/\,\varphi|^2) \lesssim \delta.$$

The above estimates are also true for higher order fluxes, as will be clear later.

In the proof, we shall see that we can prove a stronger version in which we do have L^∞ control on all the first derivatives of φ. On the final outgoing null hypersurface C_{-1}, the energy is mostly contributed by $L\varphi$ and the other components are small (measured in terms of a positive power of δ); in fact, we can show that $L\varphi$ is almost unchanged along evolution, i.e., $|L\varphi(-1, \underline{u}, \theta)| \sim |L\varphi(u_0, \underline{u}, \theta)|$. More precisely,

$$|L\varphi(-1, \underline{u}, \theta) - L\varphi(u_0, \underline{u}, \theta)| \lesssim \delta^{1/2}.$$

Since we take a short pulse datum localized in $B_{\delta^{1/2}}(\theta_0)$, roughly, we have

$$L\varphi(u_0, u, \theta) = \begin{cases} \delta^{-1/2} & \text{for } \theta \in B_{\delta^{1/2}}(\theta_0), \\ 0 & \text{for } \theta \notin B_{\delta^{1/2}}(\theta_0). \end{cases}$$

Thus, for the final surface C_{-1}, roughly, we actually have

$$L\varphi(-1, u, \theta) \sim \begin{cases} \delta^{-1/2} & \text{for } \theta \in B_{\delta^{1/2}}(\theta_0), \\ \delta^{1/2} & \text{for } \theta \notin B_{\delta^{1/2}}(\theta_0). \end{cases}$$

We then integrate those L^∞ estimates to derive the desired control on energy.

Therefore, we have a concentration phenomenon for a special class of solutions of (1.1) and we say that the solutions constructed from short pulse data localized in some small spherical sector are *strongly focused*. As remarked before, the third theorem appears to be new even for linear wave equations.

1.3. Comments on the proof

We would now like to address the motivations for and difficulties in various estimates leading to the theorem, and then give an outline of the proof.

We first explain the idea of relaxation for energy estimates. This is done in Section 4 which provides an a priori estimate with up to four derivatives for (1.1) with short pulse initial data. It is the key ingredient for the whole paper.

The proof is based on the usual vector field method. We compute the amplitudes of $L\varphi$ and $\nabla\!\!\!/\varphi$ on C_{u_0} where the data are given. Roughly speaking, the quantitative estimates look like

$$\|L\varphi\|_{L^\infty(C_{u_0})} \sim \delta^{-1/2}, \qquad \|\nabla\!\!\!/\varphi\|_{L^\infty(C_{u_0})} \sim \delta^{1/2}. \tag{1.8}$$

When one derives energy estimates for φ, a natural choice of the multiplier vector field would be L, thus, the energy flux would be

$$\int_{C_u} |L\varphi|^2 + \int_{\underline{C}_{\underline{u}}} |\nabla\!\!\!/\varphi|^2. \tag{1.9}$$

If we stick to (1.8) as ideal propagation estimates without any relaxation, we expect

$$\int_{C_u} |L\varphi|^2 \sim 1, \qquad \int_{\underline{C}_{\underline{u}}} |\nabla\!\!\!/\varphi|^2 \sim \delta.$$

Therefore, the flux term (1.9) will only yield a bound for $L\varphi$ but not for $\nabla\!\!\!/\varphi$, because the bound for $\nabla\!\!\!/\varphi$ is too small compared to that of $L\varphi$ estimated in this way. In other words, in the end, we do not expect to close the argument by the standard bootstrap method. Of course, this is due to the choice of the initial data: short pulse data do not respect the natural scaling of the wave equation!

To resolve this difficulty, one has to relax the estimates for $\nabla\!\!\!/\varphi$, namely, although the size of the initial data suggests the amplitude $\nabla\!\!\!/\varphi$ behaves like $\delta^{1/2}$, we pretend the amplitude is worse to match (1.8). Therefore, in the energy estimates, we only expect $\nabla\!\!\!/\varphi$ to behave like

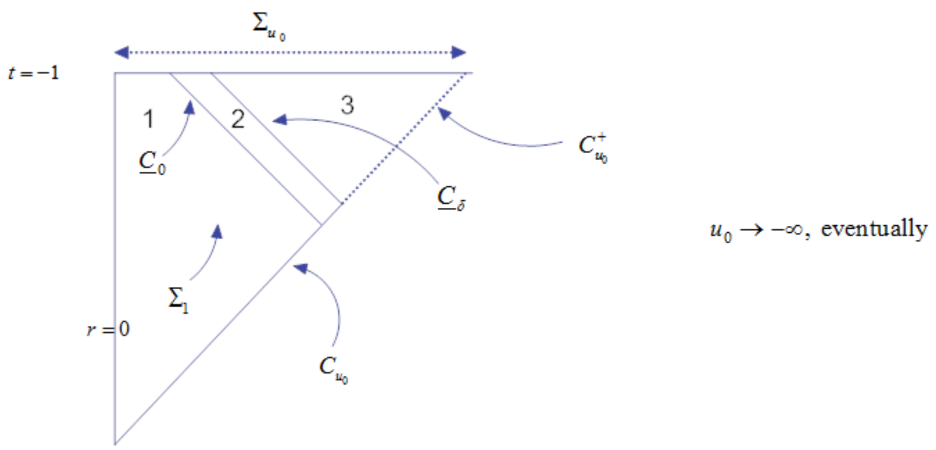
$$\|\nabla\!\!\!/\varphi\|_{L^\infty(C_u)} \sim 1. \tag{1.10}$$

It turns out that, under this weak assumption, we can still close the bootstrap argument to derive energy estimates. Moreover, once the bootstrap argument is closed, if we can afford one more derivative, we can retrieve the stronger estimates (1.8) for $\nabla\!\!\!/\varphi$. This will be proved later.

The second difficulty is the number of derivatives needed for a priori estimates. Instead of four derivatives, we may attempt to use three derivatives in Section 4, since this is still good to control the L^∞ norm of first derivatives via Sobolev inequalities. This does not work in an obvious way and the reason is as follows: when one derives estimates for third derivatives, we can use the information already obtained for the first and second derivatives, but $L\varphi$ (the worst term) still contains a third derivative term. Thus, we cannot reduce the nonlinear term to a linear one. But the situation is completely different if we use four derivatives: when we try to control fourth derivatives of the solution, we must have already obtained estimates for up to three derivatives. Thus, the control of $L\varphi$ is then independent of the fourth derivatives. Hence, this reduces the nonlinear term to the linear case where the Gronwall inequality can be used to absorb all the bad terms.

We also point out that the second difficulty is also related to the relaxation of the energy estimates. If we use only three derivatives, for some null forms, say Q_{0j}, it leads to a nonlinear term of the form $\nabla\!\!\!/^3\varphi \cdot L\varphi$. As we commented in the last paragraph, we do not have linear control on the L^∞ norm of $L\varphi$, and yet because we use relaxed estimates, the control of $\nabla\!\!\!/^3\varphi$ is not good enough to compensate for the large amplitude of $L\varphi$. This will lead to a large nonlinear term which cannot be controlled.

We now outline the proof of our Main Theorem 1. The parameter u_0 is a large negative number which will be eventually sent to $-\infty$. The following picture helps to understand the structure of the proof:



- Step 1. We prescribe an initial datum on the null hypersurface C_{u_0} where $u_0 \le \underline{u} \le \delta$. When $u_0 \le \underline{u} \le 0$, the datum is trivial, therefore the solution in Region 1 in the picture is a constant map. When $0 \leq \underline{u} \leq \delta$, the datum will be prescribed in a specific form (see Section 3 for a detailed account):

$$\varphi(\underline{u}, u_0, \theta) = \delta^{1/2}\psi_0(\underline{u}/\delta, \theta),$$

with energy approximately equal to E_0. We then show that we can construct a solution in Region 2 of the picture.

- Step 2. From the first step, we can actually show that the restrictions of the solution already constructed to $\underline{C}_\delta$ are small in energy norms. On

$$C^+_{u_0} = \{p \in C_{u_0} \mid \delta \leq \underline{u}(p), t(p) \leq -1\},$$

we extend the datum (from Step 1) by zero. Therefore, the datum is also small on $C^+_{u_0}$. We can now use $\underline{C}_\delta$ and $C^+_{u_0}$ as initial hypersurfaces to solve a small data problem to construct a solution in Region 3 of the picture.
- Step 3. We patch the solutions in Regions 1–3 to get one single solution in the above picture and then restrict it to the surface Σ_{u_0}. We then let u_0 go to $-\infty$ and use the Arzelà–Ascoli lemma to get a solution all the way up to past null infinity. The restriction to Σ_{u_0} then yields a subsequence which converges to a Cauchy datum. Finally, we can reverse and shift the time to complete the proof of Main Theorem 1.

We remark that Step 1 is the most difficult part since the datum is no longer small and we have to carefully deal with the cancelations from null forms and the profile of the data. Steps 2 and 3 are more or less standard.

We end the introduction by a heuristic discussion of Step 1 for the spherical symmetric situation. Although more effort is needed to treat the general case, the spherical symmetric case is instructive to understand the main structure of the proof. In fact, for the general case, we have to commute rotational derivatives with the main equation at least three times to obtain an a priori energy estimate. For the spherical symmetric case, it is much easier:

We assume that the datum, hence the solution, is spherical symmetric, so $\nabla\!\!\!/\varphi \equiv 0$. We can rewrite the main equation as

$$-L\underline{L}\varphi + \frac{1}{r}(L\varphi - \underline{L}\varphi) = C\underline{L}\varphi \cdot L\varphi, \tag{1.11}$$

where C is a given constant. The datum on C_{u_0} is given by (1.7) (without dependence on θ in this case), therefore, by direct computation, we obtain

$$\|L\varphi\|_{L^\infty(C_{u_0})} \lesssim \delta^{-1/2}u_0^{-1}, \qquad \|L^2\varphi\|_{L^\infty(C_{u_0})} \lesssim \delta^{-3/2}u_0^{-1},$$

and

$$\|L\varphi\|_{L^2(C_{u_0})} \lesssim 1, \qquad \|L^2\varphi\|_{L^2(C_{u_0})} \lesssim \delta^{-1}.$$

We also have an estimate on $\underline{L}\varphi$ on C_{u_0} by integrating (1.11) along L and using the fact that $\underline{L}\varphi \equiv 0$ when $\underline{u} = 0$:

$$\|\underline{L}\varphi\|_{L^\infty(C_{u_0})} \lesssim \delta^{1/2}u_0^{-2}.$$

We shall derive an a priori energy estimate for a solution φ. Therefore, we assume the existence of φ for the moment, and we would like to show that the above estimates on $L\varphi$ and $\underline{L}\varphi$ propagate to C_u ($u \leq -1$) as long as the solution exists up to C_u.

The idea is to launch a bootstrap argument. We assume that for all $u_0 \leq u \leq -1$,

$$\|L\varphi\|_{L^2(C_u)} \lesssim M, \quad \|L^2\varphi\|_{L^2(C_u)} \lesssim M\delta^{-1}, \quad \|\underline{L}\varphi\|_{L^2(C_u)} \lesssim M\delta u^{-1}, \tag{1.12}$$

where M is a large constant which may depend on φ. The aim is to show that M depends only on the initial datum of φ.

We remark that, according to the Sobolev inequality, the bootstrap assumption (1.12) has the following immediate consequence:

$$\delta^{1/2}|u|\,\|L\varphi\|_{L^\infty} + \delta^{-1/4}|u|^{3/2}\|\underline{L}\varphi\|_{L^\infty} \lesssim M. \tag{1.13}$$

We now derive energy estimates. We multiply (1.11) by $L\varphi$ and integrate on Region 2 to get

$$\int_{C_u} |L\varphi|^2 = \int_{C_{u_0}} |L\varphi|^2 + C\iint_{\mathcal{D}} L\varphi\underline{L}\varphi L\varphi + \iint_{\mathcal{D}} \frac{1}{r}\underline{L}\varphi \cdot L\varphi. \tag{1.14}$$

We remark that the trilinear term on the right hand side comes from the nonlinearity of (1.11). Thanks to (1.12) and (1.13), we can bound the double integrations in (1.14) by

$$\int_{u_0}^{u} \|\underline{L}\varphi\|_{L^\infty}\|L\varphi\|^2_{L^2(C_{u'})}\,du' + \int_{u_0}^{u}\frac{1}{|u'|}\|L\varphi\|_{L^2(C_{u'})}\|\underline{L}\varphi\|_{L^2(C_{u'})}\,du'$$
$$\lesssim \delta^{1/4}|u|^{-1/2}M^3 + \delta|u|^{-1}M^2.$$

Therefore, back to (1.14), we can easily achieve

$$\|L\varphi\|_{L^2(C_u)} \lesssim 1 + \delta^{1/8}|u|^{-1/4}M^{3/2}. \tag{1.15}$$

The constant 1 is from the initial datum. We remark that the factor $\delta^{1/8}$ on the right hand side is from $\underline{L}\varphi$. This reflects the basic structure of the null form, namely, for the nonlinear terms, there must be at least one $\underline{L}\varphi$ factor.

We then multiply (1.11) by $\underline{L}\varphi$ and integrate on Region 2 to get

$$\int_{\underline{C}_{\underline{u}}} |\underline{L}\varphi|^2 = C\iint_{\mathcal{D}} L\varphi\underline{L}\varphi\underline{L}\varphi - \iint_{\mathcal{D}} \frac{1}{r}\underline{L}\varphi \cdot L\varphi. \tag{1.16}$$

We then use (1.12) to bound the trilinear term by $\delta^{3/2}|u|^{-2}M^3$. The last term in (1.16) can be bounded by

$$\int_{u_0}^{u}\frac{\delta}{|u'|^2}\|L\varphi\|^2_{L^2(C_{u'})}\,du' + \frac{1}{\delta}\int_0^{\underline{u}}\|\underline{L}\varphi\|^2_{L^2(\underline{C}_{\underline{u}'})}\,d\underline{u}'$$
$$\lesssim \delta|u|^{-1} + \delta^{5/4}|u|^{-3/2}M^3 + \frac{1}{\delta}\int_0^{\underline{u}}\|\underline{L}\varphi\|^2_{L^2(\underline{C}_{\underline{u}'})}\,d\underline{u}'.$$

We have used (1.15) in the last inequality. According to Gronwall's inequality, we obtain

$$\|\underline{L}\varphi\|_{L^2(\underline{C}_u)} \lesssim \delta^{1/2}|u|^{-1/2} + \delta^{5/8}|u|^{-3/4}M^{3/2}. \tag{1.17}$$

The estimates for $L^2\varphi$ can be derived in a similar way. We then take δ to be sufficiently small, in view of (1.15) and (1.17), and we can improve the large constant M in (1.12) to be a universal constant depending only on the initial data. Therefore, we can close the bootstrap argument and obtain the a priori energy estimates.

2. Preliminaries

2.1. Energy estimates scheme

Let ϕ be a solution of the following nonhomogeneous wave equation on $\mathbb{R}^{3+1}$:

$$\Box\phi = \Phi. \tag{2.1}$$

We define the energy momentum tensor associated to ϕ to be

$$\mathbb{T}_{\alpha\beta}[\phi] = \nabla_\alpha\phi\nabla_\beta\phi - \tfrac{1}{2}g_{\alpha\beta}\nabla^\mu\phi\nabla_\mu\phi.$$

This tensor is symmetric and it enjoys the divergence identity

$$\nabla^\alpha\mathbb{T}_{\alpha\beta}[\phi] = \Phi\cdot\nabla_\beta\phi. \tag{2.2}$$

Given a vector field X, which is usually called a *multiplier vector field*, the associated energy currents are defined by

$$J^X_\alpha[\phi] = \mathbb{T}_{\alpha\mu}[\phi]X^\mu, \qquad K^X[\phi] = \mathbb{T}^{\mu\nu}[\phi]{}^{(X)}\pi_{\mu\nu},$$

where the deformation tensor ${}^{(X)}\pi_{\mu\nu}$ is defined by

$${}^{(X)}\pi_{\mu\nu} = \tfrac{1}{2}\mathcal{L}_X g_{\mu\nu} = \tfrac{1}{2}(\nabla_\mu X_\nu + \nabla_\nu X_\mu). \tag{2.3}$$

Thanks to (2.2), we have

$$\nabla^\alpha J^X_\alpha[\phi] = K^X[\phi] + \Phi\cdot X\phi. \tag{2.4}$$

In the null frame $\{e_1, e_2, e_3 = \underline{L}, e_4 = L\}$, we compute $\mathbb{T}_{\alpha\beta}[\phi]$ as

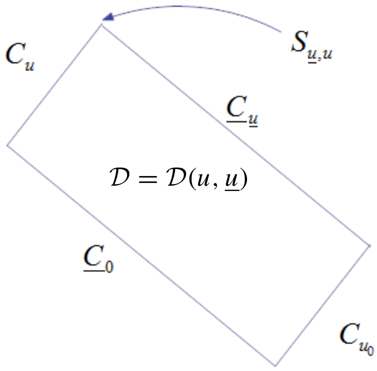
$$\mathbb{T}(L,L)[\phi] = |L\phi|^2, \qquad \mathbb{T}(L,\underline{L})[\phi] = |\nabla\!\!\!/\phi|^2, \qquad \mathbb{T}(\underline{L},\underline{L})[\phi] = |\underline{L}\phi|^2.$$

We notice that the above three terms are nonnegative and this manifests the dominant energy condition for $\mathbb{T}_{\alpha\beta}[\phi]$.

We shall use $X = \Omega$ ($\in \mathfrak{so}(3)$), L and $\underline{L}$ as multiplier vector fields; the corresponding deformation tensors and currents are computed as follows:

$$\begin{aligned} {}^{(\Omega)}\pi_{\mu\nu} &= 0, \quad {}^{(L)}\pi = \frac{2}{r}g\!\!\!/, \quad {}^{(\underline{L})}\pi = -\frac{2}{r}g\!\!\!/, \\ K^\Omega &= 0, \quad K^L = \frac{1}{r}L\phi\underline{L}\phi, \quad K^{\underline{L}} = -\frac{1}{r}L\phi\underline{L}\phi. \end{aligned} \tag{2.5}$$

where $g\!\!\!/$ is the restriction of the Minkowski metric m to the two-sphere $S_{\underline{u},u}$.



