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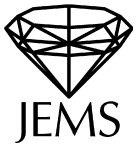

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Finite orbit decomposition of real flag manifolds

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Abstract. Let G be a connected real semisimple Lie group and H a closed connected subgroup. Let P be a minimal parabolic subgroup of G. It is shown that H has an open orbit on the flag manifold G/P if and only if it has finitely many orbits on G/P. This confirms a conjecture by T. Matsuki.

Keywords. Flag manifold, orbit decomposition, spherical subgroup

1. Introduction

Let G be a connected real semisimple Lie group and P a minimal parabolic subgroup. Let $H < G$ be a closed and connected subgroup. The following theorem was conjectured by T. Matsuki [13].

Theorem 1.1. *If there exists an open H-orbit on the real flag variety G/P then the double coset space $H\backslash G/P$ is finite.*

The purpose of this paper is to give a proof of Matsuki's conjecture. Note that the converse statement is easy: if $H\backslash G/P$ is finite, then at least one double coset must be open as a consequence of the Baire category theorem. Further we remark that the theorem becomes false if the parabolic subgroup P is not minimal. A standard counterexample is $G = \mathrm{SL}(3,\mathbb{R})$ with P a maximal parabolic and H the unipotent part of P.

In case G is a complex algebraic reductive group, the minimal parabolic P equals a Borel subgroup B of G. A complex algebraic subgroup which has an open orbit on G/B is called *spherical*. In this case the finiteness of $H\backslash G/B$ for a spherical subgroup H is a result of Brion [5] and Vinberg [14] with a simplified proof by Knop [10]. The spherical subgroups of a complex algebraic group have been classified by Krämer [11] and Brion [6], but to our knowledge there exists no such classification for G real.

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For G real and H a symmetric subgroup (that is, H is the identity component of the set of fixed points for an involution), it was shown by Wolf [15] that both the assumption and conclusion of Theorem 1.1 are always fulfilled.

Our proof of Theorem 1.1 proceeds in two steps. In the first step we reduce the assertion to the case where the real rank of G is one. The argument for this step is essentially due to Matsuki [13]. For rank one groups we then treat the cases where H is reductive or non-reductive in G separately. In case H is non-reductive, one shows that H is contained in a conjugate of P and that there are two, three or four H-orbits on G/P. For reductive H we prove a refined statement:

Theorem 1.2. *Suppose that G is of real rank one and that H is a connected reductive subgroup with an open orbit on G/P. Then there is a symmetric subgroup $H' \supset H$ such that the H'-orbit decomposition of G/P equals the H-orbit decomposition.*

This concludes the proof of Theorem 1.1 since as mentioned above, $H\backslash G/P$ is finite for all symmetric subgroups of G.

Finally we remark that the conclusion of Theorem 1.2 is false in higher real rank. For example, $H = \mathrm{SL}(2,\mathbb{R})$ diagonally embedded in the triple product $G = \mathrm{SL}(2,\mathbb{R})^3$ admits an open orbit in G/P (see [8]). Let $P = P_1^3$, where P_1 is a parabolic subgroup of $\mathrm{SL}(2,\mathbb{R})$. Then the H-orbit through the origin of G/P is one-dimensional. On the other hand, the proper symmetric subgroups containing H have the form $H' = \mathrm{SL}(2,\mathbb{R})^2$, embedded by $(x, y) \mapsto (x, x, y)$ up to permutation, and for these groups the orbit through the origin is two-dimensional.

After we finished this paper, it was brought to our attention by T. Kobayashi that the subject was previously considered by F. Bien [2], where an outline is given for a proof of Theorem 1.1. Apart from the reduction suggested by Matsuki, there is however no overlap with the current approach.

2. Reduction to the rank one case

Let us call the pair (G, H) *real spherical* provided there are open H-orbits on G/P. This means that the corresponding infinitesimal pair $(\mathfrak{g}, \mathfrak{h})$ of Lie algebras satisfies $\mathfrak{g} = \mathfrak{h} + \mathrm{Ad}(x)\mathfrak{p}$ for some $x \in G$. Since all our groups will be real, we will just say *spherical*. As is customary, we denote Lie subgroups of G by upper case roman letters and their Lie algebras by the corresponding lower case German letters.

Let (G, H) be a spherical pair. Matsuki remarked in [13, p. 813] that Theorem 1.1 holds true provided it is valid for all spherical pairs (G, H) where G is a semisimple Lie group of real rank one. The purpose of this section is to provide a proof of this remark. We follow closely the proof of [13, Theorem 4].

We assume that the assertion of Theorem 1.1 holds for all rank one groups. Let (G, H) be a spherical pair. Then there exists an open orbit in G/P, say Hx_0P.

We fix a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$, with corresponding involution θ, and a maximal abelian subspace $\mathfrak{a} \subset \mathfrak{s}$. We assume that $P \supset A$ and denote by $\Pi \subset \mathfrak{a}^*$ the set of simple roots attached to P. For $\alpha \in \Pi$ we define the parabolic subgroup $P_\alpha := Ps_\alpha P \cup P$

where $s_\alpha \in G$ is a Weyl group representative of the reflection associated to α, and write $P_\alpha = L_\alpha U_\alpha$ for its Levi decomposition relative to A. Then L_α has real rank one. Write $G = P_{\alpha_1} \cdot \ldots \cdot P_{\alpha_n}$ as a product of such parabolics. Set

$$P^i := P_{\alpha_1} \cdot \ldots \cdot P_{\alpha_i} \quad (0 \leq i \leq n)$$

with the convention that $P^0 = P$. We will prove by induction on i that $H\backslash Hx_0P^i/P$ is finite. The conclusion is reached after n steps.

