



Alex Fink · Luca Moci

## Matroids over a ring

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**Abstract.** We introduce the notion of a matroid  $M$  over a commutative ring  $R$ , assigning to every subset of the ground set an  $R$ -module according to some axioms. When  $R$  is a field, we recover matroids. When  $R = \mathbb{Z}$ , and when  $R$  is a DVR, we get (structures which contain all the data of) quasi-arithmetic matroids, and valuated matroids, i.e. tropical linear spaces, respectively.

More generally, whenever  $R$  is a Dedekind domain, we extend all the usual properties and operations holding for matroids (e.g., duality), and we explicitly describe the structure of the matroids over  $R$ . Furthermore, we compute the Tutte–Grothendieck ring of matroids over  $R$ . We also show that the Tutte quasi-polynomial of a matroid over  $\mathbb{Z}$  can be obtained as an evaluation of the class of the matroid in the Tutte–Grothendieck ring.

**Keywords.** Matroid, module over Dedekind ring, arithmetic matroid, valuated matroid, arithmetic Tutte polynomial, tropical flag Dressian, Tutte–Grothendieck ring

### 1. Introduction

The notion of a *matroid* axiomatizes the linear algebra of a list of vectors. Matroid theory has proved to be a versatile language to deal with many problems on the interface of combinatorics and algebra. In the years since 1935, when Whitney first introduced matroids, a number of enriched variants thereof have arisen, among them oriented matroids [2], valuated matroids [9], complex matroids [1], and (quasi-)arithmetic matroids [17, 6]. Each of these structures retains some information about a vector configuration, or an equivalent object, which is richer than the purely linear algebraic information that matroids retain.

As a running motivating example, let us focus on quasi-arithmetic matroids. A quasi-arithmetic matroid endows a matroid with a multiplicity function, whose values are the cardinalities of certain finite abelian groups, namely, the torsion parts of the quotients of an ambient lattice  $\mathbb{Z}^n$  by the sublattices spanned by subsets of vectors. From a list of vectors with integer coordinates one may produce objects like a toric arrangement, a partition function, and a zonotope (see [8]). In order to have a combinatorial structure from which these objects may be read off, one needs to keep track of arithmetic properties

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A. Fink: School of Mathematical Sciences, Queen Mary University of London,  
327 Mile End Road, London E1 4NS, United Kingdom; e-mail: a.fink@qmul.ac.uk

L. Moci: IMJ – Université de Paris 7, 5 rue Thomas Mann, 75205 Paris Cedex 13, France;  
e-mail: moci@math.jussieu.fr

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of the vectors, and this is what quasi-arithmetic matroids provide. (For the difference between quasi-arithmetic and arithmetic matroids, see Remark 6.4.)

It is natural to ask to what extent these generalizations of matroids can be unified under a common framework. Such a unification was sought by Dress in his program of matroids with coefficients, represented for example in his work with Wenzel [9] wherein valuated matroids are matroids with coefficients in a “fuzzy ring”.

In the present paper we suggest a different approach to such unification, by defining the notion of a *matroid  $M$  over a commutative ring  $R$* . Such an  $M$  assigns, to every subset  $A$  of a ground set, a finitely generated  $R$ -module  $M(A)$  according to some axioms (Definition 2.1). We find this definition to have multiple agreeable features. For one, by building on the well-studied setting of modules over commutative rings, we get a theory where the considerable power and development of commutative algebra can be easily brought to bear. For another, unlike arithmetic and valuated matroids, a matroid over  $R$  is not defined as a matroid decorated with extra data; there is only one axiom, and we suggest that it is comparably simple to the matroid axioms themselves. Indeed, a *realizable* matroid over  $R$  is precisely a vector configuration in a finitely generated  $R$ -module, and the axioms of a matroid over  $R$  say only that minors of at most two elements are such realizable matroids—that is, matroids are *locally* realizable matroids.

When  $R$  is a field, a matroid  $M$  over  $R$  is nothing but a matroid: the datum  $M(A)$  is a vector space, which contains only the information of its dimension, and this directly encodes the rank function of  $M$ . When  $R = \mathbb{Z}$ , every module  $M(A)$  is an abelian group, and by extracting its torsion subgroup we get a quasi-arithmetic matroid. When  $R$  is a discrete valuation ring (DVR), we may similarly extract a valuated matroid. More generally, whenever  $R$  is a Dedekind domain, we can extend the usual properties and operations holding for matroids, such as duality.

The idea of matroids over rings was suggested by certain features of the theory of quasi-arithmetic matroids. Some significant information about an integer vector configuration is lost in passing to the multiplicity function, as there exist many finite abelian groups with the same cardinality. Recording the whole structure of these groups is more desirable in several situations, for example, in developing a combinatorial intersection theory for the arrangements of subtori arising as characteristic varieties. The properties of the multiplicity function of a quasi-arithmetic matroid turn out to be just shadows of group-theoretic properties.

One of the most-loved invariants of matroids is their Tutte polynomial  $\mathbf{T}_M(x, y)$ . It thus comes as no surprise that the Tutte polynomial has been considered for generalizations of matroids as well. A quasi-arithmetic matroid  $\hat{M}$  has an associated arithmetic Tutte polynomial  $\mathbf{M}_{\hat{M}}(x, y)$ , which has proved to be a useful tool in studying toric arrangements, partition functions, zonotopes, and graphs [17, 7, 3]. More strongly, the authors of [3] define a *Tutte quasi-polynomial* of an integer vector configuration, interpolating between  $\mathbf{T}_M(x, y)$  and  $\mathbf{M}_{\hat{M}}(x, y)$ , which is no longer an invariant of the quasi-arithmetic matroid (as it depends on the groups, not just their cardinalities).

Among its properties, the Tutte polynomial of a classical matroid is the universal deletion-contraction invariant. In more algebraic language, following [4], the class of a matroid in the Tutte–Grothendieck ring for deletion-contraction relations is exactly its

Tutte polynomial. While the arithmetic Tutte polynomial and Tutte quasi-polynomial are deletion-contraction invariants, neither is universal for this property. Our generalization of the Tutte polynomial for matroids over a Dedekind ring  $R$  is also the class in the Tutte–Grothendieck ring, so it retains the universality of the usual Tutte polynomial, and we obtain the two generalizations of Tutte just mentioned as evaluations of it.

This paper is organized as follows. In Section 2 we give the basic definitions for matroids over a commutative ring, including realizability, and we explain how they generalize the classical ones.

In Section 4 we establish the existence (Definition 4.3, Proposition 4.6) and the properties of the dual of a matroid over a Dedekind domain or over a local ring (of arbitrary dimension). The former case, of Dedekind domains, is the one we focus primarily on thereafter, and we review the properties of such rings in Section 3.

In Section 5 we develop the local theory, by proving a structure theorem for matroids over a DVR (Propositions 5.2 and 5.4). We show connections with the Hall algebra and with tropical geometry. A matroid over a DVR defines a point on each Dressian, one of the tropical analogues of the Grassmannian; this is equivalent by definition to being a valuated matroid. In fact, such a matroid defines a point on the corresponding analogue of the full flag variety (Corollary 5.7).

The global theory is developed in Section 6. We describe the structure of a matroid over a Dedekind ring  $R$  in terms of that of all its localizations, whose structure was completely described in the previous section, plus some global information coming from the Picard group of  $R$  (Propositions 6.1 and 6.2). This also explains the connection between matroids over  $\mathbb{Z}$  and quasi-arithmetic matroids (Corollary 6.3).

In Section 7 we compute the Tutte–Grothendieck ring (Theorem 7.1, Corollary 7.4). In particular, given a matroid over  $\mathbb{Z}$ , we present its Tutte quasi-polynomial as an evaluation of its class in  $K(\mathbb{Z}\text{-Mat})$ .

## 2. Matroids over a ring

By  $R\text{-Mod}$  we mean the category of finitely generated  $R$ -modules over a commutative ring  $R$ . We will feel free to write “f.g.” for “finitely generated” throughout.

**Definition 2.1.** Let  $R$  be a commutative ring. A *matroid over  $R$*  on the ground set  $E$  is a function  $M$  assigning to each subset  $A \subseteq E$  a finitely generated  $R$ -module  $M(A)$  such that

(M) for every subset  $A \subseteq E$  and elements  $b, c \in E$ , there exist elements  $x = x(b, c)$  and  $y = y(b, c)$  of  $M(A)$  satisfying

$$\begin{aligned} M(A \cup \{b\}) &\cong M(A)/(x), \\ M(A \cup \{c\}) &\cong M(A)/(y), \\ M(A \cup \{b, c\}) &\cong M(A)/(x, y). \end{aligned}$$

Clearly, the choice of the modules  $M(A)$  is only relevant up to isomorphism. We regard matroids  $M$  and  $M'$  over  $R$  to be equal if they are on the same ground set  $E$  and  $M(A) \cong M'(A)$  for all  $A \subseteq E$ .

For notational concision, we will hereafter let  $M(Ab)$  abbreviate  $M(A \cup \{b\})$ ,  $M(ABC)$  stand for  $M(A \cup \{b, c\})$ , and so forth.

The case of axiom (M) where  $b = c$  is the following statement, which we separate out here as it will provide a useful waypoint in many of the proofs to come:

(M1) for every subset  $A \subseteq E$  and element  $b \in E$ , there exists an element  $x = x(b)$  of  $M(A)$  such that  $M(A \cup \{b\}) \cong M(A)/(x)$ .

A more abstract, but equivalent definition will be given in Section 2.1.

The fundamental way of producing matroids over  $R$  is from vector configurations in an  $R$ -module. Given a f.g.  $R$ -module  $N$  and a list  $X = x_1, \dots, x_n$  of elements of  $N$ , the matroid  $M_X$  of  $X$  associates to the sublist  $A$  of  $X$  the quotient

$$M_X(A) = N / \left( \sum_{x \in A} Rx \right). \quad (2.1)$$

For each  $x \in X$  there is a quotient map from  $M_X(A)$  to  $M_X(A \cup \{x\})$ , which quotients out by the image of  $Rx$  in  $M_X(A)$ . This single system of maps satisfies axiom (M): indeed, the element  $x$  depends only on  $b$ , and  $y$  only on  $c$ .

**Definition 2.2.** A matroid over  $R$  is *realizable* (or *representable*) if it has the form  $M_X$  for some list  $X$  of elements of a f.g.  $R$ -module. We call  $X$  a *realization* (*representation*) of  $M$ .

Not all matroids over rings are realizable: indeed, nonrealizable matroids in the usual sense will provide examples. Axiom (M) requires only a sort of “local” realizability. Allowing the elements  $x$  and  $y$  in the axiom to depend on both  $b$  and  $c$  is what prevents this local realizability from immediately extending to global realizability.

**Example 2.3.** The following four abelian groups do not form a matroid over  $\mathbb{Z}$ :

$$M(\emptyset) = \mathbb{Z}/8, \quad M(\{1\}) = \mathbb{Z}/2, \quad M(\{2\}) = \mathbb{Z}/2, \quad M(\{1, 2\}) = 1.$$

In axiom (M), when  $(A, b, c) = (\emptyset, 1, 2)$ , the elements  $x$  and  $y$  must both be chosen to be in the subgroup of  $\mathbb{Z}/8$  generated by 2 for the isomorphisms for  $M(\{1\})$  and  $M(\{2\})$  to hold, but then the isomorphism for  $M(\{1, 2\})$  fails.

The following four abelian groups do form a matroid over  $\mathbb{Z}$ :

$$M(\emptyset) = \mathbb{Z}/4 \oplus \mathbb{Z}/2, \quad M(\{1\}) = \mathbb{Z}/2, \quad M(\{2\}) = \mathbb{Z}/2, \quad M(\{1, 2\}) = 1.$$

Indeed, it is a realizable matroid, with a realization being given by  $N = \mathbb{Z}/4 \oplus \mathbb{Z}/2$ ,  $x_1 = (1, 0)$ ,  $x_2 = (1, 1)$ .

Our having chosen to call these objects “matroids over  $R$ ” is appropriate, as they are a generalization of matroids in the classical sense, as we show in Proposition 2.6. There is one hitch in the equivalence, corresponding to the ability to choose a vector configuration that does not span its ambient space. Accordingly, let us say that a matroid  $M$  over  $R$  is *essential* if no nontrivial projective module is a direct summand of  $M(E)$  (the term is adopted from the theory of hyperplane arrangements). Lemma 2.5 shows that very little is lost in restricting to essential matroids.

Before getting there we must generalize some standard operations on matroids. In several cases this is straightforward, but duality is conspicuously not among these: for matroid duality to work well, we must assume that  $R$  is a Dedekind domain, and so we treat it in Section 4.

Let  $M$  and  $M'$  be matroids over  $R$  on respective ground sets  $E$  and  $E'$ . We define their *direct sum*  $M \oplus M'$  on the ground set  $E \sqcup E'$  by

$$(M \oplus M')(A \sqcup A') = M(A) \oplus M'(A').$$

If  $i$  is an element of  $E$ , we define two matroids over  $R$  on the ground set  $E \setminus \{i\}$ : the *deletion* of  $i$  in  $M$ , denoted  $M \setminus i$ , by

$$(M \setminus i)(A) = M(A)$$

and the *contraction* of  $i$  in  $M$ , denoted  $M/i$ , by

$$(M/i)(A) = M(A \cup \{i\}).$$

It is easy to check that these matroids satisfy axiom (M); in fact, this axiom is entirely inherited except for  $M \oplus M'$  when one of  $b$  and  $c$  in  $E$  and the other is in  $E'$ , but these cases are clear. Since these constructions can be made without reliance on axiom (M), we will sometimes use them when speaking of a map  $\mathcal{B}(E) \rightarrow R\text{-Mod}$  which has not yet been shown to be a matroid over  $R$ .

The next fact is immediate from these definitions.

**Fact 2.4.** The class of realizable matroids is closed under minors and direct sums:

- (a) If  $M$  is realized by the vector configuration  $X$  within a module  $N$ , then  $M/A \setminus B$  is realized by the images of the vectors  $x_i$  where  $i \in E \setminus (A \cup B)$  in the quotient module  $N/(x_i : i \in A)$ .
- (b) If  $M_i$  is realized by the vector configuration  $X_i$  within a module  $N_i$ , for  $i = 1, 2$ , then  $M_1 \oplus M_2$  is realized by the configuration  $(X_1, 0) \cup (0, X_2)$  within  $N_1 \oplus N_2$ .

If  $N$  is an  $R$ -module, let the *empty matroid* for  $N$  be the matroid over  $R$  on the ground set  $\emptyset$  which maps  $\emptyset$  to  $N$ . By a *projective empty matroid* we mean an empty matroid for a projective module.

**Lemma 2.5.** Every matroid  $M$  over  $R$  is the direct sum of an essential matroid over  $R$  and a projective empty matroid.

Note that this decomposition is unique if  $R$  is a field, or if  $R$  is Dedekind (Proposition 3.3).

*Proof.* Suppose  $M$  is not essential, so that some projective module  $P$  is a direct summand of  $M(E)$ . Then in fact  $P$  is a direct summand of every module  $M(A)$ , since this property lifts back along the surjections  $M(A) \twoheadrightarrow M(Ab)$ . Therefore  $M$  is a direct sum of another matroid  $M'$  over  $R$  and the empty matroid for  $P$ . Since  $M$  is finitely generated, iterating this process with  $M'$  in place of  $M$  eventually reaches an essential matroid.  $\square$

Recall that the *corank*  $\text{cork}(A)$  of a set  $A$  in a classical matroid is  $\text{rk}(E) - \text{rk}(A)$ , where  $\text{rk}(E)$  is the rank of the matroid.

**Proposition 2.6.** *Let  $\mathbb{K}$  be a field. Essential matroids  $M$  over  $\mathbb{K}$  are equivalent to (classical) matroids. If  $M$  is an essential matroid over  $\mathbb{K}$ , then  $\dim M(A)$  is the corank of  $A$  in the corresponding classical matroid.*

*A matroid over  $\mathbb{K}$  is realizable if and only if, as a classical matroid, it is realizable over  $\mathbb{K}$ .*

*Proof.* The finitely generated modules over  $\mathbb{K}$  are the finite-dimensional  $\mathbb{K}$ -vector spaces, which are completely classified up to isomorphism by dimension. So we may replace  $M(A)$  by its  $\mathbb{K}$ -dimension without losing information.

We now check that the conditions on the dimensions of the  $M(A)$  given by axiom (M) and the “essential” condition are equivalent to the following set of rank axioms for matroids, recast in terms of a corank function  $\text{cork} : 2^E \rightarrow \mathbb{N}$ :

(C0)  $\text{cork}(E) = 0$ .

(C1) For  $A \subseteq E$  and  $b \in E \setminus A$ ,  $\text{cork}(A) - \text{cork}(Ab)$  equals 0 or 1.

(C2) For  $A \subseteq E$  and  $b \neq c \in E \setminus A$ ,

$$\text{cork}(A) + \text{cork}(Abc) \geq \text{cork}(Ab) + \text{cork}(Ac).$$

Axiom (C0) is the “essential” condition. For  $x$  in a  $\mathbb{K}$ -vector space  $V$ , the difference  $\dim V - \dim(V/\langle x \rangle)$  equals zero if  $x$  is zero and one otherwise, so that (C1) is equivalent to (M1).

Finally, in axiom (M), let the singly generated subspaces  $K = \langle x \rangle$  and  $L = \langle y \rangle$  be the respective kernels of  $M(A) \rightarrow M(Ab)$  and  $M(A) \rightarrow M(Ac)$ . Then  $M(Abc) = M(A)/(K \cup L)$ . By arranging  $K$  and  $L$  suitably, their sum  $K + L$  can be chosen to have any dimension from  $\max(\dim K, \dim L)$  to  $\dim K + \dim L$  inclusive (except those that exceed  $\dim M(A)$ ), but no others. That is, the only conditions on  $\dim M(Abc)$  in terms of the other dimensions are the monotonicity conditions

$$\dim M(Abc) \leq \min(\dim M(Ab), \dim M(Ac)),$$

and the submodularity condition

$$\dim M(A) + \dim M(Abc) \geq \dim M(Ab) + \dim M(Ac),$$

which is (C2). Since (C1) and/or (M1) implies the monotonicity conditions, we have the desired equivalence.

The realizability claim is already proved by our prior observation that a realized matroid over  $\mathbb{K}$  embodies a  $\mathbb{K}$ -vector configuration.  $\square$

Let  $R \rightarrow S$  be a map of rings. Then every matroid over  $S$  is naturally also a matroid over  $R$ . Furthermore, given such a map  $R \rightarrow S$ , the tensor product  $- \otimes_R S$  is a functor  $R\text{-Mod} \rightarrow S\text{-Mod}$ . One can use this to perform *base change* of matroids over  $R$ . If  $M$  is a matroid over  $R$ , define  $M \otimes_R S$  to be the composition of  $M$  with  $- \otimes_R S$ , so that

$$(M \otimes_R S)(A) = M(A) \otimes_R S$$

for all  $A$ . As with other uses of the tensor product, we will omit the subscript  $R$  in the notation where this causes no unclarity.

**Proposition 2.7.** *If  $M$  is a matroid over  $R$ , then  $M \otimes_R S$  is a matroid over  $S$ .*

*Proof.* Let  $0 \rightarrow K \rightarrow N \rightarrow N' \rightarrow 0$  be a short exact sequence of  $R$ -modules, with  $K$  cyclic. Tensor product being right exact, we get an exact sequence  $K \otimes S \rightarrow N \otimes S \rightarrow N' \otimes S \rightarrow 0$ , so the kernel of  $N \otimes S \rightarrow N' \otimes S$  is a quotient of the cyclic  $S$ -module  $K \otimes S$ , and is therefore cyclic. Therefore the maps from  $M(A) \otimes S$  to  $M(Ab) \otimes S$  and to  $M(Ac) \otimes S$  have cyclic kernels, and this establishes condition (M1) for  $M \otimes S$ .

Since tensor product is a left adjoint functor (to  $\text{Hom}$ ), it preserves pushouts, including the pushout from axiom (M) for  $M$ :

$$\begin{array}{ccc} M(A) & \longrightarrow & M(Ab) \\ \downarrow & \lrcorner & \downarrow \\ M(Ac) & \longrightarrow & M(Abc) \end{array}$$

This proves axiom (M) for  $M \otimes S$ . □

Two special cases of this construction will be of fundamental importance for our theory:

- (1) For every prime ideal  $\mathfrak{m}$  of  $R$ , let  $R_{\mathfrak{m}}$  be the localization of  $R$  at  $\mathfrak{m}$ . We call

$$M_{\mathfrak{m}} \doteq M \otimes_R R_{\mathfrak{m}}$$

the *localization* of  $M$  at  $\mathfrak{m}$ .

- (2) If  $R$  is a domain, let  $\text{Frac}(R)$  be the fraction field of  $R$ . Then we call

$$M_{\text{gen}} \doteq M \otimes_R \text{Frac}(R)$$

the *generic matroid* of  $M$ .

A *generic loop* or *generic coloop* of a matroid over  $R$  is a loop, respectively a coloop, of its generic matroid. Thus  $a$  is a generic coloop of  $M$  if and only if  $M(E \setminus \{a\})$  has a nontrivial projective summand.

Our approach will be much based on studying the matroid  $M$  via these localizations. The localizations, matroids over  $R_{\mathfrak{m}}$ , induce matroids over the residue fields  $R_{\mathfrak{m}}/(\mathfrak{m})$ ; the generic matroid  $M_{\text{gen}}$  is also over a field. Both constructions thus give rise to ordinary matroids (as the decomposition in 2.5 is unique).

**2.1. Towards generalizations.** Classical matroids can be defined in several, “cryptomorphic” ways, for instance by giving axioms satisfied by the bases, or circuits, or independent sets. As we have seen, our definition of matroids over rings generalizes the (co)rank function definition of matroids. This makes the following question natural.

**Question 2.8.** Are there cryptomorphic definitions of matroids over rings?

Certain axiomatizations show some promise: for example, the discussion before Remark 5.8 suggests that an axiomatization of polytopes of matroids over a DVR may be approachable.

The focus of this section is instead on the potential to generalize the “base” of the construction, away from rings. With an eye to this, we will recast the axioms in a more categorical fashion, without reference to elements. We also discuss polymatroids. To begin, matroids over rings may be taken to be instead over affine varieties:

**Example 2.9.** Let  $X$  be an affine algebraic variety, and  $R = \mathcal{O}_X$  be the ring of regular functions on  $X$ . For every point of  $X$ , the functions vanishing at it form a maximal ideal of  $R$ . Then a matroid over  $R$  can be seen as a *bundle of matroids* over  $X$ . Of course, since many of our general results hold when  $R$  is a Dedekind ring, the first case to be investigated is when  $X$  is an affine algebraic curve. In this case, by Propositions 6.1 and 6.2,  $M$  is a matroid over  $\mathcal{O}_X$  if and only if for every point of  $X$  the corresponding localization  $M_{\mathfrak{m}}$  is a matroid over  $\mathcal{O}_{X,\mathfrak{m}}$ ,  $M_{gen}$  is a matroid over  $\text{Frac}(\mathcal{O}_X)$ , and it is verified the global condition the Picard group  $\text{Pic}(X)$  stated in Proposition 6.1.

There appears to be no obstruction to patching these “bundles of matroids” in a sheaf-theoretic fashion to yield (bundles of) matroids over arbitrary schemes. Proper investigation of these is left to future work.

The next definition is Definition 2.1 recast without reference to elements, with an eye towards possible categorical generalizations; however, reference is still made to cyclic modules. We have also separated out the statement (M1).

