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Optimal observability of the multi-dimensional wave and Schrödinger equations in quantum ergodic domains

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Abstract. We consider the wave and Schrödinger equations on a bounded open connected subset Ω of a Riemannian manifold, with Dirichlet, Neumann or Robin boundary conditions whenever its boundary is nonempty. We observe the restriction of the solutions to a measurable subset ω of Ω during a time interval $[0, T]$ with $T > 0$. It is well known that, if the pair (ω, T) satisfies the Geometric Control Condition (ω being an open set), then an observability inequality holds guaranteeing that the total energy of solutions can be estimated in terms of the energy localized in $\omega \times (0, T)$.

We address the problem of the optimal location of the observation subset ω among all possible subsets of a given measure or volume fraction. A priori this problem can be modeled in terms of maximizing the observability constant, but from the practical point of view it appears more relevant to model it in terms of maximizing an average either over random initial data or over large time. This leads us to define a new notion of observability constant, either randomized, or asymptotic in time. In both cases we come up with a spectral functional that can be viewed as a measure of eigenfunction concentration. Roughly speaking, the subset ω has to be chosen so as to maximize the minimal trace of the squares of all eigenfunctions. Considering the convexified formulation of the problem, we prove a no-gap result between the initial problem and its convexified version, under appropriate quantum ergodicity assumptions, and compute the optimal value. Our results reveal intimate relations between shape and domain optimization, and the theory of quantum chaos (more precisely, quantum ergodicity properties of the domain Ω).

We prove that in 1D a classical optimal set exists only for exceptional values of the volume fraction, and in general one expects relaxation to occur and therefore classical optimal sets not to exist. We then provide spectral approximations and present some numerical simulations that fully confirm the theoretical results in the paper and support our conjectures.

Finally, we provide several remedies to nonexistence of an optimal domain. We prove that when the spectral criterion is modified to consider a weighted one in which the high frequency components are penalized, the problem has a unique classical solution determined by a finite number of low frequency modes. In particular the maximizing sequence built from spectral approximations is stationary.

Keywords. Wave equation, Schrödinger equation, observability inequality, optimal design, spectral decomposition, ergodic properties, quantum ergodicity

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1. Introduction

1.1. Problem formulation and overview of the main results

In this article we model and solve the problem of optimal observability for wave and Schrödinger equations posed on any open bounded connected subset of a Riemannian manifold, with various possible boundary conditions.

We briefly highlight the main ideas and contributions of the paper on a particular case, often arising in applications.

Assume that Ω is a given bounded open subset of \mathbb{R}^n , representing for instance a cavity in which some signals are propagating according to the wave equation

$$\partial_{tt}y = \Delta y, \quad (1)$$

with Dirichlet boundary conditions. Assume that one is allowed to place some sensors in the cavity, in order to make some measurements of the signals propagating in Ω over a certain horizon of time. We assume that we have the choice not only of the placement

of the sensors but also of their shape. The question under consideration is then the determination of the best possible shape and location of sensors, achieving the best possible observation, in some sense to be made precise.

This problem of optimal observability, inspired by the theory of inverse problems and by control-theoretical considerations, is also intimately related to those of optimal controllability and stabilization (see Section 6 for a discussion of these issues).

So far, the problem has been formulated informally and a first challenge is to state the question properly, so that the resulting problem will be both mathematically solvable and relevant in view of practical applications.

A first obvious but important remark is that, in the absence of constraints, the best policy consists in observing the solutions over the whole domain Ω . This is however clearly not reasonable, and in practice the domain covered by sensors has to be limited, due for instance to cost considerations. From the mathematical point of view, we model this basic limitation by considering, as the set of unknowns, the set of all possible measurable subsets ω of Ω that are of Lebesgue measure $|\omega| = L|\Omega|$, where $L \in (0, 1)$ is some fixed real number. Any choice of such a subset represents the sensors put in Ω , and we assume that we are able to measure the restrictions of the solutions of (1) to ω .

Modeling. Let us now model the notion of best observation. At this step it is useful to recall some well known facts on the observability of the wave equation.

For all $(y^0, y^1) \in L^2(\Omega, \mathbb{C}) \times H^{-1}(\Omega, \mathbb{C})$, there exists a unique solution $y \in C^0(0, T; L^2(\Omega, \mathbb{C})) \cap C^1(0, T; H^{-1}(\Omega, \mathbb{C}))$ of (1) such that $y(0, \cdot) = y^0(\cdot)$ and $\partial_t y(0, \cdot) = y^1(\cdot)$. Let $T > 0$.

We say that (1) is *observable* on ω in time T if there exists $C > 0$ such that¹

$$C \|(y^0, y^1)\|_{L^2 \times H^{-1}}^2 \leq \int_0^T \int_{\omega} |y(t, x)|^2 dx dt \quad (2)$$

for all $(y^0, y^1) \in L^2(\Omega, \mathbb{C}) \times H^{-1}(\Omega, \mathbb{C})$. This is the so-called observability inequality, which is of great importance in view of showing the well-posedness of some inverse problems. It is well known that within the class of C^∞ domains Ω , this observability property holds if the pair (ω, T) satisfies the *Geometric Control Condition* in Ω (see [3]), according to which every ray of geometrical optics that propagates in the cavity Ω and is reflected on its boundary $\partial\Omega$ intersects ω within time T . The *observability constant* is defined by

¹ In this inequality, and throughout the paper, we use the usual Sobolev norms. For every $u \in L^2(\Omega, \mathbb{C})$, we have $\|u\|_{L^2(\Omega, \mathbb{C})} = (\int_{\Omega} |u(x)|^2 dx)^{1/2}$. The Hilbert space $H^1(\Omega, \mathbb{C})$ is the space of functions of $L^2(\Omega, \mathbb{C})$ having a distributional derivative in $L^2(\Omega, \mathbb{C})$, endowed with the norm $\|u\|_{H^1(\Omega, \mathbb{C})} = (\|u\|_{L^2(\Omega, \mathbb{C})}^2 + \|\nabla u\|_{L^2(\Omega, \mathbb{C})}^2)^{1/2}$. The Hilbert space $H_0^1(\Omega, \mathbb{C})$ is defined as the closure in $H^1(\Omega, \mathbb{C})$ of the set of functions of class C^∞ on Ω and of compact support in the open set Ω . It is endowed with the norm $\|u\|_{H_0^1(\Omega, \mathbb{C})} = \|\nabla u\|_{L^2(\Omega, \mathbb{C})}$. The Hilbert space $H^{-1}(\Omega, \mathbb{C})$ is the dual of $H_0^1(\Omega, \mathbb{C})$ with respect to the pivot space $L^2(\Omega, \mathbb{C})$, endowed with the corresponding dual norm.

$$C_T^{(W)}(\chi_\omega) = \inf \left\{ \frac{\int_0^T \int_\Omega \chi_\omega(x) |y(t, x)|^2 dx dt}{\|(y^0, y^1)\|_{L^2 \times H^{-1}}^2} \mid (y^0, y^1) \in L^2(\Omega, \mathbb{C}) \times H^{-1}(\Omega, \mathbb{C}) \setminus \{(0, 0)\} \right\}. \quad (3)$$

It is the largest possible constant for which (2) holds. It depends both on the time T (the horizon time of observation) and on the subset ω on which the measurements are taken. Here, χ_ω stands for the characteristic function of ω .

A priori, it might appear natural to model the problem of best observability as that of maximizing the functional $\chi_\omega \mapsto C_T^{(W)}(\chi_\omega)$ over the set

$$\mathcal{U}_L = \{\chi_\omega \mid \omega \text{ is a measurable subset of } \Omega \text{ of Lebesgue measure } |\omega| = L|\Omega|\}. \quad (4)$$

This choice of model is hard to handle from the theoretical point of view, and more importantly, is not so relevant in practical issues. Let us explain these two facts.

First of all, a spectral expansion of the solutions shows the emergence of crossed terms in the functional to be minimized, which are difficult to treat. To see this, in what follows we fix a Hilbert basis $(\phi_j)_{j \in \mathbb{N}^*}$ of $L^2(\Omega, \mathbb{C})$ consisting of (real-valued) eigenfunctions of the Dirichlet–Laplacian operator on Ω , associated with the negative eigenvalues $(-\lambda_j^2)_{j \in \mathbb{N}^*}$. Then any solution y of (1) can be expanded as

$$y(t, x) = \sum_{j=1}^\infty (a_j e^{i\lambda_j t} + b_j e^{-i\lambda_j t}) \phi_j(x), \quad (5)$$

where the coefficients a_j and b_j account for initial data. It follows that

$$C_T^{(W)}(\chi_\omega) = \frac{1}{2} \inf_{\substack{(a_j), (b_j) \in \ell^2(\mathbb{C}) \\ \sum_{j=1}^\infty (|a_j|^2 + |b_j|^2) = 1}} \int_0^T \int_\omega \left| \sum_{j=1}^\infty (a_j e^{i\lambda_j t} + b_j e^{-i\lambda_j t}) \phi_j(x) \right|^2 dx dt,$$

and maximizing this functional over \mathcal{U}_L appears to be very difficult from the theoretical point of view, due to the crossed terms $\int_\omega \phi_j \phi_k dx$ measuring the interaction over ω between distinct eigenfunctions.

The second difficulty with this model is its limited relevance in practice. Indeed, the observability constant defined by (3) is deterministic and corresponds to the worst possible case. Hence, in this sense, it is a pessimistic constant. In practical applications one realizes a large number of measurements, and it may be expected that this worst case will not occur so often. Thus, one would like the observation to be optimal for most of experiments but maybe not for all of them. This leads us to consider instead an averaged version of the observability inequality over random initial data. More details will be given in Section 2.3 on the randomization procedure; in a few words, we define what we call the *randomized observability constant* by

$$C_{T, \text{rand}}^{(W)}(\chi_\omega) = \frac{1}{2} \inf_{\substack{(a_j), (b_j) \in \ell^2(\mathbb{C}) \\ \sum_{j=1}^\infty (|a_j|^2 + |b_j|^2) = 1}} \mathbb{E} \left(\int_0^T \int_\omega \left| \sum_{j=1}^\infty (\beta_{1,j}^v a_j e^{i\lambda_j t} + \beta_{2,j}^v b_j e^{-i\lambda_j t}) \phi_j(x) \right|^2 dx dt \right), \quad (6)$$

where $(\beta_{1,j}^v)_{j \in \mathbb{N}^*}$ and $(\beta_{2,j}^v)_{j \in \mathbb{N}^*}$ are two sequences of (for example) i.i.d. Bernoulli random laws on a probability space $(\mathcal{X}, \mathcal{A}, \mathbb{P})$, and \mathbb{E} is the expectation over \mathcal{X} with respect to the probability measure \mathbb{P} . It corresponds to an averaged version of the observability inequality over random initial data. Actually, we have the following result.

Theorem 1.1 (Characterization of the randomized observability constant). *For every measurable subset ω of Ω ,*

$$C_{T,\text{rand}}^{(W)}(\chi_\omega) = \frac{T}{2} \inf_{j \in \mathbb{N}^*} \int_\omega \phi_j(x)^2 dx. \tag{7}$$

It is interesting to note that always $C_T^{(W)}(\chi_\omega) \leq C_{T,\text{rand}}^{(W)}(\chi_\omega)$, and the strict inequality holds for instance in each of the following cases (see Remark 2.5 for details):

- in 1D, with $\Omega = (0, \pi)$ and Dirichlet boundary conditions, whenever T is not an integer multiple of π ;
- in multi-D, with Ω stadium-shaped, whenever ω contains an open neighborhood of the wings (in that case, we actually have $C_T^{(W)}(\chi_\omega) = 0$).

Taking all this into account we model the problem of best observability in the following more relevant and simpler way: maximize the functional

$$J(\chi_\omega) = \inf_{j \in \mathbb{N}^*} \int_\omega \phi_j(x)^2 dx \tag{8}$$

over the set \mathcal{U}_L .

The functional J can be interpreted as a criterion reflecting the concentration properties of eigenfunctions. This functional can as well be recovered by considering, instead of an averaged version of the observability inequality over random initial data, a time-asymptotic version of it. More precisely, we claim that, if the eigenvalues of the Dirichlet–Laplacian are simple (which is a generic property), then $J(\chi_\omega)$ is the largest possible constant C such that

$$C \|(y^0, y^1)\|_{L^2 \times H^{-1}}^2 \leq \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \int_\omega |y(t, x)|^2 dx dt$$

for all $(y^0, y^1) \in L^2(\Omega, \mathbb{C}) \times H^{-1}(\Omega, \mathbb{C})$ (see Section 2.5).

The derivation of this model and of the corresponding optimization problem, and the new notions of averaged observability inequalities it leads to (Section 2), constitute the first contribution of the present article.

It can be noticed that, in this model, the time T does not play any role.

It is by now well known that, in the characterization of fine observability properties of solutions of wave equations, two ingredients enter (see [40]): on the one hand, the spectral decomposition and the observability properties of eigenfunctions; on the other, the microlocal components that are driven by rays of geometric optics. The randomized observability constant takes the first spectral component into account but neglects the microlocal aspects that were annihilated, to some extent, by the randomization process. In that sense, the problem of maximizing the functional J defined by (8) is essentially a high-frequency problem.

Solving. With a view to solving the uniform optimal design problem

$$\sup_{\chi_\omega \in \mathcal{U}_L} J(\chi_\omega), \quad (9)$$

we first introduce a convexified version of the problem, by considering the convex closure of the set \mathcal{U}_L for the L^∞ weak star topology, that is, $\overline{\mathcal{U}}_L = \{a \in L^\infty(\Omega, [0, 1]) \mid \int_\Omega a(x) dx = L|\Omega|\}$. The convexified problem then consists in maximizing the functional

$$J(a) = \inf_{j \in \mathbb{N}^*} \int_\Omega a(x) \phi_j(x)^2 dx$$

over $\overline{\mathcal{U}}_L$. Clearly, a maximizer does exist. But since the functional J is not lower semi-continuous, it is not clear whether or not there may be a gap between the problem (9) and its convexified version. The analysis of this question happens to be very interesting and reveals deep connections with the theory of quantum chaos and, more precisely, with quantum ergodicity properties of Ω . We prove for instance the following result (see Section 3.2 for other related statements).

Theorem 1.2 (No-gap result and optimal value of J). *Assume that the sequence of probability measures $\mu_j = \phi_j(x)^2 dx$ converges vaguely to the uniform measure $|\Omega|^{-1} dx$ (Quantum Unique Ergodicity on the base), and that there exists $p \in (1, \infty]$ such that the sequence $(\phi_j)_{j \in \mathbb{N}^*}$ of eigenfunctions is uniformly bounded in $L^{2p}(\Omega)$. Then*

$$\sup_{\chi_\omega \in \mathcal{U}_L} J(\chi_\omega) = \max_{a \in \overline{\mathcal{U}}_L} J(a) = L$$

for every $L \in (0, 1)$. In other words, there is no gap between the problem (9) and its convexified version.

At this step, it follows from Theorems 1.1 and 1.2 that, under some spectral assumptions, the maximal possible value of $C_{T, \text{rand}}^{(W)}(\chi_\omega)$ (over the set \mathcal{U}_L) is equal to $TL/2$. Several remarks are in order.

- Except in the one-dimensional case, we are not aware of domains Ω in which the spectral assumptions of the above result are satisfied. As discussed in Section 3.3, this question is related to deep open questions in mathematical physics and semiclassical analysis such as the *QUE conjecture*.
- The spectral assumptions above are sufficient but not necessary to derive such a no-gap statement: indeed, we can prove that the result still holds true if Ω is a hypercube (with the usual eigenfunctions that are products of sine functions), or if Ω is a two-dimensional disk (with the usual eigenfunctions parametrized by Bessel functions), although, in the latter case, the eigenfunctions do not equidistribute as the eigenfrequencies increase, as illustrated by the well known whispering galleries effect (see Proposition 3.9 in Section 3.2).
- We are not aware of any example in which there is a gap between the problem (9) and its convexified version.

• It is also interesting to note that, since the spectral criterion J defined by (8) depends on the specific choice of the orthonormal basis $(\phi_j)_{j \in \mathbb{N}^*}$ of eigenfunctions of the Dirichlet–Laplacian, one can consider an intrinsic version of the problem, consisting in maximizing the spectral functional

$$J_{\text{int}}(\chi_\omega) = \inf_{\phi \in \mathcal{E}} \int_{\omega} \phi(x)^2 dx$$

over \mathcal{U}_L , where \mathcal{E} denotes the set of all normalized eigenfunctions of the Dirichlet–Laplacian. For this problem we have a result similar to the one above (Theorem 3.16 in Section 3.6).

These results show intimate connections between domain optimization and quantum ergodicity properties of Ω . Such a relation was suggested in the early work [13] concerning the exponential decay properties of dissipative wave equations.

• The result stated in the theorem above holds true as well when replacing \mathcal{U}_L with the class of Jordan measurable subsets of Ω of measure $L|\Omega|$. The proof (given in Section 3.4), based on a kind of homogenization procedure, is constructive and consists in building a maximizing sequence of subsets for the problem of maximizing J , showing that it is possible to increase the values of J by considering subsets of measure $L|\Omega|$ having an increasing number of connected components.

Nonexistence of an optimal set and remedies. The maximum of J over $\overline{\mathcal{U}}_L$ is clearly reached (in general, even in infinitely many ways, as can be seen using Fourier series, see [50]). The question of the reachability of the supremum of J over \mathcal{U}_L , that is, the existence of an optimal classical set, is a difficult question in general. In particular cases it can however be addressed using harmonic analysis. For instance in dimension one, we can prove that the supremum is reached if and only if $L = 1/2$ (and there are infinitely many optimal sets). In higher dimension, the question is completely open, and we conjecture that, for generic domains Ω and generic values of L , the supremum is not reached and hence there does not exist any optimal set. It can however be noted that, in the two-dimensional Euclidean square, if we restrict the search for optimal sets to Cartesian products of 1D subsets, then the supremum is reached if and only if $L \in \{1/4, 1/2, 3/4\}$ (see Section 4.1 for details).

In view of that, it is natural to study a finite-dimensional spectral approximation of the problem, namely the problem of maximizing the functional

$$J_N(\chi_\omega) = \min_{1 \leq j \leq N} \int_{\omega} \phi_j(x)^2 dx$$

over \mathcal{U}_L , for $N \in \mathbb{N}^*$. The existence and uniqueness of an optimal set ω^N is then not difficult to prove, as also is the Γ -convergence of J_N to J for the weak star topology of L^∞ . Moreover, the sets ω^N have a finite number of connected components, expected to increase as N increases. Several numerical simulations (provided in Section 4.2) will show the shapes of these sets; their increasing complexity (as N increases) is in accordance with the conjecture of the nonexistence of an optimal set maximizing J . It can be noted that, in the one-dimensional case, for L sufficiently small, loosely speaking, the optimal

domain ω^N for N modes is the worst possible when considering the truncated problem with $N + 1$ modes (spillover phenomenon; see [23, 50]).

This intrinsic instability is in some sense due to the fact that in the definition of the spectral criterion (8) all modes have the same weight, and the same criticism can be made on the observability inequality (2). Due to the increasing complexity of the geometry of high-frequency eigenfunctions, the optimal shape and placement problems are expected to be highly complex.

One expects the problem to be better behaved if lower frequencies are more weighted than the higher ones. It is therefore relevant to introduce a weighted version of the observability inequality (2), by considering the (equivalent) inequality

$$C_{T,\sigma}^{(W)}(\chi_\omega)(\|y^0, y^1\|_{L^2 \times H^{-1}}^2 + \sigma \|y^0\|_{H^{-1}}^2) \leq \int_0^T \int_\omega |y(t, x)|^2 dx dt,$$

where $\sigma \geq 0$ is some weight. We then have $C_{T,\sigma}^{(W)}(\chi_\omega) \leq C_T^{(W)}(\chi_\omega)$.

Considering, as before, an averaged version of this weighted observability inequality over random initial data, we get $2C_{T,\sigma,\text{rand}}^{(W)}(\chi_\omega) = T J_\sigma(\chi_\omega)$, where the weighted spectral criterion J_σ is defined by

$$J_\sigma(\chi_\omega) = \inf_{j \in \mathbb{N}^*} \sigma_j \int_\omega \phi_j(x)^2 dx$$

with $\sigma_j = \lambda_j^2 / (\sigma + \lambda_j^2)$ (an increasing sequence of positive real numbers converging to 1; see Section 4.4 for details). The truncated criterion $J_{\sigma,N}$ is then defined accordingly, by keeping only the N first modes. We then have the following result.

Theorem 1.3 (Weighted spectral criterion). *Assume that the sequence of probability measures $\mu_j = \phi_j(x)^2 dx$ converges vaguely to the uniform measure $|\Omega|^{-1} dx$, and that the sequence of eigenfunctions ϕ_j is uniformly bounded in $L^\infty(\Omega)$. Then, for every $L \in (\sigma_1, 1)$, there exists $N_0 \in \mathbb{N}^*$ such that*

$$\max_{\chi_\omega \in \mathcal{U}_L} J_\sigma(\chi_\omega) = \max_{\chi_\omega \in \mathcal{U}_L} J_{\sigma,N}(\chi_\omega) \leq \sigma_1 < L$$

for every $N \geq N_0$. In particular, the problem of maximizing J_σ over \mathcal{U}_L has a unique solution $\chi_{\omega^{N_0}}$, and moreover the set ω^{N_0} has a finite number of connected components.

As previously, note that the assumptions of the above theorem (referred to as L^∞ -QUE, as discussed further) are strong. We are however able to prove that the conclusion of Theorem 1.3 holds true in a hypercube with Dirichlet boundary conditions with the usual eigenfunctions there are of products of sine functions, although QUE is not satisfied in such a domain (see Proposition 4.20 in Section 4.4).

The theorem says that, for the problem of maximizing $J_{\sigma,N}$ over \mathcal{U}_L , the sequence of optimal sets ω^N is stationary whenever L is large enough, and ω^{N_0} is the (unique) optimal set, solving the problem of maximizing J_σ . It can be noted that the lower threshold in L depends on the chosen weights, and the numerical simulations that we will provide

indicate that this threshold is sharp in the sense that if $L < \sigma_1$ then the sequence of maximizing sets loses its stationarity feature.

In conclusion, this weighted version of our spectral criterion can be viewed as a remedy for the spillover phenomenon. Note that, of course, other more evident remedies can be discussed, such as the search for an optimal domain in a set of subdomains sharing nice compactness properties (such as having a uniform perimeter or BV norm; see Section 4.3). However, our aim is to investigate the optimization problems in the broadest classes of measurable domains and discuss the mathematical, physical and practical relevance of the criterion encoding the notion of optimal observability.

Let us finally note that all our results hold for wave and Schrödinger equations on any open bounded connected subset of a Riemannian manifold (replacing Δ with the Laplace–Beltrami operator), with various possible boundary conditions (Dirichlet, Neumann, mixed, Robin) or no boundary conditions in case the manifold is compact without boundary. The abstract framework and possible generalizations are described in Section 5.

1.2. Brief state of the art

The literature on optimal observation or sensor location problems is abundant in engineering applications (see, e.g., [35, 45, 59, 62, 65] and references therein), but the number of mathematical theoretical contributions is limited.

In engineering applications, the aim is to optimize the number, place and type of sensors in order to improve the estimation of the state of the system, and this concerns, for example, active structural acoustics, piezoelectric actuators, vibration control in mechanical structures, damage detection and chemical reactions, to name but a few. In most of these applications, however, the method consists in approximating appropriately the problem by selecting a finite number of possible optimal candidates and recasting it as a finite-dimensional combinatorial optimization problem. Among the possible approaches, the closest one to ours consists in considering truncations of Fourier expansion representations. Adopting such a Fourier point of view, the authors of [22, 23] studied optimal stabilization issues for the one-dimensional wave equation and, up to our knowledge, these are the first articles in which one can find rigorous mathematical arguments and proofs to characterize the optimal set whenever it exists, for the problem of determining the best possible shape and position of the damping subdomain of a given measure. In [4] the authors investigate the problem modeled in [59] of finding the best possible distributions of two materials (with different elastic Young modulus and different density) in a rod in order to minimize the vibration energy in the structure. For this optimal design problem in wave propagation, the authors of [4] prove existence results and provide convexification and optimality conditions. The authors of [1] also propose a convexification formulation of eigenfrequency optimization problems applied to optimal design. In [17] the authors discuss several possible criteria for optimizing the damping of abstract wave equations in Hilbert spaces, and derive optimality conditions for a certain criterion related to a Lyapunov equation. In [50] we investigated the problem (9) in the one-dimensional case. We also quote the article [51] where we study the related problem of finding the optimal location of the support of the control for the one-dimensional wave equation. We

mention our work [53] on parabolic equations (see also references therein), for which results are of a very different nature.

In this paper, we provide a complete model and mathematical analysis of the optimal observability problem, overviewed in Section 1.1. The article is structured as follows.

Section 2 is devoted to discussing and defining a relevant mathematical criterion, modeling the optimal observability problem. We first introduce the context and recall the classical observability inequality, and then using spectral considerations we introduce randomized or time asymptotic observability inequalities, to come up with a spectral criterion which is at the heart of our study.

The resulting optimal design problem is solved in Section 3, where we derive, under appropriate spectral assumptions, a no-gap result between our problem and its convexified version. To do so, we exhibit some deep relations between shape optimization and concentration properties of eigenfunctions.

The existence of an optimal set is investigated in Section 4. We study a spectral approximation of our problem, providing a maximizing sequence of optimal sets which does not converge in general. We then provide some remedies, in particular by defining a weighted spectral criterion and showing the existence and uniqueness of an optimal set.

Section 5 is devoted to generalizing all results to wave and Schrödinger equations on any open bounded connected subset of a Riemannian manifold, with various boundary conditions.

Further comments are provided in Section 6, concerning the problem of optimal shape and location of internal controllers, as well as several open problems and issues.

2. Modeling the optimal observability problem

This section is devoted to discussing and mathematically modeling the problem of maximizing the observability of wave equations. A first natural model is to settle the problem of maximizing the observability constant, but it appears that this problem is both difficult to treat from the theoretical point of view, and actually not so relevant for practice. Using spectral considerations, we will then define a spectral criterion based on averaged versions of the observability inequalities, which is better suited to model what is expected in practice.

2.1. The framework

Let $n \geq 1$, T be a positive real number and Ω be an open bounded connected subset of \mathbb{R}^n . We consider the wave equation

$$\partial_{tt}y = \Delta y \tag{10}$$

in $(0, T) \times \Omega$, with Dirichlet boundary conditions. Let ω be an arbitrary measurable subset of Ω of positive measure. Throughout the paper, the notation χ_ω stands for the characteristic function of ω . The equation (10) is said to be *observable* on ω in time T if there exists $C_T^{(W)}(\chi_\omega) > 0$ such that

$$C_T^{(W)}(\chi_\omega) \|(y^0, y^1)\|_{L^2 \times H^{-1}}^2 \leq \int_0^T \int_\omega |y(t, x)|^2 dx dt \tag{11}$$

for all $(y^0, y^1) \in L^2(\Omega, \mathbb{C}) \times H^{-1}(\Omega, \mathbb{C})$. This is the so-called *observability inequality*, relevant in inverse problems or in control theory because of its dual equivalence to controllability (see [42]). It is well known that within the class of C^∞ domains Ω , this observability property holds, roughly, if the pair (ω, T) satisfies the *Geometric Control Condition (GCC)* in Ω (see [3, 9]), according to which every geodesic ray in Ω , reflected on its boundary according to the laws of geometric optics, intersects the observation set ω within time T . In particular, if at least one ray does not reach $\bar{\omega}$ until time T then the observability inequality fails because of the existence of Gaussian beam solutions concentrated along the ray, and therefore away from the observation set (see [55]).

