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The freeness of ideal subarrangements of Weyl arrangements

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Abstract. A Weyl arrangement is the arrangement defined by the root system of a finite Weyl group. When a set of positive roots is an ideal in the root poset, we call the corresponding arrangement an ideal subarrangement. Our main theorem asserts that any ideal subarrangement is a free arrangement and that its exponents are given by the dual partition of the height distribution, which was conjectured by Sommers–Tymoczko. In particular, when an ideal subarrangement is equal to the entire Weyl arrangement, our main theorem yields the celebrated formula by Shapiro, Steinberg, Kostant, and Macdonald. The proof of the main theorem is classification-free. It heavily depends on the theory of free arrangements and thus greatly differs from the earlier proofs of the formula.

Keywords. Arrangement of hyperplanes, root system, Weyl arrangement, free arrangement, ideals, dual partition theorem

1. Introduction

Let Φ be an irreducible root system of rank ℓ and fix a simple system (or basis) $\Delta = \{\alpha_1, \dots, \alpha_\ell\}$. Let Φ^+ be the set of positive roots. Define the partial order $\geq$ on Φ^+ such that $\alpha \geq \beta$ if $\alpha - \beta \in \mathbb{Z}_{\geq 0}\alpha_1 + \cdots + \mathbb{Z}_{\geq 0}\alpha_\ell$ for $\alpha, \beta \in \Phi^+$. A subset I of Φ^+ is called an *ideal* if a positive root β satisfying $\alpha \geq \beta$ for some $\alpha \in I$ belongs to I. The *height* $\mathrm{ht}(\alpha)$ of a positive root $\alpha = \sum_{i=1}^{\ell} c_i\alpha_i$ is defined to be $\sum_{i=1}^{\ell} c_i$. Let $m = \max\{\mathrm{ht}(\alpha) \mid \alpha \in I\}$. The *height distribution* in I is a sequence of positive integers $(i_1, \dots, i_m)$, where $i_j := |\{\alpha \in I \mid \mathrm{ht}(\alpha) = j\}|$. The *dual partition* $\mathcal{DP}(I)$ of the height distribution in I is given by a multiset of ℓ integers,

$$\mathcal{DP}(I) := ((0)^{\ell - i_1}, (1)^{i_1 - i_2}, \dots, (m-1)^{i_{m-1} - i_m}, (m)^{i_m}),$$

where $(a)^b$ implies that the integer a appears exactly b times.[1]

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[1] It will follow from the inductive proof of Theorem 1.1 via the condition $q \geq p$ in Theorem 3.1 that $i_j \geq i_{j+1}$, justifying the name "partition."

For $\alpha \in \Phi^+$ let H_α denote the hyperplane orthogonal to α. For each ideal $I \subseteq \Phi^+$, define the *ideal subarrangement* $\mathcal{A}(I) := \{H_\alpha \mid \alpha \in I\}$. In particular, when $I = \Phi^+$, $\mathcal{A}(\Phi^+)$ is called the *Weyl arrangement* which is known to be a *free arrangement*. (See §2 and [9] for basic definitions and results concerning free arrangements.) Our main theorem is the following:

Theorem 1.1. *Any ideal subarrangement $\mathcal{A}(I)$ is free with exponents $\mathcal{DP}(I)$.*

Theorem 1.1 was conjectured by Sommers and Tymoczko [11] who defined and studied the ideal exponents, which is essentially the same as our $\mathcal{DP}(I)$. They also verified Theorem 1.1 when Φ is not of the type F_4, E_6, E_7 or E_8 by using the addition-deletion theorem [13]. Our proof is classification-free.

Corollary 1.2 (Steinberg [12], Kostant [5], Macdonald [6]). *The exponents of the Weyl arrangement $\mathcal{A}(\Phi^+)$ are given by $\mathcal{DP}(\Phi^+)$.*

Corollary 1.2, which was referred to as "the remarkable formula of Kostant, Macdonald, Shapiro, and Steinberg" in [2], was first discovered by A. Shapiro (unpublished). Then R. Steinberg found it independently in [12]. It was B. Kostant [5] who first proved it without using the classification by studying the principal three-dimensional subgroup of the corresponding Lie group. I. G. Macdonald [6] gave a proof using generating functions. An outline of Macdonald's proof is presented in [4, (3.20)]. G. Akyildiz and J. Carrell [1, 2] generalized the remarkable formula in a geometric setting. Theorem 1.1 is another generalization in the language of the theory of free hyperplane arrangements. Consequently, our proof, which heavily depends on the theory of free arrangements, greatly differs from the earlier proofs of the formula.

