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Affine cones over smooth cubic surfaces

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Abstract. We show that affine cones over smooth cubic surfaces do not admit non-trivial $\mathbb{G}_a$-actions.

Keywords. Affine cone, α-invariant, anticanonical divisor, cylinder, del Pezzo surface, $\mathbb{G}_a$-action, log canonical singularity

Throughout this article, we assume that all varieties considered are algebraic and defined over an algebraically closed field of characteristic 0.

1. Introduction

One of the motivations for the present article comes from the articles of H. A. Schwarz [34] and G. H. Halphen [16] in the mid-19th century, studying polynomial solutions of Brieskorn–Pham polynomial equations in three variables after L. Euler (1756), J. Liouville (1879) and so forth [12]. Meanwhile, since the mid-20th century the study of rational singularities has witnessed great development [2], [5], [25]. These two topics, one classic and the other modern, encounter each other in contemporary mathematics. For instance, there is a strong connection between the existence of a rational curve on a normal affine surface, i.e., a polynomial solution to algebraic equations, and rational singularities [15].

As an additive analogue of toric geometry, unipotent group actions, specially $\mathbb{G}_a$-actions, on varieties are attractive objects of study. Indeed, $\mathbb{G}_a$-actions have been investigated for their own sake [3], [18], [29], [35], [40]. We also observe that $\mathbb{G}_a$-actions appear in the study of rational singularities. In particular, the article [15] shows

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that a Brieskorn–Pham surface singularity is a cyclic quotient singularity if and only if the surface admits a non-trivial regular $\mathbb{G}_a$-action. Considering its 3-dimensional analogue, H. Flenner and M. Zaidenberg in 2003 proposed the following question [15, Question 2.22]:

> *Does the affine Fermat cubic threefold $x^3 + y^3 + z^3 + w^3 = 0$ in $\mathbb{A}^4$ admit a non-trivial regular $\mathbb{G}_a$-action?*

Even though it is simple-looking, this problem has stood open for 10 years. It turns out that the problem is purely geometric and can be considered in a much wider setting [19]–[22], [33].

To see the problem from a wider viewpoint, we let X be a smooth projective variety with a polarization H, where H is an ample divisor on X. The *generalized cone* over (X, H) is the affine variety defined by

$$\hat{X} = \operatorname{Spec}\Big(\bigoplus_{n\geq 0} H^0(X, \mathcal{O}_X(nH))\Big).$$

Remark 1.1. The affine variety $\hat{X}$ is the usual cone over X embedded in a projective space by the linear system $|H|$ provided that H is very ample and the image of the variety X is projectively normal.

Let S_d be a smooth del Pezzo surface of degree d and let $\hat{S}_d$ be the generalized cone over $(S_d, -K_{S_d})$. For $3 \leq d \leq 9$, the anticanonical divisor $-K_{S_d}$ is very ample and the generalized cone $\hat{S}_d$ is the affine cone in $\mathbb{A}^{d+1}$ over the smooth variety anticanonically embedded in $\mathbb{P}^d$. In particular, for $d = 3$, the cubic surface S_3 is defined by a cubic homogeneous polynomial equation $F(x, y, z, w) = 0$ in $\mathbb{P}^3$, and hence the generalized cone $\hat{S}_3$ is the affine hypersurface in $\mathbb{A}^4$ defined by the equation $F(x, y, z, w) = 0$. For $d = 2$, the generalized cone $\hat{S}_2$ is the affine cone in $\mathbb{A}^4$ over the smooth hypersurface in the weighted projective space $\mathbb{P}(1, 1, 1, 2)$ defined by a quasi-homogeneous polynomial of degree 4. For $d = 1$, the generalized cone $\hat{S}_1$ is the affine cone in $\mathbb{A}^4$ over the smooth hypersurface in the weighted projective space $\mathbb{P}(1, 1, 2, 3)$ defined by a quasi-homogeneous polynomial of degree 6 [17, Theorem 4.4].

It is natural to ask whether the affine variety $\hat{S}_d$ admits a non-trivial $\mathbb{G}_a$-action. The problem mentioned earlier is just a special case of this.

T. Kishimoto, Yu. Prokhorov and M. Zaidenberg studied this generalized problem and proved the following:

Theorem 1.2 (see [19, Theorem 3.19]). *If $4 \leq d \leq 9$, then the generalized cone $\hat{S}_d$ admits an effective $\mathbb{G}_a$-action.* □

Theorem 1.3 (see [22, Theorem 1.1]). *If $d \leq 2$, then the generalized cone $\hat{S}_d$ does not admit a non-trivial $\mathbb{G}_a$-action.* □

Their proofs make good use of a geometric property called cylindricity, which is worth studying for its own sake.

Definition 1.4 ([19]). Let M be a $\mathbb{Q}$-divisor on a smooth projective variety X. An *M-polar cylinder* in X is an open subset

$$U = X \setminus \mathrm{Supp}(D)$$

defined by an effective $\mathbb{Q}$-divisor D on X with $D \sim_{\mathbb{Q}} M$ such that U is isomorphic to $Z \times \mathbb{A}^1$ for some affine variety Z.

Kishimoto et al. show that the existence of an H-polar cylinder on X is equivalent to the existence of a non-trivial $\mathbb{G}_a$-action on the generalized cone over (X, H):

Lemma 1.5 (see [21, Corollary 3.2]). *Let H be an ample Cartier divisor on a smooth projective variety X. Suppose that the generalized cone $\hat{X}$ over (X, H) is normal. Then $\hat{X}$ admits an effective $\mathbb{G}_a$-action if and only if X contains an H-polar cylinder.* □

Remark 1.6. If X is a rational surface, then there always exists an ample Cartier divisor H on X such that $\hat{X}$ is normal and X contains an H-polar cylinder (see [19, Proposition 3.13]), which implies, in particular, that $\hat{X}$ admits an effective $\mathbb{G}_a$-action.

Indeed, what Kishimoto et al. proved in their two theorems is that a del Pezzo surface S_d has a $(-K_{S_d})$-polar cylinder if $4 \leq d \leq 9$ but no $(-K_{S_d})$-polar cylinder if $d \leq 2$.

The main result of the present article is

Theorem 1.7. *A smooth cubic surface S_3 in $\mathbb{P}^3$ does not contain any $(-K_{S_3})$-polar cylinders.*

Together with Theorems 1.2 and 1.3, this leads to the following conclusion via Lemma 1.5.