**Definition 2.10.** Let  $R$  be a commutative ring. A *matroid over  $R$*  on the ground set  $E$  is a function  $M$  assigning to each subset  $A \subseteq E$  a finitely generated  $R$ -module  $M(A)$  satisfying the following axioms:

- (M1) For any  $A \subseteq E$  and  $b \in E \setminus A$ , there exists a surjection  $M(A) \twoheadrightarrow M(A \cup \{b\})$  whose kernel is a cyclic submodule of  $M(A)$ .  
 (M2) For any  $A \subseteq E$  and  $b, c \in E$ , there exists a pushout

$$\begin{array}{ccc} M(A) & \longrightarrow & M(A \cup \{b\}) \\ \downarrow & \lrcorner & \downarrow \\ M(A \cup \{c\}) & \longrightarrow & M(A \cup \{b, c\}) \end{array}$$

where all four morphisms are surjections with cyclic kernel.



We have already seen the pushout in the proof of Proposition 2.7. Conversely, using the element-based criterion for pushouts, the fact that

$$\begin{array}{ccc}
 M(A) & \xrightarrow{\varphi} & M(Ab) \\
 \psi \downarrow & & \downarrow \psi' \\
 M(Ac) & \xrightarrow{\varphi'} & M(ABC)
 \end{array}$$

is a pushout diagram of modules can be restated as

$$M(ABC) \simeq \frac{M(Ac) \oplus M(Ab)}{\{(\psi(x), -\varphi(x)) : x \in M(A)\}}.$$

The fact that the maps are surjections implies

$$M(ABC) \simeq M(A)/(\ker \varphi, \ker \psi).$$

where by  $(\ker \varphi, \ker \psi)$  we denote the submodule of  $M(A)$  generated by the two kernels. Then the elements  $x$  and  $y$  required by axiom (M) can be chosen as generators of  $\ker \varphi$  and  $\ker \psi$ .

Realizability may also be recast; notionally, a realizable matroid is still one in which the choices in axioms (M1) and (M2) may be made globally. Let  $\mathcal{B}(E)$  be the category of the Boolean poset of subsets of  $E$ , where inclusions of sets are the morphisms.

**Definition 2.11.** A matroid  $M$  over  $R$  is *realizable* if it is the map on objects of some functor  $F : \mathcal{B}(E) \rightarrow R\text{-Mod}$ , and axioms (M1) and (M2) are satisfied by choosing the morphisms  $F(A \rightarrow Ab)$ . A *realization* of  $M$  is a choice of such an  $F$ .

Indeed, if a matroid  $M$  over  $R$  is realizable in the above sense, corresponding to the functor  $F$ , then it is also realizable as defined before: the matroid  $M_X$  of a vector configuration  $(N, X = \{x_a\})$ , where  $N$  is  $F(\emptyset)$ , and  $x_a$  is a generator of  $\ker F(\emptyset \rightarrow \{a\})$  for each  $a \in E$ . Indeed, in this above setting, the pushout axiom (M2) applied to  $F$  guarantees that equation (2.1) holds for all  $A \subseteq E$ . The converse is similarly easy.

If we remove the cyclicity requirement, we wind up with polymatroids.

**Definition 2.12.** A *polymatroid* over  $R$  on the ground set  $E$  is a function  $M$  assigning to each subset  $A \subseteq E$  a finitely generated  $R$ -module  $M(A)$  such that

(PM) for every subset  $A \subseteq E$  and elements  $b, c \in E$ , there exist submodules  $K$  and  $L$  of  $A$  satisfying

$$\begin{aligned}
 M(A \cup \{b\}) &\cong M(A)/K, \\
 M(A \cup \{c\}) &\cong M(A)/L, \\
 M(A \cup \{b, c\}) &\cong M(A)/(K + L).
 \end{aligned}$$

Axiom (PM) is also equivalent to axiom (M2) with the words “with cyclic kernel” stricken. We note that Proposition 2.6 is true for polymatroids as well, and its proof goes through *mutatis mutandis*, when the corank axiom (C1) is replaced with

(PC) For  $A \subseteq E$  and  $b \in E \setminus A$ ,  $\text{cork}(A) \geq \text{cork}(Ab)$ .

**Example 2.13.** Not every polymatroid over  $R$  satisfying (M1) is a matroid over  $R$ . For a counterexample, let  $R = \mathbb{Z}$ . There is a pushout diagram of surjections

$$\begin{array}{ccc} \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & \longrightarrow & \mathbb{Z}/2\mathbb{Z} \\ \downarrow & \lrcorner & \downarrow \\ \mathbb{Z}/4\mathbb{Z} & \longrightarrow & \mathbb{Z}/2\mathbb{Z} \end{array} \quad (2.2)$$

in which the top map has kernel  $\langle (2, 0), (0, 1) \rangle$  and the left map has kernel  $\langle (2, 1) \rangle$ . Moreover there exist surjections  $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \rightarrow \mathbb{Z}/2\mathbb{Z}$  with cyclic kernel: there are two such, one with kernel  $\langle (1, 0) \rangle$  and one with kernel  $\langle (1, 1) \rangle$ . However, neither of these maps can be fitted into a pushout diagram of surjections with groups isomorphic to (2.2); both those pushouts are the trivial group. So diagram (2.2) corresponds to a function from  $\mathcal{B}(2)$  to  $\mathbb{Z}\text{-Mod}$  that satisfies (PM) and (M1) but not (M2).

**Question 2.14.** There are various ways to axiomatize matroids using rank functions. It is possible to state the submodularity axiom for matroids as our axiom (C2) parallelling (M2), that is,

$$\text{rk}(Ab) + \text{rk}(Ac) \geq \text{rk}(A) + \text{rk}(Abc),$$

but the more usual statement of this axiom does not restrict to covers in the Boolean lattice: it asserts that

$$\text{rk}(A) + \text{rk}(B) \geq \text{rk}(A \cap B) + \text{rk}(A \cup B)$$

for all  $A, B \subseteq E$ . Similarly, the fact that rank is nondecreasing and bounded by cardinality can be framed on covers, like our (M1), or on all containments. *Is there an axiomatization of matroids over  $R$  which replaces (M1) and (M2) with axioms on all containments, respectively pairs of sets?*

Example 2.13 suggests that such an axiom system would still need to make reference to the number of generators of kernels. It is also conceivable that the axioms sought would only agree over Dedekind domains: the behaviour exhibited in Example 3.2 below for a non-Dedekind domain interferes with naïve attempts to patch pushout squares together.

### 3. Dedekind domains

The theory arising from Definition 2.1 makes a good parallel to the theory of classical matroids when  $R$  is a Dedekind domain, and this is the case we will give most attention to in the following sections. In this section we review some properties of Dedekind domains for use in following sections.

One well-behaved feature of Dedekind domains in our setting is Lemma 3.1. Example 3.2 shows that this fails in the two-dimensional setting.

**Lemma 3.1.** *Let  $R$  be a Dedekind domain. Given two  $R$ -modules  $N$  and  $N'$ , all cyclic modules that appear as kernels of surjections  $N \rightarrow N'$  are isomorphic.*

*Proof.* Suppose we have two surjections  $N \twoheadrightarrow N'$  with kernels respectively generated by elements  $x$  and  $y$  of  $N$ . We show that  $\langle x \rangle$  and  $\langle y \rangle$  are isomorphic as  $R$ -modules with the isomorphism given by  $x \mapsto y$ . It is enough to show that this map is an isomorphism after localizing at every maximal prime  $\mathfrak{m}$  of  $R$ . Now, the isomorphism class of  $\langle x \rangle_{\mathfrak{m}}$  can be read off from the other two modules in the localized exact sequence

$$0 \rightarrow \langle x \rangle_{\mathfrak{m}} \rightarrow N_{\mathfrak{m}} \rightarrow N'_{\mathfrak{m}} \rightarrow 0.$$

To be precise, if the rank of  $N_{\mathfrak{m}}$  exceeds that of  $N'_{\mathfrak{m}}$ , then  $\langle x \rangle_{\mathfrak{m}} \cong R_{\mathfrak{m}}$  is free; otherwise,  $\langle x \rangle_{\mathfrak{m}}$  is torsion and is determined up to isomorphism by its  $(R/\mathfrak{m})$ -dimension, which is the difference of the dimensions of the torsion parts of  $N_{\mathfrak{m}}$  and  $N'_{\mathfrak{m}}$ . The isomorphism class of  $\langle y \rangle_{\mathfrak{m}}$  is determined in the same way from the same data, so that  $\langle x \rangle_{\mathfrak{m}} \cong \langle y \rangle_{\mathfrak{m}}$  for all  $\mathfrak{m}$ . And since  $x$  and  $y$  are generators, the isomorphism can be taken to send  $x \mapsto y$ .  $\square$

**Example 3.2.** Let  $R = \mathbb{K}[x, y]/\langle x, y \rangle^2$ , the ring of two-dimensional first-order jets (which is imprecisely the “smallest” two-dimensional ring). Let  $N$  be the length 3  $R$ -module  $\langle x, y \rangle/\langle x^2, y^2 \rangle$ , where these  $x$  and  $y$  should be read as elements not of  $R$  but of  $\mathbb{K}[x, y]$ : thus  $N$  is isomorphic to the so-called Matlis dual of  $R$ . Then the quotients  $N/\langle x \rangle$  and  $N/\langle y \rangle$  are both isomorphic to  $\mathbb{K}$ , but their kernels  $\langle x \rangle/\langle x^2, xy^2 \rangle$  and  $\langle y \rangle/\langle x^2y, y^2 \rangle$  are not isomorphic.

We next recall some structural results about  $R$ -modules. Given an  $R$ -module  $N$ , let  $N_{\text{tors}} \subseteq N$  denote the submodule of its torsion elements, and  $N_{\text{proj}}$  denote the projective module  $N/N_{\text{tors}}$ . Then  $N$  can be described as follows.

**Proposition 3.3** ([10, exercises 19.4–6]). *Let  $R$  be a Dedekind domain. Every f.g.  $R$ -module  $N$  is the direct sum of its torsion submodule  $N_{\text{tors}}$  and of a projective module isomorphic to  $N_{\text{proj}}$ .*

*Every torsion module may be written uniquely up to isomorphism as a sum of submodules  $R/\mathfrak{m}^k$  for  $\mathfrak{m}$  a maximal prime of  $R$  and  $k \in \mathbb{Z}_{>0}$ . It may also be written uniquely as a sum of submodules  $R/I_1 \oplus \cdots \oplus R/I_m$  (its invariant factors) for a chain  $I_1 \subseteq \cdots \subseteq I_m$  of ideals of  $R$ .*

*Every nonzero projective module is uniquely isomorphic to  $R^h \oplus I$  for some  $h \geq 0$  and nonzero ideal  $I$ , up to differing isomorphic choices of  $I$ . In particular, for ideals  $I$  and  $J$ , we have  $I \oplus J \cong R \oplus (I \otimes J)$ , so that the direct sum of any two projective modules of ranks  $i, j \geq 0$  is given by*

$$(R^{i-1} \oplus I) \oplus (R^{j-1} \oplus J) \cong R^{i+j-1} \oplus (I \otimes J). \tag{3.1}$$

We recall the following definitions. The *Picard group* of  $R$ ,  $\text{Pic}(R)$ , is the group of fractional ideals of  $R$  under multiplication, modulo the subgroup of principal ideals. If  $R$  is Dedekind, then  $\text{Pic}(R)$  is isomorphic to the group of the isomorphism classes of f.g. projective modules of rank 1, with product induced by the tensor product. If  $P$  is a projective module of rank  $n$ , the exterior algebra  $\Lambda^n P$  is a f.g. projective module of rank  $\binom{n}{n} = 1$ . Its class in  $\text{Pic}(R)$  is called the *determinant* of  $P$  and denoted by  $\det(P)$ .

**Example 3.4.** The Picard group is trivial in many familiar Dedekind rings, including the cases that this paper works out in detail:  $\mathbb{Z}$  and discrete valuation rings. So we name here some examples where it is not. Among rings of integers of number fields we have examples like  $R = \mathbb{Z}[\sqrt{-5}]$ , whose Picard group is isomorphic to  $\mathbb{Z}/2\mathbb{Z}$ , with the nonidentity class necessarily represented by any nonprincipal ideal, for instance  $(2, 1 + \sqrt{-5})$ .

Another example, among rings of regular functions, is the coordinate ring  $R = \mathbb{C}[x, y]/(y^2 - x(x - 1)(x - \lambda))$  of an elliptic curve punctured at the identity, whose Picard group is isomorphic to the group of points of the (complete) elliptic curve as an abelian variety, that is, to the additive group  $\mathbb{C}$  modulo an embedded lattice  $\mathbb{Z}^2$ : the ideal  $(x - x_0, y - y_0)$  of the point  $(x_0, y_0)$  represents its class in  $\text{Pic}(R)$ .

We will also find useful a description of the *algebraic K-theory group*  $K_0(R)$  of f.g.  $R$ -modules, that is, the abelian group generated by the classes  $[N]$  of f.g.  $R$ -modules, modulo the relations  $[N] = [N'] + [N'']$  for any exact sequence

$$0 \rightarrow N' \rightarrow N \rightarrow N'' \rightarrow 0.$$

**Proposition 3.5.** *There is an isomorphism of groups*

$$\Phi : K_0(R) \rightarrow \mathbb{Z} \oplus \text{Pic}(R).$$

*Proof.* In [22, Corollary 2.6.3] the  $K$ -theory group  $K^0(R)$  of *projective*  $R$ -modules is shown to be  $\mathbb{Z} \oplus \text{Pic}(R)$ , via the map

$$[P] \mapsto (\text{rk}(P), \det(P))$$

as a consequence of formula (3.1). But since  $R$  is a regular ring, the natural homomorphism  $K^0(R) \rightarrow K_0(R)$  is an isomorphism [11, §15.1].  $\square$

In virtue of the isomorphism above, from now on we will denote by  $\det(N)$ , and name *determinant*, the class of any f.g.  $R$ -module  $N$  in the Picard group, i.e. the second summand of  $\Phi(N)$ . In the same way, by  $\text{rk}(N)$  we denote the first summand of  $\Phi(N)$ ; this coincides with the rank of  $N_{\text{proj}}$ , i.e. with the dimension of  $N \otimes \text{Frac}(R)$ .

Note in particular that  $\Phi$  extends the usual map from invertible ideals to  $\text{Pic}(R)$ .

The potential nontriviality of this summand  $\text{Pic}(R) \subseteq K_0(R)$  has global consequences for matroids over  $R$ : see Proposition 4.11 below.

## 4. Duality

One of the first notions to be demanded of a putative generalization of matroids is duality. Our construction of duality springs from the case of realized matroids, where we have Gale duality of vector configurations. Some conditions are required on  $R$  for this construction to produce a unique dual for any matroid over  $R$ . One sufficient condition will be that  $R$  is a Dedekind domain, and therefore of global dimension 1; this case will continue being our primary focus. However, dual matroids are defined in some other settings as well, for instance, when  $R$  is a local ring. We do not have a definitive answer to the following natural question:

**Question 4.1.** What is the most general class of rings  $R$  for which duality of matroids over  $R$  is well defined and correctly behaved?

**Remark 4.2.** As can be observed in the proofs to follow in this section, the construction of dual matroids over local rings relies only on the existence of minimal projective resolutions, which is a consequence of Nakayama’s lemma. Versions of Nakayama’s lemma hold in other settings as well, for instance over graded rings, suggesting that if our definitions were extended to encompass graded matroids over graded rings, these would also have duality.

We give an outline of this section here to provide some guideposts for readers less familiar with homological algebra. Duality is defined in Definition 4.3; its relationship to the Gale dual is Proposition 4.8. The construction ultimately reduces to dualizing a map of modules, in the sense of applying the functor  $\text{Hom}(-, R)$ . We produce the needed map by specifying a maximal chain of sets and composing the quotient maps provided by axiom (M); it is of course necessary to check that this independent of the choices made (Lemma 4.4), and that the resulting dual satisfies (M) as well (Theorem 4.6). In the essential case, duality is an involution (Proposition 4.10) which behaves as expected under direct sums and minors (Proposition 4.9). Our most concrete description of the dual in the Dedekind case is Corollary 4.12: as we explain after the statement, this is a generalization of the duality formula for corank functions of usual matroids. We also show in this case that certain base changes we will use later behave well under duality (Proposition 4.13).

Let  $R$  be a Dedekind domain or local ring. Let  $M$  be a matroid over a ring  $R$ , on a ground set  $E$ . For any  $A \subseteq E$  and  $b \in E \setminus A$ , the map provided by condition (M1) may be fitted into an exact sequence of the shape

$$0 \rightarrow I \rightarrow R \rightarrow M(A) \rightarrow M(Ab) \rightarrow 0 \tag{4.1}$$

where  $R/I$  is chosen isomorphic to the cyclic kernel from condition (M1), with  $I$  an ideal of  $R$ .

The next ingredient is a projective resolution of form

$$\dots \rightarrow P_2^\emptyset \rightarrow P_1^\emptyset \rightarrow P_0^\emptyset \rightarrow M(\emptyset) \rightarrow 0, \tag{4.2}$$

where  $P_0^\emptyset$  and  $P_1^\emptyset$  are projective; this can be attained from a projective resolution of  $M$ . If  $R$  is Dedekind, then there is a f.g. projective resolution of  $M(\emptyset)$  of length at most 1, so  $P_2^\emptyset$  and the terms left of it are zero; fix one of these resolutions. If  $R$  is a local ring, then all projective modules are free and minimal free resolutions exist; let (4.2) be the minimal free resolution.

From any maximal flag of subsets  $\emptyset = A_0 \subsetneq A_1 \subsetneq \dots \subsetneq A_{|A|} = A$  we obtain a composite map

$$P_0^\emptyset \rightarrow M(\emptyset) \rightarrow M(A_1) \rightarrow \dots \rightarrow M(A).$$

The kernel of this composition  $P_0^\emptyset \rightarrow M(A)$  has a filtration whose subquotients are the kernels of the individual arrows in it. The exact sequences above continue to free resolutions for these kernels, allowing us to resolve  $\ker P_0^\emptyset \rightarrow M(A)$  with a correspondingly

filtered resolution by the Horseshoe Lemma:

$$P(A)_\bullet : \dots \rightarrow P_2 \rightarrow P_1^\theta \oplus R^{|A|} \xrightarrow{d_1} P_0^\theta \rightarrow M(A) \rightarrow 0.$$

The subquotient complexes appearing in the filtration of the map  $P_2 \rightarrow P_1^\theta \oplus R^{|A|}$  in this complex are one copy of  $P_2^\theta \rightarrow P_1^\theta$ , from (4.2), and  $|A|$  copies of complexes  $I \rightarrow R$ , from (4.1).

For convenience we will give the modules in  $P(A)_\bullet$  simpler names:

$$P(A)_\bullet : \dots \rightarrow P_2(A) \rightarrow P_1(A) \xrightarrow{d_1} P_0(A) \rightarrow M(A) \rightarrow 0.$$

As usual, we write  $\vee$  for the contravariant functor  $\text{Hom}(-, R)$ .

**Definition 4.3.** Define the module  $M^*(E \setminus A)$  as the cokernel of the map dual to  $d_1$  in  $P(A)_\bullet$ , that is,

$$M^*(E \setminus A) \doteq \text{coker}(P_0(A)^\vee \xrightarrow{d_1^\vee} P_1(A)^\vee).$$

We define  $M^*$ , the dual matroid over  $R$  to  $M$ , to be the collection of these modules  $M^*(E \setminus A)$ .

**Lemma 4.4.** *If  $R$  is local or Dedekind, then the module  $M^*(E \setminus A)$  is well-defined.*

The proof will be separated according to our two cases for  $R$ .

*Proof of Lemma 4.4,  $R$  local.* Let  $\mathfrak{m}$  be the maximal ideal of  $R$ . The terms of  $P(A)_\bullet$  to the left of  $P_0(A)$  form a free resolution of the kernel  $K_1$  of the map  $P_0(A) \rightarrow M(A)$ . By Nakayama’s lemma, the minimal free resolution of  $K_1$  is a summand of this subcomplex. The rightmost term  $F_1$  of the minimal free resolution embeds into  $P_1(A)$ , and since  $P_0(A) = P_0^\theta$  is independent of the choices made in  $P(A)_\bullet$ , so is  $F_1$ . Also, we have  $P_1(A) \cong F_1 \oplus G_1$  where  $G_1$  is a free module on  $\dim_{R/\mathfrak{m}} P_1(A) - \dim_{R/\mathfrak{m}} F_1$  generators, where  $\dim_{R/\mathfrak{m}} P_1(A) = \dim_{R/\mathfrak{m}} P_1^\theta + |A|$ , so up to isomorphism  $G_1$  is independent of the choices too.

Then  $d_1$  is the sum of the canonical map  $d_1|_{F_1} : F_1 \rightarrow P_0(A)$  and the map  $G_1 \rightarrow 0$ . The dual map  $d_1^\vee$  splits correspondingly as the sum of  $(d_1|_{F_1})^\vee : P_0(A)^\vee \rightarrow F_1^\vee$  and the map  $0 \rightarrow G_1^\vee$ , so its cokernel is

$$\text{coker}((d_1|_{F_1})^\vee) \oplus G_1^\vee,$$

which is independent of the choices. □

**Lemma 4.5.** *Let  $R$  be a Dedekind domain. For any exact sequence*

$$0 \rightarrow K_2 \rightarrow Q_1 \rightarrow Q_0 \rightarrow N \rightarrow 0$$

*of  $R$ -modules with  $Q_1$  and  $Q_0$  projective, the cokernel of the induced map  $Q_0^\vee \rightarrow Q_1^\vee$  is isomorphic to  $\text{Ext}^1(N, R) \oplus \text{Hom}(K_2, R)$ .*

*Proof.* Let  $K_1$  be the kernel of  $Q_0 \rightarrow N$ . This splits the given sequence into two short exact sequences

$$0 \rightarrow K_2 \rightarrow Q_1 \rightarrow K_1 \rightarrow 0, \quad 0 \rightarrow K_1 \rightarrow Q_0 \rightarrow N \rightarrow 0,$$

which yield the following long exact sequences of  $\text{Ext}(-, R)$ :

$$\begin{aligned} 0 \rightarrow \text{Hom}(K_1, R) \rightarrow \text{Hom}(Q_1, R) \rightarrow \text{Hom}(K_2, R) \rightarrow \text{Ext}^1(K_1, R) \rightarrow 0, \\ 0 \rightarrow \text{Hom}(N, R) \rightarrow \text{Hom}(Q_0, R) \rightarrow \text{Hom}(K_1, R) \\ \rightarrow \text{Ext}^1(N, R) \rightarrow 0 \rightarrow \text{Ext}^1(K_1, R) \rightarrow 0. \end{aligned}$$

The last zero arises since  $R$  has global dimension 1, and it implies  $\text{Ext}^1(K_1, R) = 0$ . The cokernel of the composition  $\text{Hom}(Q_0, R) \rightarrow \text{Hom}(K_1, R) \rightarrow \text{Hom}(Q_1, R)$  is canonically isomorphic to an extension of the cokernels of the maps being composed, which is an extension of  $\text{Ext}^1(N, R)$  by  $\text{Hom}(K_2, R)$ . The latter is projective, so the extension can (noncanonically) be taken to be a direct sum.  $\square$

*Proof of Lemma 4.4,  $R$  Dedekind.* First of all, Lemma 3.1 implies that, given a fixed maximal flag of subsets  $\{A_i\}$  of  $A$ , there is a unique choice of the modules  $I$  in each instance of (4.1), up to isomorphism. Therefore the isomorphism class of  $P_2(A)$  is well-defined for each fixed flag.