Corollary 1.3. *Suppose that $\Phi^+ = \{\beta_1, \ldots, \beta_s\}$ with $\mathrm{ht}(\beta_1) \le \cdots \le \mathrm{ht}(\beta_s)$. Define*

$$\Phi_t := \{\beta_1, \ldots, \beta_t\} \quad (1 \le t \le s).$$

Then the arrangement $\mathcal{A}(\Phi_t)$ is free with exponents $\mathcal{DP}(\Phi_t)$.

Corollary 1.4. *For any ideal $I \subseteq \Phi^+$, the characteristic polynomial $\chi(\mathcal{A}(I), t)$ splits as*

$$\chi(\mathcal{A}(I), t) = \prod_{i=1}^{\ell}(t - d_i),$$

where $d_1, \ldots, d_\ell$ are nonnegative integers which coincide with $\mathcal{DP}(I)$.

Corollary 1.5. *For any ideal $I \subseteq \Phi^+$, let $\mathcal{A}(I)_{\mathbb{C}}$ denote the complexified arrangement of $\mathcal{A}(I)$. Then*

$$\mathrm{Poin}(M(\mathcal{A}(I)_{\mathbb{C}}), t) = \prod_{i=1}^{\ell}(1 + d_i t),$$

where $M(\mathcal{A}(I)_{\mathbb{C}})$ is the complement of $\mathcal{A}(I)_{\mathbb{C}}$ and $d_1, \ldots, d_\ell$ are nonnegative integers which coincide with $\mathcal{DP}(I)$.

The organization of this article is as follows. In §2 we review basic definitions and results about free arrangements. Then in §3 we introduce a new tool to prove the freeness of arrangements. It is called the multiple addition theorem (MAT). In §4, we verify all the three conditions in the MAT so that we may apply the MAT to prove Theorem 1.1. In §5, we complete the proof of Theorem 1.1 and its corollaries.

2. Preliminaries

In this section we review some basic concepts and results concerning free arrangements. Our standard reference is [9].

Let V be an ℓ-dimensional vector space over a field k. An *arrangement* (*of hyperplanes*) is a finite set of linear hyperplanes in V. Let $S := S(V^*)$ be the symmetric algebra of the dual space V^*. The defining polynomial $Q(\mathcal{A})$ of an arrangement $\mathcal{A}$ is

$$Q(\mathcal{A}) := \prod_{H \in \mathcal{A}} \alpha_H \in S,$$

where $\alpha_H \in V^*$ is a defining linear form of $H \in \mathcal{A}$. The derivation module $\mathrm{Der}\, S$ is the collection of all k-linear derivations from S to itself. It is a free S-module of rank ℓ. Define the module of logarithmic derivations by

$$D(\mathcal{A}) := \{\theta \in \mathrm{Der}\, S \mid \theta(\alpha_H) \in \alpha_H S \text{ for any } H \in \mathcal{A}\}.$$

We say that $\mathcal{A}$ is *free* with *exponents* $(d_1, \ldots, d_\ell)$ if $D(\mathcal{A})$ is a free S-module with a homogeneous basis $\theta_1, \ldots, \theta_\ell$ such that $\deg \theta_i = d_i$ $(i = 1, \ldots, \ell)$. In this case, we use the expression $\exp(\mathcal{A}) = (d_1, \ldots, d_\ell)$. Define the *intersection lattice* by

$$L(\mathcal{A}) := \Big\{ \bigcap_{H \in \mathcal{B}} H \;\Big|\; \mathcal{B} \subseteq \mathcal{A} \Big\}, \tag{2.1}$$

where the partial order is given by reverse inclusion. Let $V \in L(\mathcal{A})$ be the minimum. For $X \in L(\mathcal{A})$, define

$$\mathcal{A}_X := \{H \in \mathcal{A} \mid X \subseteq H\} \qquad \text{(localization)}, \tag{2.2}$$

$$\mathcal{A}^X := \{H \cap X \mid H \in \mathcal{A} \setminus \mathcal{A}_X\} \qquad \text{(restriction)}. \tag{2.3}$$