The case of $i = 0$ is clear. Assume that

$$Hx_0P^i = Hg_1P \cup \cdots \cup Hg_kP \tag{2.1}$$

for elements $g_1, \ldots, g_k \in G$ and let $\alpha = \alpha_{i+1} \in \Pi$. Then it is sufficient to show for any $g \in Hx_0P^i$ that HgP_α breaks into finitely many $H \times P$-orbits.

We shall first prove that there exists a relatively open $H \times P$-orbit in HgP_α. More precisely, we will show that for some $r = 1, \ldots, k$ we have $Hg_rP \subset HgP_\alpha$ open.

Note that $\mathfrak{p}_\alpha = \mathfrak{p} + \mathfrak{g}^{-\alpha} + \mathfrak{g}^{-2\alpha}$ and set $V := \exp(\mathfrak{g}^{-\alpha} + \mathfrak{g}^{-2\alpha})$. Then VP is an open neighborhood of $\mathbf{1}$ in P_α. As Hx_0P^i is open and contains g, we obtain an open subset $O \subset V$ with $\mathbf{1} \in O$ such that $gx \in Hx_0P^i$ for all $x \in O$. Moreover, because of (2.1) we have $O = \bigcup_{r=1}^k O_r$ with

$$O_r = \{x \in O \mid gx \in Hg_rP\},$$

and at least one of these sets has non-empty interior by Baire's theorem. Fix such an r. Then $HgxP = Hg_rP$ for every $x \in O_r$, and as O_rP has non-empty interior in VP it follows that Hg_rP has non-empty interior in HgP_α, hence is an open subset by transitivity of $H \times P$.

We can now show that HgP_α decomposes into finitely many orbits. Notice that we have $HgP_\alpha = Hg_rP_\alpha$. For simplicity we replace H by $g_rHg_r^{-1}$, and claim that if HP is open in HP_α then the latter set is a finite union of $H \times P$-orbits.

We write $\mathfrak{p}_\alpha = \mathfrak{l}_\alpha + \mathfrak{u}_\alpha$ for the Levi decomposition of the Lie algebra of P_α. Further we denote by

$$\pi_\alpha : \mathfrak{p}_\alpha \to \mathfrak{l}_\alpha$$

the projection along $\mathfrak{u}_\alpha$ and remark that this map is a Lie algebra homomorphism. Set $\mathfrak{h}_\alpha := \pi_\alpha(\mathfrak{p}_\alpha \cap \mathfrak{h})$. As HP is open we find

$$\mathfrak{h} + \mathfrak{p} = \mathfrak{h} + \mathfrak{p}_\alpha,$$

and hence $\mathfrak{p}_\alpha = (\mathfrak{p}_\alpha \cap \mathfrak{h}) + \mathfrak{p}$. In turn this implies that

$$\mathfrak{l}_\alpha = \mathfrak{h}_\alpha + (\mathfrak{l}_\alpha \cap \mathfrak{p}).$$

In other words, $(\mathfrak{l}_\alpha, \mathfrak{h}_\alpha)$ is a rank-one spherical pair. We thus find that $H_\alpha\backslash L_\alpha/(L_\alpha \cap P)$ is finite (the fact that L_α may be disconnected does not matter, because all its components intersect P non-trivially). We write

$$L_\alpha = \bigcup_{j=1}^m H_\alpha x_j (L_\alpha \cap P)$$

and claim that

$$HP_\alpha = \bigcup_{j=1}^{m} Hx_j P.$$

As $P_\alpha = L_\alpha P$, it suffices to show that $hx_j \in Hx_j P$ for all $h \in H_\alpha$ and all j. Note that $\mathfrak{h}_\alpha$ is contained in the subalgebra $(\mathfrak{p}_\alpha \cap \mathfrak{h}) + \mathfrak{u}_\alpha$ of $\mathfrak{p}_\alpha$, and hence H_α is contained in the subgroup $(P_\alpha \cap H)U_\alpha$ of P_α. It follows that

$$hx_j \in (P_\alpha \cap H)U_\alpha x_j = (P_\alpha \cap H)x_j U_\alpha \subset Hx_j P$$

as claimed.

Hence the proof of Theorem 1.1 is reduced to the following result.

Proposition 2.1. *Let G be a semisimple Lie group of real rank one, and H a connected spherical subgroup. Then the number of H-orbits on G/P is finite.*

Example: $G = \mathrm{SL}(2, \mathbb{R})$. Every one-dimensional subalgebra is conjugate to $\mathfrak{k}$, $\mathfrak{a}$ or $\mathfrak{n}$. The first two are symmetric, and in the third case finiteness of $H\backslash G/P$ follows from the Bruhat decomposition. Hence $H\backslash G/P$ is finite for every non-trivial connected subgroup H.

2.1. Simple groups

Once the real rank is one, we can easily reduce to the case that G is simple. Otherwise G is locally isomorphic to $G_1 \times K_2$ where G_1 is simple of real rank one and K_2 is compact. Then $P = P_1 \times K_2$ where $P_1 \subset G_1$ is minimal parabolic, and hence $G/P = G_1/P_1$. Moreover, if H_1 denotes the projection of H on G_1, then H-orbits on G/P are the same as H_1-orbits on G_1/P_1.

3. Non-reductive spherical subgroups

In this section we prove Proposition 2.1 for spherical subgroups which are not reductive.

