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Twists and resonance of L-functions, I

Abstract. We obtain the basic analytic properties, i.e. meromorphic continuation, polar structure and bounds for the order of growth, of all the nonlinear twists with exponents $\leq 1/d$ of the L-functions of any degree $d \geq 1$ in the extended Selberg class. In particular, this solves the resonance problem in all such cases.

Keywords. L-functions, Selberg class, twists, resonance

1. Introduction

Statement of results. Given an L-function $F(s)$ of positive degree d in the extended Selberg class $\mathcal{S}^\sharp$ (the class of Dirichlet series with meromorphic continuation and functional equation; see below for definitions and notation), in [7] we considered the *standard twist*

$$F(s, \alpha) = \sum_{n=1}^{\infty} \frac{a(n)e(-\alpha n^{1/d})}{n^s}, \quad \alpha \in \mathbb{R}, \ \alpha \neq 0, \ e(x) = e^{2\pi i x}, \qquad (1.1)$$

and obtained its main analytic properties; see [7, Theorems 1 and 2]. Precisely, writing

$$\mathrm{Spec}(F) = \{\alpha > 0 : a(n_\alpha) \neq 0\} \text{ where } n_\alpha = q d^{-d} \alpha^d \text{ and } a(n_\alpha) = \begin{cases} 0 & \text{if } n_\alpha \notin \mathbb{N}, \\ a(n_\alpha) & \text{if } n_\alpha \in \mathbb{N} \end{cases}$$

(here $q = q_F$ is the conductor of $F(s)$, see below), we proved that $F(s, \alpha)$ is an entire function if $|\alpha| \notin \mathrm{Spec}(F)$, while $F(s, \alpha)$ is meromorphic over $\mathbb{C}$ if $|\alpha| \in \mathrm{Spec}(F)$. In the latter case, $F(s, \alpha)$ has at most simple poles at the points

$$s_k = \frac{d+1}{2d} - \frac{k}{d} - i\theta_F, \quad k = 0, 1, \ldots, \qquad (1.2)$$

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and

$$\text{res}_{s=s_0} F(s, \alpha) = \frac{c_0(F)}{q^{s_0}} \frac{\overline{a(n_{|\alpha|})}}{n_{|\alpha|}^{1-s_0}}, \tag{1.3}$$

where $c_0(F) \neq 0$ is a certain constant depending only on $F(s)$, and θ_F is the internal shift of $F(s)$ (see below, and note a slight change in the definition of θ_F with respect to our previous papers). Moreover, in all cases $F(s, \alpha)$ has polynomial growth on vertical strips.

Remark 1. Actually, in [7] we considered only the case with $\alpha > 0$, but $F(s, -\alpha) = \overline{\overline{F}(\bar{s}, \alpha)}$ (see below for notation) and hence the case with $\alpha < 0$ follows at once, since $\theta_{\overline{F}} = -\theta_F$ and $c_0(\overline{F}) = \overline{c_0(F)}$.

The standard twist $F(s, \alpha)$ plays a relevant role in the Selberg class theory, and moreover is a new object in the theory of classical L-functions (i.e. the L-functions associated with algebraic, geometric and automorphic structures). For example, the properties of the standard twist were crucial in the classification of the degree 1 functions obtained in [4] and in the proof of the degree conjecture for $1 < d < 2$ given in [8]. In the latter paper we also studied the general *nonlinear twist*

$$F(s; f) = \sum_{n=1}^{\infty} \frac{a(n) e(-f(n, \boldsymbol{\alpha}))}{n^s} \tag{1.4}$$

with functions $f(n, \boldsymbol{\alpha})$ of type

$$f(n, \boldsymbol{\alpha}) = \sum_{j=0}^{N} \alpha_j n^{\kappa_j}, \quad 0 < \kappa_N < \cdots < \kappa_1 < \kappa_0, \ \alpha_j \in \mathbb{R}, \tag{1.5}$$

in the case $\kappa_0 > 1/d$ and $\alpha_0 > 0$. In [8] we could not get a full description of the analytic properties of such twists, but we obtained a useful transformation formula relating $F(s; f)$ to its dual twist $\overline{F}(s; -f^*)$. See [8, Theorem 1.1] for the precise statement, and see below for notation; again, note a slight difference in notation with respect to [8], this time concerning $\overline{F}(s; f)$.

The properties we prove in this paper for twists of type (1.4) allow us to solve, with a certain degree of generality, the *resonance problem* for the L-functions from the class $\mathcal{S}^{\sharp}$. Indeed, the resonance problem may be stated in general terms as describing under what circumstances the coefficients of an L-function, once suitably twisted and averaged, show cancellation (no resonance case) or an asymptotic behavior (resonance case). In an essentially equivalent form, the resonance problem may be stated as describing the evolution of meromorphic continuation, polar structure and order of growth of an L-function under a suitable set of twists. This is the form of the problem we deal with in this paper, and it is well known that results of the previous form can be deduced by standard methods. We shall discuss the problem in greater detail after the statement of our results.

In this paper we study the nonlinear twist (1.4) for functions of type (1.5) in the remaining cases, i.e. when $\kappa_0 = 1/d$ and at least one α_j is nonzero. For completeness, we also study the simpler case where all the exponents κ_j are negative. It turns out that

a technique different from the one we used in [7] yields a rather complete description of the analytic properties of $F(s; f)$ in these cases. The technique is closer to the one we made use of in [10] to deal with the linear twists in degree $d = 2$. Moreover, our present results provide further information in the special case (1.1) as well. We start with the new theorems in the special case, namely with general bounds on the growth of $F(s, \alpha)$ for $s \in \mathbb{C}$ (not only for s in vertical strips as in [7]), since these results will be needed in the proof of the general case. For completeness, in the statements of Theorems 1 and 2 below we also recall some of the properties already established in [7]. Since we deal with functions of degree $d > 0$, and it is known that there are no functions in $\mathcal{S}^\sharp$ with degree $0 < d < 1$ (see Conrey–Ghosh [1], and [4]), we may assume that $d \geq 1$. Denoting by $\mathcal{S}^\sharp_d$ the set of functions in $\mathcal{S}^\sharp$ with degree d, we have the following results.

Theorem 1. *Let $F \in \mathcal{S}^\sharp_d$ with $d \geq 1$ and let $\alpha \neq 0$, $|\alpha| \notin \operatorname{Spec}(F)$. Then $F(s, \alpha)$ is entire of order ≤ 1. Moreover, for every $0 < \delta < 1$ there exist $A, B, C > 0$, depending on $F(s)$ and δ, such that for every $s \in \mathbb{C}$,*

$$F(s, \alpha) \ll A^{|\sigma|}(1 + |s|)^{d|\sigma|/\delta + B} e^{C|s|^{\delta'}},$$

where $\delta' = \max(0, 2 - 1/\delta)$ and the $\ll$-constant may depend on $F(s)$, α and δ.

Theorem 2. *Let $F \in \mathcal{S}^\sharp_d$ with $d \geq 1$ and let $\alpha \neq 0$, $|\alpha| \in \operatorname{Spec}(F)$. Then $F(s, \alpha)$ is meromorphic on $\mathbb{C}$ with at most simple poles at the points s_k in (1.2), and residue at s_0 given by (1.3) for $\alpha > 0$. Moreover, for every $\delta_0 > 0$, $0 < \delta < 1$ and $\eta > 1/\delta$ there exist $A, B, C > 0$, depending on $F(s)$, δ_0, δ and η, such that for $|s - s_k| \geq \delta_0$,*

$$F(s, \alpha) \ll A^{|\sigma|}(1 + |s|)^{\eta d|\sigma| + B} e^{C(|s|^{\delta'} + |s|^{(3 - \eta\delta)/2})},$$

where $\delta' = \max(0, 2 - 1/\delta)$ and the $\ll$-constant may depend on $F(s)$, α, δ_0, δ and η.

Remark 2. Note that, if we choose e.g. $\delta = 1/2$ and $\eta = 6$, the bounds in Theorems 1 and 2 imply that $F(s, \alpha)$ has polynomial growth on vertical strips. However, the resulting bounds are essentially qualitative, although explicit values can be obtained for the constant B. In this paper we are not looking for sharp results in this respect; for example, one can immediately get sharper bounds by convexity. We refer to [7, Theorem 2] for α-uniform polynomial bounds, again quantitatively not sharp.

The proof of Theorems 1 and 2 forms the bulk of the paper. Indeed, the theorems together with Remark 5 below contain the basic bounds on the growth of $F(s, \alpha)$ needed to trigger a kind of iterative process (see Theorem 5 below), leading to the following general result.

Theorem 3. *Let $F \in \mathcal{S}^\sharp_d$ with $d \geq 1$ and let $f(n, \boldsymbol{\alpha})$ be as in (1.5) with $\kappa_0 = 1/d$ and $\alpha_1 \neq 0$. Then the nonlinear twist $F(s; f)$ in (1.4) is entire of order ≤ 1. Moreover, there exist $A, B, C > 0$ and $0 \leq \delta < 1$, depending on $F(s)$, such that for every $s \in \mathbb{C}$,*

$$F(s; f) \ll A^{|\sigma|}(1 + |s|)^{|\sigma|/\kappa_1 + B} e^{C|s|^{\delta}},$$

where the $\ll$-constant may depend on $F(s)$ and $f(n, \boldsymbol{\alpha})$.

Remark 3. At present we cannot prove in general that the twists $F(s; f)$ in Theorem 3 have polynomial growth on vertical lines; see the discussion below about the resonance problem for further information on this point.

Theorem 3 is a consequence of Theorems 1, 2 and 5, and its proof is given after the statement of Theorem 5. For completeness we state and prove a simpler and more general result concerning the nonlinear twists with negative exponents. We present this result in a form suitable to be coupled with the previous results by an iterative argument (see Remark 7 below). Let $F(s)$ be an absolutely convergent Dirichlet series for $\sigma > 1$, with meromorphic continuation to $\mathbb{C}$ and singularities contained in a horizontal strip of finite height. Moreover, let

$$f(n, \boldsymbol{\alpha}) = \sum_{j=0}^{N} \alpha_j n^{-\kappa_j}, \quad 0 < \kappa_0 < \cdots < \kappa_N, \, \alpha_j \in \mathbb{R}.$$

Denote by $\mu_F(\sigma)$ (resp. $\mu_F(\sigma; f)$) the Lindelöf μ-function of $F(s)$ (resp. $F(s; f)$), if it exists.

Theorem 4. *Let $F(s)$ and $f(n, \boldsymbol{\alpha})$ be as above. Then the twist $F(s; f)$ in (1.4) is meromorphic on $\mathbb{C}$ with singularities in the same horizontal strip as for $F(s)$, and $F(s; f)$ is entire if $F(s)$ is entire. Moreover*

$$\mu_F(\sigma; f) = \mu_F(\sigma)$$

for σ in any right half-line where $\mu_F(\sigma)$ exists.

Remark 4. From the proof of Theorem 4 it is easy to detect the location of the poles of $F(s; f)$ from the location of the poles of $F(s)$ (see (4.1)).

In order to state the results allowing the iterative process leading to Theorem 3, we need to introduce further notation. For $\rho \geq 1$ and $\tau \geq 0$ let $M(\rho, \tau)$ be the class of Dirichlet series $F(s)$, absolutely convergent for $\sigma > 1$, admitting holomorphic continuation to $|t| \geq \tau$ and for which there exist $A, B, C > 0$ and $0 \leq \delta < 1$, all depending on $F(s)$, such that for $|t| \geq \tau$,

$$F(s) \ll A^{|\sigma|}(1 + |s|)^{\rho|\sigma|+B} e^{C|s|^{\delta}}, \tag{1.6}$$

where the $\ll$-constant may depend on $F(s)$, ρ and τ. Moreover, we denote simply by $M(\rho)$ the class $M(\rho, 0)$ and write for $\lambda > 0$ and $\alpha \in \mathbb{R}$,

$$F^{\lambda}(s, \alpha) = \sum_{n=1}^{\infty} \frac{a(n)e(-\alpha n^{\lambda})}{n^s};$$

thus in particular $F(s, \alpha) = F^{1/d}(s, \alpha)$. The relevance of the class $M(\rho, \tau)$ is clarified by Theorems 1 and 2, and by the following

Remark 5. If $F \in \mathcal{S}_d^{\sharp}$ then $F \in M(d, \tau)$ for every $\tau > 0$. In fact, $F(s)$ is holomorphic except possibly at $s = 1$ (thus the entire functions $F \in \mathcal{S}_d^{\sharp}$ belong to $M(d)$). Moreover $F(s)$ is bounded for $\sigma \geq 2$, is of polynomial growth (in particular) for $-1 \leq \sigma \leq 2$ and for $\sigma \leq -1$ satisfies

$$F(s) \ll A^{|\sigma|}(1 + |s|)^{d(|\sigma|+1/2)},$$

thanks to the functional equation and Stirling's formula; see e.g. [10, Lemma 2.1].

Theorem 5. *If $F \in M(\rho, \tau)$ with $\tau > 0$, and if $0 < \lambda < 1/\rho$ and $\alpha \neq 0$, then $F^\lambda(s, \alpha)$ belongs to $M(1/\lambda)$. Moreover, if $F \in M(\rho)$ and $0 < \lambda < 1/\rho$, then $F^\lambda(s, \alpha)$ belongs to $M(\rho)$ for every $\alpha \in \mathbb{R}$.*

Remark 6. When $F \in M(\rho)$ and $0 < \lambda < 1/\rho$, the entire function $F^\lambda(s, \alpha)$ has the representation

$$F^\lambda(s, \alpha) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} F(s - \lambda k)(2\pi i \alpha)^k,$$

the series being absolutely and uniformly convergent for s in compact subsets of $\mathbb{C}$; see the proof of Theorem 5.

Proof of Theorem 3. As we already noticed, Theorem 3 is a direct consequence of Theorems 1, 2 and 5. Indeed, if $\alpha_0 \neq 0$ we start with $F(s, \alpha_0)$. Then we choose δ (and η) in Theorems 1 or 2, depending on $|\alpha_0| \notin \mathrm{Spec}(F)$ or $|\alpha_0| \in \mathrm{Spec}(F)$, in such a way that $F(s, \alpha_0)$ belongs to $M(\rho, \tau)$ for some $\tau > 0$ and $d < \rho < 1/\kappa_1$. If $\alpha_0 = 0$ we use Remark 5, asserting that $F \in M(d, \tau)$ for every $\tau > 0$. Thus, for any $\alpha_0 \in \mathbb{R}$ the function $F(s, \alpha_0)$ belongs to $M(\rho, \tau)$ with some $\tau > 0$ and $d \leq \rho < 1/\kappa_1$. Then we apply the first part of Theorem 5 with $\lambda = \kappa_1$ to $F(s, \alpha_0)$, thus deducing that

$$\sum_{n=1}^{\infty} \frac{a(n)}{n^s} e(-\alpha_0 n^{1/d} - \alpha_1 n^{\kappa_1})$$

belongs to $M(1/\kappa_1)$. Now we apply iteratively the second part of Theorem 5 with $\lambda = \kappa_2, \kappa_3, \ldots$, and Theorem 3 follows.

Remark 7. Thanks to Theorem 4, a similar, but simpler, iterative argument may be applied to any of the twists $F(s, \alpha)$ or $F(s; f)$ considered in Theorems 1–3, thus giving meromorphic continuation and polar structure (see Remark 4) of any twist of type (1.4) with

$$f(n, \boldsymbol{\alpha}) = \sum_{j=0}^{N} \alpha_j n^{\kappa_j}$$

with $\alpha_j \in \mathbb{R}$ and $\kappa_N < \cdots < \kappa_1 < \kappa_0 = 1/d$.

Remark 8. We finally remark that, as far as we know, the results in Theorems 1–3 are new also in the case of classical *L*-functions. It is interesting to note how the behavior in s of the functions involved, essentially as $\sigma \to -\infty$, appears to be critical in order to deduce the properties of their nonlinear twists. This complements the well known importance of the behavior of *L*-functions on vertical strips. A similar phenomenon already arises in [10], where the behavior of the linear twists as $\sigma \to -\infty$ is shown to give control on the shape of the Euler product; see [10, Theorem 1].

The resonance problem. Theorems 1–3 provide the basic analytic properties of the nonlinear twists $F(s; f)$ in all cases where $f(n, \boldsymbol{\alpha})$ has positive exponents $\leq 1/d$. Actually, the same holds if all exponents are $\leq 1/d$ thanks to Theorem 4 (see Remark 7), but in what follows we restrict to positive exponents for simplicity. In particular, the polar structure and the order of growth of $F(s; f)$ are determined by the above theorems for any function $F \in \mathcal{S}_d^\sharp$ in the following form:

(i) if $f(n, \boldsymbol{\alpha}) = \alpha_0 n^{1/d}$ with $|\alpha_0| \in \mathrm{Spec}(F)$, then $F(s; f)$ has a simple pole at $s = s_0$, possible simple poles at $s = s_k$ with $k \geq 1$ (see (1.2)) and polynomial growth on vertical lines;

(ii) if $f(n, \boldsymbol{\alpha}) = \alpha_0 n^{1/d}$ with $0 \neq |\alpha_0| \notin \mathrm{Spec}(F)$, then $F(s; f)$ is entire and has polynomial growth on vertical lines;

(iii) if $f(n, \boldsymbol{\alpha}) = \alpha_0 n^{1/d} + \alpha_1 n^{\kappa_1} + \cdots$ with any $\alpha_0 \in \mathbb{R}$, $\alpha_1 \neq 0$ and $\kappa_1 < 1/d$, then $F(s; f)$ is entire, and on any fixed vertical strip,

$$F(s; f) \ll \exp(|t|^\delta) \tag{1.7}$$

with some $\delta < 1$.

