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Degree three cohomological invariants of semisimple groups

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Abstract. We study the degree 3 cohomological invariants with coefficients in $\mathbb{Q}/\mathbb{Z}(2)$ of a semisimple group over an arbitrary field. A list of all invariants of adjoint groups of inner type is given.

Keywords. Semisimple groups, cohomological invariants, torsors, classifying space

1. Introduction

1a. Cohomological invariants. Let G be a linear algebraic group over a field F (of arbitrary characteristic). The notion of an *invariant* of G was defined in [9] as follows. Consider the functor

$$H^1(-, G) : \mathit{Fields}_F \to \mathit{Sets},$$

where Fields_F is the category of field extensions of F, taking a field K to the set $H^1(K, G)$ of isomorphism classes of G-torsors over Spec K. Let

$$H : \mathit{Fields}_F \to \mathit{Abelian\ Groups}$$

be another functor. An H-*invariant* of G is then a morphism of functors

$$I : H^1(-, G) \to H.$$

We denote the group of H-invariants of G by $\mathrm{Inv}(G, H)$.

An invariant $I \in \mathrm{Inv}(G, H)$ is called *normalized* if $I(X) = 0$ for the trivial G-torsor X. The normalized invariants form a subgroup $\mathrm{Inv}(G, H)_{\mathrm{norm}}$ of $\mathrm{Inv}(G, H)$ and there is a natural isomorphism

$$\mathrm{Inv}(G, H) \simeq H(F) \oplus \mathrm{Inv}(G, H)_{\mathrm{norm}}.$$

Of particular interest to us is the functor H which takes a field K/F to the Galois cohomology group $H^n(K, \mathbb{Q}/\mathbb{Z}(j))$, where the coefficients $\mathbb{Q}/\mathbb{Z}(j)$ are defined as the direct sum of the colimit over n of the Galois modules $\mu_n^{\otimes j}$, where μ_n is the Galois

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module of n^{th} roots of unity, and a p-component in the case $p = \operatorname{char}(F) > 0$ defined via logarithmic de Rham–Witt differentials (see [13, I.5.7], [14]).

We write $\operatorname{Inv}^n(G, \mathbb{Q}/\mathbb{Z}(j))$ for the group of *cohomological invariants of G of degree n with coefficients in* $\mathbb{Q}/\mathbb{Z}(j)$.

If G is connected, then $\operatorname{Inv}^1(G, \mathbb{Q}/\mathbb{Z}(j))_{\text{norm}} = 0$ (see [15, Proposition 31.15]). The degree 2 cohomological invariants with coefficients in $\mathbb{Q}/\mathbb{Z}(1)$ (equivalently, the invariants with values in the Brauer group Br) of a smooth connected group were determined in [1]:

$$\operatorname{Inv}^2(G, \operatorname{Br})_{\text{norm}} = \operatorname{Inv}^2(G, \mathbb{Q}/\mathbb{Z}(1))_{\text{norm}} \simeq \operatorname{Pic}(G).$$

In particular, for a semisimple group G we have

$$\operatorname{Inv}^2(G, \mathbb{Q}/\mathbb{Z}(1))_{\text{norm}} \simeq \widehat{C}(F),$$

where $\widehat{C}(F)$ is the character group of the kernel C of the universal cover $\widetilde{G} \to G$ by [21, Prop. 6.10].

The group of degree 3 invariants $\operatorname{Inv}^3(G, \mathbb{Q}/\mathbb{Z}(2))_{\text{norm}}$ was determined by Rost in the case when G is simply connected quasi-simple. This group is finite cyclic with a canonical generator called the *Rost invariant* (see [9, Part II]).

In the present paper, based on the results in [18], we extend Rost's result to all semisimple groups.

Theorem. *Let G be a semisimple group over a field F. Then there is an exact sequence*

$$0 \to \operatorname{CH}^2(BG)_{\text{tors}} \to H^1(F, \widehat{C}(1)) \xrightarrow{\sigma} \operatorname{Inv}^3(G, \mathbb{Q}/\mathbb{Z}(2))_{\text{norm}} \to Q(G)/\operatorname{Dec}(G) \to H^2(F, \widehat{C}(1)).$$

Here BG is the classifying space of G and $Q(G)/\operatorname{Dec}(G)$ is the group defined in Section 3c in terms of the combinatorial data associated with G (the root system, weight and root lattices).

If G is simply connected, the group $\widehat{C}$ is trivial and we obtain Rost's theorem mentioned above.

The main result has clearer form for adjoint groups G of inner type. We show that the group $\operatorname{Inv}^3(G, \mathbb{Q}/\mathbb{Z}(2))_{\text{dec}} := \operatorname{Im}(\sigma)$ of *decomposable* invariants (given by a cup-product with the degree 2 invariants), is canonically isomorphic to $\widehat{C} \otimes F^{\times}$. The factor group $\operatorname{Inv}^3(G, \mathbb{Q}/\mathbb{Z}(2))_{\text{ind}}$ of $\operatorname{Inv}^3(G, \mathbb{Q}/\mathbb{Z}(2))_{\text{norm}}$ by the decomposable invariants is nontrivial if and only if G has a simple component of type C_n or D_n (when n is divisible by 4), E_6 or E_7. If G is simple, the group of indecomposable invariants is cyclic with a canonical generator restricting to a multiple of the Rost invariant.

We will use the following notation:

F the base field, $\quad F_{\text{sep}}$ a separable closure of F, $\quad \Gamma_F = \operatorname{Gal}(F_{\text{sep}}/F)$.

For a complex A of étale sheaves on a variety X, we write $H^*(X, A)$ for the étale (hyper-)cohomology group of X with values in A.

2. Preliminaries

2a. Cohomology of BG. Let G be a connected algebraic group over a field F and let V be a generically free representation of G such that there is an open G-invariant subscheme $U \subset V$ and a G-torsor $U \to U/G$ such that $U(F) \neq \emptyset$ (see [26, Remark 1.4]).

Let H be a (contravariant) functor from the category of smooth varieties over F to the category of abelian groups. Very often the value $H(U/G)$ is independent (up to canonical isomorphism) of the choice of the representation V provided the codimension of $V \setminus U$ in V is sufficiently large. This is the case, for example, if $H = \mathrm{CH}^i$, the Chow group functor of cycles of codimension i (see [26] or [5]). We write $H(BG)$ for $H(U/G)$ and view U/G as an "approximation" for the "classifying space" BG of G.

We have the two maps $p_i^* : H(U/G) \to H((U \times U)/G)$, $i = 1, 2$, induced by the projections $p_i : (U \times U)/G \to U/G$. An element $h \in H(U/G)$ is called *balanced* if $p_1^*(h) = p_2^*(h)$. We write $H(U/G)_{\mathrm{bal}}$ for the subgroup of all balanced elements in $H(U/G)$.

Write $\mathcal{H}^n(\mathbb{Q}/\mathbb{Z}(j))$ for the Zariski sheaf on a smooth scheme X associated to the presheaf $U \mapsto H^n(U, \mathbb{Q}/\mathbb{Z}(j))$.

Let $u \in H^0_{\mathrm{Zar}}(U/G, \mathcal{H}^n(\mathbb{Q}/\mathbb{Z}(j)))_{\mathrm{bal}}$. Define an invariant $I_u \in \mathrm{Inv}^n(G, \mathbb{Q}/\mathbb{Z}(j))$ as follows (see [1]). Let X be a G-torsor over a field extension K/F. Choose a point $x \in (U/G)(K)$ such that X is isomorphic to the pull-back via x of the versal G-torsor $U \to U/G$ and set $I_u(X) = x^*(u)$, where

$$x^* : H^0_{\mathrm{Zar}}(U/G, \mathcal{H}^n(\mathbb{Q}/\mathbb{Z}(j))) \to H^0_{\mathrm{Zar}}(\mathrm{Spec}\, K, \mathcal{H}^n(\mathbb{Q}/\mathbb{Z}(j))) = H^n(K, \mathbb{Q}/\mathbb{Z}(j))$$

is the pull-back homomorphism given by $x : \mathrm{Spec}(K) \to U/G$. The fact that the element u is balanced ensures that $x^*(u)$ does not depend on the choice of the point x (see [1, Lemma 3.2]).

Write $\overline{H}^0_{\mathrm{Zar}}(U/G, \mathcal{H}^n(\mathbb{Q}/\mathbb{Z}(j)))$ for the factor group of $H^0_{\mathrm{Zar}}(U/G, \mathcal{H}^n(\mathbb{Q}/\mathbb{Z}(j)))$ by the natural image of $H^n(F, \mathbb{Q}/\mathbb{Z}(j))$.

Proposition 2.1 ([1, Corollary 3.4]). *The assignment $u \mapsto I_u$ yields an isomorphism*

$$\overline{H}^0_{\mathrm{Zar}}(U/G, \mathcal{H}^n(\mathbb{Q}/\mathbb{Z}(j)))_{\mathrm{bal}} \xrightarrow{\sim} \mathrm{Inv}^n(G, \mathbb{Q}/\mathbb{Z}(j))_{\mathrm{norm}}.$$

2b. The map α_G. Let G be a semisimple group over F and let C be the kernel of the universal cover $\widetilde{G} \to G$. For a character $\chi \in \widehat{C}(F)$ over F consider the push-out diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & C & \longrightarrow & \widetilde{G} & \longrightarrow & G & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle\chi} & & \downarrow & & \| & & \\ 1 & \longrightarrow & \mathbb{G}_{\mathrm{m}} & \longrightarrow & G' & \longrightarrow & G & \longrightarrow & 1 \end{array}$$

We define a map

$$\alpha_G : H^1(F, G) \to \mathrm{Hom}(\widehat{C}(F), \mathrm{Br}(F))$$

by $\alpha_G(\xi)(\chi) = \delta(\xi)$, where $\delta : H^1(F, G) \to H^2(F, \mathbb{G}_{\mathrm{m}}) = \mathrm{Br}(F)$ is the connecting map for the bottom row of the diagram.

Example 2.2. Let $G = \mathbf{PGL}_n$. Then $\widehat{C} = \mathbb{Z}/n\mathbb{Z}$ and the map α_G takes the class $[A] \in H^1(F, \mathbf{PGL}_n)$ of a central simple algebra A of degree n to the homomorphism $i + n\mathbb{Z} \mapsto i[A] \in \mathrm{Br}(F)$.

Let C' be the center of $\widetilde{G}$. Recall that there is the *Tits homomorphism* (see [15, Theorem 27.7])

$$\beta_{\widetilde{G}} : \widehat{C}'(F) \to \mathrm{Br}(F).$$

A central simple algebra over F representing the class $\beta_{\widetilde{G}}$ for some $\chi \in C'(F)$ is called a *Tits algebra* of G over F.

