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A cluster algebra approach to q-characters of Kirillov–Reshetikhin modules

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Abstract. We describe a cluster algebra algorithm for calculating the q-characters of Kirillov–Reshetikhin modules for any untwisted quantum affine algebra $U_q(\widehat{\mathfrak{g}})$. This yields a geometric q-character formula for tensor products of Kirillov–Reshetikhin modules. When $\mathfrak{g}$ is of type A, D, E, this formula extends Nakajima's formula for q-characters of standard modules in terms of homology of graded quiver varieties.

Keywords. Quantum affine algebra, cluster algebras, q-characters, Kirillov–Reshetikhin modules, geometric character formula

1. Introduction

Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$, and let $U_q(\widehat{\mathfrak{g}})$ be the corresponding untwisted quantum affine algebra with quantum parameter $q \in \mathbb{C}^*$ not a root of unity. The finite-dimensional complex representations of $U_q(\widehat{\mathfrak{g}})$ have been studied by many authors during the past twenty years. We refer the reader to [CP1] for a classical introduction, and to [CH, Le2] for recent surveys on this topic.

In [HL1], we started to explore some new connections between this rich representation theory and the cluster algebras of Fomin and Zelevinsky. The main result, proved in [HL1] in type A_n and D_4, and extended to any A, D, E type by Nakajima [N4], shows the existence of a tensor category $\mathscr{C}_1$ of finite-dimensional $U_q(\widehat{\mathfrak{g}})$-modules whose Grothendieck ring is a cluster algebra of the same finite Dynkin type, such that the classes of simple modules coincide with the set of cluster monomials. As a consequence, the q-characters of the simple objects of $\mathscr{C}_1$ can be computed algorithmically using the combinatorics of cluster algebras. Moreover, the Caldero–Chapoton formula for cluster expansions leads to some new geometric formulae for these characters, in terms of Euler characteristics of quiver Grassmannians.

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Unfortunately the category $\mathscr{C}_1$ is quite small. For instance it contains only three Kirillov–Reshetikhin modules for each node of the Dynkin diagram of $\mathfrak{g}$. Another limitation of the papers [HL1] and [N4] is that $\mathfrak{g}$ is assumed to be of simply laced type. In fact, the general proof of Nakajima uses in a crucial way his geometric construction of the standard $U_q(\widehat{\mathfrak{g}})$-modules [N1], which is only available in the simply laced case.

In this paper we drop the assumption of being simply laced, and we consider a much larger tensor subcategory $\mathscr{C}^-$ which contains, up to spectral shifts, all the irreducible finite-dimensional representations of $U_q(\widehat{\mathfrak{g}})$. Our first main result (Theorem 3.1) is an algorithm which calculates the q-character of an arbitrary Kirillov–Reshetikhin module in $\mathscr{C}^-$ as the result of a sequence of cluster mutations. The only input for this calculation is the initial seed of our cluster algebra $\mathscr{A}$, which is encoded in a quiver obtained from the Cartan matrix of $\mathfrak{g}$ by a simple and uniform recipe. (It may be worth noting that $\mathscr{A}$ is always a skew-symmetric cluster algebra, even when $\mathfrak{g}$ is not simply laced.)

The proof of this theorem is based on the fact that the q-characters of Kirillov–Reshetikhin modules are solutions of the corresponding T-system of Kuniba, Nakanishi and Suzuki [KNS1, N2, H]. This will come as no surprise, given the many papers already devoted to the relationships between cluster algebras and T-systems (see in particular [IIKNS], [IIKKN1], [IIKKN2]; in fact our algorithm is inspired by [GLS2, §13], where similar T-system formulas are obtained for generalized minors of symmetric Kac–Moody groups). We find it nevertheless remarkable that, by interpreting the T-system equations as appropriate cluster transformations, one is able to obtain Kirillov–Reshetikhin q-characters starting from their highest weight monomials via a procedure of successive approximations. To the best of our knowledge this simple "bootstrap" algorithm has not been noticed before, although, in retrospect, it could certainly have been formulated and proved without knowledge of cluster algebra theory.

At this stage, we should recall that long ago Frenkel and Mukhin [FM] described a completely different algorithm, which can be used for computing the q-characters of Kirillov–Reshetikhin modules [N2, H]. The advantage of our approach is that we are now in a position to apply deep results of the theory of cluster algebras and obtain new formulas for Kirillov–Reshetikhin q-characters. In [DWZ1, DWZ2], Derksen, Weyman and Zelevinsky have constructed a categorical model for a large class of cluster algebras using quivers with potentials. In particular they have proved a far-reaching generalization of the Caldero–Chapoton formula, expressing any cluster variable in terms of the F-polynomial of an associated quiver representation (see also [Pl1] for a different proof of this generalized formula). Applying this formula in our context, we get a geometric character formula for arbitrary Kirillov–Reshetikhin modules, and also for their tensor products (Theorem 4.8).

When $\mathfrak{g}$ is simply laced, and we restrict our attention to the simplest Kirillov–Reshetikhin modules and their tensor products, namely the fundamental modules and the standard modules, the quiver Grassmannians involved in our formula are homeomorphic to the projective varieties $\mathfrak{L}^\bullet(V, W)$ used by Nakajima [N3, §4] in his geometric construction of the standard modules. This suggests that the quiver Grassmannians we introduce, in connection with general Kirillov–Reshetikhin modules of not necessarily simply laced type, might be interesting new varieties.

When $\mathfrak{g}$ is a classical Lie algebra of type A, B, C, D, there exist tableau sum formulas for the q-characters of certain Kirillov–Reshetikhin modules (see [KNS2, §7] and references therein). From the geometric point of view of Theorem 4.8, these formulas can be explained by the fact that the corresponding quiver representations have a nice and regular "grid structure", and in many cases their quiver Grassmannians are reduced to points (see e.g. §§6.4–6.6).

The cluster algebra approach also suggests that our results should extend far beyond Kirillov–Reshetikhin modules. Indeed, we show (Theorem 5.1) that the cluster algebra $\mathscr{A}$ is isomorphic to the Grothendieck ring of $\mathscr{C}^-$. It is then natural to conjecture that this isomorphism maps all cluster monomials of $\mathscr{A}$ to the classes of certain simple objects of $\mathscr{C}^-$ (Conjecture 5.2), and to extend the above geometric character formula to all these simple objects (Conjecture 5.3). The results of [HL1, HL2, N4] provide some evidence supporting these conjectures in the simply laced case.

Here is a more precise outline of the paper. In Section 2 we associate with every simple Lie algebra $\mathfrak{g}$ some quivers (§2.1), from which we define a cluster algebra $\mathscr{A}$ (§2.2). We also introduce the untwisted quantum affine algebra $U_q(\widehat{\mathfrak{g}})$ (§2.3). In Section 3 we state and prove our algorithm for computing Kirillov–Reshetikhin q-characters as special cluster variables of $\mathscr{A}$. The proof uses T-systems (§3.2.1) and the notion of truncated q-characters (§3.2.2). In Section 4, we consider an algebra A defined by a quiver with potential, coming from our initial seed for $\mathscr{A}$ (§4.1). We introduce certain distinguished A-modules $K^{(i)}_{k,m}$ (§4.3), and we state our geometric formula for Kirillov–Reshetikhin q-characters in terms of Grassmannians of submodules of the $K^{(i)}_{k,m}$ (Theorem 4.8). To prove it, we calculate the g-vectors of these q-characters, regarded as cluster variables of $\mathscr{A}$, and we apply a result of Plamondon [Pl2] which allows one to reconstruct the A-module corresponding to a given cluster variable from the knowledge of its g-vector. To be in a position to apply this result, we show that the defining potential of A is rigid, and that appropriate truncations of A are finite-dimensional (Proposition 4.17). In Section 5, we prove Theorem 5.1 and we formulate Conjectures 5.2 and 5.3. The paper closes with an appendix illustrating our results with many examples.

2. Definitions and notation

2.1. Quivers

2.1.1. Cartan matrix. Let $C = (c_{ij})_{i,j\in I}$ be an indecomposable $n \times n$ Cartan matrix of finite type [Ka, §4.3]. There is a diagonal matrix $D = \mathrm{diag}(d_i \mid i \in I)$ with entries in $\mathbb{Z}_{>0}$ such that the product

$$B = DC = (b_{ij})_{i,j\in I}$$

is symmetric. We normalize D so that $\min\{d_i \mid i \in I\} = 1$, and we set $t := \max\{d_i \mid i \in I\}$. Thus

$$t = \begin{cases} 1 & \text{if } C \text{ is of type } A_n, D_n, E_6, E_7 \text{ or } E_8, \\ 2 & \text{if } C \text{ is of type } B_n, C_n \text{ or } F_4, \\ 3 & \text{if } C \text{ is of type } G_2. \end{cases}$$

It is easy to check by inspection that

$$(d_i > 1 \text{ and } c_{ij} < 0) \;\Rightarrow\; (c_{ij} = -1). \tag{1}$$

One attaches to C a Dynkin diagram δ with vertex set I [Ka, §4.7]. Since C is assumed to be indecomposable and of finite type, δ is a tree.

All the objects that we consider below depend on C, but we shall not always repeat it, nor indicate it explicitly in our notation.

Example 2.1. The Cartan matrix C of type B_3 in the Cartan–Killing classification is defined by

$$C = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -2 & 2 \end{pmatrix}.$$

We have $D = \operatorname{diag}(2, 2, 1)$ and the symmetric matrix B is given by

$$B = \begin{pmatrix} 4 & -2 & 0 \\ -2 & 4 & -2 \\ 0 & -2 & 2 \end{pmatrix}.$$

2.1.2. Infinite quiver. Set $\widetilde{V} = I \times \mathbb{Z}$. We introduce a quiver $\widetilde{\Gamma}$ with vertex set $\widetilde{V}$. The arrows of $\widetilde{\Gamma}$ are given by

$$((i, r) \to (j, s)) \;\Leftrightarrow\; (b_{ij} \neq 0 \text{ and } s = r + b_{ij}).$$

Lemma 2.2. *The quiver $\widetilde{\Gamma}$ has two isomorphic connected components.*

Proof. Let $i \in I$ be such that $d_i = 1$. For every $r \in \mathbb{Z}$ we have an arrow $(i, r) \to (i, r+2)$. Since the Dynkin diagram δ is connected, every vertex $(j, s) \in \widetilde{V}$ is connected to a vertex of the form (i, r), so $\widetilde{\Gamma}$ has at most two connected components. Moreover, since δ is a tree, any path from (i, r) to (i, s) in $\widetilde{\Gamma}$ contains as many arrows of the form $(j, p) \to (k, p + b_{jk})$ with $j \neq k$, as it contains arrows of the form $(k, t) \to (j, t + b_{kj})$. Since $b_{jk} = b_{kj}$, and since $b_{jj} \in 2\mathbb{Z}$ for every $j \in I$, it follows that if there is a path from (i, r) to (i, s) then $s - r \in 2\mathbb{Z}$. Therefore $\widetilde{\Gamma}$ has exactly two connected components. These two components are isomorphic via the map $(j, r) \mapsto (j, r + 1)$ $((j, r) \in \widetilde{V} \times \mathbb{Z})$. □

We pick one of the two isomorphic connected components of $\widetilde{\Gamma}$ and call it Γ. The vertex set of Γ is denoted by V.

2.1.3. Semi-infinite quiver. We will have to use a second labelling of the vertices of Γ. It is deduced from the first one by means of the function ψ defined by

$$\psi(i, r) = (i, r + d_i) \qquad ((i, r) \in V). \tag{2}$$

Let $W \subset I \times \mathbb{Z}$ be the image of V under ψ. We shall denote by G the same quiver as Γ but with vertices labelled by W. Write $W^- := W \cap (I \times \mathbb{Z}_{\leq 0})$. Let G^- be the full subquiver of G with vertex set W^-.

Example 2.3. The definitions of §2.1.2 and §2.1.3 are illustrated in Figures 1 and 2. We find it convenient to always display the quivers Γ in the following way. We draw all arrows of the form $(i,r) \to (i, r+b_{ii})$ vertically, going upwards. Moreover, if (i,r) and (i,s) are two vertices with $r-s \notin b_{ii}\mathbb{Z}$, we draw them in different *columns*. Hence, the quivers attached to C always have $\sum_{i\in I} d_i$ columns. Finally, the integer r determines the *altitude* of the vertex (i,r) in Γ. Therefore, since for $i \neq j$ we have $b_{ij} \leq 0$, the arrows $(i,r) \to (j, r+b_{ij})$ are represented as oblique arrows going down.

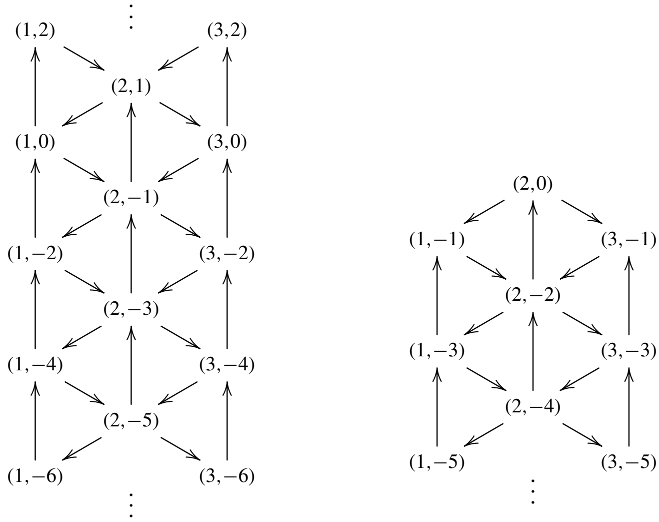


Fig. 1. The quivers Γ and G^- in type A_3.

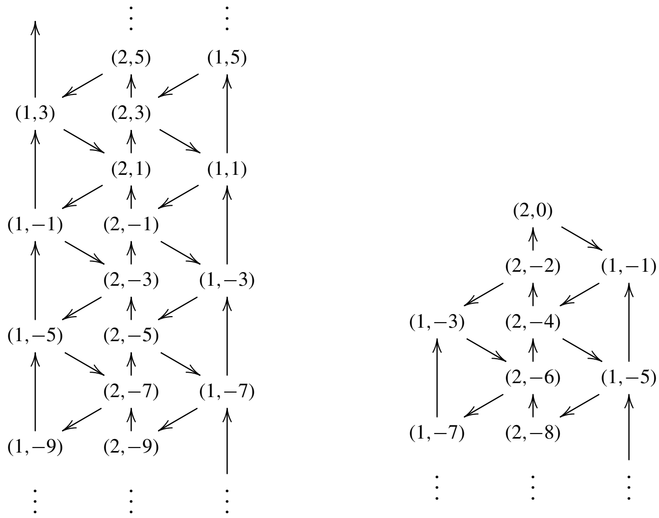


Fig. 2. The quivers Γ and G^- in type B_2.

Figure 1 displays the quivers Γ and G^- for C of type A_3. Figure 2 shows Γ and G^- for C of type B_2. In both cases we have denoted by Γ the connected component of $\widetilde{\Gamma}$ containing the vertex $(2, 1)$. For another illustration, with C of type G_2, see Figure 3. More examples can be found in the Appendix, §§6.5–6.7.

2.2. *Cluster algebras*

We refer the reader to [FZ2] and [GSV] for an introduction to cluster algebras, and for any undefined terminology.

2.2.1. Cluster algebra attached to G^-. Consider an infinite set $\mathbf{z}^- = \{z_{i,r} \mid (i,r) \in W^-\}$ of indeterminates over $\mathbb{Q}$. Let $\mathscr{A}$ be the cluster algebra defined by the initial seed $(\mathbf{z}^-, G^-)$. Thus, $\mathscr{A}$ is the $\mathbb{Q}$-subalgebra of the field $\mathbb{Q}(\mathbf{z}^-)$ of rational functions generated by all the elements obtained from some element of $\mathbf{z}^-$ via a finite sequence of seed mutations (see [GG, Definition 3.1]). Note that there are no frozen variables.

Cluster algebras of infinite rank have not received much attention up to now. (In fact, we are not aware of any paper other than [GG]; in [GG], a specific example of type A_∞ is developed, in connection with a triangulated category studied by Holm and Jorgensen [HoJo].)

For our purposes, one can always work with sufficiently large finite subseeds of the seed $(\mathbf{z}^-, G^-)$, and replace $\mathscr{A}$ by the genuine cluster subalgebras attached to them. On the other hand, statements become nicer if we formulate them in terms of the infinite rank cluster algebra $\mathscr{A}$.

2.2.2. Monomial change of variables. Let $\mathbf{Y} = \{Y_{i,r} \mid (i,r) \in W\}$ be a new set of indeterminates over $\mathbb{Q}$. Let $\mathbf{Y}^- = \{Y_{i,r} \in \mathbf{Y} \mid (i,r) \in W^-\}$. For $(i,r) \in W^-$, we perform the substitution

$$z_{i,r} = \prod_{k \geq 0,\, r+kb_{ii} \leq 0} Y_{i,\, r+kb_{ii}}. \tag{3}$$

Note that all variables on the right-hand side of (3) belong to $\mathbf{Y}^-$.

Example 2.4. If G^- is as in Figure 2, we have

$$\begin{array}{lll} z_{2,0} = Y_{2,0}, & z_{2,-2} = Y_{2,-2}Y_{2,0}, & z_{2,-4} = Y_{2,-4}Y_{2,-2}Y_{2,0}, \\ z_{2,-6} = Y_{2,-6}Y_{2,-4}Y_{2,-2}Y_{2,0}, & z_{1,-1} = Y_{1,-1}, & z_{1,-5} = Y_{1,-5}Y_{1,-1}, \\ z_{1,-9} = Y_{1,-9}Y_{1,-5}Y_{1,-1}, & z_{1,-13} = Y_{1,-13}Y_{1,-9}Y_{1,-5}Y_{1,-1}, & z_{1,-3} = Y_{1,-3}, \\ z_{1,-7} = Y_{1,-7}Y_{1,-3}, & z_{1,-11} = Y_{1,-11}Y_{1,-7}Y_{1,-3}, & \text{etc.} \end{array}$$

2.2.3. Sequence of vertices. As explained in Example 2.3, the arrows of G^- of the form $(i,r) \to (i, r + b_{ii})$ are called vertical and displayed in columns. To each column we attach an initial *label* given by the index of its top vertex (i,r), for which r is maximal among the vertices of the column.

We now form a sequence of tn columns by induction as follows. At each step we pick a column whose label (i, r) has maximal r among labels of all columns. After picking a column with label (i, r), we change its label to $(i, r - b_{ii})$. Finally, reading column after column in this ordering, from top to bottom, we get an infinite sequence $\mathscr{S}$ of vertices of G^-.

Example 2.5. If G^- is as in Figure 1, then $t = 1$, the sequence of columns consists of three columns, and we obtain the following sequence of vertices:

$$\mathscr{S} = ((2,0), (2,-2), (2,-4), \dots, (1,-1), (1,-5), (1,-9), \dots,\\ (3,-1), (3,-3), (3,-5), \dots).$$

(Here, the columns labelled $(1, -1)$ and $(3, -1)$ could be interchanged.)

If G^- is as in Figure 2, then $t = 2$, the sequence of columns consists of four columns and gives the following sequence of vertices:

$$\mathscr{S} = ((2,0), (2,-2), (2,-4), \dots, (1,-1), (1,-5), (1,-9), \dots,\\ (2,0), (2,-2), (2,-4), \dots, (1,-3), (1,-7), (1,-11), \dots).$$

Note that the column with vertices $(2, r)$ appears twice. It appears first because its initial label is $(2, 0)$. After we pick it, its label is changed to $(2, -2)$, so it appears again between the columns labelled $(1, -1)$ and $(1, -3)$.

Finally, for $(i, r) \in G^-$, we define $k_{i,r}$ to be the unique positive integer k satisfying

$$0 < kb_{ii} - |r| \le b_{ii}. \tag{4}$$

In other words, (i, r) is the kth vertex in its column, counting from the top.