These results solve the resonance problem, in the second form stated above, for all nonlinear twists with exponents $\leq 1/d$ of any function of degree $d \geq 1$ from $\mathcal{S}^\sharp$. Moreover, in cases (i) and (ii) standard techniques can be used to describe the behavior of the smoothed nonlinear exponential sums

$$S_F(x; f, \phi) = \sum_{n=1}^{\infty} a(n) e(-f(n, \boldsymbol{\alpha})) \phi(n/x), \tag{1.8}$$

where $\phi(u)$ is a smooth function on $(0, \infty)$ with compact support. More precisely, we have

$$S_F(x; f, \phi) = \sum_{k \leq K} c_k(F; f) \widetilde{\phi}(s_k) x^{s_k} + O(x^{(d-1)/2d - K/d}), \quad x \to \infty, \tag{1.9}$$

for any fixed $K \geq 0$, where $\widetilde{\phi}(s)$ is the Mellin transform of $\phi(u)$, $c_0(F; f) \neq 0$ in case (i) and $c_k(F; f) = 0$ for $k \geq 0$ in case (ii). Interesting applications would follow from suitable uniform bounds in $\boldsymbol{\alpha}$, but at present such bounds are definitely weak; see e.g. [7, Theorem 2]. Formula (1.9) follows from the estimate

$$\widetilde{\phi}(s) \ll |t|^{-h}, \quad |t| \to \infty, \tag{1.10}$$

for every fixed $h > 0$, uniformly on any fixed vertical strip. In turn, (1.10) can be obtained from the definition of $\widetilde{\phi}(s)$ by repeated partial integrations:

$$\widetilde{\phi}(s) = \frac{(-1)^h}{s(s+1) \cdots (s+h-1)} \int_0^\infty \phi^{(h)}(u) u^{s+h-1} \, du.$$

In case (iii), when only (1.7) is available, (1.10) is too weak to deal with the sums (1.8). However, in this case we can describe, again by standard techniques, the behavior of smoothed nonlinear exponential sums

$$S_F(x; f, r) = \sum_{n=1}^{\infty} a(n) e(-f(n, \boldsymbol{\alpha})) e^{-(n/x)^r}$$

with $r > 0$ arbitrary. In this case, for any fixed $K \geq 0$ we have

$$S_F(x; f, r) = \sum_{k \leq K} \frac{(-1)^k}{k!} F(-kr; f) x^{-kr} + O(x^{-(K+1)r}), \quad x \to \infty.$$

This is due to the fact that the Mellin transform of e^{-u^r} is $(1/r)\Gamma(s/r)$, and by Stirling's formula

$$\Gamma(s/r) \ll e^{-\pi|t|/2r}|t|^{c(a,b)}$$

uniformly for $|t| \geq 1$ and $a < \sigma < b$.

We omit explicit examples for Theorems 1 to 5 since one may easily construct such examples starting with any classical *L*-function. In a forthcoming paper [11] we shall study the properties and resonance of nonlinear twists of type (1.4) and (1.5) with $\kappa_0 > 1/d$.

Definitions and notation. Given a function $f(s)$ we write $\overline{f}(s) = \overline{f(\overline{s})}$; in particular, if $f(s)$ is a Dirichlet series then $\overline{f}(s)$ is the Dirichlet series with conjugate coefficients, called the *conjugate* of $f(s)$. However, when dealing with the twists of a function $F(s)$ we write

$$\overline{F}(s; f) = \sum_{n=1}^{\infty} \frac{\overline{a(n)}e(-f(n, \boldsymbol{\alpha}))}{n^s},$$

i.e. $\overline{F}(s; f)$ is the twist of the conjugate of $F(s)$. A completely analogous notation is used in the case of $F^\lambda(s, \alpha)$. A function $F(s)$ belongs to the *Selberg class* $\mathcal{S}$ if

  (i) $F(s)$ is an absolutely convergent Dirichlet series for $\sigma > 1$;
 (ii) $(s - 1)^m F(s)$ is an entire function of finite order for some integer $m \geq 0$;
(iii) $F(s)$ satisfies a functional equation of type $\Phi(s) = \omega\overline{\Phi}(1 - s)$, where $|\omega| = 1$ and

$$\Phi(s) = Q^s \prod_{j=1}^{r} \Gamma(\lambda_j s + \mu_j) F(s)$$

    with $r \geq 0$, $Q > 0$, $\lambda_j > 0$, $\Re\mu_j \geq 0$;
 (iv) the Dirichlet coefficients $a(n)$ of $F(s)$ satisfy $a(n) \ll n^\varepsilon$ for every $\varepsilon > 0$;
  (v) $\log F(s)$ is a Dirichlet series with coefficients $b(n)$ satisfying $b(n) = 0$ unless $n = p^m$, $m \geq 1$, and $b(n) \ll n^\vartheta$ for some $\vartheta < 1/2$.

The *extended Selberg class* $\mathcal{S}^\sharp$ consists of the nonzero functions satisfying only axioms (i)–(iii). The *degree*, *conductor* and *ξ-invariant* of $F \in \mathcal{S}^\sharp$ are defined respectively by

$$d_F = 2\sum_{j=1}^{r} \lambda_j, \quad q_F = (2\pi)^{d_F} Q^2 \prod_{j=1}^{r} \lambda_j^{2\lambda_j}, \quad \xi_F = 2\sum_{j=1}^{r} (\mu_j - 1/2) = \eta_F + i\theta_F d_F;$$

note the slight change in notation for θ_F, the internal shift of $F(s)$, with respect to our previous papers. We refer to Selberg [16] and Conrey–Ghosh [1], to our survey papers [5], [3], [13], [14], [15] and to our forthcoming book [12] for the basic information and results on the Selberg class.

In the next sections, the constants $A, B, C > 0$ are sufficiently large and may depend on $F(s)$ and on the parameters specified in the theorems. Moreover, such constants may also depend on other parameters appearing in the lemmas below; if this happens, it will be explicitly stated. The same applies to the constants implicit in the O- and $\ll$-notation, as well as to the constants $c, c_1, \ldots > 0$. In all cases, the value of such constants may not be the same at each occurrence. If $F \in \mathcal{S}_d^\sharp, d \geq 1$, and $\alpha > 0$ are fixed we write

$$F_X(s, \alpha) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s} e^{-z_X n^{1/d}}$$

where $X \geq X_0 > 0$ is an integer, X_0 is sufficiently large and may depend on $F(s)$ and other parameters, as for the above constants A, B, C, and

$$z_X = 2\pi\alpha\omega_X, \quad \omega_X = 1/X + i.$$

Clearly, $F_X(s, \alpha)$ converges for every $s \in \mathbb{C}$ and hence $F_X(s, \alpha)$ is an entire function,

$$\lim_{X \to \infty} F_X(s, \alpha) = F(s, \alpha), \quad \sigma > 1,$$

and by Mellin's transform we have

$$F_X(s, \alpha) = \frac{1}{2\pi i} \int_{(c)} F(s + w/d)\Gamma(w)z_X^{-w} \, dw, \quad c > \max(0, d(1 - \sigma)), \quad (1.11)$$

where $w = u + iv, z_X^{-w} = e^{-w \log z_X}$ and $\log z_X$ is meant to be the principal value. Let $\sigma \leq 3/2, c = d(3 - \sigma)$,

$$M_F = \max_{1 \leq j \leq r} (1 + |\mu_j|)^2/\lambda_j, \quad K = [3d(2 + M_F)] + 1/2$$

and $V \geq V_0 > 0$ be a parameter to be chosen in the proofs, where V_0 is sufficiently large and may depend on $F(s)$ and other parameters, as for the above constants A, B, C. Finally, let

$$\mathcal{L}_{-\infty} = (d(2 - \sigma) - i\infty, d(2 - \sigma) + iV],$$
$$\mathcal{L}_V = [d(2 - \sigma) + iV, -K + iV],$$
$$\mathcal{L}_\infty = [-K + iV, -K + i\infty).$$

Outline of the proofs. To prove Theorems 1 and 2 we start with (1.11), with the aim of letting $X \to \infty$. The integral over the negative part of the line $\sigma = c$ has good convergence properties. Hence we deform the integration as in (2.1), thus avoiding the poles of the integrand and preparing for the use of the functional equation. The first two terms in (2.1) are dealt with by Lemma A in Section 2, which holds for every $\alpha > 0$ and whose proof is based on a direct application of Stirling's formula. In the integral over $\mathcal{L}_\infty$ we apply the functional equation and expand $\overline{F}(1 - s - w/d)$, thus getting (2.4). The integral $I_X(s, y)$ in (2.4) depends on the data of $F(s)$ and is close to an incomplete hypergeometric function. We refer to [7, Section 2] for an analysis of such functions

(see Theorem 2.1 there). In particular, it turns out that the limit as $X \to \infty$ of such hypergeometric functions is meromorphic in s for every $y > 0$; moreover, it has a simple pole at $s = s_0$ (and possibly at the points $s = s_k$ with $k \geq 1$) for $y = d/\beta^{1/d}$ (see (2.3)), and is holomorphic otherwise. This behavior gives rise to the notion of $\mathrm{Spec}(F)$, and clarifies why the treatment of the function $F_X^{(3)}(s)$ in (2.1) is different depending on whether $|\alpha| \in \mathrm{Spec}(F)$ or not. As already remarked, in this paper we follow a different approach to the study of $I_X(s, y)$, leading to the new estimates in Theorems 1 and 2 required by Theorems 3 and 5. In view of Remark 1 we deal only with the case $\alpha > 0$.

When $0 < \alpha \notin \mathrm{Spec}(F)$ we have $a(n_\alpha) = 0$, and we deal separately with the integrals $I_X(s, y_n)$ in (2.4) with $n < n_\alpha$ and $n > n_\alpha$. In both cases the treatment is based on a change of the path of integration and on a careful application of the uniform version of Stirling's formula proved in [9]; see Lemma D in Section 2. This leads to Lemma B in Section 2. In turn, Lemmas A and B allow the use of Vitali's convergence theorem when $X \to \infty$, and Theorem 1 follows; see the proof of Theorem 1 in Section 2.

When $\alpha \in \mathrm{Spec}(F)$ we have the additional term $n = n_\alpha$, and the corresponding integral $I_X(s, d/\beta^{1/d})$ requires a deeper analysis. The starting point is Lemma 3.1, where the Γ-factors coming from the functional equation are transformed to a single Γ-factor plus smaller order terms. Then in a series of lemmas (from 3.2 to 3.8) we study and transform the expression of $I_X(s, d/\beta^{1/d})$ coming from Lemma 3.1, finally getting Lemma E in Section 3, where the remainder term is of the required form. Next we borrow some arguments from our previous treatment of the standard twist in [4] and [7], giving the explicit expression in (3.29) and describing the polar structure of the limit as $X \to \infty$ of the integrals $I_{X,\nu}(s)$ in Lemma E. Theorem 2 then follows from Vitali's convergence theorem as before, and from bounds of the required form for the expression in (3.29), away from its poles.

We already proved Theorem 3, and the proof of Theorem 4 follows by the Taylor expansion of $e(-f(n, \boldsymbol{\alpha}))$ and the good convergence properties of the resulting series. The proof of Theorem 5 is simpler than the proof of Theorems 1 and 2 thanks to the good convergence properties of the integral in (1.11), due to the choice $\lambda < 1/\rho$. Indeed, when $F(s)$ is entire and belongs to $M(\rho)$, we shift the line of integration in (1.11) to the left, thus getting a sum over the residues of $\Gamma(w)$ (see (5.13)). Thanks to $\lambda < 1/\rho$, as $X \to \infty$ we obtain an expression of $F^\lambda(s, \alpha)$ as a series (see (5.14)), which is nicely convergent over $\mathbb{C}$. Moreover, the required bound for $F^\lambda(s, \alpha)$ follows by plugging in the bound for $F(s - \lambda k)$ into (5.14). The case of meromorphic $F(s)$ is technically more delicate, but the good convergence properties of the integrals under consideration, again due to the choice $\lambda < 1/\rho$, are decisive in this case as well.

2. Proof of Theorem 1

We follow the notation at the end of Section 1 and suppose that $F(s)$ and α are as in Theorem 1. In view of Remark 1 of Section 1 we may assume that $\alpha > 0$. Thanks to the polynomial growth of $F(s + w/d)$ and the decay of $\Gamma(w)$ on vertical strips, and to the

location of their poles, we have

$$F_X(s, \alpha) = \frac{1}{2\pi i} \int_{\mathcal{L}_{-\infty} \cup \mathcal{L}_V \cup \mathcal{L}_\infty} F(s + w/d)\Gamma(w)z_X^{-w} \, dw = F_X^{(1)}(s) + F_X^{(2)}(s) + F_X^{(3)}(s),$$

(2.1)

say. Recalling that $\delta' = \max(0, 2 - 1/\delta)$ we have

Lemma A. *Let $\sigma \leq 3/2$ and $V \geq V_0(1 + |s|)$. Under the hypotheses of Theorem 1 and with the notation of Section 1 we have*

$$|F_X^{(1)}(s)| + |F_X^{(2)}(s)| \ll A^{|\sigma|}V^{d|\sigma|+5d}$$

uniformly for $X \geq X_0$.

Lemma B. *Let $\sigma \leq 3/2$ and $V \geq V_0(1 + |s|)^{1/\delta}$. Under the hypotheses of Theorem 1 and with the notation of Section 1 we have*

$$|F_X^{(3)}(s)| \ll A^{|\sigma|}V^{d|\sigma|+B}e^{C(|s|^{\delta'}+|s|V^{\delta-1})}$$

uniformly for $X \geq X_0$.

Before proving Lemmas A and B we recall Vitali's convergence theorem (see Titchmarsh [17, Section 5.21]), and show that Theorem 1 in an immediate consequence of Lemmas A, B and Vitali's theorem.

Lemma C (Vitali's convergence theorem). *Let $f_X(z)$ be a sequence of holomorphic functions on a region $\mathcal{D}$ and let $|f_X(z)| \leq M$ for every $X \geq X_0$ and $z \in \mathcal{D}$. Suppose that $f_X(z)$ tends to a limit, as $X \to \infty$, on a set of points having an accumulation point in $\mathcal{D}$. Then $f_X(z)$ tends to a limit uniformly in any domain $\mathcal{D}'$ whose closure is contained in $\mathcal{D}$, and hence the limit is holomorphic and bounded by M on $\mathcal{D}'$.*

Proof of Theorem 1. The result is obvious for $\sigma \geq 5/4$, while for $\sigma < 5/4$ we recall that $F_X(s, \alpha)$ is holomorphic and we apply Lemmas A and B with the choice $V = V_0(1 + |s|)^{1/\delta}$ to get

$$F_X(s, \alpha) \ll A^{|\sigma|}(1 + |s|)^{d|\sigma|/\delta+B}e^{C|s|^{\delta'}}$$

(2.2)

uniformly for $X \geq X_0$. Given any $s = \sigma + it$ with $\sigma < 5/4$ we consider the region

$$\mathcal{D}_s = \{z = x + iy \in \mathbb{C} : \sigma - 1/10 < x < 3/2, \ |y| < |t| + 1/10\}$$

and apply Lemma C with the choice $f_X(z) = F_X(z, \alpha)$ and $\mathcal{D} = \mathcal{D}_s$. Thanks to (2.2) and since $|z| \leq |s| + c$, $|x| \leq |\sigma| + c$, we may choose

$$M = KA^{|\sigma|}(1 + |s|)^{d|\sigma|/\delta+B}e^{C|s|^{\delta'}}$$

with a suitable constant $K > 0$. Recalling that $\lim_{X \to \infty} F_X(z, \alpha) = F(z, \alpha)$ for every z with $x > 1$, since $s \in \mathcal{D}_s$ we deduce from Lemma C that $F(s, \alpha)$ has holomorphic continuation to the half-plane $\sigma < 5/4$ and

$$F(s, \alpha) \ll A^{|\sigma|}(1 + |s|)^{d|\sigma|/\delta+B}e^{C|s|^{\delta'}}.$$

Theorem 1 is thus proved. $\square$

Proof of Lemma A. We start with the estimation of $F_X^{(1)}(s)$. For $w = u + iv \in \mathcal{L}_{-\infty}$ we have $u = d(2 - \sigma)$, hence $\Re(s + w/d) = 2$ and therefore $F(s + w/d) \ll 1$. Moreover, by Stirling's formula we have

$$\Gamma(w) \ll A^{|\sigma|}(1 + |w|)^{d(2-\sigma)-1/2} e^{-v \arg w},$$

and since $v \arg w = v \arctan \frac{v}{d(2-\sigma)} = |v|(\pi/2 + O((1 + |\sigma|)/|v|))$, we get

$$\Gamma(w) \ll A^{|\sigma|}(1 + |w|)^{d(2-\sigma)-1/2} e^{-\pi |v|/2}.$$

Further we have

$$|z_X^{-w}| = e^{-d(2-\sigma)\log |z_X| + v \arg z_X} \ll A^{|\sigma|} e^{v(\pi/2 - O(1/X))}.$$

Therefore for $V \geq V_0(1 + |s|)$ and $X \geq X_0$ we obtain

$$F_X^{(1)}(s) \ll A^{|\sigma|} \int_{-\infty}^{V} e^{-\pi |v|/2 + v(\pi/2 - O(1/X))} \big(1 + d(2 + |\sigma|) + |v|\big)^{d(2+|\sigma|)} dv$$

$$\ll A^{|\sigma|} \left(\int_{-V}^{V} V^{d(2+|\sigma|)} dv + \int_{-\infty}^{-V} e^{-|v|(\pi - O(1/X))} (1 + |v|)^{d(2+|\sigma|)} dv \right)$$

$$\ll A^{|\sigma|} V^{d|\sigma|+2d+1} + A^{|\sigma|} \Gamma(d(2 + |\sigma|) + 1) \ll A^{|\sigma|} \big(V^{d|\sigma|+2d+1} + (1 + |\sigma|)^{d|\sigma|}\big)$$

uniformly in X, and the first assertion of Lemma A follows.