Corollary 1.8. *Let S_d be a smooth del Pezzo surface of degree d. Then the generalized cone over $(S_d, -K_{S_d})$ admits a non-trivial regular $\mathbb{G}_a$-action if and only if $d \geq 4$.*

In particular, we here present a long-expected answer to the question raised by H. Flenner and M. Zaidenberg.

Corollary 1.9. *The affine Fermat cubic threefold $x^3 + y^3 + z^3 + w^3 = 0$ in $\mathbb{A}^4$ does not admit a non-trivial regular $\mathbb{G}_a$-action.*

The following lemma shows that having anticanonical cylinders on del Pezzo surfaces is strongly related to the log canonical thresholds of their effective anticanonical $\mathbb{Q}$-divisors.[1] It may also be one example that shows how important it is to study singularities of effective anticanonical $\mathbb{Q}$-divisors on Fano manifolds. Indeed, the proof of Theorem 1.7 is substantially based on the lemma below.

[1] An anticanonical $\mathbb{Q}$-divisor on a variety X is a $\mathbb{Q}$-divisor $\mathbb{Q}$-linearly equivalent to an anticanonical divisor of X, while an effective anticanonical divisor on X is a member of the anticanonical linear system $|-K_X|$.

Lemma 1.10. *Let S_d be a smooth del Pezzo surface of degree $d \leq 4$. Suppose that S_d contains a $(-K_{S_d})$-polar cylinder, i.e., there is an open affine subset $U \subset S_d$ and an effective anticanonical $\mathbb{Q}$-divisor D such that $U = S_d \setminus \mathrm{Supp}(D)$ and $U \cong Z \times \mathbb{A}^1$ for some smooth rational affine curve Z. Then there exists a point P on S_d such that*

- *the log pair (S_d, D) is not log canonical at P;*
- *if there exists a unique divisor T in the anticanonical linear system $|-K_{S_d}|$ such that (S_d, T) is not log canonical at P, then there is an effective anticanonical $\mathbb{Q}$-divisor D' on S_d such that*
  - *(S_d, D') is not log canonical at P;*
  - *the support of T is not contained in the support of D'.*

Proof. This follows from [19, Lemma 4.11 and proof of Lemma 4.14] (cf. [22, proof of Lemma 5.3]). Since the proof is dispersed in [19] and [22], for the convenience of the readers, we give a detailed and streamlined proof in the Appendix. □

Applying Lemma 2.2 below, we easily obtain

Corollary 1.11. *Let S_3 be a smooth del Pezzo surface of degree 3. Suppose that S_3 contains a $(-K_{S_3})$-polar cylinder. Then there is an effective anticanonical $\mathbb{Q}$-divisor D on S_3 such that*

- *the log pair (S_3, D) is not log canonical at some point P on S_3;*
- *the support of D does not contain at least one irreducible component of the tangent hyperplane section T_P of S_3 at P.*

In order to prove Theorem 1.7, it suffices to show that there is no divisor D as described in Corollary 1.11 on a smooth del Pezzo surface of degree 3. In this article, this will be done in a slightly wider setting. To be precise, we prove

Theorem 1.12. *Let S_d be a smooth del Pezzo surface of degree $d \leq 3$ and let D be an effective anticanonical $\mathbb{Q}$-divisor on S_d. Suppose that the log pair (S_d, D) is not log canonical at a point P. Then there exists a unique divisor T in the anticanonical linear system $|-K_{S_d}|$ such that (S_d, T) is not log canonical at P. Moreover, the support of D contains all the irreducible components of* $\mathrm{Supp}(T)$.

Corollary 1.13. *Let S_3 be a smooth cubic surface in $\mathbb{P}^3$ and let D be an effective anticanonical $\mathbb{Q}$-divisor on S_3. Suppose that the log pair (S_3, D) is not log canonical at a point P. Then for the tangent hyperplane section T_P at P, the log pair (S_3, T_P) is not log canonical at P and* $\mathrm{Supp}(D)$ *contains all the irreducible components of* $\mathrm{Supp}(T_P)$.

Note that Corollary 1.13 contradicts the conclusion of Corollary 1.11. This simply means that the hypothesis of Corollary 1.11 fails to be true. This shows that Theorem 1.12 implies Theorem 1.7. Moreover, we see that Theorem 1.12 recovers Theorem 1.3 through Lemma 1.10 as well.

Remark 1.14. The condition $d \leq 3$ is crucial in Theorem 1.12. Indeed, if $d \geq 4$, then the assertion of Theorem 1.12 is no longer true (see [19, proof of Theorem 3.19]). For example, consider the case when $d = 4$. There exists a birational morphism $f\colon S_4 \to \mathbb{P}^2$ such that f is the blow-up of $\mathbb{P}^2$ at five points that lie on a unique irreducible conic. Denote this conic by C. Let $\tilde{C}$ be the proper transform of the conic C on the surface S_4 and let $E_1, \dots, E_5$ be the exceptional divisors of the morphism f. Set

$$D = \frac{3}{2}\tilde{C} + \sum_{i=1}^{5} \frac{1}{2} E_i.$$

It is an effective anticanonical $\mathbb{Q}$-divisor on S_4 and the log pair (S_4, D) is not log canonical at any point P on $\tilde{C}$. Moreover, for any $T \in |-K_{S_4}|$, its support cannot be contained in the support of the divisor D.

To our surprise, Theorem 1.12 has other applications that are interesting for their own sake.

Until the end of this section, let X be a projective variety with at worst Kawamata log terminal singularities and let H be an ample divisor on X.

Definition 1.15. The *α-invariant* of the log pair (X, H) is the number defined by

$$\alpha(X, H) = \sup\left\{\lambda \in \mathbb{Q} \;\middle|\; \begin{array}{l}\text{the log pair } (X, \lambda D) \text{ is log canonical for every}\\ \text{effective } \mathbb{Q}\text{-divisor } D \text{ on } X \text{ with } D \sim_{\mathbb{Q}} H\end{array}\right\}.$$

The invariant $\alpha(X, H)$ has been studied intensively by many authors who used different notation for it [1], [6], [14], [4, §3.4], [10, Definition 3.1.1], [11, Appendix A], [38, Appendix 2]. The notation $\alpha(X, H)$ is due to G. Tian who defined $\alpha(X, H)$ in a different way [38, Appendix 2]. However, both the definitions coincide by [11, Theorem A.3]. In the case when X is a Fano variety, the invariant $\alpha(X, -K_X)$ is known as the famous α-invariant of Tian and it is denoted simply by $\alpha(X)$. The α-invariant of Tian plays an important role in Kähler geometry due to the following.