In the following proposition we relate the maps α_G and $\beta_{\widetilde{G}}$.

Proposition 2.3. *Let G be a semisimple group, X a G-torsor over F and $\chi \in \widehat{C}'(F)$, where C' is the center of the universal cover $\widetilde{G}$ of G. Let ${}^XG := \mathbf{Aut}_G(X)$ be the twist of G by X and ${}^X\widetilde{G}$ the universal cover of ${}^XG$. Then*

$$\alpha_G(X)(\chi|_C) = \beta_{{}^X\widetilde{G}}(\chi) - \beta_{\widetilde{G}}(\chi),$$

where $C \subset C'$ is the kernel of $\widetilde{G} \to G$.

Proof. By [15, §31], there exist a unique (up to isomorphism) G-torsor Y such that the twist ${}^YG = \mathbf{Aut}_G(Y)$ is quasi-split and $\alpha_G(Y)(\chi|_C) = -\beta_{\widetilde{G}}(\chi)$. If ${}^XY$ is the twist of Y by X, then $\mathbf{Aut}_{{}^XG}({}^XY) \simeq \mathbf{Aut}_G(Y)$ is quasi-split. Hence $\alpha_{{}^XG}({}^XY)(\chi|_C) = -\beta_{{}^X\widetilde{G}}(\chi)$. It follows from [15, Proposition 28.12] that $\alpha_{{}^XG}({}^XY) + \alpha_G(X) = \alpha_G(Y)$. □

2c. Admissible maps. Let G be a split simply connected group over F, and Π a set of simple roots of G.

Proposition 2.4 (cf. [10, Proposition 5.5]). *Let G be a split simply connected group over F, and C the center of G. Let Π' be a subset of Π and let G' be the subgroup of G generated by the root subgroups of all roots in Π'. Then G' is a simply connected group and $C \subset G'$ if and only if every fundamental weight w_α for $\alpha \in \Pi \setminus \Pi'$ is contained in the root lattice Λ_r of G.*

Proof. The group G' is simply connected by [22, 5.4b]. The images of the co-roots $\alpha^* : \mathbb{G}_m \to T$ for $\alpha \in \Pi'$ generate the maximal torus $T' = G' \cap T$ of G'. Therefore, the character group Ω of the torus T/T' coincides with

$$\{\lambda \in \widehat{T} : \langle \lambda, \alpha^* \rangle = 0 \text{ for all } \alpha \in \Pi'\}$$

and hence Ω is generated by the fundamental weights w_β for all $\beta \in \Pi \setminus \Pi'$. We have $\widehat{T}' = \Lambda_w/\Omega$ and $\widehat{C} = \Lambda_w/\Lambda_r$. Therefore, $C \subset G' \cap T = T'$ if and only if $\Omega \subset \Lambda_r$. □

A homomorphism $a : \widehat{C}(F) \to \mathrm{Br}(F)$ is called *admissible* if $\operatorname{ind} a(\chi)$ divides $\operatorname{ord}(\chi)$ for every $\chi \in \widehat{C}$.

Example 2.5. Suppose G is the product of split adjoint groups of type A. By Example 2.2, every admissible map belongs to the image of α_G.

Proposition 2.6. *Let G be a split adjoint group over F. Then every admissible map in* $\operatorname{Hom}(\widehat{C}(F), \operatorname{Br}(F))$ *belongs to the image of* α_G.

Proof. Let Π' be the subset of Π of all roots α such that $w_\alpha \in \Lambda_r$ and let G' be the subgroup of $\widetilde{G}$ generated by the root subgroups for all roots in Π'. Then by Proposition 2.4, G' is a simply connected group such that $C \subset G'$. Let C' be the center of G' and set $C'' := C'/C$. By Lemma 2.7 below, the top row in the commutative diagram

$$\begin{array}{ccccc}
H^1(F, G'/C) & \longrightarrow & H^1(F, G'/C') & \longrightarrow & \operatorname{Hom}(\widehat{C}''(F), \operatorname{Br}(F)) \\
\downarrow{\scriptstyle \alpha_{G'/C}} & & \downarrow{\scriptstyle \alpha_{G'/C'}} & & \| \\
\operatorname{Hom}(\widehat{C}(F), \operatorname{Br}(F)) & \hookrightarrow & \operatorname{Hom}(\widehat{C}'(F), \operatorname{Br}(F)) & \longrightarrow & \operatorname{Hom}(\widehat{C}''(F), \operatorname{Br}(F))
\end{array}$$

is exact.

Let $a \in \operatorname{Hom}(\widehat{C}(F), \operatorname{Br}(F))$ be an admissible map. Then the image a' of a in $\operatorname{Hom}(\widehat{C}'(F), \operatorname{Br}(F))$ is also admissible. Inspection shows that every component of the Dynkin diagram of G' is of type A. (A root α belongs to Π' if and only if the i^{th} row of the inverse C^{-1} of the Cartan matrix is integer, see Section 4b.) By Example 2.5, a' belongs to the image of $\alpha_{G'/C'}$. A diagram chase shows that a belongs to the image of $\alpha_{G'/C}$. The map $\alpha_{G'/C}$ is the composition of $H^1(F, G'/C) \to H^1(F, G)$ and α_G, hence a belongs to the image of α_G. □

Lemma 2.7. *Let $G_1 \to G_2$ be a central isogeny of split semisimple groups with kernel C_1. Then the sequence*

$$H^1(F, G_1) \to H^1(F, G_2) \to \operatorname{Hom}(\widehat{C}_1(F), \operatorname{Br}(F)),$$

where the second map is the composition of α_{G_2} and the restriction map on C_1, is exact.

Proof. The group C_1 is diagonalizable as G_1 is split. Let T be a split torus containing C_1 as a subgroup. The push-out diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & C_1 & \longrightarrow & G_1 & \longrightarrow & G_2 & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \chi} & & \downarrow & & \| & & \\
1 & \longrightarrow & T & \longrightarrow & G_3 & \longrightarrow & G_2 & \longrightarrow & 1
\end{array}$$

yields a commutative diagram

$$\begin{array}{ccccc}
H^1(F, G_1) & \longrightarrow & H^1(F, G_2) & \longrightarrow & \operatorname{Hom}(\widehat{C}_1(F), \operatorname{Br}(F)) \\
\downarrow & & \| & & \downarrow{\scriptstyle \chi^*} \\
H^1(F, G_3) & \longrightarrow & H^1(F, G_2) & \longrightarrow & \operatorname{Hom}(\widehat{T}(F), \operatorname{Br}(F))
\end{array}$$

The bottom row is exact as $\operatorname{Hom}(\widehat{T}(F), \operatorname{Br}(F)) = H^2(F, T)$. The left vertical arrow is surjective since $H^1(F, \operatorname{Coker}(\chi)) = 1$ by Hilbert's Theorem 90. The result follows by diagram chase. □

2d. The morphism β_f. Let G be a semisimple group, C the kernel of the universal cover $\widetilde{G} \to G$ and $f : X \to \operatorname{Spec} F$ a G-torsor. Write $\mathbb{Z}_f(1)$ for the cone of the natural morphism $\mathbb{Z}_F(1) \to Rf_*\mathbb{Z}_X(1)$ of complexes of étale sheaves over $\operatorname{Spec} F$, where $\mathbb{Z}(1) = \mathbb{G}_m[-1]$. The composition (see [18, §4])

$$\beta_f : \widehat{C} \simeq \tau_{\leq 2}\mathbb{Z}_f(1)[2] \to \mathbb{Z}_f(1)[2] \to \mathbb{Z}_F(1)[3]$$

yields a homomorphism

$$\beta_f^* : \widehat{C}(F) \to H^3(F, \mathbb{Z}_F(1)) = \operatorname{Br}(F).$$

In the following proposition we relate the maps β_f^* and α_G.

Proposition 2.8. *For a G-torsor $f : X \to \operatorname{Spec} F$, we have $\beta_f^* = \alpha_G(X)$.*

Proof. By [18, Example 6.12], the map β_f^* coincides with the connecting homomorphism for the exact sequence

$$1 \to F_{\mathrm{sep}}^{\times} \to F_{\mathrm{sep}}(X)^{\times} \to \operatorname{Div}(X_{\mathrm{sep}}) \to \widehat{C}_{\mathrm{sep}} \to 0, \tag{2.1}$$

where Div is the divisor group (recall that $\widehat{C}_{\mathrm{sep}} = \operatorname{Pic}(X_{\mathrm{sep}})$).

Consider first the case where $G = \mathbf{PGL}_n$ and $X = \operatorname{Isom}(B, M_n)$ is the variety of isomorphisms between a central simple algebra B of degree n and the matrix algebra M_n over F. We have $C = \boldsymbol{\mu}_n$ and $\widehat{C} = \mathbb{Z}/n\mathbb{Z}$. The exact sequence (2.1) for the Severi–Brauer variety S of B in place of X gives the connecting homomorphism $\mathbb{Z} \to \operatorname{Br}(F)$ that takes 1 to the class $[B]$ by [12, Theorem 5.4.10]. A natural map between the two exact sequences induced by the natural morphism $X \to S$ and Example 2.2 yields

$$\beta_f^*(\bar{1}) = [B] = \alpha_{\mathbf{PGL}_n}(X)(\bar{1}). \tag{2.2}$$

Suppose now that $G = \mathbf{PGL}_1(A)$ for a central simple algebra A of degree n. Consider the $\mathbf{PGL}_n$-torsor $Y = \operatorname{Isom}(A, M_n)$. Then G is the twist of $\mathbf{PGL}_n$ by Y. The G-torsor $Z = \operatorname{Isom}(B, A)$ is the twist of X by Y. It follows from [15, Proposition 28.12] that

$$\alpha_G(Z)(\bar{1}) = \alpha_{\mathbf{PGL}_n}(X)(\bar{1}) - \alpha_{\mathbf{PGL}_n}(Y)(\bar{1}) = [B] - [A]. \tag{2.3}$$

The group homomorphism $\mathbf{PGL}_1(B) \times \mathbf{PGL}_1(A^{\mathrm{op}}) \to \mathbf{PGL}_1(B \otimes A^{\mathrm{op}})$ takes the torsor $Z \times \operatorname{Isom}(A^{\mathrm{op}}, A^{\mathrm{op}})$ to $V := \operatorname{Isom}(B \otimes A^{\mathrm{op}}, A \otimes A^{\mathrm{op}})$. Let g and h be the structure morphisms for Z and V, respectively. It follows from (2.2) applied to β_h^* and (2.3) that

$$\beta_g^*(\bar{1}) = \beta_h^*(\bar{1}) = [B] - [A] = \alpha_G(Z)(\bar{1}). \tag{2.4}$$

Now consider the general case. By [25, Théorème 3.3], for every $\chi \in \widehat{C}(F)$, there is a central simple algebra A (of degree n) over F and a commutative diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & C & \longrightarrow & \widetilde{G} & \longrightarrow & G & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \chi} & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \boldsymbol{\mu}_n & \longrightarrow & \mathbf{SL}_1(A) & \longrightarrow & \mathbf{PGL}_1(A) & \longrightarrow & 1
\end{array}$$

A G-torsor $f : X \to \operatorname{Spec} F$ yields a $\mathbf{PGL}_1(A)$-torsor, say $k : W \to \operatorname{Spec} F$. By (2.4), we have

$$\beta_f^*(\chi) = \beta_k^*(\bar{1}) = \alpha_{\mathbf{PGL}_1(A)}(W)(\bar{1}) = \alpha_G(X)(\chi). \qquad \square$$

3. The group $\operatorname{Inv}^3(G, \mathbb{Q}/\mathbb{Z}(2))$

In this section we determine the group $\operatorname{Inv}^3(G, \mathbb{Q}/\mathbb{Z}(2))$ of degree 3 cohomological invariants of a semisimple group G.