We use $\mathcal{D}(u, \underline{u})$ to denote the space-time slab enclosed by the null hypersurfaces C_{u_0}, $\underline{C}_0$, C_u and $\underline{C}_{\underline{u}}$. We integrate (2.4) on $\mathcal{D}(u, \underline{u})$ to derive

$$\int_{C_u} \mathbb{T}[\phi](X, L) + \int_{\underline{C}_{\underline{u}}} \mathbb{T}[\phi](X, \underline{L}) = \int_{C_{u_0}} \mathbb{T}[\phi](X, L) + \int_{\underline{C}_0} \mathbb{T}[\phi](X, \underline{L}) + \iint_{\mathcal{D}(u,\underline{u})} (K^X[\phi] + \Phi \cdot X\phi).$$

where L and $\underline{L}$ are the corresponding normals of the null hypersurfaces C_u and $\underline{C}_{\underline{u}}$.

In applications, the data on $\underline{C}_0$ is always vanishing, so we have the following fundamental energy identity:

$$\int_{C_u} \mathbb{T}[\phi](X, L) + \int_{\underline{C}_{\underline{u}}} \mathbb{T}[\phi](X, \underline{L}) = \int_{C_{u_0}} \mathbb{T}[\phi](X, L) + \iint_{\mathcal{D}(u,\underline{u})} (K^X[\phi] + \Phi \cdot X\phi). \quad (2.6)$$

2.2. *Null forms*

A real valued quadratic form Q defined on $\mathbb{R}^{3+1}$ is called a *null form* if for all null vectors $\xi \in \mathbb{R}^{3+1}$, we have $Q(\xi, \xi) = 0$.

We list seven obvious examples of null forms ($\alpha \neq \beta$):

$$\begin{aligned} Q_0(\xi, \eta) &= g(\xi, \eta), \\ Q_{\alpha\beta}(\xi, \eta) &= \xi_\alpha \eta_\beta - \eta_\alpha \xi_\beta, \end{aligned} \quad (2.7)$$

where $\xi, \eta \in \mathbb{R}^{3+1}$ and $\alpha, \beta \in \{0, 1, 2, 3\}$. In fact, we can easily show that the space of all null forms on $\mathbb{R}^{3+1}$ is a real vector space and its dimension is 7. The above seven quadratic forms form a basis for that space. Thus every null form Q can be written as an $\mathbb{R}$-linear combination of basic null forms in (2.7).

Given two scalar functions ϕ, ψ and a null form $Q(\xi, \eta) = Q^{\alpha\beta}\xi_\alpha \eta_\beta$, the expression $Q(\nabla\phi, \nabla\psi)$ means $Q(\nabla\phi, \nabla\psi) = Q^{\alpha\beta}\partial_\alpha\phi\partial_\beta\psi$.

For a given rotational Killing vector field $\Omega \in \mathfrak{so}(3)$, we have

$$\Omega Q(\nabla\phi, \nabla\psi) = Q(\nabla\Omega\phi, \nabla\psi) + Q(\nabla\phi, \nabla\Omega\psi) + \widetilde{Q}(\nabla\phi, \nabla\psi), \quad (2.8)$$

where $\widetilde{Q}$ is another null form. It suffices to check this on the basic null forms in (2.7). In fact, for $\Omega = \Omega_{ij}$, one can check immediately that

$$\Omega_{ij} Q_0(\nabla\phi, \nabla\psi) = Q_0(\nabla\Omega_{ij}\phi, \nabla\psi) + Q_0(\nabla\phi, \nabla\Omega_{ij}\psi),$$
$$\Omega_{ij} Q_{\alpha\beta}(\nabla\phi, \nabla\psi) = Q_{\alpha\beta}(\nabla\Omega_{ij}\phi, \nabla\psi) + Q_{\alpha\beta}(\nabla\phi, \nabla\Omega_{ij}\psi) + \widetilde{Q}(\nabla\phi, \nabla\psi),$$

where

$$\widetilde{Q} = \delta_{i\alpha} Q_{j\beta} - \delta_{j\alpha} Q_{i\beta} + \delta_{j\beta} Q_{i\alpha} - \delta_{i\beta} Q_{j\alpha}.$$

For a vector field X, we denote

$$(Q \circ X)(\nabla\phi, \nabla\psi) = Q(\nabla X\phi, \nabla\psi) + Q(\nabla\phi, \nabla X\psi),$$

and $[Q, X] = XQ - Q \circ X$. We then have

$$[L, Q_0](\nabla\phi, \nabla\psi) = \frac{2}{r}(Q_0(\nabla\phi, \nabla\psi) + L\phi\underline{L}\psi + \underline{L}\phi L\psi),$$
$$[\underline{L}, Q_0](\nabla\phi, \nabla\psi) = -\frac{2}{r}(Q_0(\nabla\phi, \nabla\psi) + L\phi\underline{L}\psi + \underline{L}\phi L\psi),$$
$$[L, Q_{ij}](\nabla\phi, \nabla\psi) = \frac{2}{r}Q_{ij}(\nabla\phi, \nabla\psi) + \frac{1}{2r^2}\{(\underline{L}\phi - L\phi)\Omega_{ij}\psi + (L\psi - \underline{L}\psi)\Omega_{ij}\phi\},$$
$$[\underline{L}, Q_{ij}](\nabla\phi, \nabla\psi) = -\frac{2}{r}Q_{ij}(\nabla\phi, \nabla\psi) + \frac{1}{2r^2}\{(L\phi - \underline{L}\phi)\Omega_{ij}\psi + (\underline{L}\psi - L\psi)\Omega_{ij}\phi\},$$
$$[L, Q_{0i}](\nabla\phi, \nabla\psi) = \frac{1}{r}Q_{0i} - \frac{x_i}{2r^2}(\underline{L}\phi L\psi - L\phi\underline{L}\psi),$$
$$[\underline{L}, Q_{0i}](\nabla\phi, \nabla\psi) = -\frac{1}{r}Q_{0i} - \frac{x_i}{2r^2}(L\phi\underline{L}\psi - \underline{L}\phi L\psi).$$

Schematically, we write the above as

$$\begin{aligned}
[L, Q](\nabla\phi, \nabla\psi) = r^{-1}[&Q(\nabla\phi, \nabla\psi) + L\phi\underline{L}\psi + \underline{L}\phi L\psi + L\phi\nabla\!\!\!/\psi \\
&+ L\psi\nabla\!\!\!/\phi + \underline{L}\phi\nabla\!\!\!/\psi + \underline{L}\psi\nabla\!\!\!/\phi], \\
[\underline{L}, Q](\nabla\phi, \nabla\psi) = r^{-1}[&Q(\nabla\phi, \nabla\psi) + L\phi\underline{L}\psi + \underline{L}\phi L\psi + L\phi\nabla\!\!\!/\psi \\
&+ L\psi\nabla\!\!\!/\phi + \underline{L}\phi\nabla\!\!\!/\psi + \underline{L}\psi\nabla\!\!\!/\phi].
\end{aligned} \tag{2.9}$$

Besides the above algebraic properties of null forms, from the analytic point of view, in a null form $Q(\nabla\phi, \nabla\psi)$, a *bad* component is always coupled to a *good* component. To make a precise statement, we remark that in this paper, all the derivatives of φ involving the outgoing direction L (for example $L\varphi$ and $L\nabla\!\!\!/\varphi$) are *bad* since in the L^∞ norm, their size is comparable to $\delta^{-1/2}$ which is *large*; other derivatives have size at least as good as $\delta^{1/4}$ which is *small* and it is in this sense that they are good components. To see why a bad component is coupled to a good component, we use the null frame $\{e_1, e_2, e_3 = \underline{L}, e_4 = L\}$ to express the null form as follows:

$$Q(\nabla\phi,\nabla\psi) = Q^{43}L\phi\underline{L}\psi + Q^{34}\underline{L}\phi L\psi + Q^{4a}L\phi\nabla\!\!\!/_i\psi + Q^{3a}\underline{L}\phi\nabla\!\!\!/_a\psi \\ + Q^{a4}\nabla\!\!\!/_a\phi L\psi + Q^{a4}\nabla\!\!\!/_a\phi\underline{L}\psi + Q^{ab}\nabla\!\!\!/_a\phi\nabla\!\!\!/_b\psi. \tag{2.10}$$

Once again, to prove (2.10), it suffices to check it for basic null forms in (2.7).

In particular, (2.10) shows that $L\phi\cdot L\psi$ is forbidden, which is a product of two bad components. We also observe that the coefficients in (2.10) are bounded by a universal constant. Therefore, in applications, we shall bound the null form pointwise as follows:

$$|Q(\nabla\phi,\nabla\psi)| \lesssim |L\phi|\,|\underline{L}\psi| + |\underline{L}\phi|\,|L\psi| + |\nabla\!\!\!/\phi|\,|\nabla\!\!\!/\psi| \\ + (|L\phi| + |\underline{L}\phi|)|\nabla\!\!\!/\psi| + |\nabla\!\!\!/\phi|(|L\psi| + |\underline{L}\psi|). \tag{2.11}$$

2.3. *Sobolev and Gronwall's inequalities*

We first recall Sobolev inequalities on C_u, $\underline{C}_{\underline{u}}$ and $S_{\underline{u},u}$.

For any real valued function ϕ and $q \geq -1/2$, we have

$$\begin{aligned} |u|^{1/2}\|\phi\|_{L^4(S_{\underline{u},u})} &\lesssim \|L\phi\|^{1/2}_{L^2(C_u)}(\|\phi\|^{1/2}_{L^2(C_u)} + |u|^{1/2}\|\nabla\!\!\!/\phi\|^{1/2}_{L^2(C_u)}),\\ |u|^q\|\phi\|_{L^4(S_{\underline{u},u})} &\lesssim |u_0|^q\|\phi\|_{L^4(S_{\underline{u},u_0})}\\ &\quad + \big\||u|^q\underline{L}\phi\big\|^{1/2}_{L^2(\underline{C}_{\underline{u}})}\big(\big\||u|^{q-1}\phi\big\|^{1/2}_{L^2(\underline{C}_{\underline{u}})} + \big\||u|^q\nabla\!\!\!/\phi\big\|^{1/2}_{L^2(\underline{C}_{\underline{u}})}\big),\\ \|\phi\|_{L^\infty(S_{\underline{u},u})} &\lesssim |u|^{-1/2}\|\phi\|_{L^4(S_{\underline{u},u})} + |u|^{1/2}\|\nabla\!\!\!/\phi\|_{L^4(S_{\underline{u},u})}. \end{aligned} \tag{2.12}$$

The proof is based on the standard isoperimeric inequality on the unit sphere. We refer the reader to [2] for a proof.[2] We remark that in the first inequality, we assume that $\phi = 0$ on $\underline{C}_0$. This assumption is always valid when we apply the inequality in this paper. We also need a variant of the second inequality. In fact, taking $q = 0$, we derive

$$\|\phi\|_{L^4(S_{\underline{u},u})} \lesssim \|\phi\|_{L^4(S_{\underline{u},u_0})} + \|\underline{L}\phi\|^{1/2}_{L^2(\underline{C}_{\underline{u}})}\big(\big\||u|^{-1}\phi\big\|^{1/2}_{L^2(\underline{C}_{\underline{u}})} + \|\nabla\!\!\!/\phi\|^{1/2}_{L^2(\underline{C}_{\underline{u}})}\big). \tag{2.13}$$

Next, we recall the standard Gronwall inequality. Let $\phi(t)$ be a nonnegative function defined on an interval I with $t_0 \in I$. If ϕ satisfies the ordinary differential inequality

$$\frac{d}{dt}\phi \leq a\cdot\phi + b$$

with two nonnegative functions $a, b \in L^1(I)$, then for all $t \in I$,

$$\phi(t) \leq e^{A(t)}\left(\phi(t_0) + \int_{t_0}^t e^{-A(\tau)}b(\tau)\,d\tau\right),$$

where $A(t) = \int_{t_0}^t a(\tau)\,d\tau$. The proof is straightforward. And there is another version of Gronwall's inequality [8], which will be useful in the proof.

[2] Recall that in the current situation, $|u| \sim r$.

Lemma 2.1. *Let $f(x,y)$, $g(x,y)$ be positive functions defined in the rectangle $0 \leq x \leq x_0$, $0 \leq y \leq y_0$ which satisfy*

$$f(x,y) + g(x,y) \lesssim J + a\int_0^x f(x',y)\,dx' + b\int_0^y g(x,y')\,dy'$$

for some nonnegative constants a, b and J. Then, for all $0 \leq x \leq x_0$ and $0 \leq y \leq y_0$,

$$f(x,y), g(x,y) \lesssim J e^{ax+by}.$$

3. Initial data for Regions 1 and 2

Let $\widetilde{C}_{u_0}$ be a truncated light-cone defined by

$$\widetilde{C}_{u_0} = \{p \in \mathbb{R}^{3+1} \mid u(p) = u_0,\ u_0 \leq \underline{u}(p) \leq \delta\},$$

and $C_{u_0}^{[0,\delta]}$ be a truncated light-cone defined by

$$C_{u_0}^{[0,\delta]} = \{p \in \mathbb{R}^{3+1} \mid u(p) = u_0,\ 0 \leq \underline{u}(p) \leq \delta\}.$$

First of all, we require that the data of (1.1) is trivial on $\widetilde{C}_{u_0} - C_{u_0}^{[0,\delta]}$, i.e.,

$$\varphi(x) \equiv 0 \quad \text{for all } x \in \widetilde{C}_{u_0} - C_{u_0}^{[0,\delta]}.$$

Therefore, according to the weak Huygens principle, the solution of (1.1) is zero in Region 1, i.e. in the future domain of dependence of $\widetilde{C}_{u_0} - C_{u_0}^{[0,\delta]}$. That is, $\varphi(x) \equiv 0$ if $\underline{u}(x) \leq 0$ and $u(x) \geq u_0$. In particular, $\varphi \equiv 0$ on $\underline{C}_0$ up to infinite order.

Secondly, we prescribe φ on $C_{u_0}^{[0,\delta]}$ to be

$$\varphi(\underline{u}, u_0, \theta) = \frac{\delta^{1/2}}{|u_0|}\psi_0(\underline{u}/\delta, \theta), \tag{3.1}$$

where $\psi_0 : (0,1)\times\mathbb{S}^2 \to \mathbb{R}$ is a *fixed* compactly supported smooth function with L^2 norm approximately E_0. The factor $1/|u_0|$ is natural because it manifests the correct decay for free waves.

The data in the above form is called a *short pulse datum*, a name invented by Christodoulou in [2]. In his work, he prescribes the shear (more precisely, the conformal geometry) of the initial null hypersurface in a form similar to (3.1). The shear in the situation of [2] is exactly the initial data for the Einstein vacuum equation.

We remark that the above data is not small in the following sense: the derivative of the data can be extremely *large* if δ is sufficiently small. In fact, this can be easily observed once we take $\partial/\partial\underline{u}$ derivatives. We also remark that the energy flux of the data is approximately E_0 on $C_{u_0}^{[0,\delta]}$, which is bounded away from 0.

For most of the computations, we need commutator formulas and we collect them as follows:

$$\begin{aligned} &[\Omega, \nabla\!\!\!/] = 0, \quad [L, \nabla\!\!\!/] = 0, \quad [\underline{L}, \nabla\!\!\!/] = 0, \\ &[\Box, \Omega] = 0, \quad [L, \Omega] = 0, \quad [\underline{L}, \Omega] = 0, \\ &[\Box, L] = \frac{1}{r^2}(L - \underline{L}) + \frac{2}{r}\Delta\!\!\!/, \quad [\Box, \underline{L}] = \frac{1}{r^2}(\underline{L} - L) - \frac{2}{r}\Delta\!\!\!/. \end{aligned} \tag{3.2}$$

If we commute Ω with (1.1) n times, using (3.2), we have[3]

$$\Box\Omega^n\varphi = \sum_{p+q\leq n} Q(\nabla\Omega^p\varphi, \nabla\Omega^q\varphi), \tag{3.3}$$

and the Q's may be different.

We commute L, Ω with (1.1) n times, using (3.2), to derive

$$\begin{aligned} \Box L\Omega^n\varphi = &\sum_{p+q\leq n} Q(\nabla L\Omega^p\varphi, \nabla\Omega^q\varphi) + \sum_{p+q\leq n} [L, Q](\nabla\Omega^p\varphi, \nabla\Omega^q\varphi) \\ &- \frac{1}{r^2}(\underline{L}\Omega^n\varphi - L\Omega^n\varphi) + \frac{2}{r}\Delta\!\!\!/\Omega^n\varphi. \end{aligned} \tag{3.4}$$

We commute $\underline{L}$, Ω with (1.1) n times, using (3.2), to derive

$$\begin{aligned} \Box \underline{L}\Omega^n\varphi = &\sum_{p+q\leq n} Q(\nabla \underline{L}\Omega^p\varphi, \nabla\Omega^q\varphi) + \sum_{p+q\leq n} [\underline{L}, Q](\nabla\Omega^p\varphi, \nabla\Omega^q\varphi) \\ &+ \frac{1}{r^2}(\underline{L}\Omega^n\varphi - L\Omega^n\varphi) - \frac{2}{r}\Delta\!\!\!/\Omega^n\varphi. \end{aligned} \tag{3.5}$$

We remark that, thanks to (2.9), we have the following pointwise estimate which gains a factor of u:

$$\begin{aligned} &|[L, Q](\nabla\phi, \nabla\psi)| + |[\underline{L}, Q](\nabla\phi, \nabla\psi)| \lesssim \frac{1}{|u|} \times \\ &(|\nabla\!\!\!/\phi|\,|\nabla\!\!\!/\psi| + |\nabla\!\!\!/\phi|\,|L\psi| + |\nabla\!\!\!/\psi|\,|L\phi| + |\nabla\!\!\!/\phi|\,|\underline{L}\psi| + |\nabla\!\!\!/\psi|\,|\underline{L}\phi| + |L\phi|\,|\underline{L}\psi| + |L\psi|\,|\underline{L}\phi|). \end{aligned} \tag{3.6}$$

We now derive some preliminary estimates for the data on C_{u_0}. In view of (3.1), by taking derivatives in the L or $\nabla\!\!\!/$ direction, we have

$$\|L\varphi\|_{L^\infty(C_{u_0})} \leq \delta^{-1/2}|u_0|^{-1}, \quad \|\nabla\!\!\!/\varphi\|_{L^\infty(C_{u_0})} \leq \delta^{1/2}|u_0|^{-2}.$$

In fact, by taking L or $\nabla\!\!\!/$ derivatives consecutively, for $k \in \mathbb{Z}_{\geq 0}$, we immediately obtain

$$\begin{aligned} \|L\nabla\!\!\!/^k\varphi\|_{L^\infty(C_{u_0})} &\lesssim_k \delta^{-1/2}|u_0|^{-k-1}, \\ \|\nabla\!\!\!/^{k+1}\varphi\|_{L^\infty(C_{u_0})} &\lesssim_k \delta^{1/2}|u_0|^{-k-2}, \\ \|L^2\nabla\!\!\!/^k\varphi\|_{L^\infty(C_{u_0})} &\lesssim_k \delta^{-3/2}|u_0|^{-k-1}. \end{aligned} \tag{3.7}$$

[3] We shall ignore the numerical constants since they are irrelevant in this context.