In what follows, the *observability constant* $C_T^{(W)}(\chi_\omega)$ is the largest possible nonnegative constant for which the inequality (11) holds, that is,

$$C_T^{(W)}(\chi_\omega) = \inf \left\{ \frac{\int_0^T \int_\omega |y(t, x)|^2 dx dt}{\|(y^0, y^1)\|_{L^2 \times H^{-1}}^2} \mid (y^0, y^1) \in L^2(\Omega, \mathbb{C}) \times H^{-1}(\Omega, \mathbb{C}) \setminus \{(0, 0)\} \right\}. \tag{12}$$

We next discuss the question of mathematically modeling the notion of maximizing the observability of wave equations. It is a priori natural to consider the problem of maximizing the observability constant $C_T^{(W)}(\chi_\omega)$ over all possible subsets ω of Ω of Lebesgue measure $|\omega| = L|\Omega|$ for a given time $T > 0$. In the next two subsections, using spectral expansions, we discuss the difficulty and relevance of this problem, leading us to consider a more adapted spectral criterion.

2.2. Spectral expansion of solutions

From now on, we fix an orthonormal Hilbert basis $(\phi_j)_{j \in \mathbb{N}^*}$ of $L^2(\Omega, \mathbb{C})$ consisting of eigenfunctions of the Dirichlet–Laplacian on Ω , associated with the positive eigenvalues $(\lambda_j^2)_{j \in \mathbb{N}^*}$. As said in the introduction, in what follows the Sobolev norms are computed in a spectral way with respect to these eigenelements. Let $(y^0, y^1) \in L^2(\Omega, \mathbb{C}) \times H^{-1}(\Omega, \mathbb{C})$ be some arbitrary initial data. The solution $y \in C^0(0, T; L^2(\Omega, \mathbb{C})) \cap C^1(0, T; H^{-1}(\Omega, \mathbb{C}))$ of (10) such that $y(0, \cdot) = y^0(\cdot)$ and $\partial_t y(0, \cdot) = y^1(\cdot)$ can be expanded as

$$y(t, x) = \sum_{j=1}^{\infty} (a_j e^{i\lambda_j t} + b_j e^{-i\lambda_j t}) \phi_j(x), \tag{13}$$

where the sequences $(a_j)_{j \in \mathbb{N}^*}$ and $(b_j)_{j \in \mathbb{N}^*}$ belong to $\ell^2(\mathbb{C})$ and are determined in terms of the initial data (y^0, y^1) by

$$\begin{aligned} a_j &= \frac{1}{2} \left(\int_\Omega y^0(x) \phi_j(x) dx - \frac{i}{\lambda_j} \int_\Omega y^1(x) \phi_j(x) dx \right), \\ b_j &= \frac{1}{2} \left(\int_\Omega y^0(x) \phi_j(x) dx + \frac{i}{\lambda_j} \int_\Omega y^1(x) \phi_j(x) dx \right), \end{aligned} \tag{14}$$

for every $j \in \mathbb{N}^*$. Moreover, we have²

$$\|(y^0, y^1)\|_{L^2 \times H^{-1}}^2 = 2 \sum_{j=1}^{\infty} (|a_j|^2 + |b_j|^2). \tag{15}$$

It follows from (13) that

$$\int_0^T \int_{\omega} |y(t, x)|^2 dx dt = \sum_{j,k=1}^{\infty} \alpha_{jk} \int_{\omega} \phi_j(x) \phi_k(x) dx, \tag{16}$$

where

$$\alpha_{jk} = \int_0^T (a_j e^{i\lambda_j t} - b_j e^{-i\lambda_j t})(\bar{a}_k e^{-i\lambda_k t} - \bar{b}_k e^{i\lambda_k t}) dt. \tag{17}$$

The coefficients $\alpha_{jk}, (j, k) \in (\mathbb{N}^*)^2$, depend only on the initial data (y^0, y^1) , and their precise expression is given by

$$\begin{aligned} \alpha_{jk} &= \frac{2a_j \bar{a}_k}{\lambda_j - \lambda_k} \sin\left((\lambda_j - \lambda_k) \frac{T}{2}\right) e^{i(\lambda_j - \lambda_k)T/2} - \frac{2a_j \bar{b}_k}{\lambda_j + \lambda_k} \sin\left((\lambda_j + \lambda_k) \frac{T}{2}\right) e^{i(\lambda_j + \lambda_k)T/2} \\ &\quad - \frac{2b_j \bar{a}_k}{\lambda_j + \lambda_k} \sin\left((\lambda_j + \lambda_k) \frac{T}{2}\right) e^{-i(\lambda_j + \lambda_k)T/2} + \frac{2b_j \bar{b}_k}{\lambda_j - \lambda_k} \sin\left((\lambda_j - \lambda_k) \frac{T}{2}\right) e^{-i(\lambda_j - \lambda_k)T/2} \end{aligned} \tag{18}$$

whenever $\lambda_j \neq \lambda_k$, and

$$\alpha_{jj} = T(a_j \bar{a}_k + b_j \bar{b}_k) - \frac{\sin(\lambda_j T)}{\lambda_j} (a_j \bar{b}_k e^{i\lambda_j T} + b_j \bar{a}_k e^{-i\lambda_j T}) \tag{19}$$

when $\lambda_j = \lambda_k$.

Remark 2.1. In dimension one, set $\Omega = (0, \pi)$. Then $\phi_j(x) = \sqrt{2/\pi} \sin(jx)$ and $\lambda_j = j$ for every $j \in \mathbb{N}^*$. In this one-dimensional case, it can be noticed that all nondiagonal terms vanish when the time T is a multiple of 2π . Indeed, if $T = 2p\pi$ with $p \in \mathbb{N}^*$, then $\alpha_{ij} = 0$ whenever $i \neq j$, and

$$\alpha_{jj} = p\pi(|a_j|^2 + |b_j|^2) \tag{20}$$

for all $(i, j) \in (\mathbb{N}^*)^2$, and therefore

$$\int_0^{2p\pi} \int_{\omega} |y(t, x)|^2 dx dt = \sum_{j=1}^{\infty} \alpha_{jj} \int_{\omega} \sin^2(jx) dx. \tag{21}$$

Hence in that case there are no crossed terms. The optimal observability problem for this one-dimensional case was studied in detail in [50].

² Indeed, for every $u = \sum_{j=1}^{\infty} u_j \phi_j \in L^2(\Omega, \mathbb{C})$, we have $\|u\|_{L^2}^2 = \sum_{j=1}^{\infty} |u_j|^2$ and $\|u\|_{H^{-1}}^2 = \sum_{j=1}^{\infty} |u_j|^2 / \lambda_j^2$.

Using the above spectral expansions, the observability constant is given by

$$C_T^{(W)}(\chi_\omega) = \frac{1}{2} \inf_{\substack{(a_j), (b_j) \in \ell^2(\mathbb{C}) \\ \sum_{j=1}^\infty (|a_j|^2 + |b_j|^2) = 1}} \int_0^T \int_\omega \left| \sum_{j=1}^\infty (a_j e^{i\lambda_j t} - b_j e^{-i\lambda_j t}) \phi_j(x) \right|^2 dx dt, \quad (22)$$

a_j and b_j being the Fourier coefficients of the initial data, defined by (14).

Due to the crossed terms appearing in (16), the problem of maximizing $C_T^{(W)}(\chi_\omega)$ over all possible subsets ω of Ω of measure $|\omega| = L|\Omega|$ is very difficult to handle, at least from the theoretical point of view. The difficulty related to the cross terms already appears in one-dimensional problems (see [50]). Actually, this question is very much related to classical problems in nonharmonic Fourier analysis, such as the one of determining the best constants in Ingham’s inequalities (see [29, 30]).

This problem is therefore left open, but as we will see next, although it is very interesting, it is not so relevant from the practical point of view.

2.3. Randomized observability inequality

As mentioned above, the problem of maximizing the *deterministic* (classical) observability constant $C_T^{(W)}(\chi_\omega)$ defined by (12) over all possible measurable subsets ω of Ω of measure $|\omega| = L|\Omega|$ is open and is probably very difficult. However, when considering the practical problem of locating sensors in an optimal way, the optimality should rather be considered in terms of an *average* with respect to a large number of experiments. From this point of view, the observability constant $C_T^{(W)}(\chi_\omega)$, which is by definition deterministic, is expected to be pessimistic in the sense that it corresponds to the worst possible case. In practice, when carrying out a large number of experiments, it can be expected that the worst possible case does not occur very often. Having this in mind, we next define a new notion of observability inequality by considering an average over random initial data.

The observability constant defined by (12) is defined as an infimum over *all possible* (deterministic) initial data. We are going to slightly modify this definition by randomizing the initial data in some precise sense, and considering an averaged version of the observability inequality with a new (randomized) observability constant.

Consider the expression of $C_T^{(W)}(\chi_\omega)$ given by (22) in terms of spectral expansions. Following the works of N. Burq and N. Tzvetkov on nonlinear partial differential equations with random initial data (see [7, 10, 11]), which use early ideas of Paley and Zygmund (see [47]), we randomize the coefficients a_j, b_j, c_j , with respect to the initial conditions, by multiplying each of them by some well chosen random law. This random selection of all possible initial data for the wave equation (10) consists in replacing $C_T^{(W)}(\chi_\omega)$ by the randomized version

$$C_{T,\text{rand}}^{(W)}(\chi_\omega) = \frac{1}{2} \inf_{\substack{(a_j), (b_j) \in \ell^2(\mathbb{C}) \\ \sum_{j=1}^\infty (|a_j|^2 + |b_j|^2) = 1}} \mathbb{E} \left(\int_0^T \int_\omega \left| \sum_{j=1}^\infty (\beta_{1,j}^v a_j e^{i\lambda_j t} - \beta_{2,j}^v b_j e^{-i\lambda_j t}) \phi_j(x) \right|^2 dx dt \right), \quad (23)$$

where $(\beta_{1,j}^\nu)_{j \in \mathbb{N}^*}$ and $(\beta_{2,j}^\nu)_{j \in \mathbb{N}^*}$ are sequences of independent Bernoulli random variables on a probability space $(\mathcal{X}, \mathcal{A}, \mathbb{P})$, satisfying

$$\mathbb{P}(\beta_{1,j}^\nu = \pm 1) = \mathbb{P}(\beta_{2,j}^\nu = \pm 1) = 1/2 \quad \text{and} \quad \mathbb{E}(\beta_{1,j}^\nu \beta_{2,k}^\nu) = 0,$$

for all j and k in \mathbb{N}^* and every $\nu \in \mathcal{X}$. Here, \mathbb{E} stands for the expectation over the space \mathcal{X} with respect to the probability measure \mathbb{P} . In other words, instead of considering the deterministic observability inequality (11) for the wave equation (10), we consider the *randomized observability inequality*

$$C_{T,\text{rand}}^{(W)}(\chi_\omega) \|(y^0, y^1)\|_{L^2 \times H^{-1}}^2 \leq \mathbb{E} \left(\int_0^T \int_\omega |y_\nu(t, x)|^2 dx dt \right) \tag{24}$$

for all $(y^0, y^1) \in L^2(\Omega, \mathbb{C}) \times H^{-1}(\Omega, \mathbb{C})$, where y_ν denotes the solution of the wave equation with the random initial data $y_\nu^0(\cdot)$ and $y_\nu^1(\cdot)$ determined by their Fourier coefficients $a_j^\nu = \beta_{1,j}^\nu a_j$ and $b_j^\nu = \beta_{2,j}^\nu b_j$ (see (14) for the explicit relation between the Fourier coefficients and the initial data), that is,

$$y_\nu(t, x) = \sum_{j=1}^\infty (\beta_{1,j}^\nu a_j e^{i\lambda_j t} + \beta_{2,j}^\nu b_j e^{-i\lambda_j t}) \phi_j(x). \tag{25}$$

This new constant $C_{T,\text{rand}}^{(W)}(\chi_\omega)$ is called the *randomized observability constant*.

Theorem 2.2. *We have*

$$2C_{T,\text{rand}}^{(W)}(\chi_\omega) = T \inf_{j \in \mathbb{N}^*} \int_\omega \phi_j(x)^2 dx$$

for every measurable subset ω of Ω .

Proof. The proof is immediate by expanding the square in (23), using Fubini’s theorem and the fact that the random laws are independent, of zero mean and of variance 1. \square

Remark 2.3. It can be easily checked that Theorem 2.2 still holds true when considering, in the above randomization procedure, more general real random variables that are independent, have mean 0, variance 1, and superexponential decay. We refer to [7, 10] for more details on these randomization issues. Bernoulli and Gaussian random variables satisfy such appropriate assumptions. As proved in [11], for all initial data $(y^0, y^1) \in L^2(\Omega, \mathbb{C}) \times H^{-1}(\Omega, \mathbb{C})$, the Bernoulli randomization keeps the $L^2 \times H^{-1}$ norm constant, whereas Gaussian randomization generates a dense subset of $L^2(\Omega, \mathbb{C}) \times H^{-1}(\Omega, \mathbb{C})$ through the mapping $R_{(y^0, y^1)} : \nu \in \mathcal{X} \mapsto (y_\nu^0, y_\nu^1)$ provided that all Fourier coefficients of (y^0, y^1) are nonzero and the measure θ charges all open sets of \mathbb{R} . The measure $\mu_{(y^0, y^1)}$ defined as the image of \mathcal{P} by $R_{(y^0, y^1)}$ strongly depends both on the choice of the random variables and on the choice of the initial data (y^0, y^1) . Properties of these measures are established in [11].

Remark 2.4. It is easy to see that $C_{T,\text{rand}}^{(W)}(\chi_\omega) \geq C_T^{(W)}(\chi_\omega)$ for every measurable subset ω of Ω , and every $T > 0$.

Remark 2.5. As mentioned previously, the problem of maximizing the *deterministic* (classical) observability constant $C_T^{(W)}(\chi_\omega)$ defined by (12) over all possible measurable subsets ω of Ω of measure $|\omega| = L|\Omega|$ is open and is probably very difficult. For practical issues it is actually more natural to consider the problem of maximizing the *randomized* observability constant defined by (23). Indeed, when considering for instance the practical problem of locating sensors in an optimal way, the optimality should be considered in terms of an average with respect to a large number of experiments. From this point of view, the deterministic observability constant is expected to be pessimistic with respect to its randomized version. Indeed, in general it is expected that $C_{T,\text{rand}}^{(W)}(\chi_\omega) > C_T^{(W)}(\chi_\omega)$.

In dimension one, with $\Omega = (0, \pi)$ and Dirichlet boundary conditions, it follows from [50, Proposition 2] (where this one-dimensional case is studied in detail) that these strict inequalities hold if and only if T is not an integer multiple of π (note that if T is a multiple of 2π then the equalities follow immediately from Parseval’s Theorem). Note that, in the one-dimensional case, the GCC is satisfied for every $T \geq 2\pi$, and the fact that the deterministic and the randomized observability constants do not coincide is due to crossed Fourier modes in the deterministic case.

In dimension greater than one, there is a class of examples where the strict inequality holds: this is indeed the case when one is able to assert that $C_T^{(W)}(\chi_\omega) = 0$ whereas $C_{T,\text{rand}}^{(W)}(\chi_\omega) > 0$. Let us provide several examples.

An example of such a situation for the wave equation is provided by considering $\Omega = (0, \pi)^2$ with Dirichlet boundary conditions and $L = 1/2$. It is indeed proved further (see Proposition 4.2 and Remark 4.4) that the domain $\omega = \{(x, y) \in \Omega \mid x < \pi/2\}$ maximizes J over \mathcal{U}_L , and that $J(\chi_\omega) = 1/2$. Clearly, such a domain does not satisfy the Geometric Control Condition, and one has $C_T^{(W)}(\chi_\omega) = 0$, whereas $C_\infty^{(W)}(\chi_\omega) = 1/4$.

Another class of examples for the wave equation is provided by the well known Bunimovich stadium with Dirichlet boundary conditions. Setting $\Omega = R \cup W$, where R is the rectangular part and W the circular wings, it is proved in [12] that, for any open neighborhood ω of the closure of W (or even, any neighborhood ω of the vertical intervals between R and W) in Ω , there exists $c > 0$ such that $\int_\omega \phi_j(x)^2 dx \geq c$ for every $j \in \mathbb{N}^*$. It follows that $J(\chi_\omega) > 0$, whereas $C_T^{(W)}(\chi_\omega) = 0$ since ω does not satisfy the Geometric Control Condition. It can be noted that the result still holds if one replaces the wings W by any other manifold glued along R , so that Ω is a partially rectangular domain.

2.4. Conclusion: a relevant criterion

In the previous section we have shown that it is more relevant in practice to model the problem of maximizing the observability as the problem of maximizing the randomized observability constant.

Using Theorem 2.2, this leads us to consider the following spectral problem.

- Let $L \in (0, 1)$ be fixed. Maximize the spectral functional

$$J(\chi_\omega) = \inf_{j \in \mathbb{N}^*} \int_\omega \phi_j(x)^2 dx \tag{26}$$

over all possible measurable subsets ω of Ω of measure $|\omega| = L|\Omega|$.

Note that this spectral criterion is independent of T and is of diagonal nature, not involving any crossed term. However, it depends on the choice of the specific Hilbert basis $(\phi_j)_{j \in \mathbb{N}^*}$ of eigenfunctions of A , at least when the spectrum of A is not simple. We will come back to this issue in Section 3.6 by considering an intrinsic spectral criterion, where the infimum is taken over all possible normalized eigenfunctions of A .

Maximization of J will be studied in Section 3, and will lead to an unexpectedly rich field of investigations, related to quantum ergodicity properties of Ω .

Before going on with that study, let us provide another way of coming up with the spectral functional (26). In the previous section we have seen that $TJ(\chi_\omega)$ can be interpreted as a randomized observability constant, corresponding to a randomized observability inequality. We will see next that $J(\chi_\omega)$ can also be obtained by performing a time averaging procedure on the classical observability inequality.

2.5. Time asymptotic observability inequality

First of all, we claim that, for all $(y^0, y^1) \in L^2(\Omega, \mathbb{C}) \times H^{-1}(\Omega, \mathbb{C})$, the quantity

$$\frac{1}{T} \int_0^T \int_\omega |y(t, x)|^2 dx dt,$$

where $y \in C^0(0, T; L^2(\Omega, \mathbb{C})) \cap C^1(0, T; H^{-1}(\Omega, \mathbb{C}))$ is the solution of the wave equation (10) such that $y(0, \cdot) = y^0(\cdot)$ and $\partial_t y(0, \cdot) = y^1(\cdot)$, has a limit as T tends to ∞ (this fact is proved in Lemmas A.1 and A.2 further). This leads to the concept of *time asymptotic observability constant*

$$C_\infty^{(W)}(\chi_\omega) = \inf \left\{ \lim_{T \rightarrow \infty} \frac{1}{T} \frac{\int_0^T \int_\omega |y(t, x)|^2 dx dt}{\|(y^0, y^1)\|_{L^2 \times H^{-1}}^2} \mid (y^0, y^1) \in L^2(\Omega, \mathbb{C}) \times H^{-1}(\Omega, \mathbb{C}) \setminus \{(0, 0)\} \right\}. \tag{27}$$

This constant appears as the largest possible nonnegative constant for which the time asymptotic observability inequality

$$C_\infty^{(W)}(\chi_\omega) \|(y^0, y^1)\|_{L^2 \times H^{-1}}^2 \leq \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \int_\omega |y(t, x)|^2 dx dt \tag{28}$$

holds for all $(y^0, y^1) \in L^2(\Omega, \mathbb{C}) \times H^{-1}(\Omega, \mathbb{C})$.

We have the following results.

Theorem 2.6. For every measurable subset ω of Ω ,

$$2C_\infty^{(W)}(\chi_\omega) = \inf \left\{ \frac{\int_\omega \sum_{\lambda \in U} \left| \sum_{k \in I(\lambda)} c_k \phi_k(x) \right|^2 dx}{\sum_{k=1}^\infty |c_k|^2} \mid (c_j)_{j \in \mathbb{N}^*} \in \ell^2(\mathbb{C}) \setminus \{0\} \right\},$$

where U is the set of all distinct eigenvalues λ_k and $I(\lambda) = \{j \in \mathbb{N}^* \mid \lambda_j = \lambda\}$.

Corollary 2.7. $2C_\infty^{(W)}(\chi_\omega) \leq J(\chi_\omega)$ for every measurable subset ω of Ω . If the domain Ω is such that every eigenvalue of the Dirichlet–Laplacian is simple, then

$$2C_\infty^{(W)}(\chi_\omega) = \inf_{j \in \mathbb{N}^*} \int_\omega \phi_j(x)^2 dx = J(\chi_\omega)$$

for every measurable subset ω of Ω .

The proof of these results is given in the Appendix. Note that, as is well known, the assumption of the simplicity of the spectrum of the Dirichlet–Laplacian is generic with respect to the domain Ω (see e.g. [44, 63, 26]).

Remark 2.8. It follows obviously from the definitions of the observability constants that

$$\limsup_{T \rightarrow \infty} \frac{C_T^{(W)}(\chi_\omega)}{T} \leq C_\infty^{(W)}(\chi_\omega)$$

for every measurable subset ω of Ω . However, equality does not hold in general. Indeed, consider a set Ω with a smooth boundary, and a pair (ω, T) not satisfying the Geometric Control Condition. Then $C_T^{(W)}(\chi_\omega) = 0$ must hold. However, $J(\chi_\omega)$ may be positive, as already discussed in Remark 2.5 where we gave several classes of examples having this property.

3. Optimal observability under quantum ergodicity assumptions

We define

$$\mathcal{U}_L = \{\chi_\omega \mid \omega \text{ is a measurable subset of } \Omega \text{ of measure } |\omega| = L|\Omega|\}. \quad (29)$$

In Section 2, our discussions have led us to model the problem of optimal observability as

$$\sup_{\chi_\omega \in \mathcal{U}_L} J(\chi_\omega) \quad (30)$$

with

$$J(\chi_\omega) = \inf_{j \in \mathbb{N}^*} \int_\omega \phi_j(x)^2 dx,$$

where $(\phi_j)_{j \in \mathbb{N}^*}$ is a Hilbert basis of $L^2(\Omega, \mathbb{C})$ (defined in Section 2.1), consisting of eigenfunctions of Δ .

The cost functional $J(\chi_\omega)$ can be seen as a spectral energy (de)concentration criterion. For every $j \in \mathbb{N}^*$, the integral $\int_\omega \phi_j(x)^2 dx$ is the energy of the j^{th} eigenfunction restricted to ω , and the problem is to maximize the infimum over j of these energies, over all subsets ω of measure $|\omega| = L|\Omega|$.

This section is organized as follows. Section 3.1 contains some preliminary remarks and, in particular, introduces a convexified version of the problem (30). Our main results are stated in Section 3.2. They provide the optimal value of (30) under spectral assumptions on Ω , by proving moreover that there is no gap between the problem (30) and its convexified version. These assumptions are discussed in Section 3.3. Sections 3.4 and 3.5

are devoted to proving our main results. Finally, in Section 3.6 we consider an intrinsic spectral variant of (30) where, as announced in Section 2.4, the infimum is taken over all possible normalized eigenfunctions of Δ .

3.1. Preliminary remarks

Since the set \mathcal{U}_L does not have compactness properties ensuring the existence of a solution of (30), we consider the convex closure of \mathcal{U}_L for the weak star topology of L^∞ ,

$$\bar{\mathcal{U}}_L = \left\{ a \in L^\infty(\Omega, [0, 1]) \mid \int_\Omega a(x) dx = L|\Omega| \right\}. \quad (31)$$

This convexification procedure is standard in shape optimization problems where an optimal domain may fail to exist because of hard constraints (see e.g. [6]). Replacing $\chi_\omega \in \mathcal{U}_L$ with $a \in \bar{\mathcal{U}}_L$, we define a convexified formulation of the problem (30) by

$$\sup_{a \in \bar{\mathcal{U}}_L} J(a), \quad (32)$$

where

$$J(a) = \inf_{j \in \mathbb{N}^*} \int_\Omega a(x) \phi_j(x)^2 dx. \quad (33)$$

Obviously, we have

$$\sup_{\chi_\omega \in \mathcal{U}_L} \inf_{j \in \mathbb{N}^*} \int_\Omega \chi_\omega(x) \phi_j(x)^2 dx \leq \sup_{a \in \bar{\mathcal{U}}_L} \inf_{j \in \mathbb{N}^*} \int_\Omega a(x) \phi_j(x)^2 dx. \quad (34)$$

In the next section, we compute the optimal value (32) of this convexified problem and investigate the question of knowing whether the inequality (34) is strict or not. In other words we investigate whether there is a gap or not between the problem (30) and its convexified version (32).

Remark 3.1 (Comments on the choice of the topology). In our study we consider measurable subsets ω of Ω , and we endow the set $L^\infty(\Omega, \{0, 1\})$ of all characteristic functions of measurable subsets with the weak star topology. Other topologies are used in shape optimization problems, such as the Hausdorff topology. Note however that, although the Hausdorff topology has nice compactness properties, it cannot be used in our study because of the measure constraint on ω . Indeed, Hausdorff convergence does not preserve measure, and the class of admissible domains is not closed for this topology. Topologies associated with convergence in the sense of characteristic functions or in the sense of compact sets (see for instance [25, Chapter 2]) do not easily guarantee the compactness of minimizing sequences of domains, unless one restricts the class of admissible domains, imposing for example some kind of uniform regularity.

Remark 3.2. We stress that the question of the possible existence of a gap between the original problem and its convexified version is not obvious and cannot be handled with usual Γ -convergence tools, in particular because the function J defined by (33) is not lower semicontinuous for the weak star topology of L^∞ (it is however upper

semicontinuous for that topology, as an infimum of linear functions). To illustrate this fact, consider the one-dimensional case of Remark 2.1. In this specific situation, since $\phi_j(x) = \sqrt{2/\pi} \sin(jx)$ for every $j \in \mathbb{N}^*$, one has

$$J(a) = \frac{2}{\pi} \inf_{j \in \mathbb{N}^*} \int_0^\pi a(x) \sin^2(jx) dx$$

for every $a \in \overline{\mathcal{U}}_L$. Since the functions $x \mapsto \sin^2(jx)$ converge weakly to $1/2$, it clearly follows that $J(a) \leq L$ for every $a \in \overline{\mathcal{U}}_L$. Therefore, we have $\sup_{a \in \overline{\mathcal{U}}_L} J(a) = L$, and the supremum is reached for the constant function $a(\cdot) = L$. Consider the sequence of subsets ω_N of $(0, \pi)$ of measure $L\pi$ defined by

$$\omega_N = \bigcup_{k=1}^N \left(\frac{k\pi}{N+1} - \frac{L\pi}{2N}, \frac{k\pi}{N+1} + \frac{L\pi}{2N} \right)$$

for every $N \in \mathbb{N}^*$. Clearly, the sequence of functions χ_{ω_N} converges to the constant function $a(\cdot) = L$ for the weak star topology of L^∞ , but nevertheless, an easy computation shows that

$$\int_{\omega_N} \sin^2(jx) dx = \begin{cases} \frac{L\pi}{2} - \frac{N}{2j} \sin\left(\frac{jL\pi}{N}\right) & \text{if } (N+1) \mid j, \\ \frac{L\pi}{2} + \frac{1}{2j} \sin\left(\frac{jL\pi}{N}\right) & \text{otherwise,} \end{cases}$$

and hence

$$\limsup_{N \rightarrow \infty} \frac{2}{\pi} \inf_{j \in \mathbb{N}^*} \int_{\omega_N} \sin^2(jx) dx < L.$$

This simple example illustrates the difficulty in understanding the limiting behavior of the functional because of the lack of lower semicontinuity, which makes possible the occurrence of a gap in the convexification procedure. In Section 3.2, we will prove that there is no such gap under an additional geometric spectral assumption.

