JEMS

Xuwen Chen · Justin Holmer

On the Klainerman–Machedon conjecture for the quantum BBGKY hierarchy with self-interaction

Abstract. We consider the 3D quantum BBGKY hierarchy which corresponds to the N-particle Schrödinger equation. We assume the pair interaction is $N^{3\beta-1}V(N^{\beta}\cdot)$. For the interaction parameter $\beta \in (0, 2/3)$, we prove that, provided an energy bound holds for solutions to the BBKGY hierarchy, the $N \to \infty$ limit points satisfy the space-time bound conjectured by S. Klainerman and M. Machedon [45] in 2008. The energy bound was proven to hold for $\beta \in (0, 3/5)$ in [28]. This allows, in the case $\beta \in (0, 3/5)$, for the application of the Klainerman–Machedon uniqueness theorem and hence implies that the $N \to \infty$ limit of BBGKY is uniquely determined as a tensor product of solutions to the Gross–Pitaevskii equation when the N-body initial data is factorized. The first result in this direction in 3D was obtained by T. Chen and N. Pavlović [11] for $\beta \in (0, 1/4)$ and subsequently by X. Chen [15] for $\beta \in (0, 2/7)$. We build upon the approach of X. Chen but apply frequency localized Klainerman–Machedon collapsing estimates and the endpoint Strichartz estimate in the estimate of the "potential part" to extend the range to $\beta \in (0, 2/3)$. Overall, this provides an alternative approach to the mean-field program by L. Erdős, B. Schlein, and H.-T. Yau [28], whose uniqueness proof is based upon Feynman diagram combinatorics.

Keywords. BBGKY hierarchy, n-particle Schrödinger equation, Klainerman–Machedon space-time bound, quantum Kac program

Contents

X. Chen: Department of Mathematics, University of Rochester, Hylan Building,
Rochester, NY 14618, USA; e-mail: chenxuwen@math.brown.edu,
https://www.math.rochester.edu/people/faculty/xchen84/

J. Holmer: Department of Mathematics, Brown University,
151 Thayer Street, Providence, RI 02912, USA; e-mail: holmer@math.brown.edu,
http://www.math.brown.edu/˜holmer/

1. Introduction

The 3D quantum BBGKY (Bogolyubov–Born–Green–Kirkwood–Yvon) hierarchy is generated from the N-body Hamiltonian evolution $\psi_N(t) = e^{itH_N}\psi_{N,0}$ with symmetric initial datum, and the N-body Hamiltonian is given by

$$H_N = -\Delta_{\mathbf{x}_N} + \frac{1}{N}\sum_{1 \leq i < j \leq N} N^{3\beta}V(N^\beta(x_i - x_j)). \tag{1.1}$$

In the above, $t \in \mathbb{R}$, $\mathbf{x}_N = (x_1, \ldots, x_N) \in \mathbb{R}^{3N}$, $\Delta_{\mathbf{x}_N}$ denotes the standard Laplacian with respect to the variables $\mathbf{x}_N \in \mathbb{R}^{3N}$, the factor $1/N$ in (1.1) is to make sure that the interactions are proportional to the number of particles, and the pair interaction $N^{3\beta}V(N^\beta(x_i - x_j))$ is an approximation to the Dirac δ function which matches the Gross–Pitaevskii description of Bose–Einstein condensation that the many-body effect should be modeled by a strong on-site self-interaction. Since $\psi_N\overline{\psi_N}$ is a probability density, we define the marginal densities $\{\gamma_N^{(k)}(t, \mathbf{x}_k, \mathbf{x}_k')\}_{k=1}^N$ by

$$\gamma_N^{(k)}(t, \mathbf{x}_k, \mathbf{x}_k') = \int \psi_N(t, \mathbf{x}_k, \mathbf{x}_{N-k})\overline{\psi_N}(t, \mathbf{x}_k', \mathbf{x}_{N-k})\, d\mathbf{x}_{N-k}, \quad \mathbf{x}_k, \mathbf{x}_k' \in \mathbb{R}^{3k}.$$

Then $\{\gamma_N^{(k)}(t, \mathbf{x}_k, \mathbf{x}_k')\}_{k=1}^N$ is a sequence of trace class operator kernels which are symmetric, in the sense that

$$\gamma_N^{(k)}(t, \mathbf{x}_k, \mathbf{x}_k') = \overline{\gamma_N^{(k)}(t, \mathbf{x}_k', \mathbf{x}_k)},$$

and

$$\gamma_N^{(k)}(t, x_{\sigma(1)}, \ldots, x_{\sigma(k)}, x_{\sigma(1)}', \ldots, x_{\sigma(k)}') = \gamma_N^{(k)}(t, x_1, \ldots, x_k, x_1', \ldots, x_k') \tag{1.2}$$

for any permutation σ, and satisfy the 3D quantum BBGKY hierarchy of equations which written in operator form is

$$i\partial_t \gamma_N^{(k)} + [\Delta_{\mathbf{x}_k}, \gamma_N^{(k)}] = \frac{1}{N}\sum_{1 \leq i < j \leq k} [V_N(x_i - x_j), \gamma_N^{(k)}]$$

$$+ \frac{N-k}{N}\sum_{j=1}^k \operatorname{Tr}_{k+1}[V_N(x_j - x_{k+1}), \gamma_N^{(k+1)}] \tag{1.3}$$

if we do not distinguish $\gamma_N^{(k)}$ as a kernel and the operator it defines.[1] Here the operator $V_N(x)$ represents multiplication by the function $V_N(x)$, where

$$V_N(x) = N^{3\beta}V(N^\beta x), \tag{1.4}$$

---

[1] From here on, we consider only the $\beta > 0$ case. For $\beta = 0$, see [31, 32, 46, 48, 50, 36, 37, 13, 7].

and Tr_{k+1} means taking the $k+1$ trace, for example,

$$\text{Tr}_{k+1} V_N(x_j - x_{k+1})\gamma_N^{(k+1)} = \int V_N(x_j - x_{k+1})\gamma_N^{(k+1)}(t, \mathbf{x}_k, x_{k+1}; \mathbf{x}_k', x_{k+1})\, dx_{k+1}.$$

In 2008, S. Klainerman and M. Machedon implicitly made the following conjecture on the solution of the BBGKY hierarchy.

Conjecture 1 (Klainerman–Machedon [45]). *Assume the interaction parameter β is in $(0, 1]$. Suppose that the sequence $\{\gamma_N^{(k)}(t, \mathbf{x}_k, \mathbf{x}_k')\}_{k=1}^N$ is a solution to the 3D quantum BBGKY hierarchy* (1.3) *subject to the energy condition: there is a C_0 (independent of N and k) such that for any $k \geq 0$, there is an $N_0(k)$ such that*

$$\forall N \geq N_0(k), \quad \sup_{t \in \mathbb{R}} \text{Tr}\Big(\prod_{j=1}^k (1 - \Delta_{x_j})\Big)\gamma_N^{(k)} \leq C_0^k. \tag{1.5}$$

Then, for every finite time T, every limit point $\Gamma = \{\gamma^{(k)}\}_{k=1}^\infty$ of $\{\Gamma_N\}_{N=1}^\infty = \{\{\gamma_N^{(k)}\}_{k=1}^N\}_{N=1}^\infty$ in $\bigoplus_{k \geq 1} C([0, T], \mathcal{L}_k^1)$ with respect to the product topology τ_{prod} (defined in Appendix A) satisfies the space-time bound: there is a C independent of j, k such that

$$\int_0^T \|R^{(k)} B_{j,k+1}\gamma^{(k+1)}(t)\|_{L_{\mathbf{x},\mathbf{x}'}^2}\, dt \leq C^k, \tag{1.6}$$

where $\mathcal{L}_k^1$ is the space of trace class operators on $L^2(\mathbb{R}^{3k})$, $R^{(k)} = \prod_{j=1}^k (|\nabla_{x_j}| |\nabla_{x_j'}|)$, and

$$B_{j,k+1} = \text{Tr}_{k+1}[\delta(x_j - x_{k+1}), \gamma^{(k+1)}].$$

Though Conjecture 1 was not explicitly stated in [45], as we will explain after stating Theorem 1.1, the bound (1.6) is necessary to implement Klainerman–Machedon's powerful and flexible approach in the most involved part of the quantum Kac program which mathematically proves the cubic nonlinear Schrödinger equation (NLS) as the $N \to \infty$ limit of quantum N-body dynamics. Kirkpatrick–Schlein–Staffilani [43] completely solved the $\mathbb{T}^2$ version of Conjecture 1 and were the first to successfully implement such an approach. However, Conjecture 1, in the $\mathbb{R}^3$ version as stated, was fully open until recently. T. Chen and Pavlović [11] have been able to prove Conjecture 1 for $\beta \in (0, 1/4)$. In [15], X. Chen simplified and extended the result to the range of $\beta \in (0, 2/7)$. We devote this paper to proving Conjecture 1 for $\beta \in (0, 2/3)$. In particular, we surpass the self-interaction threshold,[2] namely $\beta = 1/3$. To be specific, we prove the following theorem.

Theorem 1.1 (Main theorem). *Assume the interaction parameter β is in $(0, 2/3)$ and the pair interaction V is in $L^1 \cap W^{2,(6/5)+}$. Under condition* (2.5)*, every limit point $\Gamma = \{\gamma^{(k)}\}_{k=1}^\infty$ of $\{\Gamma_N\}_{N=1}^\infty$ satisfies the Klainerman–Machedon space-time bound* (1.6).

Establishing the $N \to \infty$ limit of hierarchy (1.3) justifies the mean-field limit in Gross–Pitaevskii theory. Such an approach was first proposed by Spohn [52] and can be regarded

---

[2] We will explain why we call the $\beta > 1/3$ case *self-interaction* later in this introduction.

as a quantum version of Kac's program. We see that, as $N \to \infty$, hierarchy (1.3) formally converges to the infinite Gross–Pitaevskii hierarchy

$$i\partial_t \gamma^{(k)} + [\Delta_{\mathbf{x}_k}, \gamma^{(k)}] = \left(\int V(x)\,dx\right) \sum_{j=1}^{k} \mathrm{Tr}_{k+1}[\delta(x_j - x_{k+1}), \gamma^{(k+1)}]. \qquad (1.7)$$

When the initial data is factorized,

$$\gamma^{(k)}(0, \mathbf{x}_k, \mathbf{x}'_k) = \prod_{j=1}^{k} \phi_0(x_j)\bar{\phi}_0(x_j),$$

hierarchy (1.7) has a special solution

$$\gamma^{(k)}(t, \mathbf{x}_k; \mathbf{x}'_k) = \prod_{j=1}^{k} \phi(t, x_j)\bar{\phi}(t, x_j) \qquad (1.8)$$

if ϕ solves the cubic NLS

$$i\partial_t \phi = -\Delta_x \phi + \left(\int V(x)\,dx\right)|\phi|^2 \phi. \qquad (1.9)$$

Thus such a limit process shows that, in an appropriate sense,

$$\lim_{N\to\infty} \gamma_N^{(k)} = \prod_{j=1}^{k} \phi(t, x_j)\bar{\phi}(t, x_j),$$

hence justifies the mean-field limit.

Such a limit in 3D was first proved in a series of important papers [26, 27, 28, 29, 30] by Elgart, Erdős, Schlein, and Yau.[3] Briefly, the Elgart–Erdős–Schlein–Yau approach[4] can be described as follows:

Step A. Prove that, with respect to the topology τ_{prod} defined in Appendix A, the sequence $\{\Gamma_N\}_{N=1}^{\infty}$ is compact in the space $\bigoplus_{k\geq 1} C([0, T], \mathcal{L}_k^1)$.

Step B. Prove that every limit point $\Gamma = \{\gamma^{(k)}\}_{k=1}^{\infty}$ of $\{\Gamma_N\}_{N=1}^{\infty}$ must verify hierarchy (1.7).

Step C. Prove that, in the space in which the limit points from Step B lie, there is a unique solution to hierarchy (1.7). Thus $\{\Gamma_N\}_{N=1}^{\infty}$ is a compact sequence with only one limit point. Hence $\Gamma_N \to \Gamma$ as $N \to \infty$.

In 2007, Erdős, Schlein, and Yau obtained the first uniqueness theorem [28, Theorem 9.1] for solutions to the hierarchy (1.7). The proof is surprisingly delicate—it spans 63 pages and uses complicated Feynman diagram techniques. The main difficulty is that hierarchy (1.7) is a system of infinitely coupled equations. Briefly, [28, Theorem 9.1] is the following:[5]

---

[3] Around the same time, there was the 1D work [1].

[4] See [4, 35, 49] for different approaches.

[5] For further development in this direction, see [6, 21, 34, 40].

Theorem 1.2 (Erdős–Schlein–Yau uniqueness [28, Theorem 9.1]). *There is at most one nonnegative symmetric operator sequence $\{\gamma^{(k)}\}_{k=1}^{\infty}$ that solves hierarchy* (1.7) *subject to the energy condition*

$$\sup_{t\in[0,T]} \mathrm{Tr}\Big(\prod_{j=1}^{k}(1-\Delta_{x_j})\Big)\gamma^{(k)} \le C^k. \tag{1.10}$$

In [45], based on their null form paper [44], Klainerman and Machedon gave a different proof of the uniqueness of hierarchy (1.7) in a space different from that used in [28, Theorem 9.1]. The proof is shorter (13 pages) than the proof of [28, Theorem 9.1]. Briefly, [45, Theorem 1.1] is the following:[6]

Theorem 1.3 (Klainerman–Machedon uniqueness [45, Theorem 1.1]). *There is at most one symmetric operator sequence $\{\gamma^{(k)}\}_{k=1}^{\infty}$ that solves hierarchy* (1.7) *subject to the space-time bound* (1.6).

For special cases like (1.8), condition (1.10) is actually

$$\sup_{t\in[0,T]} \|\langle\nabla_x\rangle\phi\|_{L^2} \le C, \tag{1.11}$$

while condition (1.6) means

$$\int_0^T \big\||\nabla_x|(|\phi|^2\phi)\big\|_{L^2}\,dt \le C. \tag{1.12}$$

When ϕ satisfies NLS (1.9), both are known. In fact, due to the Strichartz estimate [41], (1.11) implies (1.12), that is, condition (1.6) seems to be a bit weaker than (1.10). As already mentioned, the proof of [45, Theorem 1.1] is considerably shorter than the proof of [28, Theorem 9.1]. It is then natural to wonder whether [45, Theorem 1.1] simplifies Step C. To answer this question it is necessary to know whether the limit points in Step B satisfy condition (1.6), that is, whether Conjecture 1 holds.

Apart from curiosity, there are realistic reasons to study Conjecture 1. While [28, Theorem 9.1] is a powerful theorem, it is difficult to adapt such an argument to other interesting and colorful settings: a different spatial dimension, a three-body interaction instead of a pair interaction, or the Hermite operator instead of the Laplacian. The last situation mentioned is physically important. On the one hand, all the known experiments of BEC use harmonic trapping to stabilize the condensate [2, 24, 23, 42, 53]. On the other hand, different trapping strength produces quantum behaviors, which do not exist in the Boltzmann limit of classical particles or in the quantum case when the trapping is missing, and have been experimentally observed [33, 54, 22, 39, 25]. The Klainerman–Machedon approach applies easily in these meaningful situations [43, 9, 14, 15, 16, 34].[7] Thus proving Conjecture 1 actually helps to advance the study of quantum many-body dynamics and the mean-field approximation in the sense that it provides a flexible and powerful tool in 3D.

---

[6] For progress in this direction, see [19].

[7] See [17, 18, 20] for progress in the case of focusing interactions.

The well-posedness theory of the Gross–Pitaevskii hierarchy (1.7) subject to general initial data also requires that the limits of the BBGKY hierarchy (1.3) lie in the space in which the space-time bound (1.6) holds. See [8, 10, 11].

As pointed out in [26], the study of the Hamiltonian (1.1) is of particular interest when $\beta \in (1/3, 1]$. The reason is the following. In physics, the initial datum $\psi_N(0)$ of the Hamiltonian evolution $e^{itH_N}\psi_N(0)$ is usually assumed to be close to the ground state of the Hamiltonian

$$H_{N,0} = -\Delta_{\mathbf{x}_N} + \omega_0^2|\mathbf{x}_N|^2 + \frac{1}{N}\sum_{1\le i<j\le N} N^{3\beta}V(N^\beta(x_i - x_j)).$$

The preparation of the available experiments and the mathematical work [47] by Lieb, Seiringer, Solovej and Yngvason confirm this assumption. Such an initial datum $\psi_N(0)$ is localized in space. We can assume all N particles are in a box of side length 1. Let the effective radius of the pair interaction V be a; then the effective radius of V_N is about a/N^β. Thus every particle in the box interacts with $(a/N^\beta)^3 \times N$ other particles. Thus, for $\beta > 1/3$ and large N, every particle interacts with only itself. This exactly matches the Gross–Pitaevskii theory that the many-body effect should be modeled by a strong on-site self-interaction. Therefore, for the mathematical justification of Gross–Pitaevskii theory, it is of particular interest to prove Conjecture 1 for self-interaction ($\beta > 1/3$) as well.

To the best of our knowledge, the main theorem (Theorem 1.1) in the current paper is the first result proving Conjecture 1 for self-interaction ($\beta > 1/3$). For $\beta \le 1/3$, the first progress on Conjecture 1 is the $\beta \in (0, 1/4)$ work [11] by T. Chen and N. Pavlović and then the $\beta \in (0, 2/7)$ work [15] by X. Chen. As a matter of fact, the main theorem (Theorem 1.1) in the current paper has already fulfilled the original intent of [45], namely, simplifying the uniqueness argument of [28], because [28] deals with $\beta \in (0, 3/5)$. Conjecture 1 for $\beta \in [2/3, 1]$ is still open.

1.1. Organization of the paper. In §2, we outline the proof of Theorem 1.1. The overall pattern follows that introduced by X. Chen [15], who obtained Theorem 1.1 for $\beta \in (0, 2/7)$. Let $P_{\le M}^{(k)}$ be the Littlewood–Paley projection defined in (2.1). Theorem 1.1 will follow once it is established that for all $M \ge 1$, there exists N_0 depending on M such that for all $N \ge N_0$,

$$\|P_{\le M}^{(k)}R^{(k)}B_{N,j,k+1}\gamma_N^{(k+1)}(t)\|_{L_T^1 L_{\mathbf{x},\mathbf{x}'}^2} \le C^k \tag{1.13}$$

where $B_{N,j,k+1}$ is defined by (2.3). By substituting the Duhamel–Born expansion, carried out to coupling level K, of the BBGKY hierarchy, this is reduced to proving analogous bounds on the free part, potential part, and interaction part, defined in §2. Each part is reduced via the Klainerman–Machedon board game. Estimates for the free part and interaction part were previously obtained by X. Chen [15] but are reproduced here for convenience in Appendix B. For the estimate of the interaction part, one takes $K = \log N$, the utility of which was first observed by T. Chen and N. Pavlović [11].

The main new achievement of our paper is the improved estimates on the potential part, which are discussed in §3. We make use of the endpoint Strichartz estimate, phrased in terms of the X_b norm, in place of the Sobolev inequality employed by X. Chen [15].