With the use of the original equation (1.1), one can further derive L^∞ estimates for derivatives of φ involving $\underline{L}$ directions. For this purpose, we first rewrite (1.1) in terms of the null frame:

$$\begin{aligned}-L\underline{L}\varphi + \Delta\!\!\!/\varphi + r^{-1}(L\varphi - \underline{L}\varphi)\\ = 2Q^{34}\underline{L}\varphi L\varphi + 2Q^{3a}\underline{L}\varphi\nabla\!\!\!/_a\varphi + 2Q^{4a}L\varphi\nabla\!\!\!/_a\varphi + Q^{ab}\nabla\!\!\!/_a\varphi\nabla\!\!\!/_b\varphi.\end{aligned} \tag{3.8}$$

To estimate $\underline{L}\varphi$, we observe that (3.8) can be written as an ODE for $\underline{L}\varphi$ as follows:[4]

$$L(\underline{L}\varphi) = a \cdot \underline{L}\varphi + b,$$

where

$$\begin{aligned}a &= -(r^{-1} + 2Q^{34}L\varphi + 2Q^{3a}\nabla\!\!\!/_a\varphi),\\ b &= r^{-1}L\varphi + \Delta\!\!\!/\varphi - 2Q^{4a}L\varphi\nabla\!\!\!/_a\varphi - Q^{ab}\nabla\!\!\!/_a\varphi\nabla\!\!\!/_b\varphi.\end{aligned}$$

According to (3.7), we have

$$\|a\|_{L^\infty(C_{u_0})} \lesssim \delta^{-1/2}|u_0|^{-1}, \qquad \|b\|_{L^\infty(C_{u_0})} \lesssim \delta^{-1/2}|u_0|^{-2}.$$

We also have

$$L|\underline{L}\varphi| \le |L(\underline{L}\varphi)| \le |a| \cdot |\underline{L}\varphi| + |b|.$$

Since $\underline{L}\varphi \equiv 0$ when $\underline{u} = 0$, by Gronwall's inequality (see Section 2.3) we have

$$|\underline{L}\varphi(u)| \lesssim e^{A(\underline{u})}\int_0^\delta e^{-A(\tau)}b(\tau)\,d\tau \lesssim \delta^{1/2}|u_0|^{-2}.$$

According to the estimate on a, for all $\tau \le \underline{u}$ we have $|A(\tau)| \lesssim \delta^{1/2}$, therefore

$$\|\underline{L}\varphi\|_{L^\infty(C_{u_0})} \lesssim \delta^{1/2}|u_0|^{-2}. \tag{3.9}$$

To estimate $\underline{L}\nabla\!\!\!/\varphi$, we first commute (1.1) with Ω, that is, take $n = 1$ in (3.3). In the null frame, we rewrite the equation as

$$-L\underline{L}\Omega\varphi + \Delta\!\!\!/\Omega\varphi + r^{-1}(L\Omega\varphi - \underline{L}\Omega\varphi) = 2Q_1(\nabla\Omega\varphi, \nabla\varphi) + Q_2(\nabla\phi, \nabla\phi).$$

We then proceed as above to obtain, by Gronwall's inequality,

$$\|\underline{L}\Omega\varphi\|_{L^\infty(C_{u_0})} \lesssim \delta^{1/2}|u_0|^{-2}.$$

Therefore, according to (1.3),

$$\|\underline{L}\nabla\!\!\!/\varphi\|_{L^\infty(C_{u_0})} \lesssim \delta^{1/2}|u_0|^{-3}. \tag{3.10}$$

Similarly, we can commute (1.1) with two and three Ω's to obtain

$$\|\underline{L}\nabla\!\!\!/^2\varphi\|_{L^\infty(C_{u_0})} \lesssim \delta^{1/2}|u_0|^{-4}, \qquad \|\underline{L}\nabla\!\!\!/^3\varphi\|_{L^\infty(C_{u_0})} \lesssim \delta^{1/2}|u_0|^{-5}. \tag{3.11}$$

[4] Since the exact numerical constants are irrelevant, we shall ignore the constants appearing in the coefficients.

Remark (Key: Relaxation of estimates). To obtain existence theorems for (1.1), we have to derive certain estimates on φ (as well as on its derivatives). Those estimates must be valid on the initial hypersurface and they should propagate along the evolution to later null hypersurfaces. For this purpose, we shall use a slightly weaker version of estimates for $\nabla\!\!\!/^k\varphi$ than those in (3.7), namely,

$$\|\nabla\!\!\!/^{k+1}\varphi\|_{L^\infty(C_{u_0})} \lesssim_k |u_0|^{-k-3/2}.$$

One expects it should be easier to prove the relaxed estimates propagating along the flow of (1.1) than the original ones in (3.7). This is precisely the *relaxation of the propagation estimates* mentioned in the introduction.

To summarize, on the initial null hypersurface C_{u_0}, with short pulse data (3.1), for up to four derivatives of φ (this is the minimal number of derivatives we need for a bootstrap argument, see next section), we have the relaxed L^∞ estimates

$$\begin{aligned} \|L\nabla\!\!\!/^k\varphi\|_{L^\infty(C_{u_0})} &\lesssim \delta^{-1/2}|u_0|^{-k-1},\\ \|\nabla\!\!\!/^{k+1}\varphi\|_{L^\infty(C_{u_0})} &\lesssim |u_0|^{-3/2-k},\\ \|\underline{L}\nabla\!\!\!/^k\varphi\|_{L^\infty(C_{u_0})} &\lesssim \delta^{1/2}|u_0|^{-2-k}, \end{aligned} \tag{3.12}$$

for $k = 0, 1, 2, 3$, as well as

$$\begin{aligned} \|L^2\varphi\|_{L^\infty(C_{u_0})} &\lesssim \delta^{-3/2}|u_0|^{-1},\\ \|L^2\nabla\!\!\!/\varphi\|_{L^\infty(C_{u_0})} &\lesssim \delta^{-3/2}|u_0|^{-2},\\ \|L^2\nabla\!\!\!/^2\varphi\|_{L^\infty(C_{u_0})} &\lesssim \delta^{-3/2}|u_0|^{-3}. \end{aligned} \tag{3.13}$$

For wave equations, we expect that the information propagating along evolution should be more or less contained in the energies of the solutions, i.e. in the L^2 norms of derivatives of φ. This heuristic leads to consider the L^2 norms of the data on C_{u_0}.

According to the L^∞ estimates in (3.12) and (3.13), we immediately obtain the following L^2 estimates (observe that the area of $C_{u_0}^{[0,\delta]}$ is comparable to δu_0^2):

$$\|L\nabla\!\!\!/^k\varphi\|_{L^2(C_{u_0})} \lesssim |u_0|^{-k}, \quad \|\nabla\!\!\!/^{k+1}\varphi\|_{L^2(C_{u_0})} \lesssim \delta^{1/2}|u_0|^{-1/2-k}, \tag{3.14}$$

for $k = 0, 1, 2, 3$, and

$$\begin{aligned} \|L^2\varphi\|_{L^2(C_{u_0})} &\lesssim \delta^{-1},\\ \|L^2\nabla\!\!\!/\varphi\|_{L^2(C_{u_0})} &\lesssim \delta^{-1}|u_0|^{-1},\\ \|L^2\nabla\!\!\!/^2\varphi\|_{L^2(C_{u_0})} &\lesssim \delta^{-1}|u_0|^{-2}. \end{aligned} \tag{3.15}$$

We remark that those L^2 estimates are also relaxed estimates. In the next section, we shall show that, up to a universal constant, the estimates in (3.14) and (3.15) (the parameter u_0 will be replaced by u) will hold on all later outgoing null hypersurfaces C_u where $u_0 \leq u \leq -1$ provided that the solution of (1.1) can be constructed up to C_u.

4. A priori estimates for up to four derivatives

This section is the technical heart of the paper. We assume that there exists a solution of (1.1) defined on the domain $\mathcal{D}_{u,\underline{u}}$ which is enclosed by the null hypersurfaces C_u, $\underline{C}_{\underline{u}}$, C_{u_0} and $\underline{C}_0$. The goal is to show that estimates (3.14) and (3.15), which are valid on C_{u_0}, also hold on C_u.

We slightly abuse the notation: we use C_u to denote $C_u^{[0,\underline{u}]}$ (i.e. $\underline{u}' \in [0,\underline{u}]$) and $\underline{C}_{\underline{u}}$ to denote $\underline{C}_{\underline{u}}^{[u_0,u]}$. We now define a family of energy norms as follows:

$$\begin{aligned}
E_1(u,\underline{u}) &= \|L\varphi\|_{L^2(C_u)} + \delta^{-1/2}|u|^{1/2}\|\nabla\!\!\!/\varphi\|_{L^2(C_u)},\\
\underline{E}_1(u,\underline{u}) &= \|\nabla\!\!\!/\varphi\|_{L^2(\underline{C}_{\underline{u}})} + \delta^{-1/2}|u|^{1/2}\|\underline{L}\varphi\|_{L^2(\underline{C}_{\underline{u}})},\\
E_2(u,\underline{u}) &= |u|\|L\nabla\!\!\!/\varphi\|_{L^2(C_u)} + \delta^{-1/2}|u|^{3/2}\|\nabla\!\!\!/^2\varphi\|_{L^2(C_u)},\\
\underline{E}_2(u,\underline{u}) &= \big\||u|\nabla\!\!\!/^2\varphi\big\|_{L^2(\underline{C}_{\underline{u}})} + \delta^{-1/2}|u|^{1/2}\big\||u|\underline{L}\nabla\!\!\!/\varphi\big\|_{L^2(\underline{C}_{\underline{u}})},\\
E_3(u,\underline{u}) &= |u|^2\|L\nabla\!\!\!/^2\varphi\|_{L^2(C_u)} + \delta^{-1/2}|u|^{5/2}\|\nabla\!\!\!/^3\varphi\|_{L^2(C_u)},\\
\underline{E}_3(u,\underline{u}) &= \big\||u|^2\nabla\!\!\!/^3\varphi\big\|_{L^2(\underline{C}_{\underline{u}})} + \delta^{-1/2}|u|^{1/2}\big\||u|^2\underline{L}\nabla\!\!\!/^2\varphi\big\|_{L^2(\underline{C}_{\underline{u}})},\\
E_4(u,\underline{u}) &= |u|^3\|L\nabla\!\!\!/^3\varphi\|_{L^2(C_u)} + \delta^{-1/2}|u|^{7/2}\|\nabla\!\!\!/^3\varphi\|_{L^2(C_u)},\\
\underline{E}_4(u,\underline{u}) &= \big\||u|^3\nabla\!\!\!/^4\varphi\big\|_{L^2(\underline{C}_{\underline{u}})} + \delta^{-1/2}|u|^{1/2}\big\||u|^3\underline{L}\nabla\!\!\!/^3\varphi\big\|_{L^2(\underline{C}_{\underline{u}})}.
\end{aligned}\tag{4.1}$$

We also need another family of norms which involves at least two null derivatives. They are defined as follows:

$$\begin{aligned}
F_2(u,\underline{u}) &= \delta\|L^2\varphi\|_{L^2(C_u)},\\
\underline{F}_2(u,\underline{u}) &= |u|^{1/2}\|\underline{L}^2\varphi\|_{L^2(\underline{C}_{\underline{u}})},\\
F_3(u,\underline{u}) &= \delta|u|\|L^2\nabla\!\!\!/\varphi\|_{L^2(C_u)},\\
\underline{F}_3(u,\underline{u}) &= |u|^{1/2}\big\||u|\underline{L}^2\nabla\!\!\!/\varphi\big\|_{L^2(\underline{C}_{\underline{u}})},\\
F_4(u,\underline{u}) &= \delta|u|^2\|L^2\nabla\!\!\!/^2\varphi\|_{L^2(C_u)},\\
\underline{F}_4(u,\underline{u}) &= |u|^{1/2}\big\||u|^2\underline{L}^2\nabla\!\!\!/^2\varphi\big\|_{L^2(\underline{C}_{\underline{u}})}.
\end{aligned}\tag{4.2}$$

We shall prove the following propagation estimates:

Main A Priori Estimates. *Assume that there exists a solution of* (1.1) *defined on the domain* $\mathcal{D}_{u^*,\underline{u}^*}$ *where* $u^* \leq -1$ *and* $0 \leq \underline{u}^* \leq \delta$. *If* δ *is sufficiently small, then for all initial data of* (1.1) *and all* $I_4 \in \mathbb{R}_{>0}$ *which satisfy*

$$\begin{aligned}
E_1(u_0,\delta) + E_2(u_0,\delta) + E_3(u_0,\delta) + E_4(u_0,\delta)&\\
+ F_2(u_0,\delta) + F_3(u_0,\delta) + F_4(u_0,\delta) &\leq I_4,
\end{aligned}\tag{4.3}$$

there is a constant $C(I_4)$, depending only on I_4 (in particular, not on δ and u_0), such that for all $u \le u^*$ and all $0 \le \underline{u} \le \underline{u}^*$,

$$\sum_{i=1}^{4}[E_i(u,\underline{u}) + \underline{E}_i(u,\underline{u})] + \sum_{j=2}^{4}[F_j(u,\underline{u}) + \underline{F}_j(u,\underline{u})] \le C(I_4). \tag{4.4}$$

The subscript 4 in I_4 denotes the number of derivatives used in the energy norms.

4.1. Bootstrap argument

To prove the Main A Priori Estimates we will perform a standard bootstrap argument. We assume that

$$\sum_{i=1}^{4}[E_i(u',\underline{u}') + \underline{E}_i(u',\underline{u}')] + \sum_{j=2}^{4}[F_j(u',\underline{u}') + \underline{F}_j(u',\underline{u}')] \le M \tag{4.5}$$

for all $u' \in [u_0, u]$ and $\underline{u}' \in [0, \underline{u}]$, where M is a sufficiently large constant. Since we have assumed the existence of the solution up to C_u and $\underline{C}_{\underline{u}}$, we can always choose such an M which may depend on φ. At the end of the current section, we will show that we can actually choose M in such a way that it depends only on the norm of the initial data but not on the profile φ. This will yield the Main A Priori Estimates.

4.2. Preliminary estimates

Under the bootstrap assumption (4.5), we first derive L^∞ estimates for first order derivatives of φ. As a byproduct, we will also obtain L^4 estimates for two derivatives of φ. For this purpose, we shall repeatedly use the Sobolev inequalities stated in Section 2.3. We remark that the Sobolev inequalities (2.12) and (2.13) have been introduced previously for the hypersurfaces C_u and $\underline{C}_{\underline{u}}$. In the rest of the paper, we indeed use Sobolev inequalities for the truncated hypersurfaces $C_u^{[0,\underline{u}]}$ and $\underline{C}_{\underline{u}}^{[u_0,u]}$. For simplicity, when we apply Sobolev inequalities, we shall use C_u and $\underline{C}_{\underline{u}}$ as shorthand for $C_u^{[0,\underline{u}]}$ and $\underline{C}_{\underline{u}}^{[u_0,u]}$.

