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Estimates on elliptic equations that hold only where the gradient is large

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Abstract. We consider a function which is a viscosity solution of a uniformly elliptic equation only at those points where the gradient is large. We prove that the Hölder estimates and the Harnack inequality, as in the theory of Krylov and Safonov, apply to these functions.

Keywords. Degenerate elliptic equations, regularity, viscosity solutions

1. Introduction

This paper is concerned with deriving estimates for functions satisfying a uniformly elliptic equation only at points where the gradient is large. For such functions, we prove a Hölder estimate together with a Harnack inequality.

Intuitively, wherever the gradient of a function u is small, the function will be Lipschitz, so we should not need any further information from the equation at those points in order to obtain a Hölder regularity result. However, there is an obvious difficulty in carrying out this proof since we do not know a priori where $|\nabla u|$ will be large and where it will be small, and these sets may be very irregular. Moreover, the proofs of regularity for elliptic equations involve integral quantities in the whole domain which are hard to obtain unless the equation holds everywhere. As an extra technical difficulty, we consider viscosity solutions which are not even differentiable a priori.

The main contribution of this paper is the way the so-called L^ε estimate is derived. We recall that deriving an L^ε estimate consists in getting a "good" estimate on the size of the superlevel set of a nonnegative supersolution that is small at least at one point. In the uniformly elliptic case, this estimate is obtained thanks to the pointwise Aleksandrov–Bakelman–Pucci estimate. Here, we proceed differently by estimating directly the measure of the set of points where the supersolution can be touched by cusps from below. This idea was inspired by [C] and [Sav], where a similar argument is carried out with paraboloids instead of cusps. We strongly believe that a proof based on applying the ABP

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estimate to the difference of the solution and a particular function, as in [CC], [KS1] or [Saf], cannot be done for the result of this paper.

Main results. In order to state our main results, the notion of supersolutions and subsolutions "for large gradients" should be made precise. We do it by introducing some extremal operators depending on the ellipticity constants λ and Λ and also on a parameter γ which measures how large the gradient should be. They coincide with the classical Pucci operators (plus first order terms) when $|\nabla u| \geq \gamma$, but provide no information otherwise. We will be dealing with merely (lower or upper) semicontinuous functions, and their gradients together with equations will be understood in the viscosity sense. For a C^2 function $u : \Omega \subset \mathbb{R}^d \to \mathbb{R}$, we consider

$$M^+(D^2u, \nabla u) = \begin{cases} \Lambda \operatorname{tr} D^2u^+ - \lambda \operatorname{tr} D^2u^- + \Lambda|\nabla u| & \text{if } |\nabla u| \geq \gamma, \\ +\infty & \text{otherwise,} \end{cases}$$

$$M^-(D^2u, \nabla u) = \begin{cases} \lambda \operatorname{tr} D^2u^+ - \Lambda \operatorname{tr} D^2u^- - \Lambda|\nabla u| & \text{if } |\nabla u| \geq \gamma, \\ -\infty & \text{otherwise.} \end{cases}$$

The main theorem of this paper is the following Hölder estimate.

Theorem 1.1 (Hölder estimate). *For any continuous function $u : \overline{B_1} \to \mathbb{R}$ such that*

$$\begin{aligned} M^-(D^2u, \nabla u) &\leq C_0 && \text{in } B_1, \\ M^+(D^2u, \nabla u) &\geq -C_0 && \text{in } B_1, \\ \|u\|_{L^\infty(B_1)} &\leq C_0, \end{aligned}$$

we have $u \in C^\alpha(B_{1/2})$ and

$$\|u\|_{C^\alpha(B_{1/2})} \leq CC_0$$

where C depends on λ, Λ, the dimension and γ/C_0, and α depends on λ, Λ and the dimension.

Remark 1.2. The constant C in Theorem 1.1 grows like $(\gamma/C_0)^\alpha$ as γ/C_0 tends to ∞. That is,

$$C(d, \lambda, \Lambda, \gamma/C_0) = \tilde{C}(d, \lambda, \Lambda)(1 + (\gamma/C_0)^\alpha).$$

Note that when $\gamma = 0$, the constant C becomes independent of C_0 and we recover the classical estimate for uniformly elliptic equations.

Our second main result is the following Harnack inequality.

Theorem 1.3 (Harnack inequality). *For any nonnegative continuous function $u : \overline{B_1} \to \mathbb{R}$ such that*

$$\begin{aligned} M^-(D^2u, \nabla u) &\leq C_0 && \text{in } B_1, \\ M^+(D^2u, \nabla u) &\geq -C_0 && \text{in } B_1, \end{aligned}$$

we have

$$\sup_{B_{1/2}} u \leq C\Big(\inf_{B_{1/2}} u + C_0\Big).$$

The constant C depends on λ, Λ, the dimension and $\gamma/(C_0 + \inf_{B_{1/2}} u)$.

We emphasize that the result stated in terms of the extremal operators M^+ and M^- is more general than a result which specifies equations of a particular form. A more classical way to write the assumption of Theorem 1.1 would be that for some uniformly elliptic measurable coefficients $a_{ij}(x)$, a bounded vector field $b_j(x)$ and a bounded function $c(x)$, the function u satisfies

$$a_{ij}(x)\partial_{ij}u + b_i(x)\partial_i u = c(x) \qquad \text{only where } |\nabla u(x)| \geq \gamma. \tag{1.1}$$

This statement is equivalent to the assumption of our theorems if u is a classical solution to the equations. Our statement with the extremal operators M^+ and M^- is more suitable for the viscosity solution framework. Note also that a bounded solution to a nonlinear equation would also satisfy our assumptions if the equation is of the form

$$F(D^2u, Du, u, x) = 0,$$

and satisfies the conditions

- $F(0, p, r, x) \leq C(r)|p|$ if $|p| \geq \gamma$.
- For every fixed p, r and x such that $|p| \geq \gamma$, $F(A, p, r, x)$ is uniformly elliptic in A.

In fact, in the case of classical solutions (or even $W^{2,d}$ solutions), this nonlinear situation is not more general than (1.1), since in particular we could obtain (1.1) by linearizing the equation.