We are done so long as every maximal flag of subsets yields the same projective module  $P_2$ . One can obtain any maximal flag of subsets from any other by successive replacements of a segment  $A_i \subsetneq A_i b \subseteq A_i bc$  with  $A_i \subsetneq A_i c \subseteq A_i bc$ , so it is sufficient to show that one such replacement does not alter  $P_2(A)$ . For any such replacement, there exists a commutative diagram as in axiom (M).

$$\begin{array}{ccc} M(A_i) & \xrightarrow{f} & M(A_i b) \\ g \downarrow & & \downarrow g' \\ M(A_i c) & \xrightarrow{f'} & M(A_i bc) \end{array}$$

Whichever of the two flags of subsets is used, these two maps correspond to two steps like (4.1) in the filtration of  $P(A)_\bullet$ . In either case, the subquotient complex of  $P(A)_\bullet$  formed from the extension formed of these two steps is a resolution of  $\ker(M(A_i) \rightarrow M(A_i bc))$  like

$$0 \rightarrow K \rightarrow R^2 \xrightarrow{d} \ker(M(A_i) \rightarrow M(A_i bc)) \rightarrow 0$$

where the labelled map  $d$  may be chosen to be  $(r, s) \mapsto rx + sy$  if  $x$  and  $y$  generate the kernels of  $f$  and  $g$ , respectively. It follows that  $K$ , and therefore the module  $P_2(A)$ , is isomorphic in the two cases.

Finally, by Lemma 4.5, since  $M^*(E \setminus A)$  depends only on the isomorphism classes of  $P_2(A)$  and  $M(A)$  itself, it is well-defined.  $\square$

**Theorem 4.6.** *If  $M$  is a matroid over  $R$ , and  $M^*$  is defined, then  $M^*$  is a matroid over  $R$  as well.*

*Proof.* Let  $A \subseteq E$  and  $b \in E \setminus A$ . In the construction of  $P(Ab)_\bullet$ , choose a maximal flag of subsets ending in  $\cdots \subseteq A \subseteq Ab$ . The construction then provides an exact sequence of complexes which, at the  $P_0$  and  $P_1$  terms, looks like

$$\begin{array}{ccccccc} 0 & \longrightarrow & P_1(A) & \longrightarrow & P_1(Ab) & \longrightarrow & R \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & P_0(A) & \longrightarrow & P_0(Ab) & \longrightarrow & 0 \end{array}$$

All these modules are projective, so dualizing all the maps preserves exactness: we have

$$\begin{array}{ccccccc} 0 & \longleftarrow & P_1(A)^\vee & \longleftarrow & P_1(Ab)^\vee & \longleftarrow & R^\vee \longleftarrow 0 \\ & & \uparrow & & \uparrow & & \uparrow \\ 0 & \longleftarrow & P_0(A)^\vee & \longleftarrow & P_0(Ab)^\vee & \longleftarrow & 0 \end{array} \tag{4.3}$$

This induces a map between the cokernels of the left two upward arrows, which is still surjective, and has kernel some quotient of  $R$ . That is, we have a surjection  $M^*(E \setminus A) \leftarrow M^*(E \setminus (Ab))$  whose kernel is a cyclic module. These maps are exactly what is needed to establish condition (M1) for  $M^*$ .

Now let  $b, c \in E \setminus A$ . Building off the maps in the pushout diagram assured by axiom (M) for  $M$ , we get a commuting square of the maps among the modules  $P_1$  constructed above:

$$\begin{array}{ccc} P_1(A) & \longrightarrow & P_1(Ab) \\ \downarrow & & \downarrow \\ P_1(Ac) & \longrightarrow & P_1(ABC) \end{array}$$

Each of these inclusions has cokernel  $R$ , and so the target splits as a direct sum. Regard the various complexes  $P(\cdot)_\bullet$  as resolutions of kernels  $\ker(P_0^\emptyset \rightarrow M(\cdot))$ . Then, taking for example the top map,  $P_1(A) \rightarrow P_1(Ab)$ , we can identify  $P_1(Ab)$  with  $P_1(A) \oplus R$ , where  $P_1(A)$  maps to  $\ker(P_0^\emptyset \rightarrow M(Ab))$  via its map to  $\ker(P_0^\emptyset \rightarrow M(A))$ , and  $R$  maps to  $\ker(P_0^\emptyset \rightarrow M(Ab))$  by sending 1 to a lift of a generator of  $\ker(M(A) \rightarrow M(Ab))$ .

Now, this lift of a generator of  $\ker(M(A) \rightarrow M(Ab))$  to  $P_0$  is also a lift of a generator of  $\ker(M(Ac) \rightarrow M(ABC))$ . The same is true with the roles of  $b$  and  $c$  reversed. So in fact the whole square of maps can be split compatibly, as

$$\begin{array}{ccc} P_1(A) & \longrightarrow & P_1(A) \oplus R \\ \downarrow & & \downarrow (x,r) \mapsto (x,r,0) \\ P_1(A) \oplus R & \xrightarrow{(x,r) \mapsto (x,0,r)} & P_1(A) \oplus R^2 \end{array}$$



Dualizing this square yields the square

$$\begin{array}{ccc}
 P_1(A)^\vee & \longleftarrow & P_1(A)^\vee \oplus R^\vee \\
 \uparrow & & \uparrow \\
 P_1(A)^\vee \oplus R^\vee & \longleftarrow & P_1(A) \oplus (R^\vee)^2
 \end{array} \tag{4.4}$$

in which all the maps are projections onto summands, which is a pushout.

Finally, the square with which we are ultimately concerned,

$$\begin{array}{ccc}
 M^*(E \setminus A) & \longleftarrow & M^*(E \setminus (Ab)) \\
 \uparrow & & \uparrow \\
 M^*(E \setminus (Ac)) & \longleftarrow & M^*(E \setminus (Abc))
 \end{array} \tag{4.5}$$

is obtained by taking the quotient of each of the modules in square (4.4) by the image of the corresponding module  $P_0(\cdot)^\vee$ . In fact all the  $P_0(\cdot)^\vee$  are isomorphic to  $(P_0^\emptyset)^\vee$ , compatibly. This remains a pushout by the universal property, as follows. Commuting maps from  $M^*(E \setminus (Ab))$  and  $M^*(E \setminus (Ac))$  to a module  $N$  lift to commuting maps to  $N$  from the upper-right and lower-left instances of  $P_1(A)^\vee \oplus R^\vee$  in (4.4), whose kernels contain  $(P_0^\emptyset)^\vee$ . Since that square is a pushout, a map  $P_1(A)^\vee \rightarrow N$  can be provided. The kernel of this map contains  $(P_0^\emptyset)^\vee$  and so it descends to a map  $M^*(E \setminus A) \rightarrow N$ . Uniqueness can be argued similarly. We have thus established axiom (M) for  $M^*$ .  $\square$

We now state a fact whose substantive content is Lemma 4.5, and the analogous local fact that the minimal free resolution is a summand of every free resolution. We have postponed it to here only so that “matroid over  $R$ ” could appear in the statement.

**Corollary 4.7.** *When it is defined,  $M^*$  is an essential matroid over  $R$ .*

*Suppose a system  $\{M'(A) : A \subseteq E\}$  of modules is constructed as in Definition 4.3 except that we allow sequence (4.2) to be an arbitrary projective resolution. Then  $M'(A)$  is the direct sum of  $M^*(A)$  and a projective empty matroid.*

*Proof.* Take first the case where  $R$  is Dedekind. The module  $P_2(\emptyset)$  is trivial, and therefore  $M^*(E) \cong M(\emptyset)_{\text{tors}}$  by Lemma 4.5.

If now we substitute for (4.2) a different projective resolution, say with second syzygy module  $K_2$ , the effect on the sequence  $P(A)_\bullet$  is to add a  $K_2$  summand to the cokernel of the dual of the differential  $d_1$ . So by Lemma 4.5 again,  $M'(A)$  will differ from

$$M^*(E \setminus A) \cong \text{Ext}^1(M(A), R) \oplus \text{Hom}(P_2(A), R) \cong M(A)_{\text{tors}} \oplus P_2(A)^\vee$$

only up to the projective summand  $\text{Hom}(K_2, R)$ .

Now suppose  $R$  is local. If  $M^*(E)$  had a nonzero projective, i.e. free, summand, say  $F$ , then the minimal projective resolution  $0 \rightarrow F \rightarrow F$  splits as a direct summand out of the complex

$$(P_0^\emptyset)^\vee \xrightarrow{d_1^\vee} (P_1^\emptyset)^\vee \rightarrow M^*(E).$$

This induces a splitting of a zero map  $F^\vee \rightarrow 0$  as a summand of the map  $d_1 : P_1^\emptyset \rightarrow P_0^\emptyset$ , but this contradicts the choice of complex (4.2) as a minimal free resolution.

If the sequence we use in place of (4.2) is

$$Q_\bullet : \dots \rightarrow Q_1 \xrightarrow{e_1^Q} Q_0 \rightarrow M(\emptyset) \rightarrow 0,$$

then the minimal free resolution  $P_\bullet^\emptyset$  splits as a summand out of  $Q_\bullet$ , and the map  $e_1^Q$  is the direct sum of  $d_1^{P^\emptyset} : P_1^\emptyset \rightarrow P_0^\emptyset$  and a surjection of free modules  $f_1 : F_1 \rightarrow F_0$ . In the construction of  $P(A)_\bullet$ , the liftings of the maps  $R \rightarrow M(A)$  in (4.1) can still be chosen to have image contained in  $P_0^\emptyset$ , so that the first differential  $d_1$  of the analogue  $P(A)_\bullet$  is again the direct sum of the original  $d_1$  and  $f_1$ . Upon dualizing, therefore, the analogue of  $d_1^\vee$  has an extra summand  $f_1^\vee$ , an injection of free modules. Its cokernel is a free module, which is independent of  $A$  and therefore splits as a matroidal direct summand.  $\square$

Our notion of duality reduces to the classical Gale duality of vector configurations in the realizable case. It was prefigured by the construction of the dual of a realizable arithmetic matroid in [6], in which the matrix transpose operation used to construct the Gale dual corresponds to dualizing our differential  $d_1$ .

**Proposition 4.8.** *If a matroid  $M$  is realizable and its dual  $M^*$  is defined, then  $M^*$  is realizable too.*

*Proof.* Let  $M$  be realized by the vector configuration  $(x_a : a \in E)$  within  $M(\emptyset)$ . Fix lifts of these vectors to vectors  $(\tilde{x}_a : a \in E)$  within  $P_0^\emptyset$ .

For a set  $A \subseteq E$  and an element  $a \in A$ , the map  $d_1^A : P_1^\emptyset \oplus R^A \rightarrow P_0^\emptyset$  appearing in the resolution  $P(A)_\bullet$  satisfies  $d_1(0, e_a) = \tilde{x}_a$ . Thus each of these maps is a restriction of the map  $d_1^E : P_1^\emptyset \oplus R^E \rightarrow P_0^\emptyset$  in the complex  $P(E)_\bullet$  to the submodule  $P_1^\emptyset \oplus R^A \subseteq P_1^\emptyset \oplus R^E$ . Let us dualize, and write  $\{e^a : a \in E\}$  for the dual standard basis of  $(R^E)^\vee$ . The map  $(d_1^A)^\vee$  factors as  $q_A \circ (d_1^E)^\vee$ , where  $q_A : (P_1^\emptyset \oplus R^E)^\vee \rightarrow (P_1^\emptyset \oplus R^A)^\vee$  is the quotient map by the submodule  $\langle e^a : A \in E \setminus A \rangle$ . Hence,  $M^*(E \setminus A) = \text{coker}((d_1^A)^\vee)$  is the quotient of  $M^*(\emptyset) = \text{coker}((d_1^E)^\vee)$  by the submodule generated by the images of the  $e^a$  for  $a \in E \setminus A$ . But we have now exactly described a vector configuration realizing  $M^*$ : the ambient module is  $M^*(\emptyset) = \text{coker}((d_1^E)^\vee)$ , and the vector labelled by  $a$  is the image of  $e^a$ .  $\square$

Matroid duality over  $R$  has the properties expected of it.

**Proposition 4.9.** *If  $M$  and  $M'$  are matroids over a ring  $R$  where matroid duality is defined, then*

- (a)  $(M \setminus i)^* = M^* / i$ .
- (b)  $(M / i)^*$  is isomorphic to  $M^* \setminus i$  plus a projective empty matroid. If  $M(\{i\})$  is a quotient of  $M(\emptyset)$  by  $R$ , then  $(M / i)^* = M^* \setminus i$ .
- (c)  $(M \oplus M')^* = M^* \oplus M'^*$ .

The extra hypothesis in part (b) plays the role of the condition that one should not contract a loop of a classical matroid. Part (a) has no corresponding condition because all dual matroids are essential, which adjusts for the discrepancy that might otherwise be expected if a coloop is deleted.

*Proof.* (a) This is immediate:  $M(A)$  equals  $(M \setminus i)(A)$  for  $A \not\ni i$ , so the chains of maps used in constructing  $P(A)_\bullet$  are identical in  $M^*$  and  $(M \setminus i)^*$ . In the former dual, the dual of  $d_1$  in  $P(A)_\bullet$  is  $M^*(E \setminus A) = (M^*/i)((E \setminus i) \setminus A)$ ; in the latter it is  $(M \setminus i)^*((E \setminus i) \setminus A)$ .

(b) The matroid  $M^* \setminus i$  over  $R$  is obtained by the construction of Definition 4.3 on the matroid  $M/i$  except that the resolution used of  $(M/i)(\emptyset) = M(\{i\})$  is not the one specified there, but rather  $P(\{i\})_\bullet$ . By Corollary 4.7,  $M^* \setminus i$  is the direct sum of  $(M/i)^*$  and a projective empty matroid, so it is enough to show that  $M^* \setminus i$  is essential, i.e. that  $M^*(E \setminus i)$  has no projective summands.

If  $M^*(E \setminus i)$  had a nonzero projective summand  $Q$ , it would pull back to a summand of  $P_1(\{i\})^\vee$ , and this can be done in such a way that  $0 \rightarrow Q \rightarrow Q \rightarrow 0$  is a summand of

$$P_0(\{i\})^\vee \rightarrow P_1(\{i\})^\vee \rightarrow M^*(E \setminus i) \rightarrow 0.$$

In the undualized complex  $P(\{i\})_\bullet$ , this would appear as a summand  $Q^\vee \rightarrow 0$  of  $d_1$ .

However, in the construction of  $P(\{i\})_\bullet$ , the fact that the map  $M(\emptyset) \rightarrow M(\{i\})$  has kernel  $R$  implies that the images  $d_1(P_1^\emptyset)$  and  $d_1(R)$  intersect in zero. Because  $d_1|_{P_1^\emptyset}$  is part of a minimal resolution, it has no direct summand lying in its kernel. Together, these imply that there is no nonzero summand  $Q$  of  $P_1(A)$  lying in  $\ker d_1$ , as was sought above.

(c) To distinguish the complexes used in the various matroids at hand, let us write the complex  $P(A)_\bullet$  for the matroid  $M$  with a subscript  $P_M(A)_\bullet$  (and similarly for the other matroids involved). For subsets  $A$  and  $A'$  of the respective ground sets of  $M$  and  $M'$ , it is easy to check from the definition that these complexes (can be taken to) split as

$$P_{M \oplus M'}(A \amalg A')_\bullet = P_M(A)_\bullet \oplus P_{M'}(A')_\bullet,$$

from which the result follows. □

**Proposition 4.10.** *If  $M$  is a matroid over a ring  $R$  for which duality is defined, then  $M$  is the direct sum of  $M^{**}$  and a projective empty matroid. In particular, if  $M$  is essential,  $M^{**} = M$ .*

*Proof.* Suppose first that  $M$  is a realizable matroid over  $R$ , and fix a realization  $(x_a : a \in E)$ . The proof of Proposition 4.8 gives a configuration of module elements realizing  $M^*$ . Now, if  $M^*(\emptyset)$  is given a projective resolution whose first map is  $(d_1^E)^\vee$ , and this resolution is used in place of (4.2) to construct a system of modules  $M^{*'}$ , Corollary 4.7 shows that  $M^{*'}$  is the direct sum of  $M^{**}$  and a projective empty matroid.

Let us write simply  $d = d_1^\emptyset : P_1^\emptyset \rightarrow P_0^\emptyset$  for the first differential in the resolution of  $M(\emptyset)$ . Then the map  $d_1^E$  is given in matrix notation, treating direct sums as spaces of column vectors, by

$$d_1^E : P_1^\emptyset \oplus R^E \xrightarrow{(d \ x)} P_0^\emptyset$$

where  $x : R^E \rightarrow P_0^\emptyset$  is determined by  $x(e_a) = \tilde{x}_a$ , this  $\tilde{x}_a$  being a lift of  $x_a$  to  $P_0^\emptyset$ . In the dual, the vector configuration which realizes  $M^*$  is the image of the standard basis of  $(R^E)^\vee$ . So when we run the duality construction the second time, the first differential of the resolution  $P(E)_\bullet$  in  $M^*$  is

$$(P_0^\emptyset)^\vee \oplus R^E \xrightarrow{\begin{pmatrix} d^\vee & 0 \\ x^\vee & 1 \end{pmatrix}} (P_1^\emptyset)^\vee \oplus (R^E)^\vee,$$

where the map  $1 : R^E \rightarrow (R^E)^\vee$  sends the primal to the dual standard basis. Therefore,  $M^{*/}$  is realized within the cokernel of

$$P_1^\emptyset \oplus R^E \xrightarrow{\begin{pmatrix} d & x \\ 0 & 1 \end{pmatrix}} P_0^\emptyset \oplus (R^E)^\vee$$

(the maps 1 and 0 being self-dual) by the image of the standard basis  $\{(0, e^a) : a \in E\}$  of the  $(R^E)^\vee$  summand on the right. By a change of basis on the target of the above map, corresponding to composition on the right by the isomorphism

$$P_0^\emptyset \oplus (R^E)^\vee \xrightarrow{\begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix}} P_0^\emptyset \oplus (R^E)^\vee,$$

we find that  $M^{*/}$  is realized within the cokernel of

$$P_1^\emptyset \oplus R^E \xrightarrow{\begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix}} P_0^\emptyset \oplus (R^E)^\vee,$$

with the realizing vector configuration the image of  $\{(\tilde{x}_a, e^a) : a \in E\}$ . But the identity map  $R^E \rightarrow (R^E)^\vee$  that is a summand of the above map can be ignored in the cokernel; that is,  $M^{*/}$  is realized in  $\text{coker } d$  by  $\{x_a : a \in E\}$ , which shows  $M^{*/} \cong M$ . We conclude that  $M$  is the direct sum of  $M^{**}$  and a projective empty matroid.

Now, let us drop the assumption that  $M$  is realizable. Axiom (M) indicates that every two-element minor of  $M$  is realizable, so by the above discussion and Proposition 4.9(a,b), the modules in the corresponding minor of  $M^{**}$  differ from those of  $M$  only up to projective summands. The rings over which we have defined matroid duality have the cancellative property for projective summands: if  $N, N', P$  are  $R$ -modules with  $P$  projective such that  $N \oplus P \cong N' \oplus P$ , then  $N \cong N'$ . So, since any two subsets  $A, B \subseteq E$ , may be linked with a chain of overlapping two-element minors on which the realizable result above may be invoked, we conclude there are fixed projective modules  $P, Q$  such that  $M(A) \oplus P \cong M^{**}(A) \oplus Q$  for any  $A \subseteq E$ . The proposition is proved by observing that, since  $M^{**}$  is essential,  $P$  must be zero; and if  $M$  is essential as well, also  $Q$  must be zero. □

The remainder of this section is dedicated to explicit formulae for the modules making up the dual matroid in the Dedekind case.

**Proposition 4.11.** *Let  $M$  be a matroid over a Dedekind domain  $R$ . The element of  $\text{Pic}(R)$*

$$\begin{aligned} \det(M) &\doteq \det(M(A)_{\text{proj}}) + \det(M^*(E \setminus A)_{\text{proj}}) + \det(M(A)_{\text{tors}}) \\ &= \det(M(A)) + \det(M^*(E \setminus A)) - \det(M(A)_{\text{tors}}) \end{aligned}$$

*is independent of the choice of  $A \subseteq E$ .*

In particular, if  $A$  is an independent set (or, more strongly, a basis) of the generic matroid of  $M$ , then by Corollary 4.7 we have  $\det(M(A)) = \det(M)$ .

*Proof.* Given a set  $A \subseteq E$ , let  $\det(M)(A)$  be the value of  $\det(M)$  computed using that choice of  $A$ . It is enough to show that, for each  $A \subseteq E$  and  $b \in E \setminus A$ ,  $\det(M)(A)$  equals  $\det(M)(Ab)$ .

Given  $A$  and  $b$ , it is true of exactly one of the two dual maps  $M(A) \rightarrow M(Ab)$  and  $M^*(E \setminus (Ab)) \rightarrow M^*(E \setminus A)$  provided by condition (M1) that the rank of the target is one less than the rank of the source. In the other map, these two ranks are equal.

If the first map has this rank drop, then its kernel must be isomorphic to  $R$ , and the exact sequence

$$0 \rightarrow R \rightarrow M(A) \rightarrow M(Ab) \rightarrow 0$$

implies that  $\det(M(A)) = \det(M(A)_{\text{proj}}) + \det(M(A)_{\text{tors}})$  equals  $\det(M(Ab)) = \det(M(Ab)_{\text{proj}}) + \det(M(Ab)_{\text{tors}})$ . Since the second map has no rank drop, its kernel is contained in the torsion submodule of its source, so  $\det(M(E \setminus A)_{\text{proj}})$  equals  $\det(M(E \setminus (Ab))_{\text{proj}})$ . Adding these equalities, we have  $\det(M)(A) = \det(M)(Ab)$ .

If instead the second map has the rank drop, then the same argument shows that  $\det(M)(A) = \det(M)(Ab)$  after exchanging  $M$  for  $M^*$  and sets  $A$  for their complements  $E \setminus A$ , and using the fact that  $M(A)_{\text{tors}} \cong M^*(E \setminus A)_{\text{tors}}$ .  $\square$

Recall that  $K_0(R) = \mathbb{Z} \oplus \text{Pic}(R)$ . Let  $\sigma : K_0(R) \rightarrow K_0(R)$  be the involution acting as the identity on the summand  $\mathbb{Z}$  and negation on the summand  $\text{Pic}(R)$ .

**Corollary 4.12.** *If  $M$  is a matroid over a Dedekind domain  $R$ , then  $M^*(E \setminus A)$  is the module whose torsion part is  $M^*(E \setminus A)_{\text{tors}} \cong M(A)_{\text{tors}}$  and whose projective part is determined by the equality*

$$[M^*(E \setminus A)_{\text{proj}}] = \sigma([M(A)] + |A| \cdot [R] - [M(\emptyset)]) \tag{4.6}$$

*in  $K_0(R)$ .*

Note that, over a field, equation (4.6) specializes to the formula for dualizing rank functions familiar from the matroid setting,

$$\text{cork}_M^*(E \setminus A) = \text{cork}_M(A) + |A| - r$$

where  $r$  is the rank of  $M$ .

*Proof of Corollary 4.12.* The assertion on the torsion parts is noted in the discussion after Lemma 4.5.

As for the projective part, we treat the summands of  $K_0(R) = \mathbb{Z} \oplus \text{Pic}(R)$  separately. In the  $\text{Pic}(R)$  summand, Proposition 4.11 implies that

$$\det(M^*(E \setminus A)_{\text{proj}}) + \det(M(A)) = \det(M) = \det(M^*(E)_{\text{proj}}) + \det(M(\emptyset)),$$

which becomes the  $\text{Pic}(R)$  part of (4.6) on noting that  $M^*$  is essential so  $M^*(E)_{\text{proj}}$  is trivial.