3.2. Optimal value of the problem

Let us first compute the optimal value of the convexified optimal design problem (32).

Lemma 3.3. *The problem (32) has at least one solution. Moreover,*

$$\sup_{a \in \overline{\mathcal{U}}_L} \inf_{j \in \mathbb{N}^*} \int_\Omega a(x) \phi_j(x)^2 dx = L, \tag{35}$$

and the supremum is reached for the constant function $a(\cdot) = L$ on Ω .

Proof. Since $J(a)$ is defined as the infimum of linear functionals that are continuous for the weak star topology of L^∞ , it is upper semicontinuous for this topology. It follows that the problem (32) has at least one solution, denoted by $a^*(\cdot)$.

In order to prove (35), we consider Cesàro means of eigenfunctions.³ Note that, since the constant function $a(\cdot) = L$ belongs to $\bar{\mathcal{U}}_L$, it follows that $\sup_{a \in \bar{\mathcal{U}}_L} J(a) \geq L$. Let us prove the converse inequality. Since

$$\sup_{a \in \bar{\mathcal{U}}_L} \inf_{j \in \mathbb{N}^*} \int_{\Omega} a(x) \phi_j(x)^2 dx = \inf_{\substack{(\alpha_j) \in \ell_1(\mathbb{R}_+) \\ \sum_{j=1}^{\infty} \alpha_j = 1}} \sum_{j=1}^{\infty} \alpha_j \int_{\Omega} a^*(x) \phi_j(x)^2 dx,$$

one gets, by considering particular choices of sequences $(\alpha_j)_{j \in \mathbb{N}^*}$,

$$\sup_{a \in \bar{\mathcal{U}}_L} \inf_{j \in \mathbb{N}^*} \int_{\Omega} a(x) \phi_j(x)^2 dx \leq \inf_{N \in \mathbb{N}^*} \frac{1}{N} \sum_{j=1}^N \int_{\Omega} a^*(x) \phi_j(x)^2 dx.$$

By [28, Theorem 17.5.7 and Corollary 17.5.8], the sequence $(N^{-1} \sum_{j=1}^N \phi_j^2)_{N \in \mathbb{N}^*}$ of Cesàro means is uniformly bounded on Ω , and converges to the constant $|\Omega|^{-1}$ uniformly on every compact subset of the open set Ω for the C^0 topology and thus weakly in $L^1(\Omega)$. As a consequence, since $a^* \in L^\infty(\Omega)$, we have

$$\inf_{N \in \mathbb{N}^*} \frac{1}{N} \sum_{j=1}^N \int_{\Omega} a^*(x) \phi_j(x)^2 dx \leq \frac{\int_{\Omega} a^*(x) dx}{|\Omega|} = L.$$

The conclusion follows. □

Remark 3.4. In general the convexified problem (32) does not admit a unique solution. Indeed, under symmetry assumptions on Ω there exist infinitely many solutions. For example, in dimension one, with $\Omega = (0, \pi)$, all solutions of (32) are given by all functions in $\bar{\mathcal{U}}_L$ whose Fourier expansion series is of the form $a(x) = L + \sum_{j=1}^{\infty} (a_j \cos(2jx) + b_j \sin(2jx))$ with coefficients $a_j \leq 0$.

It follows from (34) and (35) that $\sup_{\chi_\omega \in \mathcal{U}_L} \inf_{j \in \mathbb{N}^*} \int_{\omega} \phi_j(x)^2 dx \leq L$. The next result states that this inequality is an equality under the following spectral assumptions. Note that $\mu_j = \phi_j^2 dx$ is a probability measure for every integer j .

³ In an early version of this manuscript, we used the following two assumptions on $(\phi_j)_{j \in \mathbb{N}^*}$ in order to prove (35).

- **Weak Quantum Ergodicity (WQE) on the base.** *There exists a subsequence of the sequence of probability measures $\mu_j = \phi_j^2 dx$ converging vaguely to the uniform measure $|\Omega|^{-1} dx$.*
- **Uniform L^∞ -boundedness.** *There exists $A > 0$ such that $\|\phi_j\|_{L^\infty(\Omega)} \leq A$ for every $j \in \mathbb{N}^*$.*

Note that the two assumptions above imply, in particular, that there exists a subsequence of $(\phi_j^2)_{j \in \mathbb{N}^*}$ converging to $|\Omega|^{-1}$ for the weak star topology of $L^\infty(\Omega)$. Under these assumptions, (35) follows easily. We warmly thank Lior Silberman who indicated to us that the WQE assumption may be dropped by using a Cesàro mean argument, and Nicolas Burq for having pointed out the appropriate result of [28] used hereafter.

- **Quantum Unique Ergodicity (QUE) on the base.** The whole sequence of probability measures $\mu_j = \phi_j^2 dx$ converges vaguely to the uniform measure $|\Omega|^{-1} dx$.
- **Uniform L^p -boundedness.** There exist $p \in (1, \infty]$ and $A > 0$ such that $\|\phi_j\|_{L^{2p}(\Omega)} \leq A$ for every $j \in \mathbb{N}^*$.

We stress that these assumptions are made for a selected Hilbert basis $(\phi_j)_{j \in \mathbb{N}^*}$ of eigenfunctions. We refer to Section 3.3 for many comments on that fact from the semiclassical analysis point of view.

Theorem 3.5. Assume that $\partial\Omega$ is Lipschitz. Under QUE on the base and uniform L^p -boundedness assumptions, we have

$$\sup_{\chi_\omega \in \mathcal{U}_L} \inf_{j \in \mathbb{N}^*} \int_{\omega} \phi_j(x)^2 dx = L \quad (36)$$

for every $L \in (0, 1)$.

Theorem 3.5 is proved in Section 3.4. It follows from this result, together with Corollary 2.7 and Theorem 2.2, that the maximal value of the randomized observability constant $C_{T, \text{rand}}^{(W)}(\chi_\omega)$ over the set \mathcal{U}_L is equal to $TL/2$, and that, if the spectrum of Δ is simple, the maximal value of the time asymptotic observability constant $C_\infty^{(W)}(\chi_\omega)$ over the set \mathcal{U}_L is equal to $L/2$.

The question of knowing whether the supremum in (36) is reached (existence of an optimal set) is investigated in Section 4.1.

Remark 3.6. It follows from the proof of Theorem 3.5 that this statement holds true as well whenever the set \mathcal{U}_L is replaced with the set of all measurable subsets ω of Ω , of measure $|\omega| = L|\Omega|$, that are moreover either open with a Lipschitz boundary, or open with a bounded perimeter, or Jordan measurable (i.e., whose boundary is of measure zero).

Remark 3.7. The proof of Theorem 3.5 is constructive and provides a theoretical way of building a maximizing sequence of subsets, by implementing a kind of homogenization procedure. Moreover, this proof highlights the following interesting feature:

- It is possible to increase the value of J by considering subsets having an increasing number of connected components.

Remark 3.8. The assumptions of Theorem 3.5 are sufficient to imply (36), but they are not sharp, as proved in the next proposition.

Proposition 3.9. (i) Assume that $\Omega = (0, \pi)^2$ is a square in \mathbb{R}^2 , and consider the usual Hilbert basis of eigenfunctions of Δ made of products of sine functions. Then QUE on the base is not satisfied. However, the equality (36) holds true.

(ii) Assume that Ω is the unit disk in \mathbb{R}^2 , and consider the usual basis of eigenfunctions of Δ defined in terms of Bessel functions. Then, for every $p \in (1, \infty]$, the uniform L^p -boundedness property is not satisfied, and QUE on the base is not satisfied. However, the equality (36) holds true.

In this proposition, the result on the square could be expected, since the square is nothing but a tensorized version of the one-dimensional case (see also Remark 3.10 hereafter).

The result in the disk is more surprising, having in mind that, among the quantum limits in the disk, one can find the Dirac measure along the boundary which causes the well known phenomenon of whispering galleries. This strong concentration feature could have led to the intuition that there exists an optimal set, concentrating around the boundary; the calculations show that it is however not the case, and (36) is proved to hold.

The next subsection brings some comments on the quantum ergodicity assumptions made in these theorems.

3.3. Comments on quantum ergodicity assumptions

This subsection is organized as a series of remarks.

Remark 3.10. The assumptions of Theorem 3.5 hold true in dimension one. Indeed, it has already been mentioned that the eigenfunctions of the Dirichlet–Laplacian operator on $\Omega = (0, \pi)$ are given by $\phi_j(x) = \sqrt{2/\pi} \sin(jx)$ for every $j \in \mathbb{N}^*$. Therefore, clearly, the whole sequence (not only a subsequence) $(\phi_j^2)_{j \in \mathbb{N}^*}$ converges weakly to $1/\pi$ for the weak star topology of $L^\infty(0, \pi)$. The same property clearly holds for all other boundary conditions considered in this article.

Remark 3.11. In dimension greater than one the situation is widely open. Generally speaking, our assumptions are related to ergodicity properties of Ω . Before providing precise results, we recall the following well known definition.

- **Quantum Ergodicity (QE) on the base.** *There exists a subsequence of the sequence of probability measures $\mu_j = \phi_j^2 dx$ of density one converging vaguely to the uniform measure $|\Omega|^{-1} dx$.*

Here, *density one* means that there exists $\mathcal{I} \subset \mathbb{N}^*$ such that $\#\{j \in \mathcal{I} \mid j \leq N\}/N$ converges to 1 as $N \rightarrow \infty$. Note that QE implies WQE.⁴ It is well known that, if the domain Ω (seen as a billiard where the geodesic flow moves at unit speed and bounces at the boundary according to the geometric optics laws) is ergodic, then QE is satisfied. This is the contents of Shnirel’man’s Theorem, proved in [14, 19, 58, 68] in various contexts (manifolds with or without boundary, with a certain regularity). Actually the results proved in these references are stronger, for two reasons. Firstly, they are valid for any Hilbert basis of eigenfunctions of Δ , whereas here we make this kind of assumption only for the specific basis $(\phi_j)_{j \in \mathbb{N}^*}$ that has been fixed at the beginning of the study. Secondly, they establish that a stronger microlocal version of the QE property holds for pseudodifferential operators, in the unit cotangent bundle $S^*\Omega$ of Ω , and not just only on the configuration space Ω . Here, however, we do not need (de)concentration results in the full phase space, but only in the configuration space. This is why, following [67], we use the wording “on the base”.

Note that the vague convergence of the measures μ_j is weaker than the convergence of the functions ϕ_j^2 for the weak topology of $L^1(\Omega)$. Since Ω is bounded, the property

⁴ Note that, up to our knowledge, the notion of WQE has not been considered before, whereas QE and QUE are classical in mathematical physics.

of vague convergence in Shnirel'man's Theorem is equivalent to saying that, for a subsequence of density one, $\int_{\omega} \phi_j(x)^2 dx$ converges to $|\omega|/|\Omega|$ for every Borel measurable subset ω of Ω such that $|\partial\omega| = 0$ (this follows from the Portmanteau Theorem). In contrast, the property of convergence for the weak topology of $L^1(\Omega)$ is equivalent to saying that, for a subsequence of density one, $\int_{\omega} \phi_j(x)^2 dx$ converges to $|\omega|/|\Omega|$ for every measurable subset ω of Ω . Under the assumption that all eigenfunctions are uniformly bounded in $L^\infty(\Omega)$, both notions are equivalent.

Note that the notion of L^∞ -QE property, meaning that the above QE property holds for the weak topology of L^1 , is defined and mentioned in [67] as a delicate open problem. As said above, we stress that, under the assumption that all eigenfunctions are uniformly bounded in $L^\infty(\Omega)$, QE and L^∞ -QE are equivalent.

To the best of our knowledge, nothing seems to be known on the uniform L^p -boundedness property. As above, it follows from the Portmanteau Theorem that, under uniform L^p -boundedness (with $p > 1$), the QUE on the base property holds true for the weak topology of L^1 .

Remark 3.12. Shnirel'man's Theorem leaves open the possibility of having an exceptional subsequence of measures μ_j converging vaguely to some other measure. The QUE assumption consists in assuming that the whole sequence converges vaguely to the uniform measure. It is an important issue in quantum and mathematical physics. Note indeed that the quantity $\int_{\omega} \phi_j(x)^2 dx$ is interpreted as the probability of finding the quantum state of energy λ_j^2 in ω . We stress again that here we consider a version of QUE in the configuration space only, not in the full phase space. Moreover, we consider the QUE property for the basis $(\phi_j)_{j \in \mathbb{N}^*}$ under consideration, but not necessarily for any such basis of eigenfunctions.

QUE obviously holds true in the one-dimensional case of Remark 2.1 (see also Remark 3.2) but it does not however hold true for multi-dimensional hypercubes.

More generally, only partial results exist. The question of determining what are the possible weak limits of the μ_j 's (semiclassical measures, or quantum limits) is widely open in general. It could happen that, even in the framework of Shnirel'man's Theorem, a subsequence of density zero converges to an invariant measure like for instance a measure carried by closed geodesics (these are the so-called strong *scars*, see, e.g., [18]). Note however that, as already mentioned, here we are concerned with concentration results in the configuration space only.

The QUE on the base property, stating that the whole sequence of measures $\mu_j = \phi_j^2 dx$ converges vaguely to the uniform measure, postulates that there is no such concentration phenomenon. Note that, although rational polygonal billiards are not ergodic in the phase space, while polygonal billiards are generically ergodic (see [33]), the property QE on the base holds in any rational polygon⁵ (see [43]), and in any flat torus (see [56]). Apart from these recent results, and in spite of impressive recent results around QUE (see, e.g., the survey [57]), up to now no example of a multi-dimensional domain is known where QUE on the base holds true.

⁵ A *rational polygon* is a planar polygon whose interior is connected and simply connected and whose vertex angles are rational multiples of π .

Remark 3.13. The question of knowing whether there exists an example where there is a gap between the convexified problem (32) and the original one (30), is an open problem. We think that, if such an example exists, then the underlying geodesic flow ought to be completely integrable and have strong concentration properties. As already mentioned, in our framework we have fixed a given basis $(\phi_j)_{j \in \mathbb{N}^*}$ of eigenvectors, and we consider only the weak limits of the measures $\phi_j^2 dx$. We are not aware of any example having strong enough concentration properties to derive a gap statement.

Remark 3.14. Our results here show that shape optimization problems are intimately related to the ergodicity properties of Ω . Notice that, in the early article [13], the authors suggested such connections. They analyzed the exponential decay of solutions of damped wave equations. Their results show that the quantum effects of bouncing balls or whispering galleries play an important role in the failure of exponential decay properties. At the end of the article, the authors conjectured that such considerations could be useful in the placement and design of actuators or sensors. Our results of this section provide precise results showing these connections and new perspectives on those intuitions. In our view they are the main contribution of our article, in the sense that they point out close relations between shape optimization and ergodicity, and provide new open problems and directions for domain optimization analysis.

3.4. Proof of Theorem 3.5

In what follows, for every measurable subset ω of Ω , we set $I_j(\omega) = \int_{\omega} \phi_j(x)^2 dx$ for $j \in \mathbb{N}^*$. By definition, $J(\omega) = \inf_{j \in \mathbb{N}^*} I_j(\omega)$. Note that, from QUE on the base and from the Portmanteau Theorem (see Remark 3.11), it follows that, for every Borel measurable subset ω of Ω such that $|\omega| = L|\Omega|$ and $|\partial\omega| = 0$, one has $I_j(\omega) \rightarrow L$ as $j \rightarrow \infty$, and hence $J(\omega) \leq L$.

Let ω_0 be an open connected subset of Ω of measure $L|\Omega|$ having a Lipschitz boundary. In what follows we assume that $J(\omega_0) < L$, otherwise there is nothing to prove. By the QUE assumption, there exists an integer j_0 such that

$$I_j(\omega_0) \geq L - \frac{1}{4}(L - J(\omega_0)) \quad (37)$$

for every $j > j_0$.

Our proof below consists in implementing a kind of homogenization procedure by constructing a sequence of open subsets ω_k (starting from ω_0) having a Lipschitz boundary such that $|\omega_k| = L|\omega_k|$ and $\lim_{k \rightarrow \infty} J(\omega_k) = L$. Proving this limit is not easy and we are going to distinguish between lower and higher eigenfrequencies. For the low frequencies, we are going to prove that, by moving some mass of the initial set ω_0 according to some kind of homogenization idea, we can increase the value of J . The high frequencies will be tackled thanks to the estimate (37) implied by the QUE assumption.

Denote by $\overline{\omega_0}$ the closure of ω_0 , and by ω_0^c the complement of ω_0 in Ω . Since Ω and ω_0 have a Lipschitz boundary, it follows that ω_0 and $\Omega \setminus \omega_0$ have the δ -cone property⁶ for

⁶ We recall that an open subset Ω of \mathbb{R}^n has the δ -cone property if, for every $x \in \partial\Omega$, there exists a normalized vector ξ_x such that $C(y, \xi_x, \delta) \subset \Omega$ for every $y \in \overline{\Omega} \cap B(x, \delta)$, where $C(y, \xi_x, \delta) = \{z \in \mathbb{R}^n \mid \langle z - y, \xi \rangle \geq (\cos \delta)\|z - y\| \text{ and } 0 < \|z - y\| < \delta\}$.

some $\delta > 0$ (see [25, Theorem 2.4.7]). Consider partitions of $\overline{\omega_0}$ and ω_0^c ,

$$\overline{\omega_0} = \bigcup_{i=1}^K F_i \quad \text{and} \quad \omega_0^c = \bigcup_{i=1}^{\tilde{K}} \tilde{F}_i, \tag{38}$$

to be chosen later. As a consequence of the δ -cone property, there exists $c_\delta > 0$ and a partition $(F_i)_{1 \leq i \leq K}$ (resp. $(\tilde{F}_i)_{1 \leq i \leq \tilde{K}}$) such that, for $|F_i|$ small enough,

$$\forall i \in \{1, \dots, K\} \text{ (resp. } \forall i \in \{1, \dots, \tilde{K}\}), \frac{\eta_i}{\text{diam } F_i} \geq c_\delta \left(\text{resp. } \frac{\tilde{\eta}_i}{\text{diam } \tilde{F}_i} \geq c_\delta \right), \tag{39}$$

where η_i (resp., $\tilde{\eta}_i$) is the inradius⁷ of F_i (resp., \tilde{F}_i), and $\text{diam } F_i$ (resp., $\text{diam } \tilde{F}_i$) the diameter of F_i (resp., of \tilde{F}_i).

It is then clear that, for every $i \in \{1, \dots, K\}$ (resp., for every $i \in \{1, \dots, \tilde{K}\}$), there exists $\xi_i \in F_i$ (resp., $\tilde{\xi}_i \in \tilde{F}_i$) such that $B(\xi_i, \eta_i/2) \subset F_i \subset B(\xi_i, \eta_i/c_\delta)$ (resp., $B(\tilde{\xi}_i, \tilde{\eta}_i/2) \subset \tilde{F}_i \subset B(\tilde{\xi}_i, \tilde{\eta}_i/c_\delta)$), where $B(\xi, \eta)$ stands for the open ball centered at ξ with radius η . These features characterize a substantial family of sets (also called nicely shrinking sets), as is well known in measure theory. By continuity, the points ξ_i and $\tilde{\xi}_i$ are Lebesgue points of the functions ϕ_j^2 for every $j \leq j_0$. This implies that, for all $j \leq j_0$,

$$\int_{F_i} \phi_j(x)^2 dx = |F_i| \phi_j(\xi_i)^2 + o(|F_i|) \quad \text{as } \eta_i \rightarrow 0,$$

for every $i \in \{1, \dots, K\}$, and

$$\int_{\tilde{F}_i} \phi_j(x)^2 dx = |\tilde{F}_i| \phi_j(\tilde{\xi}_i)^2 + o(|\tilde{F}_i|) \quad \text{as } \tilde{\eta}_i \rightarrow 0,$$

for every $i \in \{1, \dots, \tilde{K}\}$. Setting $\eta = \max(\max_{1 \leq i \leq K} \text{diam } F_i, \max_{1 \leq i \leq \tilde{K}} \text{diam } \tilde{F}_i)$ and using $\sum_{i=1}^K |F_i| = |\omega_0| = L|\Omega|$ and $\sum_{i=1}^{\tilde{K}} |\tilde{F}_i| = |\omega_0^c| = (1 - L)|\Omega|$, we obtain $\sum_{i=1}^K o(|F_i|) + \sum_{i=1}^{\tilde{K}} o(|\tilde{F}_i|) = o(1)$ as $\eta \rightarrow 0$. It follows that

$$\begin{aligned} I_j(\omega_0) &= \int_{\omega_0} \phi_j(x)^2 dx = \sum_{i=1}^K |F_i| \phi_j(\xi_i)^2 + o(1), \\ I_j(\omega_0^c) &= \int_{\omega_0^c} \phi_j(x)^2 dx = \sum_{i=1}^{\tilde{K}} |\tilde{F}_i| \phi_j(\tilde{\xi}_i)^2 + o(1), \end{aligned} \tag{40}$$

for every $j \leq j_0$, as $\eta \rightarrow 0$. Note that, since ω_0^c is the complement of ω_0 in Ω ,

$$I_j(\omega_0) + I_j(\omega_0^c) = \int_{\omega_0} \phi_j(x)^2 dx + \int_{\omega_0^c} \phi_j(x)^2 dx = 1 \tag{41}$$

for every j . Setting $h_i = (1 - L)|F_i|$ and $\ell_i = L|\tilde{F}_i|$, we infer from (40) and (41) that

⁷ In other words, the largest radius of balls contained in F_i .

$$(1 - L)I_j(\omega_0) = \sum_{i=1}^K h_i \phi_j(\xi_i)^2 + o(1), \quad LI_j(\omega_0) = L - \sum_{i=1}^{\tilde{K}} \ell_i \phi_j(\tilde{\xi}_i)^2 + o(1), \quad (42)$$

for every $j \leq j_0$, as $\eta \rightarrow 0$. In what follows, we denote by V_n the Lebesgue measure of the n -dimensional unit ball. For $\varepsilon > 0$ to be chosen later, we define the perturbation ω^ε of ω_0 by

$$\omega^\varepsilon = \left(\omega_0 \setminus \bigcup_{i=1}^K \overline{B(\xi_i, \varepsilon_i)} \right) \cup \bigcup_{i=1}^{\tilde{K}} B(\tilde{\xi}_i, \tilde{\varepsilon}_i),$$

where $\varepsilon_i = \varepsilon h_i^{1/n} / |B(\xi_i, 1)|^{1/n} = \varepsilon h_i^{1/n} / V_n^{1/n}$ and $\tilde{\varepsilon}_i = \varepsilon \ell_i^{1/n} / |B(\tilde{\xi}_i, 1)|^{1/n} = \varepsilon \ell_i^{1/n} / V_n^{1/n}$. Note that it is possible to define such a perturbation provided that

$$0 < \varepsilon < \min \left(\min_{1 \leq i \leq K} \frac{\eta_i V_n^{1/n}}{h_i^{1/n}}, \min_{1 \leq i \leq \tilde{K}} \frac{\tilde{\eta}_i V_n^{1/n}}{\ell_i^{1/n}} \right).$$

It follows from the well known isodiametric inequality⁸ that $|F_i| \leq V_n(\text{diam } F_i)^n / 2^n$ for every $i \in \{1, \dots, K\}$, and $|\tilde{F}_i| \leq V_n(\text{diam } \tilde{F}_i)^n / 2^n$ for every $i \in \{1, \dots, \tilde{K}\}$, independently of the partitions considered. Set $\varepsilon_0 = \min(1, 2c_\delta)$. Using (39), we get

$$\frac{\eta_i V_n^{1/n}}{h_i^{1/n}} = \frac{\eta_i V_n^{1/n}}{(1 - L)^{1/n} |F_i|^{1/n}} \geq \frac{1}{(1 - L)^{1/n}} \frac{2\eta_i}{\text{diam } F_i} \geq \varepsilon_0$$

for every $i \in \{1, \dots, K\}$, and similarly $\tilde{\eta}_i V_n^{1/n} / \ell_i^{1/n} \geq \varepsilon_0$ for every $i \in \{1, \dots, \tilde{K}\}$. It follows that the previous perturbation is well defined for every $\varepsilon \in (0, \varepsilon_0]$. Note that, by construction,

$$\begin{aligned} |\omega^\varepsilon| &= |\omega_0| - \sum_{i=1}^K \varepsilon_i^n |B(\xi_i, 1)| + \sum_{i=1}^{\tilde{K}} \tilde{\varepsilon}_i^n |B(\tilde{\xi}_i, 1)| = |\omega_0| - \varepsilon^n \sum_{i=1}^K h_i + \varepsilon^n \sum_{i=1}^{\tilde{K}} \ell_i \\ &= |\omega_0| - \varepsilon^n (1 - L) \sum_{i=1}^K |F_i| + \varepsilon^n L \sum_{i=1}^{\tilde{K}} |\tilde{F}_i| \\ &= |\omega_0| - \varepsilon^n (1 - L)L|\Omega| + \varepsilon^n L(1 - L)|\Omega| = |\omega_0| = L|\Omega|. \end{aligned}$$

Using again the fact that ξ_i and $\tilde{\xi}_i$ are Lebesgue points of the functions ϕ_j^2 , we get

$$\int_{B(\xi_i, \varepsilon_i)} \phi_j(x)^2 dx = |B(\xi_i, \varepsilon_i)| \phi_j(\xi_i)^2 + o(|B(\xi_i, \varepsilon_i)|) \quad \text{as } \varepsilon_i \rightarrow 0,$$

for every $i \in \{1, \dots, K\}$, and

$$\int_{B(\tilde{\xi}_i, \tilde{\varepsilon}_i)} \phi_j(x)^2 dx = |B(\tilde{\xi}_i, \tilde{\varepsilon}_i)| \phi_j(\tilde{\xi}_i)^2 + o(|B(\tilde{\xi}_i, \tilde{\varepsilon}_i)|) \quad \text{as } \tilde{\varepsilon}_i \rightarrow 0,$$

⁸ The *isodiametric inequality* states that, for every compact subset K of the Euclidean space \mathbb{R}^n , we have $|K| \leq |B(0, \text{diam}(K)/2)|$.

for every $i \in \{1, \dots, \tilde{K}\}$. Since $|B(\xi_i, \varepsilon_i)| = \varepsilon^n(1-L)|F_i|$ and $|B(\tilde{\xi}_i, \tilde{\varepsilon}_i)| = \varepsilon^n L|\tilde{F}_i|$, and since $\sum_{i=1}^K |F_i| = L|\Omega|$ and $\sum_{i=1}^{\tilde{K}} |\tilde{F}_i| = (1-L)|\Omega|$, we infer that $\sum_{i=1}^K o(|B(\xi_i, \varepsilon_i)|) + \sum_{i=1}^{\tilde{K}} o(|B(\tilde{\xi}_i, \tilde{\varepsilon}_i)|) = \varepsilon^n o(1)$ as $\varepsilon \rightarrow 0$, and thus as $\eta \rightarrow 0$. It follows that

$$\begin{aligned} I_j(\omega^\varepsilon) &= \int_{\omega^\varepsilon} \phi_j(x)^2 dx = I_j(\omega_0) - \sum_{i=1}^K \int_{B(\xi_i, \varepsilon_i)} \phi_j(x)^2 dx + \sum_{i=1}^{\tilde{K}} \int_{B(\tilde{\xi}_i, \tilde{\varepsilon}_i)} \phi_j(x)^2 dx, \\ &= I_j(\omega_0) - \varepsilon^n \left(\sum_{i=1}^K h_i \phi_j(\xi_i)^2 - \sum_{i=1}^{\tilde{K}} \ell_i \phi_j(\tilde{\xi}_i)^2 \right) + \varepsilon^n o(1) \quad \text{as } \eta \rightarrow 0, \end{aligned}$$

and hence, using (42),

$$I_j(\omega^\varepsilon) = I_j(\omega_0) + \varepsilon^n(L - I_j(\omega_0)) + \varepsilon^n o(1) \quad \text{as } \eta \rightarrow 0,$$

for all $j \leq j_0$ and $\varepsilon \in (0, \varepsilon_0]$. Since $\varepsilon_0^n \leq 1$, it then follows that

$$I_j(\omega^\varepsilon) \geq J(\omega_0) + \varepsilon^n(L - J(\omega_0)) + \varepsilon^n o(1) \quad \text{as } \eta \rightarrow 0, \tag{43}$$

for all $j \leq j_0$ and $\varepsilon \in (0, \varepsilon_0]$, where the functional J is defined by (26).