We also introduce frequency localized versions of the Klainerman–Machedon collapsing estimates, allowing us to exploit the frequency localization in (1.13). Specifically, the operator $P_{\leq M}^{(k)}$ does not commute with $B_{N,j,k+1}$, but the composition $P_{\leq M_k}^{(k)} B_{N,j,k+1} P_{\sim M_{k+1}}^{(k+1)}$ enjoys better bounds if $M_{k+1} \gg M_k$. We prove the Strichartz estimate and the frequency localized Klainerman–Machedon collapsing estimates in §4. Frequency localized space-time techniques of this type were introduced by Bourgain [5, Chapter IV, §3] into the study of the well-posedness for nonlinear Schrödinger equations and other nonlinear dispersive PDE.

In [15], (1.13) is obtained without the frequency localization $P_{\leq M}^{(k)}$ for $\beta \in (0, 2/7)$. In Theorem 3.2, we prove that this estimate still holds without frequency localization for $\beta \in (0, 2/5)$ by using the Strichartz estimate alone. This already surpasses the self-interaction threshold $\beta = 1/3$. For the purpose of proving Conjecture 1, the frequency localized estimate (1.13) is equally good, but allows us to achieve higher β.

2. Proof of the main theorem

We establish Theorem 1.1 in this section. For simplicity of notation, we denote $\| \cdot \|_{L^p[0,T]L_{\mathbf{x},\mathbf{x}'}^2}$ by $\| \cdot \|_{L_T^p L_{\mathbf{x},\mathbf{x}'}^2}$, and $\| \cdot \|_{L_t^p(\mathbb{R})L_{\mathbf{x},\mathbf{x}'}^2}$ by $\| \cdot \|_{L_t^p L_{\mathbf{x},\mathbf{x}'}^2}$. Let us begin by introducing some notation for Littlewood–Paley theory. Let $P_{\leq M}^i$ be the projection onto frequencies $\leq M$ and P_M^i the analogous projections onto frequencies $\sim M$, acting on functions of $x_i \in \mathbb{R}^3$ (the ith coordinate). We take M to be a dyadic frequency range $2^\ell \geq 1$. Similarly, we define $P_{\leq M}^{i'}$ and $P_M^{i'}$, which act on the variable x_i'. Let

$$P_{\leq M}^{(k)} = \prod_{i=1}^k P_{\leq M}^i P_{\leq M}^{i'}. \tag{2.1}$$

To establish Theorem 1.1, it suffices to prove the following theorem.

Theorem 2.1. *Under the assumptions of Theorem 1.1, there exists a C (independent of k, M, N) such that for each $M \geq 1$ there exists N_0 (depending on M) such that for $N \geq N_0$,*

$$\|P_{\leq M}^{(k)} R^{(k)} B_{N,j,k+1}\gamma_N^{(k+1)}(t)\|_{L_T^1 L_{\mathbf{x},\mathbf{x}'}^2} \leq C^k \tag{2.2}$$

where

$$B_{N,j,k+1}\gamma_N^{(k+1)} = \mathrm{Tr}_{k+1}[V_N(x_j - x_{k+1}), \gamma_N^{(k+1)}]. \tag{2.3}$$

We first explain how, assuming Theorem 2.1, we can prove Theorem 1.1. When condition (1.5) holds, it has been proved in Elgart–Erdős–Schlein–Yau [26, 27, 28, 29, 30] and Kirkpatrick–Schlein–Staffilani [43] that, as trace class operators,

$$B_{N,j,k+1}\gamma_N^{(k+1)} \rightharpoonup B_{j,k+1}\gamma^{(k+1)} \quad (\text{weak}^*) \tag{2.4}$$

uniformly in t (see [43, (6.7)] or [16, (5.6)], for example). Let $\mathcal{H}_k$ be the Hilbert–Schmidt operators on $L^2(\mathbb{R}^{3k})$. Recall that the test functions for weak* convergence in $\mathcal{L}_k^1$ come

from $\mathcal{K}_k$ (the compact operators on $L^2(\mathbb{R}^{3k})$) and the test functions for weak* convergence in $\mathcal{H}_k$ come from $\mathcal{H}_k$. Thus the weak* convergence (2.4) as trace class operators implies that as Hilbert–Schmidt operators,

$$B_{N,j,k+1}\gamma_N^{(k+1)} \rightharpoonup B_{j,k+1}\gamma^{(k+1)} \quad (\text{weak*})$$

uniformly in t, because $\mathcal{H}_k \subset \mathcal{K}_k$, i.e. there are fewer test functions. Since $\mathcal{H}_k$ is reflexive, the above weak* convergence is no different from the weak convergence. Moreover, noticing that $P_{\leq M}^{(k)} R^{(k)} J$ is simply another test function if J is a test function, we know that as Hilbert–Schmidt operators,

$$P_{\leq M}^{(k)} R^{(k)} B_{N,j,k+1}\gamma_N^{(k+1)} \rightharpoonup P_{\leq M}^{(k)} R^{(k)} B_{j,k+1}\gamma^{(k+1)} \quad (\text{weak})$$

uniformly in t. Hence, by basic properties of weak convergence,

$$\|P_{\leq M}^{(k)} R^{(k)} B_{j,k+1}\gamma^{(k+1)}\|_{L_T^1 L_{\mathbf{x},\mathbf{x}'}^2} \leq \liminf_{N\to\infty} \|P_{\leq M}^{(k)} R^{(k)} B_{N,j,k+1}\gamma_N^{(k+1)}(t)\|_{L_T^1 L_{\mathbf{x},\mathbf{x}'}^2} \leq C^k.$$

Since the above holds uniformly in M, we can let $M \to \infty$ and, by the monotone convergence theorem, we obtain

$$\|R^{(k)} B_{j,k+1}\gamma^{(k+1)}\|_{L_T^1 L_{\mathbf{x},\mathbf{x}'}^2} \leq C^k,$$

which is exactly the Klainerman–Machedon space-time bound (1.6). This completes the proof of Theorem 1.1, assuming Theorem 2.1.

The rest of this paper is devoted to proving Theorem 2.1. We are going to establish estimate (2.2) for a sufficiently small T which depends on the controlling constant in condition (1.5) and is independent of k, N and M; then a bootstrap argument together with condition (1.5) gives estimate (2.2) for every finite time at the price of a larger constant C. Before we start, alert readers should keep in mind that we will mostly use the following form of condition (1.5):

$$\|S^{(k)}\gamma_N^{(k)}\|_{L_t^\infty L_{\mathbf{x},\mathbf{x}'}^2} \leq C_0^k \tag{2.5}$$

where $S^{(k)} = \prod_{j=1}^k (\langle \nabla_{x_j}\rangle\langle \nabla_{x_j'}\rangle)$, because we will be working in L^2. To see how (2.5) follows from (1.5), one simply notices that

$$\int \left| \langle\nabla_x\rangle\langle\nabla_{x'}\rangle \int \phi(x,r)\overline{\phi(x',r)}\, dr \right|^2 dx\, dx'$$

$$= \int \left| \int \langle\nabla_x\rangle\phi(x,r)\overline{\langle\nabla_{x'}\rangle\phi(x',r)}\, dr \right|^2 dx\, dx'$$

$$\leq \int \left(\int \langle\nabla_x\rangle\phi(x,r)\overline{\langle\nabla_x\rangle\phi(x,r)}\, dr \right) \left(\int \langle\nabla_{x'}\rangle\phi(x',r)\overline{\langle\nabla_{x'}\rangle\phi(x',r)}\, dr \right) dx\, dx'$$

$$= \left(\int \phi(x,r)\overline{(1-\Delta_x)\phi(x,r)}\, dx\, dr \right)^2.$$

We start the proof of Theorem 2.1 by rewriting hierarchy (1.3) as

$$
\gamma_N^{(k)}(t_k) = U^{(k)}(t_k)\gamma_{N,0}^{(k)} + \int_0^{t_k} U^{(k)}(t_k - t_{k+1})V_N^{(k)}\gamma_N^{(k)}(t_{k+1})\, dt_{k+1}
$$

$$
+ \frac{N-k}{N}\int_0^{t_k} U^{(k)}(t_k - t_{k+1})B_N^{(k+1)}\gamma_N^{(k+1)}(t_{k+1})\, dt_{k+1} \qquad (2.6)
$$

with the short-hand notation

$$
U^{(k)} = e^{it\Delta_{\mathbf{x}_k}}e^{-it\Delta_{\mathbf{x}_k'}},
$$

$$
V_N^{(k)}\gamma_N^{(k)} = \frac{1}{N}\sum_{1\le i<j\le k}[V_N(x_i - x_j), \gamma_N^{(k)}],
$$

$$
B_N^{(k+1)}\gamma_N^{(k+1)} = \sum_{j=1}^{k} B_{N,j,k+1}\gamma_N^{(k+1)}.
$$

We omit the i in front of the potential term and the interaction term so that we do not need to keep track of its exact power.

Writing out the l_cth Duhamel–Born series of $\gamma_N^{(k)}$ by iterating hierarchy (2.6) l_c times,[8] we have

$$
\gamma_N^{(k)}(t_k)
$$

$$
= U^{(k)}(t_k)\gamma_{N,0}^{(k)} + \frac{N-k}{N}\int_0^{t_k} U^{(k)}(t_k - t_{k+1})B_N^{(k+1)}U^{(k+1)}(t_{k+1})\gamma_{N,0}^{(k+1)}\, dt_{k+1}
$$

$$
+ \int_0^{t_k} U^{(k)}(t_k - t_{k+1})V_N^{(k)}\gamma_N^{(k)}(t_{k+1})\, dt_{k+1}
$$

$$
+ \frac{N-k}{N}\int_0^{t_k} U^{(k)}(t_k - t_{k+1})B_N^{(k+1)}
$$

$$
\times \int_0^{t_{k+1}} U^{(k+1)}(t_{k+1} - t_{k+2})V_N^{(k+1)}\gamma_N^{(k+1)}(t_{k+2})\, dt_{k+2}\, dt_{k+1}
$$

$$
+ \frac{N-k}{N}\frac{N-k-1}{N}
$$

$$
\times \int_0^{t_k} U^{(k)}(t_k - t_{k+1})B_N^{(k+1)}\int_0^{t_{k+1}} U^{(k+1)}(t_{k+1} - t_{k+2})B_N^{(k+2)}\gamma_N^{(k+2)}(t_{k+2})\, dt_{k+2}\, dt_{k+1}
$$

$$
= \cdots.
$$

After l_c iterations

$$
\gamma_N^{(k)}(t_k) = FP^{(k,l_c)}(t_k) + PP^{(k,l_c)}(t_k) + IP^{(k,l_c)}(t_k) \qquad (2.7)
$$

---

[8] Here, l_c stands for "level of coupling" or "length/depth of coupling". When $l_c = 0$, we recover (2.6).

where the *free part* at coupling level l_c is given by

$$
\begin{aligned}
FP^{(k,l_c)} &= U^{(k)}(t_k)\gamma_{N,0}^{(k)} \\
&\quad + \sum_{j=1}^{l_c}\Big(\prod_{l=0}^{j-1}\frac{N-k-l}{N}\Big)\int_0^{t_k}\cdots\int_0^{t_{k+j-1}}U^{(k)}(t_k-t_{k+1})B_N^{(k+1)}\cdots \\
&\quad\quad \ldots U^{(k+j-1)}(t_{k+j-1}-t_{k+j})B_N^{(k+j)}(U^{(k+j)}(t_{k+j})\gamma_{N,0}^{(k+j)})\,dt_{k+1}\cdots dt_{k+j},
\end{aligned}
$$

the *potential part* is given by

$$
\begin{aligned}
PP^{(k,l_c)} &= \int_0^{t_k}U^{(k)}(t_k-t_{k+1})V_N^{(k)}\gamma_N^{(k)}(t_{k+1})\,dt_{k+1} + \sum_{j=1}^{l_c}\Big(\prod_{l=0}^{j-1}\frac{N-k-l}{N}\Big) \\
&\quad \times \int_0^{t_k}\cdots\int_0^{t_{k+j-1}}U^{(k)}(t_k-t_{k+1})B_N^{(k+1)}\cdots U^{(k+j-1)}(t_{k+j-1}-t_{k+j})B_N^{(k+j)} \\
&\quad \times \left(\int_0^{t_{k+j}}U^{(k+j)}(t_{k+j}-t_{k+j+1})V_N^{(k+j)}\gamma_N^{(k+j)}(t_{k+j+1})\,dt_{k+j+1}\right)dt_{k+1}\cdots dt_{k+j},
\end{aligned}
$$

$$\tag{2.8}$$

and the *interaction part* is given by

$$
\begin{aligned}
IP^{(k,l_c)} &= \Big(\prod_{l=0}^{l_c}\frac{N-k-l}{N}\Big)\int_0^{t_k}\cdots\int_0^{t_{k+l_c}}U^{(k)}(t_k-t_{k+1})B_N^{(k+1)}\cdots \\
&\quad \cdots U^{(k+l_c)}(t_{k+l_c}-t_{k+l_c+1})B_N^{(k+l_c+1)}(\gamma_N^{(k+l_c+1)}(t_{k+l_c+1}))\,dt_{k+1}\cdots dt_{k+l_c+1}.
\end{aligned}
$$

By (2.7), to establish (2.2), it suffices to prove that

$$
\|P_{\le M}^{(k-1)}R^{(k-1)}B_{N,1,k}FP^{(k,l_c)}\|_{L_T^1 L_{\mathbf{x},\mathbf{x}'}^2} \le C^{k-1}, \tag{2.9}
$$

$$
\|P_{\le M}^{(k-1)}R^{(k-1)}B_{N,1,k}PP^{(k,l_c)}\|_{L_T^1 L_{\mathbf{x},\mathbf{x}'}^2} \le C^{k-1}, \tag{2.10}
$$

$$
\|P_{\le M}^{(k-1)}R^{(k-1)}B_{N,1,k}IP^{(k,l_c)}\|_{L_T^1 L_{\mathbf{x},\mathbf{x}'}^2} \le C^{k-1}, \tag{2.11}
$$

for all $k \ge 2$ and for some C and a sufficiently small T determined by the controlling constant in condition (2.5) and independent of k, N and M. We observe that $B_N^{(j)}$ has $2j$ terms inside so that each summand of $\gamma_N^{(k)}(t_k)$ contains factorially many terms ($\sim (k+l_c)!/k!$). We use the Klainerman–Machedon board game to combine them and hence reduce the number of terms that need to be treated. Define

$$
J_N^{(k,j)}(\underline{t}_{k+j})(f^{(k+j)}) = U^{(k)}(t_k-t_{k+1})B_N^{(k+1)}\cdots U^{(k+j-1)}(t_{k+j-1}-t_{k+j})B_N^{(k+j)}f^{(k+j)},
$$

where $\underline{t}_{k+j}$ means $(t_{k+1},\ldots,t_{k+j})$; then the Klainerman–Machedon board game implies the lemma below.

Lemma 2.1 (Klainerman–Machedon board game). *One can express*

$$\int_0^{t_k} \cdots \int_0^{t_{k+j-1}} J_N^{(k,j)}(\underline{t}_{k+j})(f^{(k+j)}) \, d\underline{t}_{k+j}$$

as a sum of at most 4^{j-1} terms of the form

$$\int_D J_N^{(k,j)}(\underline{t}_{k+j}, \mu_m)(f^{(k+j)}) \, d\underline{t}_{k+j},$$

that is,

$$\int_0^{t_k} \cdots \int_0^{t_{k+j-1}} J_N^{(k,j)}(\underline{t}_{k+j})(f^{(k+j)}) \, d\underline{t}_{k+j} = \sum_m \int_D J_N^{(k,j)}(\underline{t}_{k+j}, \mu_m)(f^{(k+j)}) \, d\underline{t}_{k+j}.$$

Here $D \subset [0, t_k]^j$, μ_m are maps from $\{k+1, \dots, k+j\}$ to $\{k, \dots, k+j-1\}$ satisfying $\mu_m(k+1) = k$ and $\mu_m(l) < l$ for all l, and

$$J_N^{(k,j)}(\underline{t}_{k+j}, \mu_m)(f^{(k+j)})$$
$$= U^{(k)}(t_k - t_{k+1}) B_{N,k,k+1} U^{(k+1)}(t_{k+1} - t_{k+2}) B_{N,\mu_m(k+2),k+2} \cdots$$
$$\cdots U^{(k+j-1)}(t_{k+j-1} - t_{k+j}) B_{N,\mu_m(k+j),k+j}(f^{(k+j)}).$$

Proof. Follow the proof of [45, Theorem 3.4], the Klainerman–Machedon board game, replacing $B_{j,k+1}$ by $B_{N,j,k+1}$ and noticing that $B_{N,j,k+1}$ still commutes with $e^{it\Delta_{x_i}} e^{-it\Delta_{x_i'}}$ whenever $i \neq j$. This argument reduces the number of terms by combining them. $\qquad\square$

In the rest of this paper, we establish estimate (2.10) only. The reason is the following. On the one hand, the proof of (2.10) is exactly the place that relies on the restriction $\beta \in (0, 2/3)$ in this paper. On the other hand, X. Chen has already proven estimates (2.9) and (2.11) [15, (6.3) and (6.5)] without using any frequency localization. For completeness, we include a proof of (2.9) and (2.11) in Appendix B. Before we delve into the proof of (2.10), we remark that the proof of (2.9) and (2.10) is independent of the coupling level l_c and we will take l_c to be $\log N$ for estimate (2.11).[9]

3. Estimate of the potential part

In this section, we prove estimate (2.10). To be specific, we establish the following theorem.

Theorem 3.1. *Under the assumptions of Theorem 1.1, there exists a C (independent of k, l_c, M_{k-1}, N) such that for each $M_{k-1} \geq 1$ there exists N_0 (depending on M_{k-1}) such that for $N \geq N_0$,*

$$\|P_{\leq M_{k-1}}^{(k-1)} R^{(k-1)} B_{N,1,k} PP^{(k,l_c)}\|_{L_T^1 L_{\mathbf{x},\mathbf{x}'}^2} \leq C^{k-1}$$

where $PP^{(k,l_c)}$ is given by (2.8).

---

[9] The technique of taking $k = \log N$ for estimate (2.11) was first applied by T. Chen and N. Pavlović [11].

In this section, we will employ the estimates stated and proved in Section 4. Due to the technicality of the proof of Theorem 3.1 involving Littlewood–Paley theory, we prove a simpler $\beta \in (0, 2/5)$ version first to illustrate the basic steps in establishing Theorem 3.1. We then prove Theorem 3.1 in Section 3.2.

3.1. A simpler proof in the case $\beta \in (0, 2/5)$

Theorem 3.2. *For $\beta \in (0, 2/5)$, we have*

$$\|R^{(k-1)} B_{N,1,k} PP^{(k,l_c)}\|_{L^1_T L^2_{\mathbf{x},\mathbf{x}'}} \leq C^{k-1}$$

for some C and a sufficiently small T determined by the controlling constant in condition (2.5) and independent of k, l_c and N.

Proof. The proof is divided into four steps. We will reproduce every step for Theorem 3.1 in Section 3.2.