The *Möbius function* $\mu : L(\mathcal{A}) \to \mathbb{Z}$ is characterized by

$$\mu(V) = 1, \qquad \mu(X) = -\sum_{X \subsetneq Y \subseteq V} \mu(Y).$$

Define the *characteristic polynomial* $\chi(\mathcal{A}, t)$ of $\mathcal{A}$ by

$$\chi(\mathcal{A}, t) := \sum_{X \in L(\mathcal{A})} \mu(X) t^{\dim X}.$$

Theorem 2.1 (Factorization theorem, [14, 7, 9]). *If $\mathcal{A}$ is free with* $\exp(\mathcal{A}) = (d_1, \dots, d_\ell)$, *then*

$$\chi(\mathcal{A}, t) = \prod_{i=1}^{\ell}(t - d_i).$$

Assume that $\mathcal{A}$ is a free arrangement in the complex space $V = \mathbb{C}^\ell$ with $\exp(\mathcal{A}) = (d_1, \dots, d_\ell)$. *Define the complement of $\mathcal{A}$ by*

$$M(\mathcal{A}) := V \setminus \bigcup_{H \in \mathcal{A}} H.$$

Then the Poincaré polynomial of the topological space $M(\mathcal{A})$ splits as

$$\mathrm{Poin}(M(\mathcal{A}), t) = \prod_{i=1}^{\ell}(1 + d_i t).$$

3. Multiple addition theorem

In this section, the root system Φ does not appear. The following is a variant of the addition theorem in [13], which we call the *multiple addition theorem* (MAT).

Theorem 3.1 (Multiple addition theorem (MAT)). *Let $\mathcal{A}'$ be a free arrangement with* $\exp(\mathcal{A}') = (d_1, \dots, d_\ell)$ $(d_1 \le \cdots \le d_\ell)$, *and $1 \le p \le \ell$ the multiplicity of the highest exponent, i.e.,*

$$d_1 \le \cdots \le d_{\ell-p} < d_{\ell-p+1} = \cdots = d_\ell =: d.$$

Let $H_1, \dots, H_q$ be hyperplanes with $H_i \notin \mathcal{A}'$ for $i = 1, \dots, q$. Define

$$\mathcal{A}_j'' := (\mathcal{A}' \cup \{H_j\})^{H_j} = \{H \cap H_j \mid H \in \mathcal{A}'\} \quad (j = 1, \dots, q).$$

Assume that the following three conditions are satisfied:

(1) *$X := H_1 \cap \cdots \cap H_q$ is q-codimensional.*
(2) *$X \not\subseteq \bigcup_{H \in \mathcal{A}'} H$.*
(3) *$|\mathcal{A}'| - |\mathcal{A}_j''| = d$ $(1 \le j \le q)$.*

Then $q \le p$ and $\mathcal{A} := \mathcal{A}' \cup \{H_1, \dots, H_q\}$ is free with $\exp(\mathcal{A}) = (d_1, \dots, d_{\ell-q}, (d+1)^q)$.

Proof. Assume $1 \le j \le q$. Let $\nu_j : \mathcal{A}_j'' \to \mathcal{A}'$ be a map satisfying

$$\nu_j(Y) \cap H_j = Y \quad (Y \in \mathcal{A}_j'').$$

Define a polynomial

$$b_j := Q(\mathcal{A}')/\prod_{Y \in \mathcal{A}_j''} \alpha_{\nu_j(Y)},$$

where $\alpha_{\nu_j(Y)}$ is a defining linear form of $\nu_j(Y)$. Then it is known that

$$D(\mathcal{A}')\alpha_{H_j} := \{\theta(\alpha_{H_j}) \mid \theta \in D(\mathcal{A}')\} \subseteq (\alpha_{H_j}, b_j).$$