Recall first the degree two cohomological invariants of G with coefficients in $\mathbb{Q}/\mathbb{Z}(1)$, or equivalently, the invariants with values in the Brauer group. Every character $\chi \in \widehat{C}(F)$ yields an invariant I_χ of G of degree 2 with coefficients in $\mathbb{Q}/\mathbb{Z}(1)$ defined by

$$I_\chi(X) = \alpha_G(X)(\chi_K) \in \operatorname{Br}(K).$$

By [1, Theorem 2.4], the assignment $\chi \mapsto I_\chi$ yields an isomorphism

$$\widehat{C}(F) \xrightarrow{\sim} \operatorname{Inv}^2(G, \mathbb{Q}/\mathbb{Z}(1))_{\mathrm{norm}}.$$

3a. Representation ring. (See [25].) Write $R(G)$ for the representation ring of G, i.e., $R(G)$ is the Grothendieck group of the category of finite-dimensional representations of G. As an abelian group, $R(G)$ is free with basis the isomorphism classes of irreducible representations.

Consider the weight lattice Λ of G (the character group of a maximal split torus over F_{sep}) as a Γ_F-lattice with respect to the $*$-action (see [24]). Let Γ' be the (finite) factor group of Γ_F acting faithfully on Λ. Write Δ for the semidirect product of the Weyl group W of G and Γ' with respect to the natural action of Γ' on W. The group Δ acts naturally on Λ.

Assigning the character to a representation of G, we get an injective homomorphism

$$\mathrm{ch} : R(G) \to \mathbb{Z}[\Lambda]^{\Delta}.$$

For any $\lambda \in \Lambda$ write A_λ for the corresponding Tits algebra (over the field of definition of λ) and $\Delta(\lambda)$ for the sum $\sum e^{\lambda'}$ in $\mathbb{Z}[\Lambda]^{\Delta}$, where λ' runs over the Δ-orbit of λ (we employ the exponential notation for $\mathbb{Z}[\Lambda]$). By [9, Part II, Theorem 10.11], the image of $R(G)$ in $\mathbb{Z}[\Lambda]^{\Delta}$ is generated by $\operatorname{ind}(A_\lambda) \cdot \Delta(\lambda)$ over all $\lambda \in \Lambda$.

In particular, if G is quasi-split, all Tits algebras are trivial and hence ch is an isomorphism.

Example 3.1. Consider the variety $\mathcal{X}$ of maximal tori in G and the closed subscheme $\mathcal{T} \subset G \times \mathcal{X}$ of all pairs (g, T) with $g \in T$. The generic fiber of the projection $\mathcal{T} \to \mathcal{X}$ is a maximal torus in $G_{F(\mathcal{X})}$, called the *generic maximal torus* T_{gen} of G. By [27, Theorem 1], if G is split, the decomposition group of T_{gen} coincides with the Weyl group W. It follows that if G is quasi-split, then Δ is the decomposition group of T_{gen}. Moreover, ch is an isomorphism, hence the restriction homomorphism $R(G) \to R(T_{\mathrm{gen}}) = \mathbb{Z}[\Lambda]^{\Delta}$ is an isomorphism for a quasi-split G.

3b. Root systems and invariant quadratic forms. Let $\{\alpha_1, \dots \alpha_n\}$ be a set of simple roots of an irreducible root system in a vector space V, $\{w_1, \dots, w_n\}$ the corresponding fundamental weights generating the weight lattice Λ_w, and W the Weyl group.

Consider the n-columns $\alpha := \sum \alpha_i e_i$ and $w := \sum w_i e_i$, where $\{e_i\}$ is the standard basis in $\mathbb{Z}^n$. Then $\alpha = Cw$, where $C = (c_{ij})$ is the Cartan matrix (see [2, Chapitre VI]). There is a (unique) W-invariant bilinear form on the dual space V^* such that the length of a short co-root is equal to 1. Let $D := \operatorname{diag}(d_1, \dots, d_n)$ be the diagonal matrix with d_i the length of the i^{th} co-root. Then DC is a symmetric even integer matrix (i.e., the diagonal terms are even).

Note that if A is a symmetric $n \times n$ matrix over $\mathbb{Q}$, then $\frac{1}{2} w^t A w$ is contained in $\mathit{Sym}^2(\Lambda_w)$ if and only if the matrix A is even integer.

Consider the integer quadratic form

$$q := \tfrac{1}{2} w^t DCw \in \mathit{Sym}^2(\Lambda_w)$$

on Λ_r^*, where Λ_r is the root lattice. Recall that the Weyl group W acts naturally on Λ_w.

Lemma 3.2. *The quadratic form q is W-invariant.*

Proof. Let s_i be the reflection with respect to α_i. It suffices to prove that $s_i(q) = q$. We have $s_i(w) = w - \alpha_i e_i$. Hence

$$\begin{aligned} s_i(q) &= \tfrac{1}{2}(w - \alpha_i e_i)^t DC(w - \alpha_i e_i) = q - \alpha_i e_i^t D\big(Cw - \tfrac{1}{2}\alpha_i Ce_i\big) \\ &= q - \alpha_i d_i \big(e_i^t \alpha - \tfrac{1}{2}\alpha_i e_i^t Ce_i\big) = q - \alpha_i d_i \big(\alpha_i - \tfrac{1}{2}\alpha_i c_{ii}\big) = q \end{aligned}$$

as $c_{ii} = 2$. ◻

If α_i^* is a short co-root, then $q(\alpha_i^*) = d_i = 1$ since $\langle w_j, \alpha_i^* \rangle = \delta_{ji}$. It follows that q is a (canonical) generator of the cyclic group $\mathit{Sym}^2(\Lambda_w)^W$.

Example 3.3. For the root system of type A_{n-1}, $n \geq 2$, we have $\Lambda_w = \mathbb{Z}^n/\mathbb{Z}e$, where $e = e_1 + \cdots + e_n$. The root lattice Λ_r is generated by the simple roots $\bar{e}_1 - \bar{e}_2, \bar{e}_2 - \bar{e}_3, \dots, \bar{e}_{n-1} - \bar{e}_n$. The Weyl group W is the symmetric group S_n acting naturally on Λ_w. The generator of $\mathit{Sym}^2(\Lambda_w)^W$ is the form

$$q = -\sum_{i<j} \bar{x}_i \bar{x}_j = \frac{1}{2} \sum_{i=1}^{n} \bar{x}_i^2.$$

The group $\mathit{Sym}^2(\Lambda_r)^W = \mathit{Sym}^2(\Lambda_r) \cap \mathit{Sym}^2(\Lambda_w)^W$ is also cyclic with the canonical generator a positive multiple of q.

Proposition 3.4. *Let m be the smallest positive integer such that the matrix mDC^{-1} is even integer. Then mq is a generator of $\mathit{Sym}^2(\Lambda_r)^W$.*

Proof. Rewrite q in the form $q = \frac{1}{2}(C^{-1}\alpha)^t DC(C^{-1}\alpha) = \frac{1}{2}\alpha^t DC^{-1}\alpha$. The multiple mq is contained in $\mathit{Sym}^2(\Lambda_r)$ if and only if the matrix mDC^{-1} is even integer. ◻

3c. The groups $\mathrm{Dec}(G) \subset Q(G)$**.** Let A be a lattice. Consider the *abstract total Chern class* homomorphism

$$c_\bullet : \mathbb{Z}[A] \to \mathit{Sym}^\bullet(A)[[t]]^\times$$

defined by $c_\bullet(e^a) = 1 + at$. We define the *abstract Chern class maps*

$$c_i : \mathbb{Z}[A] \to \mathit{Sym}^i(A), \qquad i \geq 0,$$

by $c_\bullet(x) = \sum_{i\geq 0} c_i(x)t^i$. Clearly, $c_0(x) = 1$,

$$c_1\Big(\sum_i e^{a_i}\Big) = \sum_i a_i, \qquad c_2\Big(\sum_i e^{a_i}\Big) = \sum_{i<j} a_i a_j,$$

c_1 is a homomorphism and

$$c_2(x + y) = c_2(x) + c_1(x)c_1(y) + c_2(y)$$

for all $x, y \in \mathbb{Z}[A]$.

If a group W acts on A, then all the c_i are W-equivariant.

Suppose that $A^W = 0$. Then c_1 is zero on $\mathbb{Z}[A]^W$ and c_2 yields a group homomorphism

$$c_2 : \mathbb{Z}[A]^W \to \mathit{Sym}^2(A)^W. \tag{3.1}$$

We write $\mathrm{Dec}(A)$ for the image of this homomorphism. The group $\mathrm{Dec}(A)$ is generated by the *decomposable* elements $\sum_{i<j} a_i a_j$, where $\{a_1, \ldots, a_n\}$ is a W-invariant subset of A. We also have

$$c_2(xy) = \mathrm{rank}(x)c_2(y) + \mathrm{rank}(y)c_2(x) \tag{3.2}$$

for all $x, y \in \mathbb{Z}[A]^W$, where $\mathrm{rank} : \mathbb{Z}[A] \to \mathbb{Z}$ is the map $e^a \mapsto 1$. If $S \subset A$ is a finite W-invariant subset, then since $\sum_{x\in S} x \in A^W = 0$, we have

$$c_2\Big(\sum_{a\in S} e^a\Big) = -\frac{1}{2}\sum_{a\in S} a^2. \tag{3.3}$$

Let G be a semisimple group over F. Recall that the weight lattice Λ is a Δ-module (see Section 3a). Note that $\Lambda^W = 0$, so we have the homomorphism (3.1) of Γ_F-modules with $A = \Lambda$.