Step I. By Lemma 2.1, we know that

$$PP^{(k,l_c)} = \int_0^{t_k} U^{(k)}(t_k - t_{k+1}) V_N^{(k)} \gamma_N^{(k)}(t_{k+1}) \, dt_{k+1}$$
$$+ \sum_{j=1}^{l_c} \left(\prod_{l=0}^{j-1} \frac{N-k-l}{N} \right) \left(\sum_m \int_D J_N^{(k,j)}(\underline{t}_{k+j}, \mu_m)(f^{(k+j)}) \, d\underline{t}_{k+j} \right) \quad (3.1)$$

where

$$f^{(k+j)} = \int_0^{t_{k+j}} U^{(k+j)}(t_{k+j} - t_{k+j+1}) V_N^{(k+j)} \gamma_N^{(k+j)}(t_{k+j+1}) \, dt_{k+j+1}, \quad (3.2)$$

and the sum $\sum_m$ has at most 4^{j-1} terms.

For the second term in (3.1), we iterate Lemma 4.2 to prove the following estimate:[10]

$$\left\| R^{(k-1)} B_{N,1,k} \int_D J_N^{(k,j)}(\underline{t}_{k+j}, \mu_m)(f^{(k+j)}) \, d\underline{t}_{k+j} \right\|_{L^1_T L^2_{\mathbf{x},\mathbf{x}'}}$$
$$\leq (CT^{1/2})^j \|R^{(k+j-1)} B_{N,\mu_m(k+j),k+j} f^{(k+j)}\|_{L^1_T L^2_{\mathbf{x},\mathbf{x}'}}. \quad (3.3)$$

In fact,

$$A := \left\| R^{(k-1)} B_{N,1,k} \int_D J_N^{(k,j)}(\underline{t}_{k+j}, \mu_m)(f^{(k+j)}) \, d\underline{t}_{k+j} \right\|_{L^1_T L^2_{\mathbf{x},\mathbf{x}'}}$$
$$= \int_0^T \left\| \int_D R^{(k-1)} B_{N,1,k} U^{(k)}(t_k - t_{k+1}) B_{N,k,k+1} \cdots dt_{k+1} \ldots dt_{k+j} \right\|_{L^2_{\mathbf{x},\mathbf{x}'}} dt_k.$$

---

[10] This also helps in proving estimates (2.9) and (2.11)—see Appendix B.

By Minkowski,

$$A \le \int_{[0,T]^{j+1}} \|R^{(k-1)} B_{N,1,k} U^{(k)}(t_k - t_{k+1}) B_{N,k,k+1} \cdots \|_{L^2_{\mathbf{x},\mathbf{x}'}} \, dt_k \, dt_{k+1} \ldots dt_{k+j},$$

and by Cauchy–Schwarz in dt_k,

$$A \le$$
$$T^{1/2} \int_{[0,T]^j} \left(\int \|R^{(k-1)} B_{N,1,k} U^{(k)}(t_k - t_{k+1}) B_{N,k,k+1} \cdots \|^2_{L^2_{\mathbf{x},\mathbf{x}'}} \, dt_k \right)^{1/2} dt_{k+1} \ldots dt_{k+j}.$$

Use Lemma 4.2 to get

$$A \le C T^{1/2} \int_{[0,T]^j} \|R^{(k)} B_{N,k,k+1} U^{(k+1)}(t_{k+1} - t_{k+2}) \cdots \|_{L^2_{\mathbf{x},\mathbf{x}'}} \, dt_{k+1} \ldots dt_{k+j}.$$

Repeating the previous steps $j - 1$ times, we reach relation (3.3).

Applying (3.3) to (3.1), we obtain

$$\|R^{(k-1)} B_{N,1,k} PP^{(k,l_c)}\|_{L^1_T L^2_{\mathbf{x},\mathbf{x}'}}$$
$$\le \left\| R^{(k-1)} B_{N,1,k} \int_0^{t_k} U^{(k)}(t_k - t_{k+1}) V_N^{(k)} \gamma_N^{(k)}(t_{k+1}) \, dt_{k+1} \right\|_{L^1_T L^2_{\mathbf{x},\mathbf{x}'}}$$
$$+ \sum_{j=1}^{l_c} 4^{j-1} (CT^{1/2})^j \|R^{(k+j-1)} B_{N,\mu_m(k+j),k+j}(f^{(k+j)})\|_{L^1_T L^2_{\mathbf{x},\mathbf{x}'}}$$
$$\le \left\| R^{(k-1)} B_{N,1,k} \int_0^{t_k} U^{(k)}(t_k - t_{k+1}) V_N^{(k)} \gamma_N^{(k)}(t_{k+1}) \, dt_{k+1} \right\|_{L^1_T L^2_{\mathbf{x},\mathbf{x}'}}$$
$$+ \sum_{j=1}^{l_c} (CT^{1/2})^j \|R^{(k+j-1)} B_{N,\mu_m(k+j),k+j}(f^{(k+j)})\|_{L^1_T L^2_{\mathbf{x},\mathbf{x}'}}.$$

Inserting a smooth cut-off $\theta(t)$ with $\theta(t) = 1$ for $t \in [-T, T]$ and $\theta(t) = 0$ for $t \in [-2T, 2T]^c$ into the above estimate, we get

$$\|R^{(k-1)} B_{N,1,k} PP^{(k,l_c)}\|_{L^1_T L^2_{\mathbf{x},\mathbf{x}'}}$$
$$\le \left\| R^{(k-1)} B_{N,1,k} \theta(t_k) \int_0^{t_k} U^{(k)}(t_k - t_{k+1}) \theta(t_{k+1}) V_N^{(k)} \gamma_N^{(k)}(t_{k+1}) \, dt_{k+1} \right\|_{L^1_T L^2_{\mathbf{x},\mathbf{x}'}}$$
$$+ \sum_{j=1}^{l_c} (CT^{1/2})^j \|R^{(k+j-1)} B_{N,\mu_m(k+j),k+j} \theta(t_{k+j})(\tilde{f}^{(k+j)})\|_{L^1_T L^2_{\mathbf{x},\mathbf{x}'}}$$

where

$$\tilde{f}^{(k+j)} = \int_0^{t_{k+j}} U^{(k+j)}(t_{k+j} - t_{k+j+1}) \big(\theta(t_{k+j+1}) V_N^{(k+j)} \gamma_N^{(k+j)}(t_{k+j+1}) \big) \, dt_{k+j+1}. \quad (3.4)$$

Step II. The X_b space (defined in §4) version of Lemma 4.2, Lemma 4.3, turns the last step into

$$\|R^{(k-1)}B_{N,1,k}PP^{(k,l_c)}\|_{L_T^1 L_{\mathbf{x},\mathbf{x}'}^2}$$
$$\leq CT^{1/2}\left\|\theta(t_k)\int_0^{t_k} U^{(k)}(t_k - t_{k+1})R^{(k)}\big(\theta(t_{k+1})V_N^{(k)}\gamma_N^{(k)}(t_{k+1})\big)\,dt_{k+1}\right\|_{X_{(1/2)+}^{(k)}}$$
$$+ C\sum_{j=1}^{l_c}(CT^{1/2})^{j+1}\|\theta(t_{k+j})R^{(k+j)}\tilde{f}^{(k+j)}\|_{X_{(1/2)+}^{(k+j)}}.$$

Step III. Recall the definition of $\tilde{f}^{(k+j)}$,

$$\tilde{f}^{(k+j)} = \int_0^{t_{k+j}} U^{(k+j)}(t_{k+j} - t_{k+j+1})\big(\theta(t_{k+j+1})V_N^{(k+j)}\gamma_N^{(k+j)}(t_{k+j+1})\big)\,dt_{k+j+1},$$

so

$$R^{(k+j)}\tilde{f}^{(k+j)}$$
$$= \int_0^{t_{k+j}} U^{(k+j)}(t_{k+j} - t_{k+j+1})R^{(k+j)}\big(\theta(t_{k+j+1})V_N^{(k+j)}\gamma_N^{(k+j)}(t_{k+j+1})\big)\,dt_{k+j+1}.$$

We then apply Lemma 4.1 to get

$$\|R^{(k-1)}B_{N,1,k}PP^{(k,l_c)}\|_{L_T^1 L_{\mathbf{x},\mathbf{x}'}^2} \leq CT^{1/2}\|R^{(k)}\big(\theta(t_{k+1})V_N^{(k)}\gamma_N^{(k)}(t_{k+1})\big)\|_{X_{-(1/2)+}^{(k)}}$$
$$+ C\sum_{j=1}^{l_c}(CT^{1/2})^{j+1}\|R^{(k+j)}\big(\theta(t_{k+j+1})V_N^{(k+j)}\gamma_N^{(k+j)}(t_{k+j+1})\big)\|_{X_{-(1/2)+}^{(k+j)}}.$$

Step IV. Now we would like to utilize Lemma 4.6. We first analyse a typical term to demonstrate the effect of Lemma 4.6. To be specific, we have

$$\big\|R^{(k)}\big(\theta(t_{k+1})V_N(x_1 - x_2)\gamma_N^{(k)}(t_{k+1})\big)\big\|_{X_{-(1/2)+}^{(k)}}$$
$$\leq \frac{C}{N}\|V_N(x_1 - x_2)\theta(t_{k+1})R^{(k)}\gamma_N^{(k)}(t_{k+1})\|_{X_{-(1/2)+}^{(k)}}$$
$$+ \frac{C}{N}\left\|(V_N)'(x_1 - x_2)\theta(t_{k+1})\left(\frac{R^{(k)}}{|\nabla_{x_1}|}\right)\gamma_N^{(k)}(t_{k+1})\right\|_{X_{-(1/2)+}^{(k)}}$$
$$+ \frac{C}{N}\left\|(V_N)''(x_1 - x_2)\theta(t_{k+1})\left(\frac{R^{(k)}}{|\nabla_{x_1}||\nabla_{x_2}|}\right)\gamma_N^{(k)}(t_{k+1})\right\|_{X_{-(1/2)+}^{(k)}}$$

by Leibniz's rule, where

$$\frac{R^{(k)}}{|\nabla_{x_1}|} = \left(\prod_{j=2}^k |\nabla_{x_j}|\right)\left(\prod_{j=1}^k |\nabla_{x_j'}|\right).$$

Applying Lemma 4.6 to each summand above yields

$$\big\| R^{(k)}\big(\theta(t_{k+1})V_N(x_1 - x_2)\gamma_N^{(k)}(t_{k+1})\big)\big\|_{X_{-(1/2)+}^{(k)}}$$

$$\leq \frac{C}{N}\|V_N\|_{L^{3+}}\|\theta(t_{k+1})R^{(k)}\gamma_N^{(k)}\|_{L_{t_{k+1}}^2 L_{\mathbf{x},\mathbf{x}'}^2}$$

$$+ \frac{C}{N}\|V_N'\|_{L^{2+}}\bigg\|\theta(t_{k+1})\langle\nabla_{x_1}\rangle^{1/2}\bigg(\frac{R^{(k)}}{|\nabla_{x_1}|}\bigg)\gamma_N^{(k)}\bigg\|_{L_{t_{k+1}}^2 L_{\mathbf{x},\mathbf{x}'}^2}$$

$$+ \frac{C}{N}\|V_N''\|_{L^{(6/5)+}}\bigg\|\theta(t_{k+1})\langle\nabla_{x_1}\rangle\langle\nabla_{x_2}\rangle\bigg(\frac{R^{(k)}}{|\nabla_{x_1}||\nabla_{x_2}|}\bigg)\gamma_N^{(k)}\bigg\|_{L_{t_{k+1}}^2 L_{\mathbf{x},\mathbf{x}'}^2}$$

$$\leq C\|S^{(k)}\gamma_N^{(k)}\|_{L_{2T}^2 L_{\mathbf{x},\mathbf{x}'}^2},$$

since $\|V_N/N\|_{L^{3+}}$ $\|V_N'/N\|_{L^{2+}}$, and $\|V_N''/N\|_{L^{(6/5)+}}$ are uniformly bounded in N for $\beta \in (0, 2/5)$. In fact,

$$\|V_N/N\|_{L^{3+}} \leq N^{2\beta-1}\|V\|_{L^{3+}},$$
$$\|V_N'/N\|_{L^{2+}} \leq N^{5\beta/2-1}\|V'\|_{L^{2+}},$$
$$\|V_N''/N\|_{L^{(6/5)+}} \leq N^{5\beta/2-1}\|V''\|_{L^{(6/5)+}},$$

where by Sobolev, $V \in W^{2,(6/5)+}$ implies $V \in L^{(6/5)+} \cap L^{6+}$ and $V' \in L^{2+}$.

Using the same idea for all the terms, we end up with

$$\|R^{(k-1)}B_{N,1,k}PP^{(k,l_c)}\|_{L_T^1 L_{\mathbf{x},\mathbf{x}'}^2}$$

$$\leq CTk^2\|S^{(k)}\gamma_N^{(k)}\|_{L_{2T}^\infty L_{\mathbf{x},\mathbf{x}'}^2} + CT^{1/2}\sum_{j=1}^{l_c}(CT^{1/2})^{j+1}(k+j)^2\|S^{(k+j)}\gamma_N^{(k+j)}\|_{L_{2T}^\infty L_{\mathbf{x},\mathbf{x}'}^2}$$

because there are k^2 terms inside $V_N^{(k)}$. Plug in condition (2.5):

$$\|R^{(k-1)}B_{N,1,k}PP^{(k,l_c)}\|_{L_T^1 L_{\mathbf{x},\mathbf{x}'}^2} \leq CTk^2C_0^k + CT^{1/2}\sum_{j=1}^{\infty}(CT^{\frac{1}{2}})^{j+1}(k+j)^2C_0^{k+j}$$

$$\leq C_0^k\Big(CTk^2 + CT^{1/2}k^2\sum_{j=1}^{\infty}(CT^{1/2})^{j+1}C_0^j + CT^{1/2}\sum_{j=1}^{\infty}(CT^{1/2})^{j+1}j^2C_0^j\Big).$$

We can then choose a T independent of k, l_c and N such that the two infinite series converge. We get

$$\|R^{(k-1)}B_{N,1,k}PP^{(k,l_c)}\|_{L_T^1 L_{\mathbf{x},\mathbf{x}'}^2} \leq C_0^k(CTk^2 + CT^{1/2}k^2 + CT^{1/2})$$

$$\leq C_0^k(CT2^k + CT^{1/2}2^k + CT^{1/2}) \leq C^{k-1}$$

for some C larger than C_0 because $k \geq 2$. This concludes the proof of Theorem 3.2. $\quad\square$

3.2. The case $\beta \in (0, 2/3)$. To make formulas shorter, let us write

$$R^{(k)}_{\leq M_k} = P^{(k)}_{\leq M_k} R^{(k)},$$

since $P^{(k)}_{\leq M_k}$ and $R^{(k)}$ are usually bundled together.

3.2.1. Step I. By (3.1),

$$\|R^{(k-1)}_{\leq M_{k-1}} B_{N,1,k} PP^{(k,l_c)}\|_{L^1_T L^2_{\mathbf{x},\mathbf{x}'}}$$
$$\leq \left\| R^{(k-1)}_{\leq M_{k-1}} B_{N,1,k} \int_0^{t_k} U^{(k)}(t_k - t_{k+1}) V_N^{(k)} \gamma_N^{(k)}(t_{k+1}) \, dt_{k+1} \right\|_{L^1_T L^2_{\mathbf{x},\mathbf{x}'}}$$
$$+ \sum_{j=1}^{l_c} \sum_m \left\| R^{(k-1)}_{\leq M_{k-1}} B_{N,1,k} \int_D J_N^{(k,j)}(\underline{t}_{k+j}, \mu_m)(f^{(k+j)}) \, d\underline{t}_{k+j} \right\|_{L^1_T L^2_{\mathbf{x},\mathbf{x}'}}$$

where $f^{(k+j)}$ is again given by (3.2) and the sum $\sum_m$ has at most 4^{j-1} terms. By Minkowski's integral inequality,

$$\left\| R^{(k-1)}_{\leq M_{k-1}} B_{N,1,k} \int_D J_N^{(k,j)}(\underline{t}_{k+j}, \mu_m)(f^{(k+j)}) \, d\underline{t}_{k+j} \right\|_{L^1_T L^2_{\mathbf{x},\mathbf{x}'}}$$
$$= \int_0^T \left\| \int_D R^{(k-1)}_{\leq M_{k-1}} B_{N,1,k} J_N^{(k,j)}(\underline{t}_{k+j}, \mu_m)(f^{(k+j)}) \, d\underline{t}_{k+j} \right\|_{L^2_{\mathbf{x},\mathbf{x}'}} dt_k$$
$$\leq \int_{[0,T]^{j+1}} \|R^{(k-1)}_{\leq M_{k-1}} B_{N,1,k} U^{(k)}(t_k - t_{k+1}) B_{N,k,k+1} \cdots \|_{L^2_{\mathbf{x},\mathbf{x}'}} \, dt_k \, d\underline{t}_{k+j} =: A.$$

By Cauchy–Schwarz in the t_k integration,

$$A \leq T^{1/2} \int_{[0,T]^j} \left(\int \|R^{(k-1)}_{\leq M_{k-1}} B_{N,1,k} U^{(k)}(t_k - t_{k+1}) B_{N,k,k+1} \cdots \|_{L^2_{\mathbf{x},\mathbf{x}'}} \, dt_k \right)^{1/2} d\underline{t}_{k+j}.$$

By Lemma 4.4,

$$A \leq$$
$$C_\varepsilon T^{1/2} \sum_{M_k \geq M_{k-1}} \left(\frac{M_{k-1}}{M_k} \right)^{1-\varepsilon} \int_{[0,T]^j} \|R^{(k)}_{\leq M_k} B_{N,k,k+1} U^{(k+1)}(t_{k+1} - t_{k+2}) \cdots \|_{L^2_{\mathbf{x},\mathbf{x}'}} \, d\underline{t}_{k+j}.$$

Iterating the previous step $j - 1$ times yields

$$A \leq (C_\varepsilon T^{1/2})^j \sum_{M_{k+j-1} \geq \cdots \geq M_k \geq M_{k-1}} \left[\left(\frac{M_{k-1}}{M_k} \frac{M_k}{M_{k+1}} \cdots \frac{M_{k+j-2}}{M_{k+j-1}} \right)^{1-\varepsilon} \right.$$
$$\left. \times \|R^{(k+j-1)}_{\leq M_{k+j-1}} B_{N,\mu_m(k+j),k+j}(f^{(k+j)})\|_{L^1_T L^2_{\mathbf{x},\mathbf{x}'}} \right]$$
$$= (C_\varepsilon T^{1/2})^j \sum_{M_{k+j-1} \geq \cdots \geq M_k \geq M_{k-1}} \left[\left(\frac{M_{k-1}}{M_{k+j-1}} \right)^{1-\varepsilon} \right.$$
$$\left. \times \|R^{(k+j-1)}_{\leq M_{k+j-1}} B_{N,\mu_m(k+j),k+j}(f^{(k+j)})\|_{L^1_T L^2_{\mathbf{x},\mathbf{x}'}} \right]$$

where the sum is over all $M_k, \ldots, M_{k+j-1}$ dyadic such that $M_{k+j-1} \geq \cdots \geq M_k \geq M_{k-1}$.