(See [13] and [9, p. 114] for example.) Let $\theta_1, \dots, \theta_\ell$ be a basis for $D(\mathcal{A}')$ with $\deg \theta_i = d_i$ $(i = 1, \dots, \ell)$ and $\deg \theta_1 \leq \cdots \leq \deg \theta_{\ell-p} = d_{\ell-p} < d$. Since

$$\deg b_j = |\mathcal{A}'| - |\mathcal{A}''_j| = d$$

by the condition (3), the above inclusion implies that

$$\theta_i \in D(\mathcal{A}) \quad (i = 1, \dots, \ell - p).$$

Define

$$\varphi_i := \theta_{\ell-i+1} \quad (i = 1, \dots, p).$$

Note that $\varphi_1, \dots, \varphi_p$ are of degree d. Again, since $\deg b_j = d$ we may express

$$\varphi_i(\alpha_{H_j}) \equiv c_{ij} b_j \bmod (\alpha_{H_j})$$

with constants c_{ij}. Let C be the $(p \times q)$-matrix $C = (c_{ij})_{i,j}$.

By the condition (2), we may choose a point $z \in X \setminus \bigcup_{H \in \mathcal{A}'} H$. Then the evaluation of $D(\mathcal{A}')$ at the point z is the tangent space $T_{V,z}$ of V at z. Thus

$$T_{V,z} = \mathrm{ev}_z(D(\mathcal{A}')) = \mathrm{ev}_z\langle \varphi_1, \dots, \varphi_p \rangle \oplus \mathrm{ev}_z\langle \theta_1, \dots, \theta_{\ell-p} \rangle.$$

Let $\pi : T_{V,z} \to T_{V,z}/T_{X,z}$ be the natural projection. Note that the definition of the matrix C shows that

$$\operatorname{rank} C = \dim \pi(\mathrm{ev}_z\langle \varphi_1, \dots, \varphi_p \rangle).$$

Since $\mathrm{ev}_z\langle \theta_1, \dots, \theta_{\ell-p} \rangle \subseteq T_{X,z}$, one has

$$\operatorname{rank} C = \dim \pi(\mathrm{ev}_z\langle \varphi_1, \dots, \varphi_p \rangle) = \dim(T_{V,z}/T_{X,z}) = q,$$

where the last equality is the condition (1). Hence $q \leq p$ and we may assume that

$$C = \begin{pmatrix} E_q \\ O \end{pmatrix}$$

by applying elementary row operations. Thus $\theta_1, \dots, \theta_{\ell-q}, \alpha_{H_1}\varphi_1, \dots, \alpha_{H_q}\varphi_q$ form a basis for $D(\mathcal{A})$. Hence $\mathcal{A}$ is a free arrangement with $\exp(\mathcal{A}) = (d_1, \dots, d_{\ell-q}, (d+1)^q)$. $\square$

4. Local heights, local-global formula and positive roots of the same height

In this section we will verify the three conditions in the MAT (Theorem 3.1). From now on we will use the notation of §1 and §2. We will often denote the Weyl arrangement $\mathcal{A}(\Phi^+)$ simply by $\mathcal{A}$. Our standard references on root systems are [3] and [4].

Let $\alpha \in \Phi^+$. Define $\mathcal{A}^\alpha$ to be the restriction of the Weyl arrangement $\mathcal{A}$ to H_α. In other words,

$$\mathcal{A}^\alpha := \mathcal{A}^{H_\alpha} = \{K \cap H_\alpha \mid K \in \mathcal{A} \setminus \{H_\alpha\}\}.$$

Then $Y \in \mathcal{A}^\alpha$ is an element of $L(\mathcal{A})$ with codim $Y = 2$.