In order to estimate $F_X^{(2)}(s)$ we recall the estimate for $F(s)$ in Remark 5 of the Introduction, which we recall in the form

$$F(s) \ll \begin{cases} A^{|\sigma|}(1 + |s|)^{d(|\sigma|+1/2)} & \text{if } \sigma \leq -1, \\ (1 + |s|)^{3d/2} & \text{if } \sigma \geq -1. \end{cases}$$

Accordingly, we split the path of integration into

$$[-K + iV, -d(1 + \sigma) + iV] \cup [-d(1 + \sigma) + iV, d(2 - \sigma) + iV],$$

thus getting

$$F_X^{(2)}(s) \ll A^{|\sigma|} \int_{-K}^{-d(1+\sigma)} V^{d(|\sigma+u/d|+1/2)} |\Gamma(u + iV) z_X^{-u-iV}| du$$

$$+ V^{3d/2} \int_{-d(1+\sigma)}^{d(2-\sigma)} |\Gamma(u + iV) z_X^{-u-iV}| du.$$

Arguing as before and taking into account the definition of V and K we have

$$|\Gamma(u + iV) z_X^{-u-iV}| \ll A^{|\sigma|} V^{u-1/2}$$

uniformly in X, hence

$$F_X^{(2)}(s) \ll A^{|\sigma|} \left(\int_{-K}^{-d(1+\sigma)} V^{d(|\sigma+u/d|+1/2)+u-1/2} du + \int_{-d(1+\sigma)}^{d(2-\sigma)} V^{(3d+2u-1)/2} du \right)$$

$$\ll A^{|\sigma|} V^{d|\sigma|+5d}$$

since $\sigma + u/d < 0$ in the first integral, and Lemma A follows. $\qquad \square$

Remark. Note that the proof of Lemma A does not depend on the hypothesis that $\alpha \notin \mathrm{Spec}(F)$, hence Lemma A holds for any $\alpha > 0$. The proof of Lemma B is definitely more delicate, and condition $\alpha \notin \mathrm{Spec}(F)$ becomes crucial.

The following uniform version of Stirling's formula (see [9]) will be repeatedly used in this paper, hence for convenience we state it as

Lemma D. *Let $N \geq 0$ be an integer, $D \geq 1$ and let $z, a \in \mathbb{C}$ satisfy*

$$\Re(z + a) \geq 0, \quad |a| \leq \tfrac{3}{5}|z|, \quad N \leq 2D|z|.$$

Then

$$\log \Gamma(z + a) = \left(z + a - \frac{1}{2}\right) \log z - z + \frac{1}{2} \log 2\pi$$

$$+ \sum_{j=1}^{N} \frac{(-1)^{j+1} B_{j+1}(a)}{j(j+1)} \frac{1}{z^j} + O\left(\frac{1}{|z|^{N+1}}\left(\left(N + \frac{|a|^2}{(N+1)^2}\right)|a|^N + D^N N!\right)\right).$$

Proof of Lemma B. For $w \in \mathcal{L}_\infty$ we have $\Re(s + w/d) = \sigma - K/d \leq 3/2 - 6 + 1/2d \leq -4$. Hence, writing

$$h(w, s) = \prod_{j=1}^{r} \frac{\Gamma(\lambda_j(1 - s) + \overline{\mu}_j - \lambda_j w/d)}{\Gamma(\lambda_j s + \mu_j + \lambda_j w/d)} \Gamma(w),$$

applying the functional equation and using the Dirichlet series expansion of $\bar{F}(1 - s - w/d)$ we get

$$F_X^{(3)}(s) = \frac{\omega Q^{1-2s}}{2\pi i} \int_{\mathcal{L}_\infty} \bar{F}(1 - s - w/d) h(w, s) Q^{-2w/d} z_X^{-w} \, dw$$

$$\ll A^{|\sigma|} \sum_{n=1}^{\infty} \frac{|a(n)|}{n^{1-\sigma}} \left| \int_{\mathcal{L}_\infty} h(w, s) \left(\frac{Q^{2/d} z_X}{n^{1/d}}\right)^{-w} dw \right|.$$

Recalling that $z_X = 2\pi\alpha\omega_X$ and $\omega_X = 1/X + i$, for convenience we write

$$I_X(s, y) = \frac{1}{2\pi i} \int_{\mathcal{L}_\infty} h(w, s)(y\omega_X)^{-w} \, dw, \quad y > 0,$$

hence recalling the definition of n_α in the Introduction and letting

$$\beta = \prod_{j=1}^{r} \lambda_j^{2\lambda_j}, \quad y_n = \left(\frac{n_\alpha}{n}\right)^{1/d} \frac{d}{\beta^{1/d}}, \tag{2.3}$$

we have

$$F_X^{(3)}(s) \ll A^{|\sigma|} \sum_{n=1}^{\infty} \frac{|a(n)|}{n^{1-\sigma}} |I_X(s, y_n)|. \tag{2.4}$$

Since $\alpha \notin \mathrm{Spec}(F)$ we may assume that $n \neq n_\alpha$ always. In the rest of the proof we obtain suitable bounds for $I_X(s, y_n)$ in the two cases $n > n_\alpha$ and $n < n_\alpha$, which, once inserted in (2.4), will prove Lemma B. Actually, in what follows we deal with $I_X(s, y)$ in the general cases $0 < y < d/\beta^{1/d}$ and $y > d/\beta^{1/d}$.

Case $0 < y < d/\beta^{1/d}$. In this case we change the path of integration from $\mathcal{L}_\infty$ to the half-line $\mathcal{L}'_\infty$ where $w = u + iV$, $-\infty < u \leq -K$, and consider the 90-degree sector $\mathcal{R}$ formed by $\mathcal{L}_\infty$ and $\mathcal{L}'_\infty$. Thanks to the choice of K, the function $h(w, s)$ is holomorphic on $\mathcal{R}$, as also is $(y\omega_X)^{-w}$. Moreover, by Cauchy's theorem applied to that sector and the arc γ_R of the circle $|w| = R$ inside the sector, as $R \to \infty$ we obtain

$$I_X(s, y) = \frac{1}{2\pi i} \int_{\mathcal{L}'_\infty} h(w, s)(y\omega_X)^{-w} \, dw.$$

Indeed, a standard application of the Stirling formula with $|w| = R$ shows that the contribution of the integral over the arc γ_R tends to 0 as $R \to \infty$.

We first deduce from Lemma D the following useful uniform bound for $|\Gamma(z + a)|$. Let $0 < \delta < 1$ and

$$\Re(z + a) \geq 0, \quad |a| \leq \min\left(|z|^\delta, \tfrac{3}{5}|z|\right), \quad |z| \geq 1/2. \tag{2.5}$$

Then for $\delta' = \max(0, 2 - 1/\delta)$ we have

$$|\Gamma(z + a)| = e^{-\Im(z+a)\arg z - \Re z}|z|^{\Re(z+a)-1/2} e^{O(1+|a|^{\delta'})}. \tag{2.6}$$

Indeed, from Lemma D with $N = 0$ and $D = 1$ we get

$$\log|\Gamma(z + a)| = \Re\{(z + a - 1/2)(\log|z| + i \arg z) - z\} + O(|a|^2/|z| + 1),$$

and (2.6) follows since the last O-term is $O(1 + |a|^{\delta'})$.

Now we proceed with the estimation of $h(w, s)$ for $w \in \mathcal{L}'_\infty$. First we use the reflection formula for the Γ function to write

$$h(w, s) = \pi^{-r} S(w, s) \prod_{j=1}^r \Gamma(\lambda_j(1-s) + \overline{\mu}_j - \lambda_j w/d)\Gamma(1 - \lambda_j s - \mu_j - \lambda_j w/d)\Gamma(w) \tag{2.7}$$

with

$$S(w, s) = \prod_{j=1}^r \sin \pi(\lambda_j s + \mu_j + \lambda_j w/d).$$

In order to estimate the first Γ-factor in the product in (2.7), we choose $z = -\lambda_j w/d = (\lambda_j/d)(|u| - iV)$ and $a = \lambda_j(1 - s) + \overline{\mu}_j$ and note that conditions (2.5) are satisfied thanks to the choice of K, provided

$$V \geq c(1 + 1/\gamma)^{1/\delta}(1 + |s|)^{1/\delta}, \quad \text{where} \quad \gamma = |\log(y\beta^{1/d}/d)| \tag{2.8}$$

and $c = c(F, \delta) > 0$ is sufficiently large. Indeed, simple computations show that $|z| \geq \lambda_j K/d \geq 3$, and

$$|z| \geq \lambda_j V/d \geq \frac{\lambda_j c}{d}(1 + |s|)^{1/\delta} \geq \max\big(|a|^{1/\delta}, \tfrac{5}{3}|a|\big),$$
$$\Re(z + a) = \lambda_j |u|/d + \lambda_j(1 - \sigma) + \Re\mu_j \geq \lambda_j K/d - \lambda_j/2 - |\mu_j| \geq 3.$$

Hence we may apply (2.6). With the above choice of z and a we have

$$\Im(z + a) = -\lambda_j(V/d + t) + O(1), \quad \arg z = -\arctan(V/|u|),$$
$$\Re z = \lambda_j |u|/d, \quad \Re(z + a) = \lambda_j(|u|/d + |\sigma|) + O(1),$$

hence from (2.6) we obtain

$$\Gamma(\lambda_j(1 - s) + \overline{\mu}_j - \lambda_j w/d)$$
$$\ll e^{-\lambda_j(V/d+t)\arctan(V/|u|)-\lambda_j|u|/d}|\lambda_j w/d|^{\lambda_j(|u|/d+|\sigma|)+B}e^{C(1+|s|^{\delta'})}. \quad (2.9)$$

To estimate the second Γ-factor in the product in (2.7) we choose again $z = -\lambda_j w/d$ and $a = 1 - \mu_j - \lambda_j s$. As before, simple computations show that (2.6) is applicable and the same formulae hold for the quantities involved. Therefore,

$$\Gamma(1 - \lambda_j s - \mu_j - \lambda_j w/d)$$
$$\ll e^{-\lambda_j(V/d+t)\arctan(V/|u|)-\lambda_j|u|/d}|\lambda_j w/d|^{\lambda_j(|u|/d+|\sigma|)+B}e^{C(1+|s|^{\delta'})}. \quad (2.10)$$

The third Γ-factor in (2.7) is estimated by first applying the reflection formula and then (2.6) (although the standard Stirling formula would suffice here) with the choice $z = |u| - iV$ and $a = 1$, thus obtaining

$$\Gamma(w) \ll \frac{e^{-\pi V}}{|\Gamma(1 + |u| - iV)|} \ll e^{-\pi V + V\arctan(V/|u|)+|u|}|w|^{-|u|-1/2}. \quad (2.11)$$

Further we have

$$S(w, s) \ll e^{\frac{\pi}{2}d(t+V/d)}, \quad (2.12)$$

and hence from (2.7) and (2.9)–(2.12), under (2.8) and recalling (2.3), we obtain

$$h(w, s) \ll A^{|\sigma|}e^{-d(V/d+t)\arctan(V/|u|)-|u|}|w|^{d(|u|/d+|\sigma|)+B}e^{\frac{\pi}{2}d(t+V/d)}\beta^{|u|/d}d^{-|u|}$$
$$\times e^{-\pi V + V\arctan(V/|u|)+|u|}|w|^{-|u|-1/2}e^{C(1+|s|^{\delta'})}$$
$$\ll A^{|\sigma|}e^{dt(\pi/2-\arctan(V/|u|))}e^{-\pi V/2}|w|^{d|\sigma|+B}\beta^{|u|/d}d^{-|u|}e^{C(1+|s|^{\delta'})}. \quad (2.13)$$

In order to estimate $I_X(s, y)$ we observe that, for $w \in \mathcal{L}'_\infty$,

$$|(y\omega_X)^{-w}| \ll |y\omega_X|^{|u|}e^{\pi V/2},$$

and hence from (2.13) we get

$$I_X(s, y) \ll A^{|\sigma|} e^{C(1+|s|^{\delta'})} \int_K^\infty \left(\frac{\beta^{1/d}}{d} y |\omega_X|\right)^u e^{dt(\pi/2-\arctan(V/u))} |w|^{d|\sigma|+B} \, du$$

$$= A^{|\sigma|} e^{C(1+|s|^{\delta'})} \left(\int_K^{V^\delta} + \int_{V^\delta}^\infty\right) \left(\frac{\beta^{1/d}}{d} y |\omega_X|\right)^u e^{dt(\pi/2-\arctan(V/u))} |w|^{d|\sigma|+B} \, du$$

$$= I_X^{(1)}(s, y) + I_X^{(2)}(s, y), \tag{2.14}$$

say; note that $(\beta^{1/d}/d)y < 1$ in this case. We deal first with $I_X^{(1)}(s, y)$, observing that for $0 < u < V^\delta$ we have $\arctan(V/u) = \pi/2 + O(V^{\delta-1})$. Therefore, if $X > c/\gamma$ with a suitable $c > 0$ then $(\beta^{1/d}/d)y|\omega_X| < 1$ and hence

$$I_X^{(1)}(s, y) \ll A^{|\sigma|} e^{C(1+|s|^{\delta'}+|s|V^{\delta-1})} V^{d|\sigma|+B} \int_K^\infty e^{-\gamma_X u} \, du$$

$$\ll A^{|\sigma|} e^{C(1+|s|^{\delta'}+|s|V^{\delta-1})} V^{d|\sigma|+B} y^K / \gamma_X, \tag{2.15}$$

where $\gamma_X = \left|\log\left(\frac{\beta^{1/d}}{d} y |\omega_X|\right)\right|$.

Concerning $I_X^{(2)}(s, y)$, again for $X > c/\gamma$ we have

$$I_X^{(2)}(s, y) \ll A^{|\sigma|} e^{C(1+|s|^{\delta'})} e^{d\pi|t|/2} \left(\int_{V^\delta}^\infty e^{-\gamma_X u} u^{d|\sigma|+B} \, du + V^{d|\sigma|+B} \int_{V^\delta}^\infty e^{-\gamma_X u} \, du\right), \tag{2.16}$$

and we use the following general result: for $1 \le \xi \le Y$ we have

$$\int_Y^\infty e^{-u} u^\xi \, du \le e^{-Y} Y^\xi (1 + \xi^2/Y). \tag{2.17}$$

Indeed, integrating by parts twice we have

$$\int_Y^\infty e^{-u} u^\xi \, du \le e^{-Y} Y^\xi + \xi e^{-Y} Y^{\xi-1} + \xi(\xi-1) \int_Y^\infty e^{-u} u^\xi \frac{du}{u^2}.$$

But the function $u \mapsto e^{-u} u^\xi$ is decreasing for $u \ge \xi$ and hence

$$\int_Y^\infty e^{-u} u^\xi \frac{du}{u^2} \le e^{-Y} Y^\xi \int_Y^\infty \frac{du}{u^2} = e^{-Y} Y^{\xi-1},$$

therefore

$$\int_Y^\infty e^{-u} u^\xi \, du \le e^{-Y} Y^\xi \left(1 + \frac{\xi}{Y} + \frac{\xi(\xi-1)}{Y}\right)$$

and (2.17) follows. We use (2.17) in the first integral in (2.16) after the change of variable $\gamma_X u \to u$, hence with the choice $Y = V^\delta \gamma_X$ and $\xi = d|\sigma| + B$. Recalling (2.8) and the definition of γ_X after (2.15) we see that

$$Y \ge c^\delta (1 + |s|) \gamma_X / \gamma \ge c^\delta (1 + |s|) \ge d|s| + B = \xi$$

if c^δ is large enough, hence

$$\int_{V^\delta}^\infty e^{-\gamma_X u} u^{d|\sigma|+B} \, du = \frac{1}{\gamma_X^{d|\sigma|+B+1}} \int_{V^\delta \gamma_X}^\infty e^{-u} u^{d|\sigma|+B} \, du$$

$$\ll \frac{e^{-V^\delta \gamma_X}(V^\delta \gamma_X)^{d|\sigma|+B}}{\gamma_X^{d|\sigma|+B+1}}(1+|\sigma|) = e^{-V^\delta \gamma_X} V^{\delta(d|\sigma|+B)} \frac{1+|\sigma|}{\gamma_X}.$$

Therefore, recalling that $\delta' < 1$, (2.16) becomes

$$I_X^{(2)}(s, y) \ll A^{|\sigma|} e^{C(1+|s|)} \left(e^{-V^\delta \gamma_X} V^{\delta(d|\sigma|+B)} \frac{1+|\sigma|}{\gamma_X} + V^{d|\sigma|+B} \frac{e^{-V^\delta \gamma_X}}{\gamma_X} \right)$$

$$\ll \frac{1}{\gamma_X} e^{C(1+|s|)-V^\delta \gamma_X} V^{d|\sigma|+B}.$$

But, as before, if c^δ is large enough we have

$$\frac{1}{2} V^\delta \gamma_X \geq \frac{1}{2} c^\delta (1+|s|) \geq C(1+|s|) \quad \text{and} \quad \frac{1}{2} V^\delta \geq \frac{c^\delta}{2\gamma}(1+|s|) \geq K,$$

hence, recalling the definition of γ_X after (2.15) and $(\beta^{1/d}/d)y|\omega_X| < 1$ for $X > c/\gamma$, we obtain

$$I_X^{(2)}(s, y) \ll \frac{1}{\gamma_X} V^{d|\sigma|+B} y^K. \tag{2.18}$$

Gathering (2.14), (2.15) and (2.18) we finally obtain, in the case under consideration, that uniformly for $X > c/\gamma$,

$$I_X(s, y) \ll A^{|\sigma|} e^{C(|s|^{\delta'}+|s|V^{\delta-1})} V^{d|\sigma|+B} y^K / \gamma, \tag{2.19}$$

since $1/\gamma_X \ll 1/\gamma$ for $X > c/\gamma$ (and the $\ll$-constant has the same features as the constant C).