Hence

$$\|R^{(k-1)}_{\leq M_{k-1}} B_{N,1,k} PP^{(k,l_c)}\|_{L^1_T L^2_{\mathbf{x},\mathbf{x}'}}$$
$$\leq \left\| R^{(k-1)}_{\leq M_{k-1}} B_{N,1,k} \int_0^{t_k} U^{(k)}(t_k - t_{k+1}) V_N^{(k)} \gamma_N^{(k)}(t_{k+1})\, dt_{k+1} \right\|_{L^1_T L^2_{\mathbf{x},\mathbf{x}'}}$$
$$+ \sum_{j=1}^{l_c} \left\{ (C_\varepsilon T^{1/2})^j \sum_{M_{k+j-1} \geq \cdots \geq M_k \geq M_{k-1}} \left[\frac{M_{k-1}^{1-\varepsilon}}{M_{k+j-1}^{1-\varepsilon}} \right. \right.$$
$$\left. \left. \times \|R^{(k+j-1)}_{\leq M_{k+j-1}} B_{N,\mu_m(k+j),k+j}(f^{(k+j)})\|_{L^1_T L^2_{\mathbf{x},\mathbf{x}'}} \right] \right\}.$$

We then insert a smooth cut-off $\theta(t)$ with $\theta(t) = 1$ for $t \in [-T, T]$ and $\theta(t) = 0$ for $t \in [-2T, 2T]^c$ into the above estimate to get

$$\|R^{(k-1)}_{\leq M_{k-1}} B_{N,1,k} PP^{(k,l_c)}\|_{L^1_T L^2_{\mathbf{x},\mathbf{x}'}}$$
$$\leq \left\| R^{(k-1)}_{\leq M_{k-1}} B_{N,1,k}\theta(t_k) \int_0^{t_k} U^{(k)}(t_k - t_{k+1})\theta(t_{k+1}) V_N^{(k)} \gamma_N^{(k)}(t_{k+1})\, dt_{k+1} \right\|_{L^1_T L^2_{\mathbf{x},\mathbf{x}'}}$$
$$+ \sum_{j=1}^{l_c} \left\{ (C_\varepsilon T^{1/2})^j \sum_{M_{k+j-1} \geq \cdots \geq M_k \geq M_{k-1}} \left[\frac{M_{k-1}^{1-\varepsilon}}{M_{k+j-1}^{1-\varepsilon}} \right. \right.$$
$$\left. \left. \times \|R^{(k+j-1)}_{\leq M_{k+j-1}} B_{N,\mu_m(k+j),k+j}(\theta(t_{k+j})\tilde{f}^{(k+j)})\|_{L^1_T L^2_{\mathbf{x},\mathbf{x}'}} \right] \right\},$$

where the sum is over all $M_k, \ldots, M_{k+j-1}$ dyadic such that $M_{k+j-1} \geq \cdots \geq M_k \geq M_{k-1}$, and $\tilde{f}^{(k+j)}$ is again defined via (3.4).

3.2.2. Step II. Using Lemma 4.5, the X_b space version of Lemma 4.4, we turn Step I into

$$\|R^{(k-1)}_{\leq M_{k-1}} B_{N,1,k} PP^{(k,l_c)}\|_{L^1_T L^2_{\mathbf{x},\mathbf{x}'}}$$
$$\leq C_\varepsilon T^{1/2} \sum_{M_k \geq M_{k-1}} \frac{M_{k-1}^{1-\varepsilon}}{M_k^{1-\varepsilon}} \left\| \int_0^{t_k} U^{(k)}(t_k - t_{k+1})\left(R^{(k)}_{\leq M_k}\theta(t_{k+1}) V_N^{(k)} \gamma_N^{(k)}(t_{k+1})\right) dt_{k+1} \right\|_{X^{(k)}_{(1/2)+}}$$
$$+ \sum_{j=1}^{l_c} (C_\varepsilon T^{1/2})^{j+1} \sum_{M_{k+j} \geq M_{k+j-1} \geq \cdots \geq M_k \geq M_{k-1}} \frac{M_{k-1}^{1-\varepsilon}}{M_{k+j}^{1-\varepsilon}} \|\theta(t_{k+j}) R^{(k+j)}_{\leq M_{k+j}}(\tilde{f}^{(k+j)})\|_{X^{(k+j)}_{(1/2)+}}.$$

3.2.3. Step III. Lemma 4.1 gives us

$$\|R^{(k-1)}_{\leq M_{k-1}} B_{N,1,k} PP^{(k,l_c)}\|_{L^1_T L^2_{\mathbf{x},\mathbf{x}'}} \leq A + B$$

where

$$A = C_\varepsilon T^{1/2} \sum_{M_k \geq M_{k-1}} \frac{M_{k-1}^{1-\varepsilon}}{M_k^{1-\varepsilon}} \| R_{\leq M_k}^{(k)} \theta(t_{k+1}) V_N^{(k)} \gamma_N^{(k)}(t_{k+1}) \|_{X_{-(1/2)+}^{(k)}}$$

and

$$B = \sum_{j=1}^{l_c} \left\{ (C_\varepsilon T^{1/2})^{j+1} \sum_{M_{k+j} \geq M_{k+j-1} \geq \cdots \geq M_k \geq M_{k-1}} \left[\frac{M_{k-1}^{1-\varepsilon}}{M_{k+j}^{1-\varepsilon}} \right. \right.$$
$$\left. \left. \times \| R_{\leq M_{k+j}}^{(k+j)} \theta(t_{k+j+1}) V_N^{(k+j)} \gamma_N^{(k+j)}(t_{k+j+1}) \|_{X_{-(1/2)+}^{(k+j)}} \right] \right\}.$$

3.2.4. Step IV. We focus for a moment on B. First, we handle the sum over $M_k \leq \cdots \leq M_{k+j-1}$ with the help of Lemma 3.1:

$$B = \sum_{j=1}^{l_c} \left\{ (C_\varepsilon T^{1/2})^{j+1} \sum_{M_{k+j} \geq M_{k-1}} \left[\frac{M_{k-1}^{1-\varepsilon}}{M_{k+j}^{1-\varepsilon}} \frac{\left(\log_2 \frac{M_{k+j}}{M_{k-1}} + j \right)^j}{j!} \right. \right.$$
$$\left. \left. \times \| R_{\leq M_{k+j}}^{(k+j)} \left(\theta(t_{k+j+1}) V_N^{(k+j)} \gamma_N^{(k+j)}(t_{k+j+1}) \right) \|_{X_{-(1/2)+}^{(k+j)}} \right] \right\}$$

We then take a $T^{j/4}$ from the front to apply Lemma 3.2 and get

$$B \lesssim C_\varepsilon T^{1/2} \sum_{j=1}^{l_c} \left\{ (C_\varepsilon T^{1/4})^j \sum_{M_{k+j} \geq M_{k-1}} \left[\frac{M_{k-1}^{1-2\varepsilon}}{M_{k+j}^{1-2\varepsilon}} \right. \right.$$
$$\left. \left. \times \| R_{\leq M_{k+j}}^{(k+j)} \left(\theta(t_{k+j+1}) V_N^{(k+j)} \gamma_N^{(k+j)}(t_{k+j+1}) \right) \|_{X_{-(1/2)+}^{(k+j)}} \right] \right\}$$

where the sum is over dyadic M_{k+j} such that $M_{k+j} \geq M_{k-1}$. Applying (4.26) yields

$$B \lesssim C_\varepsilon T^{1/2} \sum_{j=1}^{l_c} \left\{ (C_\varepsilon T^{1/4})^j (k+j)^2 \sum_{M_{k+j} \geq M_{k-1}} \left[\frac{M_{k-1}^{1-2\varepsilon}}{M_{k+j}^{1-2\varepsilon}} \min(M_{k+j}^2, N^{2\beta}) N^{\beta/2-1} \right. \right.$$
$$\left. \left. \times \| \theta(t_{k+j+1}) S^{(k+j)} \gamma_N^{(k+j)}(t_{k+j+1}) \|_{L_{t_{k+j+1}}^2 L_{\mathbf{x},\mathbf{x}'}^2} \right] \right\}.$$

Rearranging terms gives

$$B \lesssim C_\varepsilon T^{1/2} \sum_{j=1}^{l_c} \left\{ (C_\varepsilon T^{1/4})^j (k+j)^2 \| \theta(t_{k+j+1}) S^{(k+j)} \gamma_N^{(k+j)}(t_{k+j+1}) \|_{L_{t_{k+j+1}}^2 L_{\mathbf{x},\mathbf{x}'}^2} \right.$$
$$\left. \times M_{k-1}^{1-2\epsilon} N^{\beta/2-1} \sum_{M_{k+j} \geq M_{k-1}} (\cdots) \right\}$$

where

$$\sum_{M_{k+j} \geq M_{k-1}} (\cdots) = \sum_{M_{k+j} \geq M_{k-1}} \min(M_{k+j}^{1+2\epsilon}, M_{k+j}^{-1+2\epsilon} N^{2\beta}).$$

We carry out the summation in M_{k+j} by dividing into $M_{k+j} \leq N^\beta$ (for which $\min(M_{k+j}^{1+2\epsilon}, M_{k+j}^{-1+2\epsilon} N^{2\beta}) = M_j^{1+2\epsilon}$) and $M_{k+j} \geq N^\beta$ (for which $\min(M_{k+j}^{1+2\epsilon}, M_{k+j}^{-1+2\epsilon} N^{2\beta}) = M_{k+j}^{-1+2\epsilon} N^{2\beta}$). This yields

$$
\sum_{M_{k+j} \geq M_{k-1}} \min(M_{k+j}^{1+2\epsilon}, M_{k+j}^{-1+2\epsilon} N^{2\beta}) \lesssim \Big(\sum_{N^\beta \geq M_{k+j} \geq M_{k-1}} + \sum_{M_{k+j} \geq M_{k-1}, M_{k+j} \geq N^\beta} \Big)(\cdots)
$$
$$
\lesssim \sum_{N^\beta \geq M_{k+j} \geq 1} M_{k+j}^{1+2\epsilon} + \sum_{M_{k+j} \geq N^\beta} M_{k+j}^{-1+2\epsilon} N^{2\beta}
$$
$$
\lesssim N^{\beta+2\epsilon}.
$$

Hence

$$
B \lesssim C_\varepsilon T^{1/2}
$$
$$
\times \sum_{j=1}^{l_c} (C_\varepsilon T^{1/4})^j (k+j)^2 \|\theta(t_{k+j+1}) S^{(k+j)} \gamma_N^{(k+j)}(t_{k+j+1})\|_{L^2_{t_{k+j+1}} L^2_{\mathbf{x},\mathbf{x}'}} M_{k-1}^{1-2\epsilon} N^{3\beta/2-1+2\epsilon}
$$
$$
\lesssim M_{k-1}^{1-2\epsilon} N^{3\beta/2-1+2\epsilon} C_\varepsilon T^{1/2} \sum_{j=1}^{l_c} (C_\varepsilon T^{1/4})^j (k+j)^2 T^{1/2} \|S^{(k+j)} \gamma_N^{(k+j)}\|_{L^\infty_t L^2_{\mathbf{x},\mathbf{x}'}}.
$$

Via condition (2.5), this becomes

$$
B \lesssim M_{k-1}^{1-2\epsilon} N^{3\beta/2-1+2\epsilon} C_\varepsilon T \sum_{j=1}^{l_c} (C_\varepsilon T^{1/4})^j (k+j)^2 C_0^{k+j}
$$
$$
\lesssim C_0^k M_{k-1}^{1-2\epsilon} N^{3\beta/2-1+2\epsilon} C_\varepsilon T \Big(k^2 \sum_{j=1}^{\infty} (C_\varepsilon T^{1/4})^j C_0^j + \sum_{j=1}^{\infty} (C_\varepsilon T^{1/4})^j j^2 C_0^j \Big).
$$

We can then choose a T independent of M_{k-1}, k, l_c and N such that the two infinite series converge. This yields

$$
B \lesssim C^{k-1} M_{k-1}^{1-2\epsilon} N^{3\beta/2-1+2\epsilon}
$$

for some $C > C_0$. Therefore, for $\beta < 2/3$, there is a C independent of M_{k-1}, k, l_c, and N such that given M_{k-1}, there is $N_0(M_{k-1})$ which makes

$$
B \leq C^{k-1} \quad \text{for all } N \geq N_0.
$$

This completes the treatment of B for $\beta < 2/3$; and A is treated similarly (without the need to appeal to Lemmas 3.1 and 3.2 below). Thus we have completed the proof of Theorem 3.1 and hence of Theorem 2.1.

Lemma 3.1.

$$
\sum_{M_{k-1} \leq M_k \leq \cdots \leq M_{k+j-1} \leq M_{k+j}} 1 \leq \frac{\big(\log_2 \frac{M_{k+j}}{M_{k-1}} + j\big)^j}{j!},
$$

where the sum is over $M_k, \ldots, M_{k+j-1}$ dyadic such that $M_{k-1} \leq M_k \leq \cdots \leq M_{k+j-1} \leq M_{k+j}$.

Proof. This is equivalent to

$$S := \sum_{i_{k-1} \le i_k \le \cdots \le i_{k+j-1} \le i_{k+j}} 1 \le \frac{(i_{k+j} - i_{k-1} + j)^j}{j!},$$

where the sum is taken over integers $i_k, \ldots, i_{k+j-1}$ such that $i_{k-1} \le i_k \le \cdots \le i_{k+j-1} \le i_{k+j}$. We use the estimate (for $p, \ell \ge 0$)

$$\sum_{i=0}^{q} (i + \ell)^p \le \frac{(q + \ell + 1)^{p+1}}{p + 1},$$

which just follows by estimating the sum by an integral.

First, carry out the summation over i_k from i_{k-1} to i_{k+1} to obtain

$$S = \sum_{i_{k-1} \le i_{k+1} \le \cdots \le i_{k+j-1} \le i_{k+j}} \left(\sum_{i_k=i_{k-1}}^{i_{k+1}} 1 \right) \le \sum_{i_{k-1} \le i_{k+1} \le \cdots \le i_{k+j-1} \le i_{k+j}} (i_{k+1} - i_{k-1} + 1).$$

Next, carry out the summation over i_{k+1} from i_{k-1} to i_{k+2}:

$$S \le \sum_{i_{k-1} \le i_{k+2} \le \cdots \le i_{k+j-1} \le i_{k+j}} \left(\sum_{i_{k+1}=i_{k-1}}^{i_{k+2}} (i_{k+1} - i_{k-1} + 1) \right)$$

$$\le \sum_{i_{k-1} \le i_{k+2} \le \cdots \le i_{k+j-1} \le i_{k+j}} \left(\sum_{i_{k+1}=0}^{i_{k+2}-i_{k-1}} (i_{k+1} + 1) \right)$$

$$\le \sum_{i_{k-1} \le i_{k+2} \le \cdots \le i_{k+j-1} \le i_{k+j}} \frac{(i_{k+2} - i_{k-1} + 2)^2}{2}.$$

Continue in this manner $j - 2$ times to obtain the claimed bound. $\qquad\square$

Lemma 3.2. *For each $\alpha > 0$ (possibly large) and each $\epsilon > 0$ (arbitrarily small), there exists $t > 0$ (independent of M) sufficiently small such that*

$$\forall j \ge 1, \forall M, \quad \frac{t^j (\alpha \log M + j)^j}{j!} \le M^\epsilon.$$

Proof. We use the following fact: for each $\sigma > 0$ (arbitrarily small) there exists $t > 0$ sufficiently small such that

$$\forall x > 0, \quad t^x \left(\frac{1}{x} + 1 \right)^x \le e^\sigma \tag{3.5}$$

To apply this fact to prove the lemma, use Stirling's formula to obtain

$$\frac{t^j (\alpha \log M + j)^j}{j!} \le (et)^j \left(\frac{\alpha \log M + j}{j} \right)^j =: A.$$

Define x in terms of j by the formula $j = \alpha(\log M)x$. Then by (3.5),

$$A = \left[(et)^x \left(\frac{1}{x} + 1 \right)^x \right]^{\alpha \log M} \le e^{\sigma\alpha \log M} = M^{\sigma\alpha}. \qquad\square$$

4. Collapsing and Strichartz estimates

Define the norm

$$\|\alpha^{(k)}\|_{X_b^{(k)}} = \left(\int \langle \tau + |\xi_k|^2 - |\xi_k'|^2 \rangle^{2b} |\hat{\alpha}^{(k)}(\tau, \xi_k, \xi_k')|^2 \, d\tau \, d\xi_k \, d\xi_k' \right)^{1/2}.$$

We will use the case $b = (1/2)+$ of the following lemma.

Lemma 4.1. *Let $1/2 < b < 1$ and $\theta(t)$ be a smooth cut-off. Then*

$$\left\| \theta(t) \int_0^t U^{(k)}(t-s) \beta^{(k)}(s) \, ds \right\|_{X_b^{(k)}} \lesssim \|\beta^{(k)}\|_{X_{b-1}^{(k)}}. \tag{4.1}$$

Proof. The estimate reduces to the space-independent estimate

$$\left\| \theta(t) \int_0^t h(t') \, dt' \right\|_{H_t^b} \lesssim \|h\|_{H_t^{b-1}} \quad \text{for } 1/2 < b \leq 1. \tag{4.2}$$

Indeed, taking $h(t) = h_{\mathbf{x}_k \mathbf{x}_k'}(t) := U^{(k)}(-t) \beta^{(k)}(t, \mathbf{x}_k, \mathbf{x}_k')$, applying the estimate (4.2) for fixed $\mathbf{x}_k, \mathbf{x}_k'$, and then applying the $L_{\mathbf{x}_k, \mathbf{x}_k'}^2$ norm to both sides yields (4.1).

Now we prove (4.2). Let $P_{\leq 1}$ and $P_{\geq 1}$ denote the Littlewood–Paley projections onto the frequencies $|\tau| \leq 1$ and $|\tau| \geq 1$ respectively. Decompose $h = P_{\leq 1} h + P_{\geq 1} h$ and use $\int_0^t P_{\geq 1} h(t') \, dt' = \frac{1}{2} \int (\text{sgn}(t - t') + \text{sgn}(t')) P_{\geq 1} h(t') \, dt'$ to obtain the decomposition

$$\theta(t) \int_0^t h(t') \, dt' = H_1(t) + H_2(t) + H_3(t),$$

where

$$H_1(t) = \theta(t) \int_0^t P_{\leq 1} h(t') \, dt',$$
$$H_2(t) = \tfrac{1}{2} \theta(t) [\text{sgn} * P_{\geq 1} h](t) \, dt',$$
$$H_3(t) = \tfrac{1}{2} \theta(t) \int_{-\infty}^{\infty} \text{sgn}(t') P_{\geq 1} h(t') \, dt'.$$

We begin by addressing H_1. By Sobolev embedding (recall $1/2 < b \leq 1$) and the $L^p \to L^p$ boundedness of the Hilbert transform for $1 < p < \infty$,

$$\|H_1\|_{H_t^b} \lesssim \|H_1\|_{L_t^2} + \|\partial_t H_1\|_{L_t^{2/(3-2b)}}.$$

Using $\|P_{\leq 1} h\|_{L_t^\infty} \lesssim \|h\|_{H_t^{b-1}}$, we thus conclude

$$\|H_1\|_{H_t^b} \lesssim (\|\theta\|_{L_t^2} + \|\theta\|_{L_t^{2/(3-2b)}} + \|\theta'\|_{L_t^{2/3-2b}}) \|h\|_{H_t^{b-1}}.$$

Next we address H_2. By the fractional Leibniz rule,

$$\|H_2\|_{H_t^b} \lesssim \|\langle D_t \rangle^b \theta\|_{L_t^2} \|\text{sgn} * P_{\geq 1} h\|_{L_t^\infty} + \|\theta\|_{L_t^\infty} \|\langle D_t \rangle^b (\text{sgn} * P_{\geq 1} h)\|_{L_t^2}.$$

However,

$$\|\operatorname{sgn} * P_{\geq 1} h\|_{L_t^\infty} \lesssim \|\langle \tau \rangle^{-1} \hat{h}(\tau)\|_{L_\tau^1} \lesssim \|h\|_{H_t^{b-1}}.$$

On the other hand,

$$\|\langle D_t \rangle^b \operatorname{sgn} * P_{\geq 1} h\|_{L_t^2} \lesssim \|\langle \tau \rangle^b \langle \tau \rangle^{-1} \hat{h}(\tau)\|_{L_\tau^2} \lesssim \|h\|_{H_t^{b-1}}.$$

Consequently,

$$\|H_2\|_{H_t^b} \lesssim (\|\langle D_t \rangle^b \theta\|_{L_t^2} + \|\theta\|_{L_t^\infty}) \|h\|_{H_t^{b-1}}.$$

For H_3, we have

$$\|H_3\|_{H_t^b} \lesssim \|\theta\|_{H_t^b} \left\| \int_{-\infty}^\infty \operatorname{sgn}(t') P_{\geq 1} h(t')\, dt' \right\|_{L_t^\infty}.$$

However, the second term is handled via Parseval's identity:

$$\int_{t'} \operatorname{sgn}(t') P_{\geq 1} h(t')\, dt' = \int_{|\tau| \geq 1} \tau^{-1} \hat{h}(\tau)\, d\tau,$$

from which the appropriate bounds follow again by Cauchy–Schwarz.