Example 2.6. If G^- is as in Figure 2, then

$$k_{2,-2} = 2, \quad k_{1,-9} = 3.$$

2.3. Quantum affine algebras

2.3.1. The algebra $U_q(\widehat{\mathfrak{g}})$. Let $\mathfrak{g}$ be the simple Lie algebra over $\mathbb{C}$ with Cartan matrix C. We denote by α_i $(i \in I)$ the simple roots of $\mathfrak{g}$, and by ϖ_i $(i \in I)$ the fundamental weights. They are related by

$$\alpha_i = \sum_{j \in I} c_{ji} \varpi_j. \tag{5}$$

Let $h^\vee$ be the dual Coxeter number of $\mathfrak{g}$ (see [Ka, §6.1]). The values of $h^\vee$ are recalled in Table 1.

Let $\widehat{\mathfrak{g}}$ be the corresponding untwisted affine Lie algebra. Thus if $\mathfrak{g}$ has type X_n in the Cartan–Killing classification, $\widehat{\mathfrak{g}}$ has type $X_n^{(1)}$ in the Kac classification [Ka, §4.8]. Let $U_q(\widehat{\mathfrak{g}})$ be the Drinfeld–Jimbo quantum enveloping algebra of $\widehat{\mathfrak{g}}$ (see e.g. [CP1]). We regard $U_q(\widehat{\mathfrak{g}})$ as a $\mathbb{C}$-algebra with quantum parameter $q \in \mathbb{C}^*$ not a root of unity.

$\mathfrak{g}$	A_n	B_n	C_n	D_n	E_6	E_7	E_8	F_4	G_2
t	1	2	2	1	1	1	1	2	3
$h^\vee$	$n+1$	$2n-1$	$n+1$	$2n-2$	12	18	30	9	4

Table 1. Dual Coxeter numbers.

2.3.2. q-characters. Frenkel and Reshetikin [FR] have attached to every complex finite-dimensional representation of $U_q(\widehat{\mathfrak{g}})$ a q-character $\chi_q(M)$. If M is irreducible, it is determined up to isomorphism by its q-character. The irreducible finite-dimensional representations of $U_q(\widehat{\mathfrak{g}})$ have been classified by Chari and Pressley in terms of Drinfeld polynomials (see [CP1, Theorem 12.2.6]). Equivalently, irreducible finite-dimensional representations of $U_q(\widehat{\mathfrak{g}})$ can be parametrized by the highest dominant monomial of their q-character [FR], and this is the parametrization we shall use.

By definition, the q-character $\chi_q(M)$ is a Laurent polynomial with positive integer coefficients in the infinite set of variables $\mathscr{Y} = \{Y_{i,a} \mid i \in I,\ a \in \mathbb{C}^*\}$, which should be seen as a quantum affine analogue of $\{e^{\varpi_i} \mid i \in I\}$. In this paper we will be concerned only with polynomials involving the subset of variables

$$Y_{i,q^r} \qquad ((i,r) \in W).$$

For simplicity of notation, we shall therefore write $Y_{i,r}$ instead of Y_{i,q^r}. Thus our q-characters will be Laurent polynomials in the variables of the set $\mathbf{Y}$ introduced in §2.2.2.

Let m be a *dominant* monomial in the variables $Y_{i,r} \in \mathbf{Y}$, that is, a monomial with nonnegative exponents. We denote by $L(m)$ the corresponding irreducible representation of $U_q(\widehat{\mathfrak{g}})$, and by $\chi_q(m) = \chi_q(L(m))$ its q-character. For example, if m is of the form

$$m = \prod_{j=0}^{k-1} Y_{i,\,r+jb_{ii}} \qquad (i \in I,\ r \in \mathbb{Z},\ k \geq 1),$$

then $L(m)$ is called a *Kirillov–Reshetikhin module*, and usually denoted by $W_{k,r}^{(i)}$. In particular, if $k = 1$ we get a *fundamental module* $W_{1,r}^{(i)} = L(Y_{i,r})$. By convention, if $k = 0$ the module $W_{0,r}^{(i)}$ is the trivial one-dimensional module for every (i,r), and its q-character is equal to 1.

Finally, following [FR], for $(i,r) \in V$ we introduce the following quantum affine analogue of e^{α_i}:

$$A_{i,r} := Y_{i,r-d_i} Y_{i,r+d_i} \Big(\prod_{j:\, c_{ji}=-1} Y_{j,r} \Big)^{-1} \Big(\prod_{j:\, c_{ji}=-2} Y_{j,r-1} Y_{j,r+1} \Big)^{-1} \times \Big(\prod_{j:\, c_{ji}=-3} Y_{j,r-2} Y_{j,r} Y_{j,r+2} \Big)^{-1}. \tag{6}$$

Note that since $(i,r) \in V$, we have $(i, r \pm d_i) \in W$. If $c_{ji} < 0$, we also have, because of (1),

$$(j, r + c_{ji} + 1) = (j, r + d_j(c_{ji} + 1)) = (j, r + b_{ij} + d_j) \in W.$$

It follows that $A_{i,r}$ is a Laurent monomial in the variables $Y_{j,s}$ with $(j,s) \in W$.

3. An algorithm for the q-characters of Kirillov–Reshetikhin modules

3.1. Statement and examples

Let $\mathscr{A}$ be the cluster algebra defined in §2.2.1, with initial seed $\Sigma = (\mathbf{z}^-, G^-)$, and let

$$\mathscr{S} = ((i_1, r_1), (i_2, r_2), \ldots)$$

be the sequence of vertices of the quiver of $\mathscr{A}$ defined in §2.2.3. We denote by $\mu_{\mathscr{S}}(\Sigma)$ the new seed obtained after performing the sequence of mutations indexed by $\mathscr{S}$, that is, by mutating first at vertex (i_1, r_1), then at vertex (i_2, r_2), etc. More generally, for $m \geq 1$, let $\Sigma_m = \mu_{\mathscr{S}}^m(\Sigma)$ be the seed obtained from Σ after m repetitions of the mutation sequence $\mu_{\mathscr{S}}$. Let $z_{i,r}^{(m)}$ be the cluster variable of Σ_m sitting at vertex $(i, r) \in W^-$; this is a Laurent polynomial in the initial variables $z_{j,s}$, $(j, s) \in W^-$. Let $y_{i,r}^{(m)}$ be the Laurent polynomial obtained from $z_{i,r}^{(m)}$ by performing the change of variables (3) of §2.2.2; this is a Laurent polynomial in the variables $Y_{j,s}$, $(j, s) \in W^-$.

Theorem 3.1. (a) *The quiver of $\mu_{\mathscr{S}}(\Sigma)$ is equal to the quiver of Σ, that is, to G^-.*

(b) *Suppose that $m \geq h^\vee/2$. Then the $y_{i,r}^{(m)}$ are the q-characters of Kirillov–Reshetikhin modules. More precisely, for $m \geq h^\vee/2$,*

$$y_{i,r}^{(m)} = \chi_q(W_{k,\, r-2tm}^{(i)}),$$

where $k = k_{i,r}$ is defined as in §2.2.3.

Remark 3.2. It is well known that, for $p \in \mathbb{Z}$, the q-character $\chi_q(W_{k,r+p}^{(i)})$ is deduced from $\chi_q(W_{k,r}^{(i)})$ by applying the ring automorphism mapping $Y_{j,s}$ to $Y_{j,s+p}$ for every $(j, s) \in I \times \mathbb{Z}$. Therefore, modulo these straightforward automorphisms, Theorem 3.1 describes the q-characters of *all* Kirillov–Reshetikhin modules.

Remark 3.3. Although the statement of Theorem 3.1 involves an infinite seed Σ and an infinite sequence $\mathscr{S}$ of mutations, the calculation of the q-character of a given Kirillov–Reshetikhin module requires only a finite number of mutations on a finite initial segment of the semi-infinite quiver. More precisely, the proof of Theorem 3.1 will show that all the q-characters $\chi_q(W_{k,s}^{(i)})$ with $k = 1, \ldots, l$ can be calculated using $(h' + 2l - 1)h'n/2$ mutations, where $h' = \lceil h^\vee/2 \rceil$.

Example 3.4. Let $\mathfrak{g}$ be of type A_3. The quiver G^- of the initial seed is displayed in Figure 1. The initial cluster variables are

$$\begin{array}{llll}
z_{2,0} = Y_{2,0}, & z_{2,-2} = Y_{2,-2}Y_{2,0}, & z_{2,-4} = Y_{2,-4}Y_{2,-2}Y_{2,0}, & \text{etc.} \\
z_{1,-1} = Y_{1,-1}, & z_{1,-3} = Y_{1,-3}Y_{1,-1}, & z_{1,-5} = Y_{1,-5}Y_{1,-3}Y_{1,-1}, & \text{etc.} \\
z_{3,-1} = Y_{3,-1}, & z_{3,-3} = Y_{3,-3}Y_{3,-1}, & z_{3,-5} = Y_{3,-5}Y_{3,-3}Y_{3,-1}, & \text{etc.}
\end{array}$$

After the mutation sequence $\mu_{\mathscr{S}}$, the first new cluster variables are

$$
\begin{aligned}
y^{(1)}_{2,0} &= Y_{2,-2} + Y_{1,-1}Y_{3,-1}Y^{-1}_{2,0},\\
y^{(1)}_{2,-2} &= Y_{2,-4}Y_{2,-2} + Y_{1,-1}Y_{3,-1}Y_{2,-4}Y^{-1}_{2,0} + Y_{1,-3}Y_{1,-1}Y^{-1}_{2,-2}Y^{-1}_{2,0}Y_{3,-3}Y_{3,-1},\\
y^{(1)}_{2,-4} &= Y_{2,-6}Y_{2,-4}Y_{2,-2} + Y_{1,-1}Y_{3,-1}Y_{2,-6}Y_{2,-4}Y^{-1}_{2,0}\\
&\quad + Y_{1,-3}Y_{1,-1}Y_{2,-6}Y^{-1}_{2,-2}Y^{-1}_{2,0}Y_{3,-3}Y_{3,-1}\\
&\quad + Y_{1,-5}Y_{1,-3}Y_{1,-1}Y^{-1}_{2,-4}Y^{-1}_{2,-2}Y^{-1}_{2,0}Y_{3,-5}Y_{3,-3}Y_{3,-1},\\
y^{(1)}_{1,-1} &= Y_{1,-3} + Y^{-1}_{1,-1}Y_{2,-2} + Y^{-1}_{2,0}Y_{3,-1},\\
y^{(1)}_{1,-3} &= Y_{1,-5}Y_{1,-3} + Y_{1,-5}Y^{-1}_{1,-1}Y_{2,-2} + Y_{1,-5}Y^{-1}_{2,0}Y_{3,-1}\\
&\quad + Y^{-1}_{1,-3}Y^{-1}_{1,-1}Y_{2,-4}Y_{2,-2} + Y^{-1}_{1,-3}Y_{2,-4}Y^{-1}_{2,0}Y_{3,-1} + Y^{-1}_{2,-2}Y^{-1}_{2,0}Y_{3,-3}Y_{3,-1},\\
y^{(1)}_{1,-5} &= Y_{1,-7}Y_{1,-5}Y_{1,-3} + Y_{1,-7}Y_{1,-5}Y^{-1}_{1,-1}Y_{2,-2} + Y_{1,-7}Y_{1,-5}Y^{-1}_{2,0}Y_{3,-1}\\
&\quad + Y_{1,-7}Y^{-1}_{1,-3}Y^{-1}_{1,-1}Y_{2,-4}Y_{2,-2} + Y_{1,-7}Y^{-1}_{1,-3}Y_{2,-4}Y^{-1}_{2,0}Y_{3,-1}\\
&\quad + Y_{1,-7}Y^{-1}_{2,-2}Y^{-1}_{2,0}Y_{3,-3}Y_{3,-1} + Y^{-1}_{1,-5}Y^{-1}_{1,-3}Y^{-1}_{1,-1}Y_{2,-6}Y_{2,-4}Y_{2,-2}\\
&\quad + Y^{-1}_{1,-5}Y^{-1}_{1,-3}Y_{2,-6}Y_{2,-4}Y^{-1}_{2,0}Y_{3,-1} + Y^{-1}_{1,-5}Y_{2,-6}Y^{-1}_{2,-2}Y^{-1}_{2,0}Y_{3,-3}Y_{3,-1}\\
&\quad + Y^{-1}_{2,-4}Y^{-1}_{2,-2}Y^{-1}_{2,0}Y_{3,-5}Y_{3,-3}Y_{3,-1},
\end{aligned}
$$

(We omit the variables $y^{(1)}_{3,-1}$, $y^{(1)}_{3,-5}$, $y^{(1)}_{3,-5}$, since they are readily obtained from $y^{(1)}_{1,-1}$, $y^{(1)}_{1,-5}$, $y^{(1)}_{1,-5}$ via the symmetry $(1 \leftrightarrow 3)$.) After a second application of the mutation sequence $\mu_{\mathscr{S}}$, the first new cluster variables are

$$
\begin{aligned}
y^{(2)}_{2,0} &= Y_{2,-4} + Y_{1,-3}Y_{3,-3}Y^{-1}_{2,-2} + Y_{1,-3}Y^{-1}_{3,-1} + Y^{-1}_{1,-1}Y_{3,-3} + Y^{-1}_{1,-1}Y_{2,-2}Y^{-1}_{3,-1} + Y^{-1}_{2,0},\\
y^{(2)}_{2,-2} &= Y_{2,-6}Y_{2,-4} + Y_{1,-3}Y_{3,-3}Y_{2,-6}Y^{-1}_{2,-2} + Y_{1,-5}Y_{1,-3}Y^{-1}_{2,-4}Y^{-1}_{2,-2}Y_{3,-5}Y_{3,-3}\\
&\quad + Y_{1,-5}Y^{-1}_{2,0}Y^{-1}_{3,-3} + Y^{-1}_{1,-3}Y^{-1}_{2,0}Y_{3,-5} + Y^{-1}_{1,-3}Y_{2,-4}Y^{-1}_{2,0}Y^{-1}_{3,-3} + Y_{2,-6}Y^{-1}_{2,0}\\
&\quad + Y_{1,-5}Y^{-1}_{2,-4}Y^{-1}_{2,0}Y_{3,-5} + Y_{1,-3}Y_{2,-6}Y^{-1}_{3,-1} + Y_{1,-5}Y_{1,-3}Y^{-1}_{3,-3}Y^{-1}_{3,-1}\\
&\quad + Y_{1,-5}Y_{1,-3}Y^{-1}_{2,-4}Y_{3,-5}Y^{-1}_{3,-1} + Y^{-1}_{1,-1}Y_{2,-6}Y_{3,-3} + Y^{-1}_{1,-3}Y^{-1}_{1,-1}Y_{3,-5}Y_{3,-3}\\
&\quad + Y_{1,-5}Y^{-1}_{1,-1}Y^{-1}_{2,-4}Y_{3,-5}Y_{3,-3} + Y^{-1}_{1,-3}Y^{-1}_{1,-1}Y_{2,-4}Y_{2,-2}Y^{-1}_{3,-3}Y^{-1}_{3,-1}\\
&\quad + Y^{-1}_{1,-1}Y_{2,-6}Y_{2,-2}Y^{-1}_{3,-1} + Y^{-1}_{1,-3}Y^{-1}_{1,-1}Y_{2,-2}Y_{3,-5}Y^{-1}_{3,-1}\\
&\quad + Y_{1,-5}Y^{-1}_{1,-1}Y_{2,-2}Y^{-1}_{3,-3}Y^{-1}_{3,-1} + Y_{1,-5}Y^{-1}_{1,-1}Y^{-1}_{2,-4}Y_{2,-2}Y_{3,-5}Y^{-1}_{3,-1}\\
&\quad + Y^{-1}_{2,-2}Y^{-1}_{2,0},\\
y^{(2)}_{1,-1} &= Y_{1,-5} + Y^{-1}_{1,-3}Y_{2,-4} + Y^{-1}_{2,-2}Y_{3,-3} + Y^{-1}_{3,-1},\\
y^{(2)}_{1,-3} &= Y_{1,-7}Y_{1,-5} + Y_{1,-7}Y^{-1}_{1,-3}Y_{2,-4} + Y_{1,-7}Y^{-1}_{2,-2}Y_{3,-3} + Y^{-1}_{1,-5}Y^{-1}_{1,-3}Y_{2,-6}Y_{2,-4}\\
&\quad + Y^{-1}_{1,-5}Y_{2,-6}Y^{-1}_{2,-2}Y_{3,-3} + Y^{-1}_{2,-4}Y^{-1}_{2,-2}Y_{3,-5}Y_{3,-3} + Y^{-1}_{1,-5}Y_{2,-6}Y^{-1}_{3,-1}\\
&\quad + Y_{1,-7}Y^{-1}_{3,-1} + Y^{-1}_{2,-4}Y_{3,-5}Y^{-1}_{3,-1} + Y^{-1}_{3,-3}Y^{-1}_{3,-1}.
\end{aligned}
$$

Here $h^\vee/2 = 2$, so we can observe that the cluster variables obtained after performing the mutation sequence $\mu_{\mathscr{S}}$ twice are indeed q-characters of Kirillov–Reshetikhin modules, namely,

$$\begin{aligned} y_{2,0}^{(2)} &= \chi_q(Y_{2,-4}), & y_{2,-2}^{(2)} &= \chi_q(Y_{2,-6}Y_{2,-4}), & &\text{etc.,}\\ y_{1,-1}^{(2)} &= \chi_q(Y_{1,-5}), & y_{1,-3}^{(2)} &= \chi_q(Y_{1,-7}Y_{1,-5}), & &\text{etc.,}\\ y_{3,-1}^{(2)} &= \chi_q(Y_{3,-5}), & y_{3,-3}^{(2)} &= \chi_q(Y_{3,-7}Y_{3,-5}), & &\text{etc.} \end{aligned}$$

Example 3.5. Let $\mathfrak{g}$ be of type B_2. The quiver G^- of the initial seed is displayed in Figure 2. The initial cluster variables are

$$\begin{aligned} z_{2,0} &= Y_{2,0}, & z_{2,-2} &= Y_{2,-2}Y_{2,0}, & z_{2,-4} &= Y_{2,-4}Y_{2,-2}Y_{2,0}, & &\text{etc.,}\\ z_{1,-1} &= Y_{1,-1}, & z_{1,-5} &= Y_{1,-5}Y_{1,-1}, & z_{1,-9} &= Y_{1,-9}Y_{1,-5}Y_{1,-1}, & &\text{etc.,}\\ z_{1,-3} &= Y_{1,-3}, & z_{1,-7} &= Y_{1,-7}Y_{1,-3}, & z_{1,-11} &= Y_{1,-11}Y_{1,-7}Y_{1,-3}, & &\text{etc.} \end{aligned}$$