Set

$$Q(G) := \mathit{Sym}^2(\Lambda)^\Delta = (\mathit{Sym}^2(\Lambda)^W)^{\Gamma_F}$$

and write $\mathrm{Dec}(G)$ for the image of the composition

$$\tau : R(G) \xrightarrow{\mathrm{ch}} \mathbb{Z}[\Lambda]^\Delta \xrightarrow{c_2} \mathit{Sym}^2(\Lambda)^\Delta = Q(G). \tag{3.4}$$

Example 3.5. The map $\tau : R(\mathbf{SL}_n) \to Q(\mathbf{SL}_n)$ takes the class of the tautological representation to the quadratic form $\sum_{i<j} \bar{x}_i\bar{x}_j$ which is the negative of the canonical generator of $Q(\mathbf{SL}_n)$ (see Example 3.3).

It follows from Example 3.5 that if G is a quasi-simple group, then for a representation ρ of G, we have $\tau(\rho) = -N(\rho)q$, where $N(\rho)$ is the Dynkin index of ρ (see [7]). Hence the image of $\mathrm{Dec}(G)$ under τ is equal to $n_G\mathbb{Z}q$, where n_G is the gcd of the Dynkin indexes of all the representations of G. The numbers n_G for split adjoint groups G of types B_n, C_n and E_7 were computed in [7] (see also Section 4b).

A *loop* in G is a group homomorphism $\mathbb{G}_{\mathrm{m}} \to G_{\mathrm{sep}}$ over F_{sep} (see [15, §31]). By [9, Part II, §7]), the group $Q(G)$ has an intrinsic description as the group of all Γ_F-invariant quadratic integral-valued functions on the set of all loops in G. It follows that a homomorphism $G \to G'$ of semisimple groups yields a group homomorphism $Q(G') \to Q(G)$. The functoriality of the Chern class shows that this homomorphism takes $\mathrm{Dec}(G')$ into $\mathrm{Dec}(G)$.

3d. The key diagram. Let V be a generically free representation of G such that there is an open G-invariant subscheme $U \subset V$ and a G-torsor $U \to U/G$ such that $U(F) \neq \emptyset$ (see Section 2a). We assume in addition that $V \setminus U$ is of codimension at least 3.

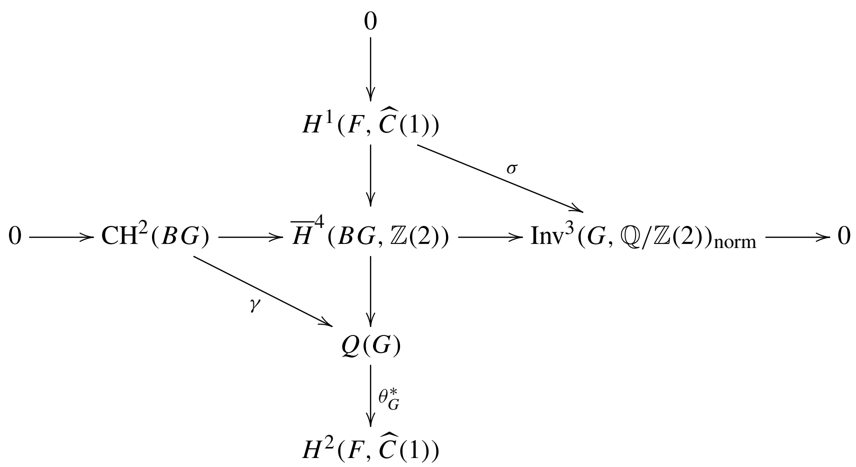
By [14, Th. 1.1], there is an exact sequence

$$0 \to \mathrm{CH}^2(U^n/G) \to \overline{H}^4(U^n/G, \mathbb{Z}(2)) \to \overline{H}^0_{\mathrm{Zar}}(U^n/G, \mathcal{H}^3(\mathbb{Q}/\mathbb{Z}(2))) \to 0$$

for every n. We can view it as an exact sequence of cosimplicial groups. The group $\mathrm{CH}^2(U^n/G)$ is independent of n, so it represents a constant cosimplicial group $\mathrm{CH}^2(BG)$. Therefore, we have an exact sequence

$$0 \to \mathrm{CH}^2(BG) \to \overline{H}^4(U/G, \mathbb{Z}(2))_{\mathrm{bal}} \to \overline{H}^0_{\mathrm{Zar}}(U/G, \mathcal{H}^3(\mathbb{Q}/\mathbb{Z}(2)))_{\mathrm{bal}} \to 0.$$

The right group in the sequence is canonically isomorphic to $\mathrm{Inv}^3(G, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{norm}}$ by Proposition 2.1, and hence is independent of V. Therefore, the middle term is also independent of V and we write $\overline{H}^4(BG, \mathbb{Z}(2))$ for $\overline{H}^4(U/G, \mathbb{Z}(2))_{\mathrm{bal}}$. Therefore, we have the exact row in the following diagram with the exact column given by [18, Theorem 5.3]:



where $\widehat{C}(1)$ is the derived tensor product $\widehat{C} \overset{L}{\otimes} \mathbb{Z}_Y(1)$ in the derived category of étale sheaves on F. Explicitly (see [18, Section 4c]),

$$\widehat{C}(1) = \mathrm{Tor}_1^{\mathbb{Z}}(\widehat{C}_{\mathrm{sep}}, F_{\mathrm{sep}}^{\times}) \oplus (\widehat{C}_{\mathrm{sep}} \otimes F_{\mathrm{sep}}^{\times})[-1].$$

Example 3.6. The group $\mathbf{SL}_n$ is special simply connected, hence we have $\widehat{C} = 0$ and $\mathrm{Inv}^3(\mathbf{SL}_n, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{norm}} = 0$. This yields isomorphisms of infinite cyclic groups

$$\gamma : \mathrm{CH}^2(B\,\mathbf{SL}_n) \xrightarrow{\sim} \overline{H}^4(B\,\mathbf{SL}_n, \mathbb{Z}(2)) \xrightarrow{\sim} Q(\mathbf{SL}_n).$$

The group $\mathrm{CH}^2(B\,\mathbf{SL}_n)$ is generated by c_2 of the tautological representation by [20, §2].

3e. The map σ. The map σ is defined as follows (see [18, §5]). Let $f : X \to \operatorname{Spec} K$ be a G-torsor over a field extension K/F, so we have a morphism $\beta_f : \widehat{C} \to \mathbb{Z}_K(1)[3]$ as in Section 2d, and therefore the composition

$$\widehat{C}(1) = \widehat{C} \overset{L}{\otimes} \mathbb{Z}_F(1) \xrightarrow{\beta_f \overset{L}{\otimes} \mathrm{Id}} (\mathbb{Z}_K(1) \overset{L}{\otimes} \mathbb{Z}_F(1))[3] \to \mathbb{Z}_K(2)[3],$$

which induces a homomorphism $H^1(F, \widehat{C}(1)) \to H^4(K, \mathbb{Z}(2)) = H^3(K, \mathbb{Q}/\mathbb{Z}(2))$. Then the value of the invariant $\sigma(\alpha)$ for an element $\alpha \in H^1(F, \widehat{C}(1))$ is equal to the image of α under this homomorphism.

Let $\chi \in \widehat{C}(F)$ and $a \in F^{\times}$. By [18, Remark 5.2], we have $\chi \cup (a) \in H^1(F, \widehat{C}(1))$, and therefore $\sigma(\chi \cup (a))$ is the invariant taking a G-torsor X over K to $\alpha_G(X)(\chi_K) \cup (a) \in H^3(K, \mathbb{Q}/\mathbb{Z}(2))$. Here the cup-product is taken with respect to the pairing

$$\mathrm{Br}(K) \otimes K^{\times} = H^2(K, \mathbb{Q}/\mathbb{Z}(1)) \otimes H^1(K, \mathbb{Z}(1)) \to H^3(K, \mathbb{Q}/\mathbb{Z}(2)).$$

3f. The map γ. We will determine the map γ in the key diagram.

Lemma 3.7. *The maps γ and $\overline{H}^4(BG, \mathbb{Z}(2)) \to Q(G)$ are functorial in G.*

Proof. In [18] the map γ is given by the composition

$$\mathrm{CH}^2(BG) \to H^4(BG, \mathbb{Z}(2)) \xrightarrow{\sim} H^3(BG, \mathbb{Z}_f(2)) \xrightarrow{\sim} H^3(BG, \tau_{\leq 3}\mathbb{Z}_f(2)) \to H^1_{\mathrm{Zar}}(BG, K_2)^{\Gamma_F} \to D(G),$$

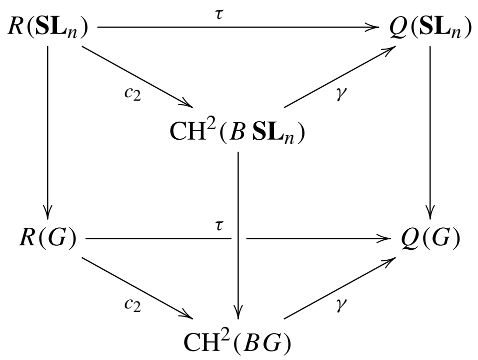
where $\mathbb{Z}_f(2)$ is the cone of $\mathbb{Z}_{BG}(2) \to Rf_*\mathbb{Z}_{EG}(2)$ for the versal G-torsor $f : EG \to BG$ and the group $D(G)$ containing $Q(G)$ is defined in [18]. The first four homomorphisms are functorial in G, and the last one is functorial as was shown in [9, p. 116] in the case G is simply connected; the proof goes through for an arbitrary semisimple G. □

Lemma 3.8. *The composition of the second Chern class map*

$$R(G) \to K_0(BG) \xrightarrow{c_2} \mathrm{CH}^2(BG)$$

with the diagonal morphism γ in the diagram coincides with the map τ in (3.4) up to sign. The image of γ coincides with $\mathrm{Dec}(G)$.

Proof. As $Q(G)$ injects when the base field gets extended, for the proof of the first statement we may assume that F is separably closed. Let $\rho : G \to \mathbf{SL}_n$ be a representation. Write $x_1, \dots, x_n$ for the characters of ρ in the weight lattice Λ. Consider the diagram



with the vertical homomorphisms induced by ρ. The vertical faces of the diagram are commutative by Lemma 3.7 and the functoriality of c_2 and the character map ch. By Example 3.5, the top map τ takes the class of the tautological representation ι of $\mathbf{SL}_n$ to a generator of $Q(\mathbf{SL}_n)$. By Example 3.6, γ in the top of the diagram is an isomorphism taking the canonical generator of $\mathrm{CH}^2(B\,\mathbf{SL}_n)$ to a generator of $Q(\mathbf{SL}_n)$. It follows that $\tau(\iota)$ and $\gamma(c_2(\iota))$ in the top face of the diagram are equal up to sign. The class of ρ in $R(G)$ is the image of τ under the left vertical homomorphism. It follows that $\tau(\rho)$ and $\gamma(c_2(\rho))$ in the bottom face of the diagram are also equal up to sign.