Collecting our estimates for H_1, H_2, and H_3, we obtain

$$\left\| \theta(t) \int_0^t h(t')\, dt' \right\|_{H_t^b} \lesssim C_\theta \|h\|_{H_t^{b-1}},$$

where

$$C_\theta = \|\theta\|_{L_t^2} + \|\theta'\|_{L_t^{2/(3-2b)}} + \|\langle D_t \rangle^b \theta\|_{L_t^2} + \|\theta\|_{L_t^{2/(3-2b)}} + \|\theta\|_{L_t^\infty}. \qquad \square$$

4.1. Various forms of collapsing estimates

Lemma 4.2. *There is a C independent of j, k, and N such that (for $f^{(k+1)}(\mathbf{x}_{k+1}, \mathbf{x}'_{k+1})$ independent of t)*

$$\|R^{(k)} B_{N,j,k+1} U^{(k+1)}(t) f^{(k+1)}\|_{L_t^2 L_{\mathbf{x},\mathbf{x}'}^2} \leq C \|V\|_{L^1} \|R^{(k+1)} f^{(k+1)}\|_{L_{\mathbf{x},\mathbf{x}'}^2}.$$

Proof. One can find this estimate in [11, (A.18)] or as a special case of [15, Theorem 7]. For more estimates of this type, see [43, 38, 12, 14, 3, 34]. $\qquad \square$

We have the following consequence of Lemma 4.2.

Lemma 4.3. *There is a C independent of j, k, and N such that (for $\alpha^{(k+1)}(t, \mathbf{x}_{k+1}, \mathbf{x}'_{k+1})$ depending on t)*

$$\|R^{(k)} B_{N,j,k+1} \alpha^{(k+1)}\|_{L_t^2 L_{\mathbf{x},\mathbf{x}'}^2} \leq C \|R^{(k+1)} \alpha^{(k+1)}\|_{X_{(1/2)+}^{(k+1)}}.$$

Proof. Let

$$f_\tau^{(k+1)}(\mathbf{x}_{k+1}, \mathbf{x}'_{k+1}) = \mathcal{F}_{t \mapsto \tau}\big(U^{(k+1)}(-t)\alpha^{(k+1)}(t, \mathbf{x}_{k+1}, \mathbf{x}'_{k+1})\big)$$

where $\mathcal{F}_{t \mapsto \tau}$ denotes the $t \mapsto \tau$ Fourier transform. Then

$$\alpha^{(k+1)}(t, \mathbf{x}_{k+1}, \mathbf{x}'_{k+1}) = \int_\tau e^{it\tau} U^{(k+1)}(t) f^{(k+1)}(\mathbf{x}_{k+1}, \mathbf{x}'_{k+1}) \, d\tau.$$

By Minkowski's inequality

$$\|R^{(k)} B_{N,j,k+1} \alpha^{(k+1)}\|_{L_t^2 L_{\mathbf{x},\mathbf{x}'}^2} \le \int_\tau \|R^{(k)} B_{N,j,k+1} U^{(k+1)}(t) f^{(k+1)}\|_{L_t^2 L_{\mathbf{x},\mathbf{x}'}^2} \, d\tau =: A.$$

By Lemma 4.2,

$$A \le \int_\tau \|R^{(k+1)} f^{(k+1)}\|_{L_{\mathbf{x},\mathbf{x}'}^2} \, d\tau.$$

For any $b > 1/2$, we write $1 = \langle\tau\rangle^{-b}\langle\tau\rangle^b$ and apply Cauchy–Schwarz in τ to obtain

$$A \le \|\langle\tau\rangle^b R^{(k+1)} f^{(k+1)}\|_{L_{\tau,\mathbf{x},\mathbf{x}'}^2} = \|R^{(k+1)} \alpha^{(k+1)}\|_{X_b^{(k+1)}}. \qquad \square$$

Lemma 4.4. *For each $\varepsilon > 0$, there is a C_ε independent of M_k, j, k, and N such that*

$$\|R^{(k)} P_{\le M_k}^{(k)} B_{N,j,k+1} U^{(k+1)}(t) f^{(k+1)}\|_{L_t^2 L_{\mathbf{x},\mathbf{x}'}^2}$$

$$\le C_\varepsilon \|V\|_{L^1} \sum_{M_{k+1} \ge M_k} \left(\frac{M_k}{M_{k+1}}\right)^{1-\varepsilon} \|R^{(k+1)} P_{\le M_{k+1}}^{(k+1)} f^{(k+1)}\|_{L_{\mathbf{x},\mathbf{x}'}^2}$$

where the sum is over M_{k+1} dyadic such that $M_{k+1} \ge M_k$.

In particular, if we drop off the projection $P_{\le M_{k+1}}^{(k+1)}$ on the right hand side, carry out the summation and let $M_k \to \infty$, we recover Lemma 4.2. This merely gives a fine structure of Lemma 4.2, but not an alternative proof.

Proof of Lemma 4.4. It suffices to take $k = 1$ and prove

$$\|R^{(1)} P_{\le M_1}^{(1)} B_{N,1,2} (R^{(2)})^{-1} U^{(2)}(t) f^{(2)}\|_{L_t^2 L_{\mathbf{x}_1 \mathbf{x}'_1}^2}$$

$$\le C_\varepsilon \|V\|_{L^1} \sum_{M_2 \ge M_1} \left(\frac{M_1}{M_2}\right)^{1-\varepsilon} \|P_{\le M_2}^{(2)} f^{(2)}\|_{L_{\mathbf{x}_2,\mathbf{x}'_2}^2} \qquad (4.3)$$

where the sum is over dyadic M_2 such that $M_2 \ge M_1$. For convenience, we take only "half" of the operator $B_{N,1,2}$: for $\alpha^{(2)}(t, x_1, x_2, x'_1, x'_2)$, define

$$(\tilde{B}_{N,1,2}\alpha^{(2)})(t, x_1, x'_1) := \int_{x_2} V_N(x_1 - x_2)\alpha^{(2)}(t, x_1, x_2, x'_1, x_2) \, dx_2.$$

Note that

$$\left(R^{(1)} P^{(1)}_{\leq M_1} \tilde{B}_{N,1,2}(R^{(2)})^{-1} U^{(2)}(t) f^{(2)}\right)^\wedge (\tau, \xi_1, \xi'_1)$$

$$= \iint_{\xi_2, \xi'_2} \chi^{(1)}_{\leq M_1} \delta(\cdots) \frac{\widehat{V_N}(\xi_2 + \xi'_2)|\xi_1|}{|\xi_1 - \xi_2 - \xi'_2| |\xi_2||\xi'_2|} \widehat{f^{(2)}}(\xi_1 - \xi_2 - \xi'_2, \xi_2, \xi'_1, \xi'_2) \, d\xi_2 \, d\xi'_2 =: I$$

where χ represents the Littlewood–Paley multiplier on the Fourier side and

$$\delta(\cdots) = \delta(\tau + |\xi_1 - \xi_2 - \xi'_2|^2 + |\xi_2|^2 - |\xi'_1|^2 - |\xi'_2|^2).$$

Divide this integral into two pieces:

$$I = \iint_{|\xi_2| \leq |\xi'_2|} (\cdots) \, d\xi_2 \, d\xi'_2 + \iint_{|\xi'_2| \leq |\xi_2|} (\cdots) \, d\xi_2 \, d\xi'_2.$$

In the first term, decompose the ξ'_2 integration into dyadic intervals, and in the second term, decompose the ξ_2 integration into dyadic intervals:

$$I = A + B$$

$$=: \left(\sum_{M_2 \geq M_1} \iint_{|\xi_2| \leq |\xi'_2|} \chi^{2'}_{M_2}(\cdots) \, d\xi_2 \, d\xi'_2 + \sum_{M_2 \geq M_1} \iint_{|\xi'_2| \leq |\xi_2|} \chi^2_{M_2}(\cdots) \, d\xi_2 \, d\xi'_2 \right)$$

$$+ \left(\iint_{|\xi_2| \leq |\xi'_2|} \chi^{2'}_{\leq M_1}(\cdots) \, d\xi_2 \, d\xi'_2 + \iint_{|\xi'_2| \leq |\xi_2|} \chi^2_{\leq M_1}(\cdots) \, d\xi_2 \, d\xi'_2 \right).$$

The A term is the one that needs elaboration. For B, we have

$$B = \iint_{|\xi_2| \leq |\xi'_2|} \chi^{(1)}_{\leq M_1} \chi^2_{\leq M_1} \chi^{2'}_{\leq M_1}(\cdots) \, d\xi_2 \, d\xi'_2$$

$$+ \iint_{|\xi_2| \leq |\xi'_2|} \chi^{(1)}_{\leq M_1} \chi^2_{\leq M_1} \chi^{2'}_{\leq M_1}(\cdots) \, d\xi_2 \, d\xi'_2,$$

and thus, by Lemma 4.2, we reach

$$\|B\|_{L^2_\tau L^2_{\xi_1 \xi'_1}} \leq C \|V\|_{L^1} \|P^{(2)}_{\leq M_1} f^{(2)}\|_{L^2_{\mathbf{x}_2 \mathbf{x}'_2}},$$

which is part of the right hand side of estimate (4.3).

We are now left with the estimate of A. Observe that, in the first integration in A, we can insert for free the projection $\chi^1_{\leq 3M_2} \chi^{1'}_{\leq M_1} \chi^2_{\leq M_2}$ onto $\widehat{f^{(2)}}$, and in the second integration, we can insert $\chi^1_{\leq 3M_2} \chi^{1'}_{\leq M_1} \chi^{2'}_{\leq M_2}$ onto $\widehat{f^{(2)}}$. Thus

$$A = \sum_{M_2 \geq M_1} \iint_{|\xi_2| \leq |\xi'_2|} \chi_{\leq 3M_2}(\xi_1 - \xi_2 - \xi'_2) \chi^{1'}_{\leq M_1} \chi^2_{\leq M_2} \chi^{2'}_{M_2}(\cdots) \, d\xi_2 \, d\xi'_2$$

$$+ \sum_{M_2 \geq M_1} \iint_{|\xi'_2| \leq |\xi_2|} \chi_{\leq 3M_2}(\xi_1 - \xi_2 - \xi'_2) \chi^{1'}_{\leq M_1} \chi^{2'}_{\leq M_2} \chi^2_{M_2}(\cdots) \, d\xi_2 \, d\xi'_2.$$

Then for each piece, we proceed as in Klainerman–Machedon [45], using Cauchy–Schwarz with respect to measures supported on hypersurfaces and applying the $L^2_{\tau \xi_1 \xi'_1}$ norm to both sides of the resulting inequality.[11] In this manner, it suffices to prove the following estimates, uniform in $\tau' = \tau - |\xi'_1|^2$:

$$\iint_{\substack{|\xi'_2| \sim M_2 \\ |\xi_2| \leq M_2}} \delta(\cdots) \frac{|\xi_1|^2}{|\xi_1 - \xi_2 - \xi'_2|^2 |\xi_2|^2 |\xi'_2|^2} \, d\xi_2 \, d\xi'_2 \leq C_\varepsilon \left(\frac{M_1}{M_2} \right)^{2(1-\varepsilon)} \tag{4.4}$$

(recall that $|\xi_1| \lesssim M_1 \ll M_2$) and also

$$\iint_{\substack{|\xi_2| \sim M_2 \\ |\xi'_2| \leq M_2}} \delta(\cdots) \frac{|\xi_1|^2}{|\xi_1 - \xi_2 - \xi'_2|^2 |\xi_2|^2 |\xi'_2|^2} \, d\xi_2 \, d\xi'_2 \leq C_\varepsilon \left(\frac{M_1}{M_2} \right)^{2(1-\varepsilon)}. \tag{4.5}$$

In both (4.4) and (4.5),

$$\delta(\cdots) = \delta(\tau' + |\xi_1 - \xi_2 - \xi'_2|^2 + |\xi_2|^2 - |\xi'_2|^2).$$

By rescaling $\xi_2 \mapsto M_2 \xi_2$ and $\xi'_2 \mapsto M_2 \xi'_2$, (4.4) and (4.5) reduce to

$$I(\tau', \xi_1) := \iint_{\substack{|\xi'_2| \sim 1 \\ |\xi_2| \leq 2}} \delta(\cdots) \frac{|\xi_1|^2}{|\xi_1 - \xi_2 - \xi'_2|^2 |\xi_2|^2 |\xi'_2|^2} \, d\xi_2 \, d\xi'_2 \leq C_\varepsilon |\xi_1|^{2(1-\varepsilon)}, \tag{4.6}$$

$$I'(\tau', \xi_1) := \iint_{\substack{|\xi_2| \sim 1 \\ |\xi'_2| \leq 2}} \delta(\cdots) \frac{|\xi_1|^2}{|\xi_1 - \xi_2 - \xi'_2|^2 |\xi_2|^2 |\xi'_2|^2} \, d\xi_2 \, d\xi'_2 \leq C_\varepsilon |\xi_1|^{2(1-\varepsilon)}, \tag{4.7}$$

respectively, for $|\xi_1| \ll 1$. To be precise, the ξ_1 in estimates (4.6) and (4.7) is ξ_1/M_2 in estimates (4.4) and (4.5). We shall obtain the upper bound $|\xi_1|^2 \log |\xi_1|^{-1}$ for both (4.6), (4.7).

First, we prove (4.7). Begin by carrying out the ξ'_2 integration to obtain

$$I'(\tau', \xi_1) = \frac{1}{2} |\xi_1|^2 \int_{1/2 \leq |\xi_2| \leq 2} \frac{H'(\tau', \xi_1, \xi_2)}{|\xi_1 - \xi_2| \, |\xi_2|^2} \, d\xi_2$$

where $H'(\tau', \xi_1, \xi_2)$ is defined as follows. Let P' be the truncated plane defined by

$$P'(\tau', \xi_1, \xi_2) = \{\xi'_2 \in \mathbb{R}^3 \mid (\xi'_2 - \lambda\omega) \cdot \omega = 0, \; |\xi'_2| \leq 2\}$$

where

$$\omega = \frac{\xi_1 - \xi_2}{|\xi_1 - \xi_2|}, \quad \lambda = \frac{\tau' + |\xi_1 - \xi_2|^2 + |\xi_2|^2}{2|\xi_1 - \xi_2|}.$$

Now let

$$H'(\tau', \xi_1, \xi_2) = \int_{\xi'_2 \in P'(\tau', \xi_1, \xi_2)} \frac{d\sigma(\xi'_2)}{|\xi_1 - \xi_2 - \xi'_2|^2 |\xi'_2|^2} \tag{4.8}$$

where the integral is computed with respect to the surface measure on P'.

---

[11] Notice that $\|\widehat{V_N}\|_{L^\infty} \leq \|V_N\|_{L^1} = \|V\|_{L^1}$, i.e. $\widehat{V_N}$ is a dummy factor.

Since $|\xi_1 - \xi_2| \sim 1$, $|\xi_2| \sim 1$, we have the following reduction:

$$I'(\tau', \xi_1) \lesssim |\xi_1|^2 \int_{1/2 \leq |\xi_2| \leq 2} H'(\tau', \xi_1, \xi_2)\, d\xi_2.$$

We now evaluate $H'(\tau', \xi_1, \xi_2)$. Introduce polar coordinates (ρ, θ) on the plane P' with respect to the "center" $\lambda\omega$, and note that

$$
\begin{aligned}
|\xi_1 - \xi_2 - \xi_2'|^2 &= \left||\xi_1 - \xi_2|\omega - \xi_2'\right|^2 = \left|(|\xi_1 - \xi_2| - \lambda)\omega - (\xi_2' - \lambda\omega)\right|^2 \\
&= (|\xi_1 - \xi_2| - \lambda)^2 + |\xi_2' - \lambda\omega|^2 = (|\xi_1 - \xi_2| - \lambda)^2 + \rho^2 \\
&= \alpha^2 + \rho^2
\end{aligned}
\tag{4.9}
$$

where

$$\alpha = |\xi_1 - \xi_2| - \lambda = \frac{|\xi_1|^2 - 2\xi_1 \cdot \xi_2 - \tau'}{2|\xi_1 - \xi_2|}.$$

Also,

$$|\xi_2'|^2 = |(\xi_2' - \lambda\omega) + \lambda\omega|^2 = |\xi_2' - \lambda\omega|^2 + \lambda^2 = \rho^2 + \lambda^2. \tag{4.10}$$

Using (4.9) and (4.10) in (4.8), we get

$$H'(\tau', \xi_1, \xi_2) = \int_0^{\sqrt{4 - \lambda^2}} \frac{2\pi\rho\, d\rho}{(\rho^2 + \alpha^2)(\rho^2 + \lambda^2)}.$$

The restriction to $0 \leq \rho \leq \sqrt{4 - \lambda^2}$ arises from the fact that P' must sit within the ball $|\xi_2'| \leq 2$. In particular, $H'(\tau, \xi_1, \xi_2) = 0$ if $|\lambda| \geq 2$ since then P' is located entirely outside the ball $|\xi_2'| \leq 2$. Since $|\lambda| \leq 2$, we have $|\alpha| \leq 3$ and $|\tau'| \leq 10$.

We consider three cases: (A) $|\lambda| \leq 1/4$ (which implies $|\alpha| \geq 1/4$), (B) $|\alpha| \leq 1/4$ (which implies $|\lambda| \geq 1/4$), and (C) $|\lambda| \geq 1/4$ and $|\alpha| \geq 1/4$. Case (C) is the easiest since clearly $|H'(\tau', \xi_1, \xi_2)| \leq C$.