After the mutation sequence $\mu_{\mathscr{S}}$, the first new cluster variables are

$$\begin{aligned} y_{2,0}^{(1)} &= Y_{2,-4} + Y_{1,-3}Y_{2,-2}^{-1},\\ y_{2,-2}^{(1)} &= Y_{2,-6}Y_{2,-4} + Y_{1,-3}Y_{2,-6}Y_{2,-2}^{-1} + Y_{1,-5}Y_{1,-3}Y_{2,-4}^{-1}Y_{2,-2}^{-1} + Y_{1,-3}Y_{1,-1}^{-1},\\ y_{2,-4}^{(1)} &= Y_{2,-8}Y_{2,-6}Y_{2,-4} + Y_{1,-3}Y_{2,-8}Y_{2,-6}Y_{2,-2}^{-1} + Y_{1,-5}Y_{1,-3}Y_{2,-8}Y_{2,-4}^{-1}Y_{2,-2}^{-1}\\ &\quad + Y_{1,-7}Y_{1,-5}Y_{1,-3}Y_{2,-6}^{-1}Y_{2,-4}^{-1}Y_{2,-2}^{-1} + Y_{1,-7}Y_{1,-3}Y_{1,-1}^{-1}Y_{2,-6}^{-1} + Y_{1,-1}^{-1}Y_{1,-3}Y_{2,-8},\\ y_{1,-1}^{(1)} &= Y_{1,-5} + Y_{1,-1}^{-1}Y_{2,-4}Y_{2,-2} + Y_{2,-4}Y_{2,0}^{-1} + Y_{1,-3}Y_{2,-2}^{-1}Y_{2,0}^{-1},\\ y_{1,-5}^{(1)} &= Y_{1,-5}Y_{1,-9} + Y_{1,-9}Y_{1,-1}^{-1}Y_{2,-4}Y_{2,-2} + Y_{1,-9}Y_{2,-4}Y_{2,0}^{-1} + Y_{1,-9}Y_{1,-3}Y_{2,-2}^{-1}Y_{2,0}^{-1}\\ &\quad + Y_{1,-5}^{-1}Y_{1,-1}^{-1}Y_{2,-8}Y_{2,-6}Y_{2,-4}Y_{2,-2} + Y_{1,-5}^{-1}Y_{2,-8}Y_{2,-6}Y_{2,-4}Y_{2,0}^{-1}\\ &\quad + Y_{1,-5}^{-1}Y_{1,-3}Y_{2,-8}Y_{2,-6}Y_{2,-2}^{-1}Y_{2,0}^{-1} + Y_{1,-3}Y_{2,-8}Y_{2,-4}^{-1}Y_{2,-2}^{-1}Y_{2,0}^{-1}\\ &\quad + Y_{1,-7}Y_{1,-3}Y_{2,-6}^{-1}Y_{2,-4}^{-1}Y_{2,-2}^{-1}Y_{2,0}^{-1},\\ y_{1,-3}^{(1)} &= Y_{1,-7} + Y_{1,-3}^{-1}Y_{2,-6}Y_{2,-4} + Y_{2,-6}Y_{2,-2}^{-1} + Y_{1,-5}Y_{2,-4}^{-1}Y_{2,-2}^{-1} + Y_{1,-1}^{-1},\\ y_{1,-7}^{(1)} &= Y_{1,-7}Y_{1,-11} + Y_{1,-11}Y_{1,-3}^{-1}Y_{2,-6}Y_{2,-4} + Y_{1,-11}Y_{2,-6}Y_{2,-2}^{-1}\\ &\quad + Y_{1,-11}Y_{1,-5}Y_{2,-4}^{-1}Y_{2,-2}^{-1} + Y_{1,-7}^{-1}Y_{1,-3}^{-1}Y_{2,-10}Y_{2,-8}Y_{2,-6}Y_{2,-4}\\ &\quad + Y_{1,-7}^{-1}Y_{2,-10}Y_{2,-8}Y_{2,-6}Y_{2,-2}^{-1} + Y_{1,-7}^{-1}Y_{1,-5}Y_{2,-10}Y_{2,-8}Y_{2,-4}^{-1}Y_{2,-2}^{-1}\\ &\quad + Y_{1,-5}Y_{2,-10}Y_{2,-6}^{-1}Y_{2,-4}^{-1}Y_{2,-2}^{-1} + Y_{1,-9}Y_{1,-5}Y_{2,-8}^{-1}Y_{2,-6}^{-1}Y_{2,-4}^{-1}Y_{2,-2}^{-1}\\ &\quad + Y_{1,-9}Y_{1,-1}^{-1}Y_{2,-8}^{-1}Y_{2,-6}^{-1} + Y_{1,-11}Y_{1,-1}^{-1} + Y_{1,-1}^{-1}Y_{2,-10}Y_{2,-6}^{-1}\\ &\quad + Y_{1,-7}^{-1}Y_{1,-1}^{-1}Y_{2,-10}Y_{2,-8} + Y_{1,-5}^{-1}Y_{1,-1}^{-1}. \end{aligned}$$

Here $h^\vee/2 = 3/2$, and we can observe that certain cluster variables are not yet q-characters of Kirillov–Reshetikhin modules. But some already are, namely

$$y_{1,-3}^{(1)} = \chi_q(Y_{1,-7}), \quad y_{1,-7}^{(1)} = \chi_q(Y_{1,-7}Y_{1,-11}), \quad \text{etc.}$$

After a second application of the mutation sequence $\mu_{\mathscr{S}}$, since $2 > 3/2$, all the new cluster variables are q-characters of Kirillov–Reshetikhin modules. For example

$$y_{2,0}^{(2)} = Y_{2,-8} + Y_{1,-7}Y_{2,-6}^{-1} + Y_{1,-3}^{-1}Y_{2,-4} + Y_{2,-2}^{-1} = \chi_q(Y_{2,-8}).$$

3.2. *Proof of Theorem 3.1*

The proof relies on two main ingredients which we shall first review, namely, T-systems and truncated q-characters.

3.2.1. T-systems. With the quantum affine algebra $U_q(\widehat{\mathfrak{g}})$ is associated a system of difference equations called a *T-system* [KNS1]. Its unknowns are denoted by

$$T_{k,r}^{(i)} \quad (i \in I,\ k \in \mathbb{N},\ r \in \mathbb{Z}).$$

We fix the initial boundary condition

$$T_{0,r}^{(i)} = 1 \quad (i \in I,\ r \in \mathbb{Z}). \tag{7}$$

If $\mathfrak{g}$ is of type A_n, D_n, E_n, the T-system equations are

$$T_{k,r+1}^{(i)}T_{k,r-1}^{(i)} = T_{k-1,r+1}^{(i)}T_{k+1,r-1}^{(i)} + \prod_{j:\, c_{ij}=-1} T_{k,r}^{(j)} \quad (i \in I,\ k \geq 1,\ r \in \mathbb{Z}). \tag{8}$$

If $\mathfrak{g}$ is not of simply laced type, the T-system equations are more complicated. They can be written in the form

$$T_{k,r+d_i}^{(i)}T_{k,r-d_i}^{(i)} = T_{k-1,r+d_i}^{(i)}T_{k+1,r-d_i}^{(i)} + S_{k,r}^{(i)} \quad (i \in I,\ k \geq 1,\ r \in \mathbb{Z}), \tag{9}$$

where $S_{k,r}^{(i)}$ is defined as follows. If $d_i \geq 2$ then

$$S_{k,r}^{(i)} = \prod_{j:\, c_{ji}=-1} T_{k,r}^{(j)} \prod_{j:\, c_{ji}\leq -2} T_{d_ik,\, r-d_i+1}^{(j)}. \tag{10}$$

If $d_i = 1$ and $t = 2$, then

$$S_{k,r}^{(i)} = \begin{cases} \displaystyle\prod_{j:\, c_{ij}=-1} T_{k,r}^{(j)} \prod_{j:\, c_{ij}=-2} T_{l,r}^{(j)}T_{l,r+2}^{(j)} & \text{if } k = 2l, \\ \displaystyle\prod_{j:\, c_{ij}=-1} T_{k,r}^{(j)} \prod_{j:\, c_{ij}=-2} T_{l+1,r}^{(j)}T_{l,r+2}^{(j)} & \text{if } k = 2l+1. \end{cases} \tag{11}$$

Finally, if $d_i = 1$ and $t = 3$, that is, if $\mathfrak{g}$ is of type G_2, then denoting by j the other vertex of δ we have $d_j = 3$ and

$$S_{k,r}^{(i)} = \begin{cases} T_{l,r}^{(j)}T_{l,r+2}^{(j)}T_{l,r+4}^{(j)} & \text{if } k = 3l, \\ T_{l+1,r}^{(j)}T_{l,r+2}^{(j)}T_{l,r+4}^{(j)} & \text{if } k = 3l+1, \\ T_{l+1,r}^{(j)}T_{l+1,r+2}^{(j)}T_{l,r+4}^{(j)} & \text{if } k = 3l+2. \end{cases} \tag{12}$$

Example 3.6. Let $\mathfrak{g}$ be of type B_2. The Cartan matrix is

$$C = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}$$

and we have $d_1 = 2$ and $d_2 = 1$. The T-system reads:

$$\begin{aligned} T^{(1)}_{k,r+2}T^{(1)}_{k,r-2} &= T^{(1)}_{k-1,r+2}T^{(1)}_{k+1,r-2} + T^{(2)}_{2k,r-1} && (k \geq 1,\ r \in \mathbb{Z}), \\ T^{(2)}_{2l,r+1}T^{(2)}_{2l,r-1} &= T^{(2)}_{2l-1,r+1}T^{(2)}_{2l+1,r-1} + T^{(1)}_{l,r}T^{(1)}_{l,r+2} && (l \geq 1,\ r \in \mathbb{Z}), \\ T^{(2)}_{2l+1,r+1}T^{(2)}_{2l+1,r-1} &= T^{(2)}_{2l,r+1}T^{(2)}_{2l+2,r-1} + T^{(1)}_{l+1,r}T^{(1)}_{l,r+2} && (l \geq 0,\ r \in \mathbb{Z}). \end{aligned}$$

It was conjectured in [KNS1], and proved in [N2] (for $\mathfrak{g}$ of type A, D, E) and [H] (general case), that the q-characters of the Kirillov–Reshetikhin modules of $U_q(\widehat{\mathfrak{g}})$ satisfy the corresponding T-system. More precisely, we have

Theorem 3.7 ([N2], [H]). *For $i \in I$, $k \in \mathbb{N}$, $r \in \mathbb{Z}$,*

$$T^{(i)}_{k,r} = \chi_q(W^{(i)}_{k,r})$$

is a solution of the T-system in the ring $\mathbb{Z}[Y^{\pm 1}_{i,r} \mid (i,r) \in I \times \mathbb{Z}]$.

3.2.2. Truncated q-characters. Let $\mathscr{C}^-$ be the full subcategory of the category of finite-dimensional $U_q(\widehat{\mathfrak{g}})$-modules whose objects have all their composition factors of the form $L(m)$ where m is a dominant monomial in the variables of $\mathbf{Y}^-$.

Lemma 3.8. *The q-character of an object in $\mathscr{C}^-$ belongs to $\mathbb{Z}[Y^{\pm 1}_{i,r} \mid Y_{i,r} \in \mathbf{Y}]$.*

Proof. A simple object of $\mathscr{C}^-$ is a quotient of a tensor product of fundamental representations of $\mathscr{C}^-$. But the q-character of a fundamental representation can be calculated by means of the Frenkel–Mukhin algorithm [FM]. At each step the algorithm produces monomials which involve only variables $Y_{i,r} \in \mathbf{Y}$. Hence the result. □

Note that for a dominant monomial m in the variables of $\mathbf{Y}^-$, the q-character $\chi_q(m)$ may contain Laurent monomials m' involving variables $Y_{i,r} \in \mathbf{Y} \setminus \mathbf{Y}^-$. Following [HL1], we define the *truncated q-character* $\chi_q^-(m)$ to be the Laurent polynomial obtained from $\chi_q(m)$ by discarding all these monomials m'. So, by definition, $\chi_q^-(m) \in \mathbb{Z}[Y^{\pm 1}_{i,r} \mid Y_{i,r} \in \mathbf{Y}^-]$.

Example 3.9. Let $\mathfrak{g}$ be of type B_2. We keep the notation of Example 3.6. The fundamental modules $L(Y_{1,-3})$ and $L(Y_{2,-4})$ have q-characters equal to

$$\begin{aligned} \chi_q(Y_{1,-3}) &= Y_{1,-3} + Y^{-1}_{1,1}Y_{2,-2}Y_{2,0} + Y_{2,-2}Y^{-1}_{2,2} + Y_{1,-1}Y^{-1}_{2,0}Y^{-1}_{2,2} + Y^{-1}_{1,3}, \\ \chi_q(Y_{2,-4}) &= Y_{2,-4} + Y_{1,-3}Y^{-1}_{2,-2} + Y^{-1}_{1,1}Y_{2,0} + Y^{-1}_{2,2}. \end{aligned}$$

The corresponding truncated q-characters are

$$\chi_q^-(Y_{1,-3}) = Y_{1,-3}, \qquad \chi_q^-(Y_{2,-4}) = Y_{2,-4} + Y_{1,-3}Y^{-1}_{2,-2}.$$

Proposition 3.10. (i) *$\mathscr{C}^-$ is a tensor category.*
(ii) *The assignment $[L(m)] \mapsto \chi_q^-(m)$ extends to an injective ring homomorphism from the Grothendieck ring $K_0(\mathscr{C}^-)$ to $\mathbb{Z}[Y_{i,r}^{\pm 1} \mid Y_{i,r} \in \mathbf{Y}^-]$.*

Proof. The argument follows the lines of [HL1, §5.2.4, §6.2]. Recall the Laurent monomials $A_{i,r}$ introduced in (6). By [FR], a Laurent monomial m' of the q-character of a simple object of $\mathscr{C}^-$ can always be written in the form $m' = mM$ where m is a dominant monomial in the variables of $\mathbf{Y}^-$, and M is a monomial in the variables $A_{i,k}^{-1}$ with $(i, k + d_i) \in W$. Note that the Y-variable appearing in $A_{i,r}$ with the highest spectral parameter is $Y_{i,r+d_i}$. It follows that $A_{i,r}^{-1}$ is a *right-negative* monomial in the sense of [FM], that is, the Y-variable with highest spectral parameter occurring in $A_{i,r}^{-1}$ has a negative exponent.

Let $L(m)$ and $L(m')$ be simple objects of $\mathscr{C}^-$, that is, m and m' are dominant monomials in the variables of $\mathbf{Y}^-$. If $L(m'')$ is a composition factor of $L(m) \otimes L(m')$, then m'' is a product of monomials of $\chi_q(m)$ and $\chi_q(m')$. So, we have $m'' = mm'M$ where M is a monomial in the variables $A_{i,r}^{-1}$. We claim that, since m'' is dominant, the spectral parameters r have to satisfy $r + d_i \le 0$. Indeed, otherwise m'' would be right-negative. Therefore, by Lemma 3.8, the monomial m'' contains only variables of $\mathbf{Y}^-$, hence $L(m'')$ is in $\mathscr{C}^-$, and $\mathscr{C}^-$ is a monoidal category. Moreover, by [CP2, Prop. 5.1], the category $\mathscr{C}^-$ is stable under duals, so it is a tensor category. This proves (i).

To prove (ii) consider now an arbitrary Laurent monomial m' of the q-character of an object of $\mathscr{C}^-$. As above, it can be written in the form $m' = mM$ where m is a dominant monomial in the variables of $\mathbf{Y}^-$, and M is a monomial in the variables $A_{i,k}^{-1}$ with $(i, k + d_i) \in W$. Now m' contains a variable $Y_{j,s} \notin \mathbf{Y}^-$ if and only if M contains a negative power of $A_{i,r}$ for some pair (i, r) such that $(i, r + d_i) \notin W^-$. So, if R denotes the subring of $\mathbb{Z}[Y_{i,r}^{\pm 1} \mid Y_{i,r} \in \mathbf{Y}]$ generated by all the monomials of the q-characters of the objects of $\mathscr{C}^-$, and if I denotes the linear span of those monomials containing a variable $Y_{j,s} \in \mathbf{Y} \setminus \mathbf{Y}^-$, we see that I is an ideal of R. Hence, if $\pi \colon R \to R/I$ is the natural projection, we can realize the truncated q-character map χ_q^- as

$$\chi_q^- = \pi \circ \chi_q,$$

which shows that χ_q^- is a ring homomorphism $K_0(\mathscr{C}^-) \to \mathbb{Z}[Y_{i,r}^{\pm 1} \mid Y_{i,r} \in \mathbf{Y}^-]$. Finally, the fact that χ_q^- is injective follows from the fact that I contains only nondominant monomials, and that two q-characters having the same dominant monomials with the same coefficients are equal. $\square$

3.2.3. Proof of the theorem. We first notice that the initial cluster variables $z_{i,r}$ are equal, after the change of variables (3), to the truncated q-characters of certain Kirillov–Reshetikhin modules, namely,

$$z_{i,r} = \prod_{k \ge 0,\, r + kb_{ii} \le 0} Y_{i,r+kb_{ii}} = \chi_q^-(W_{k_{i,r},r}^{(i)}),$$

where $k_{i,r}$ is defined as in (4). Indeed, the level of truncation is chosen so that after truncation only the highest dominant monomial of these q-characters survives.

Now, the main idea of the proof is that the quiver G^- and the mutation sequence $\mu_{\mathscr{S}}$ are designed in such a way that, at every step of the mutation sequence, the exchange relation is nothing other than a T-system equation. Let us first check this when $\mathfrak{g}$ is of rank two.

For $\mathfrak{g}$ of type A_2, the sequence of mutated quivers obtained at each step of $\mu_{\mathscr{S}}$ is shown in the Appendix, §6.1. The mutations take place at the boxed vertices. Reading the second quiver of §6.1, we see that the new cluster variable obtained after the first mutation is equal to

$$\frac{z_{2,-2}+z_{1,-1}}{z_{2,0}} = \frac{\chi_q^-(W^{(2)}_{2,-2}) + \chi_q^-(W^{(1)}_{1,-1})}{\chi_q^-(W^{(2)}_{1,0})} = \chi_q^-(W^{(2)}_{1,-2}).$$

Here we have used Theorem 3.7 and Proposition 3.10. Similarly, from the third quiver of §6.1, the new cluster variable obtained after the second mutation is equal to

$$\frac{\chi_q^-(W^{(2)}_{3,-4})\chi_q^-(W^{(2)}_{1,-2}) + \chi_q^-(W^{(1)}_{2,-3})}{\chi_q^-(W^{(2)}_{2,-2})} = \chi_q^-(W^{(2)}_{2,-4}).$$

An easy induction shows that, after every vertex of the second column has been mutated, each cluster variable of the form $\chi_q^-(W^{(2)}_{k,-2k+2})$ gets replaced by the new cluster variable $\chi_q^-(W^{(2)}_{k,-2k})$. We now continue by mutating vertices of the first column. We first get, at the top vertex,

$$\frac{\chi_q^-(W^{(1)}_{2,-3}) + \chi_q^-(W^{(2)}_{1,-2})}{\chi_q^-(W^{(1)}_{1,-1})} = \chi_q^-(W^{(1)}_{1,-3}).$$

Then, mutating at the next vertex gives

$$\frac{\chi_q^-(W^{(1)}_{3,-5})\chi_q^-(W^{(1)}_{1,-3}) + \chi_q^-(W^{(2)}_{2,-4})}{\chi_q^-(W^{(1)}_{2,-3})} = \chi_q^-(W^{(1)}_{2,-5}).$$

By induction one sees that, after every vertex of the first column has been mutated, each cluster variable of the form $\chi_q^-(W^{(1)}_{k,-2k+1})$ gets replaced by a new cluster variable $\chi_q^-(W^{(1)}_{k,-2k-1})$. Moreover, one sees that the new quiver obtained after $\mu_{\mathscr{S}}$ is nothing other than G^-. Hence we conclude that one application of $\mu_{\mathscr{S}}$ produces a seed with the same quiver, and in which every cluster variable $\chi_q^-(W^{(i)}_{k,r})$ has been replaced by $\chi_q^-(W^{(i)}_{k,r-2})$. In other words, the effect of $\mu_{\mathscr{S}}$ is merely a uniform shift of the spectral parameters r by -2.