We now choose the subdivisions (38) fine enough (that is, $\eta > 0$ small enough) so that, for every $j \leq j_0$, the remainder term $o(1)$ (as $\eta \rightarrow 0$) in (43) is bounded by $\frac{1}{2}(L - J(\omega_0))$. It follows from (43) that

$$I_j(\omega^\varepsilon) \geq J(\omega_0) + \frac{\varepsilon^n}{2}(L - J(\omega_0)) \tag{44}$$

for all $j \leq j_0$ and $\varepsilon \in (0, \varepsilon_0)$.

Let us prove that the set ω^ε still satisfies an inequality of the type (37) for ε small enough. Using the uniform L^p -boundedness property and Hölder's inequality, we have

$$\begin{aligned} |I_j(\omega^\varepsilon) - I_j(\omega_0)| &= \left| \int_{\Omega} (\chi_{\omega^\varepsilon}(x) - \chi_{\omega_0}(x)) \phi_j(x)^2 dx \right| \\ &\leq A^2 \left(\int_{\Omega} |\chi_{\omega^\varepsilon}(x) - \chi_{\omega_0}(x)|^q dx \right)^{1/q} \end{aligned}$$

for every integer j and every $\varepsilon \in (0, \varepsilon_0]$, where q is defined by $1/p + 1/q = 1$. Moreover,

$$\begin{aligned} \int_{\Omega} |\chi_{\omega^\varepsilon}(x) - \chi_{\omega_0}(x)|^q dx &= \int_{\Omega} |\chi_{\omega^\varepsilon}(x) - \chi_{\omega_0}(x)| dx = \varepsilon^n \left(\sum_{i=1}^K h_i + \sum_{i=1}^{\tilde{K}} \ell_i \right) \\ &= 2\varepsilon^n L(1-L)|\Omega|, \end{aligned}$$

and hence $|I_j(\omega^\varepsilon) - I_j(\omega_0)| \leq (2A^{2q} \varepsilon^n L(1-L)|\Omega|)^{1/q}$. Therefore, setting

$$\varepsilon_1 = \min \left(\varepsilon_0, \left(\frac{(L - J(\omega_0))^q}{2^{2q+1} A^{2q} L(1-L)|\Omega|} \right)^{1/n} \right),$$

it follows from (37) that

$$I_j(\omega^\varepsilon) \geq L - \frac{1}{2}(L - J(\omega_0)) \tag{45}$$

for all $j \geq j_0$ and $\varepsilon \in (0, \varepsilon_1]$.

Now, using the fact that $J(\omega_0) + \frac{\varepsilon^n}{2}(L - J(\omega_0)) \leq L - \frac{1}{2}(L - J(\omega_0))$ for every $\varepsilon \in (0, \varepsilon_0]$, we infer from (44) and (45) that

$$J(\omega^\varepsilon) \geq J(\omega_0) + \frac{\varepsilon^n}{2}(L - J(\omega_0)) \tag{46}$$

for all $\varepsilon \in (0, \varepsilon_1]$. In particular, this holds for ε such that $\varepsilon^n = \min(\varepsilon_0^n, C(L - J(\omega_0))^q)$ with the positive constant $C = 1/(2^{2q+1}A^{2q}L(1 - L)|\Omega|)$. For this specific value of ε , we set $\omega_1 = \omega^\varepsilon$, and hence

$$J(\omega_1) \geq J(\omega_0) + \frac{1}{2} \min(\varepsilon_0^n, C(L - J(\omega_0))^q)(L - J(\omega_0)). \tag{47}$$

Note that the constants involved in this inequality depend only on L, A and Ω . Note also that, by construction, ω_1 has the δ -cone property.

If $J(\omega_1) \geq L$ then we are done. Otherwise, we apply all the previous arguments to this new set ω_1 : using QUE, there exists an integer still denoted j_0 such that (37) holds with ω_0 replaced with ω_1 . This provides a lower bound for high frequencies. The lower frequencies $j \leq j_0$ are then handled as previously, and we end up with (44) with ω_0 replaced with ω_1 . Finally, this leads to the existence of ω_2 such that (47) holds with ω_1 replaced with ω_2 and ω_0 replaced with ω_1 .

By iteration, we construct a sequence $(\omega_k)_{k \in \mathbb{N}}$ of subsets of Ω (satisfying the δ -cone property) of measure $|\omega_k| = L|\Omega|$ satisfying, as long as $J(\omega_k) < L$,

$$J(\omega_{k+1}) \geq J(\omega_k) + \frac{1}{2} \min(\varepsilon_0^n, (L - J(\omega_k))^q)(L - J(\omega_k)).$$

If $J(\omega_k) < L$ for every integer k , then clearly the sequence $(J(\omega_k))_{k \in \mathbb{N}}$ is increasing, bounded above by L , and converges to L . This finishes the proof.

Remark 3.15. It can be noted that, in the above construction, the subsets ω_k are open, Lipschitz and of bounded perimeter. Hence, considering the problem on the class of measurable subsets ω of Ω , of measure $|\omega| = L|\Omega|$, that are moreover either open with a Lipschitz boundary, or open with a bounded perimeter, or Jordan measurable, the conclusion is still that the supremum is equal to L . This proves the assertion of Remark 3.6.

3.5. Proof of Proposition 3.9

First of all, we assume that $\Omega = (0, \pi)^2$, a square in \mathbb{R}^2 , and we consider the normalized eigenfunctions of the Dirichlet–Laplacian defined by

$$\phi_{j,k}(x, y) = \frac{2}{\pi} \sin(jx) \sin(ky), \quad \text{for all } (j, k) \in (\mathbb{N}^*)^2.$$

It is obvious that QUE on the base is not satisfied.

Let us however prove that $\sup_{\chi_\omega \in \mathcal{U}_L} J(\chi_\omega) = L$. We consider a particular subclass of measurable subsets ω of Ω defined by $\omega = \omega_1 \times \omega_2$, where ω_1 and ω_2 are measurable subsets of $(0, \pi)$. Using the Fubini Theorem, we have

$$J(\chi_\omega) = \frac{2}{\pi} \inf_{j \in \mathbb{N}^*} \int_{\omega_1} \sin^2(jx) dx \times \frac{2}{\pi} \inf_{k \in \mathbb{N}^*} \int_{\omega_2} \sin^2(ky) dy,$$

and hence, from the no-gap result in 1D (for the domain $(0, \pi)$, according to Remark 3.10), it follows that $\sup_{\chi_\omega \in \mathcal{L}_L} J(\chi_\omega) \geq |\omega_1| |\omega_2| / \pi^2 = L$, whence the result.

Assume now that $\Omega = \{x \in \mathbb{R}^2 \mid \|x\| < 1\}$ is the unit (Euclidean) disk in \mathbb{R}^2 . We consider the normalized eigenfunctions of the Dirichlet–Laplacian given by the triply indexed sequence

$$\phi_{jkm}(r, \theta) = \begin{cases} R_{0k}(r)/\sqrt{2\pi} & \text{if } j = 0, \\ R_{jk}(r)Y_{jm}(\theta) & \text{if } j \geq 1, \end{cases} \tag{48}$$

for $j \in \mathbb{N}$, $k \in \mathbb{N}^*$ and $m = 1, 2$, where (r, θ) are the usual polar coordinates. The functions $Y_{jm}(\theta)$ are defined by $Y_{j1}(\theta) = (1/\sqrt{\pi}) \cos(j\theta)$ and $Y_{j2}(\theta) = (1/\sqrt{\pi}) \sin(j\theta)$, and R_{jk} by

$$R_{jk}(r) = \sqrt{2} \frac{J_j(z_{jk}r)}{|J'_j(z_{jk})|}, \tag{49}$$

where J_j is the Bessel function of the first kind of order j , and $z_{jk} > 0$ is the k^{th} zero of J_j . The eigenvalues of the Dirichlet–Laplacian are given by the double sequence of $-z_{jk}^2$ and are of multiplicity 1 if $j = 0$, and 2 if $j \geq 1$.

A maximizing sequence for $L = 0.3$

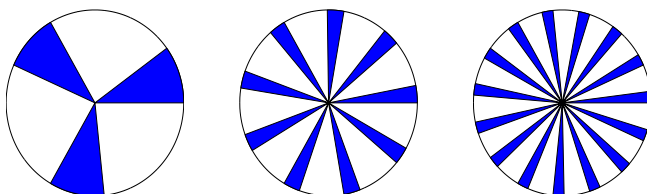


Fig. 1. Particular radial subsets.

To prove the no-gap statement, we use particular (radial) subsets ω , of the form $\omega = \{(r, \theta) \in [0, 1] \times [0, 2\pi] \mid \theta \in \omega_\theta\}$, where $|\omega_\theta| = 2L\pi$, as drawn in Figure 1. For such a subset ω , one has

$$\int_\omega \phi_{jkm}(x)^2 dx = \int_0^1 R_{jk}(r)^2 r dr \int_{\omega_\theta} Y_{jm}(\theta)^2 d\theta = \int_{\omega_\theta} Y_{jm}(\theta)^2 d\theta$$

for all $j \in \mathbb{N}^*$, $k \in \mathbb{N}^*$ and $m = 1, 2$. For $j = 0$,

$$\int_\omega \phi_{0km}(x)^2 dx = \int_0^1 R_{jk}(r)^2 r dr \int_{\omega_\theta} d\theta = |\omega_\theta|.$$

Moreover, since $L\pi = |\Omega| = \int_0^1 r dr \int_{\omega_\theta} d\theta = \frac{1}{2}|\omega_\theta|$, it follows that $|\omega_\theta| = 2L\pi$. By applying the no-gap result in 1D (clearly, it can be applied with the cosine functions as well), one has

$$\sup_{\substack{\omega_\theta \subset [0, 2\pi] \\ |\omega_\theta| = 2L\pi}} \inf_{j \in \mathbb{N}^*} \int_{\omega_\theta} \sin^2(j\theta) d\theta = \sup_{\substack{\omega_\theta \subset [0, 2\pi] \\ |\omega_\theta| = 2L\pi}} \inf_{j \in \mathbb{N}^*} \int_{\omega_\theta} \cos^2(j\theta) d\theta = L\pi.$$

Therefore, we deduce that

$$\sup_{\chi_\omega \in \mathcal{U}_L} \inf_{\substack{j \in \mathbb{N}, k \in \mathbb{N}^* \\ m \in \{1,2\}}} \int_{\omega} \phi_{jkm}(x)^2 dx = L,$$

and the conclusion follows.

3.6. An intrinsic spectral variant of the problem

The problem (26), defined in Section 2.4, whenever the spectrum of Δ is not simple, depends on the Hilbert basis $(\phi_j)_{j \in \mathbb{N}^*}$ of $L^2(\Omega, \mathbb{C})$ under consideration. In this section we assume that the eigenvalues $(\lambda_j^2)_{j \in \mathbb{N}^*}$ of $-\Delta$ are multiple, so that the choice of the basis $(\phi_j)_{j \in \mathbb{N}^*}$ enters into play.

We have already seen in Theorem 2.6 (see Section 2.3) that, in the case of multiple eigenvalues, the spectral expression for the time asymptotic observability constant is more intricate and it does not seem that our analysis can be adapted in an easy way to that case.

Moreover, recall that the criterion J defined by (26) has been motivated in Section 2.3 by means of a randomizing process on the initial data, leading to a randomized observability constant (see Theorem 2.2), but in that case this criterion depends a priori on the choice of the basis $(\phi_j)_{j \in \mathbb{N}^*}$ of eigenfunctions.

In order to get rid of this dependence, and to deal with a more intrinsic criterion, it makes sense to consider the infimum of the criteria J defined by (26) over all possible choices of orthonormal bases of eigenfunctions. This leads us to consider the following intrinsic variant of the problem.

Intrinsic uniform optimal design problem. *Maximize the functional*

$$J_{\text{int}}(\chi_\omega) = \inf_{\phi \in \mathcal{E}} \int_{\omega} \phi(x)^2 dx \tag{50}$$

over all possible subsets ω of Ω of measure $|\omega| = L|\Omega|$, where \mathcal{E} denotes the set of all normalized eigenfunctions of Δ .

Here, ‘‘intrinsic’’ means that this problem does not depend on the choice of a basis of eigenfunctions of Δ .

As in Theorem 2.2, the quantity $(T/2)J_{\text{int}}(\chi_\omega)$ can be interpreted as a constant for which the randomized observability inequality (24) for the wave equation holds, but this constant is smaller than or equal to $C_{T,\text{rand}}^{(W)}(\chi_\omega)$. Furthermore, $C_T^{(W)}(\chi_\omega) \leq (T/2)J_{\text{int}}(\chi_\omega)$. Indeed, this follows from the deterministic observability inequality applied to the particular solution $y(t, x) = e^{i\lambda t} \phi(x)$ for every eigenfunction ϕ . In brief, we have

$$C_T^{(W)}(\chi_\omega) \leq \frac{T}{2} J_{\text{int}}(\chi_\omega) \leq C_{T,\text{rand}}^{(W)}(\chi_\omega).$$

As in Section 3.1, the convexified version of the above problem is to maximize the functional

$$J_{\text{int}}(a) = \inf_{\phi \in \mathcal{E}} \int_{\Omega} a(x)\phi(x)^2 dx$$

over the set $\overline{\mathcal{U}}_L$. Obviously, this problem has at least one solution, and

$$\sup_{\chi_\omega \in \mathcal{U}_L} \inf_{\phi \in \mathcal{E}} \int_{\Omega} \chi_\omega(x) \phi(x)^2 dx \leq \sup_{a \in \overline{\mathcal{U}}_L} \inf_{\phi \in \mathcal{E}} \int_{\Omega} a(x) \phi(x)^2 dx = L,$$

the last equality being easily obtained by adapting the proof of Lemma 3.3.

The intrinsic counterpart of Theorem 3.5 is the following.

Theorem 3.16. *Assume that the uniform measure $|\Omega|^{-1} dx$ is the unique limit point of the family of probability measures $\mu_\phi = \phi^2 dx$, $\phi \in \mathcal{E}$, for the vague topology, and that the whole family of eigenfunctions in \mathcal{E} is uniformly bounded in $L^{2p}(\Omega)$ for some $p \in (1, \infty]$. Then*

$$\sup_{\chi_\omega \in \mathcal{U}_L} \inf_{\phi \in \mathcal{E}} \int_{\omega} \phi(x)^2 dx = L \quad (51)$$

for every $L \in (0, 1)$.

Proof. The proof follows the same lines as in Section 3.4, replacing the integer index j with a continuous index. The only thing to notice is the derivation of the estimate corresponding to (44). In Section 3.4, to obtain (44) from (43), we used the fact that only a finite number of terms have to be considered. Now the number of terms is infinite, but one has to consider all possible normalized eigenfunctions associated with an eigenvalue $|\lambda| \leq |\lambda_0|$. Since this set is compact for every λ_0 , there is no difficulty extending our previous proof. \square

4. Nonexistence of an optimal set and remedies

In Section 4.1 we investigate the question of the existence of an optimal set, reaching the supremum in (30). Apart from simple geometries, this question remains essentially open and we conjecture that in general there does not exist any optimal set. In Section 4.2 we study a spectral approximation of (30), by keeping only the N first modes. We establish existence and uniqueness results, and provide numerical simulations showing the increasing complexity of the optimal sets. We then investigate possible remedies to the nonexistence of an optimal set of (30). As a first remark, we consider in Section 4.3 classes of subsets with compactness properties, intended to ensure existence results for (30). Since our aim is however to investigate (measurable) domains as general as possible, in Section 4.4 we introduce a weighted variant of the observability inequality, where the weight is stronger on lower frequencies. We then come up with a weighted spectral variant of (30), for which we prove, in contrast with the previous results, that there exists a unique optimal set whenever L is large enough, and that the maximizing sequence built from a spectral truncation is stationary.

4.1. On the existence of an optimal set

In this section we comment on the problem of whether the supremum in (36) is reached or not, in the framework of Theorem 3.5. This problem remains essentially open except in several particular cases.

1D case. For the one-dimensional case already mentioned in Remarks 2.1, 3.2 and 3.10, we have the following result.

Proposition 4.1. *Assume that $\Omega = (0, \pi)$, with the usual Hilbert basis of eigenfunctions made of sine functions. Let $L \in (0, 1)$. The supremum of J over \mathcal{U}_L (which is equal to L) is reached if and only if $L = 1/2$. In that case, it is reached for all measurable subsets $\omega \subset (0, \pi)$ of measure $\pi/2$ such that ω and its symmetric image $\omega' = \pi - \omega$ are complementary in $(0, \pi)$.*

Proof. Although the proof can be found in [22] and [50], we recall it here briefly since similar arguments will be used in the proof of the forthcoming Proposition 4.2.

A subset $\omega \subset (0, \pi)$ of Lebesgue measure $L\pi$ solves (36) if and only if $\int_{\omega} \sin^2(jx) dx \geq L\pi/2$ for every $j \in \mathbb{N}^*$, that is, $\int_{\omega} \cos(2jx) dx \leq 0$. Therefore the Fourier series expansion of χ_{ω} on $(0, \pi)$ must be of the form $L + \sum_{j=1}^{\infty} (a_j \cos(2jx) + b_j \sin(2jx))$ with $a_j \leq 0$. Let $\omega' = \pi - \omega$ be the symmetric set of ω with respect to $\pi/2$. The Fourier series expansion of $\chi_{\omega'}$ is $L + \sum_{j=1}^{\infty} (a_j \cos(2jx) - b_j \sin(2jx))$. Set $g(x) = L - \frac{1}{2}(\chi_{\omega}(x) + \chi_{\omega'}(x))$ for almost every $x \in (0, \pi)$. The Fourier series expansion of g is $-\sum_{j=1}^{\infty} a_j \cos(2jx)$ with $a_j \leq 0$ for every $j \in \mathbb{N}^*$.

Assume that $L \neq 1/2$. Then the sets ω and ω' are not disjoint, and hence g is discontinuous. It then follows that $\sum_{j=1}^{\infty} a_j = -\infty$. Moreover, $\sum_{j=1}^{\infty} a_j$ is also the limit of $\sum_{k=1}^{\infty} a_k \widehat{\Delta}_n(k)$ as $n \rightarrow \infty$, where $\widehat{\Delta}_n$ is the Fourier transform of the positive function Δ_n whose graph is the triangle joining the points $(-1/n, 0)$, $(0, 2n)$ and $(1/n, 0)$ (note that Δ_n is an approximation of the Dirac measure, with integral equal to 1). This is in contradiction with the fact that

$$\int_0^{\pi} g(t)\Delta_n(t) dt = \sum_{k=1}^{\infty} a_k \widehat{\Delta}_n(k),$$

derived from Plancherel’s Theorem. □

2D square. For the two-dimensional square $\Omega = (0, \pi)^2$ studied in Proposition 3.9 we are not able to provide a complete answer to the existence question. We are however able to characterize the existence of optimal sets that are Cartesian products.

Proposition 4.2. *Assume that $\Omega = (0, \pi)^2$, with the usual basis of eigenfunctions made of products of sine functions. Let $L \in (0, 1)$. The supremum of J over the class of all possible subsets $\omega = \omega_1 \times \omega_2$ of Lebesgue measure $L\pi^2$, where ω_1 and ω_2 are measurable subsets of $(0, \pi)$, is reached if and only if $L \in \{1/4, 1/2, 3/4\}$. In that case, it is reached for all such sets ω satisfying*

$$\frac{1}{4}(\chi_{\omega}(x, y) + \chi_{\omega}(\pi - x, y) + \chi_{\omega}(x, \pi - y) + \chi_{\omega}(\pi - x, \pi - y)) = L$$

for almost all $(x, y) \in [0, \pi^2]$.

Proof. A subset $\omega \subset (0, \pi)^2$ of Lebesgue measure $L\pi^2$ is a solution of (36) if and only if $(4/\pi^2) \int_{\omega} \sin^2(jx) \sin^2(ky) dx dy \geq L$ for all $(j, k) \in (\mathbb{N}^*)^2$, that is,

$$\int_{\omega} \cos(2jx) \cos(2ky) dx dy \geq \int_{\omega} \cos(2jx) dx dy + \int_{\omega} \cos(2ky) dx dy. \tag{52}$$

Set $\ell_x = \int_0^\pi \chi_\omega(x, y) dy$ for almost every $x \in (0, \pi)$, and $\ell_y = \int_0^\pi \chi_\omega(x, y) dx$ for almost every $y \in (0, \pi)$. Letting either j or k tend to ∞ and using Fubini's Theorem in (52) leads to $\int_0^\pi \ell_x \cos(2jx) dx \leq 0$ and $\int_0^\pi \ell_y \cos(2ky) dy \leq 0$, for all $j, k \in \mathbb{N}^*$.

Now, if $\omega = \omega_1 \times \omega_2$, where ω_1 and ω_2 are measurable subsets of $(0, \pi)$, then the functions $x \mapsto \ell_x$ and $y \mapsto \ell_y$ must be discontinuous. Using arguments similar to the proof of Proposition 4.1, it follows that the functions $x \mapsto \ell_x + \ell_{\pi-x}$ and $y \mapsto \ell_y + \ell_{\pi-y}$ must be constant on $(0, \pi)$, and hence $\int_0^\pi \ell_x \cos(2jx) dx = 0$ and $\int_0^\pi \ell_y \cos(2ky) dy = 0$ for all $j, k \in \mathbb{N}^*$. By (52), it follows that $\int_\omega \cos(2jx) \cos(2ky) dx dy \geq 0$ for all $(j, k) \in (\mathbb{N}^*)^2$. The function F defined by

$$F(x, y) = \frac{1}{4}(\chi_\omega(x, y) + \chi_\omega(\pi - x, y) + \chi_\omega(x, \pi - y) + \chi_\omega(\pi - x, \pi - y))$$

for almost all $(x, y) \in (0, \pi)^2$ can only take the values 0, 1/4, 1/2, 3/4 and 1, and its Fourier series is of the form

$$L + \frac{4}{\pi^2} \sum_{j,k=1}^\infty \left(\int_\omega \cos(2ju) \cos(2kv) du dv \right) \cos(2jx) \cos(2ky),$$

and all Fourier coefficients are nonnegative. Using once again arguments similar to the proof of Proposition 4.1 (Fourier transform and Plancherel's Theorem), one shows that F must necessarily be continuous on $(0, \pi)^2$ and thus constant. The conclusion follows. \square

Remark 4.3. All the results above can obviously be generalized to multi-dimensional domains Ω that are Cartesian products of N one-dimensional sets.

Remark 4.4. According to Proposition 4.2, if $L = 1/2$, then there are infinitely many optimal sets. Four of them are drawn in Figure 2. It is interesting to note that the optimal sets on the left in the figure do not satisfy the Geometric Control Condition mentioned in Section 2.1, and that in this configuration the (classical, deterministic) observability constant $C_T^{(W)}(\chi_\omega)$ vanishes, whereas, according to the previous results, $C_{T,\text{rand}}^{(W)}(\chi_\omega) = TL/2$. This is in accordance with Remarks 2.5 and 2.8.

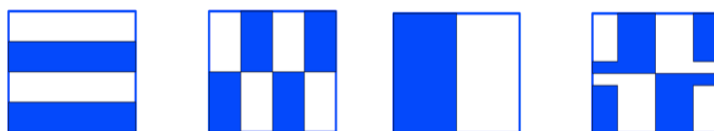


Fig. 2. $\Omega = (0, \pi)^2$, $L = 1/2$.

2D disk. In the two-dimensional disk, we are also unable to provide a complete answer to the existence question, but we can derive the following result.

Proposition 4.5. Assume that $\Omega = \{x \in \mathbb{R}^2 \mid \|x\| < 1\}$ is the unit (Euclidean) disk in \mathbb{R}^2 , with the usual Hilbert basis of eigenfunctions defined in terms of Bessel functions. Let $L \in (0, 1)$. The supremum of J (which is equal to L) over the class of all possible

subsets $\omega = \{(r, \theta) \in [0, 1] \times [0, 2\pi] \mid r \in \omega_r, \theta \in \omega_\theta\}$ such that $|\omega| = L\pi$, where ω_r is any measurable subset of $[0, 1]$ and ω_θ is any measurable subset of $[0, 2\pi]$, is reached if and only if $L = 1/2$. In that case, it is reached for all subsets $\omega = \{(r, \theta) \in [0, 1] \times [0, 2\pi] \mid \theta \in \omega_\theta\}$ of measure $\pi/2$, where ω_θ is any measurable subset of $[0, 2\pi]$ such that ω_θ and its symmetric image $\omega'_\theta = 2\pi - \omega_\theta$ are complementary in $[0, 2\pi]$.

In order to prove this result, we are going to use the explicit expression of certain semi-classical measures on the disk (weak limits of the probability measures $\phi_j^2 dx$), and not only the Dirac measure along the boundary which causes the well known phenomenon of whispering galleries.

Proof of Proposition 4.5. We consider the Hilbert basis of eigenfunctions defined in (48), with the functions R_{jk} defined by (49). Many properties of these functions are known, in particular (see [36]):

- for every $j \in \mathbb{N}$, the sequence of probability measures $r \mapsto R_{jk}(r)^2 r dr$ converges vaguely to 1 as $k \rightarrow \infty$,
- for every $k \in \mathbb{N}^*$, the sequence of probability measures $r \mapsto R_{jk}(r)^2 r dr$ converges vaguely to the Dirac at $r = 1$ as $j \rightarrow \infty$.

These convergence properties permit one to identify certain quantum limits, the second property accounting for the well known phenomenon of whispering galleries.