We start with $L\varphi$. According to Sobolev inequalities, we have

$$\begin{aligned}|u|^{1/2}\|L\varphi\|_{L^4(S_{\underline{u},u})} &\lesssim \|L^2\varphi\|_{L^2(C_u)}^{1/2}(\|L\varphi\|_{L^2(C_u)}^{1/2} + |u|^{1/2}\|L\nabla\!\!\!\!/\varphi\|_{L^2(C_u)}^{1/2})\\ &\lesssim (\delta^{-1}M)^{1/2}(M^{1/2} + |u|^{1/2}(|u|^{-1}M)^{1/2}).\end{aligned}$$

Hence,

$$\|L\varphi\|_{L^4(S_{\underline{u},u})} \lesssim \delta^{-1/2}|u|^{-1/2}M. \tag{4.6}$$

Similarly,

$$\begin{aligned}|u|^{1/2}\|L\nabla\!\!\!\!/\varphi\|_{L^4(S_{\underline{u},u})} &\lesssim \|L^2\nabla\!\!\!\!/\varphi\|_{L^2(C_u)}^{1/2}(\|L\nabla\!\!\!\!/\varphi\|_{L^2(C_u)}^{1/2} + |u|^{1/2}\|L\nabla\!\!\!\!/^2\varphi\|_{L^2(C_u)}^{1/2})\\ &\lesssim (\delta^{-1}|u|^{-1}M)^{1/2}\big((|u|^{-1}M)^{1/2} + |u|^{1/2}(|u|^{-2}M)^{1/2}\big).\end{aligned}$$

Thus,

$$\|L\nabla\!\!\!/\varphi\|_{L^4(S_{\underline{u},u})} \lesssim \delta^{-1/2}|u|^{-3/2}M. \tag{4.7}$$

Combining (4.6) and (4.7), we obtain

$$\|L\varphi\|_{L^\infty} \lesssim |u|^{-1/2}\|L\varphi\|_{L^4(S_{\underline{u},u})} + |u|^{1/2}\|L\nabla\!\!\!/\varphi\|_{L^4(S_{\underline{u},u})} \lesssim \delta^{-1/2}|u|^{-1}M. \tag{4.8}$$

We now treat $\nabla\!\!\!/\varphi$. According to Sobolev inequalities, we have

$$\begin{aligned}|u|^{1/2}\|\nabla\!\!\!/\varphi\|_{L^4(S_{\underline{u},u})} &\lesssim \|L\nabla\!\!\!/\varphi\|_{L^2(C_u)}^{1/2}(\|\nabla\!\!\!/\varphi\|_{L^2(C_u)}^{1/2} + |u|^{1/2}\|\nabla\!\!\!/^2\varphi\|_{L^2(C_u)}^{1/2})\\ &\lesssim (|u|^{-1}M)^{1/2}\big((\delta^{1/2}|u|^{-1/2}M)^{1/2} + |u|^{1/2}(\delta^{1/2}|u|^{-3/2}M)^{1/2}\big).\end{aligned}$$

Thus,

$$\|\nabla\!\!\!/\varphi\|_{L^4(S_{\underline{u},u})} \lesssim \delta^{1/4}|u|^{-5/4}M. \tag{4.9}$$

Similarly,

$$\|\nabla\!\!\!/^2\varphi\|_{L^4(S_{\underline{u},u})} \lesssim \delta^{1/4}|u|^{-9/4}M. \tag{4.10}$$

Combining (4.9) and (4.10), we obtain

$$\|\nabla\!\!\!/\varphi\|_{L^\infty} \lesssim |u|^{-1/2}\|\nabla\!\!\!/\varphi\|_{L^4(S_{\underline{u},u})} + |u|^{1/2}\|\nabla\!\!\!/^2\varphi\|_{L^4(S_{\underline{u},u})} \lesssim \delta^{1/4}|u|^{-7/4}M. \tag{4.11}$$

It remains to estimate $\underline{L}\varphi$. According to (2.13), we have

$$\|\underline{L}\varphi\|_{L^4(S_{\underline{u},u})} \lesssim \|\underline{L}\varphi\|_{L^4(S_{\underline{u},u_0})} + \|\underline{L}^2\varphi\|_{L^2(\underline{C}_{\underline{u}})}^{1/2}\big(\big\||u|^{-1}\underline{L}\varphi\big\|_{L^2(\underline{C}_{\underline{u}})}^{1/2} + \|\underline{L}\nabla\!\!\!/\varphi\|_{L^2(\underline{C}_{\underline{u}})}^{1/2}\big).$$

Since the first two terms appear in the bootstrap assumptions (or on the initial hypersurface C_{u_0}), we can control them exactly as before. For the last term, we can restrict the inequality to the part of $\underline{C}_{\underline{u}}$ where the affine parameter u' of $\underline{L}$ is in $[u_0, u]$. Thus, we have

$$\big\||u'|^{-1}\underline{L}\varphi(u', \underline{u}, \theta)\big\|_{L^2(\underline{C}_{\underline{u}})} \le |u|^{-1}\|\underline{L}\varphi\|_{L^2(\underline{C}_{\underline{u}})}.$$

The right hand side is again a term in (4.5). This allows us to derive

$$\|\underline{L}\varphi\|_{L^4(S_{\underline{u},u})} \lesssim \delta^{1/4}|u|^{-1}M. \tag{4.12}$$

Similarly,

$$\|\underline{L}\nabla\!\!\!/\varphi\|_{L^4(S_{\underline{u},u})} \lesssim \delta^{1/4}|u|^{-2}M. \tag{4.13}$$

We combine (4.12) and (4.13) to derive

$$\|\underline{L}\varphi\|_{L^\infty} \lesssim \delta^{1/4}|u|^{-3/2}M. \tag{4.14}$$

In the same way, we can derive L^4 and L^∞ estimates for two derivatives. We summarize all the estimates in the following proposition.

Proposition 4.1. *Under the bootstrap assumption* (4.5)*, we have*

$$\delta^{1/2}|u|\,\|L\varphi\|_{L^\infty}+\delta^{-1/4}|u|^{7/4}\|\nabla\!\!\!/\varphi\|_{L^\infty}+\delta^{-1/4}|u|^{3/2}\|\underline{L}\varphi\|_{L^\infty}$$
$$+\delta^{1/2}|u|^2\|L\nabla\!\!\!/\varphi\|_{L^\infty}+\delta^{-1/4}|u|^{11/4}\|\nabla\!\!\!/^2\varphi\|_{L^\infty}+\delta^{-1/4}|u|^{5/2}\|\underline{L}\nabla\!\!\!/\varphi\|_{L^\infty}\lesssim M,$$
$$\delta^{1/2}|u|^{5/2}\|L\nabla\!\!\!/^2\varphi\|_{L^4(S_{\underline{u},u})}+\delta^{-1/4}|u|^{13/4}\|\nabla\!\!\!/^3\varphi\|_{L^4(S_{\underline{u},u})}+\delta^{-1/4}|u|^3\|\underline{L}\nabla\!\!\!/^2\varphi\|_{L^4(S_{\underline{u},u})}$$
$$+\delta^{1/2}|u|^{3/2}\|L\nabla\!\!\!/\varphi\|_{L^4(S_{\underline{u},u})}+\delta^{-1/4}|u|^{9/4}\|\nabla\!\!\!/^2\varphi\|_{L^4(S_{\underline{u},u})}+\delta^{-1/4}|u|^2\|\underline{L}\nabla\!\!\!/\varphi\|_{L^4(S_{\underline{u},u})}$$
$$+\delta^{1/2}|u|^{1/2}\|L\varphi\|_{L^4(S_{\underline{u},u})}+\delta^{-1/4}|u|^{5/4}\|\nabla\!\!\!/\varphi\|_{L^4(S_{\underline{u},u})}+\delta^{-1/4}|u|\|\underline{L}\varphi\|_{L^4(S_{\underline{u},u})}\lesssim M.$$

We observe that the L^∞ estimate of $\underline{L}\varphi$ (of order $\delta^{1/4}|u|^{-3/2}$) is certainly worse than the initial estimate of $\underline{L}\varphi$ on C_{u_0} (which is of order $\delta^{1/2}|u_0|^{-2}$). To rectify this loss, we derive an L^2 estimate of $\underline{L}\varphi$ on C_u (instead of $\underline{C}_{\underline{u}}$ appearing in the definition of $\underline{E}_1(u,\underline{u})$).

Lemma 4.2. *Under the bootstrap assumption* (4.5)*, if $\delta^{1/2}M$ is sufficiently small, then*

$$\|\underline{L}\varphi\|_{L^2(C_u)}\lesssim\delta|u|^{-1}M.$$

Proof. We multiply the main equation (3.8) by $\underline{L}\varphi$ and integrate on C_u. In view of the fact that $\underline{L}\varphi\equiv 0$ on $S_{0,u}$ as well as (2.11), this leads to

$$\begin{aligned}\int_{S_{\underline{u},u}}|\underline{L}\varphi|^2\lesssim&\int_{C_u^{\underline{u}}}(r^{-1}|L\varphi|\,|\underline{L}\varphi|+|\Delta\!\!\!/\varphi|\,|\underline{L}\varphi|+|\nabla\!\!\!/\varphi|\,|\underline{L}\varphi|^2)\\&+\int_{C_u^{\underline{u}}}(|L\varphi|\,|\underline{L}\varphi|^2+|L\varphi|\,|\nabla\!\!\!/\varphi|\,|\underline{L}\varphi|+|\nabla\!\!\!/\varphi|^2|\underline{L}\varphi|),\end{aligned}\tag{4.15}$$

where the integral $\int_{C_u^{\underline{u}}}$ means $\int_0^{\underline{u}}\int_{S_{\underline{u}',u}}d\underline{u}'$. Let $f(\underline{u})^2=\int_{C_u^{\underline{u}}}(\underline{L}\varphi)^2$. We now estimate the terms on the right hand side of (4.15) one by one. For the first two terms we have

$$\int_{C_u^{\underline{u}}}r^{-1}|L\varphi|\,|\underline{L}\varphi|\lesssim|u|^{-1}f(\underline{u})M,\qquad\int_{C_u^{\underline{u}}}|\Delta\!\!\!/\varphi|\,|\underline{L}\varphi|\lesssim\delta^{1/2}|u|^{-3/2}f(\underline{u})M.$$

For the next two terms, we have

$$\int_{C_u^{\underline{u}}}|\nabla\!\!\!/\varphi|\,|\underline{L}\varphi|^2\lesssim\delta^{1/4}|u|^{-7/4}f(\underline{u})^2M,\qquad\int_{C_u^{\underline{u}}}|L\varphi|\,|\underline{L}\varphi|^2\lesssim\delta^{-1/2}|u|^{-1}f(\underline{u})^2M.$$

For the last two terms, we have

$$\int_{C_u^{\underline{u}}}|L\varphi|\,|\nabla\!\!\!/\varphi|\,|\underline{L}\varphi|\lesssim\delta^{1/4}|u|^{-7/4}f(\underline{u})M,\qquad\int_{C_u^{\underline{u}}}|\nabla\!\!\!/\varphi|^2|\underline{L}\varphi|\lesssim\delta^{3/4}|u|^{-9/4}f(\underline{u})M.$$

Back to (4.15), we have

$$\frac{d}{d\underline{u}}f(\underline{u})^2\lesssim M(\delta^{-1/2}|u|^{-1}f(\underline{u})^2+|u|^{-1}f(\underline{u})),$$

We then integrate on C_u to derive

$$f(\underline{u})\lesssim\frac{M}{|u|}\delta^{1/2}f(\underline{u})+\delta\frac{M}{|u|}.$$

Since $\delta^{1/2}M$ is sufficiently small, this completes the proof. □

4.3. Energy estimates for $E_k(u,\underline{u})$ and $\underline{E}_k(u,\underline{u})$ when $k \le 3$

We commute Ω^i with (1.1) using (3.3). We apply the scheme of Section 2.1 for this equation where we take $\phi = \Omega^i\varphi$, $i = 0, 1, 2$, and $X = L$. In view of (2.6), we have

$$\int_{C_u} |L\Omega^i\varphi|^2 + \int_{\underline{C}_{\underline{u}}} |\nabla\!\!\!/\Omega^i\varphi|^2 = \int_{C_{u_0}} |L\Omega^i\varphi|^2 + \iint_{\mathcal{D}} Q(\nabla\Omega^i\varphi, \nabla\varphi)L\Omega^i\varphi$$
$$+ \iint_{\mathcal{D}} \sum_{p+q\le i,\, p,q\neq i} Q(\nabla\Omega^p\varphi, \nabla\Omega^q\varphi)L\Omega^i\varphi + \iint_{\mathcal{D}} \frac{1}{r}\underline{L}\Omega^i\varphi\cdot L\Omega^i\varphi. \qquad (4.16)$$

We rewrite the above equations as

$$\int_{C_u} |L\Omega^i\varphi|^2 + \int_{\underline{C}_{\underline{u}}} |\nabla\!\!\!/\Omega^i\varphi|^2 = \int_{C_{u_0}} |L\Omega^i\varphi|^2 + R + S + T,$$

where R, S and T are defined in the obvious way. Before deriving the estimates, we remark that for any function ϕ, we actually have

$$\|\Omega^i\phi\|_{L^p(S_{\underline{u},u})} \sim \big\||u|^i|\nabla\!\!\!/^i\phi|\big\|_{L^p(S_{\underline{u},u})},$$

which can be easily derived from (1.3).

Let us first consider R, i.e. the second integral terms on the right hand side of all equations in (4.16). In view of (2.11), R splits into

$$R_1 = \iint_{\mathcal{D}} |\underline{L}\varphi|\,|L\Omega^i\varphi|^2, \qquad R_4 = \iint_{\mathcal{D}} |L\varphi|\,|\nabla\!\!\!/\Omega^i\varphi|\,|L\Omega^i\varphi|,$$
$$R_2 = \iint_{\mathcal{D}} |\nabla\!\!\!/\varphi|\,|L\Omega^i\varphi|^2, \qquad R_5 = \iint_{\mathcal{D}} |\nabla\!\!\!/\varphi|\,|\underline{L}\Omega^i\varphi|\,|L\Omega^i\varphi|,$$
$$R_3 = \iint_{\mathcal{D}} |L\varphi|\,|\underline{L}\Omega^i\varphi|\,|L\Omega^i\varphi|, \qquad R_6 = \iint_{\mathcal{D}} (|\underline{L}\varphi| + |\nabla\!\!\!/\varphi|)|\nabla\!\!\!/\Omega^i\varphi|\,|L\Omega^i\varphi|,$$

where $i = 0, 1, 2$. Now we bound those terms one by one.

First, we have

$$R_1 \le \int_{u_0}^{u} \|\underline{L}\varphi\|_{L^\infty}\left(\int_{C_{u'}} |L\Omega^i\varphi|^2\right) du' \lesssim \int_{u_0}^{u} \delta^{1/4}|u'|^{-3/2}MM^2\, du' \lesssim \delta^{1/4}|u|^{-1/2}M^3.$$

For R_2, we bound $\nabla\!\!\!/\varphi$ in L^∞ and then proceed exactly as above. This bound is better than R_1's and we shall use a worse one,

$$R_2 \lesssim \delta^{1/4}|u|^{-1/2}M^3.$$

Further, we have

$$\begin{aligned} R_3 &\lesssim \left(\iint_{\mathcal{D}} |L\varphi|^2|L\Omega^i\varphi|^2\right)^{1/2}\left(\iint_{\mathcal{D}} |\underline{L}\Omega^i\varphi|^2\right)^{1/2} \\ &= \left(\int_{u_0}^{u} \|L\varphi\|_{L^\infty}^2\|L\Omega^i\varphi\|_{L^2(C_{u'})}^2\, du'\right)^{1/2}\left(\int_0^{\underline{u}} \big\||u|^i\underline{L}\nabla\!\!\!/^i\varphi\big\|_{L^2(\underline{C}_{\underline{u}'})}^2\, d\underline{u}'\right)^{1/2} \\ &\lesssim \delta^{1/2}|u|^{-1}M^3. \end{aligned}$$

For R_4, since $i \leq 2$, we can use the L^4 estimates of Proposition 4.1. This is an easy but important observation since we are dealing with terms with fewer (less than 4) derivatives. And for the highest order derivative terms, we cannot use this approach. We then have (note that we bound $\nabla\!\!\!/ \Omega^i \varphi$ in L^4 instead of L^2 to gain $\delta^{1/4}$)

$$R_4 \leq \int_{u_0}^{u} \|L\varphi\|_{L^2(C_{u'})} \|\nabla\!\!\!/ \Omega^i \varphi\|_{L^4(C_{u'})} \|L\Omega^i \varphi\|_{L^4(C_{u'})} \, du' \lesssim \delta^{1/4} |u|^{-3/4} M^3.$$

R_5 and R_6 can be bounded exactly in the same way as R_4, thus,

$$R_5 \lesssim \delta^{1/2} |u|^{-5/4} M^3, \qquad R_6 \lesssim \delta^{1/4} |u|^{-3/4} M^3.$$

We now turn to S, i.e. the third integral terms on the right hand side of all equations (4.16). A general form for the integral can be written schematically as

$$S = \iint_{\mathcal{D}} |\nabla \Omega^p \varphi| \, |\nabla \Omega^q \varphi| \, |L\Omega^i \varphi|.$$

Notice that at least one ∇ in this formula is not L. Thus, we can estimate this term exactly in the same way as for R_4. This leads to

$$S \lesssim \delta^{1/4} |u|^{-1/2} M^3.$$

Finally, the last integral terms on the right hand side of all equations (4.16) is

$$T = \iint_{\mathcal{D}} \frac{1}{r} |\underline{L}\Omega^i \varphi| \, |L\Omega^i \varphi|.$$

We bound it by

$$T \lesssim \int_{u_0}^{u} \frac{1}{|u'|} \|L\Omega^i \varphi\|_{L^2(C_{u'})} \|\underline{L}\Omega^i \varphi\|_{L^2(C_{u'})} \, du' \lesssim \int_{u_0}^{u} \frac{1}{|u'|} M\delta |u'|^{-1} M \, du' \lesssim \delta |u|^{-1} M^2.$$

Putting those estimates in (4.16), in view of the size (3.14) of the initial data as well as $M \geq 1$, we obtain

$$\int_{C_u} |L\Omega^i \varphi|^2 + \int_{\underline{C}_{\underline{u}}} |\nabla\!\!\!/ \Omega^i \varphi|^2 \lesssim I_4^2 + \delta^{1/4} |u|^{-1/2} M^3.$$

Hence,

$$\|L\Omega^i \varphi\|_{L^2(C_u)} + \|\nabla\!\!\!/ \Omega^i \varphi\|_{L^2(\underline{C}_{\underline{u}})} \lesssim I_4 + \delta^{1/8} |u|^{-1/4} M^{3/2} \tag{4.17}$$

for $i = 0, 1, 2$.

Next we switch X to $\underline{L}$. In view of (2.6), we have

$$\begin{aligned}\int_{C_u} |\nabla\!\!\!/ \Omega^i \varphi|^2 + \int_{\underline{C}_{\underline{u}}} |\underline{L}\Omega^i \varphi|^2 &= \int_{C_{u_0}} |\nabla\!\!\!/ \Omega^i \varphi|^2 + \iint_{\mathcal{D}} Q(\nabla \Omega^i \varphi, \nabla \varphi) \underline{L}\Omega^i \varphi \\ &+ \iint_{\mathcal{D}} \sum_{p+q \leq i,\, p,q<i} Q(\nabla \Omega^p \varphi, \nabla \Omega^q \varphi) \underline{L}\Omega^i \varphi - \iint_{\mathcal{D}} \frac{1}{r} \underline{L}\Omega^i \varphi \cdot L\Omega^i \varphi\end{aligned} \tag{4.18}$$

for $i = 0, 1, 2$. We rewrite the above equations as

$$\int_{C_u} |\nabla\!\!\!/ \Omega^i \varphi|^2 + \int_{\underline{C}_{\underline{u}}} |\underline{L}\Omega^i \varphi|^2 = \int_{C_{u_0}} |\nabla\!\!\!/ \Omega^i \varphi|^2 + R + S + T,$$

where R, S and T are defined in the obvious way.

We start with R, i.e. the second integral terms on the right hand side of all equations (4.18). In view of (2.11), R splits into

$$R_1 = \iint_{\mathcal{D}} (|L\varphi| + |\nabla\!\!\!/ \varphi|)|\underline{L}\Omega^i \varphi|^2, \qquad R_3 = \iint_{\mathcal{D}} (|\underline{L}\varphi| + |\nabla\!\!\!/ \varphi|)|\nabla\!\!\!/ \Omega^i \varphi|\, |\underline{L}\Omega^i \varphi|,$$
$$R_2 = \iint_{\mathcal{D}} (|\nabla\!\!\!/ \varphi| + |\underline{L}\varphi|)|L\Omega^i \varphi|\, |\underline{L}\Omega^i \varphi|, \qquad R_4 = \iint_{\mathcal{D}} |L\varphi|\, |\nabla\!\!\!/ \Omega^i \varphi|\, |\underline{L}\Omega^i \varphi|.$$

We bound those terms one by one. In view of Lemma 4.2, we have

$$R_1 \leq \int_0^{\underline{u}} (\|L\varphi\|_{L^\infty} + \|\nabla\!\!\!/ \varphi\|_{L^\infty})\|\underline{L}\Omega^i \varphi\|^2_{L^2(\underline{C}_{\underline{u}'})}\, d\underline{u}' \lesssim \delta^{3/2}|u|^{-2}M^3.$$

For R_2, R_3 and R_4, in view of Proposition 4.1, we bound the three factors in the integrands in L^4, L^4 and L^2 respectively, hence

$$R_2 + R_3 + R_4 \lesssim \delta^{5/4}|u|^{-3/2}M^3.$$

We remark that for R_4, it is necessary to bound $\nabla\!\!\!/ \Omega^i \varphi$ in L^4 instead of L^2. In this way, one can gain an extra $\delta^{1/4}$.