The argument is similar for $\mathfrak{g}$ of type B_2. The sequence of mutated quivers obtained at each step of $\mu_{\mathscr{S}}$ is displayed in §6.2. Reading the second quiver of §6.2, we see that the new cluster variable obtained after the first mutation is equal to

$$\frac{z_{2,-2}+z_{1,-1}}{z_{2,0}} = \frac{\chi_q^-(W^{(2)}_{2,-2})+\chi_q^-(W^{(1)}_{1,-1})}{\chi_q^-(W^{(2)}_{1,0})} = \chi_q^-(W^{(2)}_{1,-2}).$$

Similarly, from the third quiver of §6.2, the new cluster variable obtained after the second mutation is equal to

$$\frac{\chi_q^-(W^{(2)}_{3,-4})\chi_q^-(W^{(2)}_{1,-2})+\chi_q^-(W^{(1)}_{1,-1})\chi_q^-(W^{(1)}_{1,-3})}{\chi_q^-(W^{(2)}_{2,-2})} = \chi_q^-(W^{(2)}_{2,-4}).$$

By induction, after every vertex of the second column has been mutated, each cluster variable of the form $\chi_q^-(W^{(2)}_{k,-2k+2})$ gets replaced by the new cluster variable $\chi_q^-(W^{(2)}_{k,-2k})$. We now continue by mutating vertices of the third column. We first get, at the top vertex,

$$\frac{\chi_q^-(W^{(1)}_{2,-5})+\chi_q^-(W^{(2)}_{2,-4})}{\chi_q^-(W^{(1)}_{1,-1})} = \chi_q^-(W^{(1)}_{1,-5}).$$

Then, mutating at the next vertex gives

$$\frac{\chi_q^-(W^{(1)}_{3,-9})\chi_q^-(W^{(1)}_{1,-5})+\chi_q^-(W^{(2)}_{4,-8})}{\chi_q^-(W^{(1)}_{2,-5})} = \chi_q^-(W^{(1)}_{2,-9}).$$

By induction one sees that, after every vertex of the third column has been mutated, each cluster variable of the form $\chi_q^-(W^{(1)}_{k,-4k+3})$ gets replaced by the new cluster variable $\chi_q^-(W^{(1)}_{k,-4k-1})$. For the third part of $\mu_{\mathscr{S}}$, we mutate again along the second column. One checks that after that, each cluster variable of the form $\chi_q^-(W^{(2)}_{k,-2k})$ produced after the first part of $\mu_{\mathscr{S}}$ gets replaced by $\chi_q^-(W^{(2)}_{k,-2k-2})$. Finally, the fourth part of $\mu_{\mathscr{S}}$ along the first column replaces each cluster variable of the form $\chi_q^-(W^{(1)}_{k,-4k+1})$ by the new cluster variable $\chi_q^-(W^{(1)}_{k,-4k-3})$. Moreover, one sees that the new quiver obtained after $\mu_{\mathscr{S}}$ is precisely G^-. Hence we conclude that one application of $\mu_{\mathscr{S}}$ produces a seed with the same quiver, and in which every cluster variable $\chi_q^-(W^{(i)}_{k,r})$ has been replaced by $\chi_q^-(W^{(i)}_{k,r-4})$. In other words, the effect of $\mu_{\mathscr{S}}$ is merely a uniform shift of the spectral parameters r by -4.

The argument is similar for $\mathfrak{g}$ of type G_2. The quiver G^- for this case is displayed in Figure 3, and the mutation sequence is

$$\begin{aligned}
&(2,0),(2,-2),(2,-4),\ldots,(1,-1),(1,-7),(1,-13),\ldots,\\
&(2,0),(2,-2),(2,-4),\ldots,(1,-3),(1,-9),(1,-15),\ldots,\\
&(2,0),(2,-2),(2,-4),\ldots,(1,-5),(1,-11),(1,-17),\ldots.
\end{aligned}$$

The sequence of mutated quivers obtained at each step of $\mu_{\mathscr{S}}$ is displayed in §6.3.

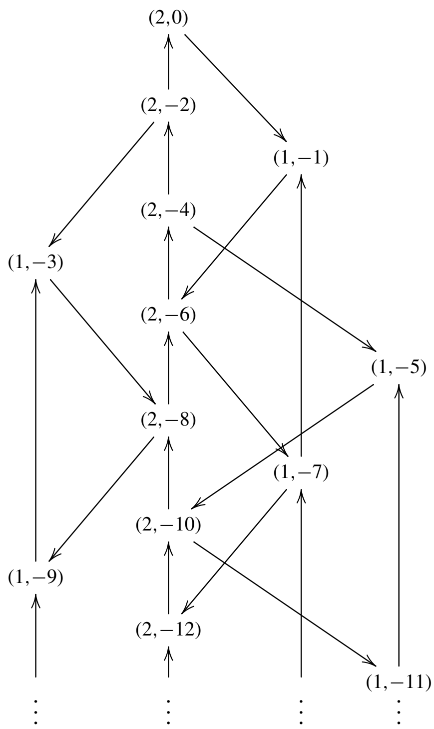


Fig. 3. The quiver G^- for $\mathfrak{g}$ of type G_2.

For a general $\mathfrak{g}$, we use a reduction to rank two. Namely, we show that mutation sequences and T-system equations are compatible with rank two reductions.

First, by construction, the sequence of vertices $\mathscr{S}$ is a union of tn columns:

$$\mathscr{S} = (\mathscr{S}_1, \dots, \mathscr{S}_{tn}),$$

where each column $\mathscr{S}_k$ is a subset of $i_k \times \mathbb{Z}_{\leq 0}$ for a certain $i_k \in I$. As above, we use $\mu_{\mathscr{S}_k}$ to denote the sequence of mutations indexed by $\mathscr{S}_k$. So we have

$$\mu_{\mathscr{S}} = \mu_{\mathscr{S}_{tn}} \circ \mu_{\mathscr{S}_{tn-1}} \circ \cdots \circ \mu_{\mathscr{S}_1}.$$

For $0 \leq k \leq tn$, we get the mutated quiver

$$\Sigma_k = (\mu_{\mathscr{S}_k} \circ \mu_{\mathscr{S}_{k-1}} \circ \cdots \circ \mu_{\mathscr{S}_1})(\Sigma).$$

For a subset $J \subset I$, let us denote by $(\Sigma_k)_J$ the subquiver of Σ_k obtained by deleting the vertices (i, r) such that $i \notin J$, and the edges whose tail or head is such a vertex. For any $i \in I$, the mutation sequence $\mu_{\mathscr{S}_k}$ does not modify $(\Sigma_k)_i$. Consequently, $(\Sigma_k)_i = (\Sigma)_i$ does not depend on k (it is a disjoint union of d_i semi-infinite linear quivers). Moreover, the mutation sequence $\mu_{\mathscr{S}_k}$ modifies only the edges whose tail (resp. head) is in $i_k \times \mathbb{Z}$ and head (resp. tail) is in $j \times \mathbb{Z}$ where $c_{i_k j} < 0$. This is because each mutation of the sequence takes place at a vertex (i_k, r) having two incoming arrows from vertices $(i_k, r \pm d_i)$ and outgoing arrows to vertices of the form (j, s) with $c_{i_k j} < 0$. Consequently, for each $i \neq j$ in I, the effect of the mutation sequence $\mu_{\mathscr{S}}$ on $(\Sigma)_{\{i,j\}}$ is the same as the effect of an iteration of the mutation sequence corresponding to the rank two Lie subalgebra of $\mathfrak{g}$ attached to $\{i, j\} \subset I$. But we have established the result for rank two Lie algebras, so this implies

$$(\mu_{\mathscr{S}}(\Sigma))_{\{i,j\}} = (\Sigma)_{\{i,j\}}.$$

As this is true for any $i \neq j$ in I, we get $\mu_{\mathscr{S}}(\Sigma) = \Sigma$.

Secondly, a T-system equation involves only a certain index $i \in I$ and the indices $j \in I$ with $c_{ij} < 0$. The T-system equations do not change by reduction, in the sense that for such a j, the powers of the factors $T^{(j)}_{l,s}$ in the second term $S^{(i)}_{k,r}$ of the right-hand side of (9) are the same as for the T-system equation associated with the rank two Lie subalgebra of $\mathfrak{g}$ attached to $\{i, j\}$. Combining this with our results above for the subquivers $(\Sigma_k)_{\{i,j\}}$, we have proved that, for a general $\mathfrak{g}$, all exchange relations of cluster variables of our mutation sequence are in fact T-system equations. Moreover, the mutation sequence $\mu_{\mathscr{S}}$ replaces the initial seed Σ by a seed with the same quiver; the cluster variables, expressed in terms of the $Y_{i,r}$ via (3), are truncated q-characters of the same Kirillov–Reshetikhin modules, the only difference being that their spectral parameters are uniformly shifted by $-2t$.

Hence, after m applications of $\mu_{\mathscr{S}}$ we will get the truncated q-characters

$$y^{(m)}_{i,r} = \chi_q^-(W^{(i)}_{k_{i,r},\, r-2tm}).$$

Now taking into account [FM, Corollary 6.14], we see that if $2tm \geq th^\vee$, then all the monomials of the q-character of $W^{(i)}_{k_{i,r},\, r-2tm}$ are lower than the level of truncation, that is,

$$\chi_q^-(W^{(i)}_{k_{i,r},\, r-2tm}) = \chi_q(W^{(i)}_{k_{i,r},\, r-2tm}).$$

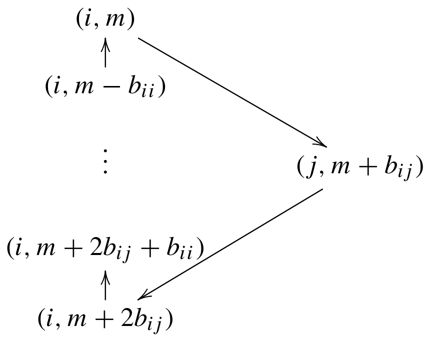
This finishes the proof of Theorem 3.1.

4. A geometric character formula for Kirillov–Reshetikhin modules

4.1. Semi-infinite quivers with potentials

Recall the map $\psi \colon V \to W$ of §2.1.3. Set $V^- := \psi^{-1}(W^-)$, and denote by Γ^- the full subquiver of Γ with vertex set V^-. Thus Γ^- is the same graph as G^-, but with a change of labelling of its vertices. (Compare for instance Figures 3 and 7.)

For every $i \neq j$ in I with $c_{ij} \neq 0$, and every (i, m) in V^-, we have in Γ^- an oriented cycle:



(13)

There are $2|b_{ij}|/b_{ii} = |c_{ij}|$ consecutive vertical up arrows, hence this cycle has length $2 + |c_{ij}|$. We define a *potential* S as the formal sum of all these oriented cycles up to cyclic permutations (see [DWZ1, §3]). This is an infinite sum, but note that a given arrow of Γ^- can only occur in a finite number of summands. Hence all the cyclic derivatives of S, defined as in [DWZ1, Definition 3.1], are finite sums of paths in Γ^-. Let R be the list of all cyclic derivatives of S. Let J denote the two-sided ideal of the path algebra $\mathbb{C}\Gamma^-$ generated by R. Following [DWZ1], we now introduce

Definition 4.1. Let A be the infinite-dimensional $\mathbb{C}$-algebra $\mathbb{C}\Gamma^-/J$.

Example 4.2. Let $\mathfrak{g}$ be of type A_3. Then Γ^- is the first graph in Figure 4. The ideal J is generated by the following seven families of linear combinations of paths, for every $m \in \mathbb{Z}_{<0}$:

$$((1, 2m), (2, 2m - 1), (1, 2m - 2)),$$
$$((3, 2m), (2, 2m - 1), (3, 2m - 2)),$$
$$((1, 2m), (1, 2m + 2), (2, 2m + 1)) + ((1, 2m), (2, 2m - 1), (2, 2m + 1)),$$
$$((3, 2m), (3, 2m + 2), (2, 2m + 1)) + ((3, 2m), (2, 2m - 1), (2, 2m + 1)),$$
$$((2, 2m - 1), (1, 2m - 2), (1, 2m)) + ((2, 2m - 1), (2, 2m + 1), (1, 2m)),$$
$$((2, 2m - 1), (3, 2m - 2), (3, 2m)) + ((2, 2m - 1), (2, 2m + 1), (3, 2m)),$$
$$((2, 2m + 1), (1, 2m), (2, 2m - 1)) + ((2, 2m + 1), (3, 2m), (2, 2m - 1)).$$

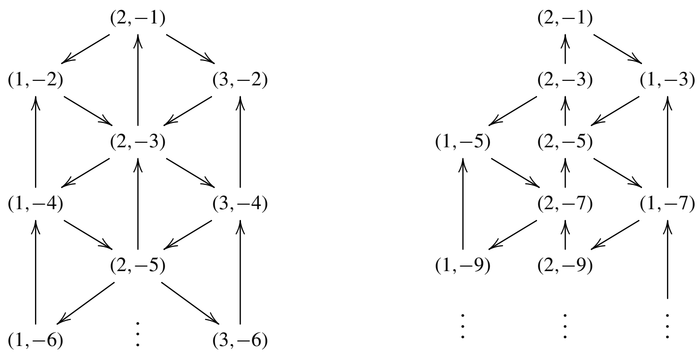


Fig. 4. The quivers Γ^- for $\mathfrak{g}$ of type A_3 and B_2.

Here, using the fact that there is at most one arrow between two vertices of Γ^-, we have denoted unambiguously paths by sequences of vertices. Thus $((1,2m),(2,2m-1),(1,2m-2))$ denotes the path of length 2 starting at $(1,2m)$, passing through $(2,2m-1)$ and ending in $(1,2m-2)$. Also, for $m=-1$, the third and fourth linear combinations of paths reduce respectively to the single paths

$$((1,-2),(2,-3),(2,-1)) \quad \text{and} \quad ((3,-2),(2,-3),(2,-1)).$$

Example 4.3. Let $\mathfrak{g}$ be of type B_2. Then Γ^- is the second graph of Figure 4. The ideal J is generated by the following four families of linear combinations of paths, for every $m \in \mathbb{Z}_{<0}$:

$$((1,2m-1),(2,2m-3),(1,2m-5)),$$
$$((1,2m-1),(1,2m+3),(2,2m+1))+((1,2m-1),(2,2m-3),(2,2m-1),(2,2m+1)),$$
$$((2,2m-3),(1,2m-5),(1,2m-1))+((2,2m-3),(2,2m-1),(2,2m+1),(1,2m-1)),$$
$$((2,2m+1),(2,2m+3),(1,2m+1),(2,2m-1))+((2,2m+1),\\ (1,2m-1),(2,2m-3),(2,2m-1)).$$

For $m=-1$ and $m=-2$ the second linear combinations of paths reduce respectively to the single paths

$$((1,-3),(2,-5),(2,-3),(2,-1)) \quad \text{and} \quad ((1,-5),(2,-7),(2,-5),(2,-3)).$$

For $m=-1$ the fourth linear combination of paths reduces to the single path

$$((2,-1),(1,-3),(2,-5),(2,-3)).$$

4.2. *F-polynomials of A-modules*

Let M be a finite-dimensional A-module, and let $e \in \mathbb{N}^{V^-}$ be a dimension vector. Let $\mathrm{Gr}_e(M)$ be the variety of submodules of M with dimension vector e. This is a projective complex variety, and we denote by $\chi(\mathrm{Gr}_e(M))$ its Euler characteristic. Following [DWZ2], consider the polynomial

$$F_M = \sum_{e\in\mathbb{N}^{V^-}} \chi(\mathrm{Gr}_e(M)) \prod_{(i,r)\in V^-} v_{i,r}^{e_{i,r}} \tag{14}$$

in the indeterminates $v_{i,r}$ ($(i,r)\in V^-$), called the *F-polynomial* of M. Note that, for $\mathrm{Gr}_e(M)$ to be nonempty, one must take e between 0 and the dimension vector of M (componentwise). Moreover, if $e=0$ or $e=\underline{\dim}(M)$, the variety $\mathrm{Gr}_e(M)$ is just a point, so F_M is a monic polynomial with constant term equal to 1.

In what follows, we shall evaluate the variables of the F-polynomials at the inverses of the variables $A_{i,r}$ introduced in (6), namely:

$$v_{i,r} := A_{i,r}^{-1} = Y_{i,r-d_i}^{-1} Y_{i,r+d_i}^{-1} \prod_{j:\,c_{ji}=-1} Y_{j,r} \prod_{j:\,c_{ji}=-2} Y_{j,r-1}Y_{j,r+1} \prod_{j:\,c_{ji}=-3} Y_{j,r-2}Y_{j,r}Y_{j,r+2}. \tag{15}$$

4.3. Generic kernels

Suppose that X and Y are A-modules such that $\mathrm{Hom}_A(X,Y)$ is finite-dimensional. Assume also that there exists $f \in \mathrm{Hom}_A(X,Y)$ such that $\mathrm{Ker}(f)$ is finite-dimensional. Then there is an open dense subset $\widetilde{O}$ of $\mathrm{Hom}_A(X,Y)$ such that the kernels of all elements of $\widetilde{O}$ are finite-dimensional. Moreover, since the map sending a homomorphism f to the F-polynomial of $\mathrm{Ker}(f)$ is constructible (see [Pa, §2]), $\widetilde{O}$ contains an open dense subset O of $\mathrm{Hom}_A(X,Y)$ such that the F-polynomials of the kernels of all elements of O coincide. We shall say that an element of O is a *generic homomorphism* from X to Y.

Let us denote by $S_{i,m}$ the one-dimensional A-module supported on $(i,m) \in V^-$. Let $I_{i,m}$ be the (infinite-dimensional) injective A-module with socle isomorphic to $S_{i,m}$. The $\mathbb{C}$-vector space $I_{i,m}$ has a basis indexed by classes modulo J of paths in Γ^- with final vertex (i,m). In particular, for every $k \geq 0$ we have in Γ^- a path

$$((i, m-kb_{ii}), (i, m-(k-1)b_{ii}), \dots, (i,m)) \tag{16}$$

of length k from $(i,m-kb_{ii})$ to (i,m), whose class modulo J is nonzero. Thus the $(i,m-kb_{ii})$-component of the dimension vector of $I_{i,m}$ is nonzero, and it follows that

$$\mathrm{Hom}_A(I_{i,m}, I_{i,m-kb_{ii}}) \neq 0 \qquad ((i,m) \in V^-,\ k \geq 0). \tag{17}$$

More precisely, $\mathrm{Hom}_A(I_{i,m}, I_{i,m-kb_{ii}})$ has finite dimension equal to the $(i,m-kb_{ii})$-component of the dimension vector of $I_{i,m}$. The next lemma will be proven in §4.5.3.

Lemma 4.4. *There exists* $f \in \mathrm{Hom}_A(I_{i,m}, I_{i,m-kb_{ii}})$ *with* $\mathrm{Ker}(f)$ *finite-dimensional.*

Because of this lemma, the following definition makes sense.

Definition 4.5. Let $K^{(i)}_{k,m}$ be the kernel of a generic A-module homomorphism from $I_{i,m}$ to $I_{i,m-kb_{ii}}$.

Example 4.6. Figures 5 and 6 show the structure of some modules $K^{(i)}_{k,m}$ in type A_3. Our convention for displaying these quiver representations is the following. We only keep the vertices of Γ^- whose corresponding vector space is nonzero, and the arrows whose corresponding linear map is nonzero. Moreover, in these small examples, almost all vertices carry a vector space of dimension 1. The only exception is the module $K^{(2)}_{2,-3}$ in Figure 6, whose vertex $(2,-3)$ carries a vector space of dimension 2. The maps associated with the

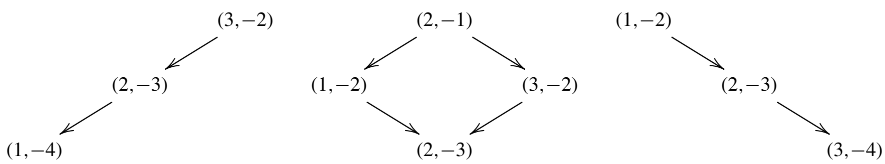


Fig. 5. The modules $K^{(1)}_{1,-4}$, $K^{(2)}_{1,-3}$, $K^{(3)}_{1,-4}$ for $\mathfrak{g}$ of type A_3.

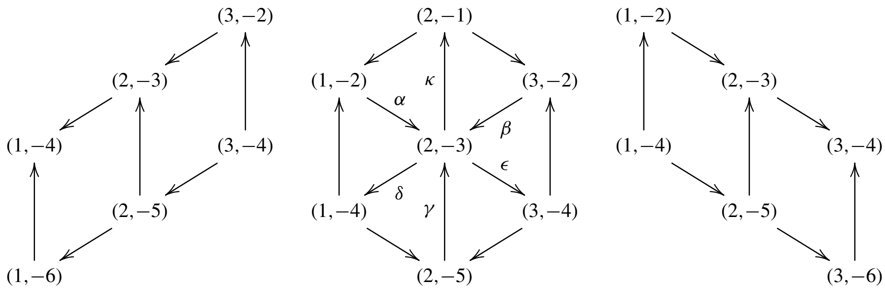


Fig. 6. The modules $K^{(1)}_{2,-4}$, $K^{(2)}_{2,-3}$, and $K^{(3)}_{2,-4}$ for $\mathfrak{g}$ of type A_3.

arrows incident to this vertex have the following matrices:

$$\alpha = \beta = \gamma = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \delta = \epsilon = \kappa = \begin{pmatrix} 0 & 1 \end{pmatrix}.$$

All other arrows carry linear maps with matrix (± 1), whose sign is easily deduced from the defining relations of A.