As mentioned above, both Theorem 1.1 and Theorem 1.3 derive from a so-called L^ε estimate (see Theorem 5.1). Its proof is based upon a method which seems to have originated in the work of Cabré [C] and continued in the work of Savin [Sav]. Such an idea has also been recently used in [AS]. The idea is to estimate the measure of the superlevel set of supersolutions by sliding some specific functions from below and estimating the measure of the set of contact points. In [Sav], and also recently in [AS], the use of the ABP estimate is bypassed by sliding paraboloids from below. In [C], X. Cabré uses the distance function squared which is a natural replacement of quadratic polynomials on a Riemannian manifold. In [AS], in order to prove the existence of a special barrier function to their equation (see [AS, Lemma 3.3]), they slide from below a barrier to a simpler equation. In the present paper, we slide cusp functions of the form $\varphi(x) = -|x|^{1/2}$.

We finally mention that we chose to state and prove results for equations with bounded (by C_0) right hand sides. We do so for the sake of clarity but, as the reader can check by following the proofs attentively, it is possible to deal with continuous right hand side f_0 in equations and get estimates which only depend on the L^d-norm of the function f_0.

Our definition of M^+ and M^- also determines the type of gradient dependence that we allow in our equations. In terms of linear equations with measurable coefficients as in (1.1), we are assuming that $b \in L^\infty$. In the uniformly elliptic case, the best known estimate depends only on $\|b\|_{L^d}$, which was obtained recently in [Saf]. We have not yet analyzed whether we can extend our result to that kind of gradient dependence. Nor have we been able to obtain a satisfactory parabolic version of our results yet.

Known results. We next explain how results stated in [DaFQ2, I, BD] are related to the ones presented in this paper.

In [BD, DaFQ2], a Harnack inequality is derived for solutions of some singular/degenerate equations. These solutions satisfy the assumptions of the Harnack inequality of Theorem 1.3.

In [I], on the one hand, a Harnack inequality and Hölder estimates are proved for functions satisfying the asumptions of this article. Unfortunately, there is a gap in the proof of the lemma corresponding to the L^ε estimate (see [I, Lemma 7]). On the other hand, an Aleksandrov–Bakelman–Pucci estimate is derived in [I]. The interested reader is also referred to [DaFQ1, J, CDDM] for other results for equations in nondivergence form and to [ACP] for equations in divergence form that are either degenerate or singular.

In [De], an equation of the following form is studied:

$$-\mathrm{tr}(A(Du, u, x)D^2u) + f(x, u, Du) = 0$$

under the assumptions that

$$\begin{aligned} &\Lambda^{-1}\lambda(p)\mathrm{I} \le A(x, r, p) \le \Lambda\lambda(p)\mathrm{I}, \\ &|f(x, r, p)| \le \tfrac{1}{2}\Lambda(1 + \lambda(p))(1 + |p|), \end{aligned}$$

where $\lambda(p) \ge \lambda_0 > 0$ for $|p| \ge \gamma$. The main theorem of that paper is a Hölder continuity result, which is proved using probabilistic techniques. Note that the assumptions of our theorems cover this situation. The most important difference between the result in [De] and ours is that in that paper the equation plays some role even where $|p|$ is small, since it is important in the proof that all the eigenvalues of A are comparable at every point.

Organization of the paper. The paper is organized as follows. In Section 2, we introduce tools that will be used in the proofs. In Section 3, we state and prove the main new lemma. It is a measure estimate satisfied by nonnegative supersolutions. In Section 5, we deduce a so-called L^ε estimate from the main new lemma. In Section 6, a Hölder estimate is derived from the L^ε estimate. The last section, Section 7, is devoted to the proof of the Harnack inequality stated above.

2. Preliminaries

2.1. Scaling

In this short subsection, we analyze how the equations involving $M^\pm$ change according to scaling. Those facts will be used repeatedly in Sections 4–7.

If u satisfies $M^+(D^2u, \nabla u) \ge A$ in Ω, then $v(x) = Ku(x_0 + rx)$ satisfies the inequality $M^+_{r,K}(D^2v, \nabla v) \ge Kr^2A$ in $x_0 + r\Omega$, where

$$M^+_{r,K}(D^2v, \nabla v) = \begin{cases} \Lambda \,\mathrm{tr}\, D^2v^+ - \lambda \,\mathrm{tr}\, D^2v^- + r\Lambda|\nabla v| & \text{if } |\nabla v| \ge rK\gamma, \\ +\infty & \text{otherwise.} \end{cases}$$

Note that if $r \le 1$ and $K \ge 1$ then $M^+_{r,K} \le M^+$. Therefore, we find in particular that $M^+(D^2v, \nabla v) \ge r^2KA$ in $x_0 + r\Omega$.

Likewise, if $M^-(D^2u, \nabla u) \le A$ in Ω, then $M^-(D^2v, \nabla v) \le r^2KA$ in $x_0 + r\Omega$.

2.2. *The growing ink-spots lemma*

In this section, we state and prove a consequence of Vitali's covering lemma. This result replaces the usual Caldéron–Zygmund decomposition [CC] when we derive the so-called L^ε estimate of Corollary 4.3. It is a statement from measure theory which is essentially the same that was used in the original work by Krylov and Safonov [KS1]. The suggestive name *growing* (or *crawling*) *ink-spots* was coined by E. M. Landis according to [KS2].

Lemma 2.1 (Growing ink-spots lemma). *Let $E \subset F \subset B_1$ be two open sets. We make the following two assumptions for some constant $\delta \in (0,1)$.*

- *If any ball $B \subset B_1$ satisfies $|B \cap E| > (1-\delta)|B|$, then $B \subset F$.*
- $|E| \leq (1-\delta)|B_1|$.

Then $|E| \leq (1 - c\delta)|F|$ for some constant c depending on the dimension only.

Proof. For every $x \in F$, since F is open, there exists some maximal ball which is contained in F and contains x. We choose one of those balls for each $x \in F$ and call it B^x.

If $B^x = B_1$ for any $x \in F$, then the result of the theorem follows immediately since $|E| \leq (1-\delta)|B_1|$, so let us assume that it is not the case.

We claim that $|B^x \cap E| \leq (1-\delta)|B^x|$. Otherwise, we could find a slightly larger ball $\tilde{B}$ containing B^x such that $|\tilde{B} \cap E| > (1-\delta)|\tilde{B}|$ and $\tilde{B} \not\subset F$, contradicting the first hypothesis.

The family of balls B^x covers the set F. By the Vitali covering lemma, we can select a finite subcollection of nonoverlapping balls $B_j := B^{x_j}$ such that $F \subset \bigcup_{j=1}^K 5B_j$.