For $X \in L(\mathcal{A})$, let $\Phi_X := \Phi \cap X^\perp$. Then Φ_X is a root system of rank codim X. Note that the positive roots in Φ_X are taken to be $\Phi^+ \cap \Phi_X$, and Φ_X may be reducible. When Φ_X is irreducible, define the *local height* of α at X by

$$\mathrm{ht}_X(\alpha) := \mathrm{ht}_{\Phi_X}(\alpha)$$

where the height on the right-hand side is taken with respect to the simple system of Φ_X corresponding to the above positive roots. When Φ_X is not irreducible, we interpret

$$\mathrm{ht}_X(\alpha) := \mathrm{ht}_\Psi(\alpha),$$

where Ψ is the irreducible component of Φ_X which contains α.

To verify the condition (3) in the MAT for ideal subarrangements, we need the following theorem together with Proposition 4.4:

Theorem 4.1 (Local-global formula for heights). *For $\alpha \in \Phi^+$, we have*

$$\mathrm{ht}_\Phi(\alpha) - 1 = \sum_{X \in \mathcal{A}^\alpha} (\mathrm{ht}_X(\alpha) - 1)\,.$$

Proof. We proceed by an ascending induction on $\mathrm{ht}_\Phi(\alpha)$. When α is a simple root, then both sides are zero. Now suppose $\mathrm{ht}_\Phi(\alpha) > 1$. Let $\alpha_1 \in \Delta$ be a simple root such that $\beta := \alpha - \alpha_1 \in \Phi^+$. Let $X_0 := H_\alpha \cap H_\beta$. Then $\{\alpha_1, \alpha, \beta\} \subseteq \Phi_{X_0}$. Set

$$C_\Phi(\alpha) := \sum_{X \in \mathcal{A}^\alpha} (\mathrm{ht}_X(\alpha) - 1)\,.$$

If we verify

$$C_1 := C_\Phi(\alpha) - C_\Phi(\beta) - 1 = 0,$$

then we will obtain

$$C_\Phi(\alpha) = C_\Phi(\beta) + 1 = \mathrm{ht}_\Phi(\beta) = \mathrm{ht}_\Phi(\alpha) - 1$$

by the induction assumption. So it remains to show $C_1 = 0$. Note that $\mathrm{ht}_{X_0}(\alpha) - \mathrm{ht}_{X_0}(\beta) = 1$, $X_0 \in \mathcal{A}^\alpha$ and $X_0 \in \mathcal{A}^\beta$. Compute

$$\begin{aligned} C_1 &= C_\Phi(\alpha) - C_\Phi(\beta) - 1 = \sum_{X \in \mathcal{A}^\alpha} (\mathrm{ht}_X(\alpha) - 1) - \sum_{Y \in \mathcal{A}^\beta} (\mathrm{ht}_Y(\beta) - 1) - 1 \\ &= \sum_{X \in \mathcal{A}^\alpha \setminus \{X_0\}} (\mathrm{ht}_X(\alpha) - 1) - \sum_{Y \in \mathcal{A}^\beta \setminus \{X_0\}} (\mathrm{ht}_Y(\beta) - 1)\,. \end{aligned} \tag{4.1}$$

Let $\mathcal{Z} := \mathcal{A}^{X_0} = \{K \cap X_0 \mid K \in \mathcal{A},\ X_0 \not\subseteq K\}$. Define

$$C_2 := \sum_{Z \in \mathcal{Z}} \Bigg(\sum_{\substack{X \in \mathcal{A}^\alpha \setminus \{X_0\} \\ X \supset Z}} (\mathrm{ht}_X(\alpha) - 1) - \sum_{\substack{Y \in \mathcal{A}^\beta \setminus \{X_0\} \\ Y \supset Z}} (\mathrm{ht}_Y(\beta) - 1) \Bigg).$$

We will show that $C_1 = C_2$. To this end, we show that in the expression of C_2, we have (A) every term in (4.1) appears and (B) each of them appears only once.

(A) We prove that every term in (4.1) appears in C_2. Let $X \in \mathcal{A}^\alpha \setminus \{X_0\}$. Let $Z := X \cap X_0 \subset X$. Then $\operatorname{codim} Z = 3$ because $X \subset H_\alpha$ and $X_0 \subset H_\alpha$. The same proof is valid for $Y \in \mathcal{A}^\beta \setminus \{X_0\}$.