Theorem 1.16 ([13], [30], [36]). *Let X be a Fano variety of dimension n with at worst quotient singularities. If $\alpha(X) > n/(n+1)$, then X admits an orbifold Kähler–Einstein metric.*

The exact values of the α-invariants of smooth del Pezzo surfaces, given below, have been obtained in [7, Theorem 1.7]. Those of del Pezzo surfaces defined over a field of positive characteristic are presented in [28, Theorem 1.6] and those of del Pezzo surfaces with du Val singularities in [8] and [32].

Theorem 1.17. *Let S_d be a smooth del Pezzo surface of degree d. Then*

$$\alpha(S_d) = \begin{cases} 1/3 & \text{if } d = 9, 7 \text{ or } d = 8 \text{ and } S_8 = \mathbb{F}_1, \\ 1/2 & \text{if } d = 6, 5 \text{ or } d = 8 \text{ and } S_8 = \mathbb{P}^1 \times \mathbb{P}^1, \\ 2/3 & \text{if } d = 4, \end{cases}$$

$$\alpha(S_3) = \begin{cases} 2/3 & \text{if } S_3 \text{ is a cubic surface in } \mathbb{P}^3 \text{ with an Eckardt point}, \\ 3/4 & \text{if } S_3 \text{ is a cubic surface in } \mathbb{P}^3 \text{ without Eckardt points}, \end{cases}$$

$$\alpha(S_2) = \begin{cases} 3/4 & \textit{if } |-K_{S_2}| \textit{ has a tacnodal curve}, \\ 5/6 & \textit{if } |-K_{S_2}| \textit{ has no tacnodal curves}, \end{cases}$$

$$\alpha(S_1) = \begin{cases} 5/6 & \textit{if } |-K_{S_1}| \textit{ has a cuspidal curve}, \\ 1 & \textit{if } |-K_{S_1}| \textit{ has no cuspidal curves}. \end{cases}$$

Remark 1.18. Theorem 1.12 also provides the exact values of the α-invariants for smooth del Pezzo surfaces of degrees ≤ 3. We here show how to extract the values from Theorem 1.12. Let ν be the greatest number such that $(S_d, \nu C)$ is log canonical for every member C in $|-K_{S_d}|$. The number ν can be easily obtained from [31, Section 3] and checked to be the same as listed in Theorem 1.17 for the α-invariant of S_d. By the definition of ν, there is an effective anticanonical divisor C on S_d such that $(S_d, \nu C)$ is log canonical but not Kawamata log terminal. This gives $\alpha(S_d) \leq \nu$.

Suppose that $\alpha(S_d) < \nu$. Then there are an effective anticanonical $\mathbb{Q}$-divisor D and a positive rational $\lambda < \nu$ such that $(S_d, \lambda D)$ is not log canonical at some point P on S_d. Since $\lambda < 1$, (S_d, D) is not log canonical at P either. By Theorem 1.12, there exists a divisor $T \in |-K_{S_d}|$ such that (S_d, T) is not log canonical at P. In addition, $\mathrm{Supp}(D)$ contains all the irreducible components of $\mathrm{Supp}(T)$.

The log pair $(S_d, \lambda T)$ is log canonical since $\lambda < \nu$. Set $D_\epsilon = (1+\epsilon)D - \epsilon T$ for every non-negative rational ϵ. Then $D_0 = D$ and D_ϵ is effective for $0 < \epsilon \ll 1$ because $\mathrm{Supp}(D)$ contains all the irreducible components of $\mathrm{Supp}(T)$. Choose the greatest ϵ such that D_ϵ is still effective. Then $\mathrm{Supp}(D_\epsilon)$ does not contain at least one irreducible component of $\mathrm{Supp}(T)$.

Since $(S_d, \lambda T)$ is log canonical at P and $(S_d, \lambda D)$ is not, $(S_d, \lambda D_\epsilon)$ is not log canonical at P (see Lemma 2.2). In particular, (S_d, D_ϵ) is not log canonical at P. However, this contradicts Theorem 1.12 since D_ϵ is an effective anticanonical $\mathbb{Q}$-divisor. Therefore, $\alpha(S_d) = \nu$.

Corollary 1.19. *Let S_d be a smooth del Pezzo surface of degree $d \leq 3$. If $d = 3$, suppose in addition that S_3 does not contain an Eckardt point. Then S_d admits a Kähler–Einstein metric.*

The problem of existence of Kähler–Einstein metrics on smooth del Pezzo surfaces was completely solved by G. Tian and S.-T. Yau [37], [39]. In particular, Corollary 1.19 follows from [37, Main Theorem].

The invariant $\alpha(X, H)$ has a global nature. It measures the singularities of effective $\mathbb{Q}$-divisors on X in a fixed $\mathbb{Q}$-linear equivalence class. F. Ambro [1] suggested a function that encodes the local behavior of $\alpha(X, H)$.

Definition 1.20 ([1]). The *α-function* α_X^H of the log pair (X, H) is a real function on X defined as follows: for $P \in X$,

$$\alpha_X^H(P) = \sup\left\{\lambda \in \mathbb{Q} \;\middle|\; \begin{array}{l} \text{the log pair } (X, \lambda D) \text{ is log canonical at } P \text{ for} \\ \text{every effective } \mathbb{Q}\text{-divisor } D \text{ on } X \text{ with } D \sim_{\mathbb{Q}} H \end{array}\right\}.$$

Lemma 1.21. *We have $\alpha(X, H) = \inf_{P \in X} \alpha_X^H(P)$.*

Proof. Easy. □

In the case when X is a Fano variety, we denote the α-function of the log pair $(X, -K_X)$ simply by α_X.

Example 1.22. One can easily see that $\alpha_{\mathbb{P}^n}(P) \leq 1/(n+1)$ for every point P on $\mathbb{P}^n$. This implies that the α-function $\alpha_{\mathbb{P}^n}$ is constant with value $1/(n+1)$ since we have $\alpha(\mathbb{P}^n) = 1/(n+1)$.