By construction, $B_j \subset F$ and $|B_j \cap E| \leq (1-\delta)|B_j|$. Thus, $|B_j \cap F \setminus E| \geq \delta |B|$. Therefore

$$|F \setminus E| \geq \sum_{j=1}^K |B_j \cap F \setminus E| \geq \sum_{j=1}^K \delta |B_j| = \frac{\delta}{5^d} \sum_{j=1}^K |5B_j| \geq \frac{\delta}{5^d}|F|.$$

The proof is finished with $c = 1/5^d$. □

3. Main new lemma

The lemma in this section is the main difference with the classical case. It is the only lemma whose proof differs substantially from the uniformly elliptic case ($\gamma = 0$).

Lemma 3.1 (A measure estimate). *There exist two small constants $\varepsilon_0 > 0$ and $\delta > 0$, and a large constant $M > 0$, such that if $\gamma \leq \varepsilon_0$, then for any lower semicontinuous function $u : B_1 \to \mathbb{R}$ such that*

$$\begin{aligned} &u \geq 0 \quad \text{in } B_1, \\ &M^-(D^2u, \nabla u) \leq 1 \quad \text{in } B_1, \\ &|\{u > M\} \cap B_1| > (1-\delta)|B_1|, \end{aligned}$$

we have $u > 1$ in $B_{1/4}$.

Remark 3.2. Amusingly enough, the values of M and ε_0 in the lemma above are absolute constants. They do not depend on λ, Λ or the dimension. But the constant δ does.

3.1. The proof for classical solutions

The proof of Lemma 3.1 is easier to understand when u is a smooth function. We will first describe the proof in this case. In the next subsection we will explain why the result holds for lower semicontinuous viscosity solutions in general.

Proposition 3.3. *The conclusion of Lemma 3.1 holds if u is a C^2 supersolution.*

Proof. For contradiction, assume that for all ε_0, δ, M, we can find u as above and such that $u(x_0) \leq 1$ for some point $x_0 \in B_{1/4}$.

Consider $U = \{u > M\} \cap B_{1/4}$. For every $x \in U$, let $y \in \overline{B_1}$ be a point where the minimum of $u(y) + 10|y-x|^{1/2}$ is achieved.

On the one hand, since $u \geq 0$ in B_1 and $x \in U \subset B_{1/4}$, we have $u(z)+10|z-x|^{1/2} > 5\sqrt{3}$ if $z \in \partial B_1$. On the other hand, $u(x_0)+10|x_0-x|^{1/2} \leq 1+5\sqrt{2} < 5\sqrt{3}$. Therefore, the minimum will never be achieved on the boundary and so $y \in B_1$. Moreover, we obtain $u(y) + 10|y-x|^{1/2} \leq 1 + 5\sqrt{2}$ and in particular $u(y) \leq 1 + 5\sqrt{2}$.

We now choose the constant M to be $M := 2 + 5\sqrt{2}$ (note that M does not depend on anything!). In this way, we know that $u(y) < M$. In particular $x \neq y$ and $|z-x|^{1/2}$ is differentiable at $z = y$.

Note that for one value of x, there could be more than one point y where the minimum is achieved. However, the value of y determines x completely since we must have

$$\nabla u(y) = 5(x-y)|y-x|^{-3/2}.$$

For convenience, set $\varphi(z) = -10|z|^{1/2}$. We thus have

$$\nabla u(y) = \nabla \varphi(y-x), \tag{3.1}$$

$$D^2 u(y) \geq D^2 \varphi(y-x). \tag{3.2}$$

The relations (3.1) and (3.2), together with $M^-(D^2u, Du) \leq 1$, imply that

$$|D^2 u(y)| \leq C\big(1 + |D^2\varphi(y-x)| + |\nabla\varphi(y-x)|\big) \tag{3.3}$$

provided that $\varepsilon_0 \leq \min_{B_{5/4}} |\nabla\varphi| = 2\sqrt{5}$. In the previous inequality, C depends on the ellipticity constants and the dimension.

Since for each value of y, there is only one value of x, we can define a map $m(y) := x$. Let $\mathcal{T}$ be the domain of m. That is, $\mathcal{T}$ is the set of values that y takes as $x \in U$. We know that $\mathcal{T} \subset \{y : u(y) < M\}$ and $m(\mathcal{T}) = U$.

Replacing $x = m(y)$ in (3.1) and applying the chain rule, we obtain

$$D^2 u(y) = D^2\varphi(y - m(y))(I - Dm(y)).$$

Solving for Dm and using the estimate (3.3), we get (in terms of Frobenius norms)

$$|Dm(y)| \leq 1 + C\frac{1 + |D^2\varphi(y-x)| + |\varphi(y-x)|}{|D^2\varphi(y-x)|} \leq C.$$

Therefore

$$(1-4^d\delta)|B_{1/4}| \le |U| = \int_{\mathcal{T}} |\det Dm(y)|\, dy \le C|\mathcal{T}|.$$

Since $\mathcal{T} \subset \{y : u(y) < M\}$, from our assumptions we obtain $|\mathcal{T}| \le \delta|B_1|$. This is a contradiction if δ is small enough (depending on the ellipticity constants and the dimension). The proof is now complete. □

3.2. *Formalizing the proof for viscosity solutions*

In this subsection, we explain how to derive Lemma 3.1 for merely lower semicontinuous viscosity supersolutions. In order to do so, we use classical inf-convolution techniques to reduce to the case of semiconcave viscosity supersolutions (Proposition 3.4). We then prove Lemma 3.1 in the semiconcave case (Proposition 3.5 below).

Proposition 3.4. *Assume Lemma 3.1 is proved for semiconcave supersolutions. Then its conclusion is also true for a lower semicontinuous supersolution u.*

Proof. Let u be a merely lower semicontinuous supersolution defined in B_1.

Let $v := \min(u, 2M)$ where M is given by Lemma 3.1 for semiconcave solutions. Note that v is still a supersolution because it is the minimum of two supersolutions. We have $0 \le v \le 2M$.

Consider the inf-convolution of v with parameter $\varepsilon > 0$:

$$v_\varepsilon(x) = \inf_{y\in B_1} \big(v(y) + (2\varepsilon)^{-1}|y-x|^2\big).$$

It is classical to prove that v_ε is still a supersolution at $x \in B_{1-\delta}$ (for $\delta > 0$) of the same equation provided that we can show that $y_x \notin B_1$.

Consider $y_x \in \overline{B_1}$ such that

$$v_\varepsilon(x) = v(y_x) + (2\varepsilon)^{-1}|y_x - x|^2 \le v(x).$$

Then

$$|y_x - x| \le 2\sqrt{\|v\|_\infty \varepsilon} = 2\sqrt{2M\varepsilon}.$$

Thus, for any $\delta > 0$, v_ε is a supersolution in $B_{1-\delta}$ provided that $2\sqrt{2M\varepsilon} < \delta$.