Lemma 3.1. *Let $\mathfrak{g}$ be a real reductive Lie algebra of real rank one, and $\mathfrak{h}$ a subalgebra which is not reductive in $\mathfrak{g}$. Then $\mathfrak{h} \subset \mathfrak{p}$ up to conjugation.*

Proof. This is an easy consequence of the main result of [4]. Let $\mathfrak{u}$ denote the unipotent radical of $\mathfrak{h}$; then $\mathfrak{u} \neq 0$ by assumption. According to [4] there is a parabolic subalgebra $\mathfrak{p}' \subset \mathfrak{g}$ such that

(1) the normalizer of $\mathfrak{u}$ is contained in $\mathfrak{p}'$, and
(2) the unipotent radical of $\mathfrak{p}'$ contains $\mathfrak{u}$.

Hence $\mathfrak{h} \subset \mathfrak{p}' \neq \mathfrak{g}$, and $\mathfrak{p}'$ is conjugate to $\mathfrak{p}$ because of rank one. □

For a rank one group G we let s be a non-trivial Weyl group representative and recall the Bruhat decomposition $G = P \cup PsP$. The following result together with Lemma 3.1 implies the conclusion of Proposition 2.1 for H non-reductive.

Lemma 3.2. *Let $\mathfrak{g}$ be a simple real rank one Lie algebra, $\mathfrak{h}$ a spherical subalgebra, and $\mathfrak{p}$ a minimal parabolic subalgebra for which $\mathfrak{h} \subset \mathfrak{p}$. Then the Bruhat decomposition $G/P = P \cup PsP$ is H-stable and there are at most four orbits of H on G/P.*

Proof. We denote by $P = MAN$ the standard Langlands decomposition of P relative to A.

The cells of the Bruhat decomposition are P-stable, hence also H-stable. In particular, the closed cell $P \in G/P$ is an H-orbit. Hence by assumption the open cell $\mathcal{O} := NsP$ admits at least one open H-orbit, and the assertion is that it then decomposes into at most three H-orbits.

We decompose $\mathfrak{h} = \mathfrak{l} \ltimes \mathfrak{n}_1$ with $\mathfrak{l}$ reductive in $\mathfrak{g}$ and $\mathfrak{n}_1$ an ideal which acts on $\mathfrak{g}$ nilpotently. As $\mathfrak{h} \subset \mathfrak{p}$ we have $\mathfrak{n}_1 \subset \mathfrak{n}$, and it is no loss of generality to assume that $\mathfrak{l} \subset \mathfrak{m} + \mathfrak{a}$. Then with $\mathfrak{m}_1 = \mathfrak{l} \cap \mathfrak{m}$ we have

$$\mathfrak{h} = \mathfrak{m}_1 + \mathbb{R}X + \mathfrak{n}_1$$

for some $X = Y + Z$ with $Y \in \mathfrak{m}$ and $Z \in \mathfrak{a}$. Since $[\mathfrak{l}, \mathfrak{l}] \subset \mathfrak{m}_1$, the element X belongs to the center of $\mathfrak{l}$ and commutes with $\mathfrak{m}_1$. Let $\mathfrak{s}_1 = \mathbb{R}X$, then both $\mathfrak{s}_1$ and $\mathfrak{m}_1$ normalize $\mathfrak{n}_1$ and we have

$$H = M_1 S_1 N_1$$

for the corresponding subgroups of G. Since $M_1 S_1 s \subset sP$ we see that the H-orbits in NsP are the sets $N_1\, \mathrm{c}_{M_1 S_1}(x) sP$ for $x \in N$, where $\mathrm{c}_g(x) = gxg^{-1}$. In particular, $N_1 sP \subset NsP$ is an H-orbit. If $N_1 = N$ we are done, hence we may assume $\mathfrak{n}_1 \subsetneq \mathfrak{n}$.

Note that $Z \neq 0$. Otherwise $X = Y$, hence $X \in \mathfrak{m}_1$ and $\mathfrak{h} = \mathfrak{m}_1 + \mathfrak{n}_1$. Then $H = M_1 N_1$ and H-orbits have the form $N_1\, \mathrm{c}_{M_1}(x) sP$. Since $N_1\, \mathrm{c}_{M_1}(x)$ cannot be open in N for any x, a contradiction is reached.

The element X acts semisimply on $\mathfrak{n}_{\mathbb{C}}$ and preserves the subspace $\mathfrak{n}_{1\mathbb{C}}$. We denote by α the indivisible positive root of $\mathfrak{a}$. Then $\mathfrak{n} = \mathfrak{g}^{\alpha} \oplus \mathfrak{g}^{2\alpha}$ and $[\mathfrak{n}, \mathfrak{n}] \subset \mathfrak{g}^{2\alpha}$. Since $Z \neq 0$, the spaces $\mathfrak{g}^{\alpha}_{\mathbb{C}}$ and $\mathfrak{g}^{2\alpha}_{\mathbb{C}}$ have no eigenvalues in common for $\mathrm{ad}(X)$. It follows that

$$\mathfrak{n}_1 = (\mathfrak{n}_1 \cap \mathfrak{g}^{\alpha}) \oplus (\mathfrak{n}_1 \cap \mathfrak{g}^{2\alpha}). \tag{3.1}$$

Let $\mathfrak{n}_0$ be the orthogonal complement of $\mathfrak{n}_1$ in $\mathfrak{n}$. Then $\mathfrak{n}_0 \neq 0$ and (3.1) implies

$$\mathfrak{n}_0 = (\mathfrak{n}_0 \cap \mathfrak{g}^{\alpha}) \oplus (\mathfrak{n}_0 \cap \mathfrak{g}^{2\alpha}). \tag{3.2}$$

It follows from (3.1) and (3.2) together with [9, Ch. IV, Lemma 6.8] that the exponential map induces a diffeomorphism of $\mathfrak{n}_0$ and the left coset space $N_1 \backslash N$. Note that $\mathfrak{n}_0$ is $M_1 S_1$-invariant. We conclude that the H-orbits in NsP correspond to the orbits of $\mathrm{Ad}(M_1 S_1)$ on $\mathfrak{n}_0$. In particular, the H-orbit $N_1 sP$ corresponds to $\{0\} \subset \mathfrak{n}_0$.