Case $y > d/\beta^{1/d}$**.** In this case we change the path of integration $\mathcal{L}_\infty$ to the half-line $\mathcal{L}_\infty''$ where $w = u + iV$, $-K \leq u < \infty$, and arguing as in the previous case we obtain

$$I_X(s, y) = \frac{1}{2\pi i} \int_{\mathcal{L}_\infty''} h(w, s)(y\omega_X)^{-w} \, dw.$$

Moreover, writing $d_0 = d \max_{1 \leq j \leq r} |\mu_j|/\lambda_j$ we find that for $-K \leq u < -d\sigma - d_0$ (recall that $\sigma \leq 3/2$)

$$\Re(\lambda_j s + \mu_j + \lambda_j w/d) < 0 \quad \text{and} \quad \Re(\lambda_j(1-s) + \overline{\mu}_j - \lambda_j w/d) > 0,$$

and we split the integral over $\mathcal{L}_\infty''$ as

$$I_X(s, y) = \frac{1}{2\pi i} \left(\int_{-K+iV}^{-d\sigma-d_0+iV} + \int_{-d\sigma-d_0+iV}^{V^\delta+iV} + \int_{V^\delta+iV}^{\infty+iV} \right) h(w, s)(y\omega_X)^{-w} \, dw$$

$$= I_X^{(3)}(s, y) + I_X^{(4)}(s, y) + I_X^{(5)}(s, y), \tag{2.20}$$

say.

The treatment of $I_X^{(3)}(s, y)$ is similar to the treatment of the previous case, in the sense that we start from (2.7) and use (2.6) to estimate the first two Γ-factors in the product (with the choice $z = -\lambda_j w/d$ and $a = \lambda_j(1-s) + \overline{\mu}_j$ or $a = 1 - \lambda_j s - \mu_j$, respectively). A computation shows that, assuming (2.8), for $w \in \mathcal{L}''_\infty$ the conditions in (2.5) are satisfied, and

$$\Gamma(\lambda_j(1-s) + \overline{\mu}_j - \lambda_j w/d) \ll e^{\lambda_j(V/d+t)\arg(-w)+\lambda_j u/d}|\lambda_j w/d|^{-\lambda_j(\sigma+u/d)+B}e^{C(1+|s|^{\delta'})}$$

$$\Gamma(1 - \lambda_j s - \mu_j - \lambda_j w/d) \ll e^{\lambda_j(V/d+t)\arg(-w)+\lambda_j u/d}|\lambda_j w/d|^{-\lambda_j(\sigma+u/d)+B}e^{C(1+|s|^{\delta'})}.$$

Moreover

$$S(w, s) \ll e^{\frac{\pi}{2}d(V/d+t)}$$

and by Stirling's formula

$$|\Gamma(w)| \ll e^{-V\arg w - u}|w|^{u-1/2}.$$

Therefore, similarly to (2.13) and observing that $\arg(-w) - \arg(w) = -\pi$, we obtain

$$h(w, s) \ll A^{|\sigma|}e^{C(1+|s|^{\delta'})}e^{dt(\pi/2+\arg(-w))}e^{-\pi V/2}|w|^{-d\sigma+B}(d/\beta^{1/d})^u,$$

and hence, since $|(y\omega_X)^{-w}| \leq (y|\omega_X|)^{-u}e^{\pi V/2}$, we get

$$I_X^{(3)}(s, y) \ll A^{|\sigma|}e^{C(1+|s|^{\delta'})}\int_{-K}^{-d\sigma-d_0} e^{dt(\pi/2+\arg(-w))}|w|^{-d\sigma+B}\left(\frac{d}{\beta^{1/d}y|\omega_X|}\right)^u du.$$

But, thanks to (2.8), $|w| \ll V$ and $\arg(-w) = -\pi/2 + O(V^{\delta-1})$. Thus, recalling the definition of γ_X after (2.15) and that $y > d/\beta^{1/d}$ in this case (and so $d/(\beta^{1/d}y|\omega_X|) < 1$), we have

$$I_X^{(3)}(s, y) \ll A^{|\sigma|}e^{C(1+|s|^{\delta'}+|s|V^{\delta-1})}V^{d|\sigma|+B}\int_{-K}^{\infty} e^{-\gamma_X u} du$$

$$\ll A^{|\sigma|}e^{C(1+|s|^{\delta'}+|s|V^{\delta-1})}V^{d|\sigma|+B}y^K/\gamma_X. \tag{2.21}$$

In order to treat $I_X^{(4)}(s, y)$ we use the reflection formula for the Γ function to write

$$h(w, s) = \pi^r \Gamma(w)\widetilde{S}(w, s)\prod_{j=1}^{r} \frac{1}{\Gamma(\lambda_j s + \mu_j + \lambda_j w/d)\Gamma(1 - \lambda_j(1-s) - \overline{\mu}_j + \lambda_j w/d)}$$

with

$$\widetilde{S}(w, s) = \prod_{j=1}^{r} \frac{1}{\sin \pi(\lambda_j(1-s-w/d) + \overline{\mu}_j)}.$$

By the factorial formula for the Γ function we rewrite $h(w, s)$ as

$$h(w, s) = \pi^r \Gamma(w)\widetilde{S}(w, s)P(s + w/d)\prod_{j=1}^{r} \frac{1}{\Gamma(a_j(s) + \lambda_j w/d)\Gamma(b_j(s) + \lambda_j w/d)} \tag{2.22}$$

where $P \in \mathbb{C}[z]$ has $\deg P = 2r\nu_0$,

$$a_j(s) = \lambda_j s + \mu_j + \nu_0, \qquad b_j(s) = 1 - \lambda_j(1-s) - \overline{\mu}_j + \nu_0,$$

and $\nu_0 = \nu_0(F) \in \mathbb{N}$ is such that for $u \geq -d\sigma - d_0$,

$$\Re(a_j(s) + \lambda_j w/d), \Re(b_j(s) + \lambda_j w/d) \geq 0.$$

Now we apply (2.6) with $z = \lambda_j w/d$ and $a = a_j(s), b_j(s)$ respectively. A computation shows that conditions (2.5) are satisfied thanks to (2.8) and to the choice of ν_0, and

$$\Gamma(a_j(s) + \lambda_j w/d) \gg e^{-\lambda_j(t+V/d)\arg w - \lambda_j u/d}|\lambda_j w/d|^{\lambda_j(\sigma+u/d)+B}e^{C(1+|s|^{\delta'})},$$

$$\Gamma(b_j(s) + \lambda_j w/d) \gg e^{-\lambda_j(t+V/d)\arg w - \lambda_j u/d}|\lambda_j w/d|^{\lambda_j(\sigma+u/d)+B}e^{C(1+|s|^{\delta'})}.$$

Moreover, thanks to (2.8) and to Stirling's formula we have

$$\widetilde{S}(w,s) \ll e^{-\frac{\pi}{2}d(V/d+t)}, \quad P(s+w/d) \ll |w|^B, \quad \Gamma(w) \ll e^{-V\arg w - u}|w|^{u-1/2},$$

hence from (2.22) we obtain

$$h(w,s) \ll A^{|\sigma|}e^{C(1+|s|^{\delta'})}e^{-\pi V/2 + dt(\arg w - \pi/2)}|w|^{-d\sigma+B}(d/\beta^{1/d})^u.$$

As before we have $|(y\omega_X)^{-w}| \leq (y|\omega_X|)^{-u}e^{\pi V/2}$, thus for $u \geq -d\sigma - d_0$,

$$h(w,s)(y\omega_X)^{-w} \ll A^{|\sigma|}e^{C(1+|s|^{\delta'})}e^{dt(\arg w - \pi/2)}|w|^{-d\sigma+B}(\beta^{1/d}y|\omega_X|/d)^{-u}. \quad (2.23)$$

But for $-d\sigma - d_0 \leq u \leq V^{\delta}$ we have $\arg w = \pi/2 + O(V^{\delta-1})$ and $|w| \ll V$, therefore recalling the definition of γ_X after (2.15) and that $y > d/\beta^{1/d}$ we obtain

$$I_X^{(4)}(s,y) \ll A^{|\sigma|}e^{C(1+|s|^{\delta'}+|s|V^{\delta-1})}V^{d|\sigma|+B}\int_{-K}^{\infty}e^{-u\gamma_X}\,du$$

$$\ll A^{|\sigma|}e^{C(1+|s|^{\delta'}+|s|V^{\delta-1})}V^{d|\sigma|+B}y^K/\gamma_X. \quad (2.24)$$

Finally, since $\arg w \leq \pi/2 + O(V^{\delta-1})$ for $u \geq V^{\delta}$, from (2.23) we get

$$I_X^{(5)}(s,y) \ll A^{|\sigma|}e^{C(1+|s|^{\delta'}+|s|V^{\delta-1})}\int_{V^{\delta}}^{\infty}e^{-u\gamma_X}(u^{d|\sigma|+B} + V^{d|\sigma|+B})\,du,$$

and by the same argument used to estimate $I_X^{(2)}(s,y)$ we obtain

$$I_X^{(5)}(s,y) \ll A^{|\sigma|}e^{C(1+|s|^{\delta'}+|s|V^{\delta-1})}V^{d|\sigma|+B}y^K/\gamma_X. \quad (2.25)$$

From (2.20), (2.21), (2.24) and (2.25) we deduce, in the case under consideration, that uniformly for $X > c/\gamma$ (see after (2.19)),

$$I_X(s,y) \ll A^{|\sigma|}e^{C(|s|^{\delta'}+|s|V^{\delta-1})}V^{d|\sigma|+B}y^K/\gamma, \quad (2.26)$$

and in view of (2.19) we infer that (2.26) holds for every $y > 0$.

We are now ready to conclude the proof of Lemma B. From (2.3), (2.4), (2.8) and (2.26) we have, uniformly for $X > c/\gamma$,

$$F_X^{(3)}(s) \ll A^{|\sigma|} e^{C(|s|^{\delta'} + |s|V^{\delta-1})} V^{d|\sigma|+B} \sum_{n=1}^{\infty} \frac{|a(n)|}{n^{1-\sigma} n^{K/d} |\log(n_\alpha/n)|}.$$

Lemma B follows since $n \neq n_\alpha$ and the series is convergent for every $\sigma \leq 3/2$.

3. Proof of Theorem 2

In this section we follow the notation of Section 1 and suppose that $F(s), \alpha, \delta_0, \delta, \eta$ are as in Theorem 2. Moreover, we always assume (including in the statement of the lemmas) that $\sigma \leq 3/2$ and $|s| \leq R$ with an arbitrary $R \geq 1$. The initial steps of the proof of Theorem 2 are identical to those of Theorem 1. Indeed, we borrow Lemma A, whose conclusions remain unchanged under the hypotheses of Theorem 2; see the remark after the proof of Lemma A. Moreover, the conclusions of Lemma B also remain unchanged under the hypotheses of Theorem 2, provided we replace $F_X^{(3)}(s)$ by $\widetilde{F}_X^{(3)}(s)$, where, recalling (2.3) and the definition of $I_X(s, y)$ before (2.3),

$$\widetilde{F}_X^{(3)}(s) = \omega Q^{1-2s} \sum_{n \neq n_\alpha} \frac{\overline{a(n)}}{n^{1-s}} I_X(s, y_n).$$

Therefore, with the notation of Section 2 we have, for $\sigma \leq 3/2$,

$$F_X(s, \alpha) = F_X^{(1)}(s) + F_X^{(2)}(s) + \widetilde{F}_X^{(3)}(s) + \omega Q^{1-2s} \frac{\overline{a(n_\alpha)}}{n_\alpha^{1-s}} I_X(s, d/\beta^{1/d}) \qquad (3.1)$$

and, provided $V \geq V_0(1 + |s|)^{1/\delta}$,

$$F_X^{(1)}(s) + F_X^{(2)}(s) + \widetilde{F}_X^{(3)}(s) \ll A^{|\sigma|} V^{d|\sigma|+B} e^{C(|s|^{\delta'} + |s|V^{\delta-1})} \qquad (3.2)$$

uniformly for $X \geq X_0$. The rest of the proof is devoted to the evaluation and estimation of $I_X(s, d/\beta^{1/d})$, which for convenience we denote simply by $I_X(s)$.

For $w \in \mathcal{L}_\infty$, as in (2.7) we write

$$h(w, s) = S(s + w/d) \Gamma(w) \prod_{j=1}^{r} \Gamma(a_j - \lambda_j w/d) \Gamma(b_j - \lambda_j w/d)$$

with

$$S(z) = \pi^{-r} \prod_{j=1}^{r} \sin \pi(\lambda_j z + \mu_j), \quad a_j = \lambda_j(1-s) + \overline{\mu}_j, \quad b_j = 1 - \mu_j - \lambda_j s.$$

We first note that a standard calculation based on the expression $\sin z = \frac{e^{iz} - e^{-iz}}{2i}$ gives for $w \in \mathcal{L}_\infty$ (recall that $v \geq V \geq |s|$)

$$S(s + w/d) = \frac{1}{(2\pi)^r} e^{-i\frac{\pi}{2}\xi_F} e^{-id\frac{\pi}{2}(s+w/d)} (1 + O(e^{-\lambda_0 v})) \qquad (3.3)$$

with some $0 < \lambda_0 \leq 1$. We recall that the H-invariants of a function $F \in \mathcal{S}^\sharp$ are defined as

$$H_F(n) = 2 \sum_{j=1}^{r} \frac{B_n(\mu_j)}{\lambda_j^{n-1}}, \quad n = 0, 1, \ldots,$$

where $B_n(x)$ is the n-th Bernoulli polynomial; see [6] for the properties of such invariants. Denoting by $R_\nu(s)$ the polynomials

$$R_\nu(s) = 2^\nu (B_{\nu+1}(a + 1/2) - B_{\nu+1}(1/2 - b))$$
$$+ \frac{d^\nu}{2} \sum_{k=0}^{\nu+1} \binom{\nu+1}{k} \left((-1)^\nu H_F(k) s^{\nu+1-k} - \overline{H_F(k)}(1-s)^{\nu+1-k} \right), \quad (3.4)$$

where $a = a(s) = \frac{d}{2}(1 - s) + \frac{1}{2}\overline{\xi_F}$ and $b = b(s) = \frac{d}{2}s + \frac{1}{2}\xi_F$, we have

Lemma 3.1. *Let $w \in \mathcal{L}_\infty$, $|s| \leq R$, $\sigma \leq 3/2$, $1 \leq M \leq DR$ and $V \geq (c_1 R)^{1/\delta}$, where $D > 0$ is arbitrary and c_1 may depend also on D. Then*

$$h(w, s) = c_0(F) \frac{\Gamma(a + 1/2 - w/2)}{\Gamma(b + 1/2 + w/2)} \Gamma(w) \left(\frac{d}{2\beta^{1/d}} \right)^{ds+w} \exp\left(\sum_{\nu=1}^{M} \frac{R_\nu(s)}{\nu(\nu+1)} \frac{1}{w^\nu} + \rho_M \right)$$

with

$$\rho_M \ll c_2^M \frac{R^{M+2}}{|w|^{M+1}} + e^{-\lambda_0 v},$$

where $c_0(F) \neq 0$, $\lambda_0 > 0$ is a certain constant, c_2 may depend also on D, and we may assume that $c_1 \geq c_2$.