Note that v_ε is semiconcave and

$$D^2 v_\varepsilon \le \varepsilon^{-1} I.$$

Since v is lower semicontinuous, it is classical to show that v_ε converges to v in the half-relaxed sense (which is exactly the same as Γ-convergence). Moreover,

$$\{u > M\} = \bigcup_{\varepsilon>0} \{v_\varepsilon > M\}.$$

Note that as $\varepsilon \to 0$, the sets $\{v_\varepsilon > M\}$ form an increasing nested collection, therefore

$$|\{u > M\}| = \lim_{\varepsilon\to 0} |\{v_\varepsilon > M\}|.$$

For ε sufficiently small, we can apply Lemma 3.1 (appropriately scaled to the ball $B_{1-\delta}$ instead of B_1) and obtain $v_\varepsilon \geq 1$ in $B_{(1-\delta)/4}$. Since $u \geq v_\varepsilon$ and δ is arbitrarily small, the proof is finished. □

Proposition 3.5. *The conclusion of Lemma 3.1 holds if u is a semiconcave viscosity supersolution.*

Proof. The main idea of the proof was already explained in Lemma 3.1 for $u \in C^2$. Here we need to work harder in order to deal with the technical difficulty that we do not assume that u is twice differentiable. Yet, the proof follows essentially the same lines.

In order to organize the proof, we highlight the main steps in bold.

We assume that we have a semiconcave function u which satisfies

$$u \geq 0 \quad \text{and} \quad M^-(D^2u, \nabla u) \leq 1 \quad \text{in } B_1.$$

We also assume that

$$\min_{B_{1/4}} u \leq 1 \quad \text{and} \quad |\{u > M\} \cap B_1| > (1-\delta)|B_1| \tag{3.4}$$

in order to obtain a contradiction.

Step 0. Analyzing the semiconcavity assumption. We assume only that $D^2u \leq C_0$ in the sense that $u(x) - C_0|x|^2/2$ is concave. This means that for every point $x_0 \in B_1$ there exists a vector $p \in \mathbb{R}^d$ (a vector in the superdifferential), which is $p = \nabla u(x_0)$ in case u is differentiable at x_0, so that

$$u(x) \leq u(x_0) + p \cdot (x - x_0) + \frac{C_0}{2}|x - x_0|^2. \tag{3.5}$$

for all $x \in B_1$.

We finally recall that by Aleksandrov theorem, the semiconcave function u is pointwise twice differentiable almost everywhere. That means that there exists a set $E \subset B_1$ of measure zero such that at every point $x \in B_1 \setminus E$, the function u is differentiable and there exists a symmetric matrix $D^2u(x)$ such that

$$u(y) = u(x) + (y-x) \cdot \nabla u(x) + \tfrac{1}{2}\langle D^2u(x)\,(y-x), (y-x)\rangle + o(|x-y|^2).$$

Moreover, we also have [HU]

$$\nabla u(y) = \nabla u(x) + D^2u(x)(y-x) + o(|x-y|),$$

where by $\nabla u(y)$ we mean any vector in the superdifferential of u at y.

Step 1. Touching u with cusps from below. As in the proof for $u \in C^2$, we define $\varphi(x) = -10|x|^{1/2}$ and $M = 2 + 5\sqrt{2}$.

Consider the open set $U = \{u > M\} \cap B_{1/4}$. From our assumption (3.4), we have $|U| > |B_{1/4}| - \delta|B_1|$, which is a significant measure for δ small. We can assume for example that $|U| \geq |B_{1/8}|$, which is a constant which depends on the dimension d only.

For every $x \in U$, we look for the point $y \in B_1$ which realizes the minimum in

$$u(y) - \varphi(y - x) = \min\{u(z) - \varphi(z - x) : z \in \overline{B_1}\}. \tag{3.6}$$

Equivalently, if we let $q(x) = \min_{z \in \overline{B_1}}(u(z) - \varphi(z - x))$, we have

$$\begin{aligned} u(y) &= \varphi(y - x) + q(x), \\ u(z) &\geq \varphi(z - x) + q(x) \quad \forall z \in B_1. \end{aligned} \tag{3.7}$$

Since $\min_{B_{1/4}} u \leq 1$, we observe that $q(x) \leq 1 - \min_{B_{1/2}} \varphi = 1 + 5\sqrt{2}$. Consequently, $y \notin \partial B_1$, since for $y \in \partial B_1$ we would have $\varphi(y - x) + q(x) < 0 \leq u(y)$. Moreover, $u(y) = \varphi(y - x) + q(x) \leq 1 + 5\sqrt{2} = M - 1$. In particular $y \notin U$ and $y \neq x$.

Since u is a semiconcave function, at the point y where it is touched from below by the smooth function φ it must be differentiable, and $\nabla u(y) = \nabla \varphi(y - x)$. A further analysis of the second derivatives of u at y is postponed until later in the proof.

Step 2. Defining the contact set $\mathcal{T}$. We define $\mathcal{T}$ as the set of contact points $y \in \overline{B}_1$ for all values of $x \in U$. In other words, for any $y \in \mathcal{T}$, there exists $x_y \in U$ such that (3.6) holds. This definition is just a rephrasing of the definition of $\mathcal{T}$ given in the proof of Lemma 3.1.

As mentioned above, we have $u(y) \leq M - 1$ for all $y \in \mathcal{T}$. Thus

$$\mathcal{T} \subset B_1 \cap \{u \leq M - 1\}.$$

Step 3. ∇u is Lipschitz on $\mathcal{T}$. Since $u(x) > M$ for all $x \in U$ and $u(y) \leq M - 1$ for all $y \in \mathcal{T}$, we must have $|y - x| > \varepsilon$ for some $\varepsilon > 0$ depending on the modulus of continuity of u. The function φ has a singularity at the origin. This constant $\varepsilon > 0$ tells us that we are evaluating $\varphi(y - x)$ away from this singularity where φ is C^2 and $|D^2\varphi| < C\varepsilon^{-3/2}$.