Less known is the convergence of the above sequence of measures when the ratio j/k is kept constant. Simple computations (see [8]) show that, when taking the limit of $R_{jk}(r)^2 r dr$ with a fixed ratio j/k , and making this ratio vary, we obtain the family of probability measures

$$\mu_s = f_s(r) dr = \frac{1}{\sqrt{1-s^2}} \frac{r}{\sqrt{r^2-s^2}} \chi_{(s,1)}(r) dr,$$

parametrized by $s \in [0, 1)$ (we can even extend to $s = 1$ by defining μ_1 as the Dirac measure at $r = 1$). It follows from the Portmanteau Theorem⁹ that

$$\sup_{a \in \overline{\mathcal{U}}_L} J(a) = \sup_{a \in \overline{\mathcal{U}}_L} \inf_{\substack{j \in \mathbb{N}, k \in \mathbb{N}^* \\ m \in \{1,2\}}} \int_0^{2\pi} \int_0^1 a(r, \theta) \phi_{jkm}(r, \theta)^2 r dr d\theta \leq \sup_{a \in \overline{\mathcal{U}}_L} K(a), \quad (53)$$

where

$$K(a) = \frac{1}{2\pi} \inf_{s \in [0,1)} \int_0^1 \int_0^{2\pi} a(r, \theta) d\theta f_s(r) dr. \quad (54)$$

Lemma 4.6. *We have $\max_{a \in \overline{\mathcal{U}}_L} K(a) = L$, and the maximum of K is reached for the constant function $a^* = L$.*

⁹ Actually to apply the Portmanteau Theorem it is required to apply the argument on every compact $[0, 1 - \varepsilon]$, thus excluding a neighborhood of $s = 1$ so as to ensure that the quantum limits under consideration are uniformly bounded in $L^{3/2}$ (for instance). Since the inequality holds for every $\varepsilon > 0$, the desired inequality follows anyway.

Proof. First, note that $K(a^* = L) = L$ and that the infimum in the definition of K is then reached for every $s \in [0, 1)$. Since K is concave (as infimum of linear functions), in order to prove that $a^* = L$ realizes the maximum it suffices to show that $\langle DK(a^*), h \rangle \leq 0$ (directional derivative) for every function h defined on Ω such that $\int_{\Omega} h(x) dx = 0$. Using Danskin's Theorem¹⁰ (see [16, 5]), we have

$$\langle DK(a^*), h \rangle = \frac{1}{2\pi} \inf_{s \in [0, 1)} \int_0^1 \int_0^{2\pi} h(r, \theta) d\theta f_s(r) dr.$$

For contradiction, assume that there exists a function h on Ω such that $\int_{\Omega} h(x) dx = 0$ and $\int_0^1 \int_0^{2\pi} h(r, \theta) d\theta f_s(r) dr > 0$ for every $s \in [0, 1)$. Then

$$\int_s^1 \int_0^{2\pi} h(r, \theta) d\theta \frac{r}{\sqrt{r^2 - s^2}} dr > 0$$

for every $s \in [0, 1)$, and integrating in s over $[0, 1)$, we get

$$\begin{aligned} 0 < \int_0^1 \int_s^1 \int_0^{2\pi} h(r, \theta) d\theta \frac{r}{\sqrt{r^2 - s^2}} dr ds &= \int_0^1 \int_0^r \frac{r}{\sqrt{r^2 - s^2}} ds \int_0^{2\pi} h(r, \theta) d\theta dr \\ &= \frac{\pi}{2} \int_0^1 r \int_0^{2\pi} h(r, \theta) d\theta dr = \frac{\pi}{2} \int_{\Omega} h(x) dx = 0, \end{aligned}$$

which is a contradiction. □

According to Proposition 3.9 (Section 3.5), we have $\sup_{\chi_{\omega} \in \mathcal{U}_L} J(\chi_{\omega}) = \max_{a \in \bar{\mathcal{U}}_L} J(a) = \max_{a \in \bar{\mathcal{U}}_L} K(a)$, where K is defined by (54). For every $a \in \bar{\mathcal{U}}_L$ (which is a function of r and θ), setting $b(r) = 2\pi^{-1} \int_0^{2\pi} a(r, \theta) d\theta$, we obtain $\int_0^1 b(r)r dr = L/2$, and clearly

$$\max_{a \in \bar{\mathcal{U}}_L} K(a) = \max_{\substack{b \in L^{\infty}(0, 1; [0, 1]) \\ \int_0^1 b(r)r dr = L/2}} K_r(b),$$

where we have set

$$K_r(b) = \inf_{s \in [0, 1)} \frac{1}{\sqrt{1 - s^2}} \int_0^1 b(r) \frac{r}{\sqrt{r^2 - s^2}} dr$$

for every $b \in L^{\infty}(0, 1)$. It follows from Lemma 4.6 that the constant function $b^* = L$ is a maximizer of K_r , and $K_r(b^*) = L$.

Lemma 4.7. *The constant function $b^* = L$ is the unique maximizer of K_r over all functions $b \in L^{\infty}(0, 1; [0, 1])$ such that $\int_0^1 b(r)r dr = L/2$.*

¹⁰ There is a small difficulty here in applying Danskin's Theorem, due to the fact that the set $[0, 1)$ is not compact. This is easily overcome by applying the slightly more general version [5, Theorem D2] of Danskin's Theorem, noting that for $a = L$ every $s \in [0, 1)$ realizes the infimum in the definition of K .

Proof. Let b be a maximizer of K_r . Then, by definition,

$$\int_s^1 b(r) \frac{r}{\sqrt{r^2 - s^2}} dr \geq L\sqrt{1 - s^2} \tag{55}$$

for every $s \in [0, 1)$. Integrating the left-hand side of (55) in s over $[0, 1)$, using the Fubini Theorem and the fact that $\int_0^1 b(r)r dr = L/2$, one gets

$$\int_0^1 \int_s^1 b(r) \frac{r}{\sqrt{r^2 - s^2}} dr ds = L\pi/4.$$

Moreover, the integral in s over $[0, 1)$ of the right-hand side of (55) is $\int_0^1 L\sqrt{1 - s^2} ds = L\pi/4$. Hence both integrals are equal, and therefore the inequality in (55) is actually an equality, that is,

$$\frac{1}{\sqrt{1 - s^2}} \int_s^1 b(r) \frac{r}{\sqrt{r^2 - s^2}} dr = L \tag{56}$$

for every $s \in [0, 1)$. Now, since the linear mapping

$$L^\infty(0, 1) \rightarrow L^\infty(0, 1), \quad b \mapsto \left(s \mapsto \frac{1}{\sqrt{1 - s^2}} \int_s^1 b(r) \frac{r}{\sqrt{r^2 - s^2}} dr \right)$$

(which is, by the way, an Abel transform) is clearly one-to-one, and since $b^* = L$ is a solution of (56), it finally follows that $b = b^*$. \square

Coming back to the problem of maximizing K over $\bar{\mathcal{U}}_L$, the following result easily follows from the above lemma.

Lemma 4.8. *An element $a \in \bar{\mathcal{U}}_L$ is a maximizer of K if and only if $(2\pi)^{-1} \int_0^{2\pi} a(r, \theta) d\theta = L$ for almost every $r \in [0, 1]$.*

Note that if $a \in \bar{\mathcal{U}}_L$ is a maximizer of J then it must be a maximizer of K .

It follows from the above lemma, from Proposition 4.1 and from the proof of Proposition 3.9 that, for $L = 1/2$, the supremum of J over \mathcal{U}_L is reached for every subset ω of the form $\omega = \{(r, \theta) \in [0, 1] \times [0, 2\pi] \mid \theta \in \omega_\theta\}$, where ω_θ is any subset of $[0, 2\pi]$ such that ω_θ and its symmetric image $\omega'_\theta = 2\pi - \omega_\theta$ are complementary in $[0, 2\pi]$.

It remains to prove that if $L \neq 1/2$ then the supremum of J over \mathcal{U}_L is not reached. Assume otherwise that $L \neq 1/2$ and $\chi_\omega \in \mathcal{U}_L$ is a maximizer of J . Then in particular

$$\inf_{j \in \mathbb{N}, k \in \mathbb{N}^*} \int_0^1 \frac{1}{\pi} \int_0^{2\pi} \chi_\omega(r, \theta) \sin^2(j\theta) d\theta R_{jk}(r)^2 r dr = L.$$

Actually this equality holds as well on replacing the sine with a cosine, but we shall not use it. Writing $\sin^2(j\theta) = \frac{1}{2} - \frac{1}{2} \cos(2j\theta)$, and noting (from Lemma 4.8) that

$$\int_0^1 \frac{1}{2\pi} \int_0^{2\pi} \chi_\omega(r, \theta) d\theta R_{jk}(r)^2 r dr = L,$$

we infer that

$$\int_0^{2\pi} \int_0^1 \chi_\omega(r, \theta) R_{jk}(r)^2 r \, dr \cos(2j\theta) \, d\theta \leq 0$$

for all $j, k \in \mathbb{N}^*$. Making use of the fact that, for every $j \in \mathbb{N}$, the sequence of probability measures $r \mapsto R_{jk}(r)^2 r \, dr$ converges vaguely to 1 as $k \rightarrow \infty$, and applying the Portmanteau Theorem, we infer that

$$\int_0^{2\pi} \int_0^1 \chi_\omega(r, \theta) \, dr \cos(2j\theta) \, d\theta \leq 0$$

for every $j \in \mathbb{N}^*$. Applying again the Fourier arguments used in the proof of Proposition 4.1, it follows that the function $\theta \mapsto \int_0^1 \chi_\omega(r, \theta) \, dr$ must be continuous (since $L \neq 1/2$). The statement of the proposition now easily follows. \square

Conjectures. In view of the results above one could expect that when Ω is the unit N -dimensional hypercube, there exist a finite number of values of $L \in (0, 1)$ such that the supremum in (36) is reached. We do not know to what extent this conjecture can be formulated for generic domains Ω .

4.2. Spectral approximation

In this section, we consider a spectral truncation of the functional J defined by (26), setting

$$J_N(\chi_\omega) = \min_{1 \leq j \leq N} \int_\omega \phi_j(x)^2 \, dx \tag{57}$$

for every $N \in \mathbb{N}^*$ and every measurable subset ω of Ω , and we consider the spectral approximation of the problem (30),

$$\sup_{\chi_\omega \in \mathcal{U}_L} J_N(\chi_\omega). \tag{58}$$

As before, the functional J_N is naturally extended to $\overline{\mathcal{U}}_L$ by

$$J_N(a) = \min_{1 \leq j \leq N} \int_\Omega a(x) \phi_j(x)^2 \, dx$$

for every $a \in \overline{\mathcal{U}}_L$. We have the following result, establishing existence, uniqueness and Γ -convergence properties.

Theorem 4.9. (i) *For every measurable subset ω of Ω , the sequence $(J_N(\chi_\omega))_{N \in \mathbb{N}^*}$ is nonincreasing and converges to $J(\chi_\omega)$.*

(ii) *We have*

$$\lim_{N \rightarrow \infty} \max_{a \in \overline{\mathcal{U}}_L} J_N(a) = \max_{a \in \overline{\mathcal{U}}_L} J(a).$$

Moreover, if $(a^N)_{n \in \mathbb{N}^}$ is a sequence of maximizers of J_N in $\overline{\mathcal{U}}_L$, then, up to a subsequence, it converges to a maximizer of J in $\overline{\mathcal{U}}_L$ in the weak star topology of L^∞ .*

(iii) For every $N \in \mathbb{N}^*$, the problem (58) has a unique solution $\chi_{\omega^N} \in \mathcal{U}_L$. Moreover, ω^N is semianalytic¹¹ and thus it has a finite number of connected components.

Proof. For every measurable subset ω of Ω , the sequence $(J_N(\chi_{\omega}))_{N \in \mathbb{N}^*}$ is clearly non-increasing and thus convergent. Note that

$$J_N(\chi_{\omega}) = \inf \left\{ \sum_{j=1}^N \alpha_j \int_{\omega} \phi_j(x)^2 dx \mid \alpha_j \geq 0, \sum_{j=1}^N \alpha_j = 1 \right\},$$

$$J(\chi_{\omega}) = \inf \left\{ \sum_{j \in \mathbb{N}^*} \alpha_j \int_{\omega} \phi_j(x)^2 dx \mid \alpha_j \geq 0, \sum_{j \in \mathbb{N}^*} \alpha_j = 1 \right\}.$$

Hence, for every $(\alpha_j)_{j \in \mathbb{N}^*} \in \ell^1(\mathbb{R}^+)$, one has

$$\sum_{j=1}^N \alpha_j \int_{\omega} \phi_j(x)^2 dx \geq J_N(\chi_{\omega}) \sum_{j=1}^N \alpha_j$$

for all $N \in \mathbb{N}^*$, and letting $N \rightarrow \infty$ yields

$$\sum_{j \in \mathbb{N}^*} \alpha_j \int_{\omega} \phi_j(x)^2 dx \geq \lim_{N \rightarrow \infty} J_N(\omega) \sum_{j \in \mathbb{N}^*} \alpha_j,$$

and thus $\lim_{N \rightarrow \infty} J_N(\chi_{\omega}) \leq J(\chi_{\omega})$. This proves (i) since always $J_N(\chi_{\omega}) \geq J(\chi_{\omega})$.

Since J_N is upper semicontinuous (and even continuous) in the L^∞ weak star topology and since $\bar{\mathcal{U}}_L$ is compact in this topology, it follows that J_N has at least one maximizer $a^N \in \bar{\mathcal{U}}_L$. Let $\bar{a} \in \bar{\mathcal{U}}_L$ be a limit point of the sequence $(a^N)_{n \in \mathbb{N}^*}$ in the L^∞ weak star topology. Then, for every $p \leq N$,

$$\sup_{a \in \bar{\mathcal{U}}_L} J(a) \leq \sup_{a \in \bar{\mathcal{U}}_L} J_N(a) = J_N(a^N) \leq J_p(a^N),$$

and letting $N \rightarrow \infty$ yields

$$\sup_{a \in \bar{\mathcal{U}}_L} J(a) \leq \lim_{N \rightarrow \infty} J_N(a^N) \leq \lim_{N \rightarrow \infty} J_p(a^N) = J_p(\bar{a})$$

for every $p \in \mathbb{N}^*$. Since $J_p(\bar{a})$ tends to $J(\bar{a}) \leq \sup_{a \in \bar{\mathcal{U}}_L} J(a)$ as $p \rightarrow \infty$, it follows that \bar{a} is a maximizer of J in $\bar{\mathcal{U}}_L$, proving (ii).

¹¹ A subset ω of a real analytic finite-dimensional manifold M is said to be *semianalytic* if it can be written in terms of equalities and inequalities of analytic functions, that is, for every $x \in \omega$, there exists a neighborhood U of x in M and $2pq$ analytic functions g_{ij}, h_{ij} (with $1 \leq i \leq p$ and $1 \leq j \leq q$) such that

$$\omega \cap U = \bigcup_{i=1}^p \{y \in U \mid g_{ij}(y) = 0 \text{ and } h_{ij}(y) > 0, j = 1, \dots, q\}.$$

We recall that such semianalytic (and more generally, subanalytic) subsets enjoy nice properties, for instance they are stratifiable in the sense of Whitney (see [21, 27]).

To prove (iii), let us now show that J_N has a unique maximizer $a^N \in \bar{\mathcal{U}}_L$, which is moreover a characteristic function. We define the simplex set

$$\mathcal{A}_N = \left\{ \alpha = (\alpha_j)_{1 \leq j \leq N} \mid \alpha_j \geq 0, \sum_{j=1}^N \alpha_j = 1 \right\}.$$

Note that

$$\min_{1 \leq j \leq N} \int_{\Omega} a(x) \phi_j(x)^2 dx = \min_{\alpha \in \mathcal{A}_N} \int_{\Omega} a(x) \sum_{j=1}^N \alpha_j \phi_j(x)^2 dx$$

for every $a \in \bar{\mathcal{U}}_L$. It follows from Sion’s minimax theorem (see [60]) that there exists $\alpha^N \in \mathcal{A}_N$ such that (a^N, α^N) is a saddle point of the bilinear functional $(a, \alpha) \mapsto \int_{\Omega} a(x) \sum_{j=1}^N \alpha_j \phi_j(x)^2 dx$ defined on $\bar{\mathcal{U}}_L \times \mathcal{A}_N$, and

$$\begin{aligned} \max_{a \in \bar{\mathcal{U}}_L} \min_{\alpha \in \mathcal{A}_N} \int_{\Omega} a(x) \sum_{j=1}^N \alpha_j \phi_j(x)^2 dx &= \min_{\alpha \in \mathcal{A}_N} \max_{a \in \bar{\mathcal{U}}_L} \int_{\Omega} a(x) \sum_{j=1}^N \alpha_j \phi_j(x)^2 dx \\ &= \max_{a \in \bar{\mathcal{U}}_L} \int_{\Omega} a(x) \sum_{j=1}^N \alpha_j^N \phi_j(x)^2 dx = \int_{\Omega} a^N(x) \sum_{j=1}^N \alpha_j^N \phi_j(x)^2 dx. \end{aligned} \tag{59}$$

Note that the eigenfunctions ϕ_j are analytic in Ω (by analytic hypoellipticity, see [46]). We claim that the (analytic) function $x \mapsto \sum_{j=1}^N \alpha_j^N \phi_j(x)^2$ is never constant on any subset of positive measure. Indeed, otherwise this function would be constant on Ω (by analyticity). We would then infer from the Dirichlet boundary conditions that the function $x \mapsto \sum_{j=1}^N \alpha_j^N \phi_j(x)^2$ vanishes on $\bar{\Omega}$, which is a contradiction.

It follows from this fact and from (59) that there exists $\lambda^N > 0$ such that $a^N(x) = 1$ if $\sum_{j=1}^N \alpha_j^N \phi_j(x)^2 \geq \lambda^N$, and $a^N(x) = 0$ otherwise, for almost every $x \in \Omega$. Hence there exists $\omega^N \in \mathcal{U}_L$ such that $a^N = \chi_{\omega^N}$. By analyticity, ω^N is semianalytic (see footnote 11) and thus has a finite number of connected components. \square

Remark 4.10. Note that Theorem 4.9(iii) can be seen as a generalization of [23, Theorem 3.1] and [48, Theorem 3.1]. We have also provided a shorter proof.

Remark 4.11. In the 1D case of $\Omega = (0, \pi)$ with Dirichlet boundary conditions, it can be proved that the optimal set ω_N maximizing J_N is the union of N intervals concentrating around equidistant points and that ω_N is actually the worst possible subset for the problem of maximizing J_{N+1} . This is the *spillover phenomenon*, observed in [23] and rigorously proved in [50].

We provide hereafter several numerical simulations based on the modal approximation described previously, which permit us to exhibit some maximizing sequences of sets.

Assume first that $\Omega = (0, \pi)^2$, with the normalized eigenfunctions of the Dirichlet–Laplacian given by $\phi_{j,k}(x_1, x_2) = (2/\pi) \sin(jx_1) \sin(kx_2)$ for all $(x_1, x_2) \in (0, \pi)^2$.

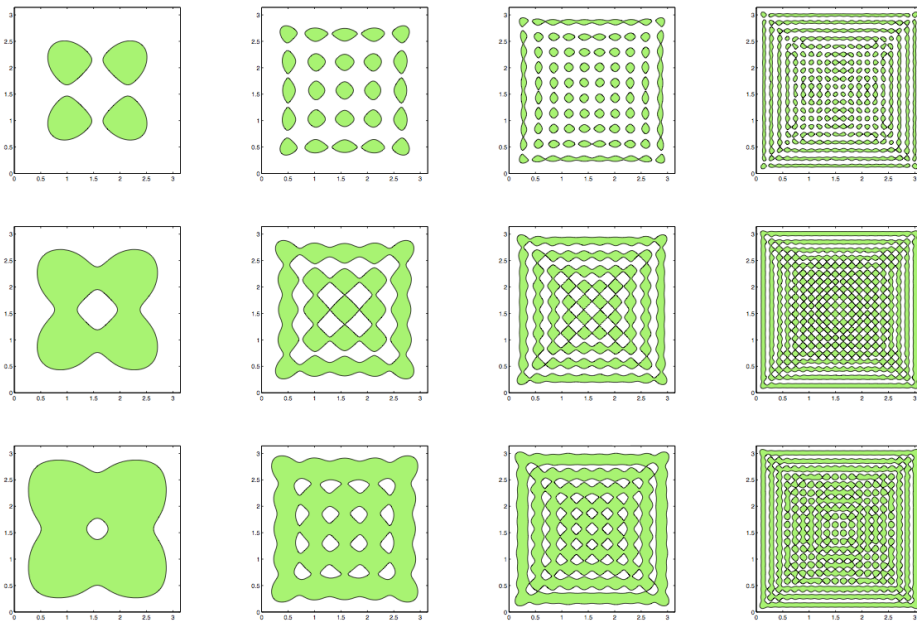


Fig. 3. $\Omega = (0, \pi)^2$, with Dirichlet boundary conditions. Row 1: $L = 0.2$; row 2: $L = 0.4$; row 3: $L = 0.6$. From left to right: $N = 2$ (4 eigenmodes), $N = 5$ (25 eigenmodes), $N = 10$ (100 eigenmodes), $N = 20$ (400 eigenmodes). The optimal domain is in green.

Let $N \in \mathbb{N}^*$. We use an interior point line search filter method to solve the spectral approximation $\sup_{\chi_\omega \in \mathcal{U}_L} J_N(\chi_\omega)$ of the problem (30), where

$$J_N(\chi_\omega) = \min_{1 \leq j, k \leq N} \int_0^\pi \int_0^\pi \chi_\omega(x_1, x_2) \phi_{j,k}(x_1, x_2)^2 dx_1 dx_2.$$

Some results are provided in Figure 3.

Assume now that $\Omega = \{x \in \mathbb{R}^2 \mid \|x\| < 1\}$, the unit Euclidean disk in \mathbb{R}^2 , with the normalized eigenfunctions of the Dirichlet–Laplacian given as before in terms of Bessel functions by (48). In Proposition 3.9, a no-gap result has been stated in this case. Some simulations are provided in Figure 4. We observe that optimal domains are radially symmetric. This is actually an immediate consequence of the uniqueness of a maximizer for the modal approximation problem stated in Theorem 4.9 and of the fact that Ω is itself radially symmetric.

4.3. A first remedy: other classes of admissible domains

According to Proposition 4.1, we know that, in the one-dimensional case, the problem (30) is ill-posed in the sense that it has no solution except for $L = 1/2$. In larger dimension, we expect a similar conclusion. One of the reasons is that the set \mathcal{U}_L defined by

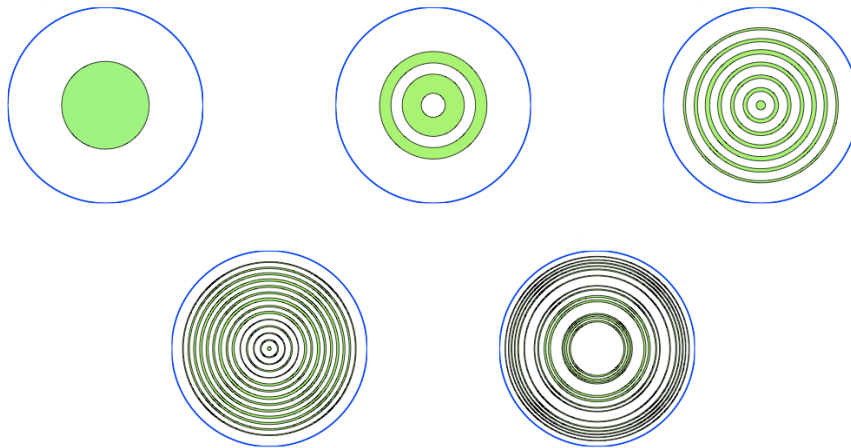


Fig. 4. $\Omega = \{x \in \mathbb{R}^2 \mid \|x\| \leq 1\}$, with Dirichlet boundary conditions, and $L = 0.2$. Optimal domain for $N = 1$ (1 eigenmode), $N = 2$ (4 eigenmodes), $N = 5$ (25 eigenmodes), $N = 10$ (100 eigenmodes) and $N = 20$ (400 eigenmodes).

(29) is not compact for the usual topologies, as discussed in Remark 3.1. To overcome this difficulty, one can define a new class of admissible sets, $\mathcal{V}_L \subset \mathcal{U}_L$, enjoying sufficient compactness properties, and to replace the problem (30) with

$$\sup_{\chi_\omega \in \mathcal{V}_L} J(\chi_\omega). \tag{60}$$

Of course, now the extremal value is not necessarily the same since the class of admissible domains has been further restricted.

To ensure the existence of a maximizer χ_{ω^*} of (60), it suffices to endow \mathcal{V}_L with a topology, finer than the weak star topology of L^∞ , for which \mathcal{V}_L is compact. Of course in this case,

$$J(\chi_{\omega^*}) = \max_{\chi_\omega \in \mathcal{V}_L} J(\chi_\omega) \leq \sup_{\chi_\omega \in \mathcal{U}_L} J(\chi_\omega).$$

This extra compactness property can be guaranteed by, for instance, considering some $\alpha > 0$, and then any of the following possible choices:

$$\mathcal{V}_L = \{\chi_\omega \in \mathcal{U}_L \mid P_\Omega(\omega) \leq \alpha\}, \tag{61}$$

where $P_\Omega(\omega)$ is the relative perimeter of ω with respect to Ω ;

$$\mathcal{V}_L = \{\chi_\omega \in \mathcal{U}_L \mid \|\chi_\omega\|_{\text{BV}(\Omega)} \leq \alpha\}, \tag{62}$$

where $\|\cdot\|_{\text{BV}(\Omega)}$ is the $\text{BV}(\Omega)$ -norm of functions of bounded variation on Ω (see for example [2]); or

$$\mathcal{V}_L = \{\chi_\omega \in \mathcal{U}_L \mid \omega \text{ satisfies the } 1/\alpha\text{-cone property}\} \tag{63}$$

(see Section 3.4, footnote 6). Naturally, the optimal set then depends on the bound α under consideration, and numerical simulations (not reported here) show that, as $\alpha \rightarrow \infty$, the family of optimal sets behaves as the maximizing sequence built in Section 4.2, so that, in particular, the number of connected components grows as α is increasing.

The point of view that we adopted in this article is however not to restrict the classes of possible subsets ω , but rather to discuss the physical relevance of the criterion under consideration. In the next subsection we consider a modification of the spectral criterion, based on physical considerations.

4.4. A second remedy: weighted observability inequalities

First, observe that, in the observability inequality (11), by definition, all modes (in the spectral expansion) have the same weight. It is however expected (and finally, observed) that the problem is difficult owing to the increasing complexity of the geometry of high frequency eigenfunctions. Moreover, measuring lower frequencies is in some sense physically different from measuring high frequencies. Therefore, it seems relevant to introduce a weighted version of the observability inequality (11), by considering the inequality

$$C_{T,\sigma}^{(W)}(\chi_\omega)(\|y^0, y^1\|_{L^2 \times H^{-1}}^2 + \sigma \|y^0\|_{H^{-1}}^2) \leq \int_0^T \int_\omega |y(t, x)|^2 dx dt, \quad (64)$$

where $\sigma \geq 0$ is some weight.

This inequality holds true under the GCC. Since the norm used at the left-hand side is stronger than the one of (11), it follows that $C_{T,\sigma}^{(W)}(\chi_\omega) \leq C_T^{(W)}(\chi_\omega)$ for every $\sigma \geq 0$.

From this weighted observability inequality (64), we can define as well the randomized observability constant and the time asymptotic observability constant (we do not provide the details), and we come up with the following result, which is the weighted version of Theorem 2.2 and of Corollary 2.7.

Proposition 4.12. *For every measurable subset ω of Ω , we have $2C_{T,\sigma,\text{rand}}^{(W)}(\chi_\omega) = TJ_\sigma(\chi_\omega)$, and moreover if every eigenvalue of Δ is simple, then $2C_{\infty,\sigma}^{(W)}(\chi_\omega) = J_\sigma(\chi_\omega)$, where*

$$J_\sigma(\chi_\omega) = \inf_{j \in \mathbb{N}^*} \frac{\lambda_j^2}{\sigma + \lambda_j^2} \int_\omega \phi_j(x)^2 dx. \quad (65)$$

It is seen from that proposition that the (initial data or time) averaging procedures do not lead to the functional J defined by (26) but to the slightly different (weighted) functional J_σ defined by (65). Let us now investigate the problem

$$\sup_{\chi_\omega \in \mathcal{U}_L} J_\sigma(\chi_\omega). \quad (66)$$

We will see that the study of (65) differs significantly from the one considered previously. Note that the sequence $(\lambda_j^2/(\sigma + \lambda_j^2))_{j \in \mathbb{N}^*}$ is increasing, and $0 < \lambda_j^2/(\sigma + \lambda_j^2) < 1$ for every $j \in \mathbb{N}^*$.