Regarding the rank, consider again the ideal  $I \cong \text{coker}(f^\vee)$  in the proof of Proposition 4.10. Since  $^\vee$  preserves rank and  $f^\vee$  is an injection, we have

$$\text{rk}(I) = \text{rk}(M(A)) - \text{rk}(M(Ab)).$$

As well,

$$\text{rk}(M^*(E \setminus Ab)) - \text{rk}(M^*(E \setminus A)) = \text{rk}(R/I) = 1 - \text{rk}(I).$$

By induction on the size of  $E \setminus A$ , we deduce that  $\text{rk}(M(A)) - \text{rk}(M^*(E \setminus A)) + |A|$  is constant, and thus always equal to its value  $\text{rk}(M(\emptyset))$  taken when  $A = \emptyset$ . This proves the part of (4.6) in the  $\mathbb{Z}$  summand.  $\square$

**Proposition 4.13.** *Let  $M$  be a matroid over a Dedekind domain  $R$ .*

- (a) *Let  $f : R \rightarrow S$  be a flat map to a Dedekind domain  $S$ . Then  $(M \otimes S)^* = M^* \otimes S$  (as matroids over  $S$ ).*
- (b) *Let  $f : R \rightarrow S$  be the quotient by a maximal ideal. Then again  $(M \otimes S)^* = M^* \otimes S$ .*

*Proof.* Since base changes are computed one module at a time, these are straightforward to check given Corollary 4.12.

For part (a), to begin, we have  $(M \otimes S)^*(A)_{\text{tors}} = (M(E \setminus A) \otimes S)_{\text{tors}}$  directly. On the other hand, since projective modules remain projective under  $- \otimes S$ , we find that  $(M^*(A) \otimes S)_{\text{tors}}$  equals  $(M^*(A)_{\text{tors}} \otimes S)_{\text{tors}}$ , which in turn is  $(M(E \setminus A)_{\text{tors}} \otimes S)_{\text{tors}} = (M(E \setminus A) \otimes S)_{\text{tors}}$ . So the torsion parts agree.

As for the projective parts, because  $f$  is flat, the induced homomorphism  $f_* : K_0(R) \rightarrow K_0(S)$  is given simply by  $f_*[N] = [N \otimes S]$ . Also, torsion modules remain torsion on tensoring with  $S$ , so that the operations  $- \otimes S$  and  $-_{\text{proj}}$  commute. Hence, by (4.6),

$$[(M \otimes S)^*(A)_{\text{proj}}] = \sigma([M(A) \otimes S] + |A| \cdot [S] - [M(\emptyset) \otimes S])$$

equals

$$\begin{aligned} [(M^* \otimes S)(A)_{\text{proj}}] &= [M^*(A)_{\text{proj}} \otimes S] = f_*[M^*(A)_{\text{proj}}] \\ &= f_*[\sigma([M(A)] + |A| \cdot [R] - [M(\emptyset)])]. \end{aligned}$$

For part (b),  $M \otimes S$  is a classical matroid, over a field, and so we need only check that the corank functions of  $(M \otimes S)^*$  and  $M^* \otimes S$  are equal. Let  $I$  be the maximal ideal such that  $S = R/I$ . For a f.g.  $R$ -module  $N$ , the  $S$ -dimension of  $N \otimes S$  is the rank of  $N_{\text{proj}}$  plus  $\dim_S \text{Tor}_1^R(N, S)$ ; the latter summand is the number of indecomposable summands of  $N$  isomorphic to  $R/I^n$  for some  $n$ . Now,

$$\text{cork}_{(M \otimes S)^*}(E \setminus A) = \text{cork}_{M \otimes S}(A) + |A| - r$$

where  $r$  is the generic rank of  $M \otimes S$ . The term  $\text{cork}_{M \otimes S}(A)$  is computed as described just above, for  $N = M(A)$ : we get  $\text{rk}(M(A)) + \dim_S \text{Tor}_1^R(M(A), S)$ . On the other hand, we know that  $M^*(E \setminus A)$  has the same projective part as  $M(A)$ ; this means  $\text{Tor}_1^R(M^*(E \setminus A), S) = \text{Tor}_1^R(M(A), S)$ . And the rank of the module  $M^*(E \setminus A)$  is  $\text{rk}(M(A)) + |A| - \text{rk}(M(\emptyset)) = \text{rk}(M(A)) + |A| - r$ , by (4.6). Therefore the dimension of  $M^*(E \setminus A)$  is

$$\text{rk}(M(A)) + |A| - r + \dim_S \text{Tor}_1^R(M^*(E \setminus A), S),$$

which agrees with  $\text{cork}_{(M \otimes S)^*}(E \setminus A)$  as required.  $\square$

### 5. Structure of matroids over a DVR

In this section and the next we record some structure theorems for matroids over  $R$  in terms of structure theorems for the modules over  $R$  themselves. Our analysis of general Dedekind domains in the next section will make much use of base changing to localizations of  $R$ , so we begin here with the local case, i.e. where  $R$  is a discrete valuation ring.

We will see that these objects have connections to tropical geometry. A matroid over a DVR (discrete valuation ring)  $R$  defines a point on each Dressian, one of the tropical analogues of the Grassmannian; this is equivalent to being a valuated matroid. (As per Remark 5.10, “discrete” appears to be inessential here, so rings familiar to tropicalists like the Puiseux series should also serve.)

For the whole of this section,  $R$  will be a DVR with maximal ideal  $\mathfrak{m}$ . We first recall the structure theory of f.g.  $R$ -modules: any indecomposable f.g.  $R$ -module is isomorphic to either  $R$  or  $R/\mathfrak{m}^n$  for some integer  $n \geq 1$ . We will sometimes formally subsume  $R$  into the latter family by writing it as  $R/\mathfrak{m}^\infty$ . So, if  $N$  is a f.g.  $R$ -module and  $i \geq 1$  is an integer, define

$$d_i(N) \doteq \dim_{R/\mathfrak{m}}(\mathfrak{m}^{i-1}N/\mathfrak{m}^iN), \quad d_{\leq i}(N) \doteq \sum_{j=1}^i d_j(N) = \dim_{R/\mathfrak{m}}(N/\mathfrak{m}^iN),$$

and for convenience  $d_i(N) = d_{\leq i}(N) = 0$  if  $i \leq 0$ . Let  $d_\bullet(N)$  denote the infinite sequence of these. We have

$$d_i(R/\mathfrak{m}^n) = \begin{cases} 1, & 0 < i \leq n, \\ 0, & i > n, \end{cases}$$

where  $n$  may be  $\infty$ . The following is a quick consequence.

**Proposition 5.1.** *Isomorphism types of f.g.  $R$ -modules are in bijection with nonincreasing infinite sequences  $d_\bullet$  of nonnegative integers indexed by positive integers, the bijection being given by*

$$N \leftrightarrow d_\bullet(N).$$

This bijection permits a straightforward identification of those isomorphism classes of modules which permit maps satisfying condition (M1).

**Proposition 5.2.** *Let  $N$  and  $N'$  be f.g.  $R$ -modules. There exists a surjection  $\phi : N \rightarrow N'$  with cyclic kernel if and only if*

$$d_i(\phi) \doteq d_i(N) - d_i(N')$$

*equals 0 or 1 for each  $i \geq 1$ .*

We can also easily extract the  $d_i(\phi)$ .

**Corollary 5.3.** *Let  $\{e_\alpha\}$  be a minimal set of generators for an f.g.  $R$ -module  $N$ , and suppose  $e_\alpha$  generates a summand isomorphic to  $R/\mathfrak{m}^{\ell_\alpha}$ , wherein  $\ell_\alpha$  may be  $\infty$ . Let  $x = \sum x_\alpha e_\alpha$  be an element of  $N$ , and  $\phi$  the canonical map  $N \rightarrow N/\langle x \rangle$ . Then  $d_\bullet(\phi)$  is the lexicographically least sequence  $d_\bullet$  such that for every  $\alpha$ ,*

$$\#\{i \leq \ell_\alpha : d_i = 0\} \leq \dim_{R/\mathfrak{m}}(\langle e_\alpha \rangle / \langle x_\alpha e_\alpha \rangle). \tag{5.1}$$

When  $\ell_\alpha$  is finite, condition (5.1) is equivalent to

$$d_{\leq \ell_\alpha} \geq \dim_{R/\mathfrak{m}}(\langle x_\alpha e_\alpha \rangle).$$

In the case that  $N$  and  $N'$  have finite length, Proposition 5.2 follows from facts about the Hall algebra [15]. Indeed, it is equivalent that  $N$  have finite length and that  $d_i(N)$  stabilize to 0 for  $i \gg 0$ . In this case  $d_i$  is a partition, and its conjugate partition is the one usually used to label  $N$ . For a cyclic module, this conjugate partition has a single row. Then, under the specialization taking the Hall polynomials to the Littlewood–Richardson coefficients, Proposition 5.2 is a consequence of the Pieri rule. (Taking this further, our foundational Lemma 3.1 is essentially the statement that all coefficients in the Pieri rule are equal to 1.)

We include a proof of the proposition nonetheless, both because we do not require finite length and because we reuse its framework in proving Corollary 5.3.

*Proof of Proposition 5.2. Necessity.* Let  $\langle x \rangle$  be the cyclic kernel of  $N \rightarrow N'$ , for  $x \in N$ . The kernel of the induced surjection  $N \otimes R/\mathfrak{m}^n \rightarrow N' \otimes R/\mathfrak{m}^n$  is

$$K_n = \langle x \rangle / (\langle x \rangle \cap \mathfrak{m}^n N).$$

The dimensions over  $R/\mathfrak{m}$  of these three modules are related by

$$d_{\leq n}(N) - d_{\leq n}(N') = \dim_{R/\mathfrak{m}} K_n,$$

and, by subtracting two such relations,

$$d_n(N) - d_n(N') = \dim_{R/\mathfrak{m}} K_n - \dim_{R/\mathfrak{m}} K_{n-1}.$$

It is clear by definition that the  $K_n$  are an increasing sequence of modules, so that  $\dim_{R/\mathfrak{m}} K_n - \dim_{R/\mathfrak{m}} K_{n-1}$  is nonnegative. On the other hand,

$$(\langle x \rangle \cap \mathfrak{m}^{n-1} N) / (\langle x \rangle \cap \mathfrak{m}^n N)$$

has length at most 1, since if  $\mathfrak{m}^i x \subseteq \mathfrak{m}^{n-1} N$  then  $\mathfrak{m}^{i+1} x \subseteq \mathfrak{m}^n N$ . But this length is  $\dim_{R/\mathfrak{m}} K_n - \dim_{R/\mathfrak{m}} K_{n-1}$ , which is thus at most 1.



*Sufficiency.* Given  $N$  and an infinite list  $\delta_i \in \{0, 1\}$  such that  $d_i(N) - \delta_i$  is also a non-increasing sequence of naturals, equal therefore to  $d_i(N')$  for a module  $N'$ , we wish to construct  $x \in N$  so that  $N/\langle x \rangle \cong N'$ .

Let  $I$  be the set of indices  $i$  for which  $\delta_i = 1$  and  $\delta_{i+1} = 0$ ; also include in  $I$  the symbol  $\infty$  if  $\delta_i = 1$  for all sufficiently large  $i$ . For each  $i \in I$ , there is a summand isomorphic to  $R/\mathfrak{m}^i$  in  $N$ . Splitting off one module of each of these isomorphism classes, we can make the identification

$$N = \bigoplus_{i \in I} R/\mathfrak{m}^i \oplus P$$

for some module  $P$ , and let  $\{e_i : i \in I\}$  be generators of the summands other than  $P$ . Let  $t \in R$  be a generator of  $\mathfrak{m}$ , and define

$$x = \sum_{k \in I} t^{k - \delta_{\leq k}} e_k,$$

where as expected  $\delta_{\leq k}$  means  $\sum_{i=1}^k \delta_i$ .

The module  $P$  will remain as a summand in  $N/\langle x \rangle$ , and we may restrict attention to the remaining summand, call it  $Q$ . Towards describing it, define the elements

$$\tilde{e}_i = \sum_{k \in I, k \geq i} t^{(k - \delta_{\leq k}) - (i - \delta_{\leq i})} e_k \in N.$$

Fix for the moment some  $i \in I$ . Let  $j = j(i)$  be the greatest index less than  $i$  such that  $\delta_j = 0$  and  $\delta_{j+1} = 1$ ; or if there is no such index let  $j = 0$ . Then we have

$$t^j \tilde{e}_i = t^{\delta_{\leq j}} x.$$

This is because  $j - \delta_{\leq j} = i - \delta_{\leq i}$  by the definition of  $i$ , so that the coefficients of  $e_k$  agree for all  $k \geq i$ ; for  $k < i$ , however, we also have  $k < j$  and thus  $k - \delta_{\leq k} + \delta_{\leq j} \geq k$ , so that the coefficient of  $e_k$  in  $t^j \tilde{e}_i$  is zero. Therefore,  $t^j \tilde{e}_i$  equals zero in  $N/\langle x \rangle$ .

However, if some  $R$ -linear combination  $y = \sum_{i \in I} r_i \tilde{e}_i \in N$  is zero in  $N/\langle x \rangle$ , then  $r_i \in \mathfrak{m}^{j(i)} N$  for each  $i$ . Otherwise, write  $y = sx$ . Let  $i$  be minimal such that  $r_i \notin \mathfrak{m}^{j(i)} N$ , and let  $j = j(i)$ . If  $y$  is expanded in terms of the  $e_k$ , then the least  $k$  such that  $e_k$  has a nonzero coefficient is  $k = i$ . Let  $i'$  be the greatest element of  $I$  less than  $i$ . Since the coefficient of  $e_{i'}$  in  $y$  is zero, the  $\mathfrak{m}$ -valuation of  $s$  must be greater than or equal to  $i' - (i' - \delta_{\leq i'}) = \delta_{\leq i'} = \delta_{\leq j}$ , in view of the definition of  $x$ . (Or, if there is no element of  $I$  less than  $i$ , then consideration of the coefficient of  $e_i$  in  $x$  yields the same conclusion.) But then the  $\mathfrak{m}$ -valuation of the coefficient of  $e_i$  in  $y$  is greater than or equal to  $(i - \delta_{\leq i}) + \delta_{\leq j} = j$ , contradicting our assumption on  $i$ .

It follows that the  $R$ -module generated by the  $\tilde{e}_i$  is isomorphic to

$$\bigoplus_{i \in I} R/\mathfrak{m}^{j(i)},$$

wherein  $\{j(i) : i \in I\}$  is the set of all indices  $j$  for which  $\delta_j = 0$  and  $\delta_{j+1} = 1$ . The elements  $\tilde{e}_i$  in fact generate  $Q$ , by a triangularity argument between the  $\tilde{e}_i$  and the  $e_i$ . We conclude that the sequences  $d_i(N) - d_i(N')$  and  $\delta_i$  are equal.  $\square$

*Proof of Corollary 5.3.* Let  $v_\alpha = \dim_{R/\mathfrak{m}}(\langle e_\alpha \rangle / \langle x_\alpha e_\alpha \rangle)$ ; this is the maximum of  $\ell_\alpha$  and the  $\mathfrak{m}$ -valuation of  $x_\alpha$ . Suppose first that  $x_\alpha = 0$  for all  $\alpha$  except for a single list  $A = \{\alpha_1, \dots, \alpha_{|A|}\}$  such that both  $(v_{\alpha_i})$  and  $(\ell_{\alpha_i} - v_{\alpha_i})$  are strictly increasing sequences. To avoid proliferation of subscripts we will write  $v_i \doteq v_{\alpha_i}$  and  $\ell_i \doteq \ell_{\alpha_i}$ .

The condition (5.1) is vacuous when  $x_\alpha = 0$ . The sequence  $d_\bullet$  that we obtain from (5.1) for the  $\alpha \in A$  is

$$0^{v_1} 1^{\ell_1 - v_1} 0^{v_2 - v_1} 1^{\ell_2 - v_2 - \ell_1 + v_1} 0^{v_3 - v_2} 1^{\ell_3 - v_3 - \ell_2 + v_2} \dots,$$

exponents indicating repetition. For this sequence, if the sufficiency argument of Proposition 5.2 is run with the same choice of generators  $\{e_\alpha\}$ , the element  $x$  produced to generate the kernel is the same one we have chosen here (up to automorphisms of the cyclic summands  $\langle e_\alpha \rangle$ ). So the corollary is proven in this case.

Now suppose two indices  $\alpha$  and  $\alpha'$  are such that

$$v_\alpha \leq v_{\alpha'} \quad \text{and} \quad \ell_\alpha - v_\alpha \geq \ell_{\alpha'} - v_{\alpha'}. \tag{5.2}$$

The inequality (5.1) holds if and only if the  $(v_\alpha + 1)$ th 0 of  $d_\bullet$ , if any, follows at least  $\ell_\alpha - v_\alpha$  1s. Hence, (5.1) for  $\alpha'$  is implied by (5.1) for  $\alpha$ , and thus  $\alpha'$  is irrelevant for computing  $d_\bullet$ . Moreover, inequalities (5.2) ensure that we may change our basis for  $N$  by adding a multiple of  $e'_\alpha$  to  $e_\alpha$ , yielding another generator  $\tilde{e}_\alpha$  of  $\langle e_\alpha \rangle$ , so that

$$x_\alpha e_\alpha + x_{\alpha'} e_{\alpha'} = \tilde{x}_\alpha \tilde{e}_\alpha$$

for some  $\tilde{x}_\alpha$  with the same  $\mathfrak{m}$ -valuation as  $x_\alpha$ .

By repeatedly making such changes of basis, we may, with no changes to the sequence  $d_\bullet$  that will be computed, assume that  $x_\alpha = 0$  for all  $\alpha$  except for a set no two of whose members  $\alpha, \alpha'$  satisfy (5.2). But such a set may be ordered so that  $(v_{\alpha_i})$  and  $(\ell_{\alpha_i} - v_{\alpha_i})$  are both strictly increasing, and this reduces to the first case.  $\square$

Having control over condition (M1), we turn to axiom (M).

**Proposition 5.4.** *Let  $M(\emptyset), M(1), M(2)$ , and  $M(12)$  be f.g.  $R$ -modules. There exist four surjections with cyclic kernels forming a pushout square*

$$\begin{array}{ccc} M(\emptyset) & \xrightarrow{\phi} & M(1) \\ \psi \downarrow & \lrcorner & \downarrow \psi' \\ M(2) & \xrightarrow{\phi'} & M(12) \end{array}$$

if and only if

(L1) *the source and target of each map satisfy the equivalent conditions of Proposition 5.2;*

(L2a) *for each  $n \geq 1$ ,*

$$d_{\leq n}(M(\emptyset)) - d_{\leq n}(M(1)) - d_{\leq n}(M(2)) + d_{\leq n}(M(12)) \geq 0;$$

(L2b) *for any  $n \geq 1$  such that  $d_n(M(1)) \neq d_n(M(2))$ , equality holds above:*

$$d_{\leq n}(M(\emptyset)) - d_{\leq n}(M(1)) - d_{\leq n}(M(2)) + d_{\leq n}(M(12)) = 0.$$

The numbering of these conditions is chosen to agree with the numbering of the axioms for a quasi-arithmetic matroid in Corollary 6.3.

Condition (L2a) asserts that  $A \mapsto -d_{\leq n}(M(A))$  is a *submodular* function.

*Proof of Proposition 5.4. Necessity.* Condition (L1) is clear from the fact that axiom (M) implies condition (M1).

Tensoring the matroid  $M$  with  $R/\mathfrak{m}^n$  gives a matroid  $M' \doteq M \otimes (R/\mathfrak{m}^n)$  over that ring. All of its modules are of finite length. Now regard these modules  $M'(A)$  as  $R/\mathfrak{m}$ -vector spaces. The maps  $M'(A) \rightarrow M'(Ab)$  given by (M1) remain surjective, and the pushout diagrams in (M) remain pushouts, since surjectivity and pushout-hood can be checked set-theoretically. Accordingly,  $M'$  can be interpreted as a polymatroid over  $R/\mathfrak{m}$ , that is, a classical polymatroid. The negative of the corank function of a polymatroid is submodular, and this is condition (L2a).

As for condition (L2b), suppose that the inequality of (L2a) were strict. Let the kernel of  $\phi$  be  $\langle x \rangle$ , and the kernel of  $\psi$  be  $\langle y \rangle$ , so that the kernel of the composite  $\psi' \circ \phi = \phi' \circ \psi$  is  $\langle x, y \rangle$ . So our assumption is

$$\dim \langle x \rangle / (\langle x \rangle \cap \mathfrak{m}^n N) + \dim \langle y \rangle / (\langle y \rangle \cap \mathfrak{m}^n N) > \dim \langle x, y \rangle / (\langle x, y \rangle \cap \mathfrak{m}^n N)$$

where all dimensions are over  $R/\mathfrak{m}$ . (Note that the nonstrict version of this inequality manifestly holds, providing another proof of (L2a).) That is, there exist  $r, s \in R$  such that  $sy - rx \in \mathfrak{m}^n N$ , but neither  $rx$  nor  $sy$  is in  $\mathfrak{m}^n N$ .

Now, suppose that  $d_n(M(\emptyset)) - d_n(M(1)) = 1$ . By the proof of Proposition 5.2, the module  $(\langle x \rangle \cap \mathfrak{m}^{n-1} N) / (\langle x \rangle \cap \mathfrak{m}^n N)$  is nontrivial, i.e. there exists  $q \in R$  such that

$$qx \in \mathfrak{m}^{n-1} N \setminus \mathfrak{m}^n N.$$

Because  $qx \in \mathfrak{m}^{n-1} N$  and  $rx \notin \mathfrak{m}^n N$ , we see that  $r$  divides  $q$  in  $R$ , say  $q = pr$ . Then

$$psy - qx = p(sy - rx) \in \mathfrak{m}^n N$$

and by adding, we get  $psy \in \mathfrak{m}^{n-1} N \setminus \mathfrak{m}^n N$ , which implies that  $d_n(M(\emptyset)) - d_n(M(2)) = 1$ . Of course the same holds with the roles of 1 and 2 in the ground set reversed, so that  $d_n(M(1)) = d_n(M(2))$ . By contradiction, (L2b) is proved.

*Sufficiency.* Suppose the modules  $M(A)$  satisfy (L1), (L2a), (L2b). By (L1), if  $f$  is one of the maps in the pushout, the sequence  $d_\bullet(f)$  has elements drawn from  $\{0, 1\}$ . Let  $I(f)$  be the set of positions  $i$  such that  $d_i(f) = 1$  and  $d_{i+1}(f) = 0$ , together with  $\infty$  if  $d_\bullet$  stabilizes at 1. We are assured of the existence of various simultaneous cyclic summands of  $M(\emptyset)$  (i.e. all participating in a single direct sum decomposition), for which we may choose generators as follows: a generator  $e_i$  with  $\langle e_i \rangle = R/\mathfrak{m}^i$  for each  $i \in I(\phi) \cup I(\psi)$ , and a generator  $\varepsilon_i$  (distinct from  $e_i$ ) with  $\langle \varepsilon_i \rangle = R/\mathfrak{m}^i$  for each  $i \in I(\phi') \cap I(\psi')$ .