Example 1.23. It is easy to see $\alpha_{\mathbb{P}^1\times\mathbb{P}^1}(P) \leq 1/2$ for every point P on $\mathbb{P}^1 \times \mathbb{P}^1$. Since $\alpha(\mathbb{P}^1 \times \mathbb{P}^1) = 1/2$ by Theorem 1.17, the α-function $\alpha_{\mathbb{P}^1\times\mathbb{P}^1}$ is constant with value $1/2$ by Lemma 1.21. Moreover, if X is a Fano variety with at most Kawamata log terminal singularities, then [11, proof of Lemma 2.21] shows that

$$\alpha_{X\times\mathbb{P}^1}(P) = \min\{1/2, \alpha_X(\mathrm{pr}_1(P))\}$$

for every point P on $X \times \mathbb{P}^1$, where $\mathrm{pr}_1 : X \times \mathbb{P}^1 \to X$ is the projection. Using the same argument as in [11, proof of Lemma 2.29], one can show that the α-function of a product of Fano varieties with at most Gorenstein canonical singularities is the pointwise minimum of the pull-backs of the α-functions of the factors.

As shown in Remark 1.18, the following can be obtained from Theorem 1.12 in a similar manner.

Corollary 1.24. *Let S_d be a smooth del Pezzo surface of degree $d \leq 3$. Then the α-function of S_d is as follows:*

$$\alpha_{S_3}(P) = \begin{cases} 2/3 & \text{if } P \text{ is an Eckardt point,} \\ 3/4 & \text{if the tangent hyperplane section at } P \text{ has a tacnode at } P, \\ 5/6 & \text{if the tangent hyperplane section at } P \text{ has a cusp at } P, \\ 1 & \text{otherwise,} \end{cases}$$

$$\alpha_{S_2}(P) = \begin{cases} 3/4 & \text{if there is an effective anticanonical divisor with a tacnode at } P, \\ 5/6 & \text{if there is an effective anticanonical divisor with a cusp at } P, \\ 1 & \text{otherwise,} \end{cases}$$

$$\alpha_{S_1}(P) = \begin{cases} 5/6 & \text{if there is an effective anticanonical divisor with a cusp at } P, \\ 1 & \text{otherwise.} \end{cases}$$

By Lemma 1.21, Corollary 1.24 implies that Theorem 1.17 holds for smooth del Pezzo surfaces of degrees at most 3. Thus, it is quite natural that we should extend Corollary 1.24 to all smooth del Pezzo surfaces in order to obtain a functional generalization of Theorem 1.17. This will be done in Section 6, where we prove

Theorem 1.25. *Let S_d be a smooth del Pezzo surface of degree $d \geq 4$. Then:*

$$\alpha_{\mathbb{P}^2}(P) = 1/3,$$

$$\alpha_{\mathbb{F}_1}(P) = 1/3, \quad \alpha_{\mathbb{P}^1\times\mathbb{P}^1}(P) = 1/2,$$

$$\alpha_{S_7}(P) = \begin{cases} 1/3 & \text{if the point } P \text{ lies on a } (-1)\text{-curve that intersects two other } (-1)\text{-curves,} \\ 1/2 & \text{otherwise,} \end{cases}$$

$$\alpha_{S_6}(P) = 1/2,$$

$$\alpha_{S_5}(P) = \begin{cases} 1/2 & \text{if there is a } (-1)\text{-curve passing through } P, \\ 2/3 & \text{otherwise,} \end{cases}$$

$$\alpha_{S_4}(P) = \begin{cases} 2/3 & \text{if } P \text{ is on a } (-1)\text{-curve,} \\ 3/4 & \text{if there is an effective anticanonical divisor that consists of two } 0\text{-curves intersecting tangentially at } P, \\ 5/6 & \text{otherwise.} \end{cases}$$

The primary statement in this article is Theorem 1.12. As explained before, it immediately implies the main result of the article, Theorem 1.7, and also recovers Theorem 1.3. Theorem 1.12 will be proved in the following way.

In Section 2, we review the results that will be used. As a warm-up, we verify Theorem 1.12 for a smooth del Pezzo surface of degree 1 (see Lemma 2.3). This is easy and instructive.

In Section 3, we establish two results about *singular* del Pezzo surfaces of degree 2 that play a role in the proof of Theorems 1.12 for smooth cubic surfaces. In addition, these two results immediately yield Theorem 1.12 for a smooth del Pezzo surface of degree 2 (see Lemma 3.4).

In Section 4, we prove Theorem 1.12 for a smooth cubic surface. This will be done by a thorough case-by-case analysis of all possible types of tangent hyperplane sections on a smooth cubic surface. Indeed, for a given point P on the smooth cubic surface, we show that every effective anticanonical $\mathbb{Q}$-divisor is log canonical at P if the tangent hyperplane section at P is log canonical at P (Lemmas 4.7–4.9), whereas we show that its support contains the support of the tangent hyperplane section at P if an effective anticanonical $\mathbb{Q}$-divisor and the tangent hyperplane section at P are not log canonical at P (see Lemmas 4.3, 4.5 and 4.6).

The proof of Lemma 4.8 deserves a separate section because it is the central and the most beautiful part of the article and it is a bit lengthy. It will be presented in Section 5.

The Appendix will deal with Lemma 1.10 for the readers' convenience.

2. Preliminaries

This section presents simple but essential tools for the article. Most of the results described here are well-known and valid in much more general settings (cf. [23]–[26]).

Let S be a projective surface with at most du Val singularities, let P be a smooth point of the surface S and let D be an effective $\mathbb{Q}$-divisor on S.

Lemma 2.1. *If the log pair (S, D) is not log canonical at the smooth point P, then*

$$\mathrm{mult}_P(D) > 1.$$

Proof. This is a well-known fact. See [26, Proposition 9.5.13], for instance. □

Write $D = \sum_{i=1}^r a_i D_i$, where D_i's are distinct prime divisors on the surface S and a_i's are positive rational numbers.

Lemma 2.2. *Let T be an effective $\mathbb{Q}$-divisor on S such that*

- *$T \sim_{\mathbb{Q}} D$ but $T \neq D$;*
- *$T = \sum_{i=1}^r b_i D_i$ for some non-negative rational numbers $b_1, \dots, b_r$.*

For every non-negative rational ϵ, set $D_\epsilon = (1+\epsilon)D - \epsilon T$. Then

(1) *$D_\epsilon \sim_{\mathbb{Q}} D$ for every $\epsilon \geq 0$;*
(2) *the set $\{\epsilon \in \mathbb{Q}_{>0} \mid D_\epsilon \text{ is effective}\}$ attains the maximum μ;*
(3) *the support of D_μ does not contain at least one component of* $\mathrm{Supp}(T)$*;*
(4) *if (S, T) is log canonical at P but (S, D) is not log canonical at P, then (S, D_μ) is not log canonical at P.*

Proof. The first assertion is obvious. For the rest we put

$$c = \max\{b_i/a_i \mid i = 1, \dots, r\}.$$

For some index k we have $c = b_k/a_k$.