Let x_1, y_1 and x_2, y_2 be two pairs of corresponding points (they are two pairs of x, y points satisfying (3.6)). Let $r = 2|y_1 - y_2|$. For any $z \in B_r(y_1)$, we use the bound of $D^2\varphi$ above and (3.7) to obtain

$$\begin{aligned} u(z) \geq \varphi(z - x_1) &\geq \varphi(y_1 - x_1) + \nabla\varphi(y_1 - x_1) \cdot (z - y_1) - C\varepsilon^{-3/2}r^2 \\ &= u(y_1) + \nabla u(y_1) \cdot (z - y_1) - C\varepsilon^{-3/2}r^2. \end{aligned}$$

In particular, for $z = y_2$,

$$u(y_2) \geq u(y_1) + \nabla u(y_1) \cdot (y_2 - y_1) - C\varepsilon^{-3/2}r^2.$$

Exchanging the roles of y_1 and y_2, we also get

$$u(y_1) \geq u(y_2) + \nabla u(y_2) \cdot (y_1 - y_2) - C\varepsilon^{-3/2}r^2.$$

Inserting this bound for $u(y_1)$ into the first inequality, we get

$$u(z) \geq u(y_2) + \nabla u(y_2) \cdot (y_1 - y_2) + \nabla u(y_1) \cdot (z - y_1) - C\varepsilon^{-3/2}r^2.$$

Moreover, from (3.5), we also have

$$u(z) \leq u(y_2) + \nabla u(y_2) \cdot (z - y_2) + Cr^2.$$

Subtracting the two inequalities above, we obtain

$$(\nabla u(y_1) - \nabla u(y_2)) \cdot (z - y_1) \leq C(\varepsilon^{-3/2} + 1)r^2.$$

Since z is an arbitrary point in $B_r(y_1)$, we conclude that $|\nabla u(y_1) - \nabla u(y_2)| \leq C(1 + \varepsilon^{-3/2})r$. That is, we have proved that ∇u is Lipschitz on $\mathcal{T}$. The estimate of the Lipschitz norm $[\nabla u]_{\mathrm{Lip}(\mathcal{T})}$ that we obtained depends on ε and consequently on the modulus of continuity of u. It is not a universal constant.

Step 4. The map $m : \mathcal{T} \to U$. As pointed out above, u must be differentiable at the point y and $\nabla u(y) = \nabla\varphi(y-x)$. Note that the value of $\nabla\varphi(y-x) = -5|y-x|^{-3/2}(y-x)$ uniquely determines the value of $y - x$. In particular, for every $y \in \mathcal{T}$, there is a unique $x \in U$ such that (3.6) holds, and that is the point x such that $\nabla u(y) = \nabla\varphi(y - x)$. Let us define $m : \mathcal{T} \to U$ as the function that maps y into x. That is, from the implicit definition

$$\nabla u(y) = \nabla\varphi(y - m(y)), \tag{3.8}$$

we deduce

$$m(y) = y - (\nabla\varphi)^{-1}\nabla u(y),$$

where by $(\nabla\varphi)^{-1}$ we mean the inverse of $\nabla\varphi$ as a function $\mathbb{R}^d \to \mathbb{R}^d$.

We have already shown that ∇u is Lipschitz on $\mathcal{T}$. Clearly, the map $(\nabla\varphi)^{-1}$ which maps $\nabla\varphi(y - x)$ to $y - x$ has a singularity for large gradients, or equivalently where $y - x$ is close to the origin. As pointed out above, we always have $|y - x| > \varepsilon$ for some $\varepsilon > 0$ depending on the modulus of continuity of u. So at least we know that on $\mathcal{T}$, $(\nabla\varphi)^{-1}$ will be a Lipschitz map (in fact smooth) with Lipschitz constant depending on ε (and consequently on the modulus of continuity of u). This implies that m is Lipschitz. Therefore, m is differentiable almost everywhere and we have the classical formula

$$|U| = \int_{\mathcal{T}} |\det Dm(y)|\, dy. \tag{3.9}$$

Step 5. A universal estimate on Dm. So far we have only estimated $|Dm(y)|$ in terms of ε. This was only a technical step to justify the expression (3.9). Now we will obtain an estimate for $|Dm(y)|$ depending only on the universal constants λ, Λ and d.

As mentioned in Step 0, u is pointwise twice differentiable except perhaps on a set E of measure zero. In particular, for all $y \in \mathcal{T} \setminus E$, we have $M^-(D^2u(y), \nabla u(y)) \leq 1$ in the classical sense and we can do the computations below.

We take γ sufficiently small in order to ensure that $|\nabla u(y)| = |\nabla\varphi(y - x)| > \gamma$ for all $y \in B_1$ and $x \in B_{1/4}$. Thus, the condition $M^-(D^2u(y), Du(y)) \leq 1$ is meaningful and we obtain

$$\begin{aligned}\lambda\, \mathrm{tr}(D^2u(y))^+ - \Lambda\, \mathrm{tr}(D^2u(y))^- \\ = M^-(D^2u(y), \nabla u(y)) + \Lambda|\nabla u(y)| \leq C(1 + |y - x|^{-1/2}).\end{aligned} \tag{3.10}$$

Moreover, from (3.7), we have $D^2u(y) \geq D^2\varphi(y-x)$. In particular the negative part of the Hessian of φ controls the Hessian of u: $(D^2u(y))^- \leq (D^2\varphi(y-x))^-$. Combining this with (3.10) we obtain

$$|D^2u(y)| \leq C\big((D^2\varphi(y-x))^- + 1 + |y-x|^{-1/2}\big) \leq C(1+|y-x|^{-3/2}).$$

We now differentiate (3.8) (recall that this is a valid computation for $y \in \mathcal{T} \setminus E$) and obtain

$$D^2u(y) = D^2\varphi(y-x)(I - Dm(y)). \tag{3.11}$$

Therefore,

$$\begin{aligned} |Dm(y)| &= D^2\varphi(y-x)^{-1}\big(D^2\varphi(y-x) - D^2u(y)\big) \\ &\leq \|D^2\varphi(y-x)^{-1}\|\,\|D^2\varphi(y-x) - D^2u(y)\| \leq C \end{aligned}$$

where C is a universal constant. For the last inequality we have used the relations $\|D^2\varphi(y-x)^{-1}\| = C|x-y|^{3/2}$ and $\|D^2\varphi(y-x) - D^2u(y)\| \leq C(1+|x-y|^{-3/2})$. Note how the dependence on $|x-y|$ cancels out. This step would not work for some other choices of φ, for example $\varphi(x) = -|x|$.