Since M acts isometrically on $\mathfrak{n}$, M_1 acts isometrically on $\mathfrak{n}_0$. Furthermore, let $X = Y + Z$ be normalized so that $\alpha(Z) = 1$ and set $s_t = \exp(tX) \in S_1$. Then for $j = 1, 2$,

$$\|\mathrm{Ad}(s_t)x\| = e^{jt}\|x\|, \qquad x \in \mathfrak{n}_0 \cap \mathfrak{g}^{j\alpha},\ t \in \mathbb{R}. \tag{3.3}$$

It follows that if $\mathcal{O}_1 \neq \{0\}$ is an $\mathrm{Ad}(M_1 S_1)$-orbit in $\mathfrak{n}_0$, then the intersection of $\mathcal{O}_1$ with every sphere in $\mathfrak{n}_0$ is non-empty and is an $\mathrm{Ad}(M_1)$-orbit.

Assume first that $\dim \mathfrak{n}_0 > 1$. Then spheres in $\mathfrak{n}_0$ are connected, and by compactness an open $\mathrm{Ad}(M_1)$-orbit is the entire sphere. Hence the open $\mathrm{Ad}(M_1 S_1)$-orbit in $\mathfrak{n}_0$ is $\mathfrak{n}_0 \setminus \{0\}$. In this case NsP decomposes into two orbits, $N_1 sP$ and its complement.

Assume finally that $\dim \mathfrak{n}_0 = 1$. In this case (3.3) implies that $\mathrm{Ad}(s_t)x = e^{jt}x$ for all $x \in \mathfrak{n}_0$ and all $t \in \mathbb{R}$ (where $j = 1$ or 2). Hence there are three orbits in $\mathfrak{n}_0$, corresponding to $\{0\}$ and the two components of its complement. □

Remark 3.1. The proof shows a bit more. The open N-orbit NsP breaks into at most three H-orbits. If we identify N with $\mathbb{R}^n$ and N_1 with $\mathbb{R}^k$, then these H-orbits are of the following type:

(1) $\mathbb{R}^n$ (one orbit, the case where $H \supset N$) when $k = n$.
(2) $\mathbb{R}^k$ and $\mathbb{R}^{n-k}\setminus\{0\} \times \mathbb{R}^k$ when $0 \leq k < n-1$.
(3) $\mathbb{R}^{n-1}$, $\mathbb{R}^+ \times \mathbb{R}^{n-1}$, $\mathbb{R}^- \times \mathbb{R}^{n-1}$ when $k = n-1$.

4. Some results in real rank one

We now turn to the case where G has real rank one and our spherical subgroup $H \subset G$ is reductive, in which case Proposition 2.1 will ultimately be deduced from Theorem 1.2. The proof of that theorem will be given after we have prepared for it through several sections. Our first preparation, Proposition 4.1, shows in this case that if $H \subset H' \subset G$ with H' symmetric, then the H-orbits in G/P coincide with the H'-orbits. The proof uses Matsuki's description of the H'-orbits on G/P.

Lemma 4.1. *Let $\mathfrak{g}$ be a real reductive Lie algebra with Cartan involution θ, and let $\mathfrak{g}_n \subset \mathfrak{g}$ be its maximal non-compact ideal. Let $\mathfrak{h} \subset \mathfrak{g}$ be a θ-stable subalgebra such that $\mathfrak{g} = \mathfrak{h} + \mathfrak{k}$. Then $\mathfrak{g}_n \subset \mathfrak{h}$.*

Proof. It follows from the assumption that $\mathfrak{s} \subset \mathfrak{h}$, and this implies the conclusion as $\mathfrak{g}_n$ is generated by $\mathfrak{s}$. □

Recall that a subalgebra $\mathfrak{h}' \subset \mathfrak{g}$ is called *symmetric* if it is the fixed points of an involution of $\mathfrak{g}$. Recall also that for every involution there exists a commuting Cartan involution. Given an involution σ, we write $\mathfrak{g} = \mathfrak{h}' + \mathfrak{q}$ for the corresponding decomposition of $\mathfrak{g}$.

Lemma 4.2. *Let $\mathfrak{g}$ be simple of real rank one and let $\mathfrak{h}'$ be a proper symmetric subalgebra defined by an involution σ commuting with θ. Let $\mathfrak{p} = \mathfrak{m} + \mathfrak{a} + \mathfrak{n}$ be a minimal parabolic subalgebra with the indicated Langlands decomposition, and assume $\mathfrak{a} \subset \mathfrak{s} \cap \mathfrak{q}$. Then*

$$\mathfrak{g} = \mathfrak{h}' + \mathfrak{p} \quad \textit{and} \quad \mathfrak{h}' \cap \mathfrak{p} \subset \mathfrak{m}.$$

Proof. This follows from [12, Theorem 3], since $\sigma(\mathfrak{n}) = \theta(\mathfrak{n})$. □

Lemma 4.3. *Let G be a simple Lie group of real rank one and let $\mathfrak{h} \subset \mathfrak{h}' \subsetneq \mathfrak{g}$ be reductive subalgebras such that*

(1) *$\mathfrak{h}$ is spherical,*
(2) *$\mathfrak{h}'$ is symmetric and defined by an involution commuting with θ.*

Then:

(3) *There exists a minimal parabolic subalgebra* $\mathfrak{p}$ *as in Lemma 4.2 with* $\mathfrak{a} \subset \mathfrak{s} \cap \mathfrak{q}$ *such that* $\mathfrak{h}' = \mathfrak{h} + (\mathfrak{h}' \cap \mathfrak{m})$ *and*

$$H' = H(H' \cap M).$$

(4) *Let* $\mathfrak{h}'_n$ *be the maximal non-compact ideal in* $\mathfrak{h}'$. *Then* $\mathfrak{h}'_n \subset \mathfrak{h}$.