The second statement follows from the first and the surjectivity of the second Chern class map $R(G) \to \mathrm{CH}^2(BG)$ (see [6, Appendix C] and [26, Corollary 3.2]). □

3g. Main theorem. The following theorem describes the group of degree 3 cohomological invariants with coefficients in $\mathbb{Q}/\mathbb{Z}(2)$ of an arbitrary semisimple group.

Theorem 3.9. *Let G be a semisimple group over a field F. Then there is an exact sequence*

$$0 \to \mathrm{CH}^2(BG)_{\mathrm{tors}} \to H^1(F, \widehat{C}(1))$$
$$\xrightarrow{\sigma} \mathrm{Inv}^3(G, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{norm}} \to Q(G)/\mathrm{Dec}(G) \xrightarrow{\theta_G^*} H^2(F, \widehat{C}(1)).$$

Proof. Follows from the key diagram above and Lemma 3.8 as $Q(G)$ is torsion-free and $H^1(F, \widehat{C}(1))$ is torsion. □

Remark 3.10. The map θ_G^* is trivial if G is split or adjoint of inner type (see [18, Proposition 4.1 and Remark 5.5]).

The exact sequence in Theorem 3.9 is functorial in G. More precisely, let $G \to G'$ be a homomorphism of semisimple groups extending to a homomorphism $C \to C'$ of the kernels of the universal covers. By Lemma 3.7, the diagram

$$\begin{array}{ccccc} H^1(F,\widehat{C}'(1)) & \xrightarrow{\sigma'} & \mathrm{Inv}^3(G',\mathbb{Q}/\mathbb{Z}(2))_{\mathrm{norm}} & \longrightarrow & Q(G')/\mathrm{Dec}(G') \\ \downarrow & & \downarrow & & \downarrow \\ H^1(F,\widehat{C}(1)) & \xrightarrow{\sigma} & \mathrm{Inv}^3(G,\mathbb{Q}/\mathbb{Z}(2))_{\mathrm{norm}} & \longrightarrow & Q(G)/\mathrm{Dec}(G) \end{array}$$

is commutative.

Write $\mathrm{Inv}^3(G,\mathbb{Q}/\mathbb{Z}(2))_{\mathrm{dec}}$ for the image of σ. We call these invariants *decomposable*. Thus, we have an exact sequence

$$0 \to \mathrm{CH}^2(BG)_{\mathrm{tors}} \to H^1(F,\widehat{C}(1)) \xrightarrow{\sigma} \mathrm{Inv}^3(G,\mathbb{Q}/\mathbb{Z}(2))_{\mathrm{dec}} \to 0.$$

We do not know if the group $\mathrm{CH}^2(BG)_{\mathrm{tors}}$ is trivial, but it is always finite:

Proposition 3.11. *The group* $\mathrm{CH}^2(BG)$ *is finitely generated. In particular,* $\mathrm{CH}^2(BG)_{\mathrm{tors}}$ *is finite.*

Proof. By [25, Théorème 3.3] and Section 3a, we have

$$\mathbb{Z}[\Lambda_r]^{\Delta} \subset R(G) \subset \mathbb{Z}[\Lambda_w].$$

The Noetherian ring $\mathbb{Z}[\Lambda_r]$ is finite over $\mathbb{Z}[\Lambda_r]^{\Delta}$, hence $\mathbb{Z}[\Lambda_r]^{\Delta}$ is Noetherian. The $\mathbb{Z}[\Lambda_r]^{\Delta}$-algebra $\mathbb{Z}[\Lambda_w]$ is finite, hence so is $R(G)$. It follows that the ring $R(G)$ is Noetherian. Let I be the kernel of the rank map $R(G) \to \mathbb{Z}$. Since I is finitely generated, the factor group $R(G)/I^2$ is finitely generated. By (3.2), the second Chern class factors through a surjective homomorphism $R(G)/I^2 \to \mathrm{CH}^2(BG)$, whence the result. □

We will show in Section 4a that the group $\mathrm{CH}^2(BG)_{\mathrm{tors}}$ is trivial if G is adjoint of inner type.

The factor group

$$\mathrm{Inv}^3(G,\mathbb{Q}/\mathbb{Z}(2))_{\mathrm{ind}} := \mathrm{Inv}^3(G,\mathbb{Q}/\mathbb{Z}(2))/\mathrm{Inv}^3(G,\mathbb{Q}/\mathbb{Z}(2))_{\mathrm{dec}}$$

is called the group of *indecomposable* invariants. Thus, we have an exact sequence

$$0 \to \mathrm{Inv}^3(G,\mathbb{Q}/\mathbb{Z}(2))_{\mathrm{ind}} \to Q(G)/\mathrm{Dec}(G) \xrightarrow{\theta_G^*} H^2(F,\widehat{C}(1)).$$

If G is simply connected quasi-simple, all decomposable invariants are trivial, and the group $\mathrm{Inv}^3(G,\mathbb{Q}/\mathbb{Z}(2)) = \mathrm{Inv}^3(G,\mathbb{Q}/\mathbb{Z}(2))_{\mathrm{ind}} \simeq Q(G)/\mathrm{Dec}(G)$ is cyclic generated by the *Rost invariant* R_G. The order of the *Rost number* n_G of R_G is determined in [9, Part II].

4. Groups of inner type

Let G be a semisimple group over F. A group G' is called an *inner form of* G if there is a G-torsor X over F such that G' is the twist of G by X, or equivalently, $G' \simeq \mathbf{Aut}_G(X)$.

The choice of the torsor X yields a canonical bijection $\varphi : H^1(K, G') \xrightarrow{\sim} H^1(K, G)$ for every field extension K/F (see [15, Proposition 8.8]). Therefore, we have an isomorphism $\operatorname{Inv}^n(G, \mathbb{Q}/\mathbb{Z}(j)) \xrightarrow{\sim} \operatorname{Inv}^n(G', \mathbb{Q}/\mathbb{Z}(j))$. Note that this isomorphism does not preserve normalized invariants as φ does not preserve trivial torsors. Precisely, φ takes the class of a trivial torsor to the class of X. We modify the isomorphism to get an isomorphism

$$\operatorname{Inv}^n(G, \mathbb{Q}/\mathbb{Z}(j))_{\mathrm{norm}} \xrightarrow{\sim} \operatorname{Inv}^n(G', \mathbb{Q}/\mathbb{Z}(j))_{\mathrm{norm}}, \tag{4.1}$$

taking an invariant I of G to an invariant I' of G' satisfying

$$I'(X') = I(\varphi(X')) - I(X).$$

4a. Decomposable invariants. Let G be a semisimple group of inner type. Then $\widehat{C}$ is a diagonalizable finite group.

Lemma 4.1. *There is a natural isomorphism* $H^1(F, \widehat{C}(1)) \simeq \widehat{C} \otimes F^\times$.

Proof. Write $\widehat{C} \simeq R/S$, where R and S are lattices. In the exact sequence

$$H^1(F, S(1)) \to H^1(F, R(1)) \to H^1(F, \widehat{C}(1)) \to H^2(F, S(1))$$

the first two terms are $S \otimes F^\times$ and $R \otimes F^\times$, respectively, and the last term is equal to $S \otimes H^2(F, \mathbb{Z}(1)) = 0$ by Hilbert's Theorem 90. The result follows. □

Recall that under the isomorphism in Lemma 4.1, the map σ in Theorem 3.9 is defined as follows. For every $\chi \in \widehat{C}$ and $a \in F^\times$, the invariant $\sigma(\chi \cup (a))$ takes a G-torsor X over a field extension K/F to $\alpha_G(X)(\chi_K) \cup (a) \in H^3(K, \mathbb{Q}/\mathbb{Z}(2))$ (see Section 3e).

Theorem 4.2. *Let G be a semisimple adjoint group of inner type over a field F. Then the homomorphism*

$$\sigma : \widehat{C} \otimes F^\times \to \operatorname{Inv}^3(G, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{dec}}$$

is an isomorphism. Equivalently, the group $\mathrm{CH}^2(BG)$ *is torsion-free.*

Proof. As G is an inner form of a split group, by (4.1) we may assume that G is split. The group $\widehat{C}$ is a direct sum of cyclic subgroups generated by $\chi_1, \dots, \chi_m$, respectively. Let $a_1, \dots, a_m \in F^\times$ be such that the element $u := \sum \chi_i \otimes a_i$ belongs to the kernel of σ. It suffices to show that $a_i \in (F^\times)^{s_i}$, where $s_i := \operatorname{ord}(\chi_i)$ for all i.

Fix an integer i. For a field extension K/F and any $\rho \in H^1(K, \mathbb{Q}/\mathbb{Z})$ of order s_i, consider the admissible map $f : \widehat{C} \to \operatorname{Br}(K(t))$ for the field $K(t)$ of rational functions over K, defined by

$$f(\chi_j) = \begin{cases} \rho \cup (t) & \text{in } \operatorname{Br}(K(t)) \text{ if } j = i, \\ 0 & \text{otherwise.} \end{cases}$$

By Proposition 2.6, there is a G-torsor X over $K(t)$ satisfying $\alpha_G(X)(\chi_j) = f(\chi_j)$ for all j. As $u \in \operatorname{Ker}(\sigma)$, we have

$$0 = \sigma(u)(X) = \sum_j \alpha_G(X)(\chi_j) \cup (a_j) = \rho \cup (t) \cup (a_i)$$

in $H^3(K(t), \mathbb{Q}/\mathbb{Z}(2))$. Taking the residue at t (see [9, Part II, Appendix A]),

$$H^3_{nr}(K(t), \mathbb{Q}/\mathbb{Z}(2)) \to H^2(K, \mathbb{Q}/\mathbb{Z}(1)) = \mathrm{Br}(K),$$

we get $\rho \cup (a_i) = 0$ in $\mathrm{Br}(K)$. By Lemma 4.3 below, we have $a \in (F^\times)^{s_i}$. □

Lemma 4.3. *Let $a \in F^\times$ and $s > 0$ be such that for every field extension K/F and every $\rho \in H^1(K, \mathbb{Q}/\mathbb{Z})$ of order s one has $\rho \cup (a) = 0$ in $H^2(K, \mathbb{Q}/\mathbb{Z}(1)) = \mathrm{Br}(K)$. Then $a \in F^{\times s}$.*

Proof. Let $H = \mathbb{Z}/s\mathbb{Z}$. Choose an H-torsor $X \to Y$ with Y smooth, $\mathrm{Pic}(X) = 0$ and $F[X]^\times = F^\times$. (For example, take an approximation of $EH \to BH$.) By [3] or [17], there is an exact sequence

$$\mathrm{Pic}(X)^H \to H^2(H, F[X]^\times) \to \mathrm{Br}(Y),$$

which yields an injective map $F^\times/F^{\times s} \to \mathrm{Br}(F(Y))$ as $H^2(H, F[X]^\times) = H^2(H, F^\times) = F^\times/F^{\times s}$ and $\mathrm{Br}(Y)$ injects into $\mathrm{Br}(F(Y))$ by [19, Corollary 2.6]. This map takes a to $\rho \cup (a)$, where $\rho \in H^1(F(Y), \mathbb{Q}/\mathbb{Z})$ corresponds to the cyclic extension $F(X)/F(Y)$. As $\rho \cup (a) = 0$ by assumption, we have $a \in F^{\times s}$. □

4b. Indecomposable invariants. In this section we compute the groups of indecomposable invariants of adjoint groups of inner type.