We now turn to S, i.e. the third integral terms on the right hand side of all equations (4.18). A general form of the integral can be written schematically as

$$S = \iint_{\mathcal{D}} |\nabla \Omega^p \varphi|\, |\nabla \Omega^q \varphi|\, |\underline{L}\Omega^i \varphi|.$$

Notice that at least one ∇ in this formula is not L. Thus, we can estimate this term by bounding the three factors in the integrands in L^2, L^4 and L^4 respectively. This leads to

$$S \lesssim \delta^{5/4}|u|^{-3/2}M^3.$$

For the last integral terms on the right hand side of all equations (4.18),

$$T = \iint_{\mathcal{D}} \frac{1}{r}|L\Omega^i \varphi|\, |\underline{L}\Omega^i \varphi|,$$

we shall use a different approach.

Remark. This is closely related to the so called *reductive structure* in Christodoulou's work [2]. Roughly speaking, at this point, one has to process the estimates in a correct order and one has to rely on the estimates derived in previous steps. The idea is as follows: Assume that we would like to bound two quantities f_1 and f_2 (they are energy quantities in [2]) and we have two inequalities: $f_1 \lesssim 1 + f_1^{1/2} f_2$ and $f_2 \lesssim 1 + f_2^{1/2}$. We should first bound f_2 using the second one (since the right hand side is sublinear) and then use this bound on f_2 to bound f_1 (since now the right hand side of the first inequality is reduced to the sublinear case). We cannot start with the first inequality because of strong nonlinearity. This is not transparent in the current work: we have already carefully chosen the order of deriving our estimates so that the estimate (for T) can be closed directly.

We bound T as follows:

$$\begin{aligned} T &\lesssim \iint_{\mathcal{D}} \frac{1}{|u'|} |L\Omega^i \varphi|\, |\underline{L}\Omega^i \varphi| \lesssim \iint_{\mathcal{D}} \left(\frac{\delta}{|u'|^2} |L\Omega^i \varphi|^2 + \frac{1}{\delta} |\underline{L}\Omega^i \varphi|^2 \right) \\ &= \int_{u_0}^{u} \frac{\delta}{|u'|^2} \|L\Omega^i \varphi\|^2_{L^2(C_{u'})}\, du' + \frac{1}{\delta} \int_0^{\underline{u}} \|\underline{L}\Omega^i \varphi\|^2_{L^2(\underline{C}_{\underline{u}'})}\, d\underline{u}'. \end{aligned}$$

The first term in the last line has already been controlled in (4.17), so we have

$$\begin{aligned} T &\lesssim \int_{u_0}^{u} \frac{\delta}{|u'|^2} (I_4^2 + \delta^{1/4} |u|^{-1/2} M^3)\, du' + \frac{1}{\delta} \int_0^{\underline{u}} \|\underline{L}\Omega^i \varphi\|^2_{L^2(\underline{C}_{\underline{u}'})}\, d\underline{u}' \\ &= \delta |u|^{-1} I_4^2 + \delta^{5/4} |u|^{-3/2} M^3 + \frac{1}{\delta} \int_0^{\underline{u}} \|\underline{L}\Omega^i \varphi\|^2_{L^2(\underline{C}_{\underline{u}'})}\, d\underline{u}'. \end{aligned}$$

In view of the size (3.14) of the initial data, we plug the above estimates into (4.18) to derive

$$\int_{C_u} |\nabla\!\!\!/ \Omega^i \varphi|^2 + \int_{\underline{C}_{\underline{u}}} |\underline{L}\Omega^i \varphi|^2 \lesssim \delta |u|^{-1} I_4^2 + \delta^{5/4} |u|^{-3/2} M^3 + \frac{1}{\delta} \int_0^{\underline{u}} \|\underline{L}\Omega^i \varphi\|^2_{L^2(\underline{C}_{\underline{u}'})}\, d\underline{u}'.$$

Since $\|\underline{L}\Omega^i \varphi\|^2_{L^2(\underline{C}_{\underline{u}})}$ also appears on the left hand side, a standard use of Gronwall's inequality removes the integral on the right hand side. This yields

$$\|\nabla\!\!\!/ \Omega^i \varphi\|_{L^2(C_u)} + \|\underline{L}\Omega^i \varphi\|_{L^2(\underline{C}_{\underline{u}})} \lesssim \delta^{1/2} |u|^{-1/2} I_4 + \delta^{5/8} |u|^{-3/4} M^{3/2}. \tag{4.19}$$

Putting (4.17) and (4.19) together, we derive the energy estimates for one derivative of φ as follows:

$$E_k(u, \underline{u}) + \underline{E}_k(u, \underline{u}) \lesssim I_4 + \delta^{1/8} |u|^{-1/4} M^{3/2} \tag{4.20}$$

for $k = 1, 2, 3$.

4.4. Energy estimates for $F_k(u,\underline{u})$ and $\underline{F}_k(u,\underline{u})$ when $k = 2, 3, 4$

We start with $F_k(u,\underline{u})$'s. By commuting L and Ω with (1.1) using (3.4), we apply the scheme of Section 2.1 for this equation where we take $\phi = L\Omega^i\varphi$ with $i = 0, 1, 2$ and $X = L$. We then have

$$\begin{aligned}\int_{C_u} |L^2\Omega^i\varphi|^2 + \int_{\underline{C}_{\underline{u}}} |\nabla\!\!\!/ L\Omega^i\varphi|^2 &= \int_{C_{u_0}} |L^2\Omega^i\varphi|^2 + \iint_{\mathcal{D}} Q(\nabla L\Omega^i\varphi, \nabla\varphi)L^2\Omega^i\varphi\\
&+ \iint_{\mathcal{D}} \sum_{p+q\le i,\, p,q<i} Q(\nabla L\Omega^p\varphi, \nabla\Omega^q\phi)L^2\Omega^i\varphi\\
&+ \iint_{\mathcal{D}} \sum_{p+q\le i} [L, Q](\nabla\Omega^p\varphi, \nabla\Omega^q\varphi)L^2\Omega^i\varphi\\
&- \iint_{\mathcal{D}}\left(\frac{1}{r^2}(\underline{L}\Omega^i\varphi - L\Omega^i\varphi)L^2\Omega^i\varphi - \frac{2}{r}\Delta\!\!\!/\Omega^i\varphi L^2\Omega^i\varphi - \frac{1}{r}\underline{L}L\Omega^i\varphi L^2\Omega^i\varphi\right). \qquad (4.21)\end{aligned}$$

We rewrite the above equations as

$$\int_{C_u} |L^2\Omega^i\varphi|^2 + \int_{\underline{C}_{\underline{u}}} |\nabla\!\!\!/ L\Omega^i\varphi|^2 = \int_{C_{u_0}} |L^2\Omega^i\varphi|^2 + R + S + T + U,$$

where R, S, T and U are defined in the obvious way.

First of all, we consider R, i.e. the second integral term on the right hand side of (4.21); in view of (2.11), it splits into

$$\begin{aligned}R_1 &= \iint_{\mathcal{D}} (|\underline{L}\varphi| + |\nabla\!\!\!/\varphi|)|L^2\Omega^i\varphi|^2,\\
R_2 &= \iint_{\mathcal{D}} (|\nabla\!\!\!/\varphi| + |L\varphi|)|\underline{L}L\Omega^i\varphi|\,|L^2\Omega^i\varphi|,\\
R_3 &= \iint_{\mathcal{D}} (|\nabla\!\!\!/\varphi| + |\underline{L}\varphi| + |L\varphi|)|\nabla\!\!\!/ L\Omega^i\varphi|\,|L^2\Omega^i\varphi|.\end{aligned}$$

We will bound the three factors in these integrands in L^∞, L^2 and L^2 respectively.

For R_1, we simply bound $|\underline{L}\varphi|$ and $|\nabla\!\!\!/\varphi|$ in L^∞ and obtain

$$R_1 \lesssim \delta^{-7/4}|u|^{-1/2}M^3.$$

For R_2, we first need L^2 estimates on $L\underline{L}\Omega^i\varphi$. According to (3.8), we have

$$\begin{aligned}\|L\underline{L}\varphi\|_{L^2(C_u)} &\lesssim \|\Delta\!\!\!/\varphi\|_{L^2(C_u)} + \|r^{-1}L\varphi\|_{L^2(C_u)} + \|r^{-1}\underline{L}\varphi\|_{L^2(C_u)}\\
&+ \|\underline{L}\varphi L\varphi\|_{L^2(C_u)} + \|\underline{L}\varphi\nabla\!\!\!/\varphi\|_{L^2(C_u)} + \|L\varphi\nabla\!\!\!/\varphi\|_{L^2(C_u)} + \big\||\nabla\!\!\!/\varphi|^2\big\|_{L^2(C_u)}\end{aligned}$$

For the quadratic terms, we bound one of them in L^∞ and the others in L^2, thus we have

$$\|L\underline{L}\varphi\|_{L^2(C_u)} \lesssim |u|^{-1}M. \qquad (4.22)$$

For $L\underline{L}\Omega^i\varphi$'s with $i = 1, 2$, we can proceed in exactly the same way to derive (we also use Sobolev inequalities)

$$\|L\underline{L}\Omega^i\varphi\|_{L^2(C_u)} \lesssim |u|^{-1}M, \quad \|L\underline{L}\Omega^{i-1}\varphi\|_{L^4(S_{\underline{u},u})} \lesssim \delta^{-1/2}|u|^{-3/2}M. \tag{4.23}$$

Therefore, we bound $|\nabla\!\!\!/\varphi|$ and $|L\varphi|$ in L^∞ and $|\underline{L}L\Omega^i\varphi|$ and $|L^2\Omega^i\varphi|$ in L^2; this yields

$$R_2 \lesssim \delta^{-3/2}|u|^{-1}M^3.$$

For R_3, similarly we have

$$R_3 \lesssim \delta^{-3/2}|u|^{-1}M^3.$$

Secondly, we consider S, i.e. the third integral term on the right hand side of (4.21). It is bounded by the sum of the following terms:

$$S_1 = \iint_{\mathcal{D}} (|\underline{L}\Omega^q\varphi| + |\nabla\!\!\!/\Omega^q\varphi|)|L^2\Omega^p\varphi|\,|L^2\Omega^i\varphi|,$$
$$S_2 = \iint_{\mathcal{D}} (|\nabla\!\!\!/\Omega^q\varphi| + |L\Omega^q\varphi|)|\underline{L}L\Omega^p\varphi|\,|L^2\Omega^i\varphi|,$$
$$S_3 = \iint_{\mathcal{D}} (|\nabla\!\!\!/\Omega^q\varphi| + |\underline{L}\Omega^q\varphi| + |L\Omega^q\varphi|)|\nabla\!\!\!/L\Omega^p\varphi|\,|L^2\Omega^j\varphi|,$$

where $p + q \leq i$ and $p, q < i$. Since the numbers of derivatives in the first factors are not saturated, we will bound the three factors in these integrands in L^4, L^4 and L^2 respectively.

First, we have

$$\begin{aligned} S_1 &\leq \int_0^{\underline{u}}\int_{u_0}^{u} (\|\nabla\!\!\!/\Omega^q\varphi\|_{L^4(S_{\underline{u}',u'})} \\ &\qquad + \|\underline{L}\Omega^q\varphi\|_{L^4(S_{\underline{u}',u'})})\|L^2\Omega^p\varphi\|_{L^4(S_{\underline{u}',u'})}\|L^2\Omega^i\varphi\|_{L^2(S_{\underline{u}',u'})}\,du'\,d\underline{u}' \\ &\leq \delta^{1/4}M\int_0^{\underline{u}}\int_{u_0}^{u}\big(|u'|^{-3/2}\|L^2\Omega^j\varphi\|_{L^2(S_{\underline{u}',u'})} \\ &\qquad + |u'|^{-1/2}\|\nabla\!\!\!/L^2\Omega^j\varphi\|_{L^2(S_{\underline{u}',u'})}\big)\|L^2\Omega^i\varphi\|_{L^2(S_{\underline{u}',u'})}\,du'\,d\underline{u}', \end{aligned}$$

where $j = q, q+1$ and we have used the following Sobolev inequalities for the last line:

$$\|L^2\Omega^j\varphi\|_{L^4(S_{\underline{u},u})} \lesssim |u|^{-1/2}\|L^2\Omega^j\varphi\|_{L^2(S_{\underline{u},u})} + |u|^{1/2}\|\nabla\!\!\!/L^2\Omega^j\varphi\|_{L^2(S_{\underline{u},u})}.$$

Thus, we have

$$\begin{aligned} S_1 &\lesssim \delta^{1/4}M\int_{u_0}^{u}\big(|u'|^{-3/2}\|L^2\Omega^j\varphi\|_{L^2(C_{u'})}\|L^2\Omega^i\varphi\|_{L^2(C_{u'})} \\ &\qquad + |u'|^{-3/2}\|L^2\Omega^{j+1}\varphi\|_{L^2(C_{u'})}\|L^2\Omega^i\varphi\|_{L^2(C_{u'})}\big)\,du' \\ &\lesssim \delta^{-7/4}|u|^{-1/2}M^3. \end{aligned}$$

For S_2 and S_3, we similarly obtain

$$S_2 + S_3 \lesssim \delta^{-3/2}|u|^{-1}M^3.$$

Thirdly, we consider T, i.e. the fourth integral terms on the right hand side of (4.21). It is bounded by

$$T = \iint_{\mathcal{D}} \frac{1}{|u|} \sum_{p+q\leq i} (|\nabla\!\!\!/ \Omega^p\varphi|\,|L\Omega^q\varphi| + |\nabla\!\!\!/ \Omega^p\varphi|\,|\underline{L}\Omega^q\varphi| + |\underline{L}\Omega^p\varphi|\,|L\Omega^q\varphi|)|L^2\Omega^i\varphi|.$$

The strategy is to control the three factors in the integrands either in L^∞, L^2 and L^2 or in L^4, L^4 and L^2 respectively, depending on whether the numbers of derivatives are saturated or not. We omit the details since the proof is exactly the same as for R and S terms. We obtain

$$T \lesssim \delta^{-3/2}|u|^{-1}M^3.$$

Finally, we consider U, i.e. the last integral term in (4.21); we simply estimate two factors in the integrands in L^2. This gives

$$U \lesssim \delta^{-1}|u|^{-1}M^2 + \delta^{-1/2}|u|^{-3/2}M^2.$$

Putting all the estimates back into (4.21), in view of the size (3.15) of the data, we have

$$\int_{C_u} |L^2\Omega^i\varphi|^2 + \int_{\underline{C}_{\underline{u}}} |\nabla\!\!\!/ L\Omega^i\varphi|^2 \lesssim \delta^{-2}I_4^2 + \delta^{-7/4}|u|^{-1/2}M^3$$

for $i = 0, 1, 2$. This is equivalent to

$$F_k(u) \lesssim I_4 + \delta^{1/8}|u|^{-1/4}M^{3/2} \tag{4.24}$$

for $k = 2, 3, 4$.

We now derive estimates for $\underline{F}_k(u, \underline{u})$'s. Since the proof can almost be repeated word for word from the proof we have just performed for $F_k(u, \underline{u})$'s, instead of giving all the details, we only sketch the idea.

By commuting $\underline{L}$, Ω with (1.1) using (3.5), one can apply the scheme of Section 2.1 for this equation by taking $\phi = \underline{L}\Omega^i\varphi$, $i = 0, 1, 2$, and $X = \underline{L}$. Thus, (2.6) reads

$$\begin{aligned}
\int_{C_u} |\nabla\!\!\!/ \underline{L}\Omega^i\varphi|^2 + \int_{\underline{C}_{\underline{u}}} |\underline{L}^2\Omega^i\varphi|^2 &= \int_{C_{u_0}} |\nabla\!\!\!/ \underline{L}\Omega^i\varphi|^2 + \iint_{\mathcal{D}} Q(\nabla\underline{L}\Omega^i\varphi, \nabla\varphi)\underline{L}^2\Omega^i\varphi \\
&+ \iint_{\mathcal{D}} \sum_{p+q\leq i,\ p,q<i} Q(\nabla\underline{L}\Omega^p\varphi, \nabla\Omega^q\varphi)\underline{L}^2\Omega^i\varphi \\
&+ \iint_{\mathcal{D}} \sum_{p+q\leq i} [\underline{L}, Q](\nabla\Omega^p\varphi, \nabla\Omega^q\varphi)\underline{L}^2\Omega^i\varphi \\
&+ \iint_{\mathcal{D}} \left(\frac{1}{r^2}(\underline{L}\Omega^i\varphi - L\Omega^i\varphi)\underline{L}^2\Omega^i\varphi - \frac{2}{r}\Delta\!\!\!/ \Omega^i\varphi\underline{L}^2\Omega^i\varphi - \frac{1}{r}L\underline{L}\Omega^i\varphi\underline{L}^2\Omega^i\varphi\right).
\end{aligned} \tag{4.25}$$

We rewrite the above equations as

$$\int_{C_u} |\nabla\!\!\!/ \underline{L}\Omega^i\varphi|^2 + \int_{\underline{C}_{\underline{u}}} |\underline{L}^2\Omega^i\varphi|^2 = \int_{C_{u_0}} |\nabla\!\!\!/ \underline{L}\Omega^i\varphi|^2 + R + S + T + U,$$

where R, S, T and U are defined in the obvious way.