Proof. Let $\mathfrak{p}_1$ be a minimal parabolic subalgebra for which $\mathfrak{h} + \mathfrak{p}_1 = \mathfrak{g}$. Then $\mathfrak{h}' + \mathfrak{p}_1 = \mathfrak{g}$ and it follows from [12, Theorems 1 and 3] that $\mathfrak{p}_1$ is H'-conjugate to a minimal parabolic subalgebra $\mathfrak{p}$ with $\mathfrak{a} \subset \mathfrak{s} \cap \mathfrak{q}$. Thus $\mathfrak{g} = \mathfrak{h} + \mathrm{Ad}(x)\mathfrak{p}$ for some $x \in H'$. Since $\mathfrak{h} \subset \mathfrak{h}'$, this implies that

$$\mathfrak{h}' = \mathfrak{h} + \mathfrak{h}' \cap \mathrm{Ad}(x)\mathfrak{p} = \mathfrak{h} + \mathrm{Ad}(x)(\mathfrak{h}' \cap \mathfrak{p}).$$

From Lemma 4.2 we find $\mathfrak{h}' \cap \mathfrak{p} = \mathfrak{h}' \cap \mathfrak{m}$, which is compact. It follows that $Hx(H' \cap M)$ is open and closed in H', hence equal to H'. Hence $H \times (H' \cap M)$ is left×right transitive on H', and (3) follows.

Since $\mathfrak{h}$ is reductive in $\mathfrak{g}$, it is reductive in $\mathfrak{h}'$. Hence some H'-conjugate $\mathfrak{h}_1$ of it is θ-stable. The conclusion in (3) is valid for $\mathfrak{h}_1$ and hence $\mathfrak{h}' = \mathfrak{h}_1 + \mathfrak{h}' \cap \mathfrak{k}$. It now follows from Lemma 4.1 that $\mathfrak{h}'_n \subset \mathfrak{h}_1$. Since $\mathfrak{h}'_n$ is an ideal, this implies $\mathfrak{h}'_n \subset \mathfrak{h}$ as well. □

Proposition 4.1. *Let* G *be a connected simple Lie group of real rank one, and let* $H \subset H'$ *be connected reductive subgroups such that* H *is spherical and* H' *is symmetric and proper in* G. *Then* H *is transitive on each* H'*-orbit in* G/P.

Proof. Choose a Cartan involution which commutes with the involution which defines H'. Let $\mathfrak{p}$ be as in Lemma 4.3. Since the real rank of G is one, it follows from Matsuki's orbit description [12] that H' has only open and closed orbits in G/P. The open orbits are of the form $H'xP$ for $x \in N_K(\mathfrak{a})$, the normalizer in K of $\mathfrak{a}$, and the closed orbits are of the form $H'yP$ with $y \in K$ such that $\mathrm{Ad}(y)(\mathfrak{a}) \subset \mathfrak{h}'$.

It follows from Lemma 4.3 that

$$H'xP = H(H' \cap M)xP = HxP$$

for $x \in N_K(\mathfrak{a})$.

Let $\mathfrak{h}'_c$ denote the ideal in $\mathfrak{h}'$ which is complementary to $\mathfrak{h}'_n$. Then $\mathfrak{h}' = \mathfrak{h}'_n \oplus \mathfrak{h}'_c$ and $H' = H'_n H'_c$. If $\mathrm{Ad}(y)(\mathfrak{a}) \subset \mathfrak{h}'$ then $\mathrm{Ad}(y)(\mathfrak{a}) \subset \mathfrak{h}'_n$ and hence $\mathrm{Ad}(y)(\mathfrak{a})$ is centralized by H'_c. It follows that $H'_c \subset yMy^{-1}$ and hence

$$H'yP = H'_n yP = HyP$$

since $H'_n \subset H$ by Lemma 4.3. □

5. Example: The Lorentzian groups

Before we treat the general case, it is instructive to see the proof of Theorem 1.2 for the case of $\mathrm{SO}_0(1, n)$ for $n \geq 2$.

Proof. We observe that $G = \mathrm{SO}_0(1,n)$ acts on $\mathbb{R}^{n+1}$. In what follows we write the elements of $x \in \mathbb{R}^{n+1}$ as $x = (x_0, x')$ with $x' \in \mathbb{R}^n$. The stabilizer $P \subset G$ of the line $\mathbb{R}(1,1,0,\dots,0) \in \mathbb{P}(\mathbb{R}^{n+1})$ is a minimal parabolic subgroup. Note that $G/P = S^{n-1}$ is an $n-1$-dimensional sphere which we shall identify with a projective quadric:

$$S^{n-1} = \{[x] \in \mathbb{P}(\mathbb{R}^{n+1}) \mid x_0^2 = \|x'\|^2 = x_1^2 + \cdots + x_n^2\}. \tag{5.1}$$