Type A_{n-1}

In the split case we have $G = \mathbf{PGL}_n$, the projective general linear group, $n \geq 2$, $\Lambda_w = \mathbb{Z}^n/\mathbb{Z}e$, where $e = e_1 + \cdots + e_n$. The root lattice is generated by the simple roots $\bar{e}_1 - \bar{e}_2, \bar{e}_2 - \bar{e}_3, \ldots, \bar{e}_{n-1} - \bar{e}_n$, $\widehat{C} = \Lambda_w/\Lambda_r \simeq \mathbb{Z}/n\mathbb{Z}$. The generator of $\mathit{Sym}^2(\Lambda_w)^W$ is the form

$$q = -\sum_{i<j} \bar{x}_i \bar{x}_j = \frac{1}{2} \sum \bar{x}_i^2.$$

The matrix D (see Section 3b) is the identity matrix I_n. The inverses of Cartan matrices here and below are taken from [4, Appendix F]:

$$C^{-1} = \frac{1}{n} \begin{pmatrix} n-1 & n-2 & n-3 & \vdots & 2 & 1 \\ n-2 & 2(n-2) & 2(n-3) & \vdots & 4 & 2 \\ n-3 & 2(n-3) & 3(n-3) & \vdots & 6 & 3 \\ \cdots & \cdots & \cdots & & \cdots & \cdots \\ 2 & 4 & 6 & \vdots & 2(n-2) & n-2 \\ 1 & 2 & 3 & \vdots & n-2 & n-1 \end{pmatrix}.$$

By Proposition 3.4,

$$Q(G) = Sym^2(\Lambda_r)^W = \begin{cases} 2n\mathbb{Z}q & \text{if } n \text{ is even,} \\ n\mathbb{Z}q & \text{if } n \text{ is odd.} \end{cases}$$

If $a := \sum_{i,j=1}^n e^{\bar{x}_i - \bar{x}_j} \in \mathbb{Z}[\Lambda_r]^W$, by (3.3) we have

$$c_2(a) = \frac{1}{2} \sum (\bar{x}_i - \bar{x}_j)^2 = n \sum \bar{x}_i^2 = 2nq \in \mathrm{Dec}(G).$$

It follows that $\mathrm{Dec}(G) = Q(G)$ if n is even.

Suppose that n is odd. If $b = \sum_{i=1}^n e^{n\bar{x}_i} \in \mathbb{Z}[\Lambda_r]^W$, we have by (3.3),

$$c_2(b) = \frac{1}{2} \sum (n\bar{x}_i)^2 = n^2 q \in \mathrm{Dec}(G).$$

As n is odd, $\gcd(2n, n^2) = n$, hence $nq \in \mathrm{Dec}(G)$ and again $\mathrm{Dec}(G) = Q(G)$.

Thus, $\mathrm{Inv}^3(G, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{ind}} = Q(G)/\mathrm{Dec}(G) = 0$.

A G-torsor is given by a central simple algebra A of degree n (here and below see [15]). The twist of G by A is the group $\mathbf{PGL}_1(A)$. The Tits classes of algebras for this group are the multiples of $[A]$ in $\mathrm{Br}(F)$. In view of Proposition 2.3 and (4.1), we have

Theorem 4.4. *Let $G = \mathbf{PGL}_1(A)$ for a central simple algebra A over F. Then*

$$\mathrm{Inv}^3(G, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{norm}} \simeq F^\times / F^{\times n}.$$

An element $x \in F^\times$ corresponds to the invariant taking a central simple algebra A' of degree n to the cup-product $([A'] - [A]) \cup (x)$.

Type B_n

In the split case we have $G = \mathbf{O}_{2n+1}^+$, the special orthogonal group, $n \geq 2$, $\Lambda_w = \mathbb{Z}^n + \mathbb{Z}e$, where $e = \frac{1}{2}(e_1 + \cdots + e_n)$, $\Lambda_r = \mathbb{Z}^n$ and $\widehat{C} \simeq \mathbb{Z}/2\mathbb{Z}$. The generator of $Sym^2(\Lambda_w)^W$ is the form $q = \frac{1}{2} \sum_i x_i^2$, and we have $D = \mathrm{diag}(1, \ldots 1, 2)$ and

$$C^{-1} = \begin{pmatrix} 1 & 1 & 1 & \vdots & 1 & 1 & 1 \\ 1 & 2 & 2 & \vdots & 2 & 2 & 2 \\ 1 & 2 & 3 & \vdots & 3 & 3 & 3 \\ \cdots & \cdots & \cdots & & \cdots & \cdots & \cdots \\ 1 & 2 & 3 & \vdots & n-2 & n-2 & n-2 \\ 1 & 2 & 3 & \vdots & n-2 & n-1 & n-1 \\ 1/2 & 1 & 3/2 & \vdots & (n-2)/2 & (n-1)/2 & n/2 \end{pmatrix}.$$

By Proposition 3.4, $Q(G) = Sym^2(\Lambda_r)^W = 2\mathbb{Z}q$.

If $a := \sum_{i=1}^{n}(e^{x_i} + e^{-x_i}) \in \mathbb{Z}[\Lambda_r]^W$, we have

$$c_2(a) = \frac{1}{2}\sum(x_i^2 + (-x_i)^2) = 2q \in \mathrm{Dec}(G).$$

It follows that $\mathrm{Dec}(G) = Q(G)$, so $\mathrm{Inv}^3(G, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{ind}} = Q(G)/\mathrm{Dec}(G) = 0$.

A G-torsor is given by the similarity class of a nondegenerate quadratic form p of dimension $2n + 1$. The twist of G by p is the special orthogonal group $\mathbf{O}^+(p)$ of the form p. The only nontrivial Tits class of algebras for this group is the class of the even Clifford algebra $C_0(p)$ of p. In view of Proposition 2.3 and (4.1), we have

Theorem 4.5. *Let $G = \mathbf{O}^+(p)$ for a nondegenerate quadratic form p of dimension $2n + 1$. Then*

$$\mathrm{Inv}^3(G, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{norm}} \simeq F^\times/F^{\times 2}.$$

An element $x \in F^\times$ corresponds to the invariant taking the similarity class of a nondegenerate quadratic form p' of dimension $2n+1$ to the cup-product $([C_0(p')]-[C_0(p)])\cup(x)$.

Type C_n

In the split case we have $G = \mathbf{PGSp}_{2n}$, the projective symplectic group, $n \geq 3$, $\Lambda_w = \mathbb{Z}^n$, Λ_r consists of all $\sum a_i e_i$ with $\sum a_i$ even, $\widehat{C} \simeq \mathbb{Z}/2\mathbb{Z}$. The generator of $\mathit{Sym}^2(\Lambda_w)^W$ is $q = \sum_i x_i^2$, and we have $D = \mathrm{diag}(2, \ldots, 2, 1)$ and

$$C^{-1} = \begin{pmatrix} 1 & 1 & 1 & \vdots & 1 & 1 & 1/2 \\ 1 & 2 & 2 & \vdots & 2 & 2 & 1 \\ 1 & 2 & 3 & \vdots & 3 & 3 & 3/2 \\ \cdots & \cdots & \cdots & & \cdots & \cdots & \cdots \\ 1 & 2 & 3 & \vdots & n-2 & n-2 & (n-2)/2 \\ 1 & 2 & 3 & \vdots & n-2 & n-1 & (n-1)/2 \\ 1 & 2 & 3 & \vdots & n-2 & n-1 & n/2 \end{pmatrix}.$$

By Proposition 3.4,

$$Q(G) = \mathit{Sym}^2(\Lambda_r)^W = \begin{cases} \mathbb{Z}q & \text{if } n \equiv 0 \bmod 4, \\ 2\mathbb{Z}q & \text{if } n \equiv 2 \bmod 4, \\ 4\mathbb{Z}q & \text{if } n \text{ is odd.} \end{cases}$$

If $a := \sum_i (e^{2x_i} + e^{-2x_i}) \in \mathbb{Z}[\Lambda_r]^W$, we have

$$c_2(a) = \sum(2x_i)^2 = 4q \in \mathrm{Dec}(G).$$

It follows that $\mathrm{Dec}(G) = Q(G)$ if n is odd.

Suppose that n is even. If $b := \sum_{i\neq j}(e^{x_i+x_j}+e^{x_i-x_j}) \in \mathbb{Z}[\Lambda_r]^W$, we have

$$c_2(b) = \frac{1}{2}\sum_{i\neq j}[(x_i-x_j)^2+(x_i+x_j)^2] = 2(n-1)q \in \mathrm{Dec}(G).$$

As n is even, $\gcd(4, 2(n-1)) = 2$, so we have $2q \in \mathrm{Dec}(G)$. On the other hand, by [9, Part II, Lemma 14.2], $\mathrm{Dec}(G) \subset 2q\mathbb{Z}$, therefore, $\mathrm{Dec}(G) = 2q\mathbb{Z}$.

It follows that

$$\mathrm{Inv}^3(G, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{ind}} = Q(G)/\mathrm{Dec}(G) = \begin{cases} (\mathbb{Z}/2\mathbb{Z})q & \text{if } n \equiv 0 \bmod 4, \\ 0 & \text{otherwise.} \end{cases}$$

A G-torsor is given by a pair (A, σ), where A is a central simple algebra of degree $2n$ and σ is a symplectic involution on A. The twist of G by (A, σ) is the projective symplectic group $\mathbf{PGSp}(A, \sigma)$. The only nontrivial Tits class of algebras for this group is the class of the algebra A. In view of Proposition 2.3 and (4.1), we have

Theorem 4.6. *Let $G = \mathbf{PGSp}(A, \sigma)$ for a central simple algebra A of degree $2n$ with symplectic involution σ. Then*

$$\mathrm{Inv}^3(G, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{dec}} \simeq F^\times/F^{\times 2}.$$

An element $x \in F^\times$ corresponds to the invariant taking a pair (A', σ') to the cup-product $([A'] - [A]) \cup (x)$.