Suppose that $c \leq 1$. Then $a_i \geq b_i$ for every i, so the divisor $D - T = \sum_{i=1}^r (a_i - b_i) D_i$ is effective. However, this is impossible since $D - T$ is non-zero and numerically trivial on a projective surface. Thus, $c > 1$, and hence $b_k > a_k$.

Set $\mu = 1/(c-1)$. Then $\mu = a_k/(b_k - a_k) > 0$ and

$$D_\mu = \frac{b_k}{b_k - a_k} D - \frac{a_k}{b_k - a_k} T = \sum_{i=1}^r \frac{b_k a_i - a_k b_i}{b_k - a_k} D_i,$$

where $b_k a_i - a_k b_i \geq 0$ by the choice of k. In particular, the divisor D_μ is effective and its support does not contain the curve D_k. Moreover, for every positive rational ϵ, $D_\epsilon = \sum_{i=1}^r (a_i + \epsilon a_i - \epsilon b_i) D_i$. If $\epsilon > \mu$, then

$$\epsilon(b_k - a_k) > \mu(b_k - a_k) = \frac{a_k}{b_k - a_k}(b_k - a_k) = a_k,$$

and hence D_ϵ is not effective. This proves the second and third assertions.

If both (S, T) and (S, D_μ) are log canonical at P, then (S, D) must be log canonical at P because $D = \frac{\mu}{1+\mu} T + \frac{1}{1+\mu} D_\mu$ and $\frac{\mu}{1+\mu} + \frac{1}{1+\mu} = 1$. □

Despite its naïve appearance, Lemma 2.2 is a handy tool. To illustrate this, we verify Theorem 1.12 for a del Pezzo surface of degree 1. This simple case also immediately follows from [7, proof of Lemma 3.1] or [22, proof of Proposition 5.1].

Lemma 2.3. *Suppose that S is a smooth del Pezzo surface of degree 1 and D is an effective anticanonical $\mathbb{Q}$-divisor on S. If the log pair (S, D) is not log canonical at the point P, then there exists a unique divisor $T \in |-K_S|$ such that (S, T) is not log canonical at P. Moreover, the support of D contains all the irreducible components of T.*

Proof. Let T be a curve in $|-K_S|$ that passes through P. Note that T is irreducible. If (S, T) is log canonical at P, then it follows from Lemma 2.2 that there exists an effective anticanonical $\mathbb{Q}$-divisor D' on S such that (S, D') is not log canonical at P and $\mathrm{Supp}(D')$ does not contain T. We then obtain $1 = T \cdot D' \geq \mathrm{mult}_P(D')$. This is impossible by Lemma 2.1. Thus, (S, T) is not log canonical at P.

Moreover, by Lemma 2.1 the divisor T is singular at P. Therefore, P is not the base point of the pencil $|-K_S|$. Consequently, such a divisor T is unique.

If the curve T is not contained in $\mathrm{Supp}(D)$, then we obtain an absurd inequality $1 = T \cdot D \geq \mathrm{mult}_P(D) > 1$ by Lemma 2.1. Therefore, $T \subset \mathrm{Supp}(D)$. □

The following is a ready-made Adjunction for our situation.

Lemma 2.4. *Suppose that the log pair (S, D) is not log canonical at the smooth point P. If a component D_j with $a_j \leq 1$ is smooth at P, then*

$$D_j \cdot \Big(\sum_{i \neq j} a_i D_i\Big) \geq \sum_{i \neq j} a_i (D_j \cdot D_i)_P > 1,$$

where $(D_j \cdot D_i)_P$ is the local intersection number of C_i and C_j at P.

Proof. This follows immediately from [24, Theorem 5.50]. □

Let $f\colon \tilde{S} \to S$ be the blow-up of the surface S at the point P with exceptional divisor E and let $\tilde{D}$ be the proper transform of D by the blow-up f. Then

$$K_{\tilde{S}} + \tilde{D} + (\mathrm{mult}_P(D) - 1)E = f^*(K_S + D).$$

The log pair (S, D) is log canonical at P if and only if $(\tilde{S}, \tilde{D} + (\mathrm{mult}_P(D) - 1)E)$ is log canonical along the curve E.

Remark 2.5. If (S, D) is not log canonical at P, then there exists a point Q on E at which $(\tilde{S}, \tilde{D} + (\mathrm{mult}_P(D) - 1)E)$ is not log canonical. Lemma 2.1 then implies

$$\mathrm{mult}_P(D) + \mathrm{mult}_Q(\tilde{D}) > 2. \tag{2.1}$$

If in addition $\mathrm{mult}_P(D) \leq 2$, then $(\tilde{S}, \tilde{D} + (\mathrm{mult}_P(D) - 1)E)$ is log canonical at every point of E other than Q. Indeed, if it is not log canonical at another point O on E, then Lemma 2.4 generates an absurd inequality

$$2 \geq \mathrm{mult}_P(D) = \tilde{D} \cdot E \geq \mathrm{mult}_Q(\tilde{D}) + \mathrm{mult}_O(\tilde{D}) > 2.$$

Notation 2.6. From now on, when we have a birational morphism of a surface denoted by a capital Roman character with a tilde onto a surface, in order to denote the proper transform of a divisor by this morphism, we will add a tilde to the same character that denotes the original divisor. For example, in a situation similar to the one preceding Remark 2.5, we use $\tilde{D}$ for the proper transform of D by f without explicit mention.

3. Del Pezzo surfaces of degree 2

Let S be a del Pezzo surface of degree 2 with at most two ordinary double points. Then the linear system $|-K_S|$ is base-point-free and induces a double cover $\pi: S \to \mathbb{P}^2$ ramified along a reduced quartic curve $R \subset \mathbb{P}^2$. Moreover, R has at most two ordinary double points. In particular, it is irreducible.

Lemma 3.1. *For an effective anticanonical $\mathbb{Q}$-divisor D on S, the log pair (S, D) is log canonical outside finitely many points on S.*

Proof. Suppose the converse. Then we may write $D = a_1C_1 + \Omega$, where C_1 is an irreducible reduced curve, a_1 is a rational number > 1 and Ω is an effective $\mathbb{Q}$-divisor whose support does not contain C_1. Since

$$2 = -K_S \cdot D = -K_S \cdot (a_1C_1 + \Omega) = -a_1K_S \cdot C_1 - K_S \cdot \Omega \geq -a_1K_S \cdot C_1 > -K_S \cdot C_1,$$

we have $-K_S \cdot C_1 = 1$. Then $\pi(C_1)$ is a line in $\mathbb{P}^2$. Thus, there exists an irreducible reduced curve C_2 on S such that $C_1 + C_2 \sim -K_S$ and $\pi(C_1) = \pi(C_2)$. Note that $C_1 = C_2$ if and only if the line $\pi(C_1)$ is an irreducible component of the branch curve R. Since R is irreducible, this is not the case. Thus, $C_1 \neq C_2$.