Let $\mathfrak{h}$ be a reductive spherical subalgebra, and let $\mathfrak{h} = \mathfrak{h}_n \oplus \mathfrak{h}_c$ be the decomposition of $\mathfrak{h}$ into ideals such that $\mathfrak{h}_n$ is non-compact and $\mathfrak{h}_c$ is compact. Since $\mathfrak{so}(1,n)$ has rank one and root multiplicity $m_{2\alpha} = 0$, the same must be true for $\mathfrak{h}_n$. Hence $\mathfrak{h}_n = \mathfrak{so}(1,p)$ for some $0 \le p \le n$. Furthermore, by conjugation of $\mathfrak{h}$ we can arrange that $\mathfrak{h}_n = \mathfrak{so}(1,p)$ is realized in the left upper corner of $\mathfrak{g} = \mathfrak{so}(1,n)$, and accordingly

$$\mathfrak{h} = \mathfrak{h}_n \oplus \mathfrak{h}_c, \qquad H = \mathrm{SO}_0(1,p) \times H_c \tag{5.2}$$

with $\mathfrak{h}_c \subset \mathfrak{so}(n-p)$ and $\mathfrak{so}(n-p)$ embedded in the lower right corner. It now follows from Proposition 4.1 that orbits on G/P for H are the same as for the symmetric subgroup $H' = \mathrm{SO}_0(1,p) \times \mathrm{SO}(n-p)$ of G. Thus the proof of Theorem 1.2 is complete for this case. □

Remark 5.1. It follows from the above that every spherical subgroup in $\mathrm{SO}_0(1,n)$ is conjugate to a subgroup of $H' = \mathrm{SO}_0(1,p) \times \mathrm{SO}(n-p)$ of the form (5.2) for some $0 \le p \le n$. Furthermore since $H' \cap M = \mathrm{SO}(n-p-1)$ in this case, it follows that such a subgroup is spherical if and only if H_c is transitive on $S^{n-p-1} = \mathrm{SO}(n-p)/\mathrm{SO}(n-p-1)$. Besides $H = H'$ this can be attained in case p satisfies certain parity conditions. A typical example is $p = n - 2k$ and

$$H = \mathrm{SO}_0(1, n-2k) \times \mathrm{SU}(k),$$

since $H_c = \mathrm{SU}(k)$ acts transitively on the spheres in $\mathbb{R}^{2k}$. For $p = n - 4k$ we can also take $H_c = \mathrm{Sp}(k)$, which again acts transitively on spheres. Besides these two series there are three exceptional cases (see [3] for the classification of transitive actions of compact Lie groups on spheres).

6. Classifications

In this section we prepare for the proof of Theorem 1.2 by showing that in the rank one case a maximal reductive subgroup is either symmetric or not spherical. This will be done by applying some results which in turn are derived from known classifications of simple Lie groups and their subgroups.

We first recall the classification of the simple real rank one Lie algebras:

$$\mathfrak{so}(1,n), \quad \mathfrak{su}(1,n), \quad \mathfrak{sp}(1,n), \quad \mathfrak{f}_4 \tag{6.1}$$

where $\mathfrak{f}_4 = \mathfrak{f}_{4(-20)}$, the real form of $\mathfrak{f}_{4\mathbb{C}}$ with maximal compact $\mathfrak{so}(9)$. In the first series, n is limited to $n \ge 2$. In the second and third series $n \ge 1$ is allowed, but as $\mathfrak{so}(1,2) \simeq \mathfrak{su}(1,1)$ and $\mathfrak{so}(1,4) \simeq \mathfrak{sp}(1,1)$ an exhaustive list is obtained by taking $n \ge 2$ in all cases.

6.1. Symmetric subalgebras

Lemma 6.1. *The symmetric pairs (excluding $\mathfrak{h} = \mathfrak{g}$ and $\mathfrak{h} = \mathfrak{k}$) for the simple real rank one Lie algebras are*

$$\begin{aligned}
&\mathfrak{g} = \mathfrak{so}(1,n), && \mathfrak{h}_m = \mathfrak{so}(1,m) \times \mathfrak{so}(n-m), && 0 < m < n,\\
&\mathfrak{g} = \mathfrak{su}(1,n), && \mathfrak{h}_m = \mathfrak{s}(\mathfrak{u}(1,m) \times \mathfrak{u}(n-m)), && 0 < m < n,\\
& && \mathfrak{h} = \mathfrak{so}(1,n), &&\\
&\mathfrak{g} = \mathfrak{sp}(1,n), && \mathfrak{h}_m = \mathfrak{sp}(1,m) \times \mathfrak{sp}(n-m), && 0 < m < n,\\
& && \mathfrak{h} = \mathfrak{u}(1,n), &&\\
&\mathfrak{g} = \mathfrak{f}_4, && \mathfrak{h}_1 = \mathfrak{so}(1,8), \quad \mathfrak{h}_2 = \mathfrak{sp}(1,2) \times \mathfrak{sp}(1), &&
\end{aligned}$$

Proof. This is seen from Berger's table [1, pp. 157–161]. □

6.2. Maximal reductive subalgebras

We are particularly interested in reductive subalgebras which are maximal. The following lemma provides the key to the reduction to symmetric pairs.