As in Section 3.1, the convexified version of this problem is

$$\sup_{a \in \bar{\mathcal{U}}_L} J_\sigma(a), \tag{67}$$

where

$$J_\sigma(a) = \inf_{j \in \mathbb{N}^*} \frac{\lambda_j^2}{\sigma + \lambda_j^2} \int_{\Omega} a(x) \phi_j(x)^2 dx. \tag{68}$$

As in Sections 3.1 and 3.2, under the assumption that there exists a subsequence of $(\phi_j^2)_{j \in \mathbb{N}^*}$ converging to $|\Omega|^{-1}$ in the weak star L^∞ topology (L^∞ -WQE property), the problem (67) has at least one solution, the supremum is equal to L , and it is reached for the constant function $a = L$.

We will next establish a no-gap result, similar to Theorem 3.5, but only valuable for nonsmall values of L . Actually, we will show that the present situation differs significantly from the previous one, in the sense that, if $\lambda_1^2/(\sigma + \lambda_1^2) < L < 1$, then the high frequency modes do not play any role in the problem (66). Before coming to that result, let us first define a truncated version of (66). For every $N \in \mathbb{N}^*$, we define

$$J_{\sigma,N}(a) = \inf_{1 \leq j \leq N} \frac{\lambda_j^2}{\sigma + \lambda_j^2} \int_{\Omega} a(x) \phi_j(x)^2 dx. \tag{69}$$

An immediate adaptation of the proof of Theorem 4.9 yields the following result.

Proposition 4.13. *For every $N \in \mathbb{N}^*$, the problem*

$$\sup_{a \in \bar{\mathcal{U}}_L} J_{\sigma,N}(a) \tag{70}$$

has a unique solution a^N , which is moreover the characteristic function of a set ω^N . Furthermore, ω^N is semianalytic (see footnote 11), and thus it has a finite number of connected components.

The main result of this section is the following.

Theorem 4.14. *Assume that the QUE on the base and uniform L^∞ -boundedness properties are satisfied for the selected Hilbert basis $(\phi_j)_{j \in \mathbb{N}^*}$ of eigenfunctions. Let $L \in (\lambda_1^2/(\sigma + \lambda_1^2), 1)$. Then there exists $N_0 \in \mathbb{N}^*$ such that*

$$\max_{\chi_\omega \in \bar{\mathcal{U}}_L} J_\sigma(\chi_\omega) = \max_{\chi_\omega \in \bar{\mathcal{U}}_L} J_{\sigma,N}(\chi_\omega) \leq \frac{\lambda_1^2}{\sigma + \lambda_1^2} < L \tag{71}$$

for every $N \geq N_0$. In particular, the problem (66) has a unique solution $\chi_{\omega^{N_0}}$, and moreover the set ω^{N_0} is semianalytic and has a finite number of connected components.

Proof. Using the same arguments as in Lemma 3.3, it is clear that the problem (67) has at least one solution, denoted by a^∞ . Let us first prove that there exists $N_0 \in \mathbb{N}^*$ such that $J_\sigma(a^\infty) = J_{\sigma,N_0}(a^\infty)$. Let $\varepsilon \in (0, L - \lambda_1^2/(\sigma + \lambda_1^2))$. It follows from the L^∞ -QUE

property that there exists $N_0 \in \mathbb{N}^*$ such that

$$\frac{\lambda_j^2}{\sigma + \lambda_j^2} \int_{\Omega} a^\infty(x) \phi_j(x)^2 dx \geq L - \varepsilon \tag{72}$$

for every $j > N_0$. Therefore,

$$\begin{aligned} J_\sigma(a^\infty) &= \inf_{j \in \mathbb{N}^*} \frac{\lambda_j^2}{\sigma + \lambda_j^2} \int_{\Omega} a^\infty(x) \phi_j(x)^2 dx \\ &= \min \left(\inf_{1 \leq j \leq N_0} \frac{\lambda_j^2}{\sigma + \lambda_j^2} \int_{\Omega} a^\infty(x) \phi_j(x)^2 dx, \inf_{j > N_0} \frac{\lambda_j^2}{\sigma + \lambda_j^2} \int_{\Omega} a^\infty(x) \phi_j(x)^2 dx \right) \\ &\geq \min(J_{\sigma, N_0}(a^\infty), L - \varepsilon) = J_{\sigma, N_0}(a^\infty), \end{aligned}$$

since $L - \varepsilon > \lambda_1^2/(\sigma + \lambda_1^2)$ and $J_{\sigma, N_0}(a^\infty) \leq \lambda_1^2/(\sigma + \lambda_1^2)$. It follows that $J_\sigma(a^\infty) = J_{\sigma, N_0}(a^\infty)$.

Let us now prove that $J_\sigma(a^\infty) = J_{\sigma, N_0}(a^{N_0})$, where a^{N_0} is the unique maximizer of J_{σ, N_0} (see Proposition 4.13). By definition of a maximizer, $J_\sigma(a^\infty) = J_{\sigma, N_0}(a^\infty) \leq J_{\sigma, N_0}(a^{N_0})$. For contradiction, assume that $J_{\sigma, N_0}(a^\infty) < J_{\sigma, N_0}(a^{N_0})$. Let us then design an admissible perturbation $a_t \in \bar{U}_L$ of a^∞ such that $J_\sigma(a_t) > J_\sigma(a^\infty)$, which will contradict the optimality of a^∞ . For every $t \in [0, 1]$, set $a_t = a^\infty + t(a^{N_0} - a^\infty)$. Since J_{σ, N_0} is concave, one gets

$$J_{\sigma, N_0}(a_t) \geq (1 - t)J_{\sigma, N_0}(a^\infty) + tJ_{\sigma, N_0}(a^{N_0}) > J_{\sigma, N_0}(a^\infty)$$

for every $t \in (0, 1]$, which means that

$$\inf_{1 \leq j \leq N_0} \frac{\lambda_j^2}{\sigma + \lambda_j^2} \int_{\Omega} a_t(x) \phi_j(x)^2 dx > \inf_{1 \leq j \leq N_0} \frac{\lambda_j^2}{\sigma + \lambda_j^2} \int_{\Omega} a^\infty(x) \phi_j(x)^2 dx \geq J_\sigma(a^\infty) \tag{73}$$

for every $t \in (0, 1]$. Moreover, since $a^{N_0}(x) - a^\infty(x) \in (-2, 2)$ for almost every $x \in \Omega$, it follows from (72) that

$$\begin{aligned} &\frac{\lambda_j^2}{\sigma + \lambda_j^2} \int_{\Omega} a_t(x) \phi_j(x)^2 dx \\ &= \frac{\lambda_j^2}{\sigma + \lambda_j^2} \int_{\Omega} a^\infty(x) \phi_j(x)^2 dx + t \frac{\lambda_j^2}{\sigma + \lambda_j^2} \int_{\Omega} (a^{N_0}(x) - a^\infty(x)) \phi_j(x)^2 dx \geq L - \varepsilon - 2t \end{aligned}$$

for every $j \geq N_0$. Let us choose t such that $0 < t < \frac{1}{2}(L - \varepsilon - \lambda_1^2/(\sigma + \lambda_1^2))$, so that the previous inequality yields

$$\frac{\lambda_j^2}{\sigma + \lambda_j^2} \int_{\Omega} a_t(x) \phi_j(x)^2 dx > \frac{\lambda_1^2}{\sigma + \lambda_1^2} \geq \frac{\lambda_1^2}{\sigma + \lambda_1^2} \int_{\Omega} a^\infty(x) \phi_1(x)^2 dx \geq J_\sigma(a^\infty) \tag{74}$$

for every $j \geq N_0$. Combining the estimate (73) on the low modes with the estimate (74) on the high modes, we conclude that

$$J_\sigma(a_t) = \inf_{j \in \mathbb{N}^*} \frac{\lambda_j^2}{\sigma + \lambda_j^2} \int_{\Omega} a_t(x) \phi_j(x)^2 dx > J_\sigma(a^\infty),$$

which contradicts the optimality of a^∞ .

Therefore $J_{\sigma, N_0}(a^\infty) = J_\sigma(a^\infty) = J_{\sigma, N_0}(a^{N_0})$, and the result follows. \square

Remark 4.15. Under the assumptions of the theorem, there is no gap between the problem (66) and its convexified formulation (67), as before. But, in contrast to the previous results, here there always exists a maximizer in the class of characteristic functions whenever L is larger than a threshold value, and moreover, this optimal set can be computed from a truncated formulation (69) for a certain value of N . In other words, the maximizing sequence $(\chi_{\omega^N})_{N \in \mathbb{N}^*}$ resulting from Proposition 4.13 is stationary. Here, the high modes play no role, whereas in the previous results all modes had the same impact. This result is due to adding the weight $\sigma \geq 0$ on the left hand-side of the observability inequalities. It can be noted that the threshold value $\lambda_1^2/(\sigma + \lambda_1^2)$, accounting for the existence of an optimal set, becomes smaller when σ increases. This is in accordance with physical intuition.

Remark 4.16. Here, if L is not too small, then there exists an optimal set (with nice regularity properties) realizing the largest possible time asymptotic and randomized observability constants. The optimal value of these constants is known to be less than L but its exact value is unknown. It is related to solving a finite-dimensional numerical optimization problem.

Remark 4.17. In the case where $L \leq \lambda_1^2/(\sigma + \lambda_1^2)$, we do not know whether there is a gap or not between the problem (66) and its convexified formulation (67). Adapting shrewdly the proof of Theorem 3.5 does not seem to allow one to derive a no-gap result. Nevertheless, using these arguments we can prove that $\sup_{\chi_\omega \in \mathcal{U}_L} J_\sigma(\chi_\omega) \geq \frac{\lambda_1^2}{\sigma + \lambda_1^2} L$.

Remark 4.18. We formulate the following two open questions.

- Under the assumptions of Theorem 4.14, does the conclusion hold true for every $L \in (0, 1)$?
- Does the statement of Theorem 4.14 still hold true under weaker ergodicity assumptions, for instance is it possible to weaken QUE to WQE (defined in footnote 3)?

Remark 4.19. The QUE assumption in Theorem 4.14 is very strong, as already discussed. It is true in the 1D case but, up to now, no example of a multi-dimensional domain satisfying such an assumption is known. Anyway, we are able to prove that the conclusion of Theorem 4.14 holds true in $\Omega = (0, \pi)^n$ as well with the usual basis made of products of sine functions.

Proposition 4.20. Assume that $\Omega = (0, \pi)^n$, with the normalized eigenfunctions of Δ given by $\phi_{j_1 \dots j_n}(x_1, \dots, x_n) = (2/\pi)^{n/2} \prod_{k=1}^n \sin(j_k x_k)$ for all $(j_1, \dots, j_n) \in (\mathbb{N}^*)^n$ and $x_i \in (0, \pi)$. There exist $L_0 \in (0, 1)$ and $N_0 \in \mathbb{N}^*$ such that

$$\max_{\chi_\omega \in \mathcal{U}_L} J_\sigma(\chi_\omega) = \max_{\chi_\omega \in \mathcal{U}_L} J_{\sigma, N}(\chi_\omega) \tag{75}$$

for all $L \in [L_0, 1)$ and $N \geq N_0$.

Proof. The proof follows the same lines as the one of Theorem 4.14. Nevertheless, the inequality (72) may not hold since QUE on the base is not satisfied here. In the specific case under consideration, (72) is replaced with the following assertion: for any $\varepsilon > 0$, there exist $N_0 \in \mathbb{N}^*$ and $L_0 \in (0, 1)$ such that

$$\frac{\lambda_{j_1 \dots j_n}^2}{\sigma + \lambda_{j_1 \dots j_n}^2} \int_{(0, \pi)^n} a(x) \phi_{j_1 \dots j_n}(x)^2 dx \geq L - \varepsilon$$

for all $a \in \bar{U}_L$, all $L \in [L_0, 1)$ and all $(j_1, \dots, j_n) \in (\mathbb{N}^*)^n$ such that $\sum_{k=1}^n j_k \geq N_0$. It then suffices to apply it to $a = a^\infty$, where a^∞ denotes a solution to the problem (67). This assertion indeed follows from the following lemma.

Lemma 4.21. Let $M > 0$ and $a \in L^\infty((0, \pi)^n, [0, M])$. Then

$$\inf_{(j_1, \dots, j_n) \in (\mathbb{N}^*)^n} \int_{(0, \pi)^n} a(x) \phi_{j_1 \dots j_n}(x)^2 dx \geq \frac{M}{\pi} F^{[n]} \left(\frac{\int_{(0, \pi)^n} a(x) dx}{M \pi^{n-1}} \right), \tag{76}$$

where $F(x) = x - \sin x$ for every $x \in [0, \pi]$ and $F^{[n]} = F \circ \dots \circ F$ (n times).

Proof. We are going to prove this lemma by induction on n . Let us first establish (76) for $n = 1$.

Let j be fixed. Clearly, the minimum of the functional $a \in L^\infty((0, \pi), [0, M]) \mapsto \int_0^\pi a(x) \sin^2(jx) dx$ is reached at $a = M\chi_\omega$, with

$$\omega = \left(0, \frac{|\omega|}{2j}\right) \cup \bigcup_{k=1}^{j-1} \left(\frac{k}{j} - \frac{|\omega|}{2j}, \frac{k}{j} + \frac{|\omega|}{2j}\right) \cup \left(1 - \frac{|\omega|}{2j}, \pi\right).$$

It is remarkable that the value of the functional over this set does not depend on j . Indeed,

$$\int_\omega \sin^2(j\pi x) dx = 2j \int_0^{|\omega|/2j} \sin^2 jx dx = 2 \int_0^{|\omega|/2} \sin^2 u du = \frac{1}{2}(|\omega| - \sin(|\omega|)).$$

Since $a = M\chi_\omega$, we have $\int_0^\pi a(x) dx = M|\omega|$, and we conclude that

$$\frac{2}{\pi} \int_0^\pi a(x) \sin^2(jx) dx \geq \frac{1}{\pi} \int_0^\pi a(x) dx - \frac{M}{\pi} \sin \frac{\int_0^\pi a(x) dx}{M} = \frac{M}{\pi} F \left(\frac{\int_0^\pi a(x) dx}{M} \right).$$

For $n \geq 2$, we prove the general formula (76) by induction on n . Let us assume that (76) has been established for integers $k \leq n - 1$, and let us prove it for $k = n$. In

the reasoning below, we use the notation $x' = (x_2, \dots, x_n)$, and thus $x = (x_1, x_n)$. Let $a \in L^\infty((0, \pi)^n, [0, M])$ be an arbitrary function. Using the Fubini Theorem, we have

$$\left(\frac{2}{\pi}\right)^n \int_{(0,\pi)^n} a(x) \prod_{k=1}^n \sin^2(j_k x_k) dx = \left(\frac{2}{\pi}\right)^{n-1} \int_{(0,\pi)^{n-1}} a_1(x') \prod_{k=2}^n \sin^2(j_k x_k) dx',$$

with $a_1(x') = (2/\pi) \int_0^\pi a(x_1, x') \sin^2(j_1 x_1) dx_1$. Since a takes its values in $[0, M]$, we see that $a_1 \in L^\infty((0, \pi), [0, M])$. From the induction assumption (formula (76) with $n - 1$), it follows that

$$\left(\frac{2}{\pi}\right)^n \int_{(0,\pi)^n} a(x) \prod_{k=1}^n \sin^2(j_k x_k) dx \geq \frac{M}{\pi} F^{[n-1]} \left(\frac{1}{M\pi^{n-2}} \int_{(0,\pi)^{n-1}} a_1(x') dx' \right). \tag{77}$$

Note that $0 \leq \frac{1}{M\pi^{n-2}} \int_{(0,\pi)^{n-1}} a_1(x') dx' \leq \pi$. Now, using the Fubini Theorem, we have

$$\int_{(0,\pi)^{n-1}} a_1(x') dx' = \frac{2}{\pi} \int_{(0,\pi)^n} a(x) \sin^2(j_1 x_1) dx = \frac{2}{\pi} \int_0^\pi b_1(x_1) \sin^2(j_1 x_1) dx_1,$$

with $b_1(x_1) = \int_{(0,\pi)^{n-1}} a(x_1, x') dx'$. Since a takes its values in $[0, M]$, it follows that $b_1 \in L^\infty((0, \pi), [0, M\pi^{n-1}])$. Therefore, applying (76) with $n = 1$, we get

$$\frac{2}{\pi} \int_0^\pi b_1(x_1) \sin^2(j_1 x_1) dx_1 \geq M\pi^{n-2} F \left(\frac{\int_0^\pi b_1(x_1) dx_1}{M\pi^{n-1}} \right),$$

and so

$$\int_{(0,\pi)^{n-1}} a_1(x') dx' \geq M\pi^{n-2} F \left(\frac{\int_{(0,\pi)^n} a(x) dx}{M\pi^{n-1}} \right). \tag{78}$$

Again, $0 \leq \frac{1}{M\pi^{n-1}} \int_{(0,\pi)^n} a(x) dx \leq \pi$. Since $F : [0, \pi] \rightarrow [0, \pi]$ is an increasing function, we infer (76) from (77) and (78). \square

Let $\varepsilon > 0$. Applying Lemma 4.21 with $M = 1$ to $a^\infty \in \bar{U}_L$, we get the existence of L_0 such that for any $L \geq L_0$ and any $(j_1, \dots, j_n) \in (\mathbb{N}^*)^n$,

$$\int_{(0,\pi)^n} a^\infty(x) \phi_{j_1 \dots j_n}(x)^2 dx \geq \frac{1}{\pi} F^{[n]}(L\pi) \geq 1 - \varepsilon,$$

since $F^{[n]}(L\pi) \rightarrow \pi$ as $L \rightarrow 1$. Moreover, since $\lambda_{j_1 \dots j_n}^2 / (\sigma + \lambda_{j_1 \dots j_n}^2) \rightarrow 1$ as $\sum_{k=1}^n j_k \rightarrow \infty$, there exists $N_0 \in \mathbb{N}^*$ such that $\lambda_{j_1 \dots j_n}^2 / (\sigma + \lambda_{j_1 \dots j_n}^2) \geq 1 - \varepsilon$ as long as $\sum_{k=1}^n j_k \geq N_0$. Therefore,

$$\frac{\lambda_{j_1 \dots j_n}^2}{\sigma + \lambda_{j_1 \dots j_n}^2} \int_{(0,\pi)^n} a^\infty(x) \phi_{j_1 \dots j_n}(x)^2 dx \geq 1 - 2\varepsilon \geq L - 2\varepsilon$$

as long as $\sum_{k=1}^n j_k \geq N_0$, and the conclusion follows. \square

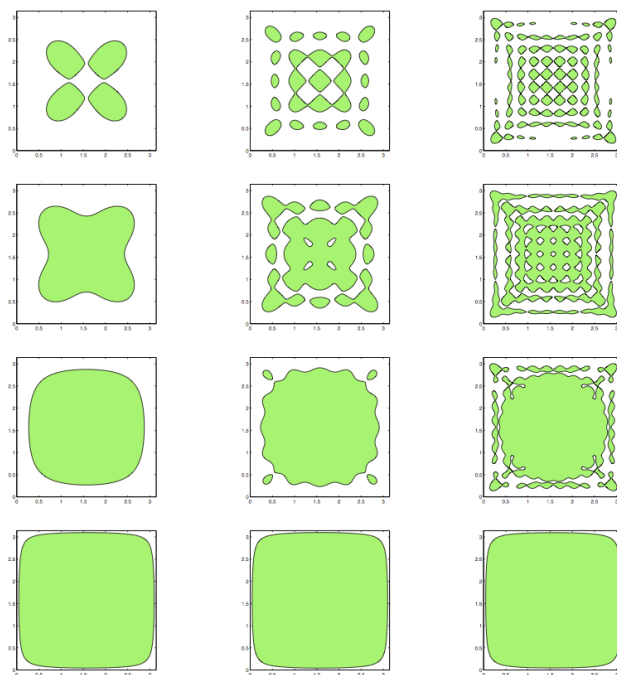


Fig. 5. $\Omega = (0, \pi)^2$, with Dirichlet boundary conditions. Row 1: $L = 0.2$; row 2: $L = 0.4$; row 3: $L = 0.6$; row 4: $L = 0.9$. From left to right: $N = 2$ (4 eigenmodes), $N = 5$ (25 eigenmodes), $N = 10$ (100 eigenmodes).

We end this section by providing several numerical simulations based on the modal approximation of this problem for the Euclidean square $\Omega = (0, \pi)^2$. Note that we are then in the framework of Remark 4.19, and hence the conclusion of Proposition 4.20 holds true. As in Section 4.2, we use an interior point line search filter method to solve the spectral approximation of the problem $\sup_{\chi_\omega \in \mathcal{U}_L} J_{N,\sigma}(\chi_\omega)$, with $\sigma = 1$. Some numerical simulations are provided in Figure 5, where the optimal domains are represented for $L \in \{0.2, 0.4, 0.6, 0.9\}$ (by row). In the first three cases, the number of connected components of the optimal set seems to increase with N . On the last row ($L = 0.9$), the numerical results illustrate the conclusion of Proposition 4.20, showing clear evidence of the stationarity feature of the maximizing sequence proved in this proposition.

5. Generalization to wave and Schrödinger equations on manifolds with various boundary conditions

In this section we show how all the previous results can be generalized to wave and Schrödinger equations posed on any bounded connected subset of a Riemannian manifold, with various possible boundary conditions. For each step of our analysis we explain what are the modifications that have to be taken into account.

General framework. Let (M, g) be a smooth n -dimensional Riemannian manifold, $n \geq 1$. Let T be a positive real number and Ω be an open bounded connected subset of M . We consider both the wave equation

$$\partial_{tt}y = \Delta_g y, \tag{79}$$

and the Schrödinger equation

$$i \partial_t y = \Delta_g y, \tag{80}$$

in $(0, T) \times \Omega$. Here, Δ_g denotes the usual Laplace–Beltrami operator on M for the metric g . If the boundary $\partial\Omega$ of Ω is nonempty, then we consider boundary conditions

$$By = 0 \quad \text{on } (0, T) \times \partial\Omega, \tag{81}$$

where B can be either:

- the usual Dirichlet trace operator, $By = y|_{\partial\Omega}$, or
- Neumann, $By = \frac{\partial y}{\partial n}|_{\partial\Omega}$, where $\frac{\partial}{\partial n}$ is the outward normal derivative on the boundary $\partial\Omega$, or
- mixed Dirichlet–Neumann, $By = \chi_{\Gamma_0} y|_{\partial\Omega} + \chi_{\Gamma_1} \frac{\partial y}{\partial n}|_{\partial\Omega}$, where $\partial\Omega = \Gamma_0 \cup \Gamma_1$ with $\Gamma_0 \cap \Gamma_1 = \emptyset$, and χ_{Γ_i} is the characteristic function of Γ_i , $i = 0, 1$, or
- Robin, $By = \frac{\partial y}{\partial n}|_{\partial\Omega} + \beta y|_{\partial\Omega}$, where β is a nonnegative bounded measurable function defined on $\partial\Omega$ such that $\int_{\partial\Omega} \beta > 0$.

Our study encompasses the case where $\partial\Omega = \emptyset$: in this case, (81) is unnecessary and Ω is a compact connected n -dimensional Riemannian manifold. The canonical Riemannian volume on M is denoted by V_g , inducing the canonical measure dV_g . Measurable sets¹² are considered with respect to the measure dV_g .

In the boundaryless and in the Neumann cases, the Laplace–Beltrami operator is not invertible on $L^2(\Omega, \mathbb{C})$ but is invertible in

$$L_0^2(\Omega, \mathbb{C}) = \left\{ y \in L^2(\Omega, \mathbb{C}) \mid \int_{\Omega} y(x) dx = 0 \right\}.$$

In what follows, the notation X stands for $L_0^2(\Omega, \mathbb{C})$ in the boundaryless and in the Neumann cases, and for $L^2(\Omega, \mathbb{C})$ otherwise. We denote by $A = -\Delta_g$ the Laplace operator defined on $D(A) = \{y \in X \mid Ay \in X \text{ and } By = 0\}$ with one of the above boundary conditions whenever $\partial\Omega \neq \emptyset$. Note that A is a selfadjoint positive operator.

For all $(y^0, y^1) \in D(A^{1/2}) \times X$, there exists a unique solution y of the wave equation (79) in the space $C^0(0, T; D(A^{1/2})) \cap C^1(0, T; X)$ such that $y(0, \cdot) = y^0(\cdot)$ and $\partial_t y(0, \cdot) = y^1(\cdot)$.

Let ω be an arbitrary measurable subset of Ω of positive measure. The equation (79) is said to be *observable* on ω in time T if there exists $C_T^{(W)}(\chi_\omega) > 0$ such that

$$C_T^{(W)}(\chi_\omega) \|(y^0, y^1)\|_{D(A^{1/2}) \times X}^2 \leq \int_0^T \int_{\omega} |\partial_t y(t, x)|^2 dV_g dt \tag{82}$$

¹² If M is the usual Euclidean space \mathbb{R}^n then $dV_g = dx$ is the usual Lebesgue measure.

for all $(y^0, y^1) \in D(A^{1/2}) \times X$. This observability inequality holds if (ω, T) satisfies the GCC in Ω .

A similar observability problem can be formulated for the Schrödinger equation (80). For every $y^0 \in D(A)$, there exists a unique solution y of (80) in the space $C^0(0, T; D(A))$ such that $y(0, \cdot) = y^0(\cdot)$. The equation (80) is said to be *observable* on ω in time T if there exists $C_T^{(S)}(\chi_\omega) > 0$ such that

$$C_T^{(S)}(\chi_\omega) \|y^0\|_{D(A)}^2 \leq \int_0^T \int_\omega |\partial_t y(t, x)|^2 dV_g dt \tag{83}$$

for every $y^0 \in D(A)$. If (ω, T^*) satisfies the GCC then the observability inequality (83) holds for every $T > 0$ (see [39]). This is so since the Schrödinger equation can be viewed as a wave equation with an infinite speed of propagation. The GCC is sufficient to ensure the observability for the Schrödinger equation, but to obtain sharp necessary and sufficient conditions is a widely open problem (see [38]). The norms that are used will be computed in a spectral way (see below).

Remark 5.1. These inequalities can be formulated in different ways by suitable choices of function spaces. For instance, the observability inequality (82) is equivalent to

$$C_T^{(W)}(\chi_\omega) \|(y^0, y^1)\|_{X \times (D(A^{1/2}))'}^2 \leq \int_0^T \int_\omega |y(t, x)|^2 dV_g dt \tag{84}$$

for all $(y^0, y^1) \in X \times (D(A^{1/2}))'$, with the same observability constants. Here the dual is considered with respect to the pivot space X . The space $(D(A^{1/2}))'$ is endowed with the norm defined by

$$\|z\|_{(D(A^{1/2}))'} = \sup_{\substack{w \in D(A^{1/2}) \\ \|w\|_{D(A^{1/2})} \leq 1}} \langle z, w \rangle_{(D(A^{1/2}))', D(A^{1/2})}.$$

For instance if $A = -\Delta$ with Dirichlet boundary conditions as considered previously, then the observability inequality (84) exactly coincides with (11); we thus recover the observability inequality that we considered up to now for wave equations with Dirichlet boundary conditions.