It is a nice exercise to check that the modules shown in Figures 5 and 6 are indeed the claimed modules $K^{(i)}_{k,m}$ (see also Example 4.7 below). For instance, one can easily see that the $(1, -6)$-component of the dimension vector of $I_{1,-4}$ is equal to 1. Hence $\mathrm{Hom}_A(I_{1,-4}, I_{1,-6})$ is of dimension 1, and $K^{(1)}_{1,-4}$ is the kernel of any nonzero homomorphism. It is also easy to see that the $(2, -5)$-component of the dimension vector of $I_{2,-3}$ is 2. In this case we have a stratification of the 2-plane $\mathrm{Hom}_A(I_{2,-3}, I_{2,-5})$ with three strata of dimension 0, 1, 2. The module $K^{(2)}_{1,-3}$ is the kernel of any homomorphism in the open stratum, that is, of any surjective homomorphism. The image of any homomorphism in the one-dimensional stratum is the unique submodule X of $I_{2,-5}$ with dimension vector given by

$$\dim(X_{i,m}) = \begin{cases} 1 & \text{if } i = 2 \text{ and } m = -5 - 2j \text{ for some } j \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

The kernel of such a homomorphism is infinite-dimensional.

Example 4.7. Assume that $\mathfrak{g}$ is of type A, D, E. In this case, the modules $K^{(i)}_{1,r}$ are closely related to the indecomposable injective modules over the preprojective algebra Λ of δ.

Consider the subalgebra $\widetilde{\Lambda}$ of A generated by the images modulo J of the arrows of Γ^- of the form $(i, m) \to (j, m-1)$, for every edge between i and j in δ, and every $(i, m) \in V^-$. In other words, if Δ^-_δ is the subquiver of Γ^- obtained by erasing all the vertical arrows $(i, m-2) \to (i, m)$, then $\widetilde{\Lambda}$ is isomorphic to the quotient of $\mathbb{C}\Delta^-_\delta$ by the two-sided ideal generated by the relations

$$\sum_{j:\, c_{ij}<0} ((i, m), (j, m-1), (i, m-2)) = 0 \qquad ((i, m) \in V^-).$$

Thus, $\widetilde{\Lambda}$ is a $\mathbb{Z}_{<0}$-graded version of Λ. We can of course regard the simple A-module $S_{i,r}$ as a $\widetilde{\Lambda}$-module. Let $H_{i,r}$ be the injective $\widetilde{\Lambda}$-module with socle $S_{i,r}$. Then $H_{i,r}$ is finite-dimensional. More precisely, for $r \le 1-h$, $H_{i,r}$ is just a graded version of the indecomposable injective Λ-module I_i with socle the one-dimensional Λ-module S_i supported on vertex i of δ. For $r > 1-h$, $H_{i,r}$ is a graded version of a submodule of I_i.

Any $\widetilde{\Lambda}$-module X can be given the structure of an A-module by letting the vertical arrows $(i, m-2) \to (i, m)$ act by 0 on X. In particular we can regard $H_{i,r}$ as a finite-dimensional A-module. Then one can check that $I_{i,r}$ has a unique submodule isomorphic to $H_{i,r}$, giving rise to a nonsplit short exact sequence

$$0 \to H_{i,r} \to I_{i,r} \to I_{i,r-2} \to 0 \qquad ((i,r) \in V^-).$$

It follows that the module $K^{(i)}_{1,m}$ is isomorphic to $H_{i,m}$. In particular, when $m \le 1-h$, $K^{(i)}_{1,m}$ is a graded version of the injective Λ-module I_i.

4.4. *A geometric character formula*

Recall the A-module $K^{(i)}_{k,r}$ defined in §4.3. We can now state our second main result.

Theorem 4.8. *Let $(i,r) \in V^-$ and $k \in \mathbb{N}$. The F-polynomial of $K^{(i)}_{k,r}$ is equal to the normalized truncated q-character of the Kirillov–Reshetikhin module $W^{(i)}_{k,\,r-(2k-1)d_i}$. More precisely,*

$$\chi_q^-(W^{(i)}_{k,\,r-(2k-1)d_i}) = \Big(\prod_{s=1}^{k} Y_{i,\,r-(2s-1)d_i}\Big) F_{K^{(i)}_{k,r}},$$

where the variables $v_{i,r}$ of the F-polynomial are evaluated as in (15).

Remark 4.9. If $r \le d_i - th^\vee$, then the truncated q-character of $W^{(i)}_{k,\,r-(2k-1)d_i}$ is equal to the complete q-character. Hence, Theorem 4.8 gives a geometric formula for the q-character of any Kirillov–Reshetikhin module (up to a spectral shift).

Remark 4.10. If M and N are two finite-dimensional A-modules, then $F_{M\oplus N} = F_M F_N$ [DWZ2, Proposition 3.2]. It follows immediately that, replacing in Theorem 4.8 the module $K^{(i)}_{k,r}$ by a direct sum of such modules, we obtain a similar geometric character formula for arbitrary tensor products of Kirillov–Reshetikhin modules. In particular, we get a geometric formula for the standard modules, which are isomorphic to tensor products of fundamental modules.

Remark 4.11. Let $\mathfrak{g}$ be of type A, D, E. Let $\mathbf{V}$ and $\mathbf{W}$ be finite-dimensional vector spaces graded by V^-. In [N1] (see also [N4]), Nakajima has introduced a graded quiver variety $\mathfrak{L}^\bullet(\mathbf{V}, \mathbf{W})$ and has endowed the sum of cohomologies

$$\bigoplus_{\mathbf{V}} H^*(\mathfrak{L}^\bullet(\mathbf{V}, \mathbf{W}))$$

with the structure of a standard $U_q(\widehat{\mathfrak{g}})$-module, with highest weight encoded by $\mathbf{W}$. It was proved by Lusztig (in the ungraded case), and by Savage and Tingley (in the graded case), that $\mathfrak{L}^\bullet(\mathbf{V}, \mathbf{W})$ is homeomorphic to a Grassmannian of submodules of an injective module over the graded preprojective algebra (see [Le2, §2.8]). Therefore, using the description of $K^{(i)}_{1,r}$ given in Example 4.7, we see that the varieties

$$\mathrm{Gr}_e\Big(\bigoplus_{(i,r)}(K^{(i)}_{1,r})^{\oplus a_{i,r}}\Big)$$

involved in our geometric q-character formula for standard modules in the simply laced case are homeomorphic to certain Nakajima varieties $\mathfrak{L}^\bullet(\mathbf{V},\mathbf{W})$. Here, the multiplicities $a_{i,r}$ are the dimensions of the graded components of $\mathbf{W}$, and we assume that $a_{i,r}=0$ if $r>1-h$. Similarly the graded dimension of $\mathbf{V}$ is encoded by the dimension vector e.

Example 4.12. Let $\mathfrak{g}$ be of type A_3. We have

$$v_{1,r}=Y^{-1}_{1,r-1}Y^{-1}_{1,r+1}Y_{2,r},\qquad v_{2,r}=Y^{-1}_{2,r-1}Y^{-1}_{2,r+1}Y_{1,r}Y_{3,r},\qquad v_{3,r}=Y^{-1}_{3,r-1}Y^{-1}_{3,r+1}Y_{2,r}.$$

We continue Example 4.6. The submodule structure of the A-modules displayed in Figure 5 is very simple. Indeed, in this case, all the nonempty varieties $\mathrm{Gr}_e(K^{(i)}_{k,r})$ are reduced to a single point, and their Euler characteristics are equal to 1. Therefore the F-polynomial reduces to a generating polynomial for the dimension vectors of the (finitely many) submodules of $K^{(i)}_{k,r}$. This yields the following well known formulas for the q-characters of the fundamental modules:

$$\begin{aligned}
\chi_q(L(Y_{1,-5})) &= Y_{1,-5}(1+v_{1,-4}+v_{1,-4}v_{2,-3}+v_{1,-4}v_{2,-3}v_{3,-2})\\
&= Y_{1,-5}+Y^{-1}_{1,-3}Y_{2,-4}+Y^{-1}_{2,-2}Y_{3,-3}+Y^{-1}_{3,-1},\\
\chi_q(L(Y_{2,-4})) &= Y_{2,-4}(1+v_{2,-3}+v_{1,-2}v_{2,-3}+v_{2,-3}v_{3,-2}+v_{1,-2}v_{2,-3}v_{3,-2}\\
&\quad +v_{1,-2}v_{2,-3}v_{3,-2}v_{2,-1})\\
&= Y_{2,-4}+Y_{1,-3}Y^{-1}_{2,-2}Y_{3,-3}+Y^{-1}_{1,-1}Y_{3,-3}+Y_{1,-3}Y^{-1}_{3,-1}\\
&\quad +Y^{-1}_{1,-1}Y_{2,-2}Y^{-1}_{3,-1}+Y^{-1}_{2,0},
\end{aligned}$$

Similarly, the A-modules shown in Figure 6 give the following Kirillov–Reshetikhin q-characters:

$$\begin{aligned}
\chi_q(L(Y_{1,-7}Y_{1,-5})) &= Y_{1,-7}Y_{1,-5}\big(1+v_{1,-4}(1+v_{1,-6}+v_{2,-3}+v_{1,-6}v_{2,-3}\\
&\quad +v_{2,-3}v_{3,-2}+v_{1,-6}v_{2,-3}v_{2,-5}+v_{1,-6}v_{2,-3}v_{3,-2}\\
&\quad +v_{1,-6}v_{2,-3}v_{2,-5}v_{3,-2}+v_{1,-6}v_{2,-3}v_{2,-5}v_{3,-2}v_{3,-4})\big),\\
\chi_q(L(Y_{2,-6}Y_{2,-4})) &= Y_{2,-6}Y_{2,-4}\big(1+v_{2,-3}(1+v_{1,-2}+v_{2,-5}+v_{3,-2}\\
&\quad +v_{1,-2}v_{2,-5}+v_{1,-2}v_{3,-2}+v_{2,-5}v_{3,-2}+v_{1,-2}v_{2,-5}v_{3,-2}\\
&\quad +v_{1,-2}v_{2,-5}v_{1,-4}+v_{1,-2}v_{3,-2}v_{2,-1}+v_{2,-5}v_{3,-2}v_{3,-4}\\
&\quad +v_{1,-2}v_{2,-5}v_{3,-2}v_{1,-4}+v_{1,-2}v_{2,-5}v_{3,-2}v_{2,-1}\\
&\quad +v_{1,-2}v_{2,-5}v_{3,-2}v_{3,-4}+v_{1,-2}v_{2,-5}v_{3,-2}v_{1,-4}v_{2,-1}\\
&\quad +v_{1,-2}v_{2,-5}v_{3,-2}v_{1,-4}v_{3,-4}+v_{1,-2}v_{2,-5}v_{3,-2}v_{3,-4}v_{2,-1}\\
&\quad +v_{1,-2}v_{2,-5}v_{3,-2}v_{1,-4}v_{2,-1}v_{3,-4}\\
&\quad +v_{1,-2}v_{2,-5}v_{3,-2}v_{1,-4}v_{2,-1}v_{3,-4}v_{2,-3})\big).
\end{aligned}$$

We omit the q-characters $\chi_q(L(Y_{3,-5}))$ and $\chi_q(L(Y_{3,-5}Y_{3,-7}))$, since they are readily obtained from $\chi_q(L(Y_{1,-5}))$ and $\chi_q(L(Y_{1,-5}Y_{1,-7}))$ via the symmetry $1 \leftrightarrow 3$.

Example 4.13. Let $\mathfrak{g}$ be of type G_2, with the long root being α_1. The quiver Γ^- is shown in Figure 7. The modules $K^{(1)}_{1,r}$ and $K^{(2)}_{1,s}$ with $r \le -10$ and $s \le -11$ have dimension 10 and 6, respectively. For instance, $K^{(1)}_{1,-10}$ and $K^{(2)}_{1,-11}$ are represented in Figure 8. In the module $K^{(1)}_{1,-10}$ the vector space sitting at vertex $(2,-7)$ has dimension 2 (all other spaces have dimension 1). The maps incident to this space are given by the following matrices (see Figure 8):

$$\alpha = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & 0 \end{pmatrix}, \quad \gamma = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \gamma' = \begin{pmatrix} 0 & 1 \end{pmatrix}.$$

The corresponding fundamental modules have dimension

$$\dim L(Y_{1,-13}) = 15, \quad \dim L(Y_{2,-12}) = 7.$$

The Grassmannians of submodules of $K^{(1)}_{1,-10}$ and $K^{(2)}_{1,-11}$ are in this case again all reduced to points, and the formula of Theorem 4.8 amounts to an enumeration of the dimension

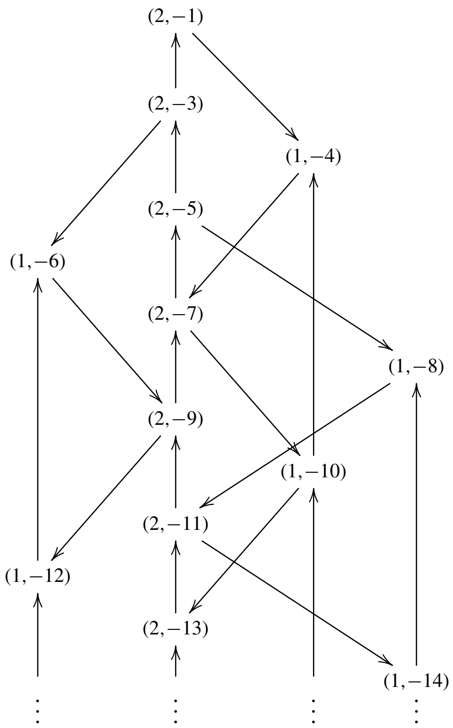


Fig. 7. The quiver Γ^- for $\mathfrak{g}$ of type G_2.

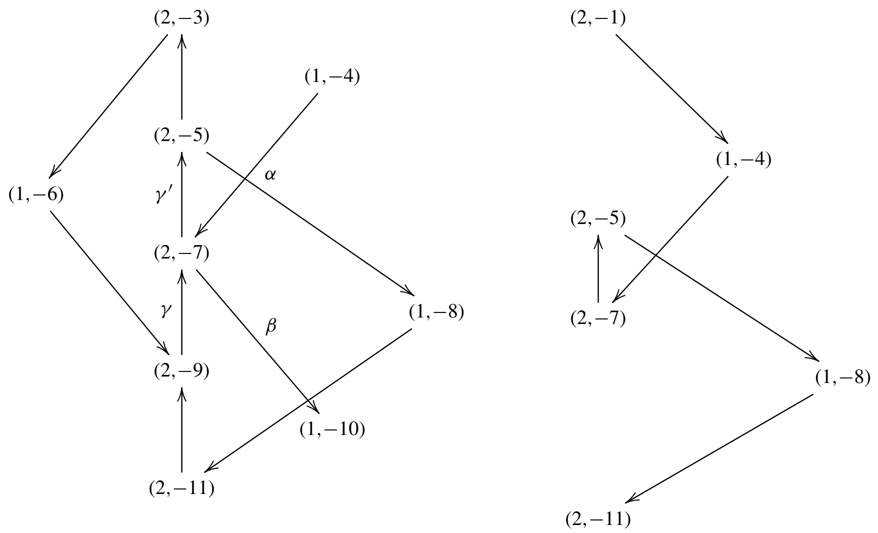


Fig. 8. The modules $K^{(1)}_{1,-10}$ and $K^{(2)}_{1,-11}$ for $\mathfrak{g}$ of type G_2.

vectors of all submodules. This gives

$$\chi_q(L(Y_{1,-13})) = Y_{1,-13}\big(1+v_{1,-10}(1+v_{2,-7}(1+v_{2,-9}(1+v_{1,-6}+v_{2,-11}+v_{1,-6}v_{2,-11} + v_{1,-6}v_{2,-3}+v_{2,-11}v_{1,-8}+v_{1,-6}v_{2,-11}v_{2,-3}+v_{1,-6}v_{2,-11}v_{1,-8} + v_{1,-6}v_{2,-11}v_{2,-3}v_{1,-8}(1+v_{2,-5}(1+v_{2,-7}(1+v_{1,-4})))))))\big),$$

$$\chi_q(L(Y_{2,-12})) = Y_{2,-12}\big(1+v_{2,-11}(1+v_{1,-8}(1+v_{2,-5}(1+v_{2,-7}(1+v_{1,-4}(1+v_{2,-1})))))\big),$$

where, following (15), we have

$$v_{1,r} = Y^{-1}_{1,r+3}Y^{-1}_{1,r-3}Y_{2,r+2}Y_{2,r}Y_{2,r-2}, \qquad v_{2,r} = Y^{-1}_{2,r+1}Y^{-1}_{2,r-1}Y_{1,r}.$$

Remark 4.14. Assuming Theorem 4.8, we can easily calculate the dimension vectors of the A-modules $K^{(i)}_{1,r}$ for $r \le d_i - th^\vee$. Indeed, by [FM, Lemma 6.8], the lowest monomial of $\chi_q(Y_{i,r-d_i})$ is equal to $Y^{-1}_{\nu(i),r-d_i+th^\vee}$, where ν is the involution of I defined by $w_0(\alpha_i) = -\alpha_{\nu(i)}$. Denote by $(d_{j,s}(K^{(i)}_{1,r}))$ the dimension vector of $K^{(i)}_{1,r}$. Then

$$Y^{-1}_{\nu(i),r-d_i+th^\vee} = Y_{i,r-d_i}\prod_{(j,s)\in V^-} v_{j,s}^{d_{j,s}(K^{(i)}_{1,r})},$$

and using (15), this equation determines the numbers $d_{j,s}(K^{(i)}_{1,r})$. In particular, if we introduce the *ungraded* dimension vector $(d_j(i))$ of $K^{(i)}_{1,r}$ by

$$d_j(i) := \sum_s d_{j,s}(K^{(i)}_{1,r}) \qquad (r \le d_i - th^\vee),$$

we can deduce from this the nice formula

$$\sum_{i,j\in I} d_j(i)\alpha_j = \sum_{\beta\in\Phi_{>0}} \beta, \tag{18}$$

where $\Phi_{>0}$ is the set of positive roots of $\mathfrak{g}$. This can be observed in Figures 5 and 8 (see also §§6.4–6.7 below). When $\mathfrak{g}$ is of type A, D, E, as explained in Remark 4.7 the modules $K^{(i)}_{1,r}$ are graded versions of the indecomposable injective modules over the preprojective algebra Λ, and formula (18) recovers a well known property of Λ.

4.5. *Proof of the theorem*

The proof relies on Theorem 3.1, and on the categorification of cluster algebras by means of quivers with potentials, developed by Derksen, Weyman and Zelevinsky [DWZ1, DWZ2]. This categorification provides (among other things) a description of cluster variables in terms of Grassmannians of submodules, which will be our key ingredient. An important additional result will be borrowed from Plamondon [Pl2].

4.5.1. F-polynomials and g-vectors of cluster variables. Recall the cluster algebra $\mathscr{A}$ of §2.2.1, with initial seed $(\mathbf{z}^-, G^-)$. Following [FZ3, (3.7)], define

$$\widehat{y}_{i,r} := \prod_{(i,r)\to(j,s)} z_{j,s} \prod_{(j,s)\to(i,r)} z_{j,s}^{-1} \qquad ((i,r)\in W^-). \tag{19}$$

Here the first (resp. second) product is over all outgoing (resp. incoming) arrows at vertex (i,r) of the graph G^-. The following result is similar to [HL1, Lemma 7.2].

Lemma 4.15. *After performing in* (19) *the change of variables* (3),

$$\widehat{y}_{i,r} = A^{-1}_{i,r-d_i} \qquad ((i,r)\in W^-),$$

where the Laurent monomials $A_{i,r}$ are given by (6).