If n is not divisible by 4*, we have* $\mathrm{Inv}^3(G, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{norm}} = \mathrm{Inv}^3(G, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{dec}}$*. If n is divisible by* 4*, the group* $\mathrm{Inv}^3(G, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{ind}}$ *is cyclic of order* 2*.*

In the case where n is divisible by 4 and $\mathrm{char}(F) \neq 2$ an invariant I of order 2 generating $\mathrm{Inv}^3(G, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{ind}}$ was constructed in [11, §4]. Thus, in this case we have

$$\mathrm{Inv}^3(G, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{norm}} = \mathrm{Inv}^3(G, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{dec}} \oplus (\mathbb{Z}/2\mathbb{Z})I \simeq F^\times/F^{\times 2} \oplus (\mathbb{Z}/2\mathbb{Z}).$$

Type D_n

In the split case we have $G = \mathbf{PGO}^+_{2n}$, the projective orthogonal group, $n \geq 4$, $\Lambda_w = \mathbb{Z}^n + \mathbb{Z}e$, where $e = \frac{1}{2}(e_1 + \cdots + e_n)$, Λ_r consists of all $\sum a_i e_i$ with $\sum a_i$ even, $\widetilde{C}$ is isomorphic to $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ if n is even and to $\mathbb{Z}/4\mathbb{Z}$ if n is odd. The generator of $\mathit{Sym}^2(\Lambda_w)^W$ is the form $q = \frac{1}{2}\sum_i x_i^2$, and

$$D = I_n, \quad C^{-1} = \begin{pmatrix} 1 & 1 & 1 & \vdots & 1 & 1/2 & 1/2 \\ 1 & 2 & 2 & \vdots & 2 & 1 & 1 \\ 1 & 2 & 3 & \vdots & 3 & 3/2 & 3/2 \\ \cdots & \cdots & \cdots & & \cdots & \cdots & \cdots \\ 1 & 2 & 3 & \vdots & n-2 & (n-2)/2 & (n-2)/2 \\ 1/2 & 1 & 3/2 & \vdots & (n-2)/2 & n/4 & (n-2)/4 \\ 1/2 & 1 & 3/2 & \vdots & (n-2)/2 & (n-2)/4 & n/4 \end{pmatrix}.$$

By Proposition 3.4,

$$Q(G) = Sym^2(\Lambda_r)^W = \begin{cases} 2\mathbb{Z}q & \text{if } n \equiv 0 \text{ mod } 4, \\ 4\mathbb{Z}q & \text{if } n \equiv 2 \text{ mod } 4, \\ 8\mathbb{Z}q & \text{if } n \text{ is odd.} \end{cases}$$

If $a := \sum_i (e^{2x_i} + e^{-2x_i}) \in \mathbb{Z}[\Lambda_r]^W$, we have

$$c_2(a) = \sum (2x_i)^2 = 8q \in \mathrm{Dec}(G).$$

It follows that $\mathrm{Dec}(G) = Q(G)$ if n is odd.

Suppose that n is even. If $b := \sum_{i \neq j} (e^{x_i + x_j} + e^{x_i - x_j}) \in \mathbb{Z}[\Lambda_r]^W$, we have

$$c_2(b) = \frac{1}{2} \sum_{i \neq j} \left[(x_i - x_j)^2 + (x_i + x_j)^2 \right] = 4(n-1)q \in \mathrm{Dec}(G).$$

As n is even, $\gcd(8, 4(n-1)) = 4$, so we have $4q \in \mathrm{Dec}(G)$. On the other hand, by [9, Part II, Lemma 15.2], $\mathrm{Dec}(G) \subset 4\mathbb{Z}q$, therefore $\mathrm{Dec}(G) = 4\mathbb{Z}q$.

It follows that

$$\mathrm{Inv}^3(G, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{ind}} = Q(G)/\mathrm{Dec}(G) = \begin{cases} (2\mathbb{Z}/4\mathbb{Z})q & \text{if } n \equiv 0 \text{ mod } 4, \\ 0 & \text{otherwise.} \end{cases}$$

A G-torsor is given by a quadruple (A, σ, f, e), where A is a central simple algebra of degree $2n$, (σ, f) is a quadratic pair on A of trivial discriminant and e an idempotent in the center of the Clifford algebra $C(A, \sigma, f)$. The twist of G by (A, σ, f, e) is the projective orthogonal group $\mathbf{PGO}^+(A, \sigma, f)$. The nontrivial Tits classes of algebras for this group are the class of the algebra A and the classes of the two components $C^{\pm}(A, \sigma, f)$ of the Clifford algebra. In view of Proposition 2.3 and (4.1), we have

Theorem 4.7. *Let $G = \mathbf{PGO}^+(A, \sigma, f)$ for a central simple algebra A of degree $2n$ with quadratic pair (σ, f) of trivial discriminant. Then*

$$\mathrm{Inv}^3(G, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{dec}} \simeq \begin{cases} (F^\times/F^{\times 2}) \oplus (F^\times/F^{\times 2}) & \text{if } n \text{ is even,} \\ F^\times/F^{\times 4} & \text{if } n \text{ is odd.} \end{cases}$$

If n is even and $x^+, x^- \in F^\times$, then the corresponding invariant takes a quadruple (A', σ', f', e') to

$$\big([C^+(A', \sigma', f')] - [C^+(A, \sigma, f)]\big) \cup (x^+) + \big([C^-(A', \sigma', f')] - [C^-(A, \sigma, f)]\big) \cup (x^-).$$

If n is even and $x \in F^\times$, then the corresponding invariant takes a quadruple (A', σ', f', e') to $([C^+(A', \sigma', f')] - [C^+(A, \sigma, f)]) \cup (x)$.

If n is not divisible by 4, we have $\mathrm{Inv}^3(G, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{norm}} = \mathrm{Inv}^3(G, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{dec}}$. If n is divisible by 4, the group $\mathrm{Inv}^3(G, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{ind}}$ is cyclic of order 2.

In the case where n is divisible by 4 and $\mathrm{char}(F) \neq 2$ we sketch below a construction of a nontrivial indecomposable invariant I of order 2 for a split adjoint group $G = \mathbf{PGO}^+_{2n}$. A G-torsor X over F is given by a triple (A, σ, e), where A is a central simple algebra over F with an orthogonal involution σ of trivial discriminant and e is a nontrivial idempotent of the center of the Clifford algebra of (A, σ) (see [15, §29F]). We need to determine the value of $I(X)$ in $H^3(F, \mathbb{Q}/\mathbb{Z}(2))$.

We have $G = \mathbf{Aut}(A, \sigma, e) = \mathbf{PGO}^+(A, \sigma)$. The exact sequence

$$1 \to \boldsymbol{\mu}_2 \to \mathbf{O}^+(A, \sigma) \to \mathbf{PGO}^+(A, \sigma) \to 1,$$

where $\mathbf{O}^+(A, \sigma)$ is the special orthogonal group, yields an exact sequence

$$H^1(F, \mathbf{O}^+(A, \sigma)) \xrightarrow{\varphi} H^1(F, \mathbf{PGO}^+(A, \sigma)) \xrightarrow{\delta} \mathrm{Br}(F).$$

The reduction method used in [11] for the construction of an indecomposable degree 3 invariant for a symplectic involution works as well in the orthogonal case. It reduces the general situation to the case $\mathrm{ind}(A) \leq 4$. In this case the algebra A is isomorphic to $M_2(B)$ for a central simple algebra B as $2n$ is divisible by 8, and hence it admits a hyperbolic involution σ'. By [15, Proposition 8.31], one of the two components of the Clifford algebra $C(A, \sigma')$ is split. Let e' be the corresponding idempotent in the center of $C(A, \sigma')$. (If both components split, then A is split by [15, Theorem 9.12], and we let e' be any of the two idempotents.)

Since $\delta(A, \sigma', e')$ is trivial, $(A, \sigma', e') = \varphi(v)$ for some $v \in H^1(F, \mathbf{O}^+(A, \sigma))$. The set $H^1(F, \mathbf{O}^+(A, \sigma))$ is described in [15, §29.27] as the set of equivalence classes of pairs $(a, x) \in A \times F$ such that a is a σ-symmetric invertible element and $x^2 = \mathrm{Nrd}(a)$. Thus, $v = (a, x)$ for such a pair (a, x) and we set $I(X) = [A] \cup (x)$.

Type E_6

We have $\widehat{C} \simeq \mathbb{Z}/3\mathbb{Z}$ and

$$D = I_6, \qquad C^{-1} = \frac{1}{3}\begin{pmatrix} 4 & 5 & 6 & 4 & 2 & 3 \\ 5 & 10 & 12 & 8 & 4 & 6 \\ 6 & 12 & 18 & 12 & 6 & 9 \\ 4 & 8 & 12 & 10 & 5 & 6 \\ 2 & 4 & 6 & 5 & 4 & 3 \\ 3 & 6 & 9 & 6 & 3 & 6 \end{pmatrix}.$$

By Proposition 3.4, $Q(G) = \mathit{Sym}^2(\Lambda_r)^W = 3\mathbb{Z}q$.

Write $\delta_i \in \mathbb{Z}[\Lambda_w]^W$ for the sum of elements in the W-orbit of e^{w_i}. We have $c_2(\delta_1) = 6q$, $c_2(\delta_2) = 24q$, $c_2(\delta_3) = 150q$ by [16, §2] and $\mathrm{rank}(\delta_1) = [W(E_6) : W(D_5)] = 27$, $\mathrm{rank}(\delta_3) = [W(E_6) : W(A_1 + A_4)] = 216$. Note that δ_2 and $\delta_1 w_3$ belong to $\mathbb{Z}[\Lambda_r]^W$. By (3.2),

$$c_2(\delta_1\delta_3) = \mathrm{rank}(\delta_1)c_2(\delta_3) + \mathrm{rank}(\delta_3)c_2(\delta_1) = 27 \cdot 150q + 216 \cdot 6q = 5346q.$$

As $\gcd(24, 5346) = 6$, we have $6q \in \mathrm{Dec}(G)$. On the other hand, $c_2(\delta_i) \in 6\mathbb{Z}q$ for all i by [16, §2], hence $\mathrm{Dec}(G) = 6\mathbb{Z}q$. Thus,

$$\mathrm{Inv}^3(G, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{ind}} = Q(G)/\mathrm{Dec}(G) = (3\mathbb{Z}/6\mathbb{Z})q.$$

Note that the exponents of the groups $\mathrm{Inv}^3(G)_{\mathrm{dec}}$ and $\mathrm{Inv}^3(G)_{\mathrm{ind}}$ are relatively prime.