Similarly, the observability inequality (83) is equivalent to

$$C_T^{(S)}(\chi_\omega) \|y^0\|_X^2 \leq \int_0^T \int_\omega |y(t, x)|^2 dV_g dt \tag{85}$$

for every $y^0 \in X$.

Spectral expansions. We fix an orthonormal Hilbert basis $(\phi_j)_{j \in \mathbb{N}^*}$ of X consisting of eigenfunctions of A on Ω , associated with the positive eigenvalues $(\lambda_j^2)_{j \in \mathbb{N}^*}$.

Remark 5.2. In the Neumann case or in the case $\partial\Omega = \emptyset$, we take $X = L_0^2(\Omega)$ to keep a uniform presentation. Otherwise in $X = L^2(\Omega)$, in those cases, we would have $\lambda_1 = 0$ (a simple eigenvalue) and $\phi_1 = 1/\sqrt{|\Omega|}$.

Remark 5.3. We have

$$D(A) = \left\{ y \in X \mid \sum_{j=1}^{\infty} \lambda_j^4 |\langle y, \phi_j \rangle_{L^2}|^2 < \infty \right\},$$

$$D(A^{1/2}) = \left\{ y \in X \mid \sum_{j=1}^{\infty} \lambda_j^2 |\langle y, \phi_j \rangle_{L^2}|^2 < \infty \right\},$$

and for every $y \in D(A)$ we set $\|y\|_{D(A)}^2 = \sum_{j=1}^{\infty} \lambda_j^4 |\langle y, \phi_j \rangle_{L^2}|^2$ and $\|y\|_{D(A^{1/2})}^2 = \sum_{j=1}^{\infty} \lambda_j^2 |\langle y, \phi_j \rangle_{L^2}|^2$.

In the case of Dirichlet boundary conditions, and if $\partial\Omega$ is C^2 , one has $D(A) = H^2(\Omega, \mathbb{C}) \cap H_0^1(\Omega, \mathbb{C})$ (endowed with the norm $\|u\|_{H^2 \cap H_0^1} = \|\Delta u\|_{L^2}$) and $D(A^{1/2}) = H_0^1(\Omega, \mathbb{C})$ (endowed with the norm $\|u\|_{H_0^1} = \|\nabla u\|_{L^2}$). For Neumann boundary conditions, one has $D(A) = \{y \in H^2(\Omega, \mathbb{C}) \mid \frac{\partial y}{\partial n}|_{\partial\Omega} = 0 \text{ and } \int_{\Omega} y(x) dx = 0\}$ and $D(A^{1/2}) = \{y \in H^1(\Omega, \mathbb{C}) \mid \int_{\Omega} y(x) dx = 0\}$. In the mixed Dirichlet–Neumann case (with $\Gamma_0 \neq \emptyset$), one has $D(A) = \{y \in H^2(\Omega, \mathbb{C}) \mid y|_{\Gamma_0} = \frac{\partial y}{\partial n}|_{\Gamma_1} = 0\}$ and $D(A^{1/2}) = H_{\Gamma_0}^1(\Omega, \mathbb{C}) = \{y \in H^1(\Omega, \mathbb{C}) \mid y|_{\Gamma_0} = 0\}$ (see e.g. [37]).

For every $y^0 \in D(A)$, the solution $y \in C^0(0, T; D(A))$ of (80) such that $y(0, \cdot) = y^0(\cdot)$ can be expanded in Fourier series as follows:

$$y(t, x) = \sum_{j=1}^{\infty} c_j e^{i\lambda_j^2 t} \phi_j(x).$$

Moreover, $\|y^0\|_{D(A)}^2 = \sum_{j=1}^{\infty} \lambda_j^4 |c_j|^2$, the sequence $(\lambda_j^2 c_j)_{j \in \mathbb{N}^*}$ being in $\ell^2(\mathbb{C})$ and determined in terms of y^0 by $c_j = \int_{\Omega} y^0(x) \phi_j(x) dV_g$ for every $j \in \mathbb{N}^*$. It follows that

$$\int_0^T \int_{\omega} |\partial_t y(t, x)|^2 dV_g dt = \sum_{j,k=1}^{\infty} \lambda_j^2 \lambda_k^2 \alpha_{jk} \int_{\omega} \phi_j(x) \phi_k(x) dV_g$$

with

$$\alpha_{jk} = c_j \bar{c}_k \int_0^T e^{i(\lambda_j^2 - \lambda_k^2)t} dt = \frac{2c_j \bar{c}_k}{\lambda_j^2 - \lambda_k^2} \sin\left(\frac{(\lambda_j^2 - \lambda_k^2)T}{2}\right) e^{i(\lambda_j^2 - \lambda_k^2)T/2}$$

whenever $j \neq k$, and $\alpha_{jj} = |c_j|^2 T$ whenever $j = k$. The observability constant is given by

$$C_T^{(S)}(\chi_{\omega}) = \inf_{\substack{(\lambda_j^2 c_j) \in \ell^2(\mathbb{C}) \\ \sum_{j=1}^{\infty} \lambda_j^4 |c_j|^2 = 1}} \int_0^T \int_{\omega} \left| \sum_{j=1}^{\infty} \lambda_j^2 c_j e^{i\lambda_j^2 t} \phi_j(x) \right|^2 dV_g dt.$$

Making, as in Section 2.3, a random selection of all possible initial data for the Schrödinger equation (80) leads to define its randomized version as

$$C_{T,\text{rand}}^{(S)}(\chi_\omega) = \inf_{\substack{(\lambda_j^2 c_j) \in \ell^2(\mathbb{C}) \\ \sum_{j=1}^{\infty} \lambda_j^4 |c_j|^2 = 1}} \mathbb{E} \left(\int_0^T \int_\omega \left| \sum_{j=1}^{\infty} \lambda_j^2 \beta_j^y c_j e^{i\lambda_j^2 t} \phi_j(x) \right|^2 dV_g dt \right),$$

where $(\beta_j^y)_{j \in \mathbb{N}^*}$ denotes a sequence of independent Bernoulli random variables on a probability space $(\mathcal{X}, \mathcal{A}, \mathbb{P})$. This corresponds to considering the randomized observability inequality

$$C_{T,\text{rand}}^{(S)}(\chi_\omega) \|y^0\|_{D(A)}^2 \leq \mathbb{E} \left(\int_0^T \int_\omega |\partial_t y_v(t, x)|^2 dV_g dt \right)$$

for every $y^0(\cdot) \in D(A)$, where y_v denotes the solution of the Schrödinger equation with the random initial data $y_v^0(\cdot)$ determined by its Fourier coefficients $c_j^y = \beta_j^y c_j$.

Theorem 2.2 then still holds in this general framework, and one has

$$2C_{T,\text{rand}}^{(W)}(\chi_\omega) = C_{T,\text{rand}}^{(S)}(\chi_\omega) = T \inf_{j \in \mathbb{N}^*} \int_\omega \phi_j(x)^2 dV_g = TJ(\chi_\omega)$$

for every measurable subset ω of Ω , where J is defined as before by (26).

The time asymptotic observability constant is defined accordingly for the Schrödinger equation by

$$C_\infty^{(S)}(\chi_\omega) = \inf \left\{ \lim_{T \rightarrow \infty} \frac{1}{T} \frac{\int_0^T \int_\omega |\partial_t y(t, x)|^2 dV_g dt}{\|y^0\|_{D(A)}^2} \mid y^0 \in D(A) \setminus \{0\} \right\}. \quad (86)$$

Corollary 2.7 holds as well, stating that $2C_\infty^{(W)}(\chi_\omega) = C_\infty^{(S)}(\chi_\omega) = J(\chi_\omega)$ whenever every eigenvalue of A is simple. Note that the spectrum of the Neumann-Laplacian is known to consist of simple eigenvalues for many choices of Ω : for instance, it is proved in [26] that this property holds for almost every polygon of \mathbb{R}^2 having N vertices.

Main results under quantum ergodicity assumptions. Theorem 3.5 is unchanged in this general framework. Several very minor things in the proof have to be (obviously) adapted to the general Riemannian setting.

Spectral approximation. It must be noted that the third point of Theorem 4.9 can hold true only if M is an analytic Riemannian manifold and if Ω has a nontrivial boundary. This assumption is indeed used at the end of the proof of this theorem, when showing by contradiction that the function $x \mapsto \sum_{j=1}^N \alpha_j^N \phi_j(x)^2$ is never constant on any subset of positive measure. The reasoning works when dealing with Dirichlet boundary conditions, but not for any other boundary condition. Actually, at this step it is required that the family $(\phi_j)_{j \in \mathbb{N}^*}$ of eigenfunctions satisfies the following geometrical property:

- *Strong Conic Independence Property.* If there exists a subset E of Ω of positive Lebesgue measure, an integer $N \in \mathbb{N}^*$, an N -tuple $(\alpha_j)_{1 \leq j \leq N} \in (\mathbb{R}_+)^N$, and $C \geq 0$ such that $\sum_{j=1}^N \alpha_j |\phi_j(x)|^2 = C$ almost everywhere on E , then $C = 0$ and $\alpha_j = 0$ for every $j \in \{1, \dots, N\}$.

This property is defined, commented on and used in [53]. We have thus to distinguish between the different boundary conditions under consideration. Up to our knowledge, the general validity of this property is an open problem. Nevertheless, this property holds true in the following cases:

- Dirichlet–Laplacian;
- mixed Dirichlet–Neumann Laplacian defined on $D(A_0) = \{y \in H^2(\Omega, \mathbb{C}) \mid \chi_{\Gamma_0} y|_{\partial\Omega} = 0\}$, with $\Gamma_0 \subset \partial\Omega$ and $\mathcal{H}^{n-1}(\Gamma_0) > 0$;
- Neumann–Laplacian on the n -dimensional square, defined on $D(A_0) = \{y \in H^2(\Omega, \mathbb{C}) \mid \int_{\Omega} y = 0 \text{ and } \frac{\partial y}{\partial n} = 0 \text{ on } \partial\Omega\}$, with the usual Hilbert basis of eigenfunctions consisting of products of cosine functions.

In all those cases, the third point of Theorem 4.9 holds true.

Numerical simulations. Some results are provided in Figure 6 for $\Omega = (0, \pi)^2$ with Neumann boundary conditions. They illustrate as well the nonstationarity feature of the maximizing sequence of optimal sets ω^N .

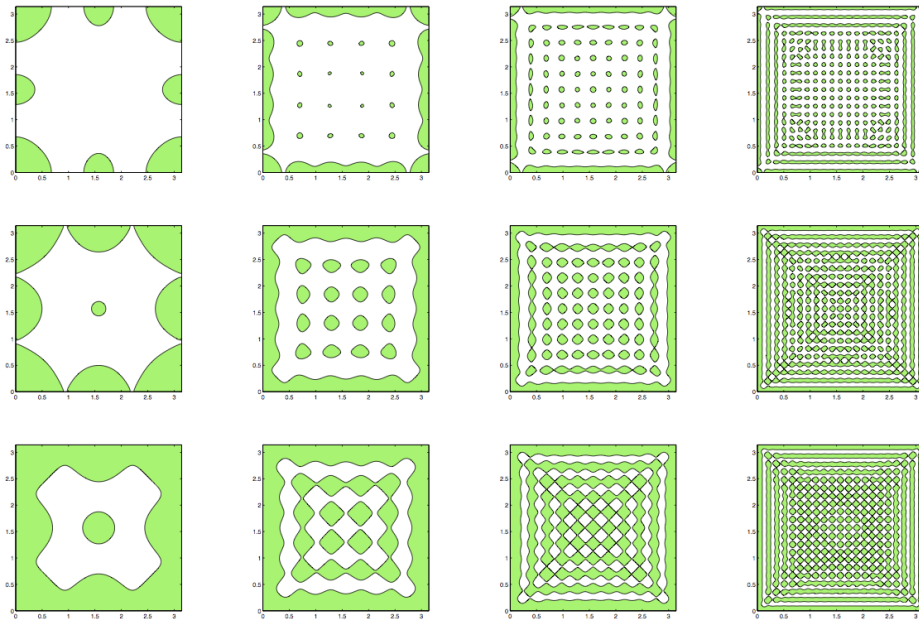


Fig. 6. $\Omega = (0, \pi)^2$, with Neumann boundary conditions. Row 1: $L = 0.2$; row 2: $L = 0.4$; row 3: $L = 0.6$. From left to right: $N = 2$ (4 eigenmodes), $N = 5$ (25 eigenmodes), $N = 10$ (100 eigenmodes), $N = 20$ (400 eigenmodes).

Remedy: weighted observability inequalities. In the general framework, the weighted versions (as discussed in Section 4.4) of the observability inequalities (82) and (83) are

$$C_{T,\sigma}^{(W)}(\chi_\omega)(\|y^0, y^1\|_{D(A^{1/2}) \times X}^2 + \sigma \|y^0\|_X^2) \leq \int_0^T \int_\omega |\partial_t y(t, x)|^2 dx dt \quad (87)$$

in the case of the wave equation, and

$$C_{T,\sigma}^{(S)}(\chi_\omega)(\|y^0\|_{D(A)}^2 + \sigma \|y^0\|_X^2) \leq \int_0^T \int_\omega |\partial_t y(t, x)|^2 dx dt \quad (88)$$

in the case of the Schrödinger equation, where $\sigma \geq 0$.

Note that, in the Dirichlet case, if $\sigma = 1$ then the inequality (87) corresponds to replacing the H_0^1 norm with the full H^1 norm defined by $\|f\|_{H^1(\Omega, \mathbb{C})} = (\|f\|_{L^2(\Omega, \mathbb{C})}^2 + \|\nabla f\|_{L^2(\Omega, \mathbb{C})}^2)^{1/2}$.

Clearly, we have $C_{T,\sigma}^{(W)}(\chi_\omega) \leq C_T^{(W)}(\chi_\omega)$ and $C_{T,\sigma}^{(S)}(\chi_\omega) \leq C_T^{(S)}(\chi_\omega)$, for every $\sigma \geq 0$.

Proposition 4.12 remains unchanged, stating that $2C_{T,\sigma,\text{rand}}^{(W)}(\chi_\omega) = C_{T,\sigma,\text{rand}}^{(S)}(\chi_\omega) = TJ_\sigma(\chi_\omega)$ for every measurable subset ω of Ω , and that $2C_{\infty,\sigma}^{(W)}(\chi_\omega) = C_{\infty,\sigma}^{(S)}(\chi_\omega) = J_\sigma(\chi_\omega)$ if moreover every eigenvalue of A is simple, where J_σ is defined by (65).

Theorem 4.14 remains in force as well. Therefore, in the general framework, the averaged versions of these weighted observability inequalities constitute a physically relevant remedy to ensure the existence and uniqueness of an optimal set.

For the sake of completeness, let us provide a numerical simulation illustrating this result. In Remark 4.19 we can also consider the domain $\Omega = \mathbb{T}^n$ (flat torus), $\Omega = (0, \pi)^n$ with Dirichlet boundary conditions, or mixed Dirichlet–Neumann boundary conditions with either Dirichlet or Neumann condition on every full edge of the hypercube, with the usual basis $(\phi_j)_{j \in \mathbb{N}^*}$ of eigenfunctions consisting of products of either sine or of cosine functions (tensorized version of the 1D case). It is then easy to see that the conclusion of Proposition 4.20 holds true in these more general cases.

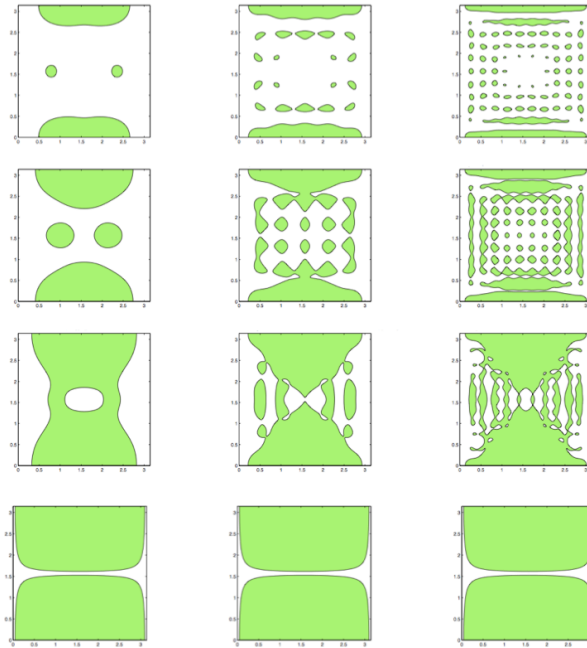


Fig. 7. $\Omega = (0, \pi)^2$, with Dirichlet boundary conditions on $\partial\Omega \cap (\{x_2 = 0\} \cup \{x_2 = \pi\})$ and Neumann boundary conditions on the rest of the boundary. Row 1: $L = 0.2$; row 2: $L = 0.4$; row 3: $L = 0.6$; row 4: $L = 0.9$. From left to right: $N = 2$ (4 eigenmodes), $N = 5$ (25 eigenmodes), $N = 10$ (100 eigenmodes).

Some numerical simulations are provided in Figure 7 (with the weight $\sigma = 1$), again clearly illustrating the stationarity feature of the maximizing sequence, as soon as L is large enough.

6. Further comments

In Section 6.1, we show how our results for the problem (30) can be extended to a natural variant of the observability inequality for Neumann boundary conditions or in the boundaryless case. In Section 6.2 we show how the problem of maximizing the observability constant is equivalent to the optimal design of a control problem and to that of controllability in which solutions are driven to rest in final time by means of a suitable control function. Section 6.3 is devoted to several open issues.

6.1. Further remarks for Neumann boundary conditions or in the boundaryless case

In the Neumann case, or in the case $\partial\Omega = \emptyset$, as explained in footnote 5.2 one has to take care of the constant (in space) solutions that can be an impediment for the observability inequality to hold. In this section, we show that, if instead of considering the observability inequalities (82) and (83), we consider the inequalities

$$C_T^{(W)}(\chi_\omega) \|(y^0, y^1)\|_{H^1 \times L^2}^2 \leq \int_0^T \int_\omega (|\partial_t y(t, x)|^2 + |y(t, x)|^2) dV_g dt \quad (89)$$

in the case of the wave equation, and

$$C_T^{(S)}(\chi_\omega) \|y^0\|_{H^2}^2 \leq \int_0^T \int_\omega (|\partial_t y(t, x)|^2 + |y(t, x)|^2) dV_g dt \quad (90)$$

in the case of the Schrödinger equation (see [61, Chapter 11] for a survey on these problems), then all our results remain unchanged.¹³

Indeed, consider initial data $(y^0, y^1) \in H^1(\Omega, \mathbb{C}) \times L^2(\Omega, \mathbb{C})$. The corresponding solution y can still be expanded as (13), except that now $(\phi_j)_{j \in \mathbb{N}^*}$ are the eigenfunctions of the Neumann–Laplacian or of the Laplace–Beltrami operator in the boundaryless case, associated with the eigenvalues $(-\lambda_j^2)_{j \in \mathbb{N}^*}$, with $\lambda_1 = 0$ and ϕ_1 being constant, equal to $1/\sqrt{|\Omega|}$. The relation (15) does not hold any more and is replaced with

$$\|(y^0, y^1)\|_{H^1 \times L^2}^2 = \sum_{j=1}^\infty (2\lambda_j^2 |a_j|^2 + 2\lambda_j^2 |b_j|^2 + |a_j + b_j|^2). \quad (91)$$

Following Section 2.3, we define the time asymptotic observability constant $C_\infty^{(W)}(\chi_\omega)$ as the largest possible nonnegative constant for which the time asymptotic observability inequality

$$C_\infty^{(W)}(\chi_\omega) \|(y^0, y^1)\|_{H^1 \times L^2}^2 \leq \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \int_\omega (|\partial_t y(t, x)|^2 + |y(t, x)|^2) dV_g dt \quad (92)$$

¹³ The norm in $H^2(\Omega, \mathbb{C})$ is given by $\|u\|_{H^2} = (\|u\|_{L^2}^2 + \|Du\|_{L^2}^2 + \|D^2u\|_{L^2}^2)^{1/2}$.

holds for all $(y^0, y^1) \in H^1(\Omega, \mathbb{C}) \times L^2(\Omega, \mathbb{C})$. Similarly, we define the randomized observability constant $C_{T,\text{rand}}^{(W)}(\chi_\omega)$ as the largest possible nonnegative constant for which the randomized observability inequality

$$C_{T,\text{rand}}^{(W)}(\chi_\omega) \|(y^0, y^1)\|_{H^1 \times L^2}^2 \leq \mathbb{E} \left(\int_0^T \int_\omega (|\partial_t y_v(t, x)|^2 + |y_v(t, x)|^2) dV_g dt \right) \quad (93)$$

holds for all $(y^0, y^1) \in H^1(\Omega, \mathbb{C}) \times L^2(\Omega, \mathbb{C})$, where y_v is defined as before by (25). The time asymptotic and randomized observability constants are defined accordingly for the Schrödinger equation. We have the following result, showing that we recover the same criterion as before.

Theorem 6.1. *Let ω be a measurable subset of Ω .*

- (i) *If the domain Ω is such that every eigenvalue of the Neumann–Laplacian is simple, then $2C_\infty^{(W)}(\chi_\omega) = C_\infty^{(S)}(\chi_\omega) = J(\chi_\omega)$.*
- (ii) *$2C_{T,\text{rand}}^{(W)}(\chi_\omega) = C_{T,\text{rand}}^{(S)}(\chi_\omega) = TJ(\chi_\omega)$.*

Proof. Following the proofs of Theorems 2.2 and 2.6, we obtain $C_{T,\text{rand}}^{(W)}(\chi_\omega) = TC_\infty^{(W)}(\chi_\omega) = T\Gamma$ with

$$\Gamma = \inf_{((a_j), (b_j)) \in (\ell^2(\mathbb{C}))^2 \setminus \{0\}} \frac{\sum_{j=1}^\infty (1 + \lambda_j^2)(|a_j|^2 + |b_j|^2) \int_\omega \phi_j(x)^2 dV_g}{\sum_{j=1}^\infty (2\lambda_j^2(|a_j|^2 + |b_j|^2) + |a_j + b_j|^2)}.$$

Let us prove that $\Gamma = \frac{1}{2}J(\chi_\omega)$. First, it is easy to see that, in the definition of Γ , it suffices to consider the infimum over real sequences (a_j) and (b_j) . Next, setting $a_j = \rho_j \cos \theta_j$ and $b_j = \rho_j \sin \theta_j$, since $|a_j + b_j|^2 = \rho_j^2(1 + \sin(2\theta_j))$, to reach the infimum one has to take $\theta_j = \pi/4$ for every $j \in \mathbb{N}^*$. It finally follows that

$$\Gamma = \inf_{\substack{(\rho_j) \in \ell^2(\mathbb{R}) \\ \sum_{j=1}^\infty \rho_j^2 = 1}} \frac{1}{2} \sum_{j=1}^\infty \rho_j^2 \int_\omega \phi_j(x)^2 dV_g = \frac{1}{2}J(\chi_\omega). \quad \square$$

6.2. Optimal shape and location of internal controllers

In this section, we investigate the question of determining the shape and location of the control domain for the wave or Schrödinger equations that minimizes the L^2 norm of the controllers realizing null controllability and its connections with the previous results on the optimal design for observability. We shall see that maximizing the observability constant is equivalent to minimizing the cost of controlling the corresponding adjoint system.

For the sake of simplicity, we will only deal with the wave equation, the Schrödinger case being easily adapted from that case. Also, without loss of generality we restrict ourselves to Dirichlet boundary conditions.

Consider the internally controlled wave equation on Ω with Dirichlet boundary conditions

$$\begin{aligned} \partial_{tt}y(t, x) - \Delta_g y(t, x) &= h_\omega(t, x), & (t, x) \in (0, T) \times \Omega, \\ y(t, x) &= 0, & (t, x) \in [0, T] \times \partial\Omega, \\ y(0, x) &= y^0(x), \quad \partial_t y(0, x) = y^1(x), & x \in \Omega, \end{aligned} \tag{94}$$

where h_ω is a control supported in $[0, T] \times \omega$ and ω is a measurable subset of Ω .

Note that (94) is well posed for all initial data $(y^0, y^1) \in H_0^1(\Omega, \mathbb{R}) \times L^2(\Omega, \mathbb{R})$ and every $h_\omega \in L^2((0, T) \times \Omega, \mathbb{R})$, and its solution y belongs to $C^0(0, T; H_0^1(\Omega, \mathbb{R})) \cap C^1(0, T; L^2(\Omega, \mathbb{R})) \cap C^2(0, T; H^{-1}(\Omega, \mathbb{R}))$.

The exact null controllability problem consists in finding a control h_ω steering system (94) to $y(T, \cdot) = \partial_t y(T, \cdot) = 0$.

It is well known that, for every subset ω of Ω of positive measure, the exact controllability problem is by duality equivalent to the fact that the observability inequality

$$C \|\phi^0, \phi^1\|_{L^2 \times H^{-1}}^2 \leq \int_0^T \int_\omega |\phi(t, x)|^2 dV_g dt \tag{95}$$

holds for all $(\phi^0, \phi^1) \in L^2(\Omega, \mathbb{R}) \times H^{-1}(\Omega, \mathbb{R})$, for a positive constant C (only depending on T and ω), where ϕ is the (unique) solution of the adjoint system

$$\begin{aligned} \partial_{tt}\phi(t, x) - \Delta_g \phi(t, x) &= 0, & (t, x) \in (0, T) \times \Omega, \\ \phi(t, x) &= 0, & (t, x) \in [0, T] \times \partial\Omega, \\ \phi(0, x) &= \phi^0(x), \quad \partial_t \phi(0, x) = \phi^1(x), & x \in \Omega. \end{aligned} \tag{96}$$

The Hilbert Uniqueness Method (HUM, see [41, 42]) provides a way to characterize the unique control solving the above exact null controllability problem and having moreover a minimal $L^2((0, T) \times \Omega, \mathbb{R})$ norm. This control is referred to as the HUM control and is characterized as follows. Define the HUM functional \mathcal{J}_ω by

$$\mathcal{J}_\omega(\phi^0, \phi^1) = \frac{1}{2} \int_0^T \int_\omega \phi(t, x)^2 dV_g dt - \langle \phi^1, y^0 \rangle_{H^{-1}, H_0^1} + \langle \phi^0, y^1 \rangle_{L^2}. \tag{97}$$

The notation $\langle \cdot, \cdot \rangle_{H^{-1}, H_0^1}$ stands for the duality bracket between $H^{-1}(\Omega, \mathbb{R})$ and $H_0^1(\Omega, \mathbb{R})$, and the notation $\langle \cdot, \cdot \rangle_{L^2}$ stands for the usual scalar product of $L^2(\Omega, \mathbb{R})$.

If (95) holds then the functional \mathcal{J}_ω has a unique minimizer (still denoted (ϕ^0, ϕ^1)) in the space $L^2(\Omega, \mathbb{R}) \times H^{-1}(\Omega, \mathbb{R})$, for all $(y^0, y^1) \in H_0^1(\Omega, \mathbb{R}) \times L^2(\Omega, \mathbb{R})$. The HUM control h_ω steering (y^0, y^1) to $(0, 0)$ in time T is then given by $h_\omega(t, x) = \chi_\omega(x)\phi(t, x)$, for almost all $(t, x) \in (0, T) \times \Omega$, where ϕ is the solution of (96) with initial data (ϕ^0, ϕ^1) minimizing \mathcal{J}_ω .