We bound R by the sum of the following terms:

$$R_1 = \iint_{\mathcal{D}} (|L\varphi| + |\nabla\!\!\!/\varphi|)|\underline{L}^2\Omega^i\varphi|^2,$$
$$R_2 = \iint_{\mathcal{D}} (|\nabla\!\!\!/\varphi| + |\underline{L}\varphi|)|L\underline{L}\Omega^i\varphi|\,|\underline{L}^2\Omega^i\varphi|,$$
$$R_3 = \iint_{\mathcal{D}} (|\nabla\!\!\!/\varphi| + |\underline{L}\varphi| + |L\varphi|)|\nabla\!\!\!/ \underline{L}\Omega^i\varphi|\,|\underline{L}^2\Omega^i\varphi|.$$

We then bound the three factors in the above integrands in L^∞, L^2 and L^2 respectively. This will yield directly

$$R_1 \lesssim \delta^{1/2}|u|^{-2}M^3, \quad R_2 \lesssim \delta^{3/4}|u|^{-5/2}M^3, \quad R_3 \lesssim \delta|u|^{-3}M^3.$$

Next, S is bounded by the sum of the following terms:

$$S_1 = \iint_{\mathcal{D}} (|L\Omega^q\varphi| + |\nabla\!\!\!/\Omega^q\varphi|)|\underline{L}^2\Omega^p\varphi|\,|\underline{L}^2\Omega^i\varphi|,$$
$$S_2 = \iint_{\mathcal{D}} (|\nabla\!\!\!/\Omega^q\varphi| + |\underline{L}\Omega^q\varphi|)|L\underline{L}\Omega^p\varphi|\,|\underline{L}^2\Omega^i\varphi|,$$
$$S_3 = \iint_{\mathcal{D}} (|\nabla\!\!\!/\Omega^q\varphi| + |\underline{L}\Omega^q\varphi| + |L\Omega^q\varphi|)|\nabla\!\!\!/ \underline{L}\Omega^p\varphi|\,|\underline{L}^2\Omega^i\varphi|,$$

where $p + q \leq i$ and $p, q < i$.

Since the numbers of derivatives in the first factors are not saturated, we can bound the three factors in these integrands in L^4, L^4 and L^2 respectively. This yields

$$S_1 \lesssim \delta^{1/2}|u|^{-2}M^3, \quad S_2 + S_3 \lesssim \delta^{3/4}|u|^{-5/2}M^3.$$

T is typically bounded by

$$\iint_{\mathcal{D}} \frac{1}{|u|} \sum_{p+q\leq i} (|\nabla\!\!\!/\Omega^p\varphi|\,|L\Omega^q\varphi| + |\nabla\!\!\!/\Omega^p\varphi|\,|\underline{L}\Omega^q\varphi| + |\underline{L}\Omega^p\varphi|\,|L\Omega^q\varphi|)|L^2\Omega^i\varphi|.$$

We control the three factors in the integrands either in L^∞, L^2 and L^2 or in L^4, L^4 and L^2 respectively and obtain

$$T \lesssim \delta^{3/4}|u|^{-5/2}M^3.$$

For U, we control the two factors in the integrands in L^2 and obtain

$$U \lesssim \delta^{1/2}|u|^{-2}M^2 + \delta|u|^{-5/2}M^2.$$

Putting all those estimates together, in view of (3.11) and the fact that $|u| \leq 1$, we derive

$$\int_{\underline{C}_{\underline{u}}} |\underline{L}^2\Omega^i\varphi|^2 \lesssim \delta^2|u_0|^{-4}I_4^2 + \delta^{1/2}|u|^{-2}M^3$$

for $i = 0, 1, 2$. This is equivalent to

$$\underline{F}_k(\underline{u}) \lesssim \delta|u_0|^{-3/2}I_4 + \delta^{1/4}|u|^{-1/2}M^{3/2} \tag{4.26}$$

for $k = 2, 3, 4$.

4.5. Estimates for $E_4(u, \underline{u})$ and $\underline{E}_4(u, \underline{u})$

First of all, we commute Ω three times with (1.1), that is, taking $n = 3$ in (3.3), this yields

$$\Box\Omega^3\varphi = \sum_{p+q\leq 3} Q(\nabla\Omega^p\varphi, \nabla\Omega^q\varphi).$$

For $E_4(u, \underline{u})$, we use the scheme of Section 2.1 for this equation by taking $\phi = \Omega^3\varphi$ and $X = L$. In view of (2.6), we have

$$\begin{aligned}\int_{C_u} |L\Omega^3\varphi|^2 + \int_{\underline{C}_{\underline{u}}} |\nabla\!\!\!/\Omega^3\varphi|^2 &= \int_{C_{u_0}} |L\Omega^3\varphi|^2 + \iint_{\mathcal{D}} L\Omega^3\varphi Q(\nabla\Omega^3\varphi, \nabla\varphi) \\ &+ \iint_{\mathcal{D}} L\Omega^3\varphi \sum_{p+q\leq 3,\, p,q<3} Q(\nabla\Omega^p\varphi, \nabla\Omega^q\varphi) + \iint_{\mathcal{D}} \frac{1}{r}\underline{L}\Omega^3\varphi \cdot L\Omega^3\varphi. \end{aligned} \tag{4.27}$$

We rewrite the above equations as

$$\int_{C_u} |L\Omega^3\varphi|^2 + \int_{\underline{C}_{\underline{u}}} |\nabla\!\!\!/\Omega^3\varphi|^2 = \int_{C_{u_0}} |L\Omega^3\varphi|^2 + R + S + T,$$

where R, S and T are defined in the obvious way.

We claim that the estimates for S and T are easy:

$$S \lesssim \delta^{1/4}|u|^{-1/2}M^3, \qquad T \lesssim \delta|u|^{-2}M^2.$$

To be more precise: since the numbers of derivatives for the integrands of S are not saturated (i.e. only one term has four derivatives), we can bound the three factors in the integrands in L^4, L^4 and L^2 respectively; for T, we simply bound the integrands in two L^2's by Hölder's inequality. The actual proof goes in the same way as in the previous section and we omit the details.

It remains to bound R. This term can be bounded by the sum of

$$\begin{aligned} R_1 &= \iint_{\mathcal{D}} (|\underline{L}\varphi| + |\nabla\!\!\!/\varphi|)|L\Omega^3\varphi|^2, \\ R_2 &= \iint_{\mathcal{D}} (|L\varphi| + |\nabla\!\!\!/\varphi|)|\underline{L}\Omega^3\varphi|\,|L\Omega^3\varphi|, \\ R_3 &= \iint_{\mathcal{D}} (|\nabla\!\!\!/\varphi| + |\underline{L}\varphi| + |L\varphi|)|\nabla\!\!\!/\Omega^3\varphi|\,|L\Omega^3\varphi|. \end{aligned}$$

Since we are dealing with top order derivative estimates, we have to estimate the last two terms in the above integrands in L^2 and the first one in L^∞. In such a way, we can easily derive

$$R_1 \lesssim \delta^{1/4}|u|^{-1/2}M^3, \qquad R_2 \lesssim \delta^{1/2}|u|^{-1}M^3.$$

The estimates for R_3 are more difficult and require the knowledge of all the estimates derived so far. Before going into details, we explain how the difficulties appear. This is intimately related to the relaxation of the propagation estimates. One may expect $\|\nabla\!\!\!/\,\Omega^3\varphi\|_{L^2(C_u)}$ behaves like $\delta^{1/2}$ in view of the initial data. But in reality, because we are using a relaxed version of propagation estimates, we automatically lose $\delta^{1/2}$. Therefore, if we treat R_3 in the same way as for R_1 and R_2, we will not get any positive power of δ and therefore we cannot close the bootstrap argument.

To get around the difficulties, we recall that in the previous sections those L^∞ estimates (say on $L\varphi$) are directly derived from the bootstrap assumptions via Sobolev inequalities. The key observation is that if we make use of the estimates derived in the previous sections instead of the bootstrap assumptions, we can indeed improve the L^∞ estimates for $L\varphi$. This improvement will be just good enough to enable us to close the argument.

We first improve the L^4 estimates for $L\Omega\varphi$. According to (4.20) and Sobolev inequalities, we have

$$\begin{aligned}
|u|^{1/2}\|L\Omega\varphi\|_{L^4(S_{\underline{u},u})} &\lesssim \|L^2\Omega\varphi\|^{1/2}_{L^2(C_u)}(\|L\Omega\varphi\|^{1/2}_{L^2(C_u)} + |u|^{1/2}\|\nabla\!\!\!/\, L\Omega\varphi\|^{1/2}_{L^2(C_u)}) \\
&\lesssim \delta^{-1/2}(I_4^{1/2} + \delta^{1/16}|u|^{-1/8}M^{3/4})(I_4^{1/2} + \delta^{1/16}|u|^{-1/8}M^{3/4}) \\
&\lesssim \delta^{-1/2}(I_4 + \delta^{1/8}|u|^{-1/4}M^{3/2}). \qquad (4.28)
\end{aligned}$$

This implies better L^∞ estimates for $L\varphi$, once again via Sobolev inequalities:

$$\begin{aligned}
\|L\varphi\|_{L^\infty} &\lesssim |u|^{-1/2}\|L\varphi\|_{L^4(S_{\underline{u},u})} + |u|^{1/2}\|L\nabla\!\!\!/\,\varphi\|_{L^4(S_{\underline{u},u})} \\
&\lesssim \delta^{-1/2}|u|^{-1}(I_4 + \delta^{1/8}|u|^{-1/4}M^{3/2}). \qquad (4.29)
\end{aligned}$$

We now proceed to bound R_3 and we only consider the main terms with $L\varphi$ (the others are much easier to control):

$$\begin{aligned}
R_3 &\leq \iint_{\mathcal{D}} (|L\varphi|^2|u'|^2|\nabla\!\!\!/\,\Omega^3\varphi|^2 + |u'|^{-2}|L\Omega^3\varphi|^2)\, du'\, d\underline{u}' \\
&\leq \int_0^\delta \delta^{-1}(I_4 + \delta^{1/8}|u|^{-1/4}M^{3/2})^2\|\nabla\!\!\!/\,\Omega^3\varphi\|^2_{L^2(\underline{C}_{\underline{u}'})}\, d\underline{u}' + \int_{u_0}^u |u'|^{-2}\|L\Omega^3\varphi\|^2_{L^2(C_{u'})}\, du'.
\end{aligned}$$

Finally, we put all the estimates together; in view of the size of the initial data on C_{u_0}, we obtain

$$\begin{aligned}
\|L\Omega^3\varphi\|^2_{L^2(C_u)} + \|\nabla\!\!\!/\,\Omega^3\varphi\|^2_{L^2(\underline{C}_{\underline{u}})} &\lesssim I_4^2 + \delta^{1/4}|u|^{-1/2}M^3 + \int_{u_0}^u |u'|^{-2}\|L\Omega^3\varphi\|^2_{L^2(C_{u'})}\, du' \\
&\quad + \int_0^\delta \delta^{-1}(I_4 + \delta^{1/8}|u|^{-1/4}M^{3/2})^2\|\nabla\!\!\!/\,\Omega^3\varphi\|^2_{L^2(\underline{C}_{\underline{u}'})}\, d\underline{u}'.
\end{aligned}$$

Thus, thanks to Gronwall's inequality,

$$|u|^3\|L\nabla\!\!\!/^3\varphi\|_{L^2(C_u)}+\big\||u^3|\nabla\!\!\!/^4\varphi\big\|_{L^2(\underline{C}_{\underline{u}})}\lesssim\big(\exp|u|^{-1}+\exp(I_4+\delta^{1/8}|u|^{-1/4}M^{3/2})^2\big)(I_4+\delta^{1/8}|u|^{-1/4}M^{3/2}).$$

Therefore, if δ is sufficiently small, we have

$$|u|^3\|L\nabla\!\!\!/^3\varphi\|_{L^2(C_u)}+\big\||u^3|\nabla\!\!\!/^4\varphi\big\|_{L^2(\underline{C}_{\underline{u}})}\lesssim I_4+\delta^{1/8}|u|^{-1/4}M^{3/2}.$$

Equivalently,

$$E_4(u,\underline{u})\lesssim I_4+\delta^{1/8}|u|^{-1/4}M^{3/2}. \tag{4.30}$$

This is the desired estimate for $E_4(u,\underline{u})$.

For $\underline{E}_4(u,\underline{u})$, we switch X to $\underline{L}$. In view of (2.6), we have

$$\begin{aligned}\int_{C_u}|\nabla\!\!\!/\Omega^3\varphi|^2+\int_{\underline{C}_{\underline{u}}}|\underline{L}\Omega^3\varphi|^2&=\int_{C_{u_0}}|\nabla\!\!\!/\Omega^3\varphi|^2+\iint_{\mathcal{D}}\underline{L}\Omega^3\varphi\, Q(\nabla\Omega^3\varphi,\nabla\varphi)\\&+\iint_{\mathcal{D}}\underline{L}\Omega^3\varphi\sum_{p+q\le 3,\ p,q<3}Q(\nabla\Omega^p\varphi,\nabla\Omega^q\varphi)-\iint_{\mathcal{D}}\frac{1}{r}\underline{L}\Omega^3\varphi\cdot L\Omega^3\varphi.\end{aligned}\tag{4.31}$$

We rewrite the above equations as

$$\int_{C_u}|\nabla\!\!\!/\Omega^3\varphi|^2+\int_{\underline{C}_{\underline{u}}}|\underline{L}\Omega^3\varphi|^2=\int_{C_{u_0}}|\nabla\!\!\!/\Omega^3\varphi|^2+R+S+T,$$

where R, S and T are defined in the obvious way.

We claim that the estimates for S are easy since the numbers of derivatives for the integrands of S are not saturated. We bound the three factors in the integrands in L^4, L^4 and L^2 respectively and we derive

$$S\lesssim\delta^{5/4}|u|^{-3/2}M^3.$$

We can bound R more or less as before. First of all, it is bounded by the sum of

$$\begin{aligned}R_1&=\iint_{\mathcal{D}}(|L\varphi|+|\nabla\!\!\!/\varphi|)|\underline{L}\Omega^3\varphi|^2,\\R_2&=\iint_{\mathcal{D}}(|\underline{L}\varphi|+|\nabla\!\!\!/\varphi|)|L\Omega^3\varphi|\,|\underline{L}\Omega^3\varphi|,\\R_3&=\iint_{\mathcal{D}}(|\nabla\!\!\!/\varphi|+|\underline{L}\varphi|+|L\varphi|)|\nabla\!\!\!/\Omega^3\varphi|\,|\underline{L}\Omega^3\varphi|.\end{aligned}$$

Once again, except for the last term in R_3, all the other terms are easy to control so we ignore them and assume

$$R_3=\iint_{\mathcal{D}}|L\varphi||\nabla\!\!\!/\Omega^3\varphi|\,|\underline{L}\Omega^3\varphi|.$$

Now, we repeat the previous argument and making use of the improved L^∞ estimates for $L\varphi$, we obtain

$$\begin{aligned} R_3 &\leq \iint_{\mathcal{D}} (|L\varphi|^2|u'|^2|\underline{L}\Omega^3\varphi|^2 + |u'|^{-2}|\nabla\!\!\!/\,\Omega^3\varphi|^2)\,du'\,d\underline{u}' \\ &\leq \int_0^\delta \delta^{-1}(I_4+\delta^{1/8}|u|^{-1/4}M^{3/2})^2\|\underline{L}\Omega^3\varphi\|^2_{L^2(\underline{C}_{\underline{u}'})}\,d\underline{u}' + \int_{u_0}^u |u'|^{-2}\|\nabla\!\!\!/\,\Omega^3\varphi\|^2_{L^2(C_{u'})}\,du'. \end{aligned}$$

For T, thanks to (4.30), we have

$$T \lesssim \delta|u|^{-1}I_4^2 + \delta^{5/4}|u|^{-3/2}M^3 + \frac{1}{\delta}\int_0^u \|\underline{L}\Omega^3\varphi\|^2_{L^2(\underline{C}_{\underline{u}'})}\,d\underline{u}'.$$

Putting all the estimates together, we have

$$\begin{aligned} \|\nabla\!\!\!/\,\Omega^3\varphi\|^2_{L^2(C_u)} + \|\underline{L}\Omega^3\varphi\|^2_{L^2(\underline{C}_{\underline{u}})} &\lesssim \delta|u|^{-1}I_4^2 + \delta^{5/4}|u|^{-3/2}M^3 \\ &\quad + \int_{u_0}^u |u'|^{-2}\|\nabla\!\!\!/\,\Omega^3\varphi\|^2_{L^2(C_{u'})}\,du' \\ &\quad + \int_0^\delta \delta^{-1}\big(1+(I_4+\delta^{1/8}|u|^{-1/4}M^{3/2})^2\big)\|\underline{L}\Omega^3\varphi\|^2_{L^2(\underline{C}_{\underline{u}'})}\,d\underline{u}'. \end{aligned}$$

Thus, thanks to Gronwall's inequality, we obtain

$$\|\nabla\!\!\!/\,\Omega^3\varphi\|_{L^2(C_u)} + \|\underline{L}\Omega^3\varphi\|_{L^2(\underline{C}_{\underline{u}})} \lesssim \delta^{1/2}|u|^{-1/2}I_4 + \delta^{5/8}|u|^{-3/4}M^{3/2}. \tag{4.32}$$

We then combine (4.30) and (4.32) to conclude

$$E_4(u) + \underline{E}_4(\underline{u}) \lesssim I_4 + \delta^{1/8}|u|^{-1/4}M^{3/2}. \tag{4.33}$$

4.6. *End of the bootstrap argument*

We add the estimates in the previous sections together; since $|u| \geq 1$, we derive

$$\sum_{i=1}^4[E_i(u)+\underline{E}_i(\underline{u})] + \sum_{j=2}^4[F_j(u)+\underline{F}_j(\underline{u})] \lesssim I_4 + \delta^{1/8}M^{3/2}.$$

By the definition of M from the bootstrap assumption (4.5), we obtain

$$M \lesssim I_4 + \delta^{1/8}M^{3/2}.$$

By choosing δ suitably small depending on I_4, we conclude that there is a constant $C(I_4)$, depending only on I_4, such that

$$\sum_{i=1}^4[E_i(u)+\underline{E}_i(\underline{u})] + \sum_{j=2}^4[F_j(u)+\underline{F}_j(\underline{u})] \leq C(I_4).$$

Thus, we have completed the proof of the Main A Priori Estimates.