Proof. We use Lemma D of Section 2 with $z = -\lambda_j w/d$, $a = a_j$ or $a = b_j$ and $N = M$ to get an asymptotic expansion of $\Gamma(a_j - \lambda_j w/d)\Gamma(b_j - \lambda_j w/d)$. Since $u = -K$ and $\sigma \leq 3/2$, and thanks to the restrictions on V and M, for $w \in \mathcal{L}_\infty$ the hypotheses of Lemma D are satisfied, hence

$$\log \Gamma(a_j - \lambda_j w/d)\Gamma(b_j - \lambda_j w/d) = (\lambda_j - 2\lambda_j(s + w/d) - 2i\Im\mu_j) \log(-w/2d)$$
$$+ (\lambda_j - 2\lambda_j(s + w/d) - 2i\Im\mu_j) \log 2\lambda_j + 2\lambda_j w/d + \log 2\pi$$
$$- \sum_{\nu=1}^{M} d^\nu \frac{B_{\nu+1}(a_j) + B_{\nu+1}(b_j)}{\nu(\nu+1)} \frac{1}{\lambda_j^\nu w^\nu} + O(\rho_{M,j})$$

where, thanks to the restriction on V,

$$\rho_{M,j} \ll \left(\frac{d}{\lambda_j |w|} \right)^{M+1} \left(\left(M + \frac{(\lambda_j |s| + B)^2}{M^2} \right)(\lambda_j |s| + B)^M + B^M M! \right) \ll c^M \frac{R^{M+2}}{|w|^{M+1}}.$$

Therefore, summing over $j = 1, \ldots, r$ we obtain

$$\log \prod_{j=1}^{r} (\Gamma(a_j - \lambda_j w/d) \Gamma(b_j - \lambda_j w/d)) = (d/2 - ds - w - id\theta_F) \log(-w/2) + c_1(F)$$

$$+ (ds + w) \log \frac{d}{2\beta^{1/d}} + w - \sum_{\nu=1}^{M} \frac{\widetilde{R}_\nu(s)}{\nu(\nu+1)} \frac{1}{w^\nu} + O\left(c^M \frac{R^{M+2}}{|w|^{M+1}}\right) \quad (3.5)$$

with a certain $c_1(F)$ and

$$\widetilde{R}_\nu(s) = d^\nu \sum_{j=1}^{r} \frac{B_{\nu+1}(a_j) + B_{\nu+1}(b_j)}{\lambda_j^\nu}.$$

Now we note that

$$\frac{\Gamma(a + 1/2 - w/2)}{\Gamma(b + 1/2 + w/2)} = \frac{\sin(\pi(b + 1/2 + w/2))}{\pi} \Gamma(a + 1/2 - w/2)\Gamma(1/2 - b - w/2),$$

and, since $w \in \mathcal{L}_\infty$ and hence $\Im(b + 1/2 + w/2) > 0$,

$$\sin(\pi(b + 1/2 + w/2)) = -\frac{1}{2i} e^{-i\pi(b+1/2+w/2)}(1 + O(e^{-\nu}))$$

$$= \frac{1}{2} e^{-i\frac{\pi}{2}\xi_F} e^{-i\frac{\pi d}{2}(s+w/d)}(1 + O(e^{-\nu})).$$

Moreover, arguing as before, from Lemma D we get

$$\log \Gamma(a+1/2-w/2)\Gamma(1/2-b-w/2) = (d/2-ds-w-id\theta_F)\log(-w/2)+w+\log 2\pi$$

$$- \sum_{\nu=1}^{M} \frac{2^\nu \left(B_{\nu+1}(a + 1/2) - B_{\nu+1}(1/2 - b)\right)}{\nu(\nu+1)} \frac{1}{w^\nu} + O\left(c^M \frac{R^{M+2}}{|w|^{M+1}}\right).$$

Hence, with a certain $c_2(F)$,

$$\log \frac{\Gamma(a+1/2-w/2)}{\Gamma(b+1/2+w/2)} = -i\frac{\pi d}{2}(s+w/d)+(d/2-ds-w-id\theta_F)\log(-w/2)+w+c_2(F)$$

$$- \sum_{\nu=1}^{M} \frac{2^\nu \left(B_{\nu+1}(a + 1/2) - B_{\nu+1}(1/2 - b)\right)}{\nu(\nu+1)} \frac{1}{w^\nu} + O\left(c^M \frac{R^{M+2}}{|w|^{M+1}} + e^{-\nu}\right). \quad (3.6)$$

Recalling the definition of $h(w, s)$ and (3.3), comparing (3.5) and (3.6) we obtain

$$\log h(w, s) = c_3(F) \log \frac{\Gamma(a + 1/2 - w/2)}{\Gamma(b + 1/2 + w/2)} \log \Gamma(w)(ds + w) \log \frac{d}{2\beta^{1/d}}$$

$$+ \sum_{\nu=1}^{M} \frac{R_\nu^*(s)}{\nu(\nu+1)} \frac{1}{w^\nu} + O\left(c^M \frac{R^{M+2}}{|w|^{M+1}} + e^{-\lambda_0 \nu}\right) \quad (3.7)$$

with a certain $c_3(F)$ and

$$R_\nu^*(s) = 2^\nu (B_{\nu+1}(a+1/2) - B_{\nu+1}(1/2-b)) - d^\nu \sum_{j=1}^r \frac{B_{\nu+1}(a_j) + B_{\nu+1}(b_j)}{\lambda_j^\nu}; \quad (3.8)$$

Lemma 3.1 follows from (3.7) as soon as we prove that $R_\nu^*(s) = R_\nu(s)$. But from the properties of the Bernoulli polynomials (see [2, Section 1.13], we have

$$\sum_{j=1}^r \frac{B_{\nu+1}(a_j) + B_{\nu+1}(b_j)}{\lambda_j^\nu} = \sum_{j=1}^r \frac{B_{\nu+1}(\lambda_j(1-s) + \overline{\mu}_j) + (-1)^{\nu+1} B_{\nu+1}(\lambda_j s + \mu_j)}{\lambda_j^\nu}$$

$$= \sum_{j=1}^r \sum_{k=0}^{\nu+1} \frac{\binom{\nu+1}{k} \lambda_j^{\nu+1-k} B_{\nu+1}(\overline{\mu}_j)(1-s)^{\nu+1-k} + (-1)^{\nu+1} \binom{\nu+1}{k} \lambda_j^{\nu+1-k} B_{\nu+1}(\mu_j) s^{\nu+1-k}}{\lambda_j^\nu}$$

$$= \sum_{k=0}^{\nu+1} \binom{\nu+1}{k} \left(\tfrac{1}{2} \overline{H_F(k)}(1-s)^{\nu+1-k} + (-1)^{\nu+1} \tfrac{1}{2} H_F(k) s^{\nu+1-k} \right)$$

$$= -\frac{1}{2} \sum_{k=0}^{\nu+1} \binom{\nu+1}{k} \left((-1)^\nu H_F(k) s^{\nu+1-k} - \overline{H_F(k)}(1-s)^{\nu+1-k} \right),$$

hence $R_\nu^*(s) = R_\nu(s)$ in view of (3.4) and (3.8); see also [10, Lemma 3.3].   □

The treatment of $\exp\left(\sum_{\nu=1}^M \frac{R_\nu(s)}{\nu(\nu+1)} \frac{1}{w^\nu} + \rho_M \right)$ in Lemma 3.1 is partly similar to the arguments in [10, Section 3], but the differences are such that we cannot simply quote the results. However, we will be a bit more sketchy in our treatment, and we shall refer to [10] whenever possible.

Lemma 3.2. *Let $R_\nu(s)$ be as in (3.4) and $1 \leq \nu \leq DR$, where $D > 0$ is arbitrary. Then*

$$R_\nu(s) \ll (c_3 R)^{\nu+1}$$

where c_3 may depend also on D.

Proof. For $d = 2$ this is essentially [10, Lemma 3.8] (notice the slightly different definition of $R_\nu(s)$ in [10]), and the proof in the general case is similar.   □

Lemma 3.3. *Under the hypotheses and with the notation of Lemma* 3.1*, and assuming in addition that $\frac{2\delta-1}{1-\delta} \leq M$, we have*

$$h(w,s) = c_0(F) \frac{\Gamma(a+1/2-w/2)}{\Gamma(b+1/2+w/2)} \Gamma(w) \left(\frac{d}{2\beta^{1/d}} \right)^{ds+w} \exp\left(\sum_{\nu=1}^M \frac{R_\nu(s)}{\nu(\nu+1)} \frac{1}{w^\nu} \right)$$

$$+ O\left(A^{|\sigma|} e^{O(R^2/V)} \frac{c_4^M R^{M+2} |\Gamma(w)|}{|w|^{M+d\sigma-K-d/2+1}} \right),$$

where A and c_4 may also depend on D.

Proof. By Lemma 3.1 we have

$$\rho_M \ll \frac{(c_2 R)^{M+2}}{(c_1 R)^{(M+1)/\delta}} + e^{-\lambda_0 v} \ll R^{M+2-(M+1)/\delta} + e^{-\lambda_0 v},$$

and thanks to the restriction $\frac{2\delta-1}{1-\delta} \le M$ we see that the exponent of R is negative. Therefore $\rho_M \ll 1$ and hence $\exp(\rho_M) = 1 + O(\rho_M)$. Moreover, for $w \in \mathcal{L}_\infty$ we have

$$e^{-\lambda_0 v} \ll e^{-\lambda_0 |w|} \ll \frac{(M+1)!}{(\lambda_0 |w|)^{M+1}} \ll \left(\frac{M+1}{\lambda_0 |w|}\right)^{M+1} \ll \frac{(cR)^{M+2}}{|w|^{M+1}}.$$

Hence in order to prove the lemma we have to show that

$$\left| \frac{\Gamma(a+1/2-w/2)}{\Gamma(b+1/2+w/2)} \left(\frac{d}{2\beta^{1/d}}\right)^{ds+w} \exp\left(\sum_{\nu=1}^{M} \frac{R_\nu(s)}{\nu(\nu+1)} \frac{1}{w^\nu}\right) \right| \ll A^{|\sigma|} \frac{e^{O(R^2/V)}}{|w|^{d\sigma-K-d/2}}. \tag{3.9}$$

But from (3.6) with $M = 1$ we obtain

$$\left| \frac{\Gamma(a+1/2-w/2)}{\Gamma(b+1/2+w/2)} \right| \ll A^{|\sigma|} |w|^{d/2-d\sigma+K} e^{O(R^2/V)}, \tag{3.10}$$

$\left(\frac{d}{2\beta^{1/d}}\right)^{ds+w} \ll A^{|\sigma|}$ and from Lemma 3.2 (observing that we may choose $c_1 \ge c_3$) we have

$$\sum_{\nu=1}^{M} \frac{|R_\nu(s)|}{\nu(\nu+1)} \frac{1}{|w|^\nu} \ll R \sum_{\nu=1}^{M} \frac{1}{\nu(\nu+1)} \left(\frac{c_3 R}{\nu}\right)^\nu \ll \frac{R^2}{V} + 1.$$

Therefore (3.9) follows and the proof of Lemma 3.3 is complete. $\qquad\square$

With the notation of Lemma 3.1 and writing

$$h_M^*(w,s) = \frac{\Gamma(a+1/2-w/2)}{\Gamma(b+1/2+w/2)} \Gamma(w) \exp\left(\sum_{\nu=1}^{M} \frac{R_\nu(s)}{\nu(\nu+1)} \frac{1}{w^\nu}\right),$$

$$I_{X.M}^*(s) = \frac{1}{2\pi i} \int_{\mathcal{L}_\infty} h_M^*(w,s)(2\omega_X)^{-w} \, dw, \tag{3.11}$$

from Lemma 3.3 we get

$$I_X(s) = c_0(F) \left(\frac{d}{2\beta^{1/d}}\right)^{ds} I_{X.M}^*(s) + O(A^{|\sigma|} e^{O(R^2/V)} (cR)^{M+2}) \tag{3.12}$$

uniformly for $X \ge X_0$, provided $V \ge (c_1 R)^{1/\delta}$ and $\max\left(\frac{2\delta-1}{1-\delta}, d|\sigma|+d/2\right) \le M \le DR$. Indeed, for $w \in \mathcal{L}_\infty$ we have

$$\Gamma(w) \ll e^{-\pi v/2} |w|^{-K-1/2} \quad \text{and} \quad \left|(d\omega_X/\beta^{1/d})^{-w}\right| \ll e^{\pi v/2},$$

hence replacing $h(w, s)$ in $I_X(s)$ by its main term in Lemma 3.3 we get an error of size

$$\ll A^{|\sigma|} e^{O(R^2/V)} c^M R^{M+2} \int_V^{\infty} \frac{dv}{v^{M+d\sigma-d/2+3/2}} \ll A^{|\sigma|} e^{O(R^2/V)} (cR)^{M+2}$$

since $M \geq d|\sigma| + d/2$, and (3.12) follows.

In order to study $h_M^*(w, s)$ we write $V_0(s) = 1$ identically and for $\mu \geq 1$,

$$V_\mu(s) = \sum_{1 \leq m \leq \mu} \frac{1}{m!} \sum_{\substack{v_1 \geq 1, \ldots, v_m \geq 1 \\ v_1 + \cdots + v_m = \mu}} \prod_{j=1}^{m} \frac{R_{v_j}(s)}{v_j(v_j + 1)}. \tag{3.13}$$

Lemma 3.4. *Under the hypotheses and with the notation of Lemma* 3.1 *we have*

$$\exp\left(\sum_{v=1}^{M} \frac{R_v(s)}{v(v+1)} \frac{1}{w^v}\right) = \sum_{\mu=0}^{M} \frac{V_\mu(s)}{w^\mu} + O\left(\frac{(c_5 R)^M}{|w|^{\delta M}} e^{O(RV^{\delta-1})}\right),$$

where c_5 may also depend on D.

Proof. Expanding the exponential and recalling (3.13) we have

$$\exp\left(\sum_{v=1}^{M} \frac{R_v(s)}{v(v+1)} \frac{1}{w^v}\right) = \sum_{\mu=0}^{M} \frac{V_\mu(s)}{w^\mu} + E$$

with

$$E = \sum_{\mu > M} \frac{V_{\mu,M}(s)}{w^\mu}, \quad V_{\mu,M}(s) = \sum_{1 \leq m \leq \mu} \frac{1}{m!} \sum_{\substack{1 \leq v_j \leq M \\ v_1 + \cdots + v_m = \mu}} \prod_{j=1}^{m} \frac{R_{v_j}(s)}{v_j(v_j + 1)}.$$

By Lemma 3.2 we obtain

$$V_{\mu,M}(s) \ll \sum_{1 \leq m \leq \mu} \frac{1}{m!} \sum_{\substack{1 \leq v_j \leq M \\ v_1 + \cdots + v_m = \mu}} (cR)^{\mu+m} \prod_{j=1}^{m} \frac{1}{v_j(v_j + 1)} \ll (cR)^\mu \sum_{m=0}^{\mu} \frac{(cR)^m}{m!},$$

hence for $\mu \leq \frac{3}{2} cR$ we have

$$V_{\mu,M}(s) \ll (cR)^{2\mu}/\mu! \tag{3.14}$$

thanks to [10, Lemma 3.6], while for $\mu > \frac{3}{2} cR$ we get

$$V_{\mu,M}(s) \ll (cR)^\mu e^{cR}.$$

Therefore

$$E \ll \sum_{M < \mu \leq 3cR/2} \frac{(cR)^{2\mu}}{|w|^\mu \mu!} + \sum_{\mu > 3cR/2} \frac{(cR)^\mu e^{cR}}{|w|^\mu} = \Sigma_1 + \Sigma_2,$$

say. But, thanks to the restrictions on w and V,

$$\Sigma_1 \ll \sum_{\mu > M} \frac{(cR)^\mu}{|w|^{\delta\mu}} \frac{(cR)^\mu}{\mu! V^{(1-\delta)\mu}} \ll \frac{(cR)^M}{|w|^{\delta M}} e^{O(RV^{\delta-1})},$$

$$\Sigma_2 \ll e^{cR} \sum_{\mu > 3cR/2} (cR/|w|)^\mu \ll e^{cR}(cR/|w|)^{3cR/2} \ll (cR/|w|)^M$$

since $M < \frac{3}{2}cR$. Lemma 3.4 is thus proved.   □

Now we transform the sum over μ in Lemma 3.4 by arguments similar to those in [10, Section 3]; see Lemmas 3.11–3.18. We start with the following variant of [10, Lemma 3.13], asserting that for $|w| \geq 2M$ and $1 \leq \mu \leq M$,

$$\frac{1}{w^\mu} = \sum_{\ell=\mu}^{M} \frac{C_{\mu,\ell}}{(w-1)\cdots(w-\ell)} + O\left(\frac{4^M M!}{(\mu-1)!}\frac{1}{|w|^{M+1}}\right), \tag{3.15}$$

where the coefficients $C_{\mu,\ell}$ satisfy

$$|C_{\mu,\ell}| \leq \frac{(\ell-1)!}{(\mu-1)!}\binom{\ell-1}{\mu-1} \tag{3.16}$$

by [10, Lemma 3.12]. Indeed, [10, Lemma 3.13] gives (3.15) with an error term

$$\ll \frac{2^M M!}{(\mu-1)!}\frac{1}{|w(w-1)\cdots(w-M)|},$$

and $|w(w-1)\cdots(w-M)| \geq (|w|-M)^{M+1} \geq (|w|/2)^{M+1}$, which yields (3.15). Let

$$A_{\mu,\ell}(s) = \sum_{k=0}^{\ell-\mu} (-1)^{\mu+\ell+k} C_{\mu+k,\ell}\binom{-\mu}{k}(s+1)^k. \tag{3.17}$$

Lemma 3.5. *Let* $1 \leq M \leq DR$ *with arbitrary* $D \geq 1$, $1 \leq \mu \leq M$ *and* $|w| \geq \max(2M, 3R+4)$. *Then with the above notation,*

$$\frac{1}{(w+s)^\mu} = \sum_{\ell=\mu}^{M} \frac{A_{\mu,\ell}(s)}{w(w+1)\cdots(w+\ell-1)} + O\left(24^M \frac{(DR)^{M-\mu+1}}{|w|^{M+1}}\right).$$

Proof. From (3.15) we have

$$\frac{(-1)^\mu}{w^\mu} = \frac{1}{(-w)^\mu} = \sum_{\ell=\mu}^{M} \frac{(-1)^\ell C_{\mu,\ell}}{(w+1)\cdots(w+\ell)} + O\left(\frac{4^M M!}{(\mu-1)!}\frac{1}{|w|^{M+1}}\right),$$

hence

$$\frac{1}{(w-1)^\mu} = \sum_{\ell=\mu}^{M} \frac{(-1)^{\ell+\mu} C_{\mu,\ell}}{w\cdots(w+\ell-1)} + O\left(\frac{8^M M!}{(\mu-1)!}\frac{1}{|w|^{M+1}}\right). \tag{3.18}$$

Since $\left|\frac{s+1}{w-1}\right| \le \frac{R+1}{|w|-1} \le \frac{1}{3}$ thanks to the hypothesis $|w| \ge 3R+4$, arguing as in [10, proof of Lemma 3.14] and starting with (3.16) there, we have

$$\frac{1}{(w+s)^\mu} = \frac{1}{(w-1)^\mu\left(1+\frac{s+1}{w-1}\right)^\mu}$$
$$= \sum_{k=0}^{M-\mu} \binom{-\mu}{k}\frac{(s+1)^k}{(w-1)^{\mu+k}} + O\left(4^M \frac{(R+1)^{M-\mu+1}}{|w|^{M+1}}\right).$$