(B) We prove that each of the terms in (A) appears only once in C_2. Let $Z_1, Z_2 \in \mathcal{Z}$ and $X \in \mathcal{A}^\alpha \setminus \{X_0\}$. Assume that $X \supset Z_1$ and $X \supset Z_2$. Then $Z_1 = X \cap X_0 = Z_2$. The same proof is valid for $Y \in \mathcal{A}^\beta \setminus \{X_0\}$.

Thus we obtain $C_1 = C_2$. It is easy to verify the local-global formula of heights directly when the root system is either A_3, B_3 or C_3. Also the local-global formula for root systems of rank two is tautologically true. Thus we may assume the local-global formula for Φ_Z with $Z \in \mathcal{Z}$ and we compute

$$\begin{aligned}
C_1 = C_2 &= \sum_{Z\in\mathcal{Z}}\Big(\sum_{\substack{X\in\mathcal{A}^\alpha\setminus\{X_0\}\\ X\supset Z}}(\mathrm{ht}_X(\alpha)-1) - \sum_{\substack{Y\in\mathcal{A}^\beta\setminus\{X_0\}\\ Y\supset Z}}(\mathrm{ht}_Y(\beta)-1)\Big)\\
&= \sum_{Z\in\mathcal{Z}}\Big(\sum_{\substack{X\in\mathcal{A}^\alpha\\ X\supset Z}}(\mathrm{ht}_X(\alpha)-1) - \sum_{\substack{Y\in\mathcal{A}^\beta\\ Y\supset Z}}(\mathrm{ht}_Y(\beta)-1) - \mathrm{ht}_{X_0}(\alpha)+\mathrm{ht}_{X_0}(\beta)\Big)\\
&= \sum_{Z\in\mathcal{Z}}\big((\mathrm{ht}_{\Phi_Z}(\alpha)-1)-(\mathrm{ht}_{\Phi_Z}(\beta)-1)-1\big) = 0. \qquad \square
\end{aligned}$$

Corollary 4.2. *For $\alpha \in \Phi^+$, we have*

$$\mathrm{ht}_\Phi(\alpha) - 1 = \left|\left\{\{\beta_1,\beta_2\} \subseteq \Phi^+ \;\middle|\; \alpha \in \mathbb{Z}_{>0}\beta_1 + \mathbb{Z}_{>0}\beta_2\right\}\right|.$$

Proof. Let $X \in \mathcal{A}^\alpha$. Then $\Psi := \Phi_X$ is a root system of rank two, (A_2, $A_1 \times A_1$, B_2 or G_2), and we may directly verify that

$$\mathrm{ht}_\Psi(\alpha) - 1 = \left|\left\{\{\beta_1,\beta_2\} \subseteq \Psi^+ \;\middle|\; \alpha \in \mathbb{Z}_{>0}\beta_1 + \mathbb{Z}_{>0}\beta_2\right\}\right|.$$

Using the local-global formula (Theorem 4.1), we compute

$$\begin{aligned}
\mathrm{ht}_\Phi(\alpha) - 1 &= \sum_{X\in\mathcal{A}^\alpha}(\mathrm{ht}_X(\alpha)-1)\\
&= \sum_{X\in\mathcal{A}^\alpha}\left|\left\{\{\beta_1,\beta_2\} \subseteq \Phi^+\cap\Phi_X \;\middle|\; \alpha \in \mathbb{Z}_{>0}\beta_1 + \mathbb{Z}_{>0}\beta_2\right\}\right|\\
&= \left|\left\{\{\beta_1,\beta_2\} \subseteq \Phi^+ \;\middle|\; \alpha \in \mathbb{Z}_{>0}\beta_1 + \mathbb{Z}_{>0}\beta_2\right\}\right|. \qquad \square
\end{aligned}$$

Remark 4.3. When the root system Φ is simply-laced, Corollary 4.2 yields

$$\mathrm{ht}_\Phi(\alpha) - 1 = \left|\left\{\{\beta_1,\beta_2\} \subseteq \Phi^+ \;\middle|\; \alpha = \beta_1 + \beta_2\right\}\right|.$$

Proposition 4.4. *Let $I \subseteq \Phi^+$ be an ideal. Fix $\alpha \in I$ with $k+1 := \mathrm{ht}(\alpha) > 1$. Define*

$$\begin{aligned}
&\mathcal{B}' := \{H_\beta \mid \beta \in I,\ \mathrm{ht}(\beta) \le k\},\\
&\mathcal{B} := \mathcal{B}' \cup \{H_\alpha\}, \qquad \mathcal{B}'' := \mathcal{B}^{H_\alpha} = \{H \cap H_\alpha \mid H \in \mathcal{B}'\}.
\end{aligned}$$

Then $|\mathcal{B}'| - |\mathcal{B}''| = k$.