Proof. Using the definition of the quiver G^-, we can rewrite (19) as

$$\widehat{y}_{i,r} = \frac{z_{i,r+b_{ii}}}{z_{i,r-b_{ii}}} \prod_{j\neq i} \frac{z_{j,r+b_{ij}+d_j-d_i}}{z_{j,r-b_{ij}+d_j-d_i}},$$

where the product is over all j's such that $c_{ij}\neq 0$. Here we use the convention that $z_{i,s}=1$ for every (i,s) with $s>0$. Using the change of variables (3), we obtain

$$\widehat{y}_{i,r} = Y^{-1}_{i,r-b_{ii}} Y^{-1}_{i,r} \prod_{j\neq i;\, c_{ij}\neq 0} Y_{r,r-d_i+b_{ij}+d_j} Y_{r,r-d_i+b_{ij}+3d_j} \cdots Y_{r,r-d_i-b_{ij}-d_j}.$$

The result then follows by comparison with (6), if we notice again that $b_{ij}+d_j = c_{ji}+1$ because of (1). □

In [FZ3] Fomin and Zelevinsky attach to every cluster variable x of $\mathscr{A}$ a polynomial F_x with integer coefficients in the set of variables $\widehat{\mathbf{y}} = \{\widehat{y}_{i,r} \mid (i,r) \in W^-\}$, and a vector $\mathbf{g}_x \in \mathbb{Z}^{(W^-)}$, such that [FZ3, Corollary 6.3]

$$x = \mathbf{z}^{\mathbf{g}_x} F_x(\widehat{\mathbf{y}}). \tag{20}$$

Note that $\mathscr{A}$ has no frozen cluster variables, so there is no denominator in (20). The polynomial F_x and the integer vector g_x are called the *F-polynomial* and *g-vector* of the cluster variable x, respectively. We refer the reader to [FZ3] for their definition.

On the other hand, it follows from the theory of q-characters that for every simple $U_q(\widehat{\mathfrak{g}})$-module $L(m)$ in the category $\mathscr{C}^-$, the truncated q-character $\chi_q^-(L(m))$ can be written as

$$\chi_q^-(L(m)) = m P_m, \tag{21}$$

where P_m is a polynomial with integer coefficients in the variables $\{A_{i,r-d_i}^{-1} \mid (i,r) \in W^-\}$. Moreover, P_m has constant term 1.

Now, by the proof of Theorem 3.1, among the cluster variables of $\mathscr{A}$, we find all the truncated q-characters of the Kirillov–Reshetikhin modules of $\mathscr{C}^-$. These are of the form $L(m)$ with

$$m = m_{k,r}^{(i)} := \prod_{j=0}^{k-1} Y_{i,r+jb_{ii}} \qquad ((i,r) \in W^-,\ r + (k-1)b_{ii} \le 0). \tag{22}$$

Proposition 4.16. *The g-vector of the truncated q-character of the Kirillov–Reshetikhin module $W_{k,r}^{(i)} = L(m_{k,r}^{(i)})$, considered as a cluster variable of $\mathscr{A}$, is given by*

$$g_{j,s} = \begin{cases} 1 & \text{if } (j,s) = (i,r), \\ -1 & \text{if } (j,s) = (i, r + kb_{ii}) \text{ and } r + kb_{ii} \le 0, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Write for short $m = m_{k,r}^{(i)}$, and denote by x the cluster variable $\chi_q^-(L(m))$. Then, comparing (20) with (21), we have

$$P_m = m^{-1} \mathbf{z}^{\mathbf{g}_x} F_x,$$

where, by Lemma 4.15, P_m and F_x are polynomials in the same variables

$$\widehat{y}_{i,r} = A_{i,r-d_i}^{-1}.$$

Since P_m has constant term 1, it follows that $m\mathbf{z}^{-\mathbf{g}_x}$ is a monomial in the variables $\widehat{y}_{i,r}$ which divides the F-polynomial F_x. But, by [FZ3, Proposition 5.2], F_x is not divisible by any $\widehat{y}_{i,r}$. So, using (3), we obtain

$$\mathbf{z}^{\mathbf{g}_x} = m = \frac{z_{i,r}}{z_{i,r+kb_{ii}}},$$

where as above, we set $z_{i,s} = 1$ if $s > 0$. □

4.5.2. Truncated algebras. Let $\ell \in \mathbb{Z}_{<0}$. Let Γ_ℓ^- be the full subquiver of Γ^- with set of vertices

$$V_\ell^- := \{(i, m) \in V^- \mid m \geq \ell\}.$$

Let S_ℓ be the corresponding truncation of the potential S, that is, S_ℓ is defined as the sum of all cycles in S which only involve vertices of V_ℓ^-. Let J_ℓ denote the two-sided ideal of $\mathbb{C}\Gamma_\ell^-$ generated by all cyclic derivatives of S_ℓ. Finally, define the *truncated algebra at height* ℓ as

$$A_\ell := \mathbb{C}\Gamma_\ell^- / J_\ell.$$

Proposition 4.17. *For every ℓ we have:*

(i) *the algebra A_ℓ is finite-dimensional;*
(ii) *the quiver with potential (Γ_ℓ^-, J_ℓ) is rigid.*

Proof. The proof is similar to [DWZ1, Example 8.7]. Let $\pi : \mathbb{C}\Gamma_\ell^- \to A_\ell$ be the natural projection. To prove (i), we show that A_ℓ is spanned by the images under π of a finite number of paths. The arrows of Γ_ℓ^- are of two types:

(a) the *vertical* arrows of the form $(i, m) \to (i, m + b_{ii})$;
(b) the *oblique* arrows of the form $(i, m) \to (j, m + b_{ij})$ provided $c_{ij} < 0$.

Let us say that a path from (i, m) to (j, s) in Γ_ℓ^- is *going up* (resp. *down*) if $m < s$ (resp. $m > s$). Note that all vertical arrows go up and all oblique arrows go down. Each oblique arrow of the boundary of Γ_ℓ^- belongs to a single cycle of the potential S_ℓ, and each interior oblique arrow belongs to exactly two cycles. Therefore each interior oblique arrow gives rise to a "commutativity relation" in A_ℓ:

$$\begin{aligned}&\pi\big((j, m + b_{ji}), (i, m + 2b_{ji}), (i, m + 2b_{ji} + b_{ii}), \ldots, (i, m - b_{ii}), (i, m)\big)\\ &\quad = -\pi\big((j, m + b_{ji}), (j, m + b_{ji} + b_{jj}), \ldots, (j, m - b_{ji} - b_{jj}), (j, m - b_{ji}), (i, m)\big).\end{aligned}$$

The path in the left-hand side consists of an oblique arrow followed by $|c_{ij}|$ vertical arrows, while the right-hand side has $|c_{ji}|$ vertical arrows followed by an oblique arrow. Let p be a path in Γ_ℓ^- with origin (i, m). Using only the above type of commutativity relations, we can bring a number of vertical arrows to the front of p and write

$$\pi(p) = \pi(p_2)\pi(p_1),$$

where p_1 is a path with origin (i, m) consisting only of vertical arrows, and p_2 is a path satisfying the following property: if q is a maximal factor of p_2 containing only vertical arrows, then q is preceded by at least one oblique arrow, say $(j, s) \to (k, s + b_{jk})$, and q contains *less* than $|c_{kj}|$ arrows. Hence q can be nontrivial only if $|c_{kj}| > 1$.

In particular, in the simply laced case, p_2 contains only oblique arrows. In that case, we can immediately conclude that all arrows of p_1 go up and all arrows of p_2 go down, so the lengths of p_1 and p_2 are both bounded by ℓ, and therefore A_ℓ is finite-dimensional.

Otherwise, if q is nontrivial and p_2 contains other vertical arrows after q, then q needs to be followed by at least *two* oblique arrows. Indeed, with the same notation as above, q consists of N vertical arrows of the form $(k, r) \to (k, r+b_{kk})$ with $1 \leq N < |c_{kj}|$. Now, by (1), the inequality $|c_{kj}| > 1$ implies $d_k = 1$ and $d_j = |b_{kj}|$. Let $(k, t) \to (l, t + b_{kl})$

be the first arrow coming after q. Then since $d_k = 1$ we have $|c_{lk}| = 1$. If this oblique arrow is followed by a vertical one $(l, t + b_{kl}) \to (l, t + b_{kl} + b_{ll})$, then we can use the commutativity relation and bring it, together with all the vertical arrows possibly following it, on top of q. In this way, we replace q by a vertical path q' followed by two consecutive oblique arrows.

One then easily checks by inspection that the subpath of p_2 containing q together with the oblique arrow preceding it and the oblique arrow following it, is going down. Therefore, by induction, p_2 can be factored into a product of paths, each of length less than $t + 2$, and all these paths go down (except possibly the last one, which might end with less than t vertical arrows). So again, the length of p_2 is bounded above, and this proves (i) in all cases.

To prove (ii), it is enough to show that every cycle of the form (13) is cyclically equivalent to an element of J_ℓ. Up to cyclic equivalence, this cycle γ can be written with origin in (i, m). Then

$$\begin{aligned}\pi(\gamma) &= \pi\big((i,m),(j,m+b_{ij}),(i,m+2b_{ij}),(i,m+2b_{ij}+b_{ii}),\dots,(i,m-b_{ii}),(i,m)\big)\\ &= \pi\big((i,m),(j,m+b_{ij}),(j,m+b_{ij}+b_{jj}),\dots,(j,m-b_{ij}-b_{jj}),(j,m-b_{ij}),(i,m)\big)\\ &= \pi\big((i,m),(i,m+b_{ii}),\dots,(i,m-2b_{ij}-b_{ii}),(i,m-2b_{ij}),(j,m-b_{ij}),(i,m)\big),\end{aligned}$$

and the last path is cyclically equivalent to

$$\big((i,m-2b_{ij}),(j,m-b_{ij}),(i,m),(i,m+b_{ii}),\dots,(i,m-2b_{ij}-b_{ii}),(i,m-2b_{ij})\big).$$

This cycle is nothing other than γ shifted vertically up by $-2b_{ij}$. Hence, iterating this process, we can replace, modulo J_ℓ and cyclic equivalence, any cycle γ of the form (13) by a similar cycle γ' sitting at the top boundary of Γ_ℓ^-. Now the upper oblique arrow of γ' does not belong to any other cycle, so it gives rise to a zero relation in A_ℓ. In other words, γ' is cyclically equivalent to an element of J_ℓ. This proves (ii). □

Remark 4.18. In the simply laced case and when $|\ell|$ is less than the Coxeter number, the algebra A_ℓ arises as the endomorphism algebra of a (finite-dimensional) rigid module over the preprojective algebra Λ associated with δ, and appears in the works of Geiss, Schröer and the second author (see [GLS1, GLS2]). This gives another proof of Proposition 4.17(i) in this case.

4.5.3. Proof of Lemma 4.4 and Theorem 4.8. Let $(i, r) \in V^-$ and $k \in \mathbb{N}$. By Theorem 3.1, the truncated q-character $\chi_q^-(W^{(i)}_{k,r-(2k-1)d_i})$ is a cluster variable x of $\mathscr{A}$. By Proposition 4.16, the g-vector of x is given by

$$g_{j,s} = \begin{cases} 1 & \text{if } (j,s) = (i, r - 2kd_i + d_i),\\ -1 & \text{if } (j,s) = (i, r + d_i),\\ 0 & \text{otherwise.}\end{cases} \tag{23}$$

Note that, since $(i, r) \in V^-$, we have $(i, r + d_i) \in W^-$. For $\ell < 0$, let $W_\ell^- := \psi(V_\ell^-)$ and $\mathbf{z}_\ell^- = \{z_{i,r} \mid (i, r) \in W_\ell^-\}$. We denote by G_ℓ^- the same quiver as Γ_ℓ^-, but with vertices labelled by W_ℓ^-. Clearly, the cluster variable x is a Laurent polynomial in the variables of $\mathbf{z}_\ell^-$ for some $\ell \ll 0$, and can be regarded as a cluster variable of the cluster algebra $\mathscr{A}_\ell$ defined by the initial seed $(\mathbf{z}_\ell^-, G_\ell^-)$. By Proposition 4.17(ii), we can apply the theory of [DWZ1, DWZ2] and deduce that the F-polynomial of x coincides with the polynomial F_M associated with a certain A_ℓ-module M. In order to identify this module, we apply [Pl2, Remark 4.1], which states in our setting that M is the kernel of a generic element of the homomorphism space between two injective A_ℓ-modules corresponding to the negative and positive components of the g-vector of x. More precisely, denote by $S_{i,m}^\ell$ the one-dimensional A_ℓ-module supported on $(i,m) \in V_\ell^-$. Let $I_{i,m}^\ell$ be the injective A_ℓ-module with socle isomorphic to $S_{i,m}^\ell$. Then, using (23) and taking into account the change of labelling $\psi\colon V_\ell^- \to W_\ell^-$ given by (2), we conclude that M is the kernel of a generic element of $\mathrm{Hom}_{A_\ell}(I_{i,r}^\ell, I_{i,r-kb_{ii}}^\ell)$.

Finally, we can identify M with the kernel of a generic homomorphism between injective A-modules. Indeed, for $m < \ell < 0$ we have a natural projection $A_m \to A_\ell$ whose kernel is generated by all arrows of Γ_m^- starting or ending at a vertex $v \in V_m^- \setminus V_\ell^-$. This induces for every $(i,r) \in V_\ell^-$ an embedding $I_{(i,r)}^\ell \to I_{(i,r)}^m$, and we can regard the A-module $I_{(i,r)}$ as the direct limit of $I_{(i,r)}^\ell$ along these maps. Since F_M is independent of $\ell \ll 0$, we see that M is also the kernel of a generic element of $\mathrm{Hom}_A(I_{i,r}, I_{i,r-kb_{ii}})$, that is, $M = K_{k,r}^{(i)}$. In particular $K_{k,r}^{(i)}$ is finite-dimensional. This proves Lemma 4.4 and finishes the proof of Theorem 4.8.

Remark 4.19. Using the same formula as (14), we can attach to the infinite-dimensional A-module $I_{i,m}$ a formal power series $F_{I_{i,m}}$ in the variables $v_{j,r}$. This series also has an interpretation in terms of quantum affine algebras. Indeed, by [HJ], the category of finite-dimensional $U_q(\widehat{\mathfrak{g}})$-modules can be seen as a subcategory of a category $\mathscr{O}$ of (possibly infinite-dimensional) representations of a Borel subalgebra of $U_q(\widehat{\mathfrak{g}})$. The q-character morphism can be extended to the Grothendieck ring of $\mathscr{O}$ (the target ring is also completed). This category contains distinguished simple representations called negative fundamental representations $L_{i,a}^-$ ($i \in I$, $a \in \mathbb{C}^*$) [HJ, Definition 3.7]. Denote by $\widetilde{\chi}_q(L_{i,a}^-)$ the normalized q-character of $L_{i,a}^-$, that is, its q-character divided by its highest weight monomial. This normalized q-character is a formal power series in the variables $A_{j,b}^{-1}$ [HJ, Theorem 6.1], and it is obtained as a limit of normalized q-characters of Kirillov–Reshetikhin modules. It is not difficult to deduce from Theorem 4.8 and Remark 4.9 that, for $m \le d_i - th^\vee$,

$$\widetilde{\chi}_q(L_{i,q^{m-d_i}}^-) = F_{I_{i,m}}.$$

This is the first geometric description of the q-character of these negative fundamental representations.

5. Beyond Kirillov–Reshetikhin modules

5.1. Grothendieck rings

Let us consider again the cluster algebra $\mathscr{A}$, with initial seed $\Sigma = (\mathbf{z}^-, G^-)$ whose cluster variables $z_{i,r}$ are given by (3). The Laurent phenomenon for cluster algebras implies that $\mathscr{A}$ is a subring of $\mathbb{Z}[Y_{i,r}^{\pm 1} \mid Y_{i,r} \in \mathbf{Y}^-]$. The following theorem gives the precise relationship between $\mathscr{A}$ and the Grothendieck ring of the category $\mathscr{C}^-$.

Theorem 5.1. *The cluster algebra $\mathscr{A}$ is equal to the image of the injective ring homomorphism from $K_0(\mathscr{C}^-)$ to $\mathbb{Z}[Y_{i,r}^{\pm 1} \mid Y_{i,r} \in \mathbf{Y}^-]$ given by $[L(m)] \mapsto \chi_q^-(m)$ (see Proposition 3.10). Hence $\mathscr{A}$ is isomorphic to the Grothendieck ring of $\mathscr{C}^-$.*

Proof. Let R^- denote the image of the homomorphism $[L(m)] \mapsto \chi_q^-(m)$. By [FR], $K_0(\mathscr{C}^-)$ is the polynomial ring in the classes of the fundamental modules of $\mathscr{C}^-$, hence R^- is the polynomial ring in the truncated q-characters $\chi_q^-(Y_{i,r})$ $(Y_{i,r} \in \mathbf{Y}^-)$. By Theorem 3.1, $\mathscr{A}$ contains all these fundamental truncated q-characters, hence $\mathscr{A}$ contains R^-.

To prove the reverse inclusion, we will use a description of the image of the q-character homomorphism as an intersection of kernels of screening operators [FR, FM]. To do this, we need to work with complete (i.e. untruncated) q-characters. So let us consider as in §3.2.2 the larger set of variables $\mathbf{Y}$. Following [FR, §7.1], for every $i \in I$, we have a linear operator S_i from the ring $\mathbb{Z}[Y_{i,r}^{\pm 1} \mid Y_{i,r} \in \mathbf{Y}]$ to a certain free module $\mathscr{Y}_i$ over this ring, which satisfies the Leibniz rule

$$S_i(xy) = x\, S_i(y) + y\, S_i(x) \quad (x, y \in \mathbb{Z}[Y_{i,r}^{\pm 1} \mid Y_{i,r} \in \mathbf{Y}]).$$

It was conjectured in [FR] and proved in [FM] that an element of $\mathbb{Z}[Y_{i,r}^{\pm 1} \mid Y_{i,r} \in \mathbf{Y}]$ is a polynomial in the q-characters $\chi_q(Y_{i,r})$ $(Y_{i,r} \in \mathbf{Y})$ if and only if it belongs to

$$\bigcap_{i\in I} \operatorname{Ker} S_i.$$

Let us now introduce an auxilliary cluster algebra $\mathscr{A}'$. It is defined using the same initial seed $(\mathbf{z}^-, G^-)$ as $\mathscr{A}$, but the initial variables of $\mathscr{A}'$ are given by the following modification of (3):

$$z'_{i,r} := \prod_{k\geq 0,\, r+kb_{ii}\leq 0} Y_{i,\, r+kb_{ii}+2th^\vee},$$

in which the spectral parameters are all shifted upwards by $2th^\vee$. By Theorem 3.1, if we apply to this initial seed of $\mathscr{A}'$ the sequence of mutations $\mu_{\mathscr{S}}$ repeated $h^\vee$ times, we will obtain a new seed Σ' with the same quiver G^-. Moreover, the cluster variable of Σ' sitting at vertex $(i, r) \in W^-$ is nothing other than the *complete* q-character $\chi_q(W^{(i)}_{k_{i,r},r})$.

Consider a cluster variable x of $\mathscr{A}$. By definition, x is obtained from Σ by a finite sequence of mutations μ_x. We want to show that x belongs to R^-. By Theorem 3.1, all cluster variables of Σ belong to R^-, so by induction on the length, we may assume that the last exchange relation of μ_x is of the form

$$xy = M_1 + M_2,$$

where y is a cluster variable of $\mathscr{A}$, M_1 and M_2 are cluster monomials of $\mathscr{A}$, and y, M_1, M_2 belong to R^-. Let us apply the same sequence of mutations μ_x in the cluster algebra $\mathscr{A}'$ to the seed Σ'. The last exchange relation will be of the form

$$x'y' = M'_1 + M'_2,$$

where y', M_1', M_2' are polynomials in the complete fundamental q-characters $\chi_q(Y_{i,r})$ ($Y_{i,r} \in \mathbf{Y}^-$). Moreover, from x', y', M_1', M_2' we recover x, y, M_1, M_2 by application of the truncation ring homomorphism. By the Laurent phenomenon [FZ1] in the cluster algebra $\mathscr{A}'$, we know that x', y', M_1', M_2' are Laurent polynomials in the variables of $\mathbf{Y}$. Since S_i is a derivation, we have

$$S_i(x'y') = x'S_i(y') + y'S_i(x') = S_i(M_1') + S_i(M_2'),$$

hence $S_i(x') = 0$ because $S_i(y') = S_i(M_1') = S_i(M_2') = 0$. It follows that x' is annihilated by all the screening operators, so x' is a polynomial in the q-characters $\chi_q(Y_{i,r})$ ($Y_{i,r} \in \mathbf{Y}^-$). This implies that x is a polynomial in the truncated q-characters $\chi_q^-(Y_{i,r})$ ($Y_{i,r} \in \mathbf{Y}^-$), that is, $x \in R^-$. □

5.2. *Conjectures*

5.2.1. Cluster monomials. In view of Theorem 5.1, it is natural to formulate some conjectures. Following [Le1], let us say that a simple $U_q(\widehat{\mathfrak{g}})$-module S is *real* if $S \otimes S$ is simple.