Inserting (3.18) in the last equation and recalling (3.17) we obtain

$$\frac{1}{(w+s)^\mu} = \sum_{\ell=\mu}^{M} \frac{A_{\mu,\ell}(s)}{w(w+1)\cdots(w+\ell-1)}$$
$$+ O\left(\sum_{k=0}^{M-\mu}\left|\binom{-\mu}{k}\right|(R+1)^k \frac{8^M M!}{(k+\mu-1)!}\frac{1}{|w|^{M+1}}\right) + O\left(4^M \frac{(R+1)^{M-\mu+1}}{|w|^{M+1}}\right);$$

we denote by E_1 and E_2 the two error terms in the last equation. Since $\left|\binom{-\mu}{k}\right| = \binom{\mu+k-1}{k}$, using [10, Lemma 3.6] we obtain

$$E_1 \ll \frac{6^M M!}{|w|^{M+1}} \sum_{k=0}^{M-\mu} \frac{(k+\mu-1)!}{k!(\mu-1)!}\frac{(R+1)^k}{(k+\mu-1)!} \ll \frac{9^M M!}{|w|^{M+1}(\mu-1)!} \sum_{k=0}^{M-\mu} \frac{(DR)^k}{k!}$$
$$\ll \frac{12^M M!}{|w|^{M+1}}\frac{(DR)^{M-\mu}}{\mu!(M-\mu)!} \ll 24^M \frac{(DR)^{M-\mu}}{|w|^{M+1}}.$$

The lemma follows by recalling the bound for E_2. $\square$

Under the hypotheses of Lemma 3.5, but with $|w| \ge \max(2M+R, 4(R+1))$, we also have

$$\frac{1}{w^\mu} = \sum_{\ell=\mu}^{M} \frac{A_{\mu,\ell}(-s)}{(w+s)\cdots(w+s+\ell-1)} + O\left(50^M \frac{(DR)^{M-\mu+1}}{|w|^{M+1}}\right). \tag{3.19}$$

Indeed, we apply Lemma 3.5 with $w+s$ in place of w and $-s$ in place of s, thus getting the main term in (3.19) plus an error satisfying (since $|w+s| \ge |w|-|s| \ge |w|/2$)

$$\ll 25^M \frac{(DR)^{M-\mu+1}}{|w+s|^{M+1}} \ll 50^M \frac{(DR)^{M-\mu+1}}{|w|^{M+1}}.$$

Recalling (3.4), (3.17) and the definition of $b = b(s)$ before Lemma 3.1, writing for $\nu \ge 0$,

$$Q_\nu(s) = \sum_{\mu=0}^{\nu} \frac{1}{2^\mu} V_\mu(s) A_{\mu,\nu}(-b-1/2), \tag{3.20}$$

and observing that $Q_0(s) = 1$ identically, we have

Lemma 3.6. *Under the hypotheses and with the notation of Lemma* 3.1 *we have*

$$\exp\left(\sum_{\nu=1}^{M} \frac{R_\nu(s)}{\nu(\nu+1)} \frac{1}{w^\nu}\right) = 1 + \sum_{\nu=1}^{M} \frac{Q_\nu(s)}{(b+1/2+w/2)\cdots(b+1/2+w/2+\nu-1)}$$
$$+ O\left(\frac{(c_6 R)^M}{|w|^{\delta M}} e^{O(RV^{\delta-1})}\right),$$

where c_6 may also depend on D.

Proof. From (3.19) applied to $\frac{1}{w^\mu} = \frac{1}{2^\mu} \frac{1}{(w/2)^\mu}$ (changing ℓ to ν) and recalling (3.20) we get

$$\sum_{\mu=0}^{M} \frac{V_\mu(s)}{w^\mu} = \sum_{\mu=0}^{M} \frac{V_\mu(s)}{2^\mu} \sum_{\nu=\mu}^{M} \frac{A_{\mu,\nu}(-b-1/2)}{(w/2+b+1/2)\cdots(w/2+b+1/2+\nu-1)}$$
$$+ O\left(c^M \sum_{\mu=0}^{M} \frac{|V_\mu(s)|}{2^\mu} \frac{(DR)^{M-\mu+1}}{|w|^{M+1}}\right)$$
$$= 1 + \sum_{\nu=1}^{M} \frac{Q_\nu(s)}{(b+1/2+w/2)\cdots(b+1/2+w/2+\nu-1)} + E,$$

say. In view of the restriction on M, for $1 \le \mu \le M$ from (3.14) we have

$$V_\mu(s) \ll (cR)^{2\mu}/\mu!;$$

hence, since $w \in \mathcal{L}_\infty$,

$$E \ll c^M \frac{(cR)^{M+1}}{|w|^{M+1}} \sum_{\mu=0}^{M} \frac{(cR)^\mu}{\mu!} \ll c^M \frac{(c^2 R)^M}{|w|^{\delta M}} \sum_{\mu=0}^{M} \frac{R^\mu}{\mu! V^{(1-\delta)\mu}} \ll c^M \frac{R^M}{|w|^{\delta M}} e^{O(RV^{\delta-1})},$$

and the lemma follows. $\qquad\square$

Since $Q_0(s) = 1$ identically, writing

$$h_M^{**}(w, s) = \frac{\Gamma(a+1/2-w/2)}{\Gamma(b+1/2+w/2)} \Gamma(w)$$
$$\times \left(1 + \sum_{\nu=1}^{M} \frac{Q_\nu(s)}{(b+1/2+w/2)\cdots(b+1/2+w/2+\nu-1)}\right)$$
$$= \sum_{\nu=0}^{M} Q_\nu(s) \frac{\Gamma(a+1/2-w/2)}{\Gamma(b+1/2+w/2+\nu)} \Gamma(w),$$
$$I_{X.M}^{**}(s) = \frac{1}{2\pi i} \int_{\mathcal{L}_\infty} h_M^{**}(w, s)(2\omega_X)^{-w} \, dw,$$

from (3.11), (3.12) and Lemma 3.6 we obtain

$$I_X(s) = c_0(F)\left(\frac{d}{2\beta^{1/d}}\right)^{ds} I^{**}_{X,M}(s) + O(A^{|\sigma|}e^{O(R^2/V+RV^{\delta-1})}(cR)^{M+2}) \qquad (3.21)$$

uniformly for $X \geq X_0$, provided $V \geq (c_1R)^{1/\delta}$ and $\max\left(\frac{2\delta-1}{1-\delta}, \frac{1}{\delta}(d|\sigma| + d/2 + 2)\right) \leq M \leq DR$. Indeed, by Lemma 3.6 we have

$$h^*_M(w,s) = h^{**}_M(w,s) + O\left((cR)^M e^{O(RV^{\delta-1})}\left|\frac{\Gamma(a+1/2-w/2)}{\Gamma(b+1/2+w/2)}\Gamma(w)\right|\frac{1}{|w|^{\delta M}}\right)$$

and, for $w \in \mathcal{L}_\infty$, $\Gamma(w) \ll e^{-\pi v/2}|w|^{-K-1/2}$ and $|(2\omega_X)^{-w}| \ll e^{\pi v/2}$. Hence, thanks to (3.10), replacing $h^*_M(w,s)$ by $h^{**}_M(w,s)$ in $I^*_{X,M}(s)$ causes an error of size

$$\ll A^{|\sigma|}(cR)^M e^{O(R^2/V+RV^{\delta-1})}\int_V^\infty v^{-(\delta M-d|\sigma|-d/2)}\,dv \ll A^{|\sigma|}(cR)^M e^{O(R^2/V+RV^{\delta-1})}$$

since the integral is convergent under the above conditions on M. Therefore, (3.21) follows from (3.12).

Lemma 3.7. *Let $Q_v(s)$ be as in (3.20) and $1 \leq v \leq DR$, where $D \geq 1$ is arbitrary. Then*

$$Q_v(s) \leq (c_7R)^{2v}/v!,$$

where c_7 may also depend on D.

Proof. From (3.17), (3.20) and the definition $b = b(s) = ds/2 + \xi_F/2$ we have

$$Q_v(s) \ll c^v \sum_{\mu=0}^{v} \frac{1}{2^\mu}|V_\mu(s)| \sum_{k=0}^{v-\mu} |C_{\mu+k,v}|\left|\binom{-\mu}{k}\right|(R+1)^k.$$

Recalling that in the proof of Lemma 3.4 we have $V_{\mu,M}(s) = V_\mu(s)$ for $\mu \leq M$ and that $M \leq DR$, from (3.14) we have

$$V_\mu(s) \ll (cR)^{2\mu}/\mu!.$$

Hence by (3.16) and $|\binom{-\mu}{k}| = \binom{\mu+k-1}{k}$ we get

$$Q_v(s) \ll c^v \sum_{\mu=0}^{v}\sum_{k=0}^{v-\mu} \frac{(v-1)!}{\mu!(\mu+k-1)!}\binom{v-1}{\mu+k-1}\binom{\mu+k-1}{k}(cR)^{2\mu+k}$$

$$\ll c^v \sum_{\mu=0}^{v}\sum_{k=0}^{v-\mu} \frac{v!}{\mu!(\mu+k)!}2^v 2^{\mu+k-1}(cR)^{2\mu+k} \ll c^v v! \sum_{\mu=0}^{v}\sum_{k=0}^{v-\mu} \frac{(cR)^{2\mu+k}}{\mu!(\mu+k)!}$$

$$\ll c^v v! \sum_{k=0}^{v} \frac{(cR)^k}{k!}\sum_{\mu=0}^{v-k}\left(\frac{(cR)^\mu}{\mu!}\right)^2 \ll c^v v! \sum_{k=0}^{v} \frac{(cR)^k}{k!}\left(\sum_{\mu=0}^{v-k}\frac{(cR)^\mu}{\mu!}\right)^2$$

thanks to $(\mu + k)! \geq \mu!k!$. Since $\nu \leq DR$ and since the last constant c may clearly be chosen larger than D, we may apply [10, Lemma 3.6], thus getting (recall that the value of c is not necessarily the same at each occurrence!)

$$Q_\nu(s) \ll c^\nu \nu! \sum_{k=0}^{\nu} \frac{(cR)^k}{k!} \left(1 + \frac{(cR)^{2(\nu-k)}}{((\nu-k)!)^2} \right),$$

and by repeated application of [10, Lemma 3.6] we have

$$Q_\nu(s) \ll c^\nu \nu! \left(1 + \frac{(cR)^\nu}{\nu!} + \sum_{k=0}^{\nu} \frac{(cR)^{2\nu-k}}{k!(\nu-k)!(\nu-k)!} \right)$$

$$\ll c^\nu \nu! + (cR)^\nu + (cR)^\nu \sum_{k=0}^{\nu} \binom{\nu}{k} \frac{(cR)^{\nu-k}}{(\nu-k)!}$$

$$\ll c^\nu \nu! + (cR)^\nu + (cR)^\nu \sum_{k=0}^{\nu} \frac{(cR)^{\nu-k}}{(\nu-k)!}$$

$$\ll c^\nu \nu! + (cR)^\nu + (cR)^\nu (1 + (cR)^\nu/\nu!) \ll (cR)^{2\nu}/\nu!$$

since $\nu \leq DR$. Lemma 3.7 is thus proved. $\qquad\square$

For $z \in \mathbb{C}$ and integer $\nu \geq 1$ we define

$$P_\nu(z) = \prod_{\substack{0 \leq j \leq \nu-1 \\ |z+j| \geq 1/2}} (z + j). \qquad (3.22)$$

Lemma 3.8. *With the above notation we have* $|P_\nu(z)| \geq 2^{-\nu}(\nu - 1)!$.

Proof. By induction. For $\nu = 1$ the right hand side equals $1/2$, while the left hand side is at least $1/2$. Since $\nu = |z - (z + \nu)| \leq |z| + |z + \nu|$, we have $|z| \geq \nu/2$ or $|z + \nu| \geq \nu/2$. Hence

$$P_{\nu+1}(z) = \begin{cases} z P_\nu(z + 1) & \text{if } |z| \geq \nu/2, \\ (z + \nu) P_\nu(z) & \text{if } |z + \nu| \geq \nu/2. \end{cases}$$

Therefore, for the appropriate $\varepsilon = 0$ or 1, from the inductive hypothesis we get

$$|P_{\nu+1}(z)| \geq \frac{\nu}{2} |P_\nu(z + \varepsilon)| \geq \frac{\nu}{2} \frac{(\nu-1)!}{2^\nu} = \frac{\nu!}{2^{\nu+1}},$$

and the lemma follows. $\qquad\square$

The proof of Theorem 2 is based on the following lemma, which summarizes the results obtained so far in this section. Let (recall the definition of a and b after (3.4))

$$I_{X,\nu}(s) = \frac{1}{2\pi i} \int_{(-K)} \frac{\Gamma(a + 1/2 - w/2)}{\Gamma(b + 1/2 + w/2 + \nu)} \Gamma(w)(2\omega_X)^{-w} \, dw. \qquad (3.23)$$

Lemma E. *Let* $-L \leq \sigma \leq 3/2$ *with* $L \geq 3/2$, $|s| \leq R$ *with* $R \geq 1$, $0 < \delta < 1$, $\eta\delta > 1$, $\delta' = \max(0, 2 - 1/\delta)$, *and* $M = [dL/\delta] + H$ *where* $H = H(F, \delta, \eta) > 0$ *is a sufficiently large integer. Then, under the hypotheses of Theorem* 2 *and with the notation in* (2.3), (3.20) *and* (3.23), *we have*

$$F_X(s, \alpha) = c(F) \frac{\overline{a(n_\alpha)}}{n_\alpha^{1-s}} q_F^{-s} (\pi d)^{ds} \sum_{\nu=0}^{M} Q_\nu(s) I_{X,\nu}(s) + O(A^L R^{\eta dL + B} e^{C(R^{\delta'} + R^{(3 - \eta\delta/2)})})$$

uniformly for $X \geq X_0$, *where* $c(F) \neq 0$.

Proof. We have to estimate the contribution to $I_X(s)$ of the part $(-K - i\infty, -K + iV)$ of the line of integration in the integrals $I_{X,\nu}(s)$ (see (3.23)). Recalling (3.22), by the factorial formula we have

$$\Gamma(b+1/2+w/2+\nu) = \Gamma(b+1/2+w/2) P_\nu(b+1/2+w/2)(b+1/2+w/2+j_0), \quad (3.24)$$

where $1 \leq j_0 \leq \nu - 1$ is such that $|b+1/2+w/2+j_0| < 1/2$, if it exists. If such a j_0 does not exist, then the factor $b + 1/2 + w/2 + j_0$ is not present in (3.24). Let $w = -K + iv$ with $v < V$. By Lemma 3.8, the reflection formula for the Γ function, the inequality $|z/\sin(z - \pi j_0)| \gg \exp(-|z|)$ for $|z| < 1/2$ and the definition of b after (3.4) we get

$$|\Gamma(b + 1/2 + w/2 + \nu)| \gg \frac{1}{|\Gamma(1/2 - b - w/2)|} \frac{(\nu - 1)!}{2^\nu} \frac{|b + 1/2 + w/2 + j_0|}{|\sin(\pi(b + 1/2 + w/2))|}$$

$$\gg \frac{1}{|\Gamma(1/2 - b - w/2)|} \frac{\nu!}{\nu 2^\nu} e^{-\pi|dt+v|/2}. \quad (3.25)$$

Moreover, recalling also the definition of a after (3.4), from Stirling's formula we have

$$\Gamma(w) \ll e^{-\pi|v|/2} |w|^{-K-1/2},$$

$$\Gamma(a + 1/2 - w/2) \ll (1 + |dt + v|/2)^{d|\sigma|/2 + B} e^{-\pi|dt+v|/4},$$

$$\Gamma(1/2 - b - w/2) \ll (1 + |dt + v|/2)^{d|\sigma|/2 + B} e^{-\pi|dt+v|/4}.$$

Hence, since $|(2\omega_X)^{-w}| = |2\omega_X|^K e^{v \arg \omega_X}$, we obtain

$$\int_{-K-i\infty}^{-K+iV} \frac{\Gamma(a + 1/2 - w/2)}{\Gamma(b + 1/2 + w/2 + \nu)} \Gamma(w)(2\omega_X)^{-w} \, dw$$

$$\ll \frac{\nu 2^\nu}{\nu!} A^{|\sigma|} \int_{-\infty}^{V} \frac{(1 + |dt + v|)^{d|\sigma| + B}}{(1 + |v|)^{K + 1/2}} e^{-\pi|v|/2 + v \arg \omega_X} \, dv$$

$$\ll \frac{\nu 2^\nu}{\nu!} A^{|\sigma|} V^{d|\sigma| + B} + \frac{\nu 2^\nu}{\nu!} A^{|\sigma|} \int_{V}^{\infty} e^{-v} v^{d|\sigma| + B} \, dv$$

$$\ll \frac{\nu 2^\nu}{\nu!} A^{|\sigma|} V^{d|\sigma| + B} \left\{ 1 + e^{-V} \left(1 + \frac{1 + |\sigma|^2}{V} \right) \right\} \ll \frac{\nu 2^\nu}{\nu!} A^{|\sigma|} V^{d|\sigma| + B} \quad (3.26)$$

from (2.17) and the fact that $V \geq (c_1 R)^{1/\delta}$. Thanks to (3.26), to the choice of M and to Lemma 3.7, the contribution of the left hand side of (3.26) to $I_X(s)$ is

$$
\ll A^{|\sigma|} V^{d|\sigma|+B} M 2^M \sum_{\nu=0}^{M} \frac{|Q_\nu(s)|}{\nu!} \ll A^L V^{d|\sigma|+B} \sum_{\nu=0}^{M} \frac{(CR)^{2\nu}}{(\nu!)^2}
$$

$$
\ll A^L V^{d|\sigma|+c} \sum_{\nu=0}^{M} \frac{(CR)^{(3-\eta\delta)\nu+(\eta\delta-1)\nu}}{(\nu!)^2}
$$

$$
\ll A^L V^{d|\sigma|+c} (cR)^{(\eta\delta-1)M} \left(\sum_{\nu=0}^{\infty} \frac{(CR)^{\nu(3-\eta\delta)/2}}{\nu!} \right)^2
$$

$$
\ll A^L V^{d|\sigma|+c} (cR)^{(\eta\delta-1)M} e^{CR^{(3-\eta\delta)/2}}. \tag{3.27}
$$