Proof. When $I = \Phi^+$ we denote the triple $(\mathcal{B}, \mathcal{B}', \mathcal{B}'')$ by $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$. Note that $\mathcal{B}''$ is a subset of $\mathcal{A}'' = \mathcal{A}^\alpha$. For $X \in \mathcal{A}''$, we will verify

$$\mathrm{ht}_X(\alpha) - 1 = \begin{cases} |\mathcal{B}_X| - 2 & \text{if } X \in \mathcal{B}'', \\ 0 & \text{otherwise}, \end{cases} \tag{4.2}$$

where $\mathcal{B}_X$ is the localization defined in (2.2). Recall the height distribution of Φ_X^+ is

$$i_1 = 2, i_2 = \cdots = i_n = 1 \quad (n = |\Phi_X^+| - 1).$$

Case 1. If $X \in \mathcal{B}''$, then $|\mathcal{B}_X| \geq 2$. Since $I_X := I \cap \Phi_X^+$ is an ideal of Φ_X^+ and $|I_X| = |\mathcal{B}_X| \geq 2$, I_X contains the simple system of Φ_X. This implies

$$I_X = \{\beta \in \Phi_X^+ \mid \mathrm{ht}_X(\beta) \leq \mathrm{ht}_X(\alpha)\} \quad \text{and} \quad |I_X| = \mathrm{ht}_X(\alpha) + 1.$$

Hence (4.2) holds in this case because

$$\mathrm{ht}_X(\alpha) - 1 = |I_X| - 2 = |\mathcal{B}_X| - 2.$$

Case 2. If $X \in \mathcal{A}'' \setminus \mathcal{B}''$, then $\mathcal{B}_X = \{H_\alpha\}$ and $I_X = \{\alpha\}$. Since I_X is an ideal of Φ_X^+, α is a simple root of Φ_X. Hence $\mathrm{ht}_X(\alpha) = 1$. This verifies (4.2).

Combining (4.2) with Theorem 4.1 we compute

$$\begin{aligned} |\mathcal{B}'| - |\mathcal{B}''| &= \sum_{X \in \mathcal{B}''} (|\mathcal{B}_X| - 2) = \sum_{X \in \mathcal{B}''} (\mathrm{ht}_X(\alpha) - 1) \\ &= \sum_{X \in \mathcal{A}''} (\mathrm{ht}_X(\alpha) - 1) = \mathrm{ht}_\Phi(\alpha) - 1 = k. \end{aligned}$$ $\square$

Remark 4.5. In particular, let $I = \Phi^+$, $\mathcal{A} = \mathcal{A}(\Phi^+)$ and let $\alpha \in \Phi^+$ be the highest root. Recall $\mathrm{ht}(\alpha) = h - 1$, where h is the Coxeter number of Φ. Then Proposition 4.4 gives a new proof of [8, Theorem 3.7]:

$$|\mathcal{A}| - |\mathcal{A}^\alpha| = 1 + |\mathcal{A}'| - |\mathcal{A}''| = h - 1$$

in the case of Weyl arrangements. This formula played a crucial role in [8].

Next we will verify the conditions (1) and (2) in the MAT. Both conditions concern positive roots of the same height. A subset A of Φ^+ is said to be an *antichain* if A is a subset of Φ^+ of mutually incomparable elements with respect to the partial order $\geq$ on Φ^+.