Lemma 6.2. *Let $\mathfrak{g}$ be a simple Lie algebra of real rank one and let $\mathfrak{h} \subset \mathfrak{g}$ be a maximal proper reductive subalgebra. Then either $\mathfrak{h}$ is a symmetric subalgebra, or*

(1) *$\mathfrak{g} = \mathfrak{sp}(1,n)$ and $\mathfrak{h}$ is conjugate to $\mathfrak{so}(1,n) \times \mathfrak{sp}(1)$, where $n > 1$,*
(2) *$\mathfrak{g} = \mathfrak{f}_4$ and $\mathfrak{h}$ is conjugate to $\mathfrak{su}(1,2) \times \mathfrak{su}(3)$,*
(3) *$\mathfrak{g} = \mathfrak{f}_4$ and $\mathfrak{h}$ is conjugate to $\mathfrak{so}(1,2) \times \mathfrak{g}_2$, where $\mathfrak{g}_2$ denotes the compact real form of $\mathfrak{g}_{2\mathbb{C}}$.*

None of the pairs in (1)–(3) *are spherical.*

Proof. It is well known that the symmetric subalgebras of a simple Lie algebra are maximal proper reductive subalgebras.

Reductive subalgebras are listed in [7, pp. 276 and 284], and it is easily seen from these lists together with the list in Lemma 6.1 that only the subalgebras in (1)–(3) are maximal and non-symmetric. The fact that these pairs are not spherical will be proved in the following subsections.

6.2.1. $(\mathfrak{g}, \mathfrak{h}) = (\mathfrak{sp}(1,n), \mathfrak{so}(1,n) \times \mathfrak{sp}(1))$ *is not spherical when* $n > 1$. If $n = 2$ then $\dim \mathfrak{h} = 6$ and $\dim(\mathfrak{g}/\mathfrak{p}) = 7$, so we may assume $n \geq 3$. As in the Lorentzian cases, we identify for $G = \mathrm{Sp}(1,n)$ the flag variety $G/P = S^{4n-1}$ with a quadric in the quaternion projective space $\mathbb{P}(V)$ where $V = \mathbb{H}^{n+1}$:

$$S^{4n-1} = \{[z] \in \mathbb{P}(V) \mid |z_0|^2 = |z_1|^2 + \cdots + |z_n|^2\}.$$

Here the action of G on V is on the left, and

$$[z] = \{zh \mid h \in \mathbb{H},\ h \neq 0\} \in \mathbb{P}(V).$$

As a representation for $\mathrm{SO}_0(1,n)$, the space V decomposes into four copies of $V_0 = \mathbb{R}^{n+1}$ with standard action. Hence the stabilizer in $H_n = \mathrm{SO}_0(1,n)$ of an element $v \in V$ is the stabilizer of four elements in V_0, hence the centralizer of an at most four-dimensional subspace in V_0. The centralizer in $\mathrm{SO}_0(1,n)$ of a four-dimensional subspace of $\mathbb{R}^{n+1}$ is conjugate to the centralizer in $\mathrm{SO}_0(1,n-3)$ of a one-dimensional subspace of $\mathbb{R}^{n-2}$. Since all non-trivial orbits of $\mathrm{SO}_0(1,n-3)$ in $\mathbb{R}^{n-2}$ have codimension one, a simple computation shows that the codimension in $\mathrm{SO}_0(1,n)$ of such a subgroup is $4n-6$. Hence orbits of H_n in V are at most of this dimension, and orbits of H in S^{4n-1} are at most of dimension $4n-3$.

6.2.2. $(\mathfrak{f}_4, \mathfrak{so}(1,2) \times \mathfrak{g}_2)$ *is not spherical.* Since $\dim G/P = 15$, it suffices to show that the subgroup $G_2 \subset K$ with Lie algebra $\mathfrak{g}_2$ has orbits in $K/M = G/P$ of dimension at most 11. Recall that $K = \mathrm{Spin}(9)$ and that we can realize K/M as the unit sphere in the 16-dimensional real spin representation V_{16} of K. This representation decomposes for the standard inclusions $\mathrm{Spin}(7) \subset \mathrm{Spin}(8) \subset \mathrm{Spin}(9)$ into a direct sum of two copies of the spin representation V_8 of $\mathrm{Spin}(7)$. Now G_2 is the isotropy subgroup of a spinor in V_8, and hence as a G_2-representation,

$$V_{16} = V_7 \oplus V_7 \oplus \mathbb{R} \oplus \mathbb{R},$$

with a 7-dimensional representation of G_2. It follows that every orbit of G_2 lies in a product $R_1 S^6 \times R_2 S^6 \subset V_7 \oplus V_7$ of spheres of radii $R_1, R_2 \geq 0$. Furthermore, the action of G_2 on $S^6 \times S^6$ is not transitive as the diagonal is invariant. Since G_2 is compact, we conclude that there are no open orbits on $S^6 \times S^6$. This proves the claim.

6.2.3. $(\mathfrak{f}_4, \mathfrak{su}(2,1) \times \mathfrak{su}(3))$ *is not spherical.* Note that $\dim H = 16$ and $\dim G/P = 15$. Let us first collect a few facts about $\mathfrak{f}_4$. We refer to [7] for more details. Consider the Jordan algebra

$$\mathrm{Herm}(3,\mathbb{O})_{2,1} = \left\{ x = \begin{pmatrix} \alpha_1 & c_3 & -\bar{c}_2 \\ \bar{c}_3 & \alpha_2 & c_1 \\ c_2 & -\bar{c}_1 & \alpha_3 \end{pmatrix} \;\middle|\; \alpha_i \in \mathbb{R},\ c_i \in \mathbb{O} \right\}. \tag{6.2}$$

The group G of automorphisms of $W := \mathrm{Herm}(3,\mathbb{O})_{2,1}$ is a real Lie group with Lie algebra $\mathfrak{f}_4$. Moreover the trace free elements $V := W_{\mathrm{tr}=0}$ is an irreducible real representation for G with a non-zero K-fixed vector. Let $v_0 \in V$ be a highest weight vector. Then $P \cdot v_0 = \mathbb{R}^+ v_0$ and we can realize the flag manifold as the image of $G \cdot v_0$ in $\mathbb{P}(V)$. According to [7, p. 275], $\mathbb{R}^\times G \cdot v_0 = \mathcal{C}$, where

$$\mathcal{C} := \{x \in V \mid x^2 = 0,\ x \neq 0\},$$

and thus $G/P = \mathbb{P}(\mathcal{C})$.