Theorem 4.8. *Let G be an adjoint group of type E_6 of inner type. Then*

$$\mathrm{Inv}^3(G, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{norm}} \simeq (F^\times/F^{\times 3}) \oplus (\mathbb{Z}/2\mathbb{Z}).$$

It follows from the computation that the pull-back of the generator of $\mathrm{Inv}^3(G)_{\mathrm{ind}}$ to $\mathrm{Inv}^3(\widetilde{G})_{\mathrm{norm}}$ is 3 times the Rost invariant $R_{\widetilde{G}}$. This was observed in [8, Proposition 7.2] in the case $\mathrm{char}(F) \neq 2$.

Type E_7

We have $\widehat{C} \simeq \mathbb{Z}/2\mathbb{Z}$ and

$$D = I_7, \qquad C^{-1} = \frac{1}{2}\begin{pmatrix} 4 & 6 & 8 & 6 & 4 & 2 & 4 \\ 6 & 12 & 16 & 12 & 8 & 4 & 8 \\ 8 & 16 & 24 & 18 & 12 & 6 & 12 \\ 6 & 12 & 18 & 15 & 10 & 5 & 9 \\ 4 & 8 & 12 & 10 & 8 & 4 & 6 \\ 2 & 4 & 6 & 5 & 4 & 3 & 3 \\ 4 & 8 & 12 & 9 & 6 & 3 & 7 \end{pmatrix}.$$

By Proposition 3.4, $Q(G) = \mathit{Sym}^2(\Lambda_r)^W = 4\mathbb{Z}q$.

We have $c_2(\delta_1) = 36q$ and $c_2(\delta_7) = 12q$ by [16, §2] and $\mathrm{rank}(\delta_7) = [W(E_7) : W(E_6)] = 56$. Note that δ_1 and δ_7^2 belong to $\mathbb{Z}[\Lambda_r]^W$.

By (3.2),

$$c_2(\delta_7^2) = 2\,\mathrm{rank}(\delta_7)c_2(\delta_7) = 2 \cdot 56 \cdot 12q = 1344.$$

As $\gcd(36, 1344) = 12$, we have $12q \in \mathrm{Dec}(G)$. On the other hand, $c_2(\delta_i) \in 12\mathbb{Z}q$ for all i by [16, §2], hence $\mathrm{Dec}(G) = 12\mathbb{Z}q$. Thus,

$$\mathrm{Inv}^3(G, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{ind}} = Q(G)/\mathrm{Dec}(G) = (4\mathbb{Z}/12\mathbb{Z})q.$$

Theorem 4.9. *Let G be an adjoint group of type E_7 of inner type. Then*

$$\mathrm{Inv}^3(G, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{norm}} \simeq (F^\times/F^{\times 2}) \oplus (\mathbb{Z}/3\mathbb{Z}).$$

It follows from the computation that the pull-back of the generator of $\mathrm{Inv}^3(G)_{\mathrm{ind}}$ to $\mathrm{Inv}^3(\widetilde{G})_{\mathrm{norm}}$ is 4 times the Rost invariant $R_{\widetilde{G}}$. This was observed in [8, Proposition 7.2] in the case $\mathrm{char}(F) \neq 3$.

Every inner semisimple group of the type G_2, F_4 or E_8 is simply connected. Then the group $\mathrm{Inv}^3(G, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{norm}}$ is of order 2, 6 and 60, respectively (see [9, Part II]).

Recall that the groups $\mathrm{Inv}^3(G)_{\mathrm{ind}}$ are all the same for all twisted forms of G. This is not the case for $\mathrm{Inv}^3(\widetilde{G})_{\mathrm{ind}} = \mathrm{Inv}^3(\widetilde{G})$. Write $\widetilde{G}_{\mathrm{gen}}$ for a "generic" twisted form of $\widetilde{G}$ (see [8, §6]). For such groups the Rost number $n_{\widetilde{G}_{\mathrm{gen}}}$ is the largest possible. Their values can be found in [9, Part II].

Theorem 4.10. *Let G be an adjoint semisimple group of inner type, and $\widetilde{G} \to G$ a universal cover. Then the map*

$$\mathrm{Inv}^3(G)_{\mathrm{ind}} \simeq \mathrm{Inv}^3(G_{\mathrm{gen}})_{\mathrm{ind}} \to \mathrm{Inv}^3(\widetilde{G}_{\mathrm{gen}})_{\mathrm{ind}} = \mathrm{Inv}^3(\widetilde{G}_{\mathrm{gen}}) = (\mathbb{Z}/n_{\tilde{G}_{\mathrm{gen}}}\mathbb{Z})R_{\tilde{G}_{\mathrm{gen}}}$$

is injective. If G is simple, the group $\mathrm{Inv}^3(G)_{\mathrm{ind}}$ is nonzero only in the following cases:

- *C_n, n is divisible by 4:* $\mathrm{Inv}^3(G)_{\mathrm{ind}} = (\mathbb{Z}/2\mathbb{Z})R_{\tilde{G}}$,
- *D_n, n is divisible by 4:* $\mathrm{Inv}^3(G)_{\mathrm{ind}} = (2\mathbb{Z}/4\mathbb{Z})R_{\tilde{G}}$,
- *E_6:* $\mathrm{Inv}^3(G)_{\mathrm{ind}} = (3\mathbb{Z}/6\mathbb{Z})R_{\tilde{G}}$,
- *E_7:* $\mathrm{Inv}^3(G)_{\mathrm{ind}} = (4\mathbb{Z}/12\mathbb{Z})R_{\tilde{G}}$.

5. Restriction to the generic maximal torus

Let G be a semisimple group over F and T_{gen} the generic maximal torus of G defined over $F(\mathcal{X})$, where $\mathcal{X}$ is the variety of maximal tori in G (see Example 3.1). We can restrict invariants of G to invariants of T_{gen} via the composition

$$\mathrm{Inv}^n(G, \mathbb{Q}/\mathbb{Z}(j)) \to \mathrm{Inv}^n(G_{F(\mathcal{X})}, \mathbb{Q}/\mathbb{Z}(j)) \xrightarrow{\mathrm{Res}} \mathrm{Inv}^n(T_{\mathrm{gen}}, \mathbb{Q}/\mathbb{Z}(j)).$$

The degree 3 invariants of algebraic tori have been studied in [1].

Suppose that G is quasi-split. Then the character group of T_{gen} is isomorphic to the weight lattice Λ with the Δ-action (see Example 3.1). The exact sequence $0 \to \Lambda \to \Lambda_w \to \widehat{C} \to 0$, Example 3.1, Theorem 3.9 and [1, Theorem 4.3] yield a diagram

$$\begin{array}{ccccc} H^1(F, \widehat{C}(1)) & \longrightarrow & \mathrm{Inv}^3(G, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{norm}} & \longrightarrow & \mathbb{Z}[\Lambda]^{\Delta}/\mathrm{Dec}(\Lambda) \\ \downarrow & & \downarrow & & \| \\ H^2(F(\mathcal{X}), \widehat{T}_{\mathrm{gen}}(1)) & \longrightarrow & \mathrm{Inv}^3(T_{\mathrm{gen}}, \mathbb{Q}/\mathbb{Z}(2))_{\mathrm{norm}} & \longrightarrow & \mathbb{Z}[\Lambda]^{\Delta}/\mathrm{Dec}(\Lambda) \end{array}$$

Theorem 5.1. *Let G be a quasi-split group over a perfect field F, and T_{gen} the generic maximal torus. Then the homomorphism*

$$\mathrm{Inv}^n(G, \mathbb{Q}/\mathbb{Z}(j)) \to \mathrm{Inv}^n(T_{\mathrm{gen}}, \mathbb{Q}/\mathbb{Z}(j))$$

is injective, i.e., every invariant of G is determined by its restriction to the generic maximal torus.

Proof. Consider the morphism $\mathcal{T} \to \mathcal{X}$ as in Example 3.1. Let V be a generically free representation of G such that there is an open G-invariant subscheme $U \subset V$ and a G-torsor $U \to U/G$. The group scheme $\mathcal{T}$ over $\mathcal{X}$ acts naturally on $U \times \mathcal{X}$. Consider the factor scheme $(U \times \mathcal{X})/\mathcal{T}$. In fact, we can view it as a variety as follows. Let T_0 be a quasi-split maximal torus in G. The Weyl group W of T_0 acts on $(U/T_0) \times (G/T_0)$ by $w(T_0u, gT_0) = (T_0wu, gw^{-1}T_0)$. Then $(U \times \mathcal{X})/\mathcal{T}$ can be viewed as the factor variety

$((U/T_0) \times (G/T_0))/W$. Note that the function field of $(U \times \mathcal{X})/\mathcal{T}$ is isomorphic to the function field of $U_{F(\mathcal{X})}/T_{\mathrm{gen}}$ over $F(\mathcal{X})$.

We claim that the natural morphism

$$f : (U \times \mathcal{X})/\mathcal{T} \to U/G$$

is surjective on K-points for any field extension K/F. A *K-point* of U/G is a G-orbit $O \subset U$ defined over K. As F is perfect, by [23, Theorem 11.1], there is a maximal torus $T \subset G$ and a T-orbit $O' \subset O$ defined over K. Then the pair (O', T) determines a point of $((U \times \mathcal{X})/\mathcal{T})(K)$ over O. The claim is proved.

It follows from the claim that the generic fiber of f has a rational point (over $F(U/G)$). Therefore, the natural homomorphism

$$H^n(F(U/G), \mathbb{Q}/\mathbb{Z}(j)) \to H^n\big(F(\mathcal{X})(U_{F(\mathcal{X})}/T_{\mathrm{gen}}), \mathbb{Q}/\mathbb{Z}(j)\big) \tag{5.1}$$

is injective.

Let $I \in \mathrm{Inv}^n(G, \mathbb{Q}/\mathbb{Z}(j))$ be an invariant with trivial restriction to T_{gen}. Let p_{gen} be the generic fiber of $p : U \to U/G$ and let q_{gen} be the generic fiber of $q : U_{F(\mathcal{X})} \to U_{F(\mathcal{X})}/T_{\mathrm{gen}}$. Then the pull-back of p_{gen} with respect to the field extension $F(\mathcal{X})(U_{F(\mathcal{X})}/T_{\mathrm{gen}})/F(U/G)$ is isomorphic to the pull-back of q_{gen} under the change of group homomorphism $T_{\mathrm{gen}} \to G$. It follows that

$$0 = \mathrm{Res}(I)(q_{\mathrm{gen}}) = I(p_{\mathrm{gen}})_{F(\mathcal{X})(U_{F(\mathcal{X})}/T_{\mathrm{gen}})}.$$

As (5.1) is injective, we have $I(p_{\mathrm{gen}}) = 0$ in $H^n(F(U/G), \mathbb{Q}/\mathbb{Z}(j))$ and hence $I = 0$ by [9, Part II, Theorem 3.3] or [1, Theorem 2.2]. □

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