4.7. Higher order derivative estimates

For higher order derivative estimates, the argument is completely analogous, in fact much simpler, because we have already closed the bootstrap argument and we can simply use an induction argument to derive estimates for each order. Therefore, we shall omit the details and only sketch the proof. We introduce a family of energy flux norms for higher order derivatives:

$$E_k(u,\underline{u}) = |u|^{k-1}\|L\nabla\!\!\!/^{k-1}\varphi\|_{L^2(C_u)} + \delta^{-1/2}|u|^{k-1/2}\|\nabla\!\!\!/^k\varphi\|_{L^2(C_u)},$$
$$\underline{E}_k(u,\underline{u}) = |u|^{k-1}\|\nabla\!\!\!/^k\varphi\|_{L^2(\underline{C}_{\underline{u}})} + \delta^{-1/2}|u|^{1/2}\big\||u|^{k-1}\underline{L}\nabla\!\!\!/^{k-1}\varphi\big\|_{L^2(\underline{C}_{\underline{u}})},$$

for all $k \geq 1$. Similar to the lower order derivatives cases, we also need a family of flux norms involving at least two null derivatives:

$$F_k(u,\underline{u}) = \delta|u|^{k-2}\|L^2\nabla\!\!\!/^{k-2}\varphi\|_{L^2(C_u)}, \quad \underline{F}_k(u,\underline{u}) = |u|^{1/2}\big\||u|^{k-2}\underline{L}^2\nabla\!\!\!/^{k-2}\varphi\big\|_{L^2(\underline{C}_{\underline{u}})},$$

for all $k \geq 2$.

To achieve higher order derivative estimates, we will perform an induction argument. The base cases have already been verified: these are indeed the Main A Priori Estimates obtained earlier, that is,

$$\sum_{i=1}^{4}[E_i(u,\underline{u}) + \underline{E}_i(u,\underline{u})] + \sum_{j=2}^{4}[F_j(u,\underline{u}) + \underline{F}_j(u,\underline{u})] \leq C(I_4),$$

where I_4 is the size of the data for up to four derivatives. The higher order estimates are formulated as follows:

Proposition 4.3. *If δ is sufficiently small which may depend only on k, for all data of* (1.1) *and all $I_{n+2} \in \mathbb{R}_{>0}$ satisfying*

$$\sum_{i=1}^{n+2} E_i(u_0,\delta) + \sum_{j=2}^{n+2} F_j(u_0,\delta) \leq I_{n+2}, \tag{4.34}$$

there exists a constant $C(I_{n+2})$, depending only on I_{n+2} (in particular, not on δ or u_0), such that

$$[E_{n+2}(u,\underline{u}) + \underline{E}_{n+2}(u,\underline{u})] + [F_{n+2}(u,\underline{u}) + \underline{F}_{n+2}(u,\underline{u})] \leq C(I_{n+2}) \tag{4.35}$$

for all $u \in [u_0, -1]$ and $\underline{u} \in [0,\delta]$ in the sense of a priori estimates.

Remark. The subscript in I_{n+2} indicates the number of derivatives needed in the energy. The small parameter δ may depend on n. In applications, since we only need the bound on at most ten derivatives of the solutions, we can ignore this dependence.

Proof of Proposition 4.3. We now sketch the proof. Once again, we make the following bootstrap assumption:

$$[E_{n+2}(u,\underline{u})+\underline{E}_{n+2}(u,\underline{u})]+[F_{n+2}(u,\underline{u})+\underline{F}_{n+2}(u,\underline{u})]\leq M \tag{4.36}$$

for all u and $\underline{u}$, where M is sufficiently large.

We proceed as before. First of all, we can derive preliminary estimates for higher order derivatives of φ, i.e. the L^∞ estimates for up to n-th order derivatives and L^4 estimates for up to $(n+1)$-th order derivatives. Those estimates come simply from Sobolev inequalities and are as follows:

$$\delta^{1/2}|u|^i\|L\nabla\!\!\!/^{i-1}\varphi\|_{L^\infty}+\delta^{-1/4}|u|^{3/4+i}\|\nabla\!\!\!/^i\varphi\|_{L^\infty}+\delta^{-1/4}|u|^{3/2}\left\||u|^{i-1}\underline{L}\nabla\!\!\!/^{i-1}\varphi\right\|_{L^\infty}\lesssim M,$$
$$\delta^{1/2}|u|^{1/2+j}\|L\nabla\!\!\!/^j\varphi\|_{L^4(S_{\underline{u},u})}+\delta^{-1/4}|u|^{5/4+j}\|\nabla\!\!\!/^{j+1}\varphi\|_{L^4(S_{\underline{u},u})}$$
$$+\delta^{-1/4}|u|^2\left\||u|^j\underline{L}\nabla\!\!\!/^j\varphi\right\|_{L^4(S_{\underline{u},u})}\lesssim M,$$

for all $i,j\in\{2,\dots,n\}$. In fact, based on the induction argument, we know that if i or j is strictly less than n, we can replace the right hand sides of the above estimates by a constant depending only on I_{n+1} instead of M.

Secondly, we can use similar arguments to those in the previous sections to obtain energy estimates; this leads to the following estimates:

$$F_{n+2}(u,\underline{u})\lesssim I_{n+2}+\delta^{1/8}|u|^{-1/4}M^{3/2},$$
$$\underline{F}_{n+2}(u,\underline{u})\lesssim I_{n+2}\delta^{1/2}+\delta^{1/8}|u|^{-1/4}M^2,$$
$$E_{n+2}(u,\underline{u})+\underline{E}_{n+2}(u,\underline{u})\lesssim I_{n+2}+\delta^{1/8}|u|^{-1/4}M^{3/2}.$$

Thus, we can complete the proof by taking a sufficiently small δ. □

5. Existence of solutions

5.1. Existence in Region 2

In this section, based on the a priori estimates of the last section, we first show that (1.1) with data prescribed on C_{u_0} where $0\leq\underline{u}\leq\delta$ in the preceding sections can be solved all the way up to $t=-1$, i.e. in Region 2. Recall that Region 2 is in the future domain of dependence of $\underline{C}_0$ and C_{u_0} (with $0\leq\underline{u}\leq\delta$) and the data on $\underline{C}_0$ is completely trivial.

To start, we use the local existence result [12] of Rendall for semilinear wave equations for characteristic data. Thus, we know that there exists a solution around S_{0,u_0}, say, defined in the region enclosed by $\underline{C}_0$, C_{u_0} and $t=u_0+\varepsilon$ with $\varepsilon\ll\delta$. Thanks to the a priori estimates, if at the beginning we assume the bound on data for at least ten derivatives, the L^∞ norms of at least up to eight derivatives of the solution are bounded by the data on $t=u_0+\varepsilon$. Therefore, we can solve the Cauchy problem with data prescribed on $t=u_0+\varepsilon$ to construct a solution in the future domain dependence of $t=u_0+\varepsilon$ whose boundary consists of the two null hypersurfaces $C_{u_0+\varepsilon}$ and $\underline{C}_\varepsilon$. Now we have two characteristic problems: for the first one, the data is prescribed on $\underline{C}_0$ and $C_{u_0+\varepsilon}$; for the second one, the data is prescribed on C_{u_0} and $\underline{C}_\varepsilon$. We can use Rendall's local existence result again to solve them around $S_{0,u_0+\varepsilon}$ and S_{ε,u_0}. In this way, we can actually push the solution to $t = u_0 + \varepsilon + \varepsilon'$ with another small ε'.

We then repeat the above process in the obvious way to push the solution all the way to $t = u_0 + \delta$. Similarly, we can then push it from $t = u_0 + \delta$ to $t = -1$. Thus, we construct a solution in the entire Region 2. We remark that this process depends crucially on the a priori estimates since the L^∞ norms of the derivatives of φ are guaranteed to be bounded.

Thus, for a finite u_0, a solution in Region 2 has been constructed.

If we restrict the above solution to $\underline{C}_\delta$, i.e. the future incoming null boundary of Region 2, it gives partially the initial data for (1.1) in Region 3. We now give a detailed description of the data on $\underline{C}_\delta$.

Proposition 5.1. *Assume we have bounds on $E_i(u_0, \delta)$ and $F_i(u_0, \delta)$ for $i \leq n+2$ for some fixed $n \geq 10$ (say $n = 10$). Then, for all $p \geq 1$ and q with $p + q \leq n-1$,*

$$\begin{aligned}
\|\nabla\!\!\!/\,\Omega^q\varphi\|_{L^\infty(\underline{C}_{\underline{u}})} &\lesssim \delta^{1/2}|u|^{-3/2} \quad \textit{for all } \underline{u} \in [0,\delta],\\
\|\underline{L}^p\Omega^q\varphi\|_{L^\infty(\underline{C}_{\underline{u}})} &\lesssim \delta^{1/2}|u|^{-p-1} \quad \textit{for all } \underline{u} \in [0,\delta],\\
\|L^p\Omega^q\varphi\|_{L^\infty(\underline{C}_\delta)} &\lesssim \delta^{1/2}|u|^{-1},
\end{aligned}$$

where $\underline{C}_{\underline{u}}$ should be understood as $\underline{C}_{\underline{u}} \cap \{u' : u_0 \leq u' \leq u\}$ and $u_0 \leq u \leq -1$.

Remark. $L^p\Omega^q\varphi$ is small only on the final hypersurface $\underline{C}_\delta$.

Proof of Proposition 5.1. First of all, by losing one derivative we can achieve a better $L^2(S_{\underline{u},u})$ estimate for $\nabla\!\!\!/\,\Omega^k\varphi$ than those in the relaxed propagation estimates. For $k \leq n$, let

$$h(\underline{u}, u) \triangleq \|\nabla\!\!\!/\,\Omega^k\varphi\|_{L^2(S_{\underline{u},u})},$$

thus,

$$\underline{L}h(\underline{u},u)^2 = \int_{S_{\underline{u},u}} \underline{L}(\nabla\!\!\!/\,\Omega^k\varphi)^2 - \frac{2}{r}\int_{S_{\underline{u},u}} (\nabla\!\!\!/\,\Omega^k\varphi)^2 \leq 2\|\underline{L}\nabla\!\!\!/\,\Omega^k\varphi\|_{L^2(S_{\underline{u},u})} \cdot h(\underline{u},u).$$

where the factor $-1/r$ in the second integral is the mean curvature of the incoming hypersurface $\underline{C}_\delta$. Therefore, we have

$$\frac{\partial}{\partial u}h(\underline{u},u) \leq \|\underline{L}\nabla\!\!\!/\,\Omega^k\varphi\|_{L^2(S_{\underline{u},u})}.$$

We now integrate to derive

$$\begin{aligned}
h(\underline{u},u) &\leq h(\underline{u},u_0) + \int_{u_0}^{u} \frac{1}{|u'|}\left\| |u'|\underline{L}\nabla\!\!\!/\,\Omega^k\varphi\right\|_{L^2(S_{\underline{u},u'})} du'\\
&\lesssim h(\underline{u},u_0) + \left(\int_{u_0}^{u} \frac{1}{|u'|^2}\,du'\right)^{1/2} \|\underline{L}\Omega^{k+1}\varphi\|_{L^2(\underline{C}_{\underline{u}})} \lesssim \delta^{1/2}|u|^{-2}.
\end{aligned}$$

For the last line, we have used the bound on the data on C_{u_0} instead of the relaxed propagation estimates. It is precisely at this point that one gains a $\delta^{1/2}$.

As a consequence, we also get improved $L^2(C_u)$ and $L^2(\underline{C}_{\underline{u}})$ estimates for $\nabla\!\!\!/\,\Omega^k\varphi$:

$$\begin{aligned}\|\nabla\!\!\!/\,\Omega^k\varphi\|_{L^2(C_u)} &= \left(\int_0^\delta \|\nabla\!\!\!/\,\Omega^k\varphi\|^2_{L^2(S_{\underline{u},u})}\,d\underline{u}\right)^{1/2} \lesssim \delta|u|^{-3/2},\\ \|\nabla\!\!\!/\,\Omega^k\varphi\|_{L^2(\underline{C}_{\underline{u}})} &= \left(\int_{u_0}^u \|\nabla\!\!\!/\,\Omega^k\varphi\|^2_{L^2(S_{\underline{u},u})}\,du'\right)^{1/2} \lesssim \delta^{1/2}|u|^{-1}.\end{aligned} \tag{5.1}$$

According to Sobolev inequalities, we then obtain

$$\|\nabla\!\!\!/\,\Omega^q\varphi\|_{L^\infty(\underline{C}_{\underline{u}})} \lesssim \delta^{1/2}|u|^{-3/2}.$$

This proves the first inequality of the proposition.

For the second inequality, we simply integrate $L(\underline{L}^{p+1}\Omega^q\varphi)$. To illustrate the idea, we only consider the case where $q = 0$. The other cases can be treated in exactly the same way. We commute $\underline{L}$ with (1.1) p times to derive

$$\begin{aligned}L|\underline{L}^{p+1}\varphi| \le |L\underline{L}^{p+1}\varphi| &\lesssim (1/r + |L\varphi| + |\nabla\!\!\!/\,\varphi|)|\underline{L}^{p+1}\varphi|\\ &\quad + (1/r + |\underline{L}\varphi| + |\nabla\!\!\!/\,\varphi|)|L\underline{L}^p\varphi| + |\nabla\!\!\!/\,\underline{L}^p\varphi|\,|\nabla\!\!\!/\,\varphi| + \text{l.o.t.},\end{aligned} \tag{5.2}$$

where the lower order terms (l.o.t.) can be determined inductively. For example, when $p = 0$,

$$\text{l.o.t.} = |\Delta\!\!\!/\,\varphi|;$$

when $p = 1$,

$$\begin{aligned}\text{l.o.t.} &\lesssim |Q(\nabla\underline{L}\varphi, \nabla\varphi)| + \frac{1}{r}|Q(\nabla\varphi, \nabla\varphi)|\\ &\quad + \frac{1}{r^2}(|L\varphi| + |\underline{L}\varphi|) + |\Delta\!\!\!/\,\underline{L}\varphi| + \frac{1}{r}|\Delta\!\!\!/\,\varphi|;\end{aligned}$$

and when $p = 2$,

$$\begin{aligned}\text{l.o.t.} &\lesssim |Q(\nabla\underline{L}^2\varphi, \nabla\varphi)| + |Q(\nabla\underline{L}\varphi, \nabla\underline{L}\varphi)| + \frac{1}{r}|Q(\nabla\underline{L}\varphi, \nabla\varphi)| + \frac{1}{r^2}|Q(\nabla\varphi, \nabla\varphi)|\\ &\quad + \frac{1}{r^2}(|L\underline{L}\varphi| + |\underline{L}^2\varphi|) + \frac{1}{r^3}(|L\varphi| + |\underline{L}\varphi|) + |\Delta\!\!\!/\,\underline{L}^2\varphi| + \frac{1}{r}|\Delta\!\!\!/\,\underline{L}\varphi| + \frac{1}{r^2}|\Delta\!\!\!/\,\varphi|.\end{aligned}$$

For $p = 0$, we have the following estimates:

$$\begin{aligned}\|\text{l.o.t.}\|_{L^\infty(\underline{C}_{\underline{u}})} &\lesssim \delta^{1/2}|u|^{-2},\\ \|(1/r + |\underline{L}\varphi| + |\nabla\!\!\!/\,\varphi|)L\varphi\|_{L^\infty(\underline{C}_{\underline{u}})} &\lesssim \delta^{-1/2}|u|^{-2},\\ |\nabla\!\!\!/\,\varphi|^2 &\lesssim \delta^{1/2}|u|^{-2}.\end{aligned}$$

We integrate along L and use Gronwall's inequality to get the correct estimate for $\underline{L}\varphi$:

$$|\underline{L}\varphi| \lesssim \delta^{1/2}|u|^{-2}.$$

Substituting this result into (5.2) leads to

$$|L\underline{L}\varphi| \lesssim \delta^{-1/2}|u|^{-2}.$$

In general, we proceed inductively. Assume

$$|\underline{L}^p\varphi| \lesssim \delta^{1/2}|u|^{-p-1}, \quad |L\underline{L}^p\varphi| \lesssim \delta^{-1/2}|u|^{-p-1}.$$

Then for $p+1$ we have

$$\begin{aligned} \|\text{l.o.t.}\|_{L^\infty(\underline{C}_u)} &\lesssim \delta^{-1/2}|u|^{-p-2}, \\ \|(1/r + |\underline{L}\varphi| + |\nabla\!\!\!/\varphi|)L\underline{L}^p\varphi\|_{L^\infty(\underline{C}_u)} &\lesssim \delta^{-1/2}|u|^{-p-2} \\ \big\| |\nabla\!\!\!/\underline{L}^p\varphi|\,|\nabla\!\!\!/\varphi| \big\|_{L^\infty(\underline{C}_u)} &\lesssim \delta^{1/2}|u|^{-p-2}. \end{aligned}$$

We then integrate along L and use Gronwall's inequality to conclude. This proves the second inequality.

The third inequality is a little surprising, since one expects that the L derivative causes a loss of $\delta^{-1/2}$, which can be seen directly from the special choice of the profile of the initial data. The idea is that the loss in δ should only result from the initial data but not from the energy estimates. Recall that the data is given by

$$\varphi(\underline{u}, u_0, \theta) = \frac{\delta^{1/2}}{|u_0|}\psi_0(\underline{u}/\delta, \theta),$$

where the $\underline{u}$-support of ψ_0 is inside (0, 1). Therefore, on C_{u_0} near S_{0,u_0}, the data is completely trivial. In particular, $(L^i\Omega^j\varphi)(u_0, \delta, \theta) \equiv 0$. We then integrate (1.1) to get estimates on $\underline{C}_\delta$.