Note that $H = \mathrm{SU}(2,1) \times \mathrm{SU}(3)$ acts naturally on V. The factor $H_n = \mathrm{SU}(2,1)$ acts by matrix conjugation. Further, the automorphism group of $\mathbb{O}$ is G_2 and $H_c = \mathrm{SU}(3)$ is the subgroup which commutes with complex multiplication on $\mathbb{O}$. We will show that every element $[x] \in \mathbb{P}(\mathcal{C})$ has an at least 2-dimensional stabilizer in H.

A straightforward matrix computation shows that if x in (6.2) satisfies $x^2 = 0$ and has trace zero, then up to multiplication by a real number,

$$x = \begin{pmatrix} |c_2|^2 & -\bar{c}_2\bar{c}_1 & -\bar{c}_2 \\ c_1c_2 & |c_1|^2 & c_1 \\ c_2 & -\bar{c}_1 & -1 \end{pmatrix} \tag{6.3}$$

with $|c_1|^2 + |c_2|^2 = 1$.

In what follows, we decompose $\mathbb{O} = \mathbb{C} + \mathbb{C}^\perp$ and regard $\mathbb{O}_I := \mathbb{C}^\perp$ as a complex vector space for the left action of $\mathbb{C}$. Then as a module for SU(3) it is equivalent to the standard complex representation on $\mathbb{C}^3$. Having said that, we write elements $x \in V$ as

$$x = x_{\mathbb{C}} + x_I$$

where $x_{\mathbb{C}} \in i\mathfrak{su}(2,1) \subset V$ and x_I is of the form

$$x_I = \begin{pmatrix} 0 & c_3 & -\bar{c}_2 \\ \bar{c}_3 & 0 & c_1 \\ c_2 & -\bar{c}_1 & 0 \end{pmatrix} \tag{6.4}$$

with $c_1, c_2, c_3 \in \mathbb{O}_I$. Note that this gives us a decomposition of H-modules.

We see from (6.3) that $x_{\mathbb{C}} \neq 0$ for all $x \in \mathcal{C}$. Hence the map

$$\mathbb{P}(\mathcal{C}) \to \mathbb{P}(i\mathfrak{su}(2,1)), \qquad [x] \mapsto [x_{\mathbb{C}}],$$

is defined. As this is an open map, the image of an open H-orbit is a non-empty open set. Since the semisimple elements in $i\mathfrak{su}(2,1)$ are dense, it suffices to consider x in (6.3) with $x_{\mathbb{C}}$ semisimple. If $x_{\mathbb{C}} \in i\mathfrak{su}(2,1)$ is semisimple, it is SU(2, 1)-conjugate to one of

$$\begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & -\alpha_1 - \alpha_2 \end{pmatrix}, \quad \begin{pmatrix} 2\alpha & 0 & 0 \\ 0 & -\alpha & \gamma i \\ 0 & \gamma i & -\alpha \end{pmatrix},$$

where $\alpha_1, \alpha_2, \alpha, \gamma \in \mathbb{R}$. However, the second case does not conform with (6.3). Hence we may assume that $x = x_{\mathbb{C}} + x_I$ with $x_{\mathbb{C}}$ diagonal and with x_I as in (6.4). It follows from (6.3) that $c_1c_2 = \bar{c}_3$.

The fact that $c_1c_2 \in \mathbb{O}_I$ implies that c_1 and c_2 are orthogonal elements. Let $\mathbb{O}_I = \mathbb{C}j \oplus \mathbb{C}l \oplus \mathbb{C}n$ in the standard notation. After application of SU(3) to x_I, it is no loss of generality to assume that $c_1 = aj$ and $c_2 = bl$ for some $a, b \in \mathbb{R}$. Then $c_3 = -abn$ and

$$x_I = \begin{pmatrix} 0 & -abn & bl \\ abn & 0 & aj \\ bl & aj & 0 \end{pmatrix}.$$

The diagonal torus $T < \mathrm{SU}(2,1)$ commutes with the diagonal matrix $x_{\mathbb{C}}$, and embedded into H via

$$t = \mathrm{diag}(t_1, t_2, t_3) \mapsto (t, t)$$

it also stabilizes x_I—note that for $z \in \mathbb{C}$ and $x \in \mathbb{O}_I$ one has $xz = \bar{z}x$. Hence the stabilizer of x in H has dimension at least 2.

This concludes the proof of Lemma 6.2. □

7. Proofs

All ingredients for the proofs have already been prepared.

7.1. Proof of Theorem 1.2

Let G be semisimple of real rank one and $H \subset G$ a connected reductive spherical subgroup. As seen in Section 2.1, we may assume G is simple.

Let $\mathfrak{h}'$ be a maximal proper reductive subalgebra which contains $\mathfrak{h}$. It follows from Lemma 6.2 that $\mathfrak{h}'$ is symmetric, and then Proposition 4.1 implies that H-orbits and H'-orbits agree on G/P.

7.2. Proof of Proposition 2.1 and Theorem 1.1

Theorem 1.2 implies the statement of Proposition 2.1 for reductive subgroups by the results of [15] or [12]. By combining with Lemmas 3.1 and 3.2 we obtain the proposition for general subgroups. This also concludes the proof of Theorem 1.1.

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