Let us consider case (B). Then

$$H'(\tau, \xi_1, \xi_2) \lesssim \int_0^2 \frac{\rho\, d\rho}{\rho^2 + \alpha^2} = \int_0^{\sqrt{2}} \frac{d\nu}{\nu + \alpha^2} = \log\left(1 + \frac{\sqrt{2}}{\alpha^2}\right).$$

Substituting back into I' yields

$$I'(\tau', \xi_1) \lesssim |\xi_1|^2 \int_{|\xi_2| \leq 2} \log\left(1 + \frac{\sqrt{2}}{\alpha^2}\right) d\xi_2.$$

Since $|\alpha| \leq \sqrt{3}$, it follows that[12]

$$
\begin{aligned}
\log\left(1 + \frac{\sqrt{2}}{\alpha^2}\right) &\leq c + \left|\log|\alpha|\right| \leq c + \left|\log\left||\xi_1|^2 - 2\xi_1 \cdot \xi_2 - \tau'\right|\right| \\
&= c + \left|\log 2\left|\xi_1 \cdot \left(\xi_2 - \frac{1}{2}\xi_1 + \frac{\tau'\xi_1}{2|\xi_1|^2}\right)\right|\right| = c + \left|\log\left|\xi_1 \cdot \left(\xi_2 - \frac{1}{2}\xi_1 + \frac{\tau'\xi_1}{2|\xi_1|^2}\right)\right|\right|.
\end{aligned}
$$

---

[12] The first step is simply: if $x \geq \delta > 0$, then $\log(1 + x) \leq \log x + \log(1 + 1/\delta)$. The second step uses $|\xi_1 - \xi_2| \sim 1$, which follows since $|\xi_1| \ll 1$ and $|\xi_2| \sim 1$.

Hence

$$I'(\tau', \xi_1) \lesssim |\xi_1|^2 \left(1 + \int_{|\xi_2| \leq 2} \left|\log \left|\xi_1 \cdot \left(\xi_2 - \frac{1}{2}\xi_1 + \frac{\tau'\xi_1}{2|\xi_1|^2}\right)\right|\right| d\xi_2\right).$$

Denote by $B(\mu, r)$ the ball of center μ and radius r. The substitution $\xi_2 \mapsto \xi_2 + \frac{1}{2}\xi_1 - \frac{\tau'\xi_1}{2|\xi_1|^2}$ yields, with $\mu = \frac{1}{2}\xi_1 - \frac{\tau'\xi_1}{2|\xi_1|^2}$,

$$I'(\tau', \xi_1) \lesssim |\xi_1|^2 \left(1 + \int_{B(\mu, 2)} \left|\log |\xi_1 \cdot \xi_2|\right| d\xi_2\right)$$

$$\lesssim |\xi_1|^2 \left(\log |\xi_1|^{-1} + \int_{B(\mu, 2)} \left|\log \left|\frac{\xi_1}{|\xi_1|} \cdot \xi_2\right|\right| d\xi_2\right).$$

By rotating coordinates so that $\xi_1/|\xi_1| = (1, 0, 0)$, and letting μ' denote the corresponding rotation of μ, we get

$$I'(\tau', \xi_1) \lesssim |\xi_1|^2 \left(\log |\xi_1|^{-1} + \int_{B(\mu', 2)} \left|\log |(\xi_2)_1|\right| d\xi_2\right)$$

where $(\xi_2)_1$ denotes the first coordinate of the vector ξ_2. Since $|\tau'| \leq 10$, it follows that $|\mu'| \lesssim |\xi_1|^{-1}$ and we finally obtain

$$I'(\tau', \xi_1) \lesssim |\xi_1|^2 \log |\xi_1|^{-1}$$

as claimed, completing Case (B).

Case (A) is similar except that we begin with the bound

$$H'(\tau', \xi_1, \xi_2) \lesssim \int_0^2 \frac{2\pi\rho \, d\rho}{\rho^2 + \lambda^2}.$$

This completes the proof of (4.7).

Next, we prove (4.6). In the integral defining $I(\tau', \xi_1)$, we have the restriction $1/2 \leq |\xi_2'| \leq 2$ and $|\xi_2| \leq 2$. Note that if $1/4 \leq |\xi_2| \leq 2$, then the argument above that provided the bound for $I'(\tau', \xi_1)$ applies. Hence it suffices to restrict to $|\xi_2| \leq 1/4$, from which it follows that $|\xi_1 - \xi_2 - \xi_2'| \sim 1$.

Begin by carrying out the ξ_2' integration to obtain

$$I(\tau', \xi_1) = \frac{1}{2}|\xi_1|^2 \int_{|\xi_2| \leq 2} \frac{H(\tau', \xi_1, \xi_2)}{|\xi_1 - \xi_2| \, |\xi_2|^2} \, d\xi_2 \tag{4.11}$$

where $H(\tau', \xi_1, \xi_2)$ is defined as follows. Let P be the truncated plane defined by

$$P(\tau', \xi_1, \xi_2) = \{\xi_2' \in \mathbb{R}^3 \mid (\xi_2' - \lambda\omega) \cdot \omega = 0, \ 1/2 \leq |\xi_2'| \leq 2\}$$

where

$$\omega = \frac{\xi_1 - \xi_2}{|\xi_1 - \xi_2|}, \quad \lambda = \frac{\tau' + |\xi_1 - \xi_2|^2 + |\xi_2|^2}{2|\xi_1 - \xi_2|}.$$

Now let

$$H(\tau', \xi_1, \xi_2) = \int_{\xi_2' \in P(\tau', \xi_1, \xi_2)} \frac{d\sigma(\xi_2')}{|\xi_1 - \xi_2 - \xi_2'|^2 |\xi_2'|^2}$$

where the integral is computed with respect to the surface measure on P. Since $|\xi_1 - \xi_2 - \xi_2'| \sim 1$ and $|\xi_2'| \sim 1$, we obtain $H(\tau', \xi_1, \xi_2) \le C$. Substituting into (4.11), we obtain

$$I(\tau', \xi_1) \lesssim |\xi_1|^2 \int_{|\xi_2| \le 1/4} \frac{d\xi_2}{|\xi_1 - \xi_2| |\xi_2|^2}$$

$$\lesssim |\xi_1|^2 \left(\int_{|\xi_2| \le 2|\xi_1|} \frac{d\xi_2}{|\xi_1 - \xi_2| |\xi_2|^2} + \int_{2|\xi_1| \le |\xi_2| \le 1/4} \frac{d\xi_2}{|\xi_1 - \xi_2| |\xi_2|^2} \right).$$

In the first integral, we change variables $\xi_2 = |\xi_1| \eta$, and in the second integral, we use the bound $|\xi_1 - \xi_2|^{-1} \le 2|\xi_2|^{-1}$ to obtain

$$I(\tau', \xi_1) \lesssim |\xi_1|^2 \left(\int_{|\eta| \le 2} \frac{d\eta}{|\xi_1/|\xi_1| - \eta| |\eta|^2} + \int_{2|\xi_1| \le |\xi_2| \le 1/4} \frac{d\xi_2}{|\xi_2|^3} \right) \lesssim |\xi_1|^2 \log |\xi_1|^{-1}.$$

This completes the proof of (4.6). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Lemma 4.5. *For each $\varepsilon > 0$, there is a C_ε independent of M_k, j, k, and N such that*

$$\| R^{(k)} P_{\le M_k}^{(k)} B_{N, j, k+1} \alpha^{(k+1)} \|_{L_t^2 L_{\mathbf{x}, \mathbf{x}'}^2}$$

$$\le C_\varepsilon \sum_{M_{k+1} \ge M_k} \left(\frac{M_k}{M_{k+1}} \right)^{1-\varepsilon} \| R^{(k+1)} P_{\le M_{k+1}}^{(k+1)} \alpha^{(k+1)} \|_{X_{(1/2)+}^{(k)}}.$$

where the sum is over dyadic M_{k+1} such that $M_{k+1} \ge M_k$.

Proof. The proof is exactly the same as deducing Lemma 4.3 from Lemma 4.2. We include the proof for completeness. Let

$$f_\tau^{(k+1)}(\mathbf{x}_{k+1}, \mathbf{x}_{k+1}') = \mathcal{F}_{t \mapsto \tau}(U^{(k+1)}(-t) \alpha^{(k+1)}(t, \mathbf{x}_{k+1}, \mathbf{x}_{k+1}'))$$

where $\mathcal{F}_{t \mapsto \tau}$ denotes the $t \mapsto \tau$ Fourier transform. Then

$$\alpha^{(k+1)}(t, \mathbf{x}_{k+1}, \mathbf{x}_{k+1}') = \int_\tau e^{it\tau} U^{(k+1)}(t) f^{(k+1)}(\mathbf{x}_{k+1}, \mathbf{x}_{k+1}') \, d\tau.$$

By Minkowski's inequality

$$\| R^{(k)} P_{\le M_k}^{(k)} B_{N, j, k+1} \alpha^{(k+1)} \|_{L_t^2 L_{\mathbf{x}, \mathbf{x}'}^2} \le \int_\tau \| R^{(k)} P_{\le M_k}^{(k)} B_{N, j, k+1} U^{(k+1)}(t) f^{(k+1)} \|_{L_t^2 L_{\mathbf{x}, \mathbf{x}'}^2} \, d\tau$$

$$=: I.$$

By Lemma 4.4,

$$I \le C_\varepsilon \sum_{M_{k+1} \ge M_k} \left(\frac{M_k}{M_{k+1}} \right)^{1-\varepsilon} \int_\tau \| R^{(k+1)} P_{\le M_{k+1}}^{(k+1)} f^{(k+1)} \|_{L_{\mathbf{x}, \mathbf{x}'}^2} \, d\tau.$$

For any $b > 1/2$, we write $1 = \langle\tau\rangle^{-b}\langle\tau\rangle^{b}$ and apply Cauchy–Schwarz in τ to obtain

$$
\begin{aligned}
I &\leq C_\varepsilon \sum_{M_{k+1} \geq M_k} \left(\frac{M_k}{M_{k+1}}\right)^{1-\varepsilon} \|\langle\tau\rangle^b R^{(k+1)} P^{(k+1)}_{\leq M_{k+1}} f^{(k+1)}\|_{L^2_{\tau,\mathbf{x},\mathbf{x}'}} \\
&= C_\varepsilon \sum_{M_{k+1} \geq M_k} \left(\frac{M_k}{M_{k+1}}\right)^{1-\varepsilon} \|R^{(k+1)} P^{(k+1)}_{\leq M_{k+1}} \alpha^{(k+1)}\|_{X^{(k)}_{(1/2)+}}. \qquad\qquad \square
\end{aligned}
$$

4.2. A Strichartz estimate

Lemma 4.6. *Assume $\gamma^{(k)}(t, \mathbf{x}_k, \mathbf{x}'_k)$ satisfies the symmetry condition* (1.2). *Let*

$$
\beta^{(k)}(t, \mathbf{x}_k, \mathbf{x}'_k) = V(x_i - x_j)\gamma^{(k)}(t, \mathbf{x}_k, \mathbf{x}'_k). \tag{4.12}
$$

Then we have the estimates

$$
\|\beta^{(k)}\|_{X^{(k)}_{-(1/2)+}} \lesssim \|V\|_{L^{(6/5)+}_x} \|\langle\nabla_{x_i}\rangle\langle\nabla_{x_j}\rangle\gamma^{(k)}\|_{L^2_t L^2_{\mathbf{x},\mathbf{x}'}}, \tag{4.13}
$$

$$
\|\beta^{(k)}\|_{X^{(k)}_{-(1/2)+}} \lesssim \|V\|_{L^{3+}_x} \|\gamma^{(k)}\|_{L^2_t L^2_{\mathbf{x},\mathbf{x}'}}, \tag{4.14}
$$

$$
\|\beta^{(k)}\|_{X^{(k)}_{-(1/2)+}} \lesssim \|V\|_{L^{2+}_x} \|\langle\nabla_{x_i}\rangle^{1/2}\gamma^{(k)}\|_{L^2_t L^2_{\mathbf{x},\mathbf{x}'}}. \tag{4.15}
$$

Proof. It suffices to prove the assertion for $k = 2$. Since we will be need to deal with Fourier transforms in only selected coordinates, we introduce the following notation: $\mathcal{F}_0$ denotes Fourier transform in t, $\mathcal{F}_j$ denotes Fourier transform in x_j, and $\mathcal{F}_{j'}$ denotes Fourier transform in x'_j. Fourier transforms in multiple coordinates will denoted as combined subscripts – for example, $\mathcal{F}_{01'} = \mathcal{F}_0\mathcal{F}_{1'}$ denotes the Fourier transform in t and x'_1.[13]

We start by splitting $\gamma^{(2)}$ into

$$
\gamma^{(2)} = \gamma^{(2)}_{|\xi_1|\geq|\xi_2|} + \gamma^{(2)}_{|\xi_2|\geq|\xi_1|}.
$$

Below we treat

$$
\beta^{(2)}_{|\xi_2|\geq|\xi_1|} = V(x_1 - x_2)\gamma^{(2)}_{|\xi_2|\geq|\xi_1|}
$$

since the $|\xi_1| \geq |\xi_2|$ case is similar. Let T denote the translation operator

$$
(Tf)(x_1, x_2) = f(x_1 + x_2, x_2).
$$

Suppressing the x'_1, x'_2 dependence, we have

$$
(\mathcal{F}_{12}T\beta^{(2)}_{|\xi_2|\geq|\xi_1|})(t, \xi_1, \xi_2) = (\mathcal{F}_{12}\beta^{(2)}_{|\xi_2|\geq|\xi_1|})(t, \xi_1, \xi_2 - \xi_1). \tag{4.16}
$$

---

[13] We are going to apply the endpoint Strichartz estimate on the nontransformed coordinates. We do not know the origin of such a technique, although it was also used by the first author in [13, Lemma 6].

Also

$$e^{-2it\xi_1 \cdot \xi_2}(\mathcal{F}_{12}T\beta^{(2)}_{|\xi_2| \geq |\xi_1|})(t, \xi_1, \xi_2) = \mathcal{F}_1\big[(\mathcal{F}_2 T\beta^{(2)}_{|\xi_2| \geq |\xi_1|})(t, x_1 - 2t\xi_2, \xi_2)\big](\xi_1). \quad (4.17)$$

Now

$$
\begin{aligned}
(\mathcal{F}_{012}\beta^{(2)}_{|\xi_2| \geq |\xi_1|})&(\tau - |\xi_2|^2 + 2\xi_1 \cdot \xi_2, \xi_1, \xi_2 - \xi_1) \\
&= (\mathcal{F}_{012}T\beta^{(2)}_{|\xi_2| \geq |\xi_1|})(\tau - |\xi_2|^2 + 2\xi_1 \cdot \xi_2, \xi_1, \xi_2) \qquad \text{by (4.16)} \\
&= \mathcal{F}_0\big[e^{it|\xi_2|^2} e^{-2it\xi_1 \cdot \xi_2}(\mathcal{F}_{12}T\beta^{(2)}_{|\xi_2| \geq |\xi_1|})(t, \xi_1, \xi_2)\big](\tau) \\
&= \mathcal{F}_0\big[e^{it|\xi_2|^2} \mathcal{F}_1\big[(\mathcal{F}_2 T\beta^{(2)}_{|\xi_2| \geq |\xi_1|})(t, x_1 - 2t\xi_2, \xi_2)\big](\xi_1)\big](\tau) \quad \text{by (4.17)} \\
&= \mathcal{F}_{01}\big[e^{it|\xi_2|^2}(\mathcal{F}_2 T\beta^{(2)}_{|\xi_2| \geq |\xi_1|})(t, x_1 - 2t\xi_2, \xi_2)\big](\tau, \xi_1). \qquad (4.18)
\end{aligned}
$$

By changing variables $\xi_2 \mapsto \xi_2 - \xi_1$ and then $\tau \mapsto \tau - |\xi_2|^2 + 2\xi_1 \cdot \xi_2$, we obtain

$$
\begin{aligned}
\|\beta^{(2)}_{|\xi_2| \geq |\xi_1|}&\|_{X^{(2)}_{-(1/2)+}} \\
&= \|(\beta^{(2)}_{|\xi_2| \geq |\xi_1|})\widehat{}(\tau, \xi_1, \xi_2, \xi_1', \xi_2')\langle \tau + |\xi_1|^2 + |\xi_2|^2 - |\xi_1'|^2 - |\xi_2'|^2 \rangle^{-(1/2)+}\|_{L^2_{\tau\xi_1\xi_2\xi_1'\xi_2'}} \\
&= \|(\beta^{(2)}_{|\xi_2| \geq |\xi_1|})\widehat{}(\tau - |\xi_2|^2 + 2\xi_1 \cdot \xi_2, \xi_1, \xi_2 - \xi_1, \xi_1', \xi_2') \\
&\qquad\qquad\qquad\qquad \cdot \langle \tau + 2|\xi_1|^2 - |\xi_1'|^2 - |\xi_2'|^2 \rangle^{-(1/2)+}\|_{L^2_{\tau\xi_1\xi_2\xi_1'\xi_2'}}.
\end{aligned}
$$

Applying the dual Strichartz (see (4.20) below) shows that

$$
\begin{aligned}
\|\beta^{(2)}_{|\xi_2| \geq |\xi_1|}&\|_{X^{(2)}_{-(1/2)+}} \\
&\lesssim \|\mathcal{F}_{01}^{-1}[(\mathcal{F}_{012}\beta^{(2)}_{|\xi_2| \geq |\xi_1|})(\tau - |\xi_2|^2 + 2\xi_1 \cdot \xi_2, \xi_1, \xi_2 - \xi_1)](t, x_1)\|_{L^2_{\xi_2}L^2_t L^{(6/5)+}_{x_1} L^2_{x_1' x_2'}} =: A.
\end{aligned}
$$

Utilizing (4.18) and then changing variable $x_1 \mapsto x_1 + 2t\xi_2$ yields

$$
\begin{aligned}
A &= \|(\mathcal{F}_2 T\beta^{(2)}_{|\xi_2| \geq |\xi_1|})(t, x_1 - 2t\xi_2, \xi_2)\|_{L^2_t L^2_{\xi_2} L^{(6/5)+}_{x_1} L^2_{x_1' x_2'}} \\
&= \|(\mathcal{F}_2 T\beta^{(2)}_{|\xi_2| \geq |\xi_1|})(t, x_1, \xi_2)\|_{L^2_t L^2_{\xi_2} L^{(6/5)+}_{x_1} L^2_{x_1' x_2'}}.
\end{aligned}
$$

Now note that from (4.12), we have

$$(\mathcal{F}_2 T\beta^{(2)}_{|\xi_2| \geq |\xi_1|})(t, x_1, \xi_2) = V(x_1)(\mathcal{F}_2 T\gamma^{(2)}_{|\xi_2| \geq |\xi_1|})(t, x_1, \xi_2).$$

It follows that

$$\|(\mathcal{F}_2 T \beta^{(2)}_{|\xi_2| \geq |\xi_1|})(t, x_1, \xi_2)\|_{L^2_t L^2_{\xi_2} L^{(6/5)+}_{x_1} L^2_{x'_1 x'_2}}$$

$$= \|V(x_1)(\|(\mathcal{F}_2 T \gamma^{(2)}_{|\xi_2| \geq |\xi_1|})(t, x_1, \xi_2)\|_{L^2_{x'_1 x'_2}})\|_{L^2_t L^2_{\xi_2} L^{(6/5)+}_{x_1}}$$

$$\leq \|V\|_{L^{(6/5)+}} \|(\mathcal{F}_2 T \gamma^{(2)}_{|\xi_2| \geq |\xi_1|})(t, x_1, \xi_2)\|_{L^2_t L^2_{\xi_2} L^{\infty}_{x_1} L^2_{x'_1 x'_2}}$$

$$\leq \|V\|_{L^{(6/5)+}} \|(\mathcal{F}_2 T \gamma^{(2)}_{|\xi_2| \geq |\xi_1|})(t, x_1, \xi_2)\|_{L^2_t L^2_{\xi_2 x'_1 x'_2} L^{\infty}_{x_1}}$$

$$\lesssim \|V\|_{L^{(6/5)+}} \|\langle \nabla_{x_1} \rangle^2 (\mathcal{F}_2 T \gamma^{(2)}_{|\xi_2| \geq |\xi_1|})(t, x_1, \xi_2)\|_{L^2_t L^2_{\xi_2 x'_1 x'_2} L^2_{x_1}} =: B \quad (4.19)$$

by Sobolev in x_1. Move the $d\xi_2 \, dx'_1 \, dx'_2$ integration to the inside and apply Plancherel in $\xi_2 \mapsto x_2$ to obtain

$$B = \|V\|_{L^{(6/5)+}} \|\langle \nabla_{x_1} \rangle^2 T \gamma^{(2)}_{|\xi_2| \geq |\xi_1|}\|_{L^2_t L^2_{\mathbf{x}, \mathbf{x}'}} = \|V\|_{L^{(6/5)+}} \|\langle \nabla_{x_1} \rangle^2 \gamma^{(2)}_{|\xi_2| \geq |\xi_1|}\|_{L^2_t L^2_{\mathbf{x}, \mathbf{x}'}}.$$

Recall that the ξ_2 frequency dominates in $\gamma^{(2)}_{|\xi_2| \geq |\xi_1|}$, and thus

$$B \lesssim \|V\|_{L^{(6/5)+}} \|\langle \nabla_{x_1} \rangle \langle \nabla_{x_2} \rangle \gamma^{(2)}_{|\xi_2| \geq |\xi_1|}(t, \mathbf{x}_2, \mathbf{x}'_2)\|_{L^2_t L^2_{\mathbf{x}, \mathbf{x}'}}$$

$$\lesssim \|V\|_{L^{(6/5)+}} \|\langle \nabla_{x_1} \rangle \langle \nabla_{x_2} \rangle \gamma^{(2)}(t, \mathbf{x}_2, \mathbf{x}'_2)\|_{L^2_t L^2_{\mathbf{x}, \mathbf{x}'}}.$$

This proves estimate (4.13). Using Hölder exponents $(3+, 2, (6/5)+)$ and $(2+, 3, (6/5)+)$ in (4.19) yields estimates (4.14) and (4.15). Their proofs are easier in the sense that there is no need to split $\gamma^{(2)}$.