Conjecture 5.2. *In the above identification of the cluster algebra $\mathscr{A}$ with the ring of truncated q-characters of $\mathscr{C}^-$, the cluster monomials get identified with the truncated q-characters of the real simple modules of $\mathscr{C}^-$.*

When $\mathfrak{g}$ is of type A, D, E, Conjecture 5.2 is essentially equivalent to [HL1, Conjecture 13.2]. But the initial seed used here is different and allows a direct connection between cluster expansions and (truncated) q-characters.

5.2.2. Geometric q-character formulas. Using the methods and tools of §4, we can translate Conjecture 5.2 into a new conjectural geometric formula for the (truncated) q-character of a real simple module of $\mathscr{C}^-$.

Let m be a dominant monomial in the variables $Y_{i,r} \in \mathbf{Y}^-$. Using the change of variables (3), which we can express as

$$Y_{i,r} = \frac{z_{i,r}}{z_{i,r+b_{ii}}} \qquad ((i,r) \in W^-),$$

(where we understand $z_{i,s} = 1$ if $s > 0$), we can rewrite

$$m = \mathbf{z}^{g(m)} := \prod_{(i,r)\in W^-} z_{i,r}^{g_{i,r}(m)}.$$

Let us call the integer vector $g(m) \in \mathbb{Z}^{(W^-)}$ the *g-vector* of $L(m)$. Following §4.3, let us attach to m the A-module $K(m)$ defined as the kernel of a generic A-module homomorphism from the injective A-module $I(m)^-$ to the injective A-module $I(m)^+$, where

$$I(m)^+ = \bigoplus_{g_{i,r}(m)>0} I_{i,r-d_i}^{\oplus g_{i,r}(m)}, \qquad I(m)^- = \bigoplus_{g_{i,r}(m)<0} I_{i,r-d_i}^{\oplus |g_{i,r}(m)|}.$$

Finally, define the F-polynomial $F_{K(m)}$ of $K(m)$ as in §4.2. We can now state the following conjectural generalization of Theorem 4.8.

Conjecture 5.3. *Suppose that $L(m)$ is an irreducible real $U_q(\widehat{\mathfrak{g}})$-module in $\mathscr{C}^-$. Then the truncated q-character of $L(m)$ is equal to*

$$\chi_q^-(L(m)) = m F_{K(m)},$$

where the variables $v_{i,r}$ of the F-polynomial are evaluated as in (15).

Example 5.4. Let $\mathfrak{g}$ be of type A_3. Take $m = Y_{1,-7}Y_{2,-4}$. We have

$$I(m)^+ = I_{1,-8} \oplus I_{2,-5}, \qquad I(m)^- = I_{1,-6} \oplus I_{2,-3}.$$

The module $K(m)$ has dimension 7 and is displayed in Figure 9. Using for instance the fact that $L(m)$ is a minimal affinization (in the sense of [C]), we can compute its q-character. We find

$$\begin{aligned}\chi_q(L(Y_{1,-7}Y_{2,-4})) = Y_{1,-7}Y_{2,-4}\big(&1 + v_{1,-6} + v_{2,-3} + v_{1,-6}v_{2,-3} + v_{1,-2}v_{2,-3}\\
&+ v_{2,-3}v_{3,-2} + v_{1,-6}v_{1,-2}v_{2,-3} + v_{1,-6}v_{2,-3}v_{3,-2}\\
&+ v_{1,-6}v_{2,-3}v_{2,-5} + v_{1,-2}v_{2,-3}v_{3,-2} + v_{1,-6}v_{1,-2}v_{2,-5}v_{2,-3}\\
&+ v_{1,-6}v_{1,-2}v_{2,-3}v_{3,-2} + v_{1,-6}v_{2,-5}v_{2,-3}v_{3,-2}\\
&+ v_{1,-2}v_{2,-5}v_{2,-3}v_{3,-2} + v_{1,-6}v_{1,-2}v_{2,-5}v_{2,-3}v_{3,-2}\\
&+ v_{1,-6}v_{1,-2}v_{2,-3}v_{2,-1}v_{3,-2} + v_{1,-6}v_{2,-5}v_{2,-3}v_{3,-4}v_{3,-2}\\
&+ v_{1,-6}v_{1,-2}v_{2,-5}v_{2,-3}v_{3,-4}v_{3,-2}\\
&+ v_{1,-6}v_{1,-2}v_{2,-5}v_{2,-3}v_{2,-1}v_{3,-2}\\
&+ v_{1,-6}v_{1,-2}v_{2,-5}v_{2,-3}v_{2,-1}v_{3,-4}v_{3,-2}\big),\end{aligned}$$

in agreement with Conjecture 5.3.

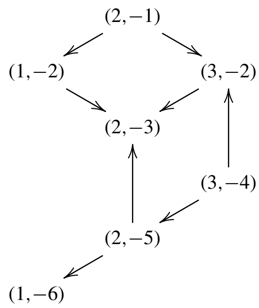
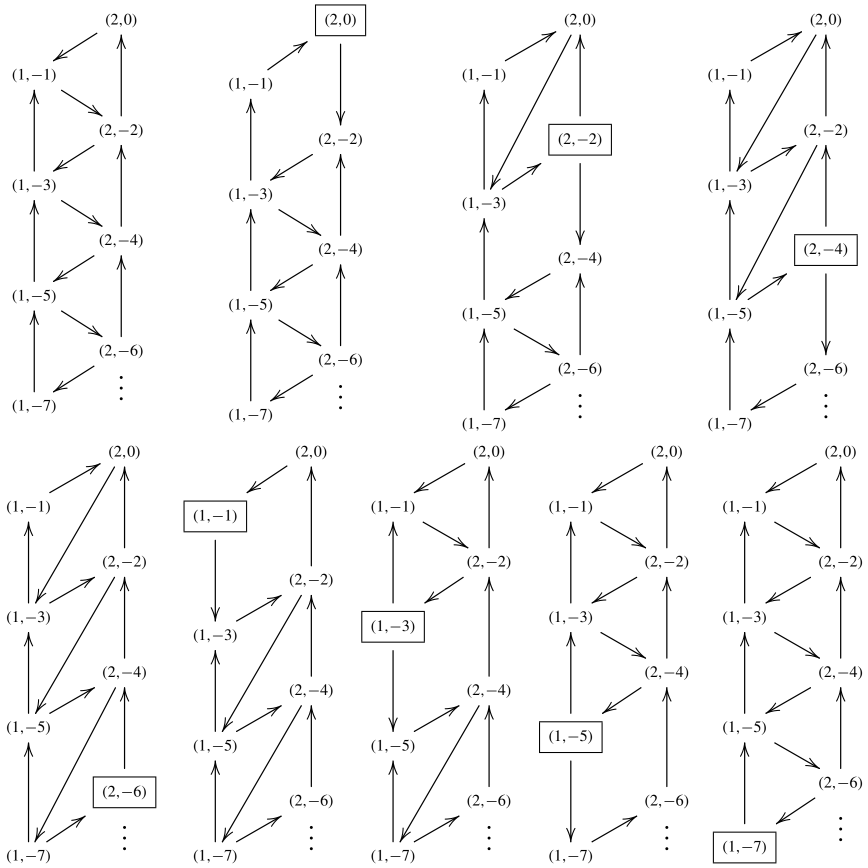


Fig. 9. The A-module $K(m)$ for $m = Y_{1,-7}Y_{2,-4}$ in type A_3.

6. Appendix

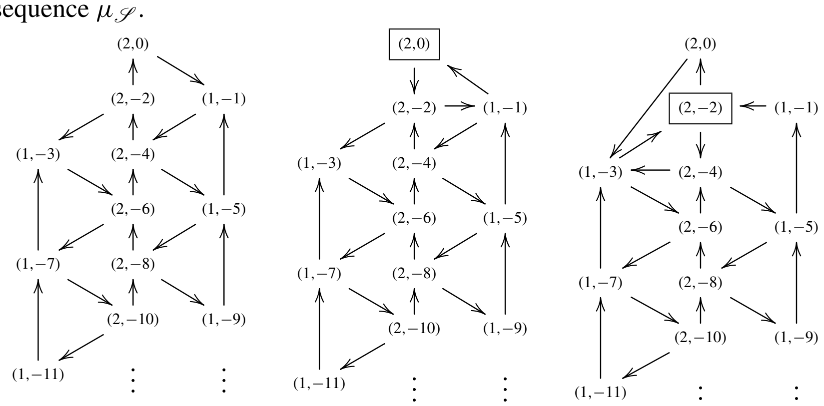
6.1. Mutation sequence in type A_2

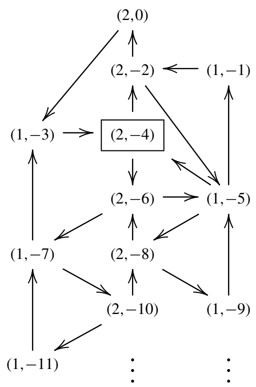
We display the sequence of mutated quivers obtained from G^- at each step of the mutation sequence $\mu_{\mathscr{S}}$. The first quiver is G^-, and in the next quivers the box indicates at which vertex a mutation has been performed.



6.2. *Mutation sequence in type B_2*

We display the sequence of mutated quivers obtained from G^- at each step of the mutation sequence $\mu_{\mathscr{S}}$.

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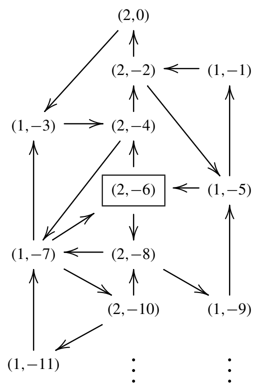
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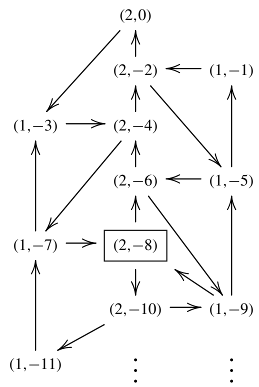
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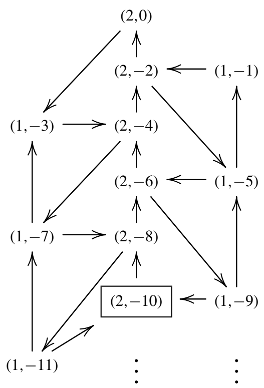
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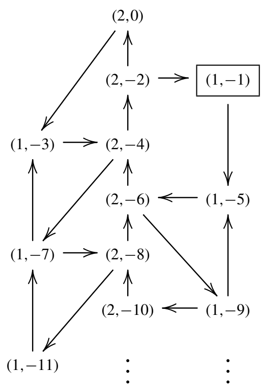
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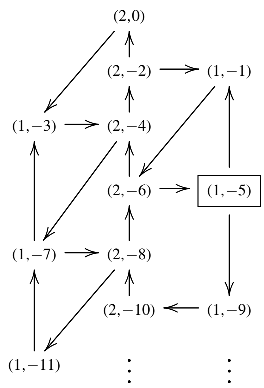
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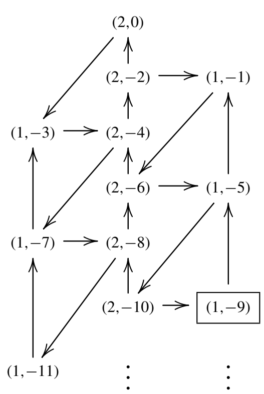
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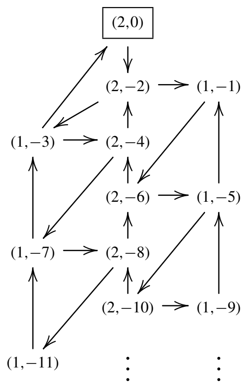
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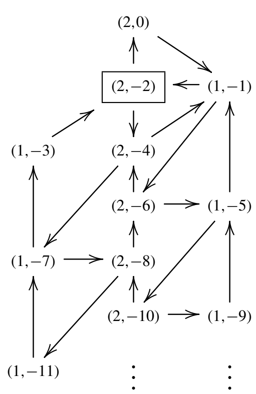
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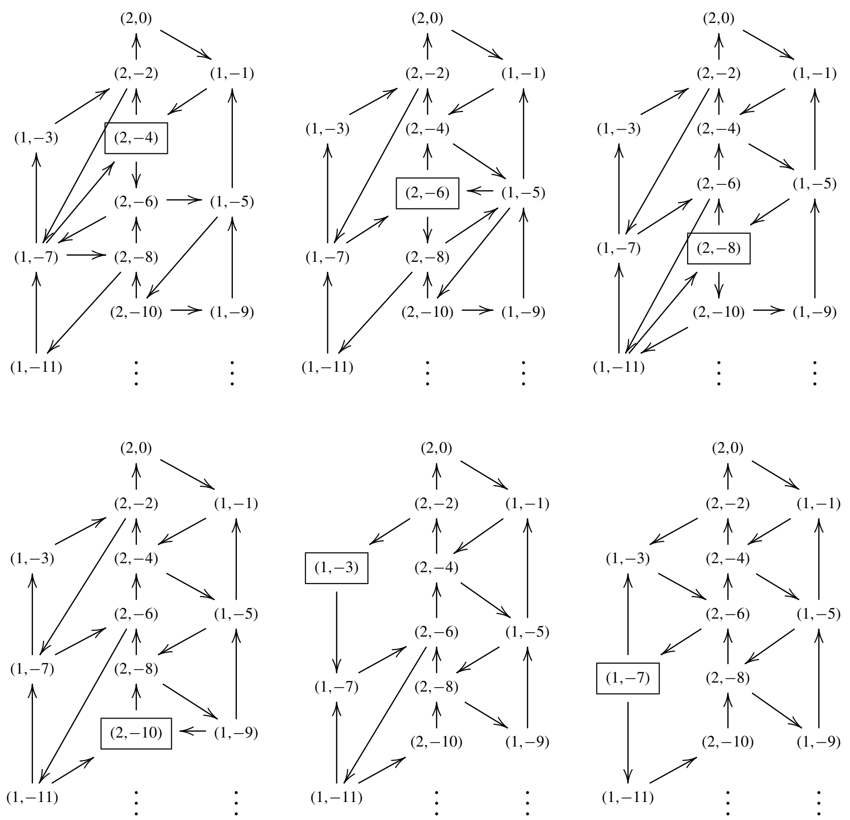
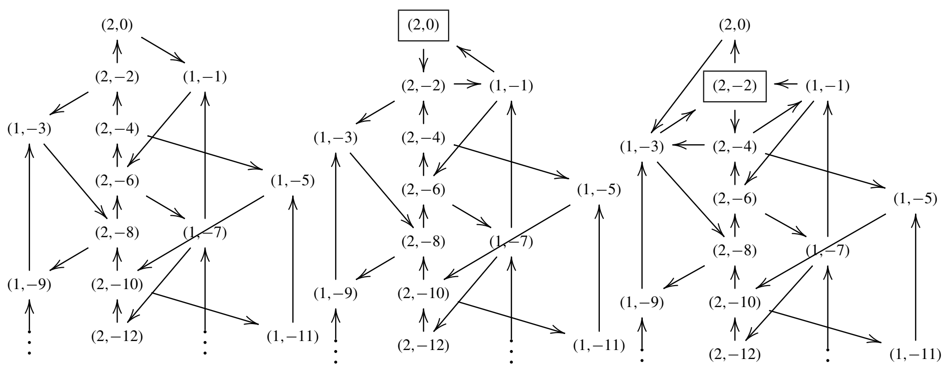
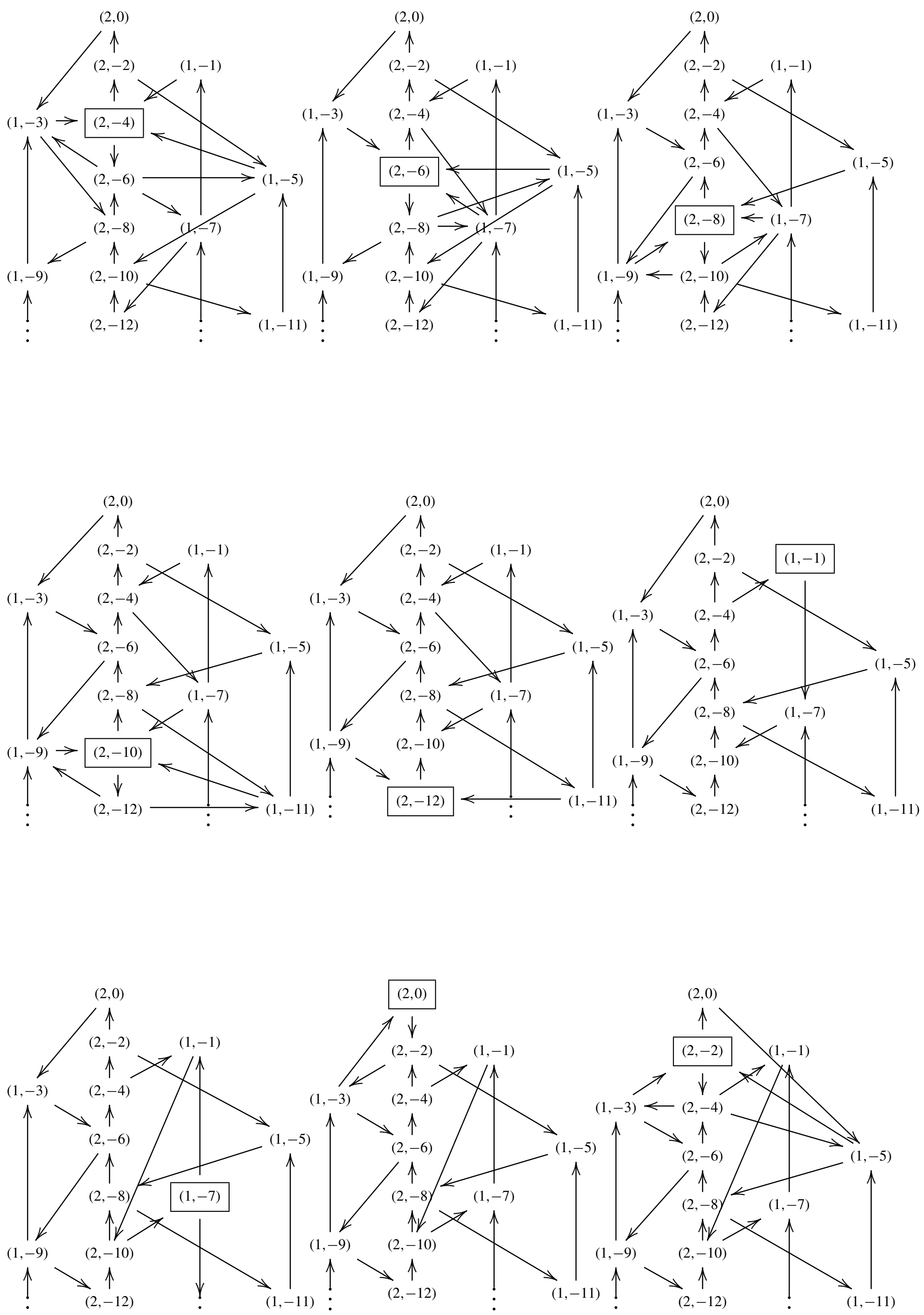


6.3. Mutation sequence in type G_2

We display the sequence of mutated quivers obtained from G^- at each step of the mutation sequence $\mu_{\mathscr{S}}$.