Note that $C_1^2 = C_2^2$ because C_1 and C_2 are interchanged by the biregular involution of S induced by the double cover π. Thus,

$$2 = (-K_S)^2 = (C_1 + C_2)^2 = 2C_1^2 + 2C \cdot C_2,$$

which implies that $C_1 \cdot C_2 = 1 - C_1^2$. Since C_1 and C_2 are smooth rational curves, we easily obtain $C_1^2 = C_2^2 = -1 + k/2$, where k is the number of singular points of S that lie on C_1.

Now we write $D = a_1C_1 + a_2C_2 + \Gamma$, where a_2 is a non-negative rational number and Γ is an effective $\mathbb{Q}$-divisor whose support contains neither C_1 nor C_2. Then

$$\begin{aligned} 1 &= C_1 \cdot (a_1C_1 + a_2C_2 + \Gamma) = a_1C_1^2 + a_2C_1 \cdot C_2 + C_1 \cdot \Gamma \\ &\geq a_1C_1^2 + a_2C_1 \cdot C_2 = a_1C_1^2 + a_2(1 - C_1^2). \end{aligned}$$

Similarly, from $C_2 \cdot D = 1$, we obtain $1 \geq a_2C_1^2 + a_1(1 - C_1^2)$. The two inequalities imply that $a_1 \leq 1$ and $a_2 \leq 1$ since $C_1^2 = -1 + k/2$, $k = 0, 1, 2$. Since $a_1 > 1$ by assumption, this is a contradiction. □

The following two lemmas can be verified in much the same way as [7, Lemma 3.5]. Nevertheless we present their proofs since we should carefully deal with singular points on S that have been considered neither in [7] nor in [22].

Lemma 3.2. *For any effective anticanonical $\mathbb{Q}$-divisor D on S, the log pair (S, D) is log canonical at every point outside the ramification divisor of the double cover π.*

Proof. Suppose that (S, D) is not log canonical at a point P whose image by π lies outside R.

Let H be a general curve in $|-K_S|$ that passes through P. Since $\pi(P) \notin R$, the surface S is smooth at P. Then

$$2 = H \cdot D \geq \mathrm{mult}_P(H)\mathrm{mult}_P(D) \geq \mathrm{mult}_P(D),$$

and hence $\mathrm{mult}_P(D) \leq 2$.

Let $f\colon \tilde{S} \to S$ be the blow-up of the surface S at P. We have

$$K_{\tilde{S}} + \tilde{D} + (\mathrm{mult}_P(D) - 1)E = f^*(K_S + D),$$

where E is the exceptional curve of the blow-up f. Then Remark 2.5 gives a unique point Q on E such that $(\tilde{S}, \tilde{D} + (\mathrm{mult}_P(D) - 1)E)$ is not log canonical at Q on E.

Since $\pi(P) \notin R$, there exists a unique reduced but possibly reducible curve $C \in |-K_S|$ that passes through P and whose proper transform $\tilde{C}$ passes through Q. Note that C is smooth at P. Since (S, C) is log canonical at P, Lemma 2.2 enables us to assume that the support of D does not contain at least one irreducible component of C.

If the curve C is irreducible, then

$$2 - \mathrm{mult}_P(D) = 2 - \mathrm{mult}_P(C)\mathrm{mult}_P(D) = \tilde{C} \cdot \tilde{D} \geq \mathrm{mult}_Q(\tilde{C})\mathrm{mult}_Q(\tilde{D}) = \mathrm{mult}_Q(\tilde{D}).$$

This contradicts (2.1). Thus, C must be reducible.

We may then write $C = C_1 + C_2$, where C_1 and C_2 are irreducible smooth curves that intersect at two points. Without loss of generality we may assume that the curve C_1 is not contained in the support of D. The point P must belong to C_2: otherwise we would have

$$1 = D \cdot C_1 \geq \mathrm{mult}_P(D) > 1.$$

We set $D = aC_2 + \Omega$, where a is a non-negative rational number and Ω is an effective $\mathbb{Q}$-divisor whose support does not contain C_2. Then

$$1 = C_1 \cdot D = \big(2 - \tfrac{1}{2}k\big)a + C_1 \cdot \Omega \geq \big(2 - \tfrac{1}{2}k\big)a,$$

where k is the number of singular points of S on C_1. On the other hand, the log pair $(\tilde{S}, a\tilde{C}_2 + \tilde{\Omega} + (\mathrm{mult}_P(D) - 1)E)$ is not log canonical at Q, where $a \leq 1$ by Lemma 3.1. We then obtain

$$\big(2 - \tfrac{1}{2}k\big)a = \tilde{C}_2 \cdot \big(\tilde{\Omega} + (\mathrm{mult}_P(D) - 1)E\big) > 1$$

from Lemma 2.4. This is a contradiction. □

Lemma 3.3. *For a smooth point P of S with $\pi(P) \in R$, let T_P be the unique divisor in $|-K_S|$ that is singular at P. If the log pair (S, T_P) is log canonical at P, then for any effective anticanonical $\mathbb{Q}$-divisor D on S the log pair (S, D) is log canonical at P.*

Proof. Suppose that (S, D) is not log canonical at P. Applying Lemma 2.2 to the log pairs (S, D) and (S, T_P), we may assume that $\mathrm{Supp}(D)$ does not contain at least one irreducible component of T_P. Thus, if the divisor T_P is irreducible, then Lemma 2.1 gives an absurd inequality

$$2 = T_P \cdot D \geq \mathrm{mult}_P(T_P)\mathrm{mult}_P(D) \geq 2\mathrm{mult}_P(D) > 2$$

since T_P is singular at P. Hence, T_P must be reducible.

We may then write $T_P = T_1 + T_2$, where T_1 and T_2 are smooth rational curves. Note that P is one of the intersection points of T_1 and T_2. Without loss of generality, we may assume that T_1 is not contained in the support of D. Then

$$1 = T_1 \cdot D \geq \mathrm{mult}_P(T_1)\mathrm{mult}_P(D) = \mathrm{mult}_P(D) > 1$$

by Lemma 2.1, a contradiction. □

Lemmas 3.2 and 3.3 yield the following result.