The HUM operator Γ_ω is then defined by

$$\Gamma_\omega : H_0^1(\Omega, \mathbb{R}) \times L^2(\Omega, \mathbb{R}) \rightarrow L^2((0, T) \times \Omega, \mathbb{R}), \quad (y^0, y^1) \mapsto h_\omega.$$

With this definition, it is a priori natural to model the problem of determining the best control domain as the problem of minimizing the norm of the operator Γ_ω ,

$$\|\Gamma_\omega\| = \sup \left\{ \frac{\|h_\omega\|_{L^2((0,T)\times\Omega,\mathbb{R})}}{\|(y^0, y^1)\|_{H_0^1\times L^2}} \mid (y^0, y^1) \in H_0^1(\Omega, \mathbb{R}) \times L^2(\Omega, \mathbb{R}) \setminus \{(0, 0)\} \right\}, \tag{98}$$

over the set \mathcal{U}_L . The following result holds and can be proved using Fourier expansion. We refer to [51] for the details of the proof in the one-dimensional case. Note that the proof in the multi-dimensional case is exactly the same.

Proposition 6.2. *Let $T > 0$ and let ω be measurable subset of Ω . If $C_T^{(W)}(\chi_\omega) > 0$ then $\|\Gamma_\omega\| = 1/C_T^{(W)}(\chi_\omega)$, and if $C_T^{(W)}(\chi_\omega) = 0$, then $\|\Gamma_\omega\| = \infty$.*

This result illustrates the well known duality between controllability and observability, and moreover shows that, for the optimal design control problem, one has

$$\inf_{\chi_\omega \in \mathcal{U}_L} \|\Gamma_\omega\| = \left(\sup_{\chi_\omega \in \mathcal{U}_L} C_T^{(W)}(\chi_\omega) \right)^{-1}. \tag{99}$$

Therefore the problem of minimizing $\|\Gamma_\omega\|$ is equivalent to the problem of maximizing the observability constant over \mathcal{U}_L .

However, as discussed previously, it is more relevant to maximize the randomized observability constant $C_{T,\text{rand}}(\chi_\omega)$ defined by (23) (see Section 2.3). It is therefore natural to identify the dual optimal design problem at the control level.

The corresponding control problem reads

$$\begin{aligned} \partial_{tt}y(t, x) - \Delta_g y(t, x) &= \sum_{j \geq 1} f_j(t) \int_\omega \phi_j^2(u) du \phi_j(x), & (t, x) \in (0, T) \times \Omega, \\ y(t, x) &= 0, & (t, x) \in [0, T] \times \partial\Omega, \\ y(0, x) &= y^0(x), \quad \partial_t y(0, x) = y^1(x), & x \in \Omega. \end{aligned} \tag{100}$$

Note that in this problem the control is of lumped type, determined by the time-dependent functions $\{f_j(t)\}_{j \geq 1} \in L^2(0, T; \ell^2)$ that are weighted by the constant $\int_\omega \phi_j(x)^2 dx$, which underlines the need of choosing ω so that the uniform spectral observability property holds, and acting on the profiles ϕ_j of the eigenfunctions of the Laplacian.

Note in particular that, as a consequence of the randomization process at the observability level, the controls are distributed everywhere in the domain, through the profiles given by the eigenfunctions of the Laplacian.

This issue will be investigated with further detail in a forthcoming article, together with the corresponding consequences at the level of stabilization.

6.3. Open problems

We provide here a list of open problems.

Optimal stabilization domain. Similar important problems can be addressed as well for stabilization issues. For instance, when considering the wave equation with a local damping,

$$\partial_{tt}y = \Delta y - 2k\chi_\omega y_t, \quad (101)$$

with $k > 0$, one can address the question of determining the best possible damping domain ω (in the class \mathcal{U}_L), achieving for instance (if possible) the largest possible exponential decay rate.

This question was investigated in [22] in the one-dimensional case. The following is known. First, if $k \rightarrow \infty$ then the decay rate tends to 0 (overdamping phenomenon). Second, as proved in [15], if the set ω has a finite number of connected components and if k is small enough, then, at the first order, the decay rate is determined by the spectral abscissa which is of the order of $k \inf_{j \in \mathbb{N}^*} \int_\omega \sin^2(jx) dx$. Therefore, in the 1D case, for k small maximizing the decay rate is equivalent to the problem (30) in 1D (however, with the additional restriction that χ_ω is in BV).

Note that, even in 1D, except for those two asymptotic regimes in which k is small or large, the problem of maximizing the decay rate over \mathcal{U}_L is completely open.

The issue is of course much more complex in the multi-dimensional case. See [49] for a much easier situation with constant damping.

Generally speaking, the exponential stability property of (101) is equivalent to the observability property of the corresponding conservative wave equation (1) (see [20]). Note however that this general statement, “observability implies stabilization”, does not yield explicit decay rates for the dissipative semigroup.

The ultimate dependence of the decay rate on the amplitude of the dissipative potential (k) and the geometry of ω is rather complex, as proved in [40]. In fact, the exponential decay rate $\tau(\omega)$ does not coincide in general with the negative of the spectral abscissa $S(\omega)$ since it is the minimum of this real number and of a geometric quantity depending on the average time spent by geodesics crossing ω (see [24] for a study of this geometric quantity in the square).

It is an interesting open problem to study the maximization of this geometric criterion over the set \mathcal{U}_L .

It can be noted that the fact that $\tau(\omega) \leq -S(\omega)$ and that in multi-D the strict inequality may hold, is similar to the fact, underlined in Remark 2.5, that $C_T^{(W)}(\chi_\omega) \leq C_{T,\text{rand}}^{(W)}(\chi_\omega)$ and that the strict inequality may hold.

As in the context of control, randomized observability implies weaker stabilization results (see [54] for precise results).

Maximization of the deterministic observability constant. As discussed in Section 2.3, the problem of maximizing the (usual) deterministic observability constant $C_T^{(W)}(\chi_\omega)$ (defined by (12)) over \mathcal{U}_L is open, and is difficult due to the crossed terms appearing in the spectral expansion. It can be noted that the convexified version of this problem, consisting in maximizing $C_T^{(W)}(a)$ over $\overline{\mathcal{U}}_L$, obviously has some solutions, and again here the question of a gap, and the question of knowing whether the supremum is reached over \mathcal{U}_L (existence of a classical optimal set), are open. Even a truncated version of this criterion is an open problem, that is, the problem of maximizing the lowest eigenvalue

of the Gramian matrix whose j, k entry is $\int_{\omega} \phi_j(x) \phi_k(x) dx$. An interesting problem is to investigate, theoretically or numerically, the sequence of maximizing subsets for this truncated problem. Even in 1D, this problem is open.

As noted in Remark 2.1, in the one-dimensional case and if T is an integer multiple of 2π , the crossed terms disappear and the Gramian matrix is diagonal; but if T is not an integer multiple of 2π then owing to the crossed terms the functional cannot be handled easily. Similar difficulties due to crossed terms are encountered in the open problem of determining the best constants in Ingham's inequality (see [29]), according to which, for every $\gamma > 0$ and every $T > 2\pi/\gamma$, there exist $C_1(T, \gamma), C_2(T, \gamma) > 0$ such that for every sequence $(\lambda_n)_{n \in \mathbb{N}^*}$ of real numbers satisfying $|\lambda_{n+1} - \lambda_n| \geq \gamma$ for every $n \in \mathbb{N}^*$, we have

$$C_1(T, \gamma) \sum_{n \in \mathbb{N}^*} |a_n|^2 \leq \int_0^T \left| \sum_{n \in \mathbb{N}^*} a_n e^{i\lambda_n t} \right|^2 dt \leq C_2(T, \gamma) \sum_{n \in \mathbb{N}^*} |a_n|^2$$

for all $(a_n)_{n \in \mathbb{N}^*} \in \ell^2(\mathbb{C})$ (see, e.g., [30, 32, 34, 64]).

Dependence on time. Instead of maximizing the observability constant over \mathcal{U}_L , for a fixed time T , one can think of running the optimization also over time.

Before setting this problem, let us make the following remark in 1D. Setting $\Omega = [0, \pi]$ (with Dirichlet boundary conditions), it is clear that if $T \geq 2\pi$ then the observability inequality (11) is satisfied for every subset ω of positive measure. However, 2π is not the smallest possible time for a given specific choice of ω . For instance, if ω is a subinterval of $[0, \pi]$ then the smallest possible time for which the observability inequality holds is $2 \operatorname{diam}((0, \pi) \setminus \omega)$. The question of determining this minimal time is nontrivial if, instead of an interval, the set ω is, for instance, a fractal set. We state the following problem (not only in 1D but also in general): given $L \in (0, 1)$, does there exist a time $T_L > 0$ such that the observability inequality (11) holds for all $\omega \in \mathcal{U}_L$ and $T \geq T_L$?

Having in mind this open question, it is interesting to investigate the problem of maximizing the functional $(\chi_{\omega}, T) \mapsto C_T(\chi_{\omega})$ over $\mathcal{U}_L \times (0, \infty)$. Similar questions arise when the observation set is not cylindrical but rather a measurable space-time set having a certain fixed measure. For such problems the existence of a maximizer is easy to derive when considering their convexified version, but the question of proving a no-gap result is nontrivial and has not been studied. Also, it is interesting to investigate whether or not the supremum is reached in the class of classical sets.

Nonexistence of an optimal set. In Section 4.1, using harmonic analysis we have proved that, in 1D, the supremum of J over \mathcal{U}_L is reached if and only if $L = 1/2$ (and there are infinitely many optimal sets). In the Euclidean square the question is open, but if the supremum is considered only over Cartesian products of 1D subsets, then it is reached if and only if $L \in \{1/4, 1/2, 3/4\}$. In general, the question of the existence of an optimal set is completely open, and we expect that the supremum is not reached for generic domains Ω and generic values of L .

This conjecture is in accordance with the observed increasing complexity of the sequence of optimal sets ω^N that solve the problem of maximizing the truncated spectral

criterion J_N . An interesting question occurs here. In the (certainly) nongeneric case where an optimal set does exist (as in 1D for $L = 1/2$ where there are infinitely many optimal sets), what is the limit of the sets ω^N ? More precisely, can it happen that ω^N converges to a set of fractal type? The study of [52], done for fixed initial data, indicates that it might be the case. The question is however completely open.

Note that in 1D for L sufficiently small, the spillover phenomenon was proved to occur, according to which the optimal set ω^N maximizing J_N is, loosely speaking, the worst possible one for the problem of maximizing J_{N+1} . Proving this fact in a more general context is an open problem.

Moreover, note that J_N is defined as a truncation of the functional J , keeping the N first modes. It would be interesting to consider similar optimal design problems running for instance over initial data whose Fourier coefficients satisfy a uniform exponential decreasing property. Another possibility is to truncate the Fourier series and keep only the modes whose index is between two integers N_1 and N_2 .

Weighted observability inequalities as a remedy. Intending to provide a physically relevant remedy to the problem of nonexistence of an optimal set, in Section 4.4 we introduced a weighted version of the observability inequality, which is however equivalent to the classical one. We proved that if $L > \lambda_1^2/(\sigma + \lambda_1^2)$ then there exists a unique optimal set, which is moreover the limit of the stationary sequence of optimal set ω^N of the truncated criterion. Our simulations indicate that this threshold in L is sharp. It is an open question to investigate the situation where $L \leq \lambda_1^2/(\sigma + \lambda_1^2)$: is there a gap or not between the problem and its convexified version? Is the supremum over \mathcal{U}_L reached or not?

Quantum ergodicity assumptions. In Theorem 3.5, we assumed the strong QUE on the base and uniform L^p -boundedness properties. As discussed in Section 3.3, except in 1D, up to now no domain is known where these assumptions hold true. The property QUE is related to a well known conjecture in mathematical physics. With the example of the disk (Proposition 3.9), we have seen that these assumptions are however not sharp.

Theorem 4.14, providing the existence of an optimal set for the weighted version of the problem, holds true under L^∞ -QUE on the base. The example of the hypercube (Proposition 4.20) shows that these assumptions are not sharp.

Weakening the sufficient assumptions of these three results is a completely open problem.

Note that, concerning the quantum ergodicity assumptions that we used, and the discussion in Section 3.3, we used the current state of the art in mathematical physics. The model that we used throughout, based on averaging either with respect to time or with respect to random initial data, leads to a spectral criterion whose solving requires a good knowledge of quantum ergodicity properties which are not well known presently. The problem remains of finding more robust models in which the solving of an optimal design problem would not require such a fine knowledge of the eigenelements. For instance, it is likely that the microlocal methods used in [3] in order to provide an almost necessary and sufficient condition for observability to hold (the Geometric Control Condition) in terms of geometric rays, should allow one to identify classes of domains where the constant is governed by a finite number of modes.

In brief, it is an open question to model the optimal design problems under consideration (possibly, based on the notion of geometric rays as discussed above) in such a way that the resulting problem will be both physically and mathematically relevant, and will not require, for its solving, sufficient assumptions so strong as the ones considered here.

Other models. In this article we have modeled and studied the optimal observability problem for wave and Schrödinger equations. It can be noted that, using the randomization procedure or the time averaging procedure that we have introduced for the observability inequalities, the spectral criterion J considered throughout can be derived as well for many other conservative models; however, nothing is then known on the probability measures $\mu_j = \phi_j^2 dx$ where the ϕ_j are the eigenfunctions of the underlying operator. As we have seen, even for the Laplacian the quantum ergodicity properties are widely unknown, and the situation is even more open for other operators.

For parabolic models the situation seems to be different. The randomization leads to a weighted spectral criterion similar to J_σ , but with the sequence of weights σ_j increasing to ∞ (whereas here it was increasing to 1). Because of that, in contrast to the results of the present article, it is expected that an optimal set does exist, under mild assumptions. We refer to [53] for results in that direction.

Also, for such other models, the previously raised questions—optimal shape and location of internal controllers; maximization of the deterministic observability constant—are open problems as well.

Appendix: Proof of Theorem 2.6 and of Corollary 2.7

For the convenience of the reader, we first prove Theorem 2.6 in the particular case where all the eigenvalues of Δ are simple (it corresponds exactly to the proof of Corollary 2.7), and then we give a generalization to the case of multiple eigenvalues.

From (13), we have $y(t, x) = \sum_{j=1}^\infty y_j(t, x)$ with

$$y_j(t, x) = (a_j e^{i\lambda_j t} + b_j e^{-i\lambda_j t}) \phi_j(x). \tag{102}$$

Without loss of generality, we consider initial data $(y^0, y^1) \in L^2(\Omega, \mathbb{C}) \times H^{-1}(\Omega, \mathbb{C})$ such that $\|(y^0, y^1)\|_{L^2 \times H^{-1}}^2 = 2$, in other words, $\sum_{j \in \mathbb{N}^*} (|a_j|^2 + |b_j|^2) = 1$ (using (15)).

Setting

$$\Sigma_T(a, b) = \frac{1}{T} \frac{\int_0^T \int_\omega |y(t, x)|^2 dx dt}{\|(y^0, y^1)\|_{L^2 \times H^{-1}}^2} = \frac{1}{2T} \int_0^T \int_\omega |y(t, x)|^2 dx dt,$$

we write, for an arbitrary $N \in \mathbb{N}^*$,

$$\begin{aligned} \Sigma_T(a, b) = \frac{1}{T} \int_0^T \int_\omega & \left(\left| \sum_{j=1}^N y_j(t, x) \right|^2 + \left| \sum_{k=N+1}^\infty y_k(t, x) \right|^2 \right. \\ & \left. + 2\Re \left(\sum_{j=1}^N y_j(t, x) \sum_{k=N+1}^\infty \bar{y}_k(t, x) \right) \right) dx dt. \end{aligned} \tag{103}$$

Using the assumption that the spectrum of Δ consists of simple eigenvalues, we have the following result.

Lemma A.1. *With the notation above, we have*

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \int_{\omega} \left| \sum_{j=1}^N y_j(t, x) \right|^2 dx dt = \sum_{j=1}^N (|a_j|^2 + |b_j|^2) \int_{\omega} \phi_j(x)^2 dx.$$

Proof. Since the sum is finite we can invert the infimum (which is a minimum) and the limit. Now, we write

$$\begin{aligned} \frac{1}{T} \int_0^T \int_{\omega} \left| \sum_{j=1}^N y_j(t, x) \right|^2 dx dt \\ = \frac{1}{T} \sum_{j=1}^N \alpha_{jj} \int_{\omega} \phi_j(x)^2 dx + \frac{1}{T} \sum_{j=1}^N \sum_{\substack{k=1 \\ k \neq j}}^N \alpha_{jk} \int_{\omega} \phi_j(x) \phi_k(x) dx, \end{aligned}$$

where α_{jk} is defined by (17). Using (18) and (19), we get $\lim_{T \rightarrow \infty} \alpha_{jj}/T = |a_j|^2 + |b_j|^2$ for every $j \in \mathbb{N}^*$, and since the spectrum of Δ consists of simple eigenvalues,

$$|\alpha_{jk}| \leq \frac{4 \max_{1 \leq j, k \leq N} (\lambda_j, \lambda_k)}{|\lambda_j^2 - \lambda_k^2|} \tag{104}$$

whenever $j \neq k$. The conclusion follows easily. □

Let us now estimate the remaining terms

$$\begin{aligned} R &= \frac{1}{T} \int_0^T \int_{\omega} \left| \sum_{j=N+1}^{\infty} y_j(t, x) \right|^2 dx dt, \\ \delta &= \frac{1}{T} \Re e \left(\int_0^T \int_{\omega} \sum_{j=1}^N y_j(t, x) \sum_{k=N+1}^{\infty} \bar{y}_k(t, x) dx dt \right), \end{aligned}$$

of the right-hand side of (103).

Estimate of R. Using the fact that the ϕ_j 's form a Hilbert basis, we get

$$\begin{aligned} R &\leq \frac{1}{T} \int_0^T \int_{\Omega} \left| \sum_{j=N+1}^{\infty} y_j(t, x) \right|^2 dx dt = \frac{1}{T} \sum_{j=N+1}^{\infty} \int_0^T |a_j e^{i\lambda_j t} - b_j e^{-i\lambda_j t}|^2 dt \\ &= \frac{1}{T} \sum_{j=N+1}^{\infty} \left(T(|a_j|^2 + |b_j|^2) - \frac{1}{\lambda_j} \Re e \left(a_j \bar{b}_j \frac{e^{2i\lambda_j T} - 1}{i} \right) \right) \end{aligned}$$

and finally

$$R \leq \left(1 + \frac{1}{\lambda_N T} \right) \sum_{j=N+1}^{\infty} (|a_j|^2 + |b_j|^2). \tag{105}$$

Estimate of δ . Using (18) and the fact that $\lambda_j \neq \lambda_k$ for every $j \in \{1, \dots, N\}$ and every $k \geq N + 1$, we have $|\delta| \leq \frac{2}{T}(S_1^N + S_2^N + S_3^N + S_4^N)$ with

$$\begin{aligned} S_1^N &= \left| \sum_{j=1}^N \sum_{k=N+1}^{\infty} \frac{1}{\lambda_j - \lambda_k} a_j \bar{a}_k e^{i(\lambda_j - \lambda_k)T/2} \sin\left((\lambda_j - \lambda_k)\frac{T}{2}\right) \int_{\omega} \phi_j(x) \phi_k(x) dx \right|, \\ S_2^N &= \left| \sum_{j=1}^N \sum_{k=N+1}^{\infty} \frac{1}{\lambda_j + \lambda_k} a_j \bar{b}_k e^{i(\lambda_j + \lambda_k)T/2} \sin\left((\lambda_j + \lambda_k)\frac{T}{2}\right) \int_{\omega} \phi_j(x) \phi_k(x) dx \right|, \\ S_3^N &= \left| \sum_{j=1}^N \sum_{k=N+1}^{\infty} \frac{1}{\lambda_j + \lambda_k} b_j \bar{a}_k e^{-i(\lambda_j + \lambda_k)T/2} \sin\left((\lambda_j + \lambda_k)\frac{T}{2}\right) \int_{\omega} \phi_j(x) \phi_k(x) dx \right|, \\ S_4^N &= \left| \sum_{j=1}^N \sum_{k=N+1}^{\infty} \frac{1}{\lambda_j - \lambda_k} b_j \bar{b}_k e^{-i(\lambda_j - \lambda_k)T/2} \sin\left((\lambda_j - \lambda_k)\frac{T}{2}\right) \int_{\omega} \phi_j(x) \phi_k(x) dx \right|. \end{aligned}$$

Let us estimate S_1^N . We write

$$S_1^N = \left| \sum_{j=1}^N a_j \int_{\omega} \phi_j(x) \sum_{k=N+1}^{\infty} \frac{\bar{a}_k}{\lambda_j - \lambda_k} e^{i(\lambda_j - \lambda_k)T/2} \sin\left((\lambda_j - \lambda_k)\frac{T}{2}\right) \phi_k(x) dx \right|,$$

and, using the Cauchy–Schwarz inequality and the fact that the integral of a nonnegative function over ω is smaller than the integral of the same function over Ω , one gets

$$\begin{aligned} S_1^N &\leq \sum_{j=1}^N |a_j| \left(\int_{\Omega} \left| \sum_{k=N+1}^{\infty} \frac{\bar{a}_k}{\lambda_j - \lambda_k} e^{i(\lambda_j - \lambda_k)T/2} \sin\left((\lambda_j - \lambda_k)\frac{T}{2}\right) \phi_k(x) \right|^2 dx \right)^{1/2} \\ &= \sum_{j=1}^N |a_j| \left(\sum_{k=N+1}^{\infty} \frac{|a_k|^2}{(\lambda_j - \lambda_k)^2} \sin^2\left((\lambda_j - \lambda_k)\frac{T}{2}\right) \right)^{1/2}. \end{aligned}$$

The last equality is established by expanding the square of the sum inside the integral, and by using the fact that the ϕ_k 's are orthonormal in $L^2(\Omega)$. Since the spectrum of Δ consists of simple eigenvalues (assumed to form an increasing sequence), we infer that $\lambda_k - \lambda_j \geq \lambda_{N+1} - \lambda_N$ for all $j \in \{1, \dots, N\}$ and $k \geq N + 1$, and since $\sum_{j=1}^{\infty} |a_j|^2 \leq 1$, it follows that

$$S_1^N \leq \frac{1}{\lambda_{N+1} - \lambda_N} \sum_{j=1}^N |a_j| \left(\sum_{k=N+1}^{\infty} |a_k|^2 \right)^{1/2} \leq \frac{N}{\lambda_{N+1} - \lambda_N}.$$

The same arguments lead to the estimates $S_2^N \leq N/\lambda_N$, $S_3^N \leq N/\lambda_N$, $S_4^N \leq N/(\lambda_{N+1} - \lambda_N)$, and therefore

$$|\delta| \leq \frac{4N}{T} \left(\frac{1}{\lambda_N} + \frac{1}{\lambda_{N+1} - \lambda_N} \right). \tag{106}$$

Now, combining Lemma A.1 with the estimates (105) and (106) shows that for every $\varepsilon > 0$, there exist $N_\varepsilon \in \mathbb{N}^*$ and $T(\varepsilon, N_\varepsilon) > 0$ such that if $N \geq N_\varepsilon$ and $T \geq T(\varepsilon, N_\varepsilon)$, then

$$\left| \Sigma_T(a, b) - \sum_{j=1}^N (|a_j|^2 + |b_j|^2) \int_\omega \phi_j(x)^2 dx \right| \leq \varepsilon.$$

As an immediate consequence, and using the obvious fact that, for every $\eta > 0$, there exists $N_\eta \in \mathbb{N}^*$ such that if $N \geq N_\eta$ then

$$\left| \sum_{j=1}^\infty (|a_j|^2 + |b_j|^2) \int_\omega \phi_j(x)^2 dx - \sum_{j=1}^N (|a_j|^2 + |b_j|^2) \int_\omega \phi_j(x)^2 dx \right| \leq \eta,$$

one deduces that $\lim_{T \rightarrow \infty} \Sigma_T(a, b) = \sum_{j=1}^\infty (|a_j|^2 + |b_j|^2) \int_\omega \phi_j(x)^2 dx$. Thus, we have proved the following lemma, which improves the statement of Lemma A.1.

Lemma A.2. Denoting by a_j and b_j the Fourier coefficients of (y^0, y^1) defined by (14), we have

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \int_\omega |y(t, x)|^2 dx dt = \sum_{j=1}^\infty (|a_j|^2 + |b_j|^2) \int_\omega \phi_j(x)^2 dx.$$

Corollary 2.7 follows by noting that

$$\inf_{\substack{(a_j), (b_j) \in \ell^2(\mathbb{C}) \\ \sum_{j=1}^\infty (|a_j|^2 + |b_j|^2) = 1}} \sum_{j=1}^\infty (|a_j|^2 + |b_j|^2) \int_\omega \phi_j(x)^2 dx = \inf_{j \in \mathbb{N}^*} \int_\omega \phi_j(x)^2 dx.$$

To finish the proof, we now explain how the arguments above can be generalized to the case of multiple eigenvalues. In particular, the statement of Lemma A.1 is adapted in the following way.

Lemma A.3. Using the previous notation, one has

$$\begin{aligned} \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \int_\omega \left| \sum_{j=1}^N y_j(t, x) \right|^2 dx dt \\ = \sum_{\substack{\lambda \in U \\ \lambda \leq \lambda_N}} \int_\omega \left(\left| \sum_{k \in I(\lambda)} \lambda_k a_k \phi_k(x) \right|^2 + \left| \sum_{k \in I(\lambda)} \lambda_k b_k \phi_k(x) \right|^2 \right) dx. \end{aligned}$$

Proof. Following the proof of Lemma A.1, simple computations show that

$$\begin{aligned} \frac{1}{T} \int_0^T \int_\omega \left| \sum_{j=1}^N y_j(t, x) \right|^2 dx dt &= \frac{1}{T} \sum_{\lambda \in U} \sum_{(j,k) \in I(\lambda)^2} \alpha_{jk} \int_\omega \phi_j(x) \phi_k(x) dx \\ &\quad + \frac{1}{T} \sum_{\substack{(\lambda, \mu) \in U^2 \\ \lambda \neq \mu}} \sum_{\substack{j \in I(\lambda) \\ k \in I(\mu)}} \alpha_{jk} \int_\omega \phi_j(x) \phi_k(x) dx, \end{aligned}$$

where

$$\lim_{T \rightarrow \infty} \frac{\alpha_{jk}}{T} = \begin{cases} a_j \bar{a}_k + b_j \bar{b}_k & \text{if } (j, k) \in I(\lambda)^2, \\ 0 & \text{if } j \in I(\lambda), k \in I(\mu), \text{ with } (\lambda, \mu) \in U^2 \text{ and } \lambda \neq \mu. \end{cases}$$

The conclusion of the lemma follows. \square

Noting that the previous estimates on R and δ are still valid and that

$$\begin{aligned} \inf_{\substack{(a_j), (b_j) \in \ell^2(\mathbb{C}) \\ \sum_{j=1}^{\infty} (|a_j|^2 + |b_j|^2) = 1}} \sum_{\lambda \in U} \int_{\omega} \left(\left| \sum_{k \in I(\lambda)} a_k \phi_k(x) \right|^2 + \left| \sum_{k \in I(\lambda)} b_k \phi_k(x) \right|^2 \right) dx \\ = \inf_{\substack{(c_k)_{k \in \mathbb{N}^*} \in \ell^2(\mathbb{C}) \\ \sum_{k=1}^{\infty} |c_k|^2}} \int_{\omega} \sum_{\lambda \in U} \left| \sum_{k \in I(\lambda)} c_k \phi_k(x) \right|^2 dx, \end{aligned}$$

Theorem 2.6 follows.

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