Thus, we have obtained $|Dm| \leq C$ almost everywhere in $\mathcal{T}$, for a universal constant C. We can insert this estimate in (3.9) and obtain

$$|U| \leq \int_{\mathcal{T}} C^d\,dy = C^d|\mathcal{T}|.$$

This gives a lower bound for the measure of the set $\mathcal{T}$ of contact points. Thus, $|\mathcal{T}| \geq \delta|B_1|$ for some $\delta > 0$. Since $\mathcal{T} \subset \{u \leq M-1\}$, we obtain a contradiction with (3.4) and finish the proof. □

4. A barrier function and the doubling property

Consider the barrier function $b(x) = |x|^{-p}$. Assume initially that $\gamma = 0$. We compute, for $x \in B_2 \setminus \{0\}$,

$$\begin{aligned} M^-(D^2b, \nabla b) &= \lambda p(p+1)|x|^{-p-2} - \Lambda(d-1)p|x|^{-p-2} - \Lambda p|x|^{-p-1} \\ &= p|x|^{-p-2}\,(\lambda(p+1) - \Lambda(d-1) - \Lambda|x|) \\ &\geq p|x|^{-p-2}\,(\lambda(p+1) - \Lambda(d+1)) \\ &\geq p|x|^{-p-2} \qquad \text{if } p \text{ is large enough.} \end{aligned}$$

Thus, the function $b(x) = |x|^{-p}$ is a subsolution of the Pucci relation $M^-(D^2b, \nabla b) \geq 0$ in $B_2 \setminus \{0\}$ with $\gamma = 0$. Likewise, it will be a subsolution of $M^-(D^2b, \nabla b) \geq 0$ in $B_2 \setminus \{0\}$ provided that γ is chosen smaller than the minimum norm of its gradient.

Using this barrier function, we prove the following doubling property for lower bounds of supersolutions.

Lemma 4.1 (Doubling property for supersolutions). *There exists a small constant $\varepsilon_0 > 0$ depending on λ, Λ and the dimension such that if $u \geq 0$ is a supersolution of $\mathcal{M}^-(D^2u, \nabla u) \leq 1$ in B_2 and $u > M$ in $B_{1/4}$ for some large constant M, then $u > 1$ in B_1.*

Remark 4.2. The constant M depends on λ, Λ, γ and the dimension.

Proof of Lemma 4.1. We compare the function u with

$$B(x) := M \frac{|x|^{-p} - 2^{-p}}{2 \cdot 4^p}.$$

We choose $M \geq 1$ sufficiently large that both $B \geq 1$ and $|\nabla B| \geq \gamma$ in B_1.

We have

$$\begin{aligned} \mathcal{M}^-(D^2B, \nabla B) &\geq \frac{M}{2 \cdot 4^p} \mathcal{M}^-(D^2b, \nabla b) \\ &\geq \frac{M}{2 \cdot 4^p} p 2^{-p-2} \geq 2 \quad \text{for } M \text{ large enough.} \end{aligned}$$

Moreover, $B = 0$ on ∂B_2 and $B < M$ in $\partial B_{1/4}$. Therefore $B \leq u$ in the ring $B_2 \setminus B_{1/4}$ (this is the comparison principle between the viscosity supersolution u and the classical subsolution B, which follows directly from the definition of viscosity solution).

Therefore, $u \geq B \geq 1$ in B_1. Moreover, for $\varepsilon = \min_{B_{1/4}}(u/M - 1)$ we also have $u \geq (1 + \varepsilon)M > 1$ in B_1, which finishes the proof. □

Combining Lemmas 3.1 and 4.1, we obtain

Corollary 4.3. *There exist small constants $\varepsilon_0 > 0$ and $\delta > 0$, and a large constant $M > 0$, such that if $\gamma \leq \varepsilon_0$, then for any continuous function $u : B_2 \to \mathbb{R}$ such that*

$$\begin{aligned} &u \geq 0 \quad \text{in } B_2, \\ &\mathcal{M}^-(D^2u, \nabla u) \leq 1 \quad \text{in } B_2, \\ &|\{u > M\} \cap B_1| > (1 - \delta)|B_1|, \end{aligned}$$

we have $u > 1$ in B_1.

Remark 4.4. Note that the constant M in Corollary 4.3 is the product of the two constants M in Lemmas 3.1 and 4.1.

Proof of Corollary 4.3. Let M_1 and M_2 be the constants from Lemmas 3.1 and 4.1, respectively. Then the function $v = u/M_2$ satisfies the assumption of Lemma 3.1 for $M_2 \geq 1$ (which can be assumed without loss of generality). We conclude that $v > 1$ in $B_{1/4}$, i.e. $u > M_2$ in $B_{1/4}$. We can then apply Lemma 4.1 to get $u > 1$ in B_1. □

The following corollary is just a scaled version of the above result.

Corollary 4.5. *There exist small constants $\varepsilon_0 > 0$ and $\delta > 0$, and a large constant $M > 0$, such that if $\gamma \le \varepsilon_0$, then for any $r \le 1$ and $\kappa \ge 1$, and any continuous function $u : \overline{B_r} \to \mathbb{R}$ such that*

$$u \ge 0 \quad \text{in } B_r,$$
$$M^-(D^2u, \nabla u) \le \kappa \quad \text{in } B_r,$$
$$|\{u > \kappa M\} \cap B_{r/2}| > (1-\delta)|B_{r/2}|,$$

we have $u > \kappa$ in $B_{r/2}$.

Proof. The scaled function $u_r(x) = u(rx/2)/\kappa$ satisfies the scaled condition

$$M^-_{r/2,\kappa}(D^2u_r, \nabla u_r) \le r^2 \le 1 \quad \text{in } B_2.$$

We remark that u_r satisfies a stronger condition since γ can be replaced by the smaller value $\kappa^{-1} r\gamma$. So we can apply Corollary 4.3 to u_r and obtain the result. □

5. The L^ε estimate

Combining Corollary 4.3 with Lemma 2.1, we obtain the L^ε estimate.

Theorem 5.1 (L^ε estimate). *There exist small constants $\varepsilon_0 > 0$ and $\varepsilon > 0$ such that if $\gamma \le \varepsilon_0$, then for any lower semicontinuous function $u : B_2 \to \mathbb{R}$ such that*

$$u \ge 0 \quad \text{in } B_2,$$
$$M^-(D^2u, \nabla u) \le 1 \quad \text{in } B_2,$$
$$\inf_{B_1} u \le 1,$$

we have

$$|\{u > t\} \cap B_1| \le Ct^{-\varepsilon} \quad \text{for all } t > 0.$$

Remark 5.2. This estimate is referred to as the L^ε *estimate* since it yields an estimate on $\int_{B_1} u^\varepsilon(x)\,dx$ (depending on C only).