Lemma 4.6 (Panyushev [10, Proposition 2.10]). *Let Φ be a root system of rank ℓ and Δ be a simple system of Φ. Suppose that ℓ positive roots $\beta_1, \ldots, \beta_\ell$ form an antichain. Then $\Delta = \{\beta_1, \ldots, \beta_\ell\}$. In particular, $\beta_1, \ldots, \beta_\ell$ are linearly independent.*

Proposition 4.7. *Assume that $\beta_1, \ldots, \beta_q$ are distinct positive roots of the same height $k + 1$. Define*

$$X := \bigcap_{i=1}^{q} H_{\beta_i}.$$

Then

(1) *X is q-codimensional, and*
(2) *$X \not\subseteq \bigcup_{\alpha \in \Phi^+, \, \mathrm{ht}(\alpha) \leq k} H_\alpha$.*

Proof. (1) Since $\beta_1, \ldots, \beta_q$ are distinct positive roots of the same height, they form an antichain. Apply Lemma 4.6.

(2) Since $\beta_1, \ldots, \beta_q \in \Phi_X$ form an antichain and rank $\Phi_X = q$, Lemma 4.6 implies that they form the simple system of Φ_X. Assume that $X \subseteq H_\alpha$ with $\mathrm{ht}(\alpha) \leq k$. Then $\alpha \in \Phi_X$. So α can be expressed as a linear combination of $\beta_1, \ldots, \beta_q$ with nonnegative integer coefficients. As the heights of $\beta_1, \ldots, \beta_q$ are all $k+1$, this is a contradiction. □

5. Proof of Theorem 1.1

In this section we will complete the proof of Theorem 1.1 and its corollaries, and make a final remark.

Proof of Theorem 1.1. We use induction on

$$\mathrm{ht}(I) := \max\{\mathrm{ht}(\alpha) \mid \alpha \in I\}.$$

When $\mathrm{ht}(I) = 1$, $\mathcal{A}(I)$ is a Boolean arrangement. Hence there is nothing to prove.

Assume that $k+1 := \mathrm{ht}(I) > 1$. Define

$$I_j := \{\alpha \in I \mid \mathrm{ht}(\alpha) \leq j\}.$$

By definition, I_j is also an ideal for any $j \leq k+1$. By the induction hypothesis, Theorem 1.1 holds true for $I_1, \ldots, I_k$. In particular, $\mathcal{A}(I_k)$ is free with exponents

$$\exp(\mathcal{A}(I_k)) = (d_1, \ldots, d_\ell)$$

which coincide with $\mathcal{DP}(I_k)$. If we set $p := |I_k \setminus I_{k-1}|$, then the induction hypothesis shows that

$$d_1 \leq \cdots \leq d_{\ell-p} < d_{\ell-p+1} = \cdots = d_\ell = k.$$

Let $\{\beta_1, \ldots, \beta_q\} := I_{k+1} \setminus I_k$. Let $H_i := H_{\beta_i}$ and define $X := H_1 \cap \cdots \cap H_q$. Then Proposition 4.7 shows that codim $X = q$, and

$$X \nsubseteq \bigcup_{H \in \mathcal{A}(I_k)} H.$$

Also, Proposition 4.4 shows that $|\mathcal{A}(I_k)| - |(\mathcal{A}(I_k) \cup \{H_j\})^{H_j}| = k$ for any j. Hence all of the conditions (1)–(3) in the MAT are satisfied. Now apply the MAT to $\mathcal{A}(I) = \mathcal{A}(I_k) \cup \{H_1, \ldots, H_q\}$. □

Corollary 1.3 holds true because the set Φ_t is an ideal. Applying Theorem 2.1 to the ideal arrangement $\mathcal{A}(I)$, we get Corollaries 1.4 and 1.5.

Remark 5.1. Note that the product $\mathcal{A}_1 \times \mathcal{A}_2$ of two free arrangements $\mathcal{A}_1$ and $\mathcal{A}_2$ is again free, and $\exp(\mathcal{A}_1 \times \mathcal{A}_2)$ is the disjoint union of $\exp(\mathcal{A}_1)$ and $\exp(\mathcal{A}_2)$ by [9, Proposition 4.28]. Thus it is not hard to see that Theorem 1.1 and its corollaries hold true for all finite root systems including the reducible ones.

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