Lemma 3.4. *Suppose that a del Pezzo surface S of degree 2 is smooth. Let D be an effective anticanonical $\mathbb{Q}$-divisor on S. Suppose that the log pair (S, D) is not log canonical at a point P. Then there exists a unique divisor $T \in |-K_S|$ such that (S, T) is not log canonical at P. The support of D contains all the irreducible components of T. The divisor T is either an irreducible rational curve with a cusp at P or a union of two (-1)-curves meeting tangentially at P.*

Proof. By Lemma 3.2, the point $\pi(P)$ lies on R. Then there is a unique curve $T \in |-K_S|$ that is singular at P. By Lemma 3.3, (S, T) is not log canonical at P.

Suppose that the support of D does not contain an irreducible component of T. Then the proof of Lemma 3.3 works verbatim to yield a contradiction.

The last assertion immediately follows from [31, Proposition 3.2]. □

Lemma 3.4 shows that Theorem 1.12 holds for a smooth del Pezzo surface of degree 2.

4. Cubic surfaces

In the present section we prove Theorem 1.12. Lemmas 2.3 and 3.4 show that Theorem 1.12 holds for del Pezzo surfaces of degrees 1 and 2, respectively. Thus, to complete the proof, we let S be a smooth cubic surface in $\mathbb{P}^3$ and let D be an effective anticanonical $\mathbb{Q}$-divisor on S.

Lemma 4.1. *The log pair (S, D) is log canonical outside finitely many points.*

Proof. Suppose not. Then we may write $D = aC + \Omega$, where C is an irreducible curve, a is a rational number > 1 and Ω is an effective $\mathbb{Q}$-divisor whose support does not contain C. Then

$$3 = -K_S \cdot (aC + \Omega) = -aK_S \cdot C - K_S \cdot \Omega \geq -aK_S \cdot C > -K_S \cdot C.$$

This implies that C is either a line or an irreducible conic.

Suppose that C is a line. Let Z be a general irreducible conic on S with $Z+C \sim -K_S$. Since Z is general, it is not contained in the support of D. We then obtain

$$2 = Z \cdot D = Z \cdot (aC + \Omega) = 2a + Z \cdot \Omega \geq 2a.$$

This contradicts our assumption.

Suppose that C is an irreducible conic. Then there exists a unique line L on S such that $L + C \sim -K_S$. Write $D = aC + bL + \Gamma$, where b is a non-negative rational number and Γ is an effective $\mathbb{Q}$-divisor whose support contains neither C nor L. Then

$$1 = L \cdot D = L \cdot (aC + bL + \Gamma) = 2a - b + L \cdot \Gamma \geq 2a - b.$$

On the other hand,

$$2 = C \cdot D = C \cdot (aC + bL + \Gamma) = 2b + C \cdot \Gamma \geq 2b.$$

Combining these two inequalities, we obtain $a \leq 1$. This contradicts our assumption again. □

For a point P on S, let T_P be the tangent hyperplane section of S at P. This is the unique anticanonical divisor that is singular at P. The curve T_P is reduced but it may be reducible.

In order to prove Theorem 1.12 we must show that (S, D) is log canonical at P provided that one of the following two conditions is satisfied:

- (S, T_P) is log canonical at P;
- (S, T_P) is not log canonical at P but $\mathrm{Supp}(D)$ does not contain at least one irreducible component of T_P.

The log pair (S, T_P) is log canonical at P if and only if P is an ordinary double point of T_P (see [31, Proposition 3.2]). Thus, (S, T_P) is log canonical at P if and only if T_P is one of the following curves: an irreducible cubic curve with one ordinary double point, a union of three coplanar lines that do not intersect at one point, or a union of a line and a conic that intersect transversally at two points.

Overall, we must consider the following cases:

(a) T_P is a union of three lines that intersect at P (Eckardt point);
(b) T_P is a union of a line and a conic that intersect tangentially at P;
(c) T_P is an irreducible cubic curve with a cusp at P;
(d) T_P is an irreducible cubic curve with one ordinary double point;
(e) T_P is a union of three coplanar lines that do not intersect at one point;
(f) T_P is a union of a line and a conic that intersect transversally at two points.

We consider these cases one by one in separate lemmas (Lemmas 4.3 and 4.5–4.9). We however present the detailed proof of Lemma 4.8 in Section 5 to improve the readability of this section. These lemmas together imply Theorem 1.12.

For simplicity, set $m = \mathrm{mult}_P(D)$.

Lemma 4.2. *If the log pair (S, D) is not log canonical at the point P, then the support of D contains all the lines on S passing through P.*

Proof. Let L be a line passing through P that is not contained in the support of D. Then $1 = L \cdot D \geq m$ implies that (S, D) is log canonical at P by Lemma 2.1. □

Lemma 4.3 ([19, Lemma 4.13]). *Suppose that the tangent hyperplane section T_P consists of three lines intersecting at P. If the support of D does not contain at least one of the three lines, then the log pair (S, D) is log canonical at P.*

Proof. This immediately follows from Lemma 4.2. □

From now on, let $f : \tilde{S} \to S$ be the blow-up of the cubic surface S at P. In addition, let E be the exceptional curve of f. We then have

$$K_{\tilde{S}} + \tilde{D} + (m-1)E = f^*(K_S + D).$$

Note that (S, D) is log canonical at P if and only if $(\tilde{S}, \tilde{D} + (m-1)E)$ is log canonical along the exceptional divisor E.

Remark 4.4. If there is a line passing through P, then $\tilde{S}$ is not a del Pezzo surface but a weak del Pezzo surface of degree 2, i.e., $K_{\tilde{S}}^2 = 2$ and $-K_{\tilde{S}}$ is nef and big. The proper transforms of the lines passing through P will be (-2)-curves on $\tilde{S}$. All the (-2)-curves on $\tilde{S}$ are disjoint from each other and they come from the lines passing through P on S. By contracting these (-2)-curves we obtain a birational morphism $g : \tilde{S} \to \bar{S}$. Then $\bar{S}$ is a del Pezzo surface of degree 2 with ordinary double points. In particular, the linear system $|-K_{\bar{S}}|$ induces a double cover $\pi : \bar{S} \to \mathbb{P}^2$ ramified along a quartic curve $R \subset \mathbb{P}^2$. The (-2)-curves on $\tilde{S}$ are contracted to the ordinary double points on $\bar{S}$. Therefore, the number of ordinary double points on $\bar{S}$ is the number of lines passing through P on S. Since we have at most two lines passing through P, the surface $\bar{S}$ has at most two ordinary double points, and hence the quartic curve R must be an irreducible curve with at most two ordinary double points.