It remains to prove the following dual Strichartz estimate (here $\sigma^{(2)}(t, x_1, x'_1, x'_2)$, note that the x_2 coordinate is missing):

$$\|\langle \tau + 2|\xi_1|^2 - |\xi'_1|^2 - |\xi'_2|^2 \rangle^{-(1/2)+} \hat{\sigma}^{(2)}(\tau, \xi_1, \xi'_1, \xi'_2)\|_{L^2_\tau L^2_{\xi_1 \xi'_1 \xi'_2}}$$

$$\lesssim \|\sigma^{(2)}\|_{L^2_t L^{(6/5)+}_{x_1} L^2_{x'_1 x'_2}}. \quad (4.20)$$

The estimate (4.20) is dual to the equivalent estimate

$$\|\sigma^{(2)}\|_{L^2_t L^{6-}_{x_1} L^2_{x'_1 x'_2}} \lesssim \|\langle \tau + 2|\xi_1|^2 - |\xi'_1|^2 - |\xi'_2|^2 \rangle^{(1/2)-} \hat{\sigma}^{(2)}(\tau, \xi_1, \xi'_1, \xi'_2)\|_{L^2_\tau L^2_{\xi_1 \xi'_1 \xi'_2}}. \quad (4.21)$$

To prove (4.21), we prove

$$\|\sigma^{(2)}\|_{L^2_t L^6_{x_1} L^2_{x'_1 x'_2}} \lesssim \|\langle \tau + 2|\xi_1|^2 - |\xi'_1|^2 - |\xi'_2|^2 \rangle^{(1/2)+} \hat{\sigma}^{(2)}(\tau, \xi_1, \xi'_1, \xi'_2)\|_{L^2_\tau L^2_{\xi_1 \xi'_1 \xi'_2}}. \quad (4.22)$$

The estimate (4.21) follows from the interpolation of (4.22) and the trivial equality

$$\|\sigma^{(2)}\|_{L^2_t L^2_{x_1} L^2_{x'_1 x'_2}} = \|\langle \tau + 2|\xi_1|^2 - |\xi'_1|^2 - |\xi'_2|^2 \rangle^0 \hat{\sigma}^{(2)}(\tau, \xi_1, \xi'_1, \xi'_2)\|_{L^2_\tau L^2_{\xi_1 \xi'_1 \xi'_2}}.$$

Thus proving (4.20) is reduced to proving (4.22), which we do now. Let

$$\phi_\tau(x_1, x_1', x_2) := \mathcal{F}_0[U^1(-2t)U^{1'}(-t)U^{2'}(-t)\sigma^{(2)}(t, x_1, x_1', x_2')](\tau). \qquad (4.23)$$

Then ϕ_τ is independent of t and

$$\sigma^{(2)}(t, x_1, x_1', x_2') = \int e^{it\tau} U^1(2t)U^{1'}(t)U^{2'}(t)\phi_\tau(x_1, x_1', x_2') \, d\tau.$$

Thus

$$\begin{aligned}
\|\sigma^{(2)}\|_{L_t^2 L_{x_1}^6 L_{x_1' x_2'}^2} &\lesssim \int_\tau \|U^{1'}(t)U^{2'}(t)U^1(2t)\phi_\tau(x_1, x_1', x_2')\|_{L_t^2 L_{x_1}^6 L_{x_1' x_2'}^2} \, d\tau \\
&\lesssim \int_\tau \|U^1(2t)\phi_\tau(x_1, x_1', x_2')\|_{L_t^2 L_{x_1}^6 L_{x_1' x_2'}^2} \, d\tau \\
&\lesssim \int_\tau \|U^1(2t)\phi_\tau(x_1, x_1', x_2')\|_{L_{x_1' x_2'}^2 L_t^2 L_{x_1}^6} \, d\tau =: A.
\end{aligned}$$

Now apply Keel–Tao's [41] endpoint Strichartz estimate to obtain

$$\begin{aligned}
A &\lesssim \int_\tau \|\phi_\tau(x_1, x_1', x_2')\|_{L_{x_1' x_2'}^2 L_{x_1}^2} \, d\tau \\
&\lesssim \|\langle\tau\rangle^{(1/2)+} \phi_\tau(x_1, x_1', x_2')\|_{L_\tau^2 L_{x_1 x_1' x_2'}^2} \\
&= \|\langle\tau + 2|\xi_1|^2 - |\xi_1'|^2 - |\xi_2'|^2\rangle^{(1/2)+} \hat\sigma^{(2)}(\tau, \xi_1, \xi_1', \xi_2')\|_{L_{\tau\xi_1\xi_1'\xi_2'}^2}
\end{aligned}$$

by (4.23), which completes the proof of (4.22). $\qquad\qquad\qquad\qquad\qquad\Box$

Corollary 4.1. *Let*

$$\beta^{(k)}(t, \mathbf{x}_k, \mathbf{x}_k') = N^{3\beta-1} V(N^\beta(x_i - x_j))\gamma^{(k)}(t, \mathbf{x}_k, \mathbf{x}_k').$$

Then for $N \geq 1$, we have

$$\left\||\nabla_{x_i}||\nabla_{x_j}|\beta^{(k)}\right\|_{X_{-(1/2)+}^{(k)}} \lesssim N^{5\beta/2-1} \|\langle\nabla_{x_i}\rangle\langle\nabla_{x_j}\rangle\gamma^{(k)}\|_{L_t^2 L_{\mathbf{x},\mathbf{x}'}^2}, \qquad (4.24)$$

$$\|\beta^{(k)}\|_{X_{-(1/2)+}^{(k)}} \lesssim N^{\beta/2-1} \|\langle\nabla_{x_i}\rangle\langle\nabla_{x_j}\rangle\gamma^{(k)}\|_{L_t^2 L_{\mathbf{x},\mathbf{x}'}^2}. \qquad (4.25)$$

Consequently (with $R_{\leq M}^{(k)} = P_{\leq M}^{(k)} R^{(k)}$),

$$\|R_{\leq M}^{(k)} \beta^{(k)}\|_{X_{-(1/2)+}^{(k)}} \lesssim N^{\beta/2-1} \min(M^2, N^{2\beta}) \|S^{(k)}\gamma^{(k)}\|_{L_t^2 L_{\mathbf{x},\mathbf{x}'}^2}. \qquad (4.26)$$

Proof. Estimate (4.24) follows by applying either (4.13), (4.14), or (4.15) according to whether two derivatives, no derivatives, or one derivative, respectively, land on $N^{3\beta-1}V(N^\beta(x_i - x_j))$.

Estimate (4.25) follows by applying (4.13).

Finally, (4.26) follows from (4.24) and (4.25), as follows. Let

$$Q = \prod_{\substack{1 \le \ell \le k \\ \ell \ne i, j}} |\nabla_{x_\ell}|.$$

Then

$$\|R_{\le M}^{(k)}\beta^{(k)}\|_{X_{-(1/2)+}^{(k)}} \le M^2 \|Q\beta^{(k)}\|_{X_{-(1/2)+}^{(k)}}.$$

The Q operator passes directly onto $\gamma^{(k)}$, and one applies (4.25) to obtain

$$\|R_{\le M}^{(k)}\beta^{(k)}\|_{X_{-(1/2)+}^{(k)}} \lesssim N^{\beta/2-1}M^2\|S^{(k)}\gamma^{(k)}\|_{L_t^2 L_{\mathbf{x,x'}}^2}. \tag{4.27}$$

On the other hand,

$$\|R_{\le M}^{(k)}\beta^{(k)}\|_{X_{-(1/2)+}^{(k)}} \le \|Q|\nabla_{x_i}||\nabla_{x_j}|\beta^{(k)}\|_{X_{-(1/2)+}^{(k)}}.$$

The Q operator passes directly on $\gamma^{(k)}$, and one applies (4.24) to obtain

$$\|R_{\le M}^{(k)}\beta^{(k)}\|_{X_{-(1/2)+}^{(k)}} \lesssim N^{5\beta/2-1}\|S^{(k)}\gamma^{(k)}\|_{L_t^2 L_{\mathbf{x,x'}}^2}. \tag{4.28}$$

Combining (4.27) and (4.28), we obtain (4.26). $\square$

Appendix A. The topology on the density matrices

In this appendix, we define a topology τ_{prod} on the density matrices, as was previously done in [26, 31, 27, 28, 29, 30, 43, 9, 14, 15, 16].

Denote the spaces of compact operators and trace class operators on $L^2(\mathbb{R}^{3k})$ by $\mathcal{K}_k$ and $\mathcal{L}_k^1$, respectively. Then $(\mathcal{K}_k)' = \mathcal{L}_k^1$. Since $\mathcal{K}_k$ is separable, we select a dense countable subset $\{J_i^{(k)}\}_{i\ge 1} \subset \mathcal{K}_k$ in the unit ball of $\mathcal{K}_k$ (so $\|J_i^{(k)}\|_{\text{op}} \le 1$ where $\|\cdot\|_{\text{op}}$ is the operator norm). For $\gamma^{(k)}, \tilde{\gamma}^{(k)} \in \mathcal{L}_k^1$, we then define a metric d_k on $\mathcal{L}_k^1$ by

$$d_k(\gamma^{(k)}, \tilde{\gamma}^{(k)}) = \sum_{i=1}^{\infty} 2^{-i}|\text{Tr}\, J_i^{(k)}(\gamma^{(k)} - \tilde{\gamma}^{(k)})|.$$

A uniformly bounded sequence $\gamma_N^{(k)} \in \mathcal{L}_k^1$ converges to $\gamma^{(k)} \in \mathcal{L}_k^1$ in the weak* topology if and only if

$$\lim_N d_k(\gamma_N^{(k)}, \gamma^{(k)}) = 0.$$

For fixed $T > 0$, let $C([0, T], \mathcal{L}_k^1)$ be the space of functions of $t \in [0, T]$ with values in $\mathcal{L}_k^1$ which are continuous with respect to the metric d_k. On $C([0, T], \mathcal{L}_k^1)$, we define the metric

$$\hat{d}_k(\gamma^{(k)}(\cdot), \tilde{\gamma}^{(k)}(\cdot)) = \sup_{t \in [0, T]} d_k(\gamma^{(k)}(t), \tilde{\gamma}^{(k)}(t)).$$

We can then define a topology τ_{prod} on the space $\bigoplus_{k \geq 1} C([0, T], \mathcal{L}_k^1)$ to be the product of the topologies generated by the metrics $\hat{d}_k$ on $C([0, T], \mathcal{L}_k^1)$.

Appendix B. Proof of estimates (2.9) and (2.11)

Proof of (2.9). Applying Lemma 2.1 and estimate (3.3) to the free part of $\gamma_N^{(2)}$, we obtain

$$\|R^{(k-1)} B_{N,1,k} FP^{(k,l_c)}\|_{L_T^1 L_{\mathbf{x},\mathbf{x}'}^2}$$

$$\leq CT^{1/2} \|R^{(k)} \gamma_{N,0}^{(k)}\|_{L_{\mathbf{x},\mathbf{x}'}^2}$$

$$+ \sum_{j=1}^{l_c} \sum_m \left\| R^{(k-1)} B_{N,1,k} \int_D J_N^{(k,j)}(\underline{t}_{k+j}, \mu_m)(U^{(k+j)}(t_{k+j}) \gamma_{N,0}^{(k+j)}) \, d\underline{t}_{k+j} \right\|_{L_T^1 L_{\mathbf{x},\mathbf{x}'}^2}$$

$$\leq CT^{1/2} \|R^{(k)} \gamma_{N,0}^{(k)}\|_{L_{\mathbf{x},\mathbf{x}'}^2}$$

$$+ \sum_{j=1}^{l_c} \sum_m (CT^{1/2})^j \|R^{(k+j-1)} B_{N,\mu_m(k+j),k+j} U^{(k+j)}(t_{k+j}) \gamma_{N,0}^{(k+j)}\|_{L_T^1 L_{\mathbf{x},\mathbf{x}'}^2}$$

$$\leq CT^{1/2} \|R^{(k)} \gamma_{N,0}^{(k)}\|_{L_{\mathbf{x},\mathbf{x}'}^2} + \sum_{j=1}^{\infty} 4^{j-1} (CT^{1/2})^{j+1} \|R^{(k+j)} \gamma_{N,0}^{(k+j)}\|_{L_{\mathbf{x},\mathbf{x}'}^2}$$

Use condition (2.5) to get

$$\|R^{(k-1)} B_{N,1,k} FP^{(k,l_c)}\|_{L_T^1 L_{\mathbf{x},\mathbf{x}'}^2} \leq CT^{1/2} C_0^k + \sum_{j=1}^{\infty} 4^{j-1} (CT^{1/2})^{j+1} C_0^{k+j}$$

$$\leq C_0^k \Big(CT^{1/2} + \sum_{j=1}^{\infty} 4^{j-1} (CT^{1/2})^{j+1} C_0^j \Big).$$

We can choose a T independent of k, l_c and N such that the series in the above estimate converges. We then have

$$\|R^{(k-1)} B_{N,1,k} FP^{(k,l_c)}\|_{L_T^1 L_{\mathbf{x},\mathbf{x}'}^2} \leq C_0^k (CT^{1/2} + C) \leq C^{k-1}$$

for some C larger than C_0. Thus, we have shown estimate (2.9). $\qquad\square$

Proof of (2.11). We proceed as in the proof of estimate (2.9) and end up with

$$\|R^{(k-1)}B_{N,1,k}IP^{(k,l_c)}\|_{L_T^1 L_{\mathbf{x},\mathbf{x}'}^2}$$
$$\leq \sum_m \left\| R^{(k-1)}B_{N,1,k}\int_D J_N^{(k,l_c+1)}(\underline{t}_{k+l_c+1},\mu_m)(\gamma_N^{(k+l_c+1)}(t_{k+l_c+1}))\,d\underline{t}_{k+l_c+1} \right\|_{L_T^1 L_{\mathbf{x},\mathbf{x}'}^2}$$
$$\leq \sum_m (CT^{1/2})^{l_c}\|R^{(k+l_c)}B_{N,\mu_m(k+l_c+1),k+l_c+1}\gamma_N^{(k+l_c+1)}(t_{k+l_c+1})\|_{L_T^1 L_{\mathbf{x},\mathbf{x}'}^2}.$$

We then investigate

$$\|R^{(k+l_c)}B_{N,\mu_m(k+l_c+1),k+l_c+1}\gamma_N^{(k+l_c+1)}(t_{k+l_c+1})\|_{L_T^1 L_{\mathbf{x},\mathbf{x}'}^2}.$$

Setting $\mu_m(k+l_c+1)=1$ for simplicity and looking at $\tilde{B}_{N,1,k+l_c+1}$, we have

$$R^{(k+l_c)}\tilde{B}_{N,1,k+l_c+1}\gamma_N^{(k+l_c+1)}(t)$$
$$= R^{(k+l_c)}\int V_N(x_1-x_{k+l_c+1})\gamma_N^{(k+l_c+1)}(t,\mathbf{x}_{k+l_c},x_{k+l_c+1};\mathbf{x}'_{k+l_c},x_{k+l_c+1})\,dx_{k+l_c+1}$$
$$= I + II$$

with I and II given by the product rule:

$$I = \int V_N'(x_1-x_{k+l_c+1})\left(\frac{R^{(k+l_c)}}{|\nabla_{x_1}|}\right)\gamma_N^{(k+l_c+1)}(t,\mathbf{x}_{k+l_c},x_{k+l_c+1};\mathbf{x}'_{k+l_c},x_{k+l_c+1})\,dx_{k+l_c+1},$$
$$II = \int V_N(x_1-x_{k+l_c+1})(R^{(k+l_c)}\gamma_N^{(k+l_c+1)})(t,\mathbf{x}_{k+l_c},x_{k+l_c+1};\mathbf{x}'_{k+l_c},x_{k+l_c+1})\,dx_{k+l_c+1},$$

where we wrote

$$\frac{R^{(k+l_c)}}{|\nabla_{x_1}|} = \left(\prod_{j=2}^{k+l_c}|\nabla_{x_j}|\right)\left(\prod_{j=1}^{k+l_c}|\nabla_{x'_j}|\right).$$

Then

$$\int |R^{(k+l_c)}\tilde{B}_{N,1,k+l_c+1}\gamma_N^{(k+l_c+1)}(t)|^2\,d\mathbf{x}_{k+l_c}\,d\mathbf{x}'_{k+l_c}$$
$$= \int |I+II|^2\,d\mathbf{x}_{k+l_c}\,d\mathbf{x}'_{k+l_c}$$
$$\leq C\int |I|^2\,d\mathbf{x}_{k+l_c}\,d\mathbf{x}'_{k+l_c} + C\int |II|^2\,d\mathbf{x}_{k+l_c}\,d\mathbf{x}'_{k+l_c}.$$