If we now recall that $q_F = (2\pi)^d Q^2 \beta$ and write $c(F) = \omega Q c_0(F)$, then Lemma E follows from (3.1), (3.2), (3.21), (3.23) and (3.27), by choosing $V = (cR)^{1/\delta}$. $\qquad\square$

Now we show how Theorem 2 follows from Lemma E and some of the arguments in our previous treatment of the standard twist in [4] and [7]. Under the hypotheses of Lemma E and with the notation there we write

$$
H_X(s, \alpha) = F_X(s, \alpha) - c(F) \frac{\overline{a(n_\alpha)}}{n_\alpha^{1-s}} q_F^{-s} (\pi d)^{ds} \sum_{\nu=0}^{M} Q_\nu(s) I_{X,\nu}(s)
$$

and note that for $\sigma > 1$ the limit

$$
\lim_{X \to \infty} H_X(s, \alpha) = H(s, \alpha)
$$

exists. Indeed, clearly $F_X(s, \alpha)$ tends to $F(s, \alpha)$ for $\sigma > 1$, while the treatment of the limit of the integrals $I_{X,\nu}(s)$ as $X \to \infty$ is borrowed from our previous papers on this subject; see in particular [4, Theorem 5.1] in the case $d = 1$, and [7, Lemma 2.4] in the general case. Note, however, that [7, Lemma 2.4] deals with a normalized situation where s, λ_j and μ_j mean, in our present notation,

$$
ds - \frac{d-1}{2}, \quad \frac{\lambda_j}{d}, \quad \mu_j + \frac{\lambda_j}{2}(1 - 1/d),
$$

respectively (see [7, discussion before and after (2.2) and (2.3), pp. 320–321]). As a consequence we have

$$
H(s, \alpha) = F(s, \alpha) - c(F) \frac{\overline{a(n_\alpha)}}{n_\alpha^{1-s}} q_F^{-s} (\pi d)^{ds} \sum_{\nu=0}^{M} Q_\nu(s) I_\nu(s), \tag{3.28}
$$

where, recalling that $a = a(s) = \frac{d}{2}(1-s) + \frac{1}{2}\overline{\xi_F}$ and $b = b(s) = \frac{d}{2}s + \frac{1}{2}\xi_F$ (see after (3.4)),

$$I_\nu(s) = -\sum_{k=0}^{K} \frac{1}{k!} \frac{\Gamma(a + 1/2 + k/2)}{\Gamma(b + 1/2 - k/2 + \nu)} \left(\frac{i}{2}\right)^{-k} + \sqrt{\pi} \frac{\Gamma(a + 1/2)\Gamma(b - a - 1/2 + \nu)}{\Gamma(-a)\Gamma(b + \nu)\Gamma(1/2 + b + \nu)}$$

$$- i\sqrt{\pi} \frac{\Gamma(a + 1)\Gamma(b - a - 1/2 + \nu)}{\Gamma(1/2 - a)\Gamma(b + \nu)\Gamma(1/2 + b + \nu)}$$

$$= -I_\nu^{(1)}(s) + \sqrt{\pi} I_\nu^{(2)}(s) - i\sqrt{\pi} I_\nu^{(3)}(s), \tag{3.29}$$

say. [4, Theorem 5.1] deals only with $I_0(s)$, while [7, Lemma 2.4] deals with all $I_\nu(s)$, denoted by $\Gamma_{K,\nu}(s)$ there. From Lemma E and Vitali's convergence theorem (see Lemma C and the proof of Theorem 1 in Section 2), we therefore infer that the limit function $H(s, \alpha)$ exists and is holomorphic for $|s| \leq R$ and $-L \leq \sigma \leq 3/2$, and satisfies

$$H(s, \alpha) \ll A^L R^{\eta d L + B} e^{C(R^{\delta'} + R^{(3 - \eta\delta)/2})}. \tag{3.30}$$

Since R and L are arbitrary, from (3.28) and (3.29) we see in particular that $F(s, \alpha)$ has meromorphic continuation to $\mathbb{C}$. Moreover, from [7, Lemma 2.5] (recall the normalization there), $F(s, \alpha)$ is holomorphic over $\mathbb{C}$ except possibly for simple poles at the points s_k in (1.2) satisfying (1.3), coming from the term $\Gamma(b - a - 1/2 + \nu)$ in (3.29). Our next goal is therefore estimating the last term in (3.28) away from such poles.

Before starting the estimation of such a term, we remark that we shall always assume that s is δ_0-away from any pole which might arise during the estimation. We call such poles *potential poles*, and we shall deal with them at the end of the proof. Moreover, we always assume that $0 \leq k \leq K$ and $|s| \leq R$. We deal first with $I_\nu^{(1)}(s)$, in a similar way to the proof of Lemma E. Indeed, following (3.24) we have

$$\Gamma(b + 1/2 - k/2 + \nu) = \Gamma(b + 1/2 - k/2) P_\nu(b + 1/2 - k/2)(b + 1/2 - k/2 - j_0(k)),$$

where $1 \leq j_0(k) \leq \nu - 1$ is such that $|b + 1/2 - w/2 - j_0(k)| < 1/2$, if it exists. Hence similarly to (3.25) we get

$$|\Gamma(b + 1/2 - k/2 + \nu)| \gg \frac{1}{|\Gamma(1/2 - b + k/2)|} \frac{\nu!}{\nu 2^\nu} e^{-\pi d |t|/2},$$

and so

$$I_\nu^{(1)}(s) \ll \frac{\nu 2^\nu}{\nu!} \sum_{k=0}^{K} \frac{1}{k!} |\Gamma(a + 1/2 + k/2)\Gamma(1/2 - b + k/2)| e^{\pi d |t|/2}.$$

Using (2.6), away from the potential poles we have

$$|\Gamma(a + 1/2 + k/2)\Gamma(1/2 - b + k/2)| \ll A^L R^{dL + B} e^{-\pi d |t|/2},$$

therefore under the above mentioned assumptions we have

$$I_\nu^{(1)}(s) \ll \frac{A^L}{\nu!} R^{dL + B}. \tag{3.31}$$

In order to estimate $I_\nu^{(2)}(s)$ we apply the duplication formula, the factorial formula and recall (3.22), thus getting

$$\Gamma(b+\nu)\Gamma(1/2+b+\nu) = \frac{2^{-ds-\xi_F-2\nu}}{\sqrt{\pi}}\Gamma(ds+\xi_F+2\nu)$$

$$= \frac{2^{-ds-\xi_F-2\nu}}{\sqrt{\pi}}\Gamma(ds+\xi_F+\nu)P_\nu(ds+\xi_F+\nu)(ds+\xi_F+j_0(\nu)),$$

where $\nu \le j_0(\nu) \le 2\nu-1$ is such that $|ds+\xi_F+j_0(\nu)| < 1/2$, if it exists. Hence by the reflection formula we obtain

$$I_\nu^{(2)}(s) = \sqrt{\pi}\,2^{ds+\xi_F+2\nu}\frac{\sin(\pi(ds/2-d/2-\overline{\xi_F}/2))}{(ds+\xi_F+j_0(\nu))P_\nu(ds+\xi_F+\nu)}$$

$$\times\frac{\Gamma((d+1)/2-ds/2+\overline{\xi_F}/2)\Gamma(1+d/2-ds/2+\overline{\xi_F}/2)\Gamma(ds-(d+1)/2+id\theta_F+\nu)}{\Gamma(ds+\xi_F+\nu)},$$

(3.32)

and by Lemma 3.8 we get, away from potential poles,

$$I_\nu^{(2)}(s) \ll \frac{A^L c^\nu}{\nu!}e^{\pi d|t|/2}|\Gamma((d+1)/2-ds/2+\overline{\xi_F}/2)\Gamma(1+d/2-ds/2+\overline{\xi_F}/2)|$$

$$\times\left|\frac{\Gamma(ds-(d+1)/2+id\theta_F)}{\Gamma(ds+\xi_F)}\right|\prod_{j=0}^{\nu-1}\left|\frac{ds-(d+1)/2+id\theta_F+j}{ds+\xi_F+j}\right|.$$

Since each factor of the last product is bounded away from the potential poles we have

$$\prod_{j=0}^{\nu-1}\left|\frac{ds-(d+1)/2+id\theta_F+j}{ds+\xi_F+j}\right| \ll c^\nu,$$

and by Stirling's formula

$$|\Gamma((d+1)/2-ds/2+\overline{\xi_F}/2)\Gamma(1+d/2-ds/2+\overline{\xi_F}/2)| \ll e^{-\pi d|t|/2}R^{dL+B}.$$

Moreover, by the reflection formula and Stirling's formula we get

$$\left|\frac{\Gamma(ds-(d+1)/2+id\theta_F)}{\Gamma(ds+\xi_F)}\right| \ll \left|\frac{\Gamma(1-ds-\xi_F)}{\Gamma((d+3)/2-ds-id\theta_F)}\right| \ll R^B.$$

Therefore, from (3.32) and the above bounds we conclude that under the above assumptions

$$I_\nu^{(2)}(s) \ll \frac{c^\nu}{\nu!}A^L R^{dL+B}. \tag{3.33}$$

Similar computations show that the same bound holds for $I_\nu^{(3)}(s)$ as well, hence from (3.29), (3.31) and (3.33) we obtain

$$I_\nu(s) \ll \frac{c^\nu}{\nu!}A^L R^{dL+B}$$

for $0 \leq v \leq M$ and s as in Lemma E, away from the potential poles. Therefore from Lemma 3.7 we have

$$c(F)\frac{\overline{a(n_\alpha)}}{n_\alpha^{1-s}}q_F^{-s}(\pi d)^{ds}\sum_{v=0}^{M}Q_v(s)I_v(s) \ll A^L R^{dL+B}\sum_{v=0}^{M}\frac{(CR)^{2v}}{(v!)^2}$$

$$\ll A^L R^{\eta dL+B}\sum_{v=0}^{M}\frac{(CR)^{2v-(\eta-1)dL}}{(v!)^2},$$

and since $(\eta-1)dL \geq (\eta-1)(\delta M - c) \geq \delta(\eta-1)v - c$ we obtain

$$\sum_{v=0}^{M}\frac{(CR)^{2v-(\eta-1)dL}}{(v!)^2} \ll R^c \sum_{v=0}^{M}\frac{(CR)^{(2-(\eta-1)\delta)v}}{(v!)^2} \ll R^c\Big(\sum_{v=0}^{\infty}\frac{(CR)^{(1-(\eta-1)\delta/2)v}}{(v!)^2}\Big)^2$$

$$\ll R^c e^{CR^{1-(\eta-1)\delta/2}} \ll R^c e^{CR^{(3-\eta\delta)/2}},$$

in view of $1-(\eta-1)\delta/2 \leq (3-\eta\delta)/2$. As a consequence, away from the potential poles we have

$$c(F)\frac{\overline{a(n_\alpha)}}{n_\alpha^{1-s}}q_F^{-s}(\pi d)^{ds}\sum_{v=0}^{M}Q_v(s)I_v(s) \ll A^L R^{\eta dL+B}e^{CR^{(3-\eta\delta)/2}}. \tag{3.34}$$

Hence from (3.28), (3.30) and (3.34) we finally obtain

$$F(s,\alpha) \ll A^L R^{\eta dL+B}e^{C(R^{\delta'}+R^{(3-\eta\delta)/2})} \tag{3.35}$$

away from the potential poles, provided $|s| \leq R$ and $-L \leq \sigma \leq 3/2$. However, we already know from [7, Theorem 1] that $F(s,\alpha)$ is holomorphic over $\mathbb{C}$ except possibly at $s = s_k, k = 0, 1, \ldots$. Hence, thanks to the maximum modulus principle, the bound (3.35) holds for every s outside discs of radius δ_0 around the points s_k, and Theorem 2 follows.

4. Proof of Theorem 4

Let $f(n, \boldsymbol{\alpha})$ be as in Theorem 4 and define

$$\mathcal{A}_f = \Big\{\omega = \sum_{v=0}^{N} m_v \kappa_v : m_v \in \mathbb{Z}, \ m_v \geq 0\Big\},$$

$$A(\omega) = \sum_{\substack{m_0 \geq 0,\ldots,m_N \geq 0 \\ \sum_{v=0}^{N} m_v \kappa_v = \omega}} \frac{\prod_{v=0}^{N}(-2\pi i \alpha_v)^{m_v}}{m_0! \cdots m_N!};$$

note that $\mathcal{A}_f$ has no accumulation points in $\mathbb{R}$. We have

Lemma 4.1. *We have the bounds*

$$\sum_{\omega \in \mathcal{A}_f,\ \omega \leq x} 1 \ll x^{N+1} \quad and \quad A(\omega) \ll e^{-c\omega \log \omega} \quad with \quad c = c(f) > 0,$$

and the series $\sum_{\omega \in \mathcal{A}_f} A(\omega)$ is absolutely convergent.

Proof. The first bound is trivial, since for every ν we have $m_\nu \leq 1 + x/\kappa_\nu$, and hence the number of possible $\omega \in \mathcal{A}_f$ up to x is $\ll \prod_{\nu=0}^{N}(1 + x/\kappa_\nu) \ll x^{N+1}$. To prove the second bound we write $r(\omega)$ for the number of representations $\sum_{\nu=0}^{N} m_\nu \kappa_\nu = \omega$, and M_ν for the maximum of the m_ν occurring in such representations. Recalling that the value of the constants c may not be the same at each occurrence we have

$$
\begin{aligned}
A(\omega) &\ll \frac{r(\omega) \prod_{\nu=0}^{N}(2\pi|\alpha_\nu| + 1)^{M_\nu}}{[\omega/\kappa_N]!} \ll \frac{\omega^{N+1} \prod_{\nu=0}^{N}(2\pi|\alpha_\nu| + 1)^{\omega/\kappa_\nu}}{[\omega/\kappa_N]!} \\
&\ll \frac{\omega^{N+1} c^\omega}{[\omega/\kappa_N]!} \ll \frac{c^\omega}{\omega^{\omega/\kappa_N}},
\end{aligned}
$$

and the second bound follows. Finally, the absolute convergence of the series follows by partial summation from the first two bounds. $\qquad\square$

Let now $F(s)$ be as in Theorem 4. Then, expanding the exponential, thanks to the good convergence properties of the series involved we get, for $\sigma > 1$,

$$
\begin{aligned}
F(s; f) &= \sum_{n=1}^{\infty} \frac{a(n)}{n^s} e(-f(n, \boldsymbol{\alpha})) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s} \prod_{\nu=0}^{N} e^{-2\pi i \alpha_\nu n^{-\kappa_\nu}} \\
&= \sum_{n=1}^{\infty} \frac{a(n)}{n^s} \sum_{m_0 \geq 0, \ldots, m_N \geq 0} \frac{\prod_{\nu=0}^{N}(-2\pi i \alpha_\nu n^{-\kappa_\nu})^{m_\nu}}{m_0! \cdots m_N!} \\
&= \sum_{m_0 \geq 0, \ldots, m_N \geq 0} \frac{\prod_{\nu=0}^{N}(-2\pi i \alpha_\nu)^{m_\nu}}{m_0! \cdots m_N!} \sum_{n=1}^{\infty} \frac{a(n)}{n^{s + \sum_{\nu=0}^{N} m_\nu \kappa_\nu}} \\
&= \sum_{\omega \in \mathcal{A}_f} A(\omega) F(s + \omega), \qquad\qquad\qquad\qquad\qquad (4.1)
\end{aligned}
$$

the last series being clearly absolutely convergent for $\sigma > 1$ thanks to Lemma 4.1. Moreover, for any given $s \in \mathbb{C}$ we have $\sigma + \omega \geq 2$ for all but finitely many values of $\omega \in \mathcal{A}_f$, hence such a series is absolutely and uniformly convergent on compact subsets of $\mathbb{C}$, except at the points $s = s_0 - \omega$ where s_0 is a pole of $F(s)$ and $\omega \in \mathcal{A}_f$ is such that $A(\omega) \neq 0$. Therefore $F(s; f)$ is meromorphic on $\mathbb{C}$, its poles are contained in the same horizontal strip of $F(s)$, and $F(s; f)$ is entire if $F(s)$ is entire. Further, let σ be fixed and note that by the above observation we may clearly write

$$
F(s; f) = F(s) + \sum_{\substack{\omega \in \mathcal{A}_f \\ 0 < \omega \leq \omega_0}} A(\omega) F(s + \omega) + B(s) \qquad\qquad (4.2)
$$

where $\omega_0 = \omega_0(\sigma)$ is such that $\sigma + \omega_0 \leq 3/2$ and $B(s)$ is bounded as $|t| \to \infty$. Thanks to the properties of the Lindelöf μ-function of Dirichlet series, in particular the fact that it is strictly decreasing on any interval where it is positive, (4.2) immediately shows that $\mu_F(\sigma) = \mu_F(\sigma; f)$, and Theorem 4 is proved.