Proof of Theorem 5.1. In order to prove the result, we will prove the equivalent inequality

$$|\{u > M^k\} \cap B_1| \le \tilde{C} M^{-\varepsilon k}$$

where M is the constant from Corollary 4.5 and $\varepsilon > 0$ has to be properly chosen.

Let $A_k := \{u > M^k\} \cap B_1$, which are open sets. Since $\inf_{B_1} u \le 1$, from Corollary 4.3 we obtain $|A_1| \le (1-\delta)|B_1|$. Since $A_k \subset A_1$ for all $k > 1$, we also have $|A_k| \le (1-\delta)|B_1|$ for all k.

We note that Corollary 4.5, with $\kappa = M^k$, says that every time a ball $B \subset B_1$ satisfies $|B \cap A_{k+1}| > (1-\delta)|B|$, then $B \subset A_k$. Using Lemma 2.1, we obtain

$$|A_{k+1}| \le (1 - c\delta)|A_k|,$$

and therefore, by induction, $|A_k| \le (1-c\delta)^{k-1}(1-\delta)|B_1| = \tilde{C} M^{-\varepsilon k}$, where $-\varepsilon = \log(1-c\delta)/\log M$ and $\tilde{C} = (1-c\delta)^{-1}(1-\delta)|B_1|$. □

The following lemma is a scaled version of Theorem 5.1.

Lemma 5.3 (Scaled L^ε estimate). *There exist small constants $\tilde{\varepsilon}_0 > 0$, $\varepsilon_1 > 0$ and $\theta > 0$ such that if $\gamma \le \tilde{\varepsilon}_0$, then for any $r \le 1$ and $\alpha \in (0,1)$, and any lower semicontinuous function $u : \overline{B_{2r}} \to \mathbb{R}$ such that*

$$\begin{aligned} &u \ge 0 \quad \text{in } B_{2r},\\ &M^-(D^2u, \nabla u) \le \varepsilon_1 \quad \text{in } B_{2r},\\ &|\{u > r^\alpha\} \cap B_r| \ge \tfrac{1}{2}|B_r|, \end{aligned}$$

we have $u > \varepsilon_1 r^\alpha$ in B_r.

Remark 5.4. We shall see that $\tilde{\varepsilon}_0 = \varepsilon_0\varepsilon_1$ where ε_0 is given by Lemma 5.1.

Proof of Lemma 5.3. Let τ be the universal constant such that $C\tau^{-\varepsilon} < |B_1|/2$, where C and ε are the constants of Theorem 5.1. Consider the function $\tilde{u}(x) = \tau r^{-\alpha} u(rx)$. It has the properties

$$\begin{aligned} &\tilde{u} \ge 0 \quad \text{in } B_2,\\ &M^-(D^2\tilde{u}, \nabla \tilde{u}) \le \tau r^{2-\alpha}\varepsilon_1 \quad \text{in } B_2,\\ &|\{\tilde{u} > \tau\} \cap B_1| \ge \tfrac{1}{2}|B_1| > C\tau^{-\varepsilon}, \end{aligned}$$

with $\tau r^{1-\alpha}\gamma$ instead of γ.

Let us choose $\varepsilon_1 = \tau^{-1}$. Since $r \le 1$, we have

$$M^-(D^2\tilde{u}, \nabla\tilde{u}) \le 1 \quad \text{in } B_2.$$

We now apply Theorem 5.1 to find that $\tilde{u} > 1$ in B_1 if $\tau r^{1-\alpha}\gamma \le \varepsilon_0$. We just have to choose $\tilde{\varepsilon}_0 = \varepsilon_0\tau^{-1} = \varepsilon_0\varepsilon_1$ since $r^{1-\alpha} \le 1$. Scaling back, we obtain $u > \varepsilon_1 r^\alpha$ in B_r. ◻

6. Hölder continuity

In this section, we derive the Hölder estimates of Theorem 1.1 from the (scaled) L^ε estimate.

Proof of Theorem 1.1. We start by normalizing the solution u. Let

$$v(x) = \frac{u(\rho x)}{C_0(1+\varepsilon_1^{-1})},$$

where $\rho \le 1$ and ε_1 is the constant from Lemma 5.3. The function v satisfies the estimates

$$\begin{aligned} &M^-(D^2v, \nabla v) \le \varepsilon_1 \quad \text{in } B_1,\\ &M^+(D^2v, \nabla v) \ge -\varepsilon_1 \quad \text{in } B_1,\\ &\|v\|_{L^\infty(B_1)} \le 1, \end{aligned}$$

with γ replaced by $\frac{\rho}{C_0(1+\varepsilon_1^{-1})}\gamma$. Thus, we pick $\rho \le 1$ such that

$$\frac{\rho\gamma}{C_0(1+\varepsilon_1^{-1})} \le \tilde{\varepsilon}_0,$$

where $\tilde{\varepsilon}_0$ is given by Lemma 5.3. It is enough to choose

$$\rho = \min\left(1, \frac{\tilde{\varepsilon}_0 C_0 (1 + \varepsilon_1^{-1})}{\gamma}\right).$$

Let $a_k = \min_{B_{2^{-k}}} v$ and $b_k = \max_{B_{2^{-k}}} v$. We will prove that for some $\alpha > 0$,

$$b_k - a_k \leq 2 \times 2^{-\alpha k}. \tag{6.1}$$

For $k = 0$, the statement is obvious since $b_0 \leq \|v\|_{L^\infty(B_1)}$ and $a_0 \geq -\|v\|_{L^\infty(B_1)}$, thus $b_0 - a_0 \leq 2$. Now we proceed by induction.

Assume that $b_k - a_k \leq 2 \times 2^{-\alpha k}$ and let us prove that $b_{k+1} - a_{k+1} \leq 2 \times 2^{-\alpha(k+1)}$. If $b_k - a_k \leq 2 \times 2^{-\alpha(k+1)}$, then we are done since $b_{k+1} - a_{k+1} \leq b_k - a_k$. Hence, we can assume that $(b_k - a_k)/2 \geq 2^{-\alpha(k+1)}$.

Let $m_k = (a_k + b_k)/2$. Then we have either $|\{v > m_k\} \cap B_{2^{-k-1}}| \geq |B_{2^{-k-1}}|/2$ or $|\{v \leq m_k\} \cap B_{2^{-k-1}}| \geq |B_{2^{-k-1}}|/2$. In the first case we will prove that a_{k+1} is larger than a_k, whereas in the second case we will show that b_{k+1} is smaller than b_k.