To illustrate the above idea, we now prove

$$\|L\varphi\|_{L^\infty(\underline{C}_\delta)} \lesssim \delta^{1/2}|u|^{-1}.$$

We rewrite (1.1) as

$$|\underline{L}L\varphi + r^{-1}L\varphi| \le |\underline{L}\varphi|\,|L\varphi| + |\nabla\!\!\!/\varphi|\,|L\varphi| + \text{l.o.t.}$$

This can be viewed as an ODE for $L\varphi$ on $\underline{C}_\delta$ with trivial data on S_{δ,u_0}. We observe that $\nabla\!\!\!/^i\varphi$ and $\underline{L}\varphi$ are all of size $\delta^{1/2}|u|^{-3/2}$ and the l.o.t. is of size $\delta^{1/2}|u|^{-5/2}$. Therefore,

$$|\underline{L}L\varphi + r^{-1}L\varphi| \lesssim \delta^{1/2}|u|^{-3/2}|L\varphi| + \delta^{1/2}|u|^{-5/2}.$$

In order to use Gronwall's inequality, we first multiply both sides by r, yields

$$|\underline{L}(rL\varphi)| \lesssim \delta^{1/2}|u|^{-3/2}(r|L\varphi|) + \delta^{1/2}|u|^{-5/2}.$$

We then use Gronwall's inequality to obtain

$$\|L\varphi\|_{L^\infty(\underline{C}_\delta)} \lesssim \delta^{1/2}|u|^{-1}, \quad \|L\underline{L}\varphi\|_{L^\infty(\underline{C}_\delta)} \lesssim \delta^{1/2}|u|^{-2}.$$

In the same way, by induction, we deduce

$$\|L^p\varphi\|_{L^\infty(\underline{C}_\delta)} \lesssim \delta^{1/2}|u|^{-1}, \quad \|\underline{L}L^p\varphi\|_{L^\infty(\underline{C}_\delta)} \lesssim \delta^{-1/2}|u|^{-2}.$$

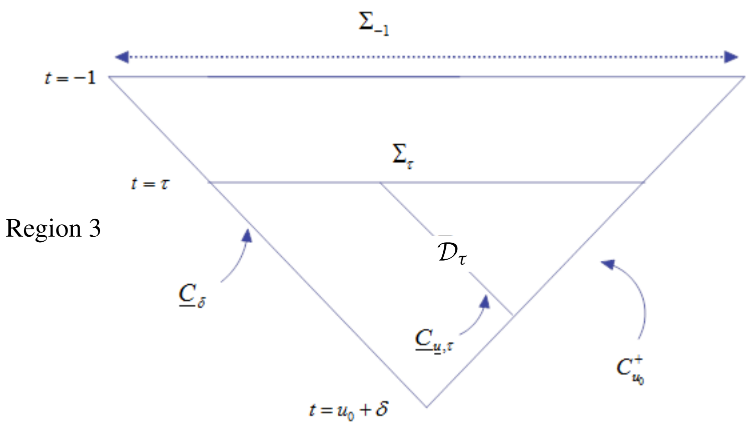
When $q \ge 1$, we can proceed in the same manner. This completes the proof. □

5.2. *Existence in Region 3*

To show the existence of a solution for (1.1), we have to solve a small data problem with data prescribed on $\underline{C}_\delta$ and $C^+_{u_0}$. The data on $\underline{C}_\delta$ is induced from the solution in Region 2, and the smallness of δ leads to the smallness of the data; the data on $C^+_{u_0}$ is simply an extension by zero of the short pulse datum prescribed on C_{u_0}, since $L\varphi$ and all higher order derivatives of φ on S_{δ,u_0} are small (we have seen this in the proof of Proposition 5.1); the data on $C^+_{u_0}$ is also small.

We now prove a theorem similar to the classical small data results [6] and [7] of Klainerman. The approach we are going to use is inspired by the harmonic gauge based proof of nonlinear stability of Minkowski space-time from [10] and [11] by Lindblad and Rodnianski. Since all the arguments are more or less well-known and scattered in the literature, we only sketch the key estimates.

The following picture will be helpful to see the structure of the proof:



Let Γ denote one of the following vector fields:

$$\Gamma \in \{\partial/\partial t, \partial/\partial x_i, \Omega_{ij}, \Omega_{0i}, S \mid i = 1, 2, 3\},$$

where

$$\Omega_{ij} = -\Omega_{ji} = x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i},$$

$$\Omega_{0i} = \Omega_{i0} = t\frac{\partial}{\partial x_i} + x_i \frac{\partial}{\partial t},$$

$$S = t\frac{\partial}{\partial t} + r\frac{\partial}{\partial r} = \underline{u}L + u\underline{L}.$$

For φ, we define the k-th weighted energy on Σ_t as follows:

$$E_k^{(\mathrm{w})}(\varphi)(t) = \left(\sum_{|\alpha|\le k}\int_{\Sigma_t}\left(|\partial_t\Gamma^\alpha\varphi|^2 + \sum_{i=1}^{3}|\partial_i\Gamma^\alpha\varphi|^2\right)\right)^{1/2},$$

where $\Sigma_\tau = \{t = \tau\} \cap$ Region 2.

We also use $\partial\phi$ to denote all possible derivatives, i.e. $\partial\phi \in \{\nabla\!\!\!/\phi, L\phi, \underline{L}\phi\}$; we write $\bar\partial\phi$ for good derivatives, i.e. $\bar\partial\phi \in \{\nabla\!\!\!/\phi, \underline{L}\phi\}$.

Therefore, the classical Klainerman–Sobolev inequalities imply

$$\|\partial\phi\|_{L^\infty(\Sigma_t)} \lesssim \frac{1}{(1+|u|)(1+|\underline{u}|)^{1/2}} E_3^{(\mathrm{w})}(\phi).$$

When we commute Γ's with (1.1) k times ($k = 8$ suffices), we have

$$\Box\Gamma^k\varphi = F_k \tag{5.3}$$

where

$$F_k = \sum_{|\beta|+|\gamma|\leq k} Q(\nabla\Gamma^\beta\varphi, \nabla\Gamma^\gamma\varphi),$$

where the Q's are null forms and we have ignored the irrelevant numerical constants.

Once the following a priori estimates have been established, the rest of the proof will be routine. So we only give the details for the estimates.

Proposition 5.2. *For all $k \leq 8$, assume that the standard energy fluxes on $\underline{C}_\delta$ for (5.3) are all bounded above by ε where ε is a small positive constant. Moreover, assume that such a solution of (1.1) exists in $\mathcal{D}_t$ where $u_0+\delta \leq t \leq -1$ and $\mathcal{D}_t$ is the region below Σ_t. If ε is sufficiently small, there is a universal constant C_0 such that*

$$\sum_{k=0}^{8} \sup_{\tau\in[u_0+\delta,t]} E_k^{(w)}(\varphi)(\tau) \leq C_0\varepsilon.$$

Proof. First, using the Killing vector field $\partial/\partial t$ in $\mathcal{D}_t$, we have the standard energy estimate

$$\int_{\Sigma_t}\Big(|\partial_t\Gamma^k\varphi|^2 + \sum_{i=1}^{3}|\partial_i\Gamma^k\varphi|^2\Big) \leq \varepsilon^2 + 2\int_{u_0+\delta}^{t}\int_{\Sigma_\tau}|F_k|\,|\partial_t\Gamma^k\varphi|. \tag{5.4}$$

Secondly, let $\underline{C}_{\underline{u},t}$ be the part of $\underline{C}_{\underline{u}}$ in $\mathcal{D}_t$. We can then apply the standard energy estimate in the region bounded by $\underline{C}_{\underline{u},t}$, $\underline{C}_\delta$, Σ_t and $C^+_{u_0}$ to obtain

$$\int_{\underline{C}_{\underline{u},t}}|\bar\partial\Gamma^k\varphi|^2 \leq \varepsilon^2 + 2\int_{u_0+\delta}^{t}\int_{\Sigma_\tau}|F_k|\,|\partial_t\Gamma^k\varphi|.$$

We multiply the above inequality by $1/(1+|\underline{u}|)^{1+\kappa}$ with $\kappa > 0$ and integrate over $\underline{u} \in [\delta, t-u_0]$ to get

$$\iint_{\mathcal{D}_t}\frac{|\bar\partial\Gamma^k\varphi|^2}{(1+|\underline{u}|)^{1+\kappa}} = \int_\delta^{t-u_0}\int_{\underline{C}_{\underline{u},t}}\frac{|\bar\partial\Gamma^k\varphi|^2}{(1+|\underline{u}|)^{1+\kappa}} \lesssim \varepsilon^2 + \int_{u_0+\delta}^{t}\int_{\Sigma_\tau}|F_k|\,|\partial_t\Gamma^k\varphi|.$$

Combining this with (5.4), we obtain the following estimates which will serve as the main tool for the rest of the proof:

$$\sum_{|\alpha|\leq 8}\int_{\Sigma_t}|\partial\Gamma^\alpha\varphi|^2 + \sum_{|\alpha|\leq 8}\iint_{\mathcal{D}_t}\frac{|\bar\partial\Gamma^\alpha\varphi|^2}{(1+|\underline{u}|)^{1+\kappa}} \lesssim \varepsilon^2 + \sum_{|\alpha|\leq 8,\,k\leq 8}\int_{u_0+\delta}^{t}\int_{\Sigma_\tau}|F_k|\,|\partial_t\Gamma^\alpha\varphi|. \tag{5.5}$$

For a bootstrap argument, we now make the following assumption:

$$\sum_{k=0}^{8} \sup_{\tau\in[u_0+\delta,t]} E_k^{(\mathrm{w})}(\varphi)(\tau) \leq M\varepsilon,$$

where M is a large constant.

Since F_k's are linear combinations of null forms, the nonlinear terms on the right hand side of (5.5) can be bounded by $N_1 + N_2$ where

$$N_1 = \sum_{|\beta|+|\gamma|\leq 8,\, k\leq 8,\, |\gamma|\leq 4} \int_{u_0+\delta}^{t} \int_{\Sigma_\tau} |\partial\Gamma^\beta\varphi|\,|\bar\partial\Gamma^\gamma\varphi|\,|\partial_t\Gamma^\alpha\varphi|,$$

$$N_2 = \sum_{|\beta|+|\gamma|\leq 8,\, k\leq 8,\, |\beta|\leq 4} \int_{u_0+\delta}^{t} \int_{\Sigma_\tau} |\partial\Gamma^\beta\varphi|\,|\bar\partial\Gamma^\gamma\varphi|\,|\partial_t\Gamma^\alpha\varphi|.$$

The control of N_1 is slightly easier; we use Klainerman–Sobolev inequalities for $|\bar\partial\Gamma^\gamma\varphi|$. Most importantly, recall that the Klainerman–Sobolev inequalities improve the estimate by a factor of $1/|u|^{1/2}$ for good derivatives. This yields

$$\begin{aligned} N_1 &\lesssim \sum_{|\beta|+|\gamma|\leq 8,\, k\leq 8,\, |\gamma|\leq 4} \int_{u_0+\delta}^{t} \frac{M\varepsilon}{|u|^{3/2}} \left(\int_{\Sigma_\tau} |\partial\Gamma^\beta\varphi|\,|\partial_t\Gamma^\alpha\varphi|\right) d\tau \\ &\lesssim \sum_{|\beta|+|\gamma|\leq 8,\, k\leq 8,\, |\gamma|\leq 4} \int_{u_0+\delta}^{t} \frac{M\varepsilon}{|u|^{3/2}} (M\varepsilon)^2\, d\tau \lesssim M^3\varepsilon^3. \end{aligned}$$

To control N_2, we would like to use the second term in (5.5). To this end, we first use the Cauchy–Schwarz inequality:

$$\begin{aligned} N_2 &\lesssim \frac{1}{C} \sum_{|\beta|+|\gamma|\leq 8,\, k\leq 8,\, |\beta|\leq 4} \int_{u_0+\delta}^{t} \int_{\Sigma_\tau} \frac{|\bar\partial\Gamma^\alpha\varphi|^2}{(1+|\underline{u}|)^{1+\kappa}} \\ &\quad + C \sum_{|\beta|+|\gamma|\leq 8,\, k\leq 8,\, |\beta|\leq 4} \int_{u_0+\delta}^{t} \int_{\Sigma_\tau} (1+|\underline{u}|)^{1+\kappa} |\partial\Gamma^\beta\varphi|^2 |\partial_t\Gamma^\alpha\varphi|^2. \end{aligned}$$

We can choose a large C so that the first term can be absorbed by the left hand side of (5.5); for the second term, we can use Klainerman–Sobolev inequalities to control $|\partial\Gamma^\beta\varphi|^2$ and the energy norms to control the rest; this yields

$$N_2 \lesssim M^4\varepsilon^4.$$

We put all these estimates back into (5.5) to obtain

$$\sum_{|\alpha|\leq 8} \int_{\Sigma_t} |\partial\Gamma^\alpha\varphi|^2 \lesssim \varepsilon^2 + M^3\varepsilon^3 + M^4\varepsilon^4.$$

We can then take a sufficiently small ε to close the argument. □

The parameter ε is proportional to $\delta^{1/2}$. If δ is sufficiently small, we have thus constructed a solution in Region 3.

5.3. Existence from past null infinity

We will let u_0 go to $-\infty$ so that the null hypersurface C_{u_0} will approximate the past null infinity.

We choose a decreasing sequence $\{u_{0,i}\}$ in such a way that $u_{0,i} \to -\infty$ and we solve the Goursat problem for (1.1) with initial data on $C_{u_{0,i}}$. We emphasize that the choice of the seed datum ψ_0 is the same for all $u_{0,i}$'s. For each $u_{0,i}$, we obtain a unique smooth solution φ_i defined in the region

$$\mathcal{D}_i = \{p \in \mathbb{R}^{3+1} \mid t(p) \leq -1,\ u(p) \geq u_{0,i}\}.$$

Moreover, by Sobolev inequalities, for all $k \leq 8$, there exists a constant C_0 independent of i such that

$$\|\varphi_i\|_{C^k(\mathcal{D}_i)} \leq C_0.$$

Thanks to the lemma of Arzelà–Ascoli, we can extract a subsequence, still denoted by $\{\varphi_i\}$, that converges uniformly on any compact subset of $\{(t, x) \in \mathbb{R}^{3+1} \mid t \leq -1\}$. We denote the limit by φ and this is a classical solution of (1.1) all the way down to past null infinity.

To prove Main Theorem 2 (which implies Main Theorem 1 in a straightforward way), it remains to show the uniqueness from past null infinity. Suppose φ and ϕ were two classical solutions for (1.1) with the same scattering data (1.5). Then

$$\Box(\varphi - \phi) = Q(\nabla\varphi, \nabla\varphi) - Q(\nabla\phi, \nabla\phi) \triangleq F(\nabla\varphi, \nabla\phi), \tag{5.6}$$

with

$$\lim_{u_0 \to -\infty} |u_0(\varphi - \phi)|_{u=u_0} = 0.$$

For $\tau \leq -1$, let

$$E(\tau) = \int_{t=\tau} \Big(|\partial_t(\varphi - \phi)|^2 + \sum_{i=1}^{3} |\partial_i(\varphi - \phi)|^2\Big).$$

In view of Remark 1.2, we have

$$\lim_{t \to -\infty} E(t) = 0,$$

i.e. $E(t) = o(1)$. According to the standard energy estimate, we have

$$E(t) \leq E(2t) + \int_{2t}^{t} \Big(\int_{\Sigma_\tau} |F(\nabla\varphi, \nabla\phi)|\, |\partial_t(\varphi - \phi)|\Big)\, d\tau. \tag{5.7}$$

We observe that at least one factor in the quadratic form $F(\nabla\varphi, \nabla\phi)$ is a good term, i.e. it is either $\bar{\partial}\varphi$, $\bar{\partial}\phi$ or $\bar{\partial}(\varphi - \phi)$. Its L^∞ norm is bounded by $O(|t|^{-3/2})$. Therefore, (5.7) becomes

$$\begin{aligned} E(t) &\leq E(2t) + C_0 \int_{2t}^{t} |\tau|^{-3/2} E(\tau)\, d\tau \leq E(2t) + C_0 |t|^{-1/2} \sup_{\tau \in [2t, t]} E(\tau) \\ &\leq E(2t) + C_0^2 |t|^{-1/2}, \end{aligned} \tag{5.8}$$

where C_0 is the size of $E(t)$, which is actually small. We can iterate this estimate to derive (notice that the $o(1)$ below is independent of k)

$$E(t) \leq E(2^k \cdot t) + \Big(\sum_{j=0}^{k-1} 2^{-j/2}\Big) C_0^2 |t|^{-1/2}, \quad \forall k \in \mathbb{Z}_{>0}.$$

We then let $k \to \infty$ and we improve the decay $E(t) = o(1)$ to

$$E(t) \leq \frac{C_0^2}{1 - 2^{-1/2}} |t|^{-1/2}. \tag{5.9}$$

We substitute (5.9) into (5.8) to iterate again, which gives

$$E(t) \leq E(2t) + \frac{C_0^3}{2(1 - 2^{-1/2})} |t|^{-1}.$$

We repeat the above dyadic iteration and further improve the decay of $E(t)$ to

$$E(t) \leq \frac{C_0^3}{1 - 2^{-1/2}} |t|^{-1}.$$

Repeating the whole procedure we obtain, for all $k \in \mathbb{Z}_{>0}$,

$$E(t) \leq \frac{C_0^{3+k}}{(1 - 2^{-1/2}) \prod_{j=0}^{k} |1 + j/2|} |t|^{-1-k/2}.$$

Letting $k \to \infty$ implies $E(t) = 0$, which proves the uniqueness.

5.4. *Proof of Main Theorem 3*

Main Theorem 3 is an easy consequence of the estimates derived previously. We first recall that u_0 is a fixed number. From the proof of Proposition 5.1, we know that, for all $\theta \in \mathbb{S}^2$, $\underline{u} \in [0, \delta]$ and $u \in [u_0, -1]$, we have

$$\begin{aligned}
|L\varphi(u, \underline{u}, \theta) - L\varphi(u_0, \underline{u}, \theta)| &\lesssim \delta^{1/2}, \\
|\nabla\!\!\!/ \varphi(u, \underline{u}, \theta) - \nabla\!\!\!/ \varphi(u_0, \underline{u}, \theta)| &\lesssim \delta^{1/2}, \\
|\underline{L}\varphi(u, \underline{u}, \theta) - \underline{L}\varphi(u_0, \underline{u}, \theta)| &\lesssim \delta^{1/2}.
\end{aligned}$$

Since we take short pulse data localized in $B_{\delta^{1/2}}(\theta_0)$, we simply integrate the above inequalities and use the pointwise information on the initial data on C_{u_0} to get

$$\int_{C_u - C_u^o} (|L\varphi|^2 + |\nabla\!\!\!/ \varphi|^2) \lesssim \delta^2,$$

$$\left| \int_{C_u^o} (|L\varphi|^2 + |\nabla\!\!\!/ \varphi|^2) - \int_{C_{u_0}^o} (|L\varphi|^2 + |\nabla\!\!\!/ \varphi|^2) \right| \lesssim \delta.$$

Moreover, on $\underline{C}_\delta$, we have seen that almost no energy has radiated from it. In terms of estimates, this means $\underline{E}_1(u, \delta) \lesssim \delta^{1/2}$. This demonstrates the strong focusing phenomenon and completes the proof of Main Theorem 3.

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