5. Proof of Theorem 5

Suppose first that $F \in M(\rho, \tau)$, $\tau > 0$, and $0 < \lambda < 1/\rho$. As in Sections 2 and 3, we may assume that $\alpha > 0$ and $\sigma \leq 3/2$; we deal first with the case $t > 0$. Let $u_0 = (2 - \sigma)/\lambda$, $v_0 = \tau/\lambda$,

$$\mathcal{L}_{-\infty} = (u_0 - i\infty, u_0 + iv_0), \qquad \mathcal{L}_{v_0} = (u_0 + iv_0, -\infty + iv_0),$$

and, using the notation in the proof of Theorem 1, write

$$F_X^\lambda(s, \alpha) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s} e^{-z_X n^\lambda}.$$

Since $F(s)$ is holomorphic for $|t| \geq \tau$, and $\Im(s + \lambda w) > \tau$ when $v \geq v_0$, by Mellin's transform and arguments already used in the proof of Theorem 1 we have

$$F_X^\lambda(s, \alpha) = \frac{1}{2\pi i} \int_{(u_0)} F(s + \lambda w)\Gamma(w)z_X^{-w}\, dw$$

$$= \frac{1}{2\pi i} \left(\int_{\mathcal{L}_{-\infty}} + \int_{\mathcal{L}_{v_0}} \right) F(s + \lambda w)\Gamma(w)z_X^{-w}\, dw = F_X^{(1)}(s) + F_X^{(2)}(s), \quad (5.1)$$

say. Clearly,

$$F_X^{(1)}(s) \ll \int_{-\infty}^{v_0} |\Gamma(u_0 + iv)z_X^{-u_0 - iv}|\, dv$$

and for $w \in \mathcal{L}_{-\infty}$ we have

$$\log|\Gamma(w)| = \Re((w - 1/2)\log w - w) + O(1)$$

$$= (u_0 - 1/2)\log|w| - v \arctan\frac{\lambda v}{2 - \sigma} + O(1 + |\sigma|)$$

$$= (u_0 - 1/2)\log|w| - |v|\left(\pi/2 + O\left(\frac{|\sigma| + 1}{|v|}\right)\right) + O(1 + |\sigma|)),$$

therefore

$$\Gamma(w) \ll A^{|\sigma|}|w|^{u_0 - 1/2} e^{-\pi|v|/2}.$$

Moreover $|z_X^{-w}| = |z_X|^{-u_0} e^{v \arg z_X} \ll A^{|\sigma|} e^{v(\pi/2 + O(1/X))}$, hence

$$F_X^{(1)}(s) \ll A^{|\sigma|} \int_{-\infty}^{v_0} |w|^{u_0 - 1/2} e^{-\pi|v|/2 + \pi v/2 + O(|v|/X)}\, dv$$

$$\ll A^{|\sigma|} + A^{|\sigma|} \int_0^{\infty} |u_0 + iv|^{u_0 - 1/2} e^{-v(\pi + O(1/X))}\, dv$$

$$\ll A^{|\sigma|} + A^{|\sigma|}(1 + |\sigma|)^{|\sigma|/\lambda + B} + A^{|\sigma|} \int_0^{\infty} (1 + v)^{u_0 - 1/2} e^{-v(\pi + O(1/X))}\, dv$$

$$\ll A^{|\sigma|} + A^{|\sigma|}(1 + |\sigma|)^{|\sigma|/\lambda + B} + A^{|\sigma|}\Gamma(|\sigma|/\lambda + B) \ll A^{|\sigma|}(1 + |s|)^{|\sigma|/\lambda + B} \quad (5.2)$$

uniformly for $X \geq X_0$.

Recalling (1.6), for $w \in \mathcal{L}_{v_0}$ we have

$$F(s + \lambda w) \ll A^{|\sigma + \lambda u|}(1 + |s + \lambda w|)^{\rho|\sigma + \lambda u| + B} e^{C|s + \lambda w|^{\delta}}$$
$$\ll A^{|\sigma| + |\lambda u|}(1 + |s + \lambda w|)^{\rho|\sigma| - \lambda \rho u + B} e^{C(|s|^{\delta} + |u|^{\delta})},$$

and also, thanks to the reflection formula and Stirling's formula,

$$\Gamma(w) \ll \frac{1}{|\Gamma(1 - w)|} \ll A^{|u|}(1 + |u|)^{-|u|}, \quad z_X^{-w} \ll A^{|u|}.$$

Therefore

$$F_X^{(2)}(s) \ll A^{|\sigma|} e^{C|s|^{\delta}} \int_{-\infty}^{u_0} A^{|u|}(1 + |s + \lambda w|)^{\rho|\sigma| - \lambda \rho u + B}(1 + |u|)^{-|u|} \, du$$
$$\ll A^{|\sigma|} e^{C|s|^{\delta}} \int_{-\infty}^{-|s|} A^{|u|}(1 + |u|)^{-(1 - \lambda \rho)|u| + \rho|\sigma| + B} \, du$$
$$+ A^{|\sigma|} e^{C|s|^{\delta}}(1 + |s|)^{\rho|\sigma| + B} \int_{-|s|}^{u_0} A^{|u|} \frac{(1 + |s|)^{-\lambda \rho u}}{(1 + |u|)^{|u|}} \, du$$
$$\ll A^{|\sigma|} e^{C|s|^{\delta}}(1 + |s|)^{\rho|\sigma| + B} \left(1 + \int_1^{|s|} A^u \frac{(1 + |s|)^{\lambda \rho u}}{u^u} \, du\right)$$
$$\ll A^{|\sigma|}(1 + |s|)^{\rho|\sigma| + B} e^{C(|s|^{\delta} + |s|^{\lambda \rho})} \tag{5.3}$$

uniformly for $X \geq X_0$, since the integrand in the last but one row is maximal (essentially) when $u = c|s|^{\lambda \rho}$. Hence from (5.1)–(5.3) we find that for $\sigma \leq 3/2$ and $t > 0$,

$$F_X^{\lambda}(s, \alpha) \ll A^{|\sigma|}(1 + |s|)^{|\sigma|/\lambda + B} e^{C(|s|^{\delta} + |s|^{\lambda \rho})} \tag{5.4}$$

uniformly for $X \geq X_0$.

The case $t \leq 0$ is partly similar to the previous case, thus we will be more concise; all the estimates below hold uniformly for $X \geq X_0$. Given $F \in M(\rho, \tau)$ we write $u_0 = (2 - \sigma)/\lambda$, $v_0 = (\tau - t)/\lambda$ and consider again the half-lines $\mathcal{L}_{-\infty}$ and $\mathcal{L}_{v_0}$ (with the new value of v_0). Since $\Im(s + \lambda w) \geq \tau$ for $v \geq v_0$, as in the previous case we have

$$F_X^{\lambda}(s, \alpha) = \frac{1}{2\pi i} \left(\int_{\mathcal{L}_{-\infty}} + \int_{\mathcal{L}_{v_0}}\right) F(s + \lambda w) \Gamma(w) z_X^{-w} \, dw = F_X^{(3)}(s) + F_X^{(4)}(s), \tag{5.5}$$

say, and a similar argument gives, uniformly for $X \geq X_0$,

$$F_X^{(3)}(s) \ll A^{|\sigma|}(1 + |s|)^{|\sigma|/\lambda + B}. \tag{5.6}$$

Recalling again (1.6), for $w \in \mathcal{L}_{v_0}$ we have

$$F(s + \lambda w) \ll A^{|\sigma| + |\lambda u|}(1 + |\sigma + \lambda u|)^{\rho|\sigma + \lambda u| + B} e^{C|\sigma + \lambda u|^{\delta}}.$$

Moreover, for $w \in \mathcal{L}_{v_0}$ with $u \geq 0$ from Stirling's formula we get

$$|\Gamma(w)z_X^{-w}| \ll A^{|\sigma|}(1+|s|)^u e^{-u} e^{-v_0 \arg w} e^{v_0 \arg z_X}$$
$$\ll A^{|\sigma|}(1+|s|)^u e^{-v_0(u/v_0 + \arctan(v_0/u))} e^{\pi v_0/2} \ll A^{|\sigma|}(1+|s|)^u$$

since $x + \arctan(1/x) \geq \pi/2$ for $x > 0$. Hence the contribution to $F_X^{(4)}(s)$ coming from the part of $\mathcal{L}_{v_0}$ with $u \geq 0$ is

$$\ll A^{|\sigma|} \int_0^{u_0} (1+|s|)^{\rho|\sigma|+u(1-\lambda\rho)+B} \, du \ll A^{|\sigma|}(1+|s|)^{|\sigma|/\lambda+B}. \tag{5.7}$$

For $w \in \mathcal{L}_{v_0}$ with $u < 0$ we have, again by the reflection and Stirling formulae,

$$|\Gamma(w)z_X^{-w}| \ll e^{-\pi v_0} \frac{1}{|\Gamma(1-w)|} A^{|u|} e^{\pi v_0/2} \ll |w|^u A^{|u|}.$$

Hence the contribution to $F_X^{(4)}(s)$ coming from the part of $\mathcal{L}_{v_0}$ with $-c(1+|\sigma|) \leq u \leq 0$ (here $c > 1$ is an arbitrary constant) is

$$\ll A^{c|\sigma|}(1+|s|)^{\rho|\sigma|+B} \int_0^{c(1+|\sigma|)} \frac{(1+|\sigma|)^{\lambda\rho u}}{|w|^u} \, du$$
$$\ll A^{c|\sigma|}(1+|s|)^{\rho|\sigma|+B} \max_{0 \leq u \leq c(1+|\sigma|)} \frac{(1+|\sigma|)^{\lambda\rho u}}{|w|^u} \ll A^{c|\sigma|}(1+|s|)^{\rho|\sigma|+B}, \tag{5.8}$$

since (recall that $\lambda < 1/\rho$)

$$\max_{0 \leq u \leq c(1+|\sigma|)} \left(\frac{(1+|\sigma|)^{\lambda\rho}}{|w|} \right)^u \leq 1 + \max_{0 \leq u \leq (1+|\sigma|)^{\lambda\rho}} \left(\frac{(1+|\sigma|)^{\lambda\rho}}{|w|} \right)^u$$
$$\ll (1+|\sigma|)^{\lambda\rho(1+|\sigma|)^{\lambda\rho}} \ll A^{|\sigma|}.$$

Finally, the contribution to $F_X^{(4)}(s)$ coming from the part of $\mathcal{L}_{v_0}$ with $u < -c(1+|\sigma|)$ is

$$\ll A^{|\sigma|} \int_{c(1+|\sigma|)}^{\infty} A^u u^{\rho|\sigma|+(\lambda\rho-1)u+B} \, du = A^{|\sigma|} \int_{c(1+|\sigma|)}^{\infty} e^{f(u)} \frac{du}{u^2},$$

where $f(u) = u \log A + (\rho|\sigma| + B) \log u - (1-\lambda\rho)u \log u + 2$. But then $f'(u) = -(1-\lambda\rho)\log u + O(1)$, therefore $f'(u) < 0$ for $u > c(1+|\sigma|)$ if c is large enough, and hence this contribution is

$$\ll A^{|\sigma|} e^{f(c(1+|\sigma|))} \ll A^{|\sigma|}(1+|s|)^{\rho|\sigma|+B}, \tag{5.9}$$

with such a choice of c. From (5.7)–(5.9) we obtain

$$F_X^{(4)}(s) \ll A^{|\sigma|}(1+|s|)^{|\sigma|/\lambda+B}, \tag{5.10}$$

hence from (5.5), (5.6) and (5.10) we conclude that for $\sigma \leq 3/2$ and $t \leq 0$,

$$F_X^{\lambda}(s,\alpha) \ll A^{|\sigma|}(1+|s|)^{|\sigma|/\lambda+B} \tag{5.11}$$

uniformly for $X \geq X_0$.

From (5.4) and (5.11) we get, for $\sigma \le 3/2$,

$$F_X^{\lambda}(s, \alpha) \ll A^{|\sigma|}(1 + |s|)^{|\sigma|/\lambda + B} e^{C(|s|^{\delta} + |s|^{\lambda \rho})} \tag{5.12}$$

uniformly for $X \ge X_0$. Hence the argument based on Vitali's convergence theorem in the proof of Theorem 1 shows that $F^{\lambda}(s, \alpha)$ is entire and satisfies the bound in (5.12), therefore $F^{\lambda}(s, \alpha)$ belongs to $M(1/\lambda)$ and the first part of Theorem 5 is proved.

Suppose now that $F \in M(\rho)$ and $0 < \lambda < 1/\rho$; again we may assume that $\alpha > 0$. In this case $F(s)$ is entire, and the argument is simpler. Starting with the usual integral representation of $F_X^{\lambda}(s, \alpha)$ (recall that X is a large integer) and shifting the line of integration to $u = -X - 1/2$, thanks to the decay of the Γ function on vertical strips we obtain

$$\begin{aligned} F_X^{\lambda}(s, \alpha) &= \frac{1}{2\pi i} \int_{(u_0)} F(s + \lambda w)\Gamma(w) z_X^{-w} \, dw \\ &= \sum_{k=0}^{X} \frac{(-1)^k}{k!} F(s - \lambda k) z_X^k + O\left(\int_{(-X-1/2)} |F(s + \lambda w)\Gamma(w) z_X^{-w} \, dw| \right). \end{aligned} \tag{5.13}$$

Next we show that, as $X \to \infty$, the last integral tends to 0 and the resulting series is suitably convergent, thus it represents an entire function. This provides the analytic continuation and series representation of $F^{\lambda}(s, \alpha)$ over the whole complex plane, since $F_X^{\lambda}(s, \alpha) \to F^{\lambda}(s, \alpha)$ for $\sigma > 1$. Moreover, we get suitable bounds for that series, showing that $F^{\lambda}(s, \alpha)$ belongs to $M(\rho)$ and thus closing the proof of Theorem 5.

Once again thanks to (1.6), for $u = -X - 1/2$ we have

$$F(s + \lambda w) \ll A^{|\sigma| + \lambda(X + 1/2)}(|s| + \lambda |w|)^{\rho(|\sigma| + \lambda X) + B} e^{C(|s| + \lambda |w|)^{\delta}},$$

and moreover (reflection and Stirling formulae)

$$\begin{aligned} \Gamma(w) &\ll e^{-\pi |v|} \frac{1}{|\Gamma(1 - w)|} \ll e^{-\pi |v| + |v| \, |\arg(-w)|} |w|^{-X-1} e^X, \\ z_X^{-w} &\ll |z_X|^{X + 1/2} e^{v \arg z_X}. \end{aligned}$$

Hence

$$F(s + \lambda w)\Gamma(w) z_X^{-w} \ll A^{|\sigma| + X}(|s| + \lambda |w|)^{\rho(|\sigma| + \lambda X) + B} |w|^{-X-1} e^{C(|s| + \lambda |w|)^{\delta}} e^{-c(X)|v|}$$

where $c(X) \gg 1/X$. Let $K \subset \mathbb{C}$ be compact, $s \in K$ and $X \ge X_0(K, F, \rho, \lambda)$. Then the above bound becomes

$$F(s + \lambda w)\Gamma(w) z_X^{-w} \ll A^X |w|^{-(1 - \lambda \rho)X + B} e^{C|w|^{\delta} - c(X)|v|} \ll A^X X^{-(1 - \lambda \rho)X} e^{C|v|^{\delta} - c|v|/X}$$

uniformly for $s \in K$ (constants may now also depend on K), and therefore

$$\int_{(-X-1/2)} |F(s + \lambda w)\Gamma(w) z_X^{-w} \, dw| \ll A^X X^{-(1 - \lambda \rho)X} \to 0$$

as $X \to \infty$ since $\lambda < 1/\rho$. Finally, we show that the series

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{k!} F(s - \lambda k)(2\pi i \alpha)^k, \tag{5.14}$$

obtained from (5.13) as $X \to \infty$, is absolutely and uniformly convergent for $s \in K$ and, at the same time, we give the required bounds to show that $F^\lambda(s, \alpha)$ belongs to $M(\rho)$. Indeed, from (1.6) we get

$$\sum_{k=0}^{\infty} \frac{1}{k!} |F(s - \lambda k)|(2\pi\alpha)^k \ll \sum_{k=0}^{\infty} \frac{1}{k!} A^{|\sigma|+\lambda k}(|s| + \lambda k)^{\rho(|\sigma|+\lambda k)+B} e^{C(|s|+\lambda k)^\delta}(2\pi\alpha)^k, \tag{5.15}$$

which proves the absolute and uniform convergence since the term $k!$ dominates the terms in the numerator, again thanks to the fact that $\lambda < 1/\rho$. Moreover, for $|s|$ sufficiently large we split the series in the right hand side of (5.15) into $S_1 + S_2$, where S_1 is the sum with $k \le |s|/\lambda$ and S_2 with $k > |s|/\lambda$. But since $\lambda < 1/\rho$,

$$S_1 \ll A^{|\sigma|} e^{C|s|^\delta}(1 + |s|)^{\rho|\sigma|+B} \sum_{k \le |s|/\lambda} \frac{(C|s|^{\rho\lambda})^k}{k!} \le A^{|\sigma|}(1 + |s|)^{\rho|\sigma|+B} e^{C(|s|^\delta + |s|^{\lambda\rho})}$$

$$\ll A^{|\sigma|}(1 + |s|)^{\rho|\sigma|+B} e^{C|s|^\delta}$$

with a suitable $\delta < 1$, and

$$S_2 \ll A^{|\sigma|} \sum_{k > |s|/\lambda} \frac{e^k}{k^k} A^k k^{\rho\lambda k} e^{Ck^\delta} \ll A^{|\sigma|} \sum_{k > |s|/\lambda} \frac{A^k}{k^{(1-\lambda\rho)k}} \ll A^{|\sigma|},$$

and the result follows.

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