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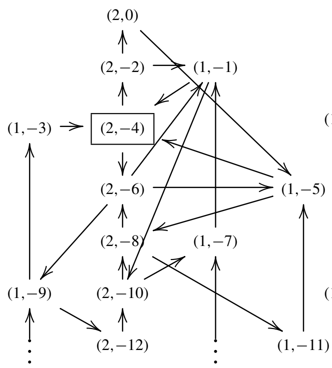
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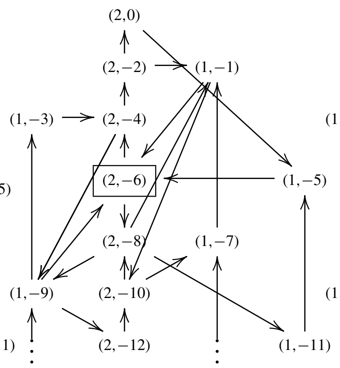
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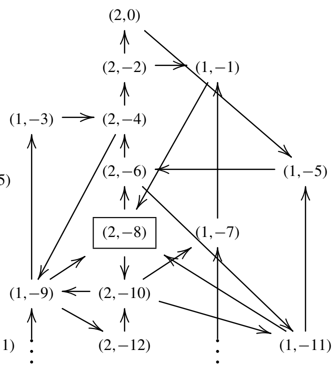
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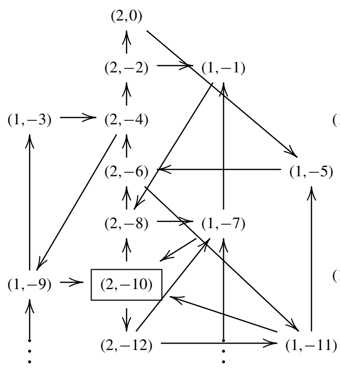
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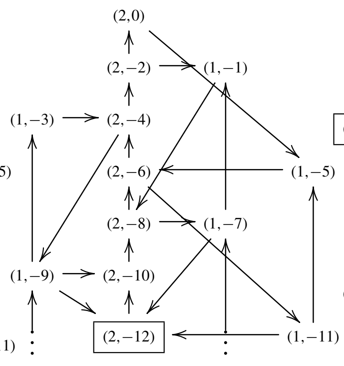
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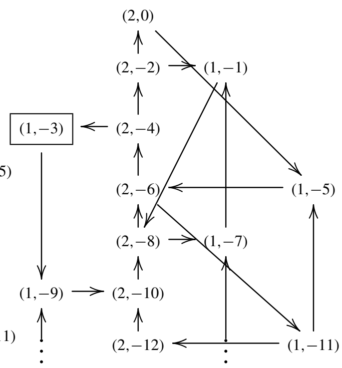
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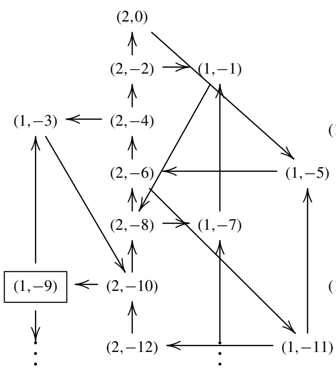
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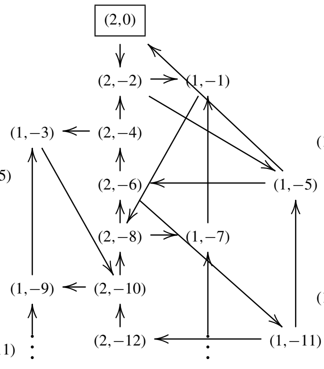
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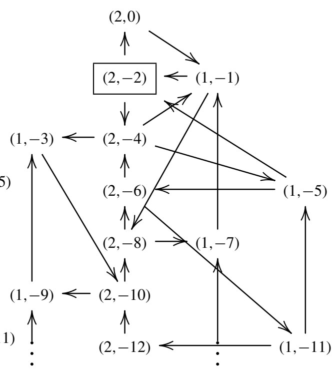
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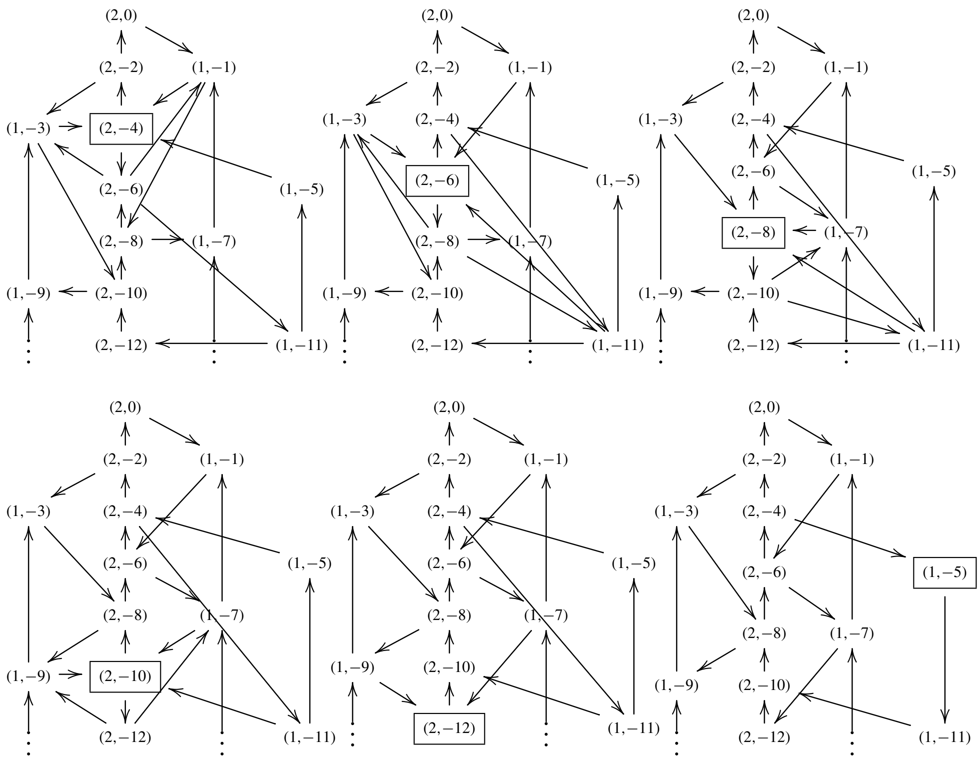
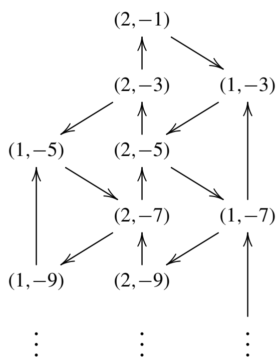


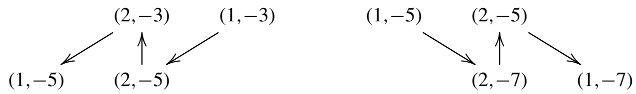
6.4. *Examples of A-modules for* $\mathfrak{g}$ *of type* B_2

We describe some A-modules $K_{k,m}^{(i)}$ for $\mathfrak{g}$ of type B_2. The quiver Γ^- is

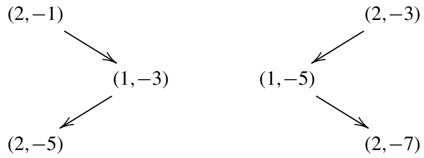


Following the convention of Example 4.6, unless otherwise specified, in the following figures the vertices carry one-dimensional spaces, and the arrows carry linear maps with matrix (± 1).

The modules $K^{(1)}_{1,-5}$ and $K^{(1)}_{1,-7}$ are



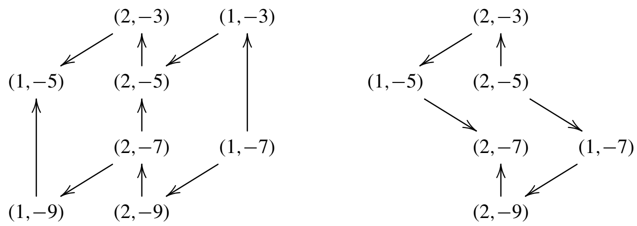
The modules $K^{(2)}_{1,-5}$ and $K^{(2)}_{1,-7}$ are



Applying Theorem 4.8, we recover the following well known formulas for the q-characters of the fundamental $U_q(\widehat{\mathfrak{g}})$-modules:

$$\chi_q(L(Y_{1,-7})) = Y_{1,-7}\big(1 + v_{1,-5}(1 + v_{2,-3}(1 + v_{2,-5}(1 + v_{1,-3})))\big),$$
$$\chi_q(L(Y_{2,-6})) = Y_{2,-6}\big(1 + v_{2,-5}(1 + v_{1,-3}(1 + v_{2,-1}))\big).$$

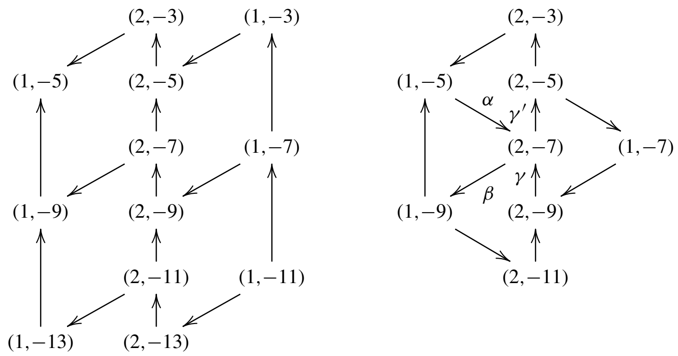
The modules $K^{(1)}_{2,-5}$ and $K^{(2)}_{2,-7}$ are



They correspond under Theorem 4.8 to the Kirillov–Reshetikhin modules

$$W^{(1)}_{2,-11} = L(Y_{1,-11}Y_{1,-7}) \quad \text{and} \quad W^{(2)}_{2,-10} = L(Y_{2,-10}Y_{2,-8}).$$

The modules $K^{(1)}_{3,-5}$ and $K^{(2)}_{3,-7}$ are



In $K^{(2)}_{3,-7}$, the vertex $(2,-7)$ carries a two-dimensional vector space. The linear maps carried by the adjacent arrows have the following matrices:

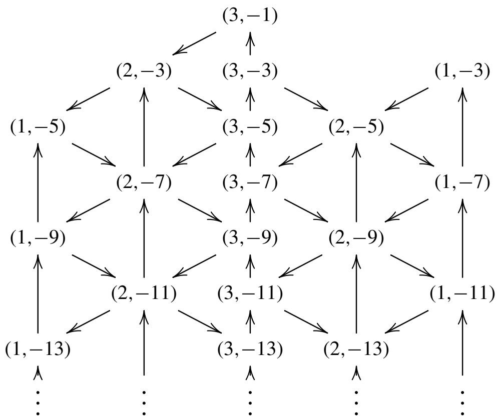
$$\alpha = \gamma = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \beta = \gamma' = \begin{pmatrix} 0 & 1 \end{pmatrix}.$$

They correspond under Theorem 4.8 to the Kirillov–Reshetikhin modules

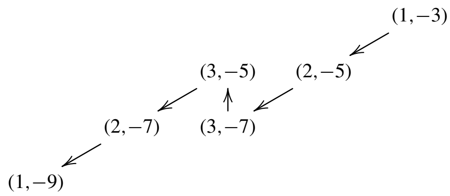
$$W^{(1)}_{3,-15} = L(Y_{1,-15}Y_{1,-11}Y_{1,-7}) \quad \text{and} \quad W^{(2)}_{3,-12} = L(Y_{2,-12}Y_{2,-10}Y_{2,-8}).$$

6.5. Examples of A-modules for $\mathfrak{g}$ of type B_3

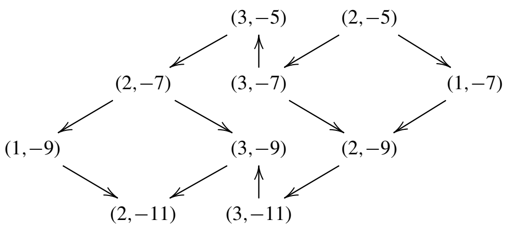
Let $\mathfrak{g}$ be of type B_3, with the short root being α_3. The quiver Γ^- is

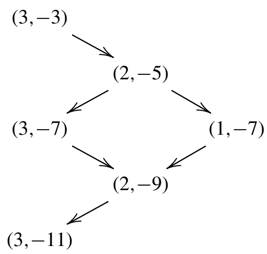


The module $K^{(1)}_{1,-9}$ is



The modules $K^{(2)}_{1,-11}$ and $K^{(3)}_{1,-11}$ are

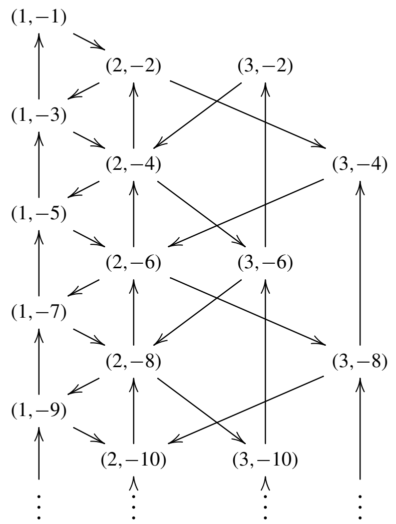




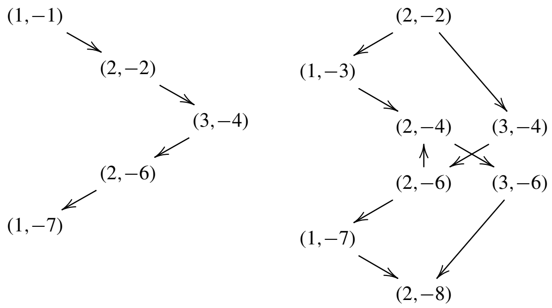
The corresponding fundamental $U_q(\widehat{\mathfrak{g}})$-modules are $L(Y_{1,-11})$, $L(Y_{2,-13})$, and $L(Y_{3,-12})$, of respective dimensions 7, 22, and 8.

6.6. *Examples of A-modules for* $\mathfrak{g}$ *of type* C_3

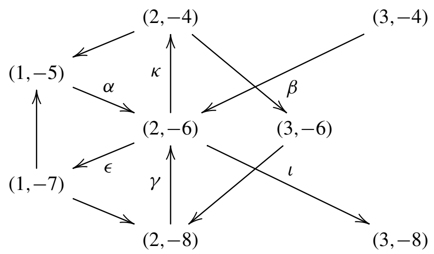
Let $\mathfrak{g}$ be of type C_3, with the long root being α_3. The quiver Γ^- is



The modules $K^{(1)}_{1,-7}$ and $K^{(2)}_{1,-8}$ are



The module $K^{(3)}_{1,-8}$ is



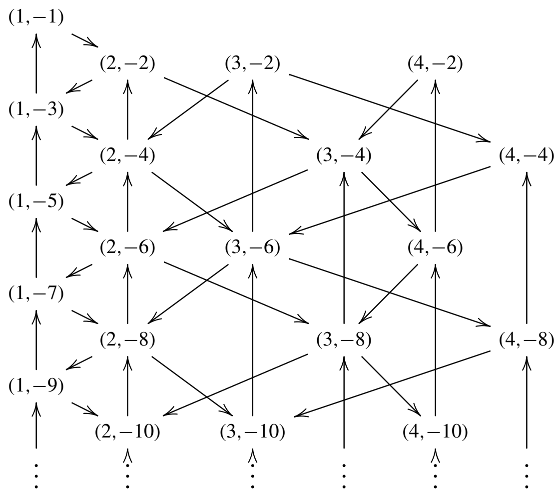
Here, the vector space sitting at vertex $(2, -6)$ has dimension 2. The maps incident to this space are given by the following matrices:

$$\alpha = \begin{pmatrix}1\\0\end{pmatrix}, \quad \beta = \begin{pmatrix}0\\1\end{pmatrix}, \quad \gamma = \begin{pmatrix}1\\0\end{pmatrix}, \quad \epsilon = \begin{pmatrix}0 & 1\end{pmatrix}, \quad \kappa = \begin{pmatrix}0 & 1\end{pmatrix}, \quad \iota = \begin{pmatrix}1 & 0\end{pmatrix}.$$

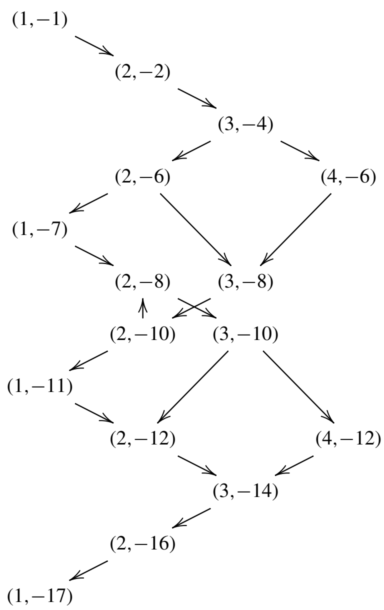
The corresponding fundamental $U_q(\widehat{\mathfrak{g}})$-modules are $L(Y_{1,-8})$, $L(Y_{2,-10})$, and $L(Y_{3,-10})$, of respective dimensions 6, 14, and 14.

6.7. Examples of A-modules for $\mathfrak{g}$ of type F_4

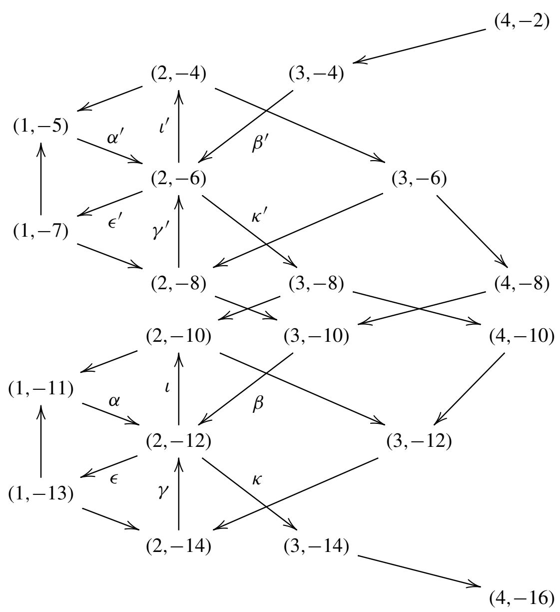
Let $\mathfrak{g}$ be of type F_4. We label the simple roots $\alpha_1, \alpha_2, \alpha_3, \alpha_4$, so that the short simple roots are α_1 and α_2. The quiver Γ^- is



The module $K^{(1)}_{1,-17}$ is



The module $K^{(4)}_{1,-16}$ is



Here, the vector spaces sitting at vertex $(2,-6)$ and $(2,-12)$ have dimension 2. The maps incident to these spaces are given by the following matrices:

$$\alpha = \begin{pmatrix}1\\0\end{pmatrix}, \quad \kappa = \begin{pmatrix}1 & 0\end{pmatrix}, \quad \beta = \begin{pmatrix}0\\1\end{pmatrix}, \quad \epsilon = \begin{pmatrix}0 & 1\end{pmatrix}, \quad \gamma = \begin{pmatrix}1\\0\end{pmatrix}, \quad \iota = \begin{pmatrix}0 & 1\end{pmatrix},$$
$$\alpha' = \begin{pmatrix}1\\0\end{pmatrix}, \quad \kappa' = \begin{pmatrix}1 & 0\end{pmatrix}, \quad \beta' = \begin{pmatrix}0\\1\end{pmatrix}, \quad \epsilon' = \begin{pmatrix}0 & 1\end{pmatrix}, \quad \gamma' = \begin{pmatrix}1\\0\end{pmatrix}, \quad \iota' = \begin{pmatrix}0 & 1\end{pmatrix}.$$

The corresponding fundamental $U_q(\widehat{\mathfrak{g}})$-modules are $L(Y_{1,-18})$ and $L(Y_{4,-18})$, of respective dimensions 26, and 53.

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