To estimate the first term, we first use Cauchy–Schwarz in dx_{k+l_c+1}:

$$
\int |I|^2 \, d\mathbf{x}_{k+l_c} \, d\mathbf{x}'_{k+l_c}
$$

$$
\leq \int d\mathbf{x}_{k+l_c} \, d\mathbf{x}'_{k+l_c} \left(\int |V'_N(x_1 - x_{k+l_c+1})|^2 \, dx_{k+l_c+1} \right)
$$

$$
\times \left(\int \left| \left(\frac{R^{(k+l_c)}}{|\nabla_{x_1}|} \right) \gamma_N^{(k+l_c+1)}(t, \mathbf{x}_{k+l_c}, x_{k+l_c+1}; \mathbf{x}'_{k+l_c}, x_{k+l_c+1}) \right|^2 dx_{k+l_c+1} \right)
$$

$$
\leq N^{5\beta} \|V'\|_{L^2}^2 \int d\mathbf{x}_{k+l_c} \, d\mathbf{x}'_{k+l_c}
$$

$$
\times \left(\int |S^{(k+l_c)} \gamma_N^{(k+l_c+1)}(t, \mathbf{x}_{k+l_c}, x_{k+l_c+1}; \mathbf{x}'_{k+l_c}, x_{k+l_c+1})|^2 \, dx_{k+l_c+1} \right)
$$

where $V \in W^{2,(6/5)+}$ implies that $V \in H^1$ by Sobolev. The trace theorem then gives

$$
\int |I|^2 \, d\mathbf{x}_{k+l_c} \, d\mathbf{x}'_{k+l_c} \leq C N^{5\beta} \|V'\|_{L^2}^2 \int d\mathbf{x}_{k+l_c} \, d\mathbf{x}'_{k+l_c}
$$

$$
\times \left(\int |S^{(k+l_c+1)} \gamma_N^{(k+l_c+1)}(t, \mathbf{x}_{k+l_c}, x_{k+l_c+1}; \mathbf{x}'_{k+l_c}, x'_{k+l_c+1})|^2 \, dx_{k+l_c+1} \, dx'_{k+l_c+1} \right)
$$

$$
= C N^{5\beta} \|V'\|_{L^2}^2 \|S^{(k+l_c+1)} \gamma_N^{(k+l_c+1)}\|_{L_T^\infty L_{\mathbf{x},\mathbf{x}'}^2}^2.
$$

Estimating the second term in the same manner, we get

$$
\int |II|^2 \, d\mathbf{x}_{k+l_c} \, d\mathbf{x}'_{k+l_c} = \int d\mathbf{x}_{k+l_c} \, d\mathbf{x}'_{k+l_c}
$$

$$
\times \left| \int V_N(x_1 - x_{k+l_c+1})(R^{(k+l_c)} \gamma_N^{(k+l_c+1)})(t, \mathbf{x}_{k+l_c}, x_{k+l_c+1}; \mathbf{x}'_{k+l_c}, x_{k+l_c+1}) \, dx_{k+l_c+1} \right|^2
$$

$$
\leq C N^{3\beta} \|V\|_{L^2}^2 \|S^{(k+l_c+1)} \gamma_N^{(k+l_c+1)}\|_{L_T^\infty L_{\mathbf{x},\mathbf{x}'}^2}^2,
$$

Accordingly,

$$
\int |R^{(k+l_c)} \tilde{B}_{N,1,k+l_c+1} \gamma_N^{(k+l_c+1)}(t)|^2 \, d\mathbf{x}_{k+l_c} \, d\mathbf{x}'_{k+l_c}
$$

$$
\leq C N^{5\beta} \|V\|_{H^1}^2 \|S^{(k+l_c+1)} \gamma_N^{(k+l_c+1)}\|_{L_T^\infty L_{\mathbf{x},\mathbf{x}'}^2}^2.
$$

Hence

$$
\|R^{(k-1)} B_{N,1,k} IP^{(k,l_c)}\|_{L_T^1 L_{\mathbf{x},\mathbf{x}'}^2}
$$

$$
\leq \sum_m (CT^{1/2})^{l_c} \|R^{(k+l_c)} B_{N,\mu_m(k+l_c+1),k+l_c+1} \gamma_N^{(k+l_c+1)}(t_{k+l_c+1})\|_{L_T^1 L_{\mathbf{x},\mathbf{x}'}^2}
$$

$$
\leq C 4^{l_c} (CT^{1/2})^{l_c} T (C N^{5\beta/2} \|V\|_{H^1} \|S^{(k+l_c+1)} \gamma_N^{(k+l_c+1)}\|_{L_T^\infty L_{\mathbf{x},\mathbf{x}'}^2}).
$$

Put in condition (2.5), it becomes

$$\|R^{(k-1)}B_{N,1,k}IP^{(k,l_c)}\|_{L_T^1 L_{\mathbf{x},\mathbf{x}'}^2} \le CT(CT^{1/2})^{l_c} N^{5\beta/2} C_0^{k+l_c+1}$$
$$= C_0^k[CT(CT^{1/2})^{l_c} N^{5\beta/2} C_0^{l_c+1}].$$

Replacing the constants C and C_0 inside the bracket with some larger constant and grouping the terms, we have

$$\|R^{(k-1)}B_{N,1,k}IP^{(k,l_c)}\|_{L_T^1 L_{\mathbf{x},\mathbf{x}'}^2} \le C_0^k[(T^{1/2})^{2+l_c} N^{5\beta/2} C^{l_c}].$$

As in [11, 15], we take the coupling level $l_c = \log N$ to deal with what is inside the bracket:

$$\|R^{(k-1)}B_{N,1,k}IP^{(k,l_c)}\|_{L_T^1 L_{\mathbf{x},\mathbf{x}'}^2} \le CC_0^k[(T^{1/2})^{2+\log N} N^{5\beta/2} N^c].$$

Notice that there is no k inside the bracket. Selecting T such that $T \le e^{-(5\beta+2C)}$ ensures that

$$(T^{1/2})^{\log N} N^{5\beta/2} N^c \le 1,$$

and hence

$$\|R^{(k-1)}B_{N,1,k}IP^{(k,l_c)}\|_{L_T^1 L_{\mathbf{x},\mathbf{x}'}^2} \le CC_0^k \le C^{k-1}$$

with a C larger than C_0 and independent of k and N. Thus, we have finished the proof of estimate (2.11). □

References

[1] Adami, R., Golse, F., Teta, A.: Rigorous derivation of the cubic NLS in dimension one. J. Statist. Phys. **127**, 1193–1220 (2007) Zbl 1118.81021 MR 2331036

[2] Anderson, M. H., Ensher, J., Matthews, M. R., Wieman, C. E., Cornell, E. A.: Observation of Bose–Einstein condensation in a dilute atomic vapor. Science **269**, 198–201 (1995)

[3] Beckner, W.: Multilinear embedding—convolution estimates on smooth submanifolds. Proc. Amer. Math. Soc. **142**, 1217–1228 (2014) Zbl 1297.35011 MR 3162244

[4] Benedikter, N., Oliveira, G., Schlein, B.: Quantitative derivation of the Gross–Pitaevskii equation. Comm. Pure Appl. Math. **68**, 1399–1482 (2015) Zbl 1320.35318 MR 3366749

[5] Bourgain, J.: Global Solutions of Nonlinear Schrödinger Equations. Amer. Math. Soc. Colloq. Publ. 46, Amer. Math. Soc., Providence, RI (1999) Zbl 0933.35178 MR 1691575

[6] Chen, L., Lee, J. O., Schlein, B.: Rate of convergence towards Hartree dynamics. J. Statist. Phys. **144**, 872–903 (2011) Zbl 1227.82046 MR 2826623

[7] Chen, T., Hainzl, C., Pavlović, N., Seiringer, R.: Unconditional uniqueness for the cubic Gross–Pitaevskii hierarchy via quantum de Finetti. Comm. Pure Appl. Math. **68**, 1845–1884 (2015) Zbl 1326.35332 MR 3385343

[8] Chen, T., Pavlović, N.: On the Cauchy problem for focusing and defocusing Gross–Pitaevskii hierarchies. Discrete Contin. Dynam. Systems **27**, 715–739 (2010) Zbl 1190.35207 MR 2600687

[9] Chen, T., Pavlović, N.: The quintic NLS as the mean field limit of a boson gas with three-body interactions. J. Funct. Anal. **260**, 959–997 (2011) Zbl 1213.35368 MR 2747009

[10] Chen, T., Pavlović, N.: A new proof of existence of solutions for focusing and defocusing Gross–Pitaevskii hierarchies. Proc. Amer. Math. Soc. **141**, 279–293 (2013) Zbl 1260.35197 MR 2988730

[11] Chen, T., Pavlović, N.: Derivation of the cubic NLS and Gross–Pitaevskii hierarchy from manybody dynamics in $d = 3$ based on spacetime norms. Ann. Henri Poincaré **15**, 543–588 (2014) Zbl 06290593 MR 3165917

[12] Chen, X.: Classical proofs of Kato type smoothing estimates for the Schrödinger equation with quadratic potential in $\mathbb{R}^{n+1}$ with application. Differential Integral Equations **24**, 209–230 (2011) Zbl 1240.35429 MR 2757458

[13] Chen, X.: Second order corrections to mean field evolution for weakly interacting bosons in the case of three-body interactions. Arch. Ration. Mech. Anal. **203**, 455–497 (2012) Zbl 1256.35099 MR 2885567

[14] Chen, X.: Collapsing estimates and the rigorous derivation of the 2D cubic nonlinear Schrödinger equation with anisotropic switchable quadratic traps. J. Math. Pures Appl. **98**, 450–478 (2012) Zbl 1251.35144 MR 2968164

[15] Chen, X.: On the rigorous derivation of the 3D cubic nonlinear Schrödinger equation with a quadratic trap. Arch. Ration. Mech. Anal. **210**, 365–408 (2013) Zbl 1294.35132 MR 3101788

[16] Chen, X., Holmer, J.: On the rigorous derivation of the 2D cubic nonlinear Schrödinger equation from 3D quantum many-body dynamics. Arch. Ration. Mech. Anal. **210**, 909–954 (2013) Zbl 1288.35429 MR 3116008

[17] Chen, X., Holmer, J.: Focusing quantum many-body dynamics: the rigorous derivation of the 1D focusing cubic nonlinear Schrödinger equation. Arch. Ration. Mech. Anal., doi:10.1007/s00205-016-0970-6

[18] Chen, X., Holmer, J.: Focusing quantum many-body dynamics II: the rigorous derivation of the 1D focusing cubic nonlinear Schrödinger equation from 3D. arXiv:1407.8457 (2014)

[19] Chen, X., Holmer, J.: Correlation structures, many-body scattering processes and the derivation of the Gross–Pitaevskii hierarchy. Int. Math. Res. Notices 2015, doi:10.1093/imrn/rnv228

[20] Chen, X., Holmer, J.: The rigorous derivation of the 2D cubic focusing NLS from quantum many-body evolution. arXiv:1508.07675 (2015)

[21] Chen, X., Smith, P.: On the unconditional uniqueness of solutions to the infinite radial Chern–Simons–Schrödinger hierarchy. Anal. PDE **7**, 1683–1712 (2014) Zbl 1307.35273 MR 3293448

[22] Cladé, P., Ryu, C., Ramanathan, A., Helmerson, K., Phillips, W. D.: Observation of a 2D Bose gas: From thermal to quasicondensate to superfluid. Phys. Rev. Lett. **102**, 170401 (2009)

[23] Cornish, S. L., Claussen, N. R., Roberts, J. L., Cornell, E. A., Wieman, C. E.: Stable ^{85}Rb Bose–Einstein condensates with widely turnable interactions. Phys. Rev. Lett. **85**, 1795–1798 (2000)

[24] Davis, K. B., Mewes, M.-O., Andrews, M. R., van Druten, N. J., Durfee, D. S., Kurn, D. M., Ketterle, W.: Bose–Einstein condensation in a gas of sodium atoms. Phys. Rev. Lett. **75**, 3969–3973 (1995)

[25] Desbuquois, R., Chomaz, L., Yefsah, T., Léonard, J., Beugnon, J., Weitenberg, C., Dalibard, J.: Superfluid behaviour of a two-dimensional Bose gas. Nature Physics **8**, 645–648 (2012)

[26] Elgart, A., Erdős, L., Schlein, B., Yau, H. T.: Gross–Pitaevskii equation as the mean field limit of weakly coupled bosons. Arch. Ration. Mech. Anal. **179**, 265–283 (2006) Zbl 1086.81035 MR 2209131

[27] Erdős, L., Schlein, B., Yau, H. T.: Derivation of the Gross–Pitaevskii hierarchy for the dynamics of Bose–Einstein condensate. Comm. Pure Appl. Math. **59**, 1659–1741 (2006) Zbl 1122.82018 MR 2257859

[28] Erdős, L., Schlein, B., Yau, H. T.: Derivation of the cubic non-linear Schrödinger equation from quantum dynamics of many-body systems. Invent. Math. **167**, 515–614 (2007) Zbl 1123.35066 MR 2276262

[29] Erdős, L., Schlein, B., Yau, H. T.: Rigorous derivation of the Gross–Pitaevskii equation with a large interaction potential. J. Amer. Math. Soc. **22**, 1099–1156 (2009) Zbl 1207.82031 MR 2525781

[30] Erdős, L., Schlein, B., Yau, H. T.: Derivation of the Gross–Pitaevskii equation for the dynamics of Bose–Einstein condensate. Ann. of Math. **172**, 291–370 (2010) Zbl 1204.82028 MR 2680421

[31] Erdős, L., Yau, H. T.: Derivation of the nonlinear Schrödinger equation from a many body Coulomb system. Adv. Theor. Math. Phys. **5**, 1169–1205 (2001) Zbl 1014.81063 MR 1926667

[32] Fröhlich, J., Knowles, A., Schwarz, S.: On the mean-field limit of bosons with Coulomb two-body interaction. Comm. Math. Phys. **288**, 1023–1059 (2009) Zbl 1177.82016 MR 2504864

[33] Görlitz, A., Vogels, J. M., Leanhardt, A. E., Raman, C., Gustavson, T. L., Abo-Shaeer, J. R., Chikkatur, A. P., Gupta, S., Inouye, S., Rosenband, T., Ketterle, W.: Realization of Bose–Einstein condensates in lower dimensions. Phys. Rev. Lett. **87**, 130402 (2001)

[34] Gressman, P., Sohinger, V., Staffilani, G.: On the uniqueness of solutions to the periodic 3D Gross–Pitaevskii hierarchy. J. Funct. Anal. **266**, 4705–4764 (2014) Zbl 1297.35215 MR 3170216

[35] Grillakis, M. G., Machedon, M.: Pair excitations and the mean field approximation of interacting bosons, I. Comm. Math. Phys. **324**, 601–636 (2013) Zbl 1277.82034 MR 3117522

[36] Grillakis, M. G., Machedon, M., Margetis, D.: Second order corrections to mean field evolution for weakly interacting bosons. I. Comm. Math. Phys. **294**, 273–301 (2010) Zbl 1208.82030 MR 2575484

[37] Grillakis, M. G., Machedon, M., Margetis, D.: Second order corrections to mean field evolution for weakly interacting bosons. II. Adv. Math. **228**, 1788–1815 (2011) Zbl 1226.82033 MR 2824569

[38] Grillakis, M. G., Margetis, D.: A priori estimates for many-body Hamiltonian evolution of interacting boson system. J. Hyperbolic Differential Equations **5**, 857–883 (2008) Zbl 1160.35357 MR 2475483

[39] Hadzibabic, Z., Krüger, P., Cheneau, M., Battelier, B., Dalibard, J.: Berezinskii–Kosterlitz–Thouless crossover in a trapped atomic gas. Nature **441**, 1118–1121 (2006)

[40] Hong, Y., Taliaferro, K., Xie, Z.: Unconditional uniqueness of the cubic Gross–Pitaevskii hierarchy with low regularity. SIAM J. Math. Anal. **47**, 3314–3341 (2015) Zbl 06484946 MR 3395127

[41] Keel, M., Tao, T.: Endpoint Strichartz estimates. Amer. J. Math. **120**, 955–980 (1998) Zbl 0922.35028 MR 1646048

[42] Ketterle, W., J. van Druten, N.: Evaporative cooling of trapped atoms. Adv. Atomic Molecular Optical Phys. **37**, 181–236 (1996)

[43] Kirkpatrick, K., Schlein, B., Staffilani, G.: Derivation of the two-dimensional nonlinear Schrödinger equation from many body quantum dynamics. Amer. J. Math. **133**, 91–130 (2011) Zbl 1208.81080 MR 2752936

[44] Klainerman, S., Machedon, M.: Space-time estimates for null forms and the local existence theorem. Comm. Pure Appl. Math. **46**, 1221–1268 (1993)   Zbl 0803.35095   MR 1231427

[45] Klainerman, S., Machedon, M.: On the uniqueness of solutions to the Gross–Pitaevskii hierarchy. Comm. Math. Phys. **279**, 169–185 (2008)   Zbl 1147.37034   MR 2377632

[46] Knowles, A., Pickl, P.: Mean-field dynamics: singular potentials and rate of convergence. Comm. Math. Phys. **298**, 101–138 (2010)   Zbl 1213.81180   MR 2657816

[47] Lieb, E. H., Seiringer, R., Solovej, J. P., Yngvason, J.: The Mathematics of the Bose Gas and Its Condensation. Birkhäuser, Basel (2005)   Zbl 1104.82012   MR 2143817

[48] Michelangeli, A., Schlein, B.: Dynamical collapse of boson stars. Comm. Math. Phys. **311**, 645–687 (2012)   Zbl 1242.85007   MR 2909759

[49] Pickl, P.: A simple derivation of mean field limits for quantum systems. Lett. Math. Phys. **97**, 151–164 (2011)   Zbl 1242.81150   MR 2821235

[50] Rodnianski, I., Schlein, B.: Quantum fluctuations and rate of convergence towards mean field dynamics. Comm. Math. Phys. **291**, 31–61 (2009)   Zbl 1186.82051   MR 2530155

[51] Sohinger, V.: A rigorous derivation of the defocusing cubic nonlinear Schrödinger equation on $\mathbb{T}^3$ from the dynamics of many-body quantum systems. Ann. Inst. H. Poincaré Anal. Non Linéaire **32**, 1337–1365 (2015)   Zbl 1328.35220   MR 3425265

[52] Spohn, H.: Kinetic equations from Hamiltonian dynamics. Rev. Mod. Phys. **52**, 569–615 (1980)   MR 0578142

[53] Stamper-Kurn, D. M., Andrews, M. R., Chikkatur, A. P., Inouye, S., Miesner, H.-J., Stenger, J., Ketterle, W.: Optical confinement of a Bose–Einstein condensate. Phys. Rev. Lett. **80**, 2027–2030 (1998)

[54] Stock, S., Hadzibabic, Z., Battelier, B., Cheneau, M., Dalibard, J.: Observation of phase defects in quasi-two-dimensional Bose–Einstein condensates. Phys. Rev. Lett. **95**, 190403 (2005)