Let  $t \in R$  be a generator of  $\mathfrak{m}$  and define two elements  $x$  and  $y$  in  $M(\emptyset)$  by

$$x = \sum_{i \in I(\phi)} t^{i-d_{\leq i}(\phi)} e_i,$$

$$y = \sum_{i \in I(\psi)} t^{i-d_{\leq i}(\psi)} e_i + \sum_{i \in I(\psi') \setminus (I(\psi) \cup I(\phi))} t^{i-d_{\leq i}(\psi')} \varepsilon_i.$$

Let  $\phi$  be the quotient map on  $M(\emptyset)$  by  $\langle x \rangle$ ,  $\psi$  the quotient map by  $\langle y \rangle$ , and  $\phi'$  and  $\psi'$  the maps completing this to a pushout. We must check that the images of these maps, i.e. the quotients  $M(\emptyset)/\langle x \rangle$ ,  $M(\emptyset)/\langle y \rangle$ , and  $M(\emptyset)/\langle x, y \rangle$  have the isomorphism types they should.

The element  $x$  is the same one we constructed in the proof of Proposition 5.2, so  $M(\emptyset)/\langle x \rangle \cong M(1)$ . Next consider  $y$ . By (L2a), we have  $d_{\leq i}(\psi') \leq d_{\leq i}(\psi)$  for any  $i$ , and therefore  $i - d_{\leq i}(\psi') \geq i - d_{\leq i}(\psi)$ . Among the first  $i$  entries of  $d_{\bullet}(\psi)$  there are exactly  $i - d_{\leq i}(\psi)$  zeroes, and hence at most  $i - d_{\leq i}(\psi')$  of them. Therefore, by Corollary 5.3, the presence of the second sum in the definition of  $y$  does not affect the isomorphism type of  $M(\emptyset)/\langle y \rangle$ , and parallel to the  $x$  case we have  $M(\emptyset)/\langle y \rangle \cong M(2)$ .

Lastly, we wish to show that  $M(1)$  modulo the image  $\phi(y)$  is isomorphic to  $M(12)$ . As our set of generators of  $M(1)$  we will use the images of the generators we have defined for  $M(\emptyset)$ , except replacing the  $e_i$  for  $i \in I(\phi)$  with the  $\tilde{e}_i$  defined in the proof of Proposition 5.2. From the definition of the  $\tilde{e}_i$ , we obtain the following base change formulae: for  $i \in I(\phi)$ , if  $i'$  is the minimum element of  $I(\phi)$  exceeding  $i$ , and  $j$  is the unique index with  $i < j < i'$  such that  $d_j(\phi) = 0$  and  $d_{j+1}(\phi) = 1$ , then

$$e_i = \tilde{e}_i - t^{(i' - \delta_{\leq i'}) - (i - \delta_{\leq i})} \tilde{e}_{i'} = \tilde{e}_i - t^{j-i} \tilde{e}_{i'}.$$

If there is no  $i'$  then  $e_i = \tilde{e}_i$ .

If  $i$  is in  $I(\psi')$ , the latter sum in the definition of  $y$  includes a term  $t^{i-d_{\leq i}(\psi')} \varepsilon_i$ , which is also a term in our expansion of  $\phi(y)$ , unless  $i \in I(\psi)$  and  $i \notin I(\phi)$ . The latter noninclusion implies that the image of  $e_i$  is still one of the elements in our set of generators for  $M(1)$ . We also have  $d_i(\psi) = 1$  and  $d_{i+1}(\psi) = 0$ , while either  $d_i(\phi) = 0$  or  $d_{i+1}(\phi) = 1$ . In the former case, (L2b) immediately implies  $d_{\leq i}(\psi) = d_{\leq i}(\psi')$ ; in the latter case, it implies  $d_{\leq i+1}(\psi) = d_{\leq i+1}(\psi')$ , from which the statement with  $i$  in place of  $i+1$  follows. Therefore, in either case the former sum in the definition of  $y$  includes a term  $t^{i-d_{\leq i}(\psi)} e_i = t^{i-d_{\leq i}(\psi')} e_i$ , which is also a term in  $\phi(y)$ . These terms of  $\phi(y)$  establish inequalities on the sequence  $d_{\bullet}(\psi)$ , of the sort described in Corollary 5.3, which  $d_n(M(1)) - d_n(M(12))$  is the lexicographically first sequence satisfying. To complete the proof, we need only check that the terms of  $\phi(y)$  we have not yet accounted for introduce no incompatible inequalities (much like we just checked for  $y$ ).

The remaining terms of  $\phi(y)$  are those corresponding to terms of  $y$  in the first sum which we have not treated yet, in  $\langle e_i \rangle$  for  $i \in I(\psi) \setminus I(\psi')$ . Now, any such  $i$  is in  $I(\phi)$ . To justify this, we have  $(d_i(\psi), d_{i+1}(\psi)) = (1, 0)$ , while  $(d_i(\psi'), d_{i+1}(\psi'))$  is some other two-bit sequence. The easiest case is  $(d_i(\psi'), d_{i+1}(\psi')) = (0, 1)$ . Then  $(d_i(\phi), d_{i+1}(\phi)) = (1, 0)$  because

$$d_n(\phi) + d_n(\psi') = d_n(M(\emptyset)) - d_n(M(12)) = d_n(\psi) + d_n(\phi')$$

for all  $n$ , and the  $d_n$  are in  $\{0, 1\}$ . Next suppose  $(d_i(\psi'), d_{i+1}(\psi')) = (0, 0)$ . Then  $d_i(\phi) = 1$  and  $d_i(\phi') = 0$ . Since property (L2a) holds at  $i-1$ , the equality in this property must be strict at  $i$ , and therefore it would contradict property (L2b) if  $d_{i+1}(\phi)$  were 1. The argument is similar with 0 and 1 exchanged and sequences reversed if  $(d_i(\psi'), d_{i+1}(\psi')) = (1, 1)$ . Therefore  $i \in I(\phi)$  as claimed. It follows that our remaining terms of  $\phi(y)$  are terms containing generators of the form  $\tilde{e}_i$ .

For any index  $i \in I(\psi) \cap I(\phi)$ , let  $k \leq i$  be minimal and  $\ell \geq i$  maximal (possibly  $\infty$ ) such that  $d_j(\phi) = d_j(\psi)$  for all  $k < j \leq \ell$ . Summing over this range, we have

$$\begin{aligned} \sum_{\substack{i \in I(\psi) \cap I(\phi) \\ k < i \leq \ell}} t^{i-d_{\leq i}(\psi)} e_i &= \sum_{\substack{i \in I(\psi) \cap I(\phi) \\ k < i \leq \ell}} t^{i-d_{\leq i}(\psi)} \tilde{e}_i - t^{i'-d_{\leq i'}(\psi)} e_{i'} \\ &= t^{i_0-d_{\leq i_0}(\psi)} \tilde{e}_{i_0} - t^{i'_1-d_{\leq i'_1}(\psi)} e_{i'_1} \end{aligned}$$

where  $i_0$  is the least element of  $I(\psi) \cap I(\phi) \cap (k, \ell]$  and  $i_1$  is the greatest; if  $i'_1$  does not exist, the term containing it above simply drops. Property (L2b) applies at  $n = k$ , so by property (L2a), we must have  $d_i(\psi') = 0$  for all  $k < i \leq i_0$ . Therefore the inequality of Corollary 5.3 for the term  $t^{i_0-d_{\leq i_0}(\psi)} \tilde{e}_{i_0}$  is satisfied by the sequence  $d_n(M(1)) - d_n(M(12))$ . Likewise, if there is an  $i'_1$ , then  $\ell$  is finite, and property (L2b) applies at  $n = \ell$ , whence property (L2a) implies that  $d_i(\psi') = 1$  for all  $i_1 < i \leq \ell$ . Therefore the inequality of Corollary 5.3 for the term  $t^{i'_1-d_{\leq i'_1}(\psi)} e_{i'_1}$  is satisfied by the sequence  $d_n(M(1)) - d_n(M(12))$  as well. This accounts for the last of the terms of  $\phi(y)$ , and at last we conclude by that corollary that  $M(1)/\langle \phi(y) \rangle \cong M(12)$ , as desired.  $\square$

By the time we come to three-element matroids over  $R$ , there are already nontrivial conditions on the functions  $d_{\leq n}$  beyond their negatives being submodular.

**Proposition 5.5.** *Let  $M$  be a matroid over  $R$  on the ground set  $[3]$ , and let  $n$  be a natural or  $\infty$ . Then, among the three quantities*

$$d_{\leq n}(M(1)) + d_{\leq n}(M(23)), \quad d_{\leq n}(M(2)) + d_{\leq n}(M(13)), \quad d_{\leq n}(M(3)) + d_{\leq n}(M(12)),$$

*the minimum is achieved at least twice.*

*Proof.* If  $M'$  is a two-element matroid over  $R$ , let  $s_{\leq n}(M)$  denote the alternating sum appearing in conditions (L2a,b). The matroid  $M$  has six minors with two elements. By adding

$$d_{\leq n}(M(\emptyset)) - d_{\leq n}(M(1)) - d_{\leq n}(M(2)) - d_{\leq n}(M(3))$$

to the three quantities in the proposition, we obtain the three values  $s_{\leq n}(M \setminus a)$  for the deletions; by adding instead

$$d_{\leq n}(M(123)) - d_{\leq n}(M(12)) - d_{\leq n}(M(13)) - d_{\leq n}(M(23)),$$

we recover the three values  $s_{\leq n}(M/a)$  for the contractions. So it is equivalent to prove that either of these sets of three attains its minimum multiple times.

We use induction on  $n$ . As base case we take  $n = 0$ , and see that  $s_{\leq 0}(M') = 0$  for any  $M'$ . So let  $n > 0$ . Suppose first, as  $A$  varies over subsets of  $[3]$ , that  $d_n(M(A))$  depends only on  $|A|$ . In this case, the three sums of form

$$d_n(M(\emptyset)) - d_n(M(a)) - d_n(M(b)) + d_n(M(ab))$$

are equal, as of course are the three sums of form

$$d_n(M(c)) - d_n(M(ac)) - d_n(M(bc)) + d_n(M(abc)).$$

Therefore the differences  $s_{\leq n}(M \setminus a) - s_{\leq n-1}(M \setminus a)$  are all equal, and the induction step succeeds.

So suppose this is not the case, and there are two sets  $A$  and  $B$  with  $|A| = |B| \in \{1, 2\}$  for which  $d_n(M(A)) \neq d_n(M(B))$ . We will proceed assuming that  $|A| = |B| = 1$ ; the argument for the other case is exactly analogous (in fact, the two cases are exchanged by replacing  $M$  by its dual). Since there are only two possible values for  $d_n(M(A))$  with  $|A| = 1$ , namely  $d_n(M(\emptyset))$  and  $d_n(M(\emptyset)) - 1$ , two of the  $d_n(M(A))$  with  $|A| = 1$  are equal and are unequal to the third. Without loss of generality suppose  $d_n(M(1)) = d_n(M(2)) \neq d_n(M(3))$ . By condition (L2b), it follows that  $s_{\leq n}(M \setminus 1) = s_{\leq n}(M \setminus 2) = 0$ . Since  $s_{\leq n}(M \setminus 3)$  is nonnegative by condition (L2a), this completes the induction for finite  $n$ .

Finally, the case  $n = \infty$  holds because if any  $d_{\leq \infty}(A)$  is finite, then  $d_{\leq n}(A)$  must be eventually constant and equal to  $d_{\leq \infty}(A)$ . If the minimum in the proposition is not  $\infty$ , there is nothing to prove; if this minimum is finite, the claim follows on replacing  $n$  by a sufficiently large finite number.  $\square$

Suppose we are given a matroid  $M$  over  $R$  with ground set  $E$ . For  $A \subseteq E$ , define  $p_A$  to be  $d_{\leq n}(M(A))$ . Apply Proposition 5.5 to all three-element minors of  $M$ ; the result can be restated to say that the *tropicalizations* of the relations

$$p_{Ab}p_{Acd} - p_{Ac}p_{Abd} + p_{Ad}p_{Abc} = 0 \tag{5.3}$$

hold of the numbers  $p_\bullet$ , where we continue abbreviating  $A \cup \{b, c\}$  as  $Abc$  and similarly.

For background on tropical geometry, see [16]. We say a bare minimum here: tropicalization is a procedure transforming algebraic varieties to tropical varieties, combinatorial “shadows” thereof, which are the sets of points on which the tropicalizations of all elements of their ideal of defining equations vanish. In our situation without a valued field, the tropicalization of a polynomial  $f = \sum_{a \in A} c_a x^a$  in variables  $x_1, \dots, x_d$  is said to vanish at those points  $(x_i)$  where, of linear forms  $\sum_i a_i x_i$  corresponding to the monomials in  $f$ , the minimum value is attained by two or more of the forms.

The relations (5.3) are among the Plücker relations for the full flag variety (of type  $A$ ). A Plücker relation is a quadratic relation among the  $p_\bullet$  arising from the straightening algorithm for Young tableaux. The full flag variety has a tropical analogue, the *tropical flag Dressian* [12] cut out by the *tropical Plücker relations*, which arise from tropicalizing those Plücker relations with the fewest terms. To be precise, the tropical Plücker relations arise from the Plücker relations of the form

$$\sum_{i \in T \setminus S} \pm p_{S \cup \{i\}} p_{T \setminus \{i\}} = 0 \tag{5.4}$$

where  $S$  and  $T$  are subsets of  $E$  satisfying  $|S| + 1 \leq |T| - 1$ . When one considers only those  $p_A$  with  $|A| = r$  one can restrict to the relations with  $|S| = r - 1$  and  $|T| = r + 1$ . These relations define the *Dressian*  $\text{Dr}(r, n)$ , which is one Grassmannian-like space in tropical geometry. It is the parameter space for tropical linear spaces [14]. That is, there is a tropical linear space determined by  $(p_A : |A| = r)$  if and only if this point lies on the Dressian. Observe that if  $|S| + 1 = |T| - 1$  and  $S \subseteq T$ , the relation (5.4) vacuously says  $0 = 0$ , so the smallest tropical Plücker relations have three terms.

Considering all the tropical Plücker relations yields the flag Dressian, which is to the full flag variety as the Dressian is to the Grassmannian: while points in the Dressian are tropical linear spaces, points in the flag Dressian are full flags of tropical linear spaces, where a flag is defined by satisfying incidence conditions.

**Proposition 5.6.** *Define  $p_A = d_{\leq n}(M(A))$ , where  $M$  is a matroid over  $R$ , and  $n$  is a natural or  $\infty$ . Then the collection of  $p_A$  for all subsets  $A \subseteq E$  satisfies every tropical Plücker relation.*

**Corollary 5.7.** *The collection of  $p_A$  in Proposition 5.6 gives a point on the flag Dressian.*

In particular, for every  $0 < r < n$ , the point  $(p_A : |A| = r)$  lies on the Dressian  $\text{Dr}(r, n)$ . As another equivalent formulation, if  $M$  is a matroid over  $R$ , then in the regular subdivision of the hypersimplex  $\text{conv}\{\sum_{i \in A} e_i\}$  wherein the height of vertex  $A$  is  $p_A = d_{\leq n}(M(A))$ , then all maximal faces of this subdivision are matroid polytopes. For more on these correspondences see [20] and [16, Section 4.4].

**Remark 5.8.** In general,  $M$  is not determined by the collection  $(p_A : A \subseteq E)$ , for any fixed  $n$ . For instance, they do not distinguish the two one-element matroids  $M(\emptyset) = R/(\mathfrak{m}^2)$ ,  $M(1) = R/\mathfrak{m}$  and  $M(\emptyset) = R/\mathfrak{m} \oplus R/\mathfrak{m}$ ,  $M(1) = R/\mathfrak{m}$ . However,  $M$  is determined by the whole family of tuples  $(p_A : A \subseteq E)$  as  $n$  varies (geometrically, by a map from a tropical ray into each Dressian).

**Corollary 5.9.** *Let  $M$  be a matroid over a DVR  $(R, \mathfrak{m})$ . Then the function  $A \mapsto \dim_{R/\mathfrak{m}} M(A)$  makes the generic matroid of  $M$  into a valuated matroid, in the sense of Dress and Wenzel [9].*

To be precise, our sign convention is the opposite of the one adopted in [9]; for perfect agreement we would have to negate this function. But our sign convention is frequently adopted in tropical geometry (see e.g. [16]).

*Proof of Corollary 5.9.* Choose  $n \gg 0$  sufficiently larger than the greatest length of any finite length summand of a module  $M(A)$ . The lengths of  $M(A) \otimes R/\mathfrak{m}^n$  for  $A$  not a spanning set of the generic matroid are sufficiently greater than these lengths when  $A$  is a spanning set, since  $A$  is a spanning set of the generic matroid if and only if  $M(A)$  has  $R$  as a summand.

The axiom of Dress and Wenzel for the valuation  $v$  of a valuated matroid is that, given bases  $A$  and  $B$  and  $a \in A \setminus B$ , there exists  $b \in B \setminus A$  such that  $A \setminus \{a\} \cup \{b\}$  and  $B \cup \{a\} \setminus \{b\}$  are bases, and such that

$$v(A) + v(B) \geq v(A \setminus \{a\} \cup \{b\}) + v(B \cup \{a\} \setminus \{b\}).$$

The fact that the sets on the right hand side are bases follows from our choice of  $n$ , and the inequality is immediate from the minimum in the Plücker relation

$$\sum_{b \in (B \setminus A) \cup \{a\}} \pm p_{A \setminus \{a\} \cup \{b\}} p_{B \cup \{a\} \setminus \{b\}} = 0$$

being attained multiply, since  $p_A p_B$  is one of its terms. □

**Remark 5.10.** We expect that matroids over the ring of integers in the Puiseux series,  $R = \bigcup_{n \geq 1} \mathbb{K}[[t^{1/n}]]$ , should directly produce tropical objects with coordinates in  $\mathbb{Q}$ , when the length of  $R/(t^a)$  is taken as  $a$  for  $a \in \mathbb{Q}$ ; and that it is possible to use other valued rings similarly. Everywhere we have assumed  $R$  is a Dedekind domain, we expect it is sufficient to let  $R$  be a *Prüfer domain*, that is, a ring all of whose localizations at primes are valuation rings, but not necessarily discrete (which is to say Noetherian). Verifying this, and extending those parts of the theory which have relied on Noetherianity, is left for future work.

*Proof of Proposition 5.6.* In any Plücker relation with  $|A_e| = 1$ , the constraint  $|A_e| + |B_e| = |B \setminus A| + 1$  implies that  $B_e$  is all of  $B \setminus A$ . So, once  $A$  and  $B$  are chosen, there is just one exchange relation for each one-element subset (i.e. element) of  $A \setminus B$ .

We will proceed by induction on  $|A \setminus B| + |B \setminus A|$ . If  $A$  is a subset of  $B$ , or if  $|A \setminus B| = |B \setminus A| = 1$ , there is no nontrivial Plücker relation. Thus the first nontrivial case is  $|A \setminus B| = 1$  and  $|B \setminus A| = 2$ , and this is equation (5.3), which we have established as a base case.

The second nontrivial case is  $|A \setminus B| = |B \setminus A| = 2$ , and we again handle this case separately. In this case, let  $F = A \cap B$ , which equals  $B_f$  and is one element short of  $A_f$ . Suppose without loss of generality that  $(A \setminus B) \cup (B \setminus A) = \{1, 2, 3, 4\}$ . Then the tropicalized Plücker relation to be proved involves the three terms

$$p_{F12} + p_{F34}, \quad p_{F13} + p_{24}, \quad p_{F14} + p_{F23}.$$

Consider the six sums  $p_{F \cup S_1} + p_{F \cup S_2} + p_{F \cup S_3}$ , where  $S_1, S_2$ , and  $S_3$  are subsets of  $\{1, 2, 3, 4\}$  of respective sizes 2, 2, and 1, whose union is  $\{1, 2, 3, 4\}$ , and such that 1 is the unique element appearing twice. There are six of these sums (not twelve, because the sum is the same even if  $S_1$  and  $S_2$  are exchanged). Among them there are three sums in which  $p_{F12}$  appears. The remaining two summands in these sums are the tropicalized terms of a Plücker relation (5.3), so their minimum is attained twice. The same goes for the sums in which  $p_{F13}$  appears, or  $p_{F14}$ .

From there, it follows that if the minimum value of all six of these sums were not attained by, say,  $(S_1, S_2, S_3) = (13, 14, 2)$ , it would be attained at both  $(13, 24, 1)$  and  $(23, 14, 1)$ , and then by subtracting the common  $p_{F \cup \{1\}}$  we would be finished. Accordingly, and by symmetry permuting  $\{2, 3, 4\}$ , we may assume that  $(13, 14, 2)$ ,  $(12, 14, 3)$ , and  $(12, 13, 4)$  all attain the minimum. In particular, they are all equal, and we rearrange to

$$p_{F12} - p_{F2} = p_{F13} - p_{F3} = p_{F14} - p_{F4}.$$

The same argument can be repeated with any of the elements of  $\{2, 3, 4\}$  taking the place of 1. So if none of those gives the relation sought, we may conclude  $p_{Fij} - p_{Fj} = p_{Fik} - p_{Fk}$  for every  $i, j, k$  in  $\{1, 2, 3, 4\}$ . But then

$$(p_{F12} - p_{F2}) + (p_{F34} - p_{F4}) = (p_{F14} - p_{F4}) + (p_{F23} - p_{F2})$$

so that  $p_{F12} + p_{F34} = p_{F14} + p_{F23}$ , and by symmetry  $p_{F13}p_{F24}$  is equal to both of these as well. This finishes the case  $|A \setminus B| = |B \setminus A| = 2$ .



We finally proceed to the remaining cases, where  $|B \setminus A| > 2$ . For convenience, write  $C = A_e \cup B_e$ . It is reasonable to do this because, when  $|A_e| = 1$ , there is only one distinct Plücker relation for  $C$ : different partitions of it back into  $A_e$  and  $B_e$  of the correct sizes yield the same relation.

Let  $c_1 \neq c_2$  be elements of  $C$ . Let  $D$  be a single-element subset of  $A_f \setminus B$ , if that set is nonempty, and let  $D = \emptyset$  otherwise. Consider the set  $P$  of triples  $S = (S_1, S_2, S_3)$  of sets where  $S_1, S_2$ , and  $S_3$  are subsets of  $C$  with  $|S_1| = 1, |S_2| = |C| - 2, |S_3| = |C| - 1$ , and such that the multiset union of  $S_1, S_2$ , and  $S_3$  contains  $c_1$  and  $c_2$  with multiplicity 1, and each element of  $C \setminus \{c_1, c_2\}$  with multiplicity 2. To each  $S \in P$  associate the sum

$$\sigma(S) \doteq p_{A_f \cup S_1} + p_{B_f \cup D \cup S_2} + p_{B_f \cup S_3}.$$

A triple  $S \in P$  is determined by two elements of  $C$ , namely the unique element  $a$  of  $S_1$  and the unique element  $b$  of  $C \setminus S_3$ . We write  $S_{a,b}$  for this triple, and observe that  $P$  contains exactly those  $S_{a,b}$  with either  $a = b$  or  $a \notin \{c_1, c_2\}, b \in \{c_1, c_2\}$ .

In particular,  $P$  contains three elements  $S_{a,c}$  for each  $a \in C \setminus \{c_1, c_2\}$ . The sums  $\sigma(S_{a,c})$  for these elements are the constant  $p_{A_f \cup \{a\}}$  plus the tropicalizations of the three terms of an instance of (5.3) if  $D$  is empty, respectively the three terms of a Plücker relation where the sets corresponding to  $A \setminus B$  and  $B \setminus A$  each have size 2 if  $D$  is a singleton. Accordingly, the minimum value of  $\sigma(S_{a,c})$  as  $c$  varies is attained twice.