Assume we are in the first case, i.e. $|\{v > m_k\} \cap B_{2^{-k-1}}| \geq |B_{2^{-k-1}}|/2$. We apply Lemma 5.3 to $v - a_k$ with $r = 2^{-k-1}$ to obtain $v - a_k \geq \varepsilon_1 2^{-(k+1)\alpha}$ for some universal $\varepsilon_1 > 0$. Therefore, $a_{k+1} \geq a_k + \varepsilon_1 2^{-(k+1)\alpha}$. In particular

$$b_{k+1} - a_{k+1} \leq b_k - a_k - \varepsilon_1 2^{-(k+1)\alpha} \leq (2^{\alpha+1} - \varepsilon_1) 2^{-(k+1)\alpha} \leq 2 \times 2^{-(k+1)\alpha}$$

as soon as α is chosen small enough that $2^{\alpha+1} \leq 2 + \varepsilon_1$.

Assume now we are in the second case. We argue similarly by applying Lemma 5.3 to $b_k - v$.

The estimate (6.1) implies that v is C^α at the origin, with

$$|v(x) - v(0)| \leq 4|x|^\alpha$$

for all $x \in B_1$. Scaling back to the function u, this means that for all $x \in B_\rho$,

$$|u(x) - u(0)| \leq 4(1 + \varepsilon_1^{-1}) \rho^{-\alpha} C_0 |x|^\alpha \leq C C_0 |x|^\alpha$$

where $C = C(\lambda, \Lambda, d, \gamma/C_0)$. By a standard translation and covering argument, we conclude that $u \in C^\alpha(B_{1/2})$ and

$$[u]_{C^{0,\alpha}(B_{1/2})} \leq \tilde{C} C_0$$

where $\tilde{C}$ differs from C by a universal constant. The proof is now complete. □

7. Harnack inequality

This section is devoted to the derivation of a Harnack inequality.

Proof of Theorem 1.3. We first reduce the problem to $C_0 = 1$ and $\inf_{B_{1/2}} u \leq 1$ by replacing u with $u/(C_0 + \inf_{B_{1/2}} u)$. In particular, γ is replaced with $\gamma/(C_0 + \inf_{B_{1/2}} u)$.

Let $\beta > 0$ and let $h_t(x) = t(3/4 - |x|)^{-\beta}$ be defined in $B_{3/4}$. We consider the minimum value of t such that $h_t \geq u$ in $B_{3/4}$. The objective of the proof is to show that this value of t cannot be too large. If $t \leq 1$, we are done. Hence, we further assume that $t \geq 1$.

Since t is chosen to be the *minimum* value such that $h_t \geq u$, there must exist some $x_0 \in B_{3/4}$ such that $h_t(x_0) = u(x_0)$. Let $r = (3/4 - |x_0|)/2$. That is, $2r$ is the distance from x_0 to $\partial B_{3/4}$. Let $H_0 := h_t(x_0) = t(2r)^{-\beta} \geq 1$.

We will estimate the measure of the set $\{u \geq H_0/2\} \cap B_r(x_0)$ in two different ways. We will get a contradiction if t is too large.

Let us start by an upper bound of the measure. From Theorem 5.1, properly rescaled,

$$|\{u > H_0/2\} \cap B_r(x_0)| \leq |\{u > H_0/2\} \cap B_{3/4}| \leq C H_0^{-\varepsilon} = C t^{-\varepsilon} (2r)^{\beta\varepsilon}. \tag{7.1}$$

Let us now obtain a lower bound. Let μ be the small universal constant and β be a large universal constant such that

$$M\left(\left(\frac{2-\mu}{2}\right)^{-\beta} - 1\right) \leq \frac{1}{2}, \qquad \frac{(\mu r)^2}{\left(\frac{2-\mu}{2}\right)^{-\beta} - 1} \leq 1,$$
$$\frac{(\mu r)\gamma}{\left(\frac{2-\mu}{2}\right)^{-\beta} - 1} \leq \varepsilon_0, \qquad \beta \geq \frac{d}{\varepsilon},$$

where M and ε_0 are the constants from Corollary 4.3 and ε comes from Theorem 5.1. The reader can check that choosing

$$\beta = \varepsilon^{-1} \max(d, \gamma)$$

and μ small enough that

$$\mu \leq \frac{\gamma}{\varepsilon_0}, \qquad \frac{\log(1 + \mu\gamma/\varepsilon)}{-\log(1 - \mu/2)} \leq \frac{\gamma}{\varepsilon} \quad \text{and} \quad (1 - \mu/2)^{-\beta} - 1 \leq \frac{1}{2M},$$

we get the four desired inequalities.

The maximum of u in the ball $B_{\mu r}(x_0)$ is at most the maximum of h_t, which equals $t(2r - \mu r)^{-\beta} = \left(\frac{2-\mu}{2}\right)^{-\beta} H_0$. Let us define

$$v(x) = \frac{\left(\frac{2-\mu}{2}\right)^{-\beta} H_0 - u(x_0 + \mu r x)}{\left(\left(\frac{2-\mu}{2}\right)^{-\beta} - 1\right) H_0}.$$

Note that $v(0) = 1$, v is nonnegative in B_1 and satisfies

$$M^-(D^2 v, \nabla v) \leq \frac{(\mu r)^2}{\left(\left(\frac{2-\mu}{2}\right)^{-\beta} - 1\right) H_0} \leq 1 \qquad \text{in } B_1,$$
$$M^+(D^2 v, \nabla v) \geq -\frac{(\mu r)^2}{\left(\left(\frac{2-\mu}{2}\right)^{-\beta} - 1\right) H_0} \geq -1 \quad \text{in } B_1$$

with γ replaced by

$$\frac{(\mu r)\gamma}{H_0\big(\big(\frac{2-\mu}{2}\big)^{-\beta}-1\big)} \le \varepsilon_0$$

(because of the choice of μ and β).

We can apply Corollary 4.3 (in fact, its contrapositive) and obtain

$$|\{v \le M\} \cap B_{1/2}| \ge \delta |B_{1/2}|.$$

In terms of the original function u, this is an estimate of a set where u is larger than

$$H_0\left(\left(\frac{2-\mu}{2}\right)^{-\beta} - M\left(\left(\frac{2-\mu}{2}\right)^{-\beta} - 1\right)\right) \ge \frac{H_0}{2},$$

because of the choice of μ and β. Thus, we obtain the estimate

$$|\{u \ge H_0/2\} \cap B_{\mu r}(x_0)| \ge \delta |B_{\mu r}|.$$

Together with (7.1), this implies that t is bounded above (since $\beta \ge d/\varepsilon$). □

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