Similarly,  $P$  contains  $|C| - 1$  elements  $S_{c,b}$  for each  $b \in \{c_1, c_2\}$ . The sums  $\sigma(S_{c,b})$  for these elements are the constant  $p_{B_f \cup C \setminus \{b\}}$  added to tropicalizations of the terms in another Plücker relation, where  $c_1$  has been removed from whichever of  $A$  and  $B$  it was in, and  $D \subseteq A$  has been added to  $B$ . This Plücker relation is one of those covered by the inductive hypothesis, since we have shrunk the symmetric difference of  $A$  and  $B$ . So the minimum value of  $\sigma(S_{c,b})$  as  $c$  varies is also attained twice.

Once more, for the  $|C|$  elements of  $P$  of the form  $S_{c,c}$ , each sum  $\sigma(S_{c,c})$  is the constant  $p_{B_f \cup C \setminus \{c_1, c_2\}}$  plus the tropicalization of a term in the Plücker relation whose tropical vanishing we are concerned with. So our objective is to show that the minimum value of  $\sigma(S_{c,c})$  is attained twice.

Now let  $c_1$  and  $c_2$  be chosen so that the number of pairs  $a \neq b$  for which  $\sigma(S_{a,b})$  attains the minimum value  $x = \min_{S \in P} \sigma(S)$  is as small as possible. We will prove that the minimum value of  $\sigma(S_{c,c})$  is indeed attained twice. Suppose not. We then claim that there exists  $a \notin \{c_1, c_2\}$  such that  $\sigma(S_{a,c_1}) = \sigma(S_{a,c_2}) = x$ . If this were false, choose  $a \neq b$  so that  $\sigma(S_{a,b}) = x$  (this must be possible, because if  $\sigma(S_{a,b}) = x$  only when  $a = b$  then the minimum of either  $S_{a,c}$  or  $S_{c,b}$  as  $c$  varies, whichever is appropriate, is attained just once.) By the structure of  $P$ ,  $b$  must be  $c_1$  or  $c_2$ ; without loss of generality let it be  $c_1$ . That is, we are assuming  $\sigma(S_{a,c_1}) = x$ . Then by assumption  $\sigma(S_{a,c_2}) > x$ , so by the three-term Plücker relation,  $\sigma(S_{a,a}) = x$ . Moreover there must exist  $a' \neq a$  such that  $\sigma(S_{a',c_1}) = x$ , by the other Plücker relation. If  $a' = c_1$  then we have a contradiction with our first assumption (that the minimum is attained twice); otherwise we repeat for  $a'$  the argument we made for  $a$  and have a contradiction with our second.

Thus, we have  $\sigma(S_{a,c_1}) = \sigma(S_{a,c_2}) = x$ . Now, by assumption, at least one  $c \in \{c_1, c_2\}$  has  $\sigma(S_{c,c}) > x$ ; without loss of generality let it be  $c_1$ . Let  $P'$  be defined like  $P$  except

using  $\{a, c_2\}$  where  $P$  uses  $\{c_1, c_2\}$ . For each  $S = (S_1, S_2, S_3)$  in  $P$  such that  $a \in S_2$ , there is a corresponding  $S' = (S_1, S_2 \setminus \{a\} \cup \{c_1\}, S_3)$  in  $P'$ , with

$$\sigma(S) - \sigma(S') = p_{B_f \cup D \cup S_2} - p_{B_f \cup D \cup S_2 \setminus \{a\} \cup \{c_1\}}.$$

Therefore, if  $S$  attains the minimum value  $x$  of  $\sigma(S)$  over  $P$ , then  $S'$  attains the minimum value of  $\sigma(S')$  over  $P'$  (unless this new minimum value is strictly less than  $x - p_{B_f \cup D \cup S_2} - p_{B_f \cup D \cup S_2 \setminus \{a\} \cup \{c_1\}}$ , in which case only one  $S_{a,b} \in P'$  attains it, which contradicts our choice of  $c_1$  and  $c_2$ ). But in  $\{S_{a,b} \in P : a \neq b\}$  we have two elements  $S_{a,c_1}$  and  $S_{a,c_2}$  without counterparts in  $P'$ , both of which attain the minimum, whereas in  $\{S_{a,b} \in P' : a \neq b\}$  we have the counterpart of  $S_{c_1,c_1}$ , which does not attain the minimum. So this is also a contradiction to our choice of  $c_1$  and  $c_2$ , and our Plücker relation is proved in this case, completing the proof.  $\square$

## 6. Global structure of matroids over a Dedekind domain

Throughout this section  $R$  will be a Dedekind domain. Let us recall that given an  $R$ -module  $N$ , we will denote by  $\det(N)$  its class in the Picard group, as defined in Section 3. Understanding the local ring case, we can now give a necessary and sufficient condition for which pairs of modules can occur in condition (M1).

**Proposition 6.1.** *Let  $N$  and  $N'$  be f.g.  $R$ -modules. There exists a surjection  $N \rightarrow N'$  with cyclic kernel if and only if there exists such a surjection  $N_{\mathfrak{m}} \rightarrow N'_{\mathfrak{m}}$  after localizing at each maximal prime  $\mathfrak{m}$  of  $R$ , and*

- if  $\operatorname{rk}(N) - \operatorname{rk}(N') = 0$  then  $\det(N_{\text{proj}}) = \det(N'_{\text{proj}})$ , whereas
- if  $\operatorname{rk}(N) - \operatorname{rk}(N') = 1$  then  $\det(N) = \det(N')$ .

To test whether surjections exist in the localizations, we have the criterion in Proposition 5.2.

*Proof of Proposition 6.1. Necessity.* Localization is a base change, so preserves condition (M1). If  $\operatorname{rk}(N) = \operatorname{rk}(N')$ , then the kernel of  $N \rightarrow N'$  is contained in  $N_{\text{tors}}$ , so that  $N_{\text{proj}} \cong N'_{\text{proj}}$ , and so their classes are equal. If  $\operatorname{rk}(N) = \operatorname{rk}(N') + 1$  then the kernel of  $N \rightarrow N'$  must be a cyclic rank 1  $R$ -module, which up to isomorphism is  $R$ . Therefore  $\det(N) = \det(R) + \det(N') = \det(N')$  by the definition of  $K_0(R)$ .

*Sufficiency.* Note first that  $\operatorname{rk}(N) - \operatorname{rk}(N') \in \{0, 1\}$ , because the same is true in every localization. Suppose that  $\operatorname{rk}(N) = \operatorname{rk}(N')$ . Then  $\det(N_{\text{proj}}) = \det(N'_{\text{proj}})$  implies  $N_{\text{proj}} \cong N'_{\text{proj}}$ , by Proposition 3.3. Moreover,  $N_{\text{tors}}$  and  $N'_{\text{tors}}$  are the direct sums of their localizations. The kernel of each of the given maps  $N_{\mathfrak{m}} \rightarrow N'_{\mathfrak{m}}$  is contained in the torsion  $(N_{\mathfrak{m}})_{\text{tors}} = (N_{\text{tors}})_{\mathfrak{m}}$ , so a map  $(N_{\text{tors}})_{\mathfrak{m}} \rightarrow (N'_{\text{tors}})_{\mathfrak{m}}$  is induced. The direct sum of all these maps is a map  $N_{\text{tors}} \rightarrow N'_{\text{tors}}$  which is still a surjection; its kernel is a sum of cyclic modules with disjoint supports, which is still cyclic. Taking the direct sum with an isomorphism  $N_{\text{proj}} \rightarrow N'_{\text{proj}}$  yields the requisite map  $N \rightarrow N'$ .

Now suppose that  $\text{rk}(N) = \text{rk}(N') + 1$ . In this case, construct a set function  $M : \mathcal{B}(1) \rightarrow R\text{-Mod}$  so that  $M(\emptyset) = N'_{\text{tors}}$  and  $M(1) = N_{\text{tors}}$ . Note that  $M^*(\emptyset)$  and  $M^*(1)$  both have rank 0 and therefore trivial projective part. Moreover, there exist localized surjections with cyclic kernel  $M(\emptyset)_{\mathfrak{m}} \rightarrow M(1)_{\mathfrak{m}}$  for each  $\mathfrak{m}$ . This is by Proposition 4.10(c) (or because localization is flat), because  $M(\emptyset)_{\mathfrak{m}}$  and  $M(1)_{\mathfrak{m}}$  are the modules of a one-element matroid over  $R_{\mathfrak{m}}$ , the dual of the matroid built from  $N_{\mathfrak{m}} \rightarrow N'_{\mathfrak{m}}$ .

Using the previous case, there exists a surjection with cyclic kernel  $M(\emptyset) \rightarrow M(1)$ . That is,  $M$  is a matroid over  $R$ . By Proposition 4.10(a), dualizing this matroid and taking the direct sum with the empty matroid for  $N'_{\text{proj}}$  yields a one-matroid over  $R$  whose objects are  $N$  and  $N'$ . There thus exists a surjection with cyclic kernel  $N \rightarrow N'$ .  $\square$

For a complete description of the structure of matroids over  $R$  we must of course treat axiom (M). It turns out there are no (ring-theoretically) global conditions on such squares, and thus on matroids over  $R$ , further to those imposed by condition (M1).

**Proposition 6.2.** *Let  $M(\emptyset)$ ,  $M(1)$ ,  $M(2)$ , and  $M(12)$  be f.g.  $R$ -modules. There exist four surjections with cyclic kernels forming a pushout square*

$$\begin{array}{ccc} M(\emptyset) & \longrightarrow & M(1) \\ \downarrow & \lrcorner & \downarrow \\ M(2) & \longrightarrow & M(12) \end{array}$$

if and only if the same is true after localizing at each maximal prime  $\mathfrak{m}$ , and the source and target of each map satisfy the equivalent conditions of Proposition 6.1.

*Proof. Necessity.* Trivial in view of Proposition 6.1 and the fact that pushout squares localize to pushout squares.

*Sufficiency.* Fix a pushout square for each localization; label its maps as follows:

$$\begin{array}{ccc} M(\emptyset)_{\mathfrak{m}} & \xrightarrow{f_{\mathfrak{m}}} & M(1)_{\mathfrak{m}} \\ g_{\mathfrak{m}} \downarrow & \lrcorner & \downarrow g'_{\mathfrak{m}} \\ M(2)_{\mathfrak{m}} & \xrightarrow{f'_{\mathfrak{m}}} & M(12)_{\mathfrak{m}} \end{array}$$

It is enough to construct two maps  $M(\emptyset) \rightarrow M(1)$  and  $M(\emptyset) \rightarrow M(2)$  which localize correctly everywhere, for then we may choose  $M(12)$  to be their pushout, since pushouts localize to pushouts.

Suppose first that one of  $M(1)$  and  $M(2)$  has the same rank as  $M(\emptyset)$ , without loss of generality that  $M(2)$  does. By Proposition 6.1, we may construct a map  $\phi : M(\emptyset) \rightarrow M(1)$  so that there exist isomorphisms  $i_{\mathfrak{m}}(A)$  for each prime  $\mathfrak{m}$  and  $A = \emptyset, 1$  making the squares

$$\begin{array}{ccc} M(\emptyset)_{\mathfrak{m}} & \xrightarrow{\phi_{\mathfrak{m}}} & M(1)_{\mathfrak{m}} \\ i_{\mathfrak{m}}(\emptyset) \downarrow & & \downarrow i_{\mathfrak{m}}(1) \\ M(\emptyset)_{\mathfrak{m}} & \xrightarrow{f_{\mathfrak{m}}} & M(1)_{\mathfrak{m}} \end{array}$$

commute. Now, by the proof of Proposition 6.1, we construct  $M(0) \rightarrow M(2)$  as the direct sum of the restriction of the given  $g_m$  to the torsion submodule of  $M(\emptyset)_m$ , and the identity map  $M(0)_{\text{proj}} \rightarrow M(0)_{\text{proj}}$ . It changes nothing to precompose each of these restrictions of  $g_m$  with the corresponding  $i_m(\emptyset)$ . Doing this yields a commutative square

$$\begin{array}{ccc} M(\emptyset)_m & \xrightarrow{\psi_m} & M(2)_m \\ i_m(\emptyset) \downarrow & & \downarrow \\ M(\emptyset)_m & \xrightarrow{g_m} & M(2)_m \end{array}$$

and pasting this square to the last one shows that we have constructed the two maps  $M(\emptyset) \rightarrow M(1)$  and  $M(\emptyset) \rightarrow M(2)$  which localize as desired.

The remaining case is the one in which the ranks of  $M(1)$  and  $M(2)$  are both less than that of  $M(\emptyset)$ . In this case, like the second case of Proposition 6.1, we will proceed via dualization, and then via a similar argument. In brief, we may first construct a map  $M^*(2) \rightarrow M^*(12)$  which localizes correctly, up to intertwining with some isomorphisms. Then since the map  $M^*(1) \rightarrow M^*(12)$  does not involve a rank drop, we may construct it as a direct sum of localizations on the torsion parts using the same isomorphisms. This gives us a diagram  $M^*(1) \rightarrow M^*(12) \leftarrow M^*(2)$  which localizes correctly at every maximal prime. Finally, we may temporarily insert any suitable module in place of  $M^*(\emptyset)$ , for instance the pullback, and then dualize the resulting matroid over  $R$ . Discarding the ersatz  $M(12)$  gives us maps  $M(2) \leftarrow M(\emptyset) \rightarrow M(1)$  which localize correctly, as desired.  $\square$

**6.1. Quasi-arithmetic matroids.** If  $M$  is a matroid over  $\mathbb{Z}$ , then we can define a corank function of  $M$  as the corank function of the generic matroid  $M \otimes_{\mathbb{Z}} \mathbb{Q}$  described above, that is,

$$\text{cork}(A) = \dim M(A)_{\text{proj}}.$$

We also define

$$m(A) \doteq |M(A)_{\text{tors}}|.$$

**Corollary 6.3.** *The triple  $(E, \text{cork}, m)$  is a quasi-arithmetic matroid, i.e.  $m$  satisfies the following properties:*

- (A1) *Let  $A \subseteq E$  and  $b \in E$ . If  $b$  is dependent on  $A$ , then  $m(A \cup \{b\})$  divides  $m(A)$ ; otherwise  $m(A)$  divides  $m(A \cup \{b\})$ .*
- (A2b) *If  $A \subseteq B \subseteq E$  and  $B$  is a disjoint union  $B = A \cup F \cup T$  such that for all  $A \subseteq C \subseteq B$  we have  $\text{rk}(C) = \text{rk}(A) + |C \cap F|$ , then*

$$m(A) \cdot m(B) = m(A \cup F) \cdot m(A \cup T).$$

*Furthermore it has the following property:*

- (A2a) *If  $A, B \subseteq E$  and  $\text{rk}(A \cup B) + \text{rk}(A \cap B) = \text{rk}(A) + \text{rk}(B)$ , then  $m(A) \cdot m(B)$  divides  $m(A \cup B) \cdot m(A \cap B)$ .*

*Proof.* Since  $\text{Pic}(\mathbb{Z})$  is trivial, this is immediate from Propositions 5.2, 5.4, 6.1 and 6.2.  $\square$

This corollary establishes that matroids over  $\mathbb{Z}$  recover many of the essential features of the second author's theory of *arithmetic matroids* from [6].

**Remark 6.4.** An arithmetic matroid is an object satisfying all the axioms of a quasi-arithmetic matroid plus a further one, namely the positivity property (P) of [3]. The axioms of quasi-algebraic matroids are arithmetic ones, pertaining to integer divisibility, whereas (P) has geometric motivation. To be precise, (P) is included as an axiom in order to force positivity of the arithmetic Tutte polynomial  $M_A(x, y)$ . Its geometric nature is that the numbers whose positivity it demands are, in the realizable case, numbers of components of certain strata in the corresponding toric arrangement. Its coefficients also have two natural but nontrivial combinatorial interpretations [6, 3].

The additional property (A2a) appeared in an earlier choice of the axioms [6, first arXiv version].

In fact quasi-arithmetic matroids and matroids over  $\mathbb{Z}$  are not truly equivalent, in that the information contained in the latter is richer: it retains isomorphism classes of torsion groups, not just their cardinalities.

## 7. The Tutte–Grothendieck ring

In this section we continue to let  $R$  be a Dedekind domain. All matroids over  $R$  in this section are essential. The word “matroid” will mean “matroid over  $R$ ” from here through the end of the proof of Lemma 7.8, except when we speak of a generic matroid.

As we defined the operations of deletion and contraction in Section 2, any element may be deleted or contracted. However, if  $a \in E$  is a generic coloop, then  $M \setminus a$  is not essential, so we will disallow these deletions here. Dually, we will exclude the case of contracting a generic loop.

Essentially following Brylawski [4], define the *Tutte–Grothendieck ring* of matroids over  $R$ , which we here denote  $K(R\text{-Mat})$ , to be the ring whose underlying abelian group is generated by a symbol  $\mathbf{T}_M$  for each unlabelled essential matroid  $M$  over  $R$  with nonempty ground set, modulo the relations

$$\mathbf{T}_M = \mathbf{T}_{M \setminus a} + \mathbf{T}_{M/a}$$

whenever  $a$  is not a generic loop or coloop; and whose multiplication is given by linear extension from the relation

$$\mathbf{T}_M \cdot \mathbf{T}_{M'} = \mathbf{T}_{M \oplus M'}.$$

By “unlabelled”, we mean that we consider two matroids  $M$  and  $M'$  over  $R$  to be identical if there is a bijection  $\sigma : E \xrightarrow{\sim} E'$  of their ground sets such that  $M(A) \cong M'(\sigma(A))$  for each subset  $A$  of  $E$ .

The ring  $K(R\text{-Mat})$  turns out to be best understood in terms of the monoid ring of the monoid of  $R$ -modules under direct sum, as in Theorem 7.1 below. This however only identifies a ring which  $K(R\text{-Mat})$  injects into; the precise description of the image is given in Corollary 7.4.

Define  $\mathbb{Z}[R\text{-Mod}]$  to be the ring with a  $\mathbb{Z}$ -linear basis  $\{u^N\}$  with an element  $u^N$  for each f.g.  $R$ -module  $N$  up to isomorphism, and product given by  $u^N u^{N'} = u^{N \oplus N'}$ .

**Theorem 7.1.** *The Tutte–Grothendieck ring injects into*

$$\mathbb{Z}[R\text{-Mod}] \otimes \mathbb{Z}[R\text{-Mod}],$$

*in such a way that for every matroid  $M$  over  $R$ ,*

$$\mathbf{T}_M \mapsto \sum_{A \subseteq E} X^{M(A)} Y^{M^*(E \setminus A)} \quad (7.1)$$

*where  $\{X^N\}$  and  $\{Y^N\}$  are the respective bases of the two tensor factors  $\mathbb{Z}[R\text{-Mod}]$ .*

As a point of notation, we will allow ourselves the abbreviation  $(XY)^N$  for  $X^N Y^N$ .

We immediately compare Theorem 7.1 with the case of matroids over a field, where the Tutte–Grothendieck invariant is the familiar Tutte polynomial  $\mathbf{T}_M$ ; in Section 7.1 we will relate it to other known invariants. If  $R$  is a field, then  $\mathbb{Z}[R\text{-Mod}]$  is the univariate polynomial ring  $\mathbb{Z}[u]$ , and then  $\mathbb{Z}[R\text{-Mod}] \otimes \mathbb{Z}[R\text{-Mod}]$  is, appropriately, a bivariate polynomial ring. If we call the generators of the two tensor factors  $x - 1$  and  $y - 1$  rather than  $X$  and  $Y$ , then (7.1) in fact gives the classical Tutte polynomial, since  $\dim M(A)$  is the corank of  $A$  and  $\dim M^*(E \setminus A)$  is its nullity.

**Remark 7.2.** We have excluded empty matroids from the definition of  $K(R\text{-Mat})$  because there are no linear relations relating them to matroids with nonempty ground set: the unique element in a matroid on one element, whence one might get a relation, must be a loop or coloop. Thus, constructing the Tutte–Grothendieck ring in the presence of zero-element matroids would yield a ring which would, in a minimal fashion, fail to be a domain or to inject into  $\mathbb{Z}[R\text{-Mod}] \otimes \mathbb{Z}[R\text{-Mod}]$ . Applying (7.1) to a matroid  $M$  on zero elements yields the monomial  $X^{M(\emptyset)} Y^{M^*(\emptyset)}$ . But sums of such monomials can also be achieved as sums of polynomials  $\mathbf{T}_M$  for nonempty  $M$ , and these cannot be equal in  $K(R\text{-Mat})$ . However, if classes  $\mathbf{T}_M$  for empty matroids are defined via (7.1), these classes behave correctly under the multiplication of  $K(R\text{-Mat})$ .

Since decomposing a matroid  $M$  over a ring into  $M \setminus i$  and  $M / i$  is not a unique decomposition in the sense of [4], and the irreducibles for direct sum are not all single-element matroids, Theorem 7.1 does not follow directly from the bidecomposition methods of [4], and we must prove it by hand.

For the proof it will be useful to have some explicit understanding of the ring  $\mathbb{Z}[R\text{-Mod}] \otimes \mathbb{Z}[R\text{-Mod}]$ . Proposition 3.3 implies that  $\mathbb{Z}[R\text{-Mod}]$  has one generator  $u^{R/\mathfrak{m}^k}$  for each maximal ideal  $\mathfrak{m}$  and integer  $k > 0$ , and no relations involving these, together with one generator  $u^P$  for each rank 1 projective module  $P$ , among which there are many relations. Indeed, the subring  $\mathbb{Z}[u^P]$  embeds in the group ring of the Picard group with one more variable  $u^R$  adjoined.

*Proof of Theorem 7.1.* To be concise, let  $S$  be the ring  $\mathbb{Z}[R\text{-Mod}] \otimes \mathbb{Z}[R\text{-Mod}]$ . To keep distinct the objects which we have not yet proven isomorphic, let  $[M]$  represent the class of  $M$  in  $K(R\text{-Mat})$ , reserving  $\mathbf{T}_M$  for the element of  $S$  defined in (7.1).

Consider the map  $\mathbf{T} : K(R\text{-Mat}) \rightarrow S$  given by  $\mathbf{T}([M]) = \mathbf{T}_M$ . We find that  $\mathbf{T}$  is a homomorphism of rings, because the deletion–contraction relations and multiplicativity relations hold among the various  $\mathbf{T}_M$ . Both of these are straightforward to check, and

correspond to easy operations on (7.1). The deletion-contraction relation on an element  $a$  is proved by splitting the sum into one sum containing the terms with  $a \notin A$  and another containing the terms with  $a \in A$ . Multiplicativity under direct sum is proved by expanding the product of (7.1) for  $M$  and  $M'$ , and collecting into a single sum over  $A \sqcup A' \subseteq E \sqcup E'$ .

With that, we come to the involved part of the proof, which is to show  $\mathbf{T}$  is an injection. Our approach will be to construct a family  $\mathcal{I}$  of matroids  $M$  whose polynomials  $\mathbf{T}_M$  are linearly independent in  $S$ , and use deletion-contraction relations to expand every matroid in terms of  $\mathcal{I}$ . This will allow every linear relation among the  $\mathbf{T}_M$  to be lifted to a relation among the  $[M]$  via expansion in terms of  $\mathcal{I}$ , proving injectivity. We will moreover be able to conclude that the image of  $\mathbf{T}$  is the span of the images of the matroids in  $\mathcal{I}$ .

As we use the deletion-contraction relations, we will make frequent use of induction on the size of the ground set. In fact, our main technique will be to embed a matroid  $M$  as a minor of another,  $M'$ , and then relate  $M$  to another minor of  $M'$  of the same size plus a collection of smaller minors. But if the ground set has size 1, this will not be as useful: the unique element of a 1-element matroid is necessarily either a loop or a coloop, hence we cannot get construct a deletion-contraction relation involving a smaller matroid as a minor. This will be our base case, and require a different argument. We have broken out the arguments expanding these matroids in terms of  $\mathcal{I}$  into two lemmas, Lemma 7.6 and Lemma 7.8.

The following construction is relevant to both cases. Linearly extend the divisibility relation on the ideals of  $R$  to a total order  $\leq$  such that for ideals  $I, J, K, I \leq J$  implies  $IK \leq JK$ . For each class  $\mathcal{E} \in \text{Pic}(R)$ , let  $N_{\mathcal{E}}$  equal  $R/I$ , where  $I$  is the  $\leq$ -least ideal of  $R$  whose determinant is  $\mathcal{E}^{-1}$ . This produces a fixed cyclic torsion module  $N_{\mathcal{E}}$  representing each class  $\mathcal{E} \in \text{Pic}(R)$ . Note that every submodule  $N'$  of  $N_{\mathcal{E}}$  is also the representative of its own class,  $N' = N_{[N']}$ . Define the single-element matroid  $L_{\mathcal{E}}$  by  $L_{\mathcal{E}}(\emptyset) = N_{\mathcal{E}}$  and  $L_{\mathcal{E}}(1) = 0$ . The dual matroid  $L_{\mathcal{E}}^*$  therefore has  $L_{\mathcal{E}}^*(\emptyset) = R$  and  $L_{\mathcal{E}}^*(1) = N_{\mathcal{E}}$ . ( $L$  and  $L^*$  can be taken to stand for “loop” and “coloop”.) Also, let  $\emptyset_N$  be the empty matroid associated to a torsion  $R$ -module  $N$ .

For a torsion module  $N$ , define a second sort of loop  $K_N$  by taking  $K_N(\emptyset) = N$  and  $K_N(1)$  to be the quotient of  $N$  by its largest invariant factor.

We construct the set  $\mathcal{I}$  as follows:

$$\begin{aligned} \mathcal{I} = \{ & K_N : N \text{ is torsion} \} \\ & \cup \{ \emptyset_N \oplus L_{\mathcal{E}} \oplus L_0^{\oplus a} : N \text{ is torsion, } \mathcal{E} \in \text{Pic}(R), a \geq 0 \} \\ & \cup \{ \emptyset_N \oplus L_{\mathcal{F}}^* \oplus (L_0^*)^{\oplus b} : N \text{ is torsion, } \mathcal{F} \in \text{Pic}(R), b \geq 0 \} \\ & \cup \{ \emptyset_N \oplus L_{\mathcal{E}} \oplus L_0^{\oplus a} \oplus L_{\mathcal{F}}^* \oplus (L_0^*)^{\oplus b} : N \text{ is torsion, } \mathcal{E}, \mathcal{F} \in \text{Pic}(R), a, b \geq 0 \}. \end{aligned} \tag{7.2}$$

To analyze linear relations in  $\mathcal{I}$ , we give the ring  $S$  a monomial order wherein, if  $P$  and  $Q$  are rank 1 projective modules, then  $X^P$  is greater than  $Y^Q$ , which in turn is greater than  $X^N$  or  $Y^N$  for any torsion module  $N$ .

Then if  $M$  is a matroid with a unique basis, as all the matroids in  $\mathcal{I}$  are, the initial term of  $\mathbf{T}_M$  is the term contributed to the sum in (7.1) by the complement of the unique

basis of  $M$ . For the matroid

$$M = \emptyset_N \oplus L_{\mathcal{E}} \oplus L_0^{\oplus a} \oplus L_{\mathcal{F}}^* \oplus (L_0^*)^{\oplus b}$$

the complement of the unique basis is sent to  $N \oplus P \oplus R^b$ , where  $P$  is the rank 1 projective module whose determinant is  $\mathcal{F} \in \text{Pic}(R)$ . For the dual of this matroid, the analogous module is  $N \oplus Q \oplus R^a$  where  $Q$  is the rank 1 projective whose determinant is  $\mathcal{E}$ . Therefore the initial term of  $\mathbf{T}_M$  is  $X^{N \oplus P \oplus R^b} Y^{N \oplus Q \oplus R^a}$ . Similarly, if instead of  $M$  we had taken one of the matroids from the previous two lines in  $\mathcal{I}$ , the initial term would be just  $X^{N \oplus P \oplus R^b}$  or  $Y^{N \oplus Q \oplus R^a}$ , respectively. All the monomials in these three classes are distinct.

Finally, the initial term of  $\mathbf{T}_{K_N}$  is  $Y^Q$  times monomials corresponding to torsion modules, for some rank 1 projective  $Q$ . It follows that any nontrivial  $\mathbb{Z}$ -linear relation among the classes of elements of  $\mathcal{I}$  may contain only these matroids and others of smaller leading terms: that is, it may involve only one-element matroids whose unique element is a loop.

Temporarily let  $\mathcal{I}_1$  be the set of the matroids  $K_N$ , and  $\mathcal{I}_2$  the set of matroids of form  $\emptyset_N \oplus L_{\mathcal{E}}$ , so that together every matroid in  $\mathcal{I}$  whose unique element is a loop is in  $\mathcal{I}_1$  or  $\mathcal{I}_2$ . Suppose there was a nontrivial linear dependence among the classes of these matroids. (The sets  $\mathcal{I}_1$  and  $\mathcal{I}_2$  share some elements, but since we have taken their union as *sets*, this is not a problem.) The class of each matroid in  $\mathcal{I}_1 \cup \mathcal{I}_2$  is of the form  $(XY)^N + Y^Q (XY)^{N'}$  where  $N$  and  $N'$  are torsion  $R$ -modules, and  $Q$  is a rank 1 projective module. Moreover, there is only one element of  $\mathcal{I}_1$  and one of  $\mathcal{I}_2$  with a given value of  $N$ . Therefore, if there is any linear relation, there must be a minimal one of the form

$$\sum_{j=1}^k [M_{1,j}] - \sum_{j=1}^k [M_{2,j}] = 0 \tag{7.3}$$

where  $M_{i,j} \in \mathcal{I}_i$ , all  $M_{i,j}(\emptyset)$  have the same determinant in  $\text{Pic}(R)$ , and  $M_{1,j}(\emptyset) = M_{2,j}(\emptyset)$ . The equality also implies that the product of the annihilators in  $R$  of the kernels of  $M_{1,j}(\emptyset) \rightarrow M_{1,j}(1)$  equals the corresponding product of annihilators for the kernels of  $M_{2,j}(\emptyset) \rightarrow M_{2,j}(1)$ . All of these annihilators have the same determinant. The latter product is  $I^k$ , where  $I$  is the annihilator of  $N_{\mathcal{E}}$ . Therefore, at least one of the ideals in the former product must be less than or equal to  $I$  in  $\leq$  order. But  $I$  is the  $\leq$ -minimal ideal of its class, and so these ideals must all equal  $I$ , so that all coefficients on the left side of (7.3) are zero and the relation is trivial. Thus  $\mathcal{I}$  is dependent, as claimed.  $\square$

We may now describe the image of  $K(R\text{-Mat})$  within the ring  $\mathbb{Z}[R\text{-Mod}] \otimes \mathbb{Z}[R\text{-Mod}]$ . Two constraints on the monomials that may appear can be extracted from Corollary 4.12.

**Corollary 7.3.** (a) *If  $X^N Y^{N'}$  is a term of  $\mathbf{T}_M$ , then  $N_{\text{tors}} = N'_{\text{tors}}$ .*  
 (b) *Consider the ring homomorphism*

$$\det : \mathbb{Z}[R\text{-Mod}] \otimes \mathbb{Z}[R\text{-Mod}] \rightarrow \mathbb{Z}[u^{\mathcal{E}} : \mathcal{E} \in \text{Pic}(R)]$$

*given by  $\det(X^N) = u^{[N] - [N_{\text{tors}}]}$  and  $\det(Y^N) = u^{[N]}$ . Then  $\det(\mathbf{T}_M)$  is a scalar multiple of  $u^{\det(M)}$ .*



*Proof.* Part (a) is immediate from the preservation of torsion parts under duality in Corollary 4.12, and part (b) from the equality

$$\det(M) = \det(M(A)) + \det(M^*(E \setminus A)) - \det(M(A)_{\text{tors}}). \quad \square$$

Examination of the classes of matroids in  $\mathcal{I}$  in the proof of Theorem 7.1 shows that they span the subring of polynomials compatible with Corollary 7.3(a). Thus we deduce the following.

**Corollary 7.4.** *If empty matroids are included, the ring  $K(R\text{-Mat})$  is the subring of  $\mathbb{Z}[R\text{-Mod}] \otimes \mathbb{Z}[R\text{-Mod}]$  generated by the symbols  $X^P$  and  $Y^P$  as  $P$  ranges over rank 1 projective modules, and  $(XY)^N$  as  $N$  ranges over torsion modules.*

We now set out the main substance of the proof of Theorem 7.1. The following subsidiary lemma will afford us useful flexibility for manipulating single-element matroids in proving Lemma 7.6.

**Lemma 7.5.** *Let  $\mathcal{E}$  be a class in  $\text{Pic}(R)$ , and  $P$  a finite set of maximal primes of  $R$ . There exists a cyclic torsion  $R$ -module  $N$  whose support is disjoint from  $P$  such that  $\det(N) = \mathcal{E}$ .*

*Proof.* This is a restatement of a standard lemma on ideal factorizations (see e.g. [5, Corollary 4.9]). In the notation of that corollary, we let  $\mathfrak{a}$  be any ideal of determinant  $\mathcal{E}$  and let  $\mathfrak{b}$  be the product of the members of  $P$ , and the module we seek is  $N = R/\mathfrak{c}$ .  $\square$

**Lemma 7.6.** *If  $M$  is a one-element matroid satisfying any one of the following, then the class  $[M] \in K(R\text{-Mat})$  lies in the span of the classes of matroids in the set  $\mathcal{I}$  of (7.2).*

- (a)  $M(\emptyset) = P \oplus N$  and  $M(1) = N \oplus C$ , where  $P$  is rank 1 projective,  $N$  is torsion and  $C$  is cyclic, and the supports of  $N$  and  $C$  are disjoint.
- (b)  $M(\emptyset) = N \oplus C$  and  $M(1) = N$ , where  $N$  is torsion and  $C$  is cyclic, and the supports of  $N$  and  $C$  are disjoint.
- (c)  $M(\emptyset) = P \oplus N'$ ,  $M(1) = N$ , where  $P$  is rank 1 projective,  $N$  is torsion, and  $N'$  is the quotient of  $N$  by its largest invariant factor.
- (d) Either  $M$  or  $M^*$  sends  $\emptyset$  to  $P \oplus T \oplus N'$  and  $\{1\}$  to  $T \oplus N$ , where  $P$  is rank 1 projective,  $N$  is torsion,  $N'$  is the quotient of  $N$  by its largest invariant factor, and the support of  $T$  is disjoint from that of  $N$ .
- (e) Either  $M$  or  $M^*$  sends  $\emptyset$  to  $N \oplus T$  and  $\{1\}$  to  $T$ , where  $N$  is cyclic and  $T$  is torsion.
- (f)  $M$  is any one-element matroid.

*Proof.* A matroid  $M'$  on two elements whose generic matroid is the uniform matroid  $U_{1,2}$  gives rise to a linear relation among its four one-element minors,

$$[M' \setminus 1] + [M' / 1] = [M'] = [M' \setminus 2] + [M' / 2]. \quad (7.4)$$

We will use this to prove the cases of the lemma sequentially, reducing each to a linear combination of matroids in  $\mathcal{I}$  and matroids in previous cases. For visibility we will specify these  $M'$  by drawing the commutative square

$$\begin{array}{ccc}
 M'(\emptyset) & \longrightarrow & M'(1) \\
 \downarrow & & \downarrow \\
 M'(2) & \longrightarrow & M'(12)
 \end{array}$$

In each case  $M'$  can be checked to be a matroid by Proposition 6.2. The nonlocal conditions reduce to checking that  $\det(M'(1)) = \det(M'(2))$ .

(a) Let  $N'$  be the quotient of  $N$  by its largest invariant factor. First, suppose that the support of  $C$  is disjoint from the supports of  $N$  and  $L_{[C]}(1)$ . In that case, the following square specifies a matroid  $M'$ . The modules  $N \oplus C$  and  $N' \oplus L_C(1)$  have the same determinant by construction. And because of the assumption on supports, modulo each maximal ideal  $\mathfrak{m}$ , either the top map has kernel  $R_{\mathfrak{m}}$  and the right one is trivial, or the same is true of the left and bottom maps respectively. This ensures that the localizations of the square are pushouts.

$$\begin{array}{ccc}
 P \oplus N & \longrightarrow & N \oplus L_{[C]}(1) \\
 \downarrow & & \downarrow \\
 N \oplus C & \longrightarrow & N'
 \end{array}$$

The left minor  $M' \setminus 1$  is the matroid we are interested in; the bottom minor  $M' / 2$  and the right minor  $M' / 1$  are both among the matroids  $K_N$  in  $\mathcal{I}$ ; and the top minor  $M' \setminus 2$  is among the  $\emptyset_N \oplus L_{\mathcal{F}}^*$ . So the relation (7.4) proves the result in this case.

Next, if we lack the support assumptions on  $C$ , we are assured the existence of a cyclic module  $C'$  of support disjoint from  $L_{[C]}(1)$  and  $N$ , with the same determinant as  $C$ , by Lemma 7.5. In this case, we repeat the argument with the following square, which can similarly be checked to give a matroid  $M''$ :

$$\begin{array}{ccc}
 P \oplus N & \longrightarrow & N \oplus C' \\
 \downarrow & & \downarrow \\
 N \oplus C & \longrightarrow & N'
 \end{array}$$

Now  $M'' \setminus 1$  is the matroid of interest, the minors  $M'' / 2$  and  $M'' / 1$  are among the matroids  $K_N$ , and  $M'' \setminus 2$  is in the span of  $\mathcal{I}$  by the last paragraph. Therefore, using (7.4) again, we have proved case (a).

(b) Let  $R/I$  be the largest invariant factor of  $N$ . Let  $J$  be a nonzero ideal contained in  $I$  chosen so that  $\det(J) = -\det(C)$ , and the supports of  $R/J$  and  $L_{[C]}(1)$  are disjoint; this exists by Lemma 7.5. Then  $R/J$  is the largest invariant factor of  $N \oplus R/J$ . Having done this, both of the following squares give matroids, where  $P$  is a suitably chosen rank 1 projective module:

$$\begin{array}{ccc}
 M' : & P \oplus N & \longrightarrow & N \oplus L_{[C]}(1) \\
 & \downarrow & & \downarrow \\
 & N \oplus R/J & \longrightarrow & N \\
 \\
 M'' : & P \oplus N & \longrightarrow & N \oplus C \\
 & \downarrow & & \downarrow \\
 & N \oplus R/J & \longrightarrow & N
 \end{array}$$

Subtracting the relation (7.4) for the two matroids, we express the class of  $M''/1$ , which is the matroid of interest, as a linear combination of the classes of  $M''\setminus 2$ ,  $M'/1$ , and  $M'\setminus 2$ . But, of these,  $M''\setminus 2$  is one of the matroids appearing in part (a),  $M'/1$  is of form  $\emptyset_N \oplus L_{\mathcal{E}}$ , and  $M'\setminus 2$  is of form  $\emptyset_N \oplus L_{\mathcal{F}}^*$ . This proves case (b).

(c) Use Lemma 7.5 to produce a cyclic module  $C$  whose determinant is  $\det(N) - \det(N')$  and whose support is disjoint from that of  $N$ . Then the following square gives a matroid  $M'$ :

$$\begin{array}{ccc} P \oplus N' & \longrightarrow & N' \oplus C \\ \downarrow & & \downarrow \\ N & \longrightarrow & N' \end{array}$$

Here,  $M'/2$  is of form  $\emptyset_N \oplus L_{\mathcal{E}}$ , and  $M'/1$  is covered by case (b) of the lemma, and  $M'\setminus 2$  is covered by case (a). So (7.4) proves case (c).

At this point, we pause to take note that the matroids of form  $\emptyset_N \oplus L_{\mathcal{E}}$  and  $\emptyset_N \oplus L_{\mathcal{E}}^*$  and the matroids encompassed by this last case (c) are the duals of the  $K_N$ . These are all the matroids in  $\mathcal{I}$  of one element. Moreover, in equation (7.4), dualizing  $M'$  dualizes the four minors. Therefore, for the rest of this proof, arguing that a class  $[M]$  is in the linear span of classes of matroids in  $\mathcal{I}$  will imply the same for the class  $[M^*]$  of the dual.

(d) As stated just above, it is sufficient to treat the case where  $M$  is as described, not its dual. For this we use induction on the number of invariant factors of the torsion module  $T$ . If it has none, it is the zero module and we are in case (a). Otherwise, let  $T'$  be the quotient of  $T$  by its largest invariant factor.

Let  $N''$  be the quotient of  $N'$  by its largest invariant factor. Lemma 7.5 gives cyclic modules  $C$  and  $D$  whose determinants take the necessary values, and whose supports are disjoint from the supports of other appearing modules as necessary, in order for the following squares to specify matroids  $M'$  and  $M''$ ,

$$\begin{array}{ccc} M' : P \oplus N' \oplus T & \longrightarrow & N' \oplus T \oplus C \\ \downarrow & & \downarrow \\ N \oplus T & \longrightarrow & N' \oplus T' \end{array} \quad \begin{array}{ccc} M'' : P \oplus N'' \oplus T' & \longrightarrow & N' \oplus T \oplus C \\ \downarrow & & \downarrow \\ N' \oplus T' \oplus D & \longrightarrow & N' \oplus T' \end{array}$$

Here  $M'/1$  equals  $M''/1$ , and these can be cancelled out of the two corresponding invocations of (7.4), leaving a linear relation among the six other minors. Of these,  $M'\setminus 1$  is the matroid of interest.  $M''\setminus 1$  is the matroid to which we will apply the induction hypothesis: when applying it we take the new module  $T$  to be  $T'$ , which has one invariant factor fewer than (the old)  $T$ , and the new modules  $N' \rightarrow N$  to be  $N'' \rightarrow N' \oplus D$ . The remaining minors are dealt with:  $M'/2$  is among the  $K_N$ ,  $M''/2$  is dealt with in case (b) of this lemma,  $M'\setminus 2$  in case (a), and  $M''\setminus 2$  in case (c). Therefore the induction goes through and we have proved case (d).

(e) Again we may assume  $M$  (not  $M^*$ ) is as described. We use induction on the maximum  $k$  such that, for some maximal prime  $\mathfrak{m}$  contained in the support of  $N$ , there are  $k$  cyclic summands of  $T_{\mathfrak{m}}$  longer than  $N_{\mathfrak{m}}$ . If  $k = 0$ , then  $N$  is the largest invariant factor of the part of  $N \oplus T$  with the same support, and  $M$  falls under case (d).

Otherwise, let  $F$  be the direct sum of the localizations of the largest invariant factor of  $T$  of length exceeding the corresponding localization of  $N$ , and let  $T'$  be  $T/F$ . Lemma 7.5 provides cyclic modules  $C$  and  $D$  so that the following squares are matroids (for the top maps of the squares, this is where the fact that  $\dim_{R/m} F_m \geq \dim_{R/m} N_m$  is used):

$$\begin{array}{ccc}
 M' : P \oplus N & \longrightarrow & T \oplus C \\
 \downarrow & & \downarrow \\
 N \oplus T & \longrightarrow & T
 \end{array}
 \quad
 \begin{array}{ccc}
 M'' : P \oplus N & \longrightarrow & T \oplus C \\
 \downarrow & & \downarrow \\
 N \oplus T' \oplus D & \longrightarrow & T'
 \end{array}$$

Again,  $M'/2$  equals  $M''/2$ , and two invocations of (7.4) give a linear relation among the remaining minors. Of these  $M'\setminus 2$  is the matroid of interest,  $M''\setminus 2$  is covered by the inductive hypothesis, and all of  $M'/1$ ,  $M''/1$ ,  $M'\setminus 1$ , and  $M''\setminus 1$  fall under case (d). This proves case (e).

(f) Either  $M$  or  $M^*$  is of global rank 0, and we may assume it is  $M$ . Here we use one further induction. Let  $k$  be the maximum, over maximal primes  $m$ , of the number of times 01 or 10 appear as substrings of the sequence  $d_\bullet(\phi_m)$  associated in Section 5 to the map  $\phi_m$  in the localized matroid  $M \otimes R_m$ . As a base case, if  $k \leq 1$ , then each  $\phi_m$ , and therefore  $\phi$ , is a quotient by a cyclic summand; this is case (e).

Otherwise, let  $N$  be the quotient of  $M(\emptyset)$  by its largest invariant factor. With  $C$  provided by Lemma 7.5 as usual, we have a matroid  $M'$  given by

$$\begin{array}{ccc}
 P \oplus N & \longrightarrow & M(1) \oplus C \\
 \downarrow & & \downarrow \\
 M(\emptyset) & \longrightarrow & M(1)
 \end{array}$$

The minor  $M'/2$  is  $M$ . The minor  $M'\setminus 2$  is covered by the induction hypothesis: if  $\psi$  is the map in this matroid, then for each  $m$  the sequence  $d_\bullet(\psi_m)$  is obtained from  $d_\bullet(\phi_m)$  by replacing the final infinite run of 0s by 1s, so one of the substrings 10 is lost. The minors  $M'/1$  and  $M'\setminus 1$  are both handled by case (e). This proves case (f) and finishes our discussion of one-element matroids.  $\square$

We approach the reduction of matroids on several elements to our basis in Lemma 7.8 below, in several steps as we did in Lemma 7.6. The bulk of our discussion here will pertain to matroids with one generic basis; in the terminology of [6], these are called *molecules* (since for matroids over a field all molecules are direct sums of *atoms*, i.e. one-element matroids). First we state two subsidiary technical lemmas.

**Lemma 7.7.** *If  $M$  is a molecule on a ground set  $E$  and  $a$  a generic coloop in it, then  $M(A)_{\text{proj}} = M(E \setminus a)_{\text{proj}} \oplus M(Aa)_{\text{proj}}$  for every  $A \subseteq E \setminus \{a\}$ .*

*Proof.* This is clear for  $A = E \setminus \{a\}$ , so by induction on the size of the complement of  $A$  we need only establish the statement for  $A$  given the statement for  $Ab$ . The rank drop between  $M(A)$  and  $M(Ab)$  equals that between  $M(Aa)$  and  $M(Aab)$ . If this rank drop is

zero, then we are done because  $M(Ab)_{\text{proj}} = M(A)_{\text{proj}}$  and  $M(Aab)_{\text{proj}} = M(Aa)_{\text{proj}}$ . If the rank drop is one, then given maps making a pushout square

$$\begin{array}{ccc} M(A) & \xrightarrow{\phi} & M(Ab) \\ \downarrow & & \downarrow \\ M(Aa) & \xrightarrow{\phi'} & M(Aab) \end{array}$$

the kernels of  $\phi$  and  $\phi'$  are isomorphic projective modules; call one of them  $P$ . Then, in  $K_0(R)$ , we have

$$[M(A)] = [M(Ab)] \oplus [\ker \phi] = [P] \oplus [M(Aab)] \oplus [\ker \phi'] = [P] \oplus [M(Aa)]$$

and the  $K$ -class of a projective module determines it. □

**Lemma 7.8.** *If  $M$  is a matroid on a ground set  $E$  satisfying any of the following, then the class  $[M] \in K(R\text{-Mat})$  lies in the span of the classes of the matroids in  $\mathcal{I}$ .*

- (a)  $M$  is a direct sum of one-element matroids.
- (b) The only generic basis of  $M$  is  $\emptyset$ .
- (c)  $M$  is a molecule.
- (d)  $M$  is arbitrary.

*Proof.* As in the last proof, we will manipulate our matroid  $M$  by fabricating larger matroids in which  $M$  appears as a minor. But we have more room to manoeuvre, as matroids on fewer elements than  $M$  may also appear in the deletion-contraction relations, and induction on the ground set size shows that the classes of these are in the span of  $\mathcal{I}$ .

(a) For this step of the argument we will use another induction, on the number of one-element summands of a matroid  $M$  which are not  $L_0$  or  $L_0^*$ , possibly excluding one summand of each of the forms  $L_{\mathcal{E}}$  and  $L_{\mathcal{F}}^*$ . (There may also be a summand which is an empty matroid for a torsion module; this will be inert and have no effect on our argument.) In the base case,  $M$  is a direct sum of some copies of  $L_0$ , possibly a single  $L_{\mathcal{E}}$ , some copies of  $L_0^*$ , possibly a single  $L_{\mathcal{E}}^*$ , and some empty matroid  $\emptyset_N$ ; this is an element of  $\mathcal{I}$ .

As inductive step, we will use deletion-contraction relations to increase the number of such summands in two ways, one of which applies to any direct sum of at least two one-element matroids. One of our constructions will replace a direct sum of two one-element summands of the same generic rank by a matroid of form  $L_{\mathcal{E}} \oplus L_0$  or  $L_{\mathcal{F}}^* \oplus L_0^*$ . The other will replace a direct sum of two one-element summands of unequal generic ranks with some  $L_{\mathcal{E}} \oplus L_{\mathcal{F}}^*$ .

For the former construction suppose we have a two-element molecule  $N$ , without loss of generality having two coloops, which is a summand of  $M$ ; write  $M = N \oplus K$ . For convenience suppose the ground set of  $N$  is  $\{1, 2\}$ .

The basis of  $N$  is  $\emptyset$ , and  $N(\emptyset)$  has the form  $P \oplus R \oplus T$ , where  $P$  is a rank 1 projective module and  $T$  is some torsion module. Fix maps  $\phi : N(\emptyset) \rightarrow N(1)$ ,  $\psi : N(\emptyset) \rightarrow N(2)$ . By making the nonfree analogue of a change of basis in this splitting  $P \oplus R$  if necessary, we can suppose that neither of the saturations of  $\ker \phi$  nor  $\ker \psi$  is contained in  $R$  or  $P$ .

Now embed  $N$  in a realizable matroid  $N'$  on  $\{1, 2, 3, 4\}$  so that  $N = N' \setminus \{3, 4\}$ , the map  $N'(\emptyset) \rightarrow N(3)$  is the quotient map  $P \oplus R \oplus T \rightarrow L_{[P]}(1) \oplus R \oplus T$  on the first factor, the map  $N'(\emptyset) \rightarrow N(4)$  is the quotient  $P \oplus R \oplus T \rightarrow P \oplus T$  on the second, and the rest of  $N'$  is completed by taking pushouts.

By construction, none of the kernels of the maps with source  $N'(\emptyset)$  in this realization has its saturation contained in another such saturation, so that the quotient of  $N'(\emptyset)$  by the sum of two such kernels has rank 0. Thus, the generic matroid of  $N'$  is  $U_{2,4}$ . Thus, the direct sum  $M' = N' \oplus K$  is  $U_{2,4}$  plus a molecule. We will use deletion-contraction relations to break  $M'$  down in two ways, the knowledge of the generic matroid of  $M'$  assuring us that we are not choosing loops or coloops. On the one hand, use (in sequence) the elements 3 of  $M'$ , 4 of  $M' \setminus 3$ , 1 of  $M' \setminus 3/4$ , 1 of  $M'/3$ , and 2 of  $M' \setminus 1/3$ . On the other, use the elements 1 of  $M'$ , 2 of  $M' \setminus 1$ , 3 of  $M' \setminus 1/2$ , 3 of  $M'/1$ , and 4 of  $M' \setminus 3/1$ . This gives us equalities of classes in the Grothendieck ring:

$$\begin{aligned} & [M' \setminus 3, 4] + [M' \setminus 1, 3/4] + [M' \setminus 3/1, 4] + [M' \setminus 1, 2/3] + [M' \setminus 1/2, 3] + [M'/1, 3] \\ &= [M'] \\ &= [M' \setminus 1, 2] + [M' \setminus 1, 3/2] + [M' \setminus 1/2, 3] + [M' \setminus 3, 4/1] + [M' \setminus 3/1, 4] + [M'/1, 3]. \end{aligned}$$

The term  $[M'/1, 3]$  cancels, and all of the remaining terms aside from  $[M' \setminus 3, 4]$  and  $[M' \setminus 1, 2]$  are classes of matroids on fewer elements, so they are in the span of the classes of  $\mathcal{I}$  by our top-level induction. The matroid  $M' \setminus 3, 4$  is our original  $M$ . Finally,  $M' \setminus 1, 2$  has more summands than  $M$  which are  $L_0$  or  $L_0^*$ : there is a new such summand in  $M' \setminus 1, 2$  on the element 4. So it is covered by one of our inductions as well.

Turning to the latter construction, we will in fact need to invoke a second induction, on the rank of the generic matroid of  $M$ , that is the size of its generic basis. We set this up decreasingly, so the base case is when  $M$  has only coloops: in this case,  $M$  has no loop and this construction cannot in fact apply.

Continuing, we suppose  $M$  has a two-element summand  $N$ , say on the ground set  $\{1, 2\}$ , which is itself the sum of a matroid  $N_1$  on its coloop 1, and a matroid  $N_2$  on its loop 2. Again we write  $M = N \oplus K$ .

By choosing any maps and computing the pushout, we may construct a matroid  $\tilde{N}_2$  on the ground set  $\{2, 4\}$  where  $\tilde{N}_2(\emptyset) = N_2(\emptyset)$ ,  $\tilde{N}_2(2) = N_2(2)$ , and  $\tilde{N}_2(4) = N_2(\emptyset)_{\text{tors}} \oplus L_{[P]}(1)$  where  $P = N_2(\emptyset)_{\text{proj}}$ . Its generic matroid will be  $U_{1,2}$ . With the dual of this construction we also construct a matroid  $\tilde{N}_1$  on the ground set  $\{1, 3\}$  with generic matroid  $U_{1,2}$ , where  $\tilde{N}_1(3) = N_1(\emptyset)$ ,  $\tilde{N}_1(13) = N_1(1)$ , and  $\tilde{N}_1(1) = N_1(1) \oplus L_{\mathcal{E}}(1)$  where  $\mathcal{E} = [N_1(\emptyset)] - [N_1(1)]$ .

We will construct  $N'$  as a perturbation of  $\tilde{N} \doteq \tilde{N}_1 \oplus \tilde{N}_2$ , as follows. Fix realizations of  $\tilde{N}_1$  and  $\tilde{N}_2$ , so that the induced realization of  $\tilde{N}$  provides four maps  $\phi_1, \dots, \phi_4$  with cyclic kernel from the module  $\tilde{N}(\emptyset)$ , corresponding respectively to the atoms  $1, \dots, 4$  covering  $\emptyset$  in  $\mathcal{B}(4)$ . The kernels of  $\phi_1$  and  $\phi_3$  are both contained in  $\tilde{N}_1(\emptyset)$ , while the kernels of  $\phi_2$  and  $\phi_4$  are contained in  $\tilde{N}_2(\emptyset)$ ; all of them are isomorphic to  $R$  as  $R$ -modules. The module  $\tilde{N}_1(\emptyset)$  is the direct sum of a projective rank 1 summand  $P$ , and a torsion module. There exists an injection  $\psi : P \hookrightarrow \ker \phi_2 \cap \ker \phi_4$ . This can be composed with the embedding  $\ker \phi_2 \cap \ker \phi_4 \subseteq \tilde{N}(\emptyset)$  and summed with zero maps on the other

summands to produce a map  $\psi : N(\emptyset) \rightarrow N(\emptyset)$ . The map  $\text{id} + \psi : N(\emptyset) \rightarrow N(\emptyset)$  is then “upper triangular” and hence an automorphism. Let  $x$  be a generator of  $\ker \phi_3$ , and define a new map  $\phi'_3$  from  $\tilde{N}(\emptyset)$  to be the quotient by the submodule  $\langle x + \psi(x) \rangle$ . Finally, let  $N'$  be the matroid on the ground set  $\{1, 2, 3, 4\}$  with  $N'(\emptyset) = \tilde{N}(\emptyset)$  and whose other modules and maps are induced as pushouts of  $\phi_1, \phi_2, \phi'_3$ , and  $\phi_4$ .

Our perturbation of  $\phi_3$  to  $\phi'_3$  has arranged that  $\text{cork}_{N'}(13) = 0$ . On the other hand, if  $3 \notin A$  then  $N'(A)$  is unchanged from  $\tilde{N}(A)$ ; if  $A$  contains 3 and one of 2 or 4 but not 1 then  $N'(A) \cong \tilde{N}(A)$  by construction of  $\psi$ ; and  $N'(3) \cong \tilde{N}(3)$  as well, since  $\text{id} + \psi$  is an automorphism. In particular the generic matroid of  $N$  is the rank 2 matroid on  $\{1, 2, 3, 4\}$  with no loops whose only nontrivial parallelism class is  $\{2, 4\}$ .

Let  $M' = N' \oplus K$ . We have deletion-contraction relations giving the following equalities:

$$\begin{aligned}
 & [M' \setminus 3, 4] + [M' \setminus 3/4] + [M' \setminus 1, 2/3] + [M' \setminus 1/2, 3] + [M' / 1, 3] \\
 &= [M'] = [M' \setminus 1, 2] + [M' \setminus 1/2] + [M' \setminus 3, 4/1] + [M' \setminus 3/1, 4] + [M' / 1, 3].
 \end{aligned}$$

The term  $[M' / 1, 3]$  cancels, and the two terms before it in each line are matroids on fewer elements. The matroid  $M' \setminus 3/4$  is our original  $M$ , since  $M$  was the same minor of  $\tilde{N} \oplus K$  and we have not altered the relevant modules in it. The matroid  $M' \setminus 1/2$ , for the analogous reason, is the direct sum of  $L_{[P]}^*$  on the element 4,  $L_E$  on the element 3, and an empty matroid, so  $M' \setminus 1/2$  improves on the quantity counted in the induction we introduced at the start of this case (a). The remaining matroids,  $M' \setminus 3, 4$  and  $M' \setminus 1, 2$ , are also direct sums of one-element matroids, and they both have generic rank 2, so they are covered by our latest-introduced induction. Altogether, this finishes case (a).

(b) We will use induction on the number of elements of  $E$  which are not the ground set of a one-element direct summand. The base case is part (a).

We construct a matroid  $M'$  on  $E \sqcup \{\eta\}$  which will agree in most of its modules with the direct sum of  $M$  and a loop  $\emptyset \mapsto 0, \{\eta\} \mapsto 0$ . In particular  $M' / \eta$  will be  $M$ . We let  $M'(\emptyset)$  be obtained from  $M(\emptyset)$  by replacing its largest invariant factor with a projective module with the same determinant. For each  $b \in E$ , we use Lemma 7.5 to produce a cyclic module  $C(b)$  of support disjoint from any module in  $M$  and so that  $[C(b)] + [M(b)] = [M(\emptyset)]$  in  $\text{Pic}(R)$ , and then set  $M'(b) = M(b) \oplus C(b)$ . In any other case set  $M'(A) = M(A \setminus \eta)$ , where  $\eta$  is not necessarily in  $A$ .

Our choices of  $M'(\emptyset)$  and the modules  $M'(b)$  for singletons are exactly as is needed so that all the pairs  $M'(\emptyset), M'(b)$  satisfy the  $K$ -theoretic condition of Proposition 6.1. For the other covering relations of subsets of  $E \sqcup \{\eta\}$ , both modules are rank 0 so the  $K$ -theoretic condition is trivially satisfied. The localization conditions are essentially inherited from  $M$ . Since the summands  $C(b)$  have supports disjoint from any of the other modules under consideration, they do not interfere in this respect. The alteration we have made to  $M'(\emptyset)$  replaces a final infinite string of 0s by 1s in the sequences  $d_\bullet$  associated to the maps  $M'(\emptyset)_m \rightarrow M'(b)_m$ ; the resulting sequence is still of the sort allowed by Proposition 5.2. These same facts about the localizations also suffice to establish Proposition 6.2, in which only the local considerations of Proposition 5.4 are relevant.

The generic matroid of  $M'$  is  $U_{1,|E|+1}$ . Therefore, no deletion of  $M'$  with more than one element is a molecule, and we may freely use the deletion-contraction relation on such deletions. Let  $a$  be any element of  $E$ . Splitting  $M$  into three minors by deletion-contraction in two ways, we have

$$[M' \setminus \eta \setminus a] + [M' \setminus \eta / a] + [M' / \eta] = [M'] = [M' \setminus a / \eta] + [M' \setminus a / \eta] + [M' / a],$$

so that, cancelling the common deletion,

$$[M' \setminus \eta / a] + [M' / \eta] = [M' \setminus a / \eta] + [M' / a].$$

Here, the minor  $M' / \eta$  is our matroid  $M$  of interest. The matroid  $M' / a$  has a one-element summand with ground set  $\{\eta\}$  together with whichever one-element summands  $a$  had, so it is subsumed by our induction hypothesis. The other two matroids are on fewer elements. This proves case (b).

(c) Here we use induction on the number of generic coloops and on the size of the number of elements which do not generate single-element direct summands.

Suppose that  $a$  is a generic coloop of  $M$ . Then  $M(E \setminus a)$  has a rank 1 projective summand, call it  $P$ . By Lemma 2.5, the empty matroid  $\emptyset_P$  for  $P$  splits as a direct summand of  $M \setminus a$ . Name the other direct summand  $N$ .

Let  $C$  be a cyclic module which is sufficiently large that every cyclic summand of a module appearing in  $M$  is isomorphic to a quotient of  $C$ , and such that  $[P] = [C]$  in  $\text{Pic}(R)$ . Let  $M'$  be a system of  $R$ -modules so that  $M' \setminus \eta = M$ ;  $M' / \eta \setminus a = N \oplus \emptyset_C$ ; and  $M' / \eta / a = M / a$ . That is,  $M'$  is obtained from the direct sum  $\tilde{M}$  of  $M$  and the one-element matroid  $\emptyset \mapsto 0$ ,  $\{\eta\} \mapsto 0$  by replacing a summand  $P$  by  $C$  at every set containing  $\eta$  but not  $a$ . We will show that  $M'$  is a matroid using Propositions 6.1 and 6.2.

For Proposition 6.2, since  $\tilde{M}$  is a matroid, we need only check that the replacements of  $P$  by  $C$  do not interfere with the condition to be checked in Proposition 5.4. If  $\mathfrak{m}$  is a maximal prime, then the sequences  $d_\bullet(\tilde{M}(A))$  and  $d_\bullet(M'(A))$  are of course identical if no replacement has taken place, and if one has, they differ only in that  $d_i(M'(A)) = d_i(\tilde{M}(A)) - 1$  for all  $i \geq k$ , where  $k$  is such that every sequence  $d_\bullet(M(B))$  is constant from the  $k$ th position on. Replacing  $P$  by  $C$  cannot cause any difference  $d_i(M'(A)) - d_i(M'(Ab))$  to leave the range  $\{0, 1\}$ : if this difference were to be 2 then  $b$  must be  $\eta$ , and if it were to be  $-1$  then  $b$  must be  $a$ , but neither of these situations occurs in the construction. The replacement also does not change the quantity on the left side of the displays in (L2a) and (L2b) for any two-element minor  $M''$  of  $M'$ , and hence does not undermine the truth of these conditions, unless the ground set of  $M''$  is  $\{a, \eta\}$ , in which case that quantity is incremented. But in this event, by construction, the equality of (L2b) is attained in the corresponding minor of  $\tilde{M}$  for  $d_{\leq k}$ , and so (L2b) is still true of  $M''$ .

For Proposition 6.1, all that remains to check are the equalities of determinants. There are two cases to consider which are not inherited from  $M$  or  $N \oplus \emptyset_C$ . One involves  $M'(A)$  and  $M'(A\eta)$  for  $\eta \notin A$  and  $a \notin A$ , where the rank drop is 1, and  $M'(A) = P \oplus N(A)$  and  $M'(A\eta) = C \oplus N(A)$  have the same determinant by choice of  $C$ . The other involves  $M'(A)$  and  $M'(Aa)$  for  $\eta \in A$  and  $a \notin A$ , where the rank drop is 0. In this case Lemma 7.7 gives  $M(A)_{\text{proj}} = P \oplus M(Aa)_{\text{proj}}$ . Then  $M'(A)_{\text{proj}} = (C \oplus M(Aa))_{\text{proj}} = M(Aa)_{\text{proj}}$  and  $M'(Aa)_{\text{proj}} = M(Aa)_{\text{proj}}$  agree.



Thus  $M'$  is a matroid. In its generic matroid, all elements are loops or coloops except for  $\eta$  and  $a$  which generate a uniform matroid  $U_{1,2}$ , so we have deletion-contraction relations

$$[M' \setminus \eta] + [M' / \eta] = [M'] = [M' \setminus a] + [M' / a].$$

In this relation  $M' \setminus \eta$  is our  $M$ . The matroid  $M' / \eta$  has a one-element direct summand on the ground set  $\{a\}$ , so is encompassed by our second induction; the matroids  $M' \setminus a$  and  $M' / a$  have a greater number of coloops than  $M$ , so are encompassed by our first. This proves case (c).

(d) Repeatedly using deletion-contraction to break up any matroid with at least two bases, on any element which is not a loop or coloop, expresses the class of any matroid as a sum of classes of molecules.  $\square$

**7.1. Arithmetic Tutte polynomial and quasi-polynomial.** In this subsection,  $M$  is a matroid over  $\mathbb{Z}$ . We show that the arithmetic Tutte polynomial and the Tutte quasi-polynomial are images of  $\mathbf{T}_M$  under ring homomorphisms.

Since  $\mathbb{Z}[\mathbb{Z}\text{-Mod}] \otimes \mathbb{Z}[\mathbb{Z}\text{-Mod}] \simeq \mathbb{Z}[X, Y]$ , we have

$$\mathbf{T}_M = \sum_{A \subseteq E} (X^R)^{\text{cork}_M(A)} (Y^R)^{\text{nullity}_M(A)} (XY)^{M(A)_{\text{tors}}},$$

where we use the notation  $\text{nullity}_M(A) = \text{cork}_{M^*}(E \setminus A) = \dim M^*(E \setminus A)$ .

We may define a specialization of  $\mathbf{T}_M$  by evaluating  $X^R$  at  $(x - 1)$ ,  $Y^R$  at  $(y - 1)$ , and  $(XY)^N$  at the cardinality of  $N$  for each torsion module  $N$ . This specialization is the arithmetic Tutte polynomial  $\mathbf{M}_{\hat{M}}(x, y)$  of the quasi-arithmetic matroid  $\hat{M}$  defined by  $M$ :

$$\mathbf{M}_{\hat{M}}(x, y) = \sum_{A \subseteq E} m(A)(x - 1)^{\text{rk}(E) - \text{rk}(A)} (y - 1)^{|A| - \text{rk}(A)},$$

where  $m(A) = |M(A)_{\text{tors}}|$ . This polynomial proved to have several applications to toric arrangements, partition functions, zonotopes, and graphs with labelled edges (see [17], [6]). Notice that an ordinary matroid  $\tilde{M}$  can be trivially made into an arithmetic matroid  $\hat{M}$  by setting all the multiplicities to be equal to 1, and then  $\mathbf{M}_{\hat{M}}(x, y)$  is nothing but the classical Tutte polynomial  $\mathbf{T}_{\tilde{M}}(x, y)$ .

Clearly, the polynomial  $\mathbf{M}_{\hat{M}}(x, y)$  is not the universal deletion-contraction invariant of  $\hat{M}$ . For instance, the ordinary Tutte polynomial  $\mathbf{T}_{\tilde{M}}(x, y)$  of the matroid  $\tilde{M}$  obtained from  $\hat{M}$  by forgetting of its arithmetic data is also a deletion-contraction invariant of  $\hat{M}$ , which is not determined by  $\mathbf{M}_{\hat{M}}(x, y)$ . This fact led the authors of [3] to define a *Tutte quasi-polynomial*  $\mathbf{Q}_M(x, y)$ , interpolating between  $\mathbf{T}_{\tilde{M}}(x, y)$  and  $\mathbf{M}_{\hat{M}}(x, y)$ . This invariant is stronger, but still not universal, and more importantly, it is not an invariant of the arithmetic matroid, as it depends on the groups  $M(A)_{\text{tors}}$  and not just on their cardinalities. We will now show that  $\mathbf{Q}_M(x, y)$  is actually an invariant of the matroid over  $\mathbb{Z}$ , and write explicitly how to compute it from the universal invariant.

For every positive integer  $q$ , let us define a function  $V_q$  as follows:

$$V_q((XY)^{\mathbb{Z}/p^k}) = \begin{cases} 1 & \text{if } p^k \text{ divides } q, \\ p^{k-j} & \text{if } 0 \leq j < k \text{ is maximal such that } p^j \text{ divides } q. \end{cases}$$

We will extend this to define  $V_q((XY)^N)$  multiplicatively for any torsion abelian group  $N$ . Then we define a specialization of  $\mathbf{T}_M$  to the ring of quasipolynomials by specializing  $X^R$  to  $x - 1$ ,  $Y^R$  to  $(y - 1)$ , and  $(XY)^N$  to  $V_{(x-1)(y-1)}((XY)^N)$ . This gives

$$\begin{aligned} \mathbf{Q}_M(x, y) &\doteq \sum_{A \subseteq E} (x - 1)^{\text{cork}_{M(A)}} (y - 1)^{\text{nullity}_{M(A)}} V_{(x-1)(y-1)}((XY)^{M(A)_{\text{tors}}}) \\ &= \sum_{A \subseteq E} \frac{|M(A)_{\text{tors}}|}{|(x - 1)(y - 1)M(A)_{\text{tors}}|} (x - 1)^{\text{rk}(E) - \text{rk}(A)} (y - 1)^{|A| - \text{rk}(A)}. \end{aligned}$$

Since  $(q + |G|)G = qG$  for any finite group  $G$ , the function  $\mathbf{Q}_M(x, y)$  is a quasi-polynomial in  $q = (x - 1)(y - 1)$ . In particular, if  $|M(A)_{\text{tors}}|$  divides  $(x - 1)(y - 1)$ , then the group  $(x - 1)(y - 1)M(A)_{\text{tors}}$  is trivial and  $\mathbf{Q}_M(x, y)$  coincides with  $\mathbf{M}_{\hat{M}}(x, y)$ ; while if  $|M(A)_{\text{tors}}|$  is coprime with  $(x - 1)(y - 1)$ , then  $\mathbf{Q}_M(x, y)$  coincides with  $\mathbf{T}_{\hat{M}}(x, y)$ . Then in some sense  $\mathbf{Q}_M(x, y)$  interpolates between the two polynomials.

Notice that while  $\mathbf{M}_{\hat{M}}$  and  $\mathbf{T}_{\hat{M}}(x, y)$  only depend on the induced quasi-arithmetic matroid  $\hat{M}$ ,  $\mathbf{T}_M$  and  $\mathbf{Q}_M(x, y)$  are indeed invariants of the matroid  $M$  over  $\mathbb{Z}$ . Also the *chromatic quasi-polynomial* and the *flow quasi-polynomial* defined in [3] are actually invariants of the matroid over  $\mathbb{Z}$ : by [3, Theorem 9.1] they are specializations of  $\mathbf{Q}_M(x, y)$ , and hence of the universal invariant  $\mathbf{T}_M$ .

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