Lemma 4.5. *Suppose that the tangent hyperplane section T_P consists of a line and a conic intersecting tangentially at the point P. If the support of D does not contain both the line and the conic, then the log pair (S, D) is log canonical at P.*

Proof. Suppose that (S, D) is not log canonical at P. Let L and C be the line and the conic, respectively, such that $T_P = L + C$. By Lemma 4.2, we may assume that C is not contained in the support of D but L is in that support. We write $D = aL + \Omega$, where a is a positive rational number and Ω is an effective $\mathbb{Q}$-divisor whose support contains neither L nor C. We have $m \leq C \cdot D = 2$.

Note that the three curves $\tilde{L}$, $\tilde{C}$ and E meet at one point transversally. Since $m \leq 2$, we have the unique point Q on E defined in Remark 2.5. The point Q does not belong to $\tilde{C}$, and hence not to $\tilde{L}$ either. Indeed, otherwise

$$2 - m = \tilde{C} \cdot (a\tilde{L} + \tilde{\Omega}) \geq a + \mathrm{mult}_Q(\tilde{\Omega}) = \mathrm{mult}_Q(\tilde{D}).$$

This contradicts (2.1).

Let $g\colon \tilde{S} \to \bar{S}$ be the contraction defined in Remark 4.4. Note that $g(\tilde{L})$ is the ordinary double point of the surface $\bar{S}$. Set $\bar{\Omega} = g(\tilde{\Omega})$, $\bar{E} = g(E)$, $\bar{C} = g(\tilde{C})$ and $\bar{Q} = g(Q)$. Then

$\pi(\bar{E}) = \pi(\bar{C})$ since $\bar{E}+\bar{C}$ is an anticanonical divisor on $\bar{S}$. The point $\pi(\bar{Q})$ lies outside R because Q lies outside $\tilde{C}$. Since the divisor $\bar{\Omega}+(m-1)\bar{E}$ is $\mathbb{Q}$-linearly equivalent to $-K_{\bar{S}}$ by our construction, Lemma 3.2 shows that $(\bar{S}, \bar{\Omega}+(m-1)\bar{E})$ is log canonical at $\bar{Q}$. On the other hand, it is not log canonical at $\bar{Q}$ since g is an isomorphism in a neighborhood of Q. This is a contradiction. □

Lemma 4.6. *Suppose that the tangent hyperplane section T_P is an irreducible cubic curve with a cusp at P. If T_P is not contained in the support of D, then the log pair (S, D) is log canonical at P.*

Proof. Suppose that (S, D) is not log canonical at P. From the inequality

$$3 = T_P \cdot D \geq m \cdot \mathrm{mult}_P(T_P) = 2m,$$

we obtain $m \leq 3/2$. Then we have the unique point Q on E defined in Remark 2.5.

The surface $\tilde{S}$ is a smooth del Pezzo surface of degree 2. The linear system $|-K_{\tilde{S}}|$ induces a double cover $\pi : \tilde{S} \to \mathbb{P}^2$ ramified along a smooth quartic curve $R \subset \mathbb{P}^2$. Then the integral divisor $E + \tilde{T}_P$ is linearly equivalent to $-K_{\tilde{S}}$, and hence $\pi(E) = \pi(\tilde{T}_P)$ is a line in $\mathbb{P}^2$. Moreover, $\tilde{T}_P$ tangentially meets E at a single point. The point $\pi(Q)$ lies on R if and only if Q is the intersection point of E and $\tilde{T}_P$.

Applying Lemma 3.2 to $(\tilde{S}, \tilde{D} + (m-1)E)$, we see that $\pi(Q) \in R$ because this log pair is not log canonical at Q and the divisor $\tilde{D} + (m-1)E$ is $\mathbb{Q}$-linearly equivalent to $-K_{\tilde{S}}$. The point Q therefore lies on the curve $\tilde{T}_P$. Then from (2.1) we obtain

$$3 - 2m = \tilde{T}_P \cdot \tilde{D} \geq \mathrm{mult}_Q(\tilde{D}) > 2 - m.$$

This contradicts Lemma 2.1. □

For the remaining three cases, we show that the hypothesis of Theorem 1.12 is never fulfilled, so that Theorem 1.12 is true.

Lemma 4.7. *If the tangent hyperplane section T_P is an irreducible cubic curve with a node at P, then the log pair (S, D) is log canonical at P.*

Proof. Suppose that (S, D) is not log canonical at P. The surface $\tilde{S}$ is a smooth del Pezzo surface of degree two. Since $\tilde{D} + (m-1)E \sim_{\mathbb{Q}} -K_{\tilde{S}}$ and $(\tilde{S}, \tilde{D} + (m-1)E)$ is not log canonical at some point Q on E, it follows from Lemma 3.4 that there must be an anticanonical divisor H on $\tilde{S}$ that has either a tacnode or a cusp at Q.

If H has a tacnode at Q, then it consists of the exceptional divisor E and another (-1)-curve L meeting E tangentially at Q. Then $f(H)$ is an effective anticanonical divisor on S that has a cusp at P and is distinct from T_P. This is impossible.

If H has a cusp at Q, then H must be irreducible. However, this is impossible since H is singular at Q and $E \cdot H = 1$. □

Lemma 4.8. *Suppose that the tangent hyperplane section T_P consists of three lines one of which does not pass through the point P. Then the log pair (S, D) is log canonical at P.*

Proof. This proof is the central and most beautiful part of the proof of Theorem 1.12. Since it is a bit lengthy, it will be presented separately in Section 5. □

Lemma 4.9. *Suppose that the tangent hyperplane section T_P consists of a line and a conic intersecting transversally. Then the log pair (S, D) is log canonical at the point P.*

Proof. We write $T_P = L + C$, where L is a line and C is an irreducible conic that intersect L transversally at P. Suppose that (S, D) is not log canonical at P.

By Lemmas 2.2 and 4.2, we may assume that C is not contained in the support of D but L is. We write $D = aL + \Omega$, where a is a positive rational number and Ω is an effective $\mathbb{Q}$-divisor whose support contains neither L nor C.

We have the unique point Q on E defined in Remark 2.5 since $m \leq D \cdot C = 2$.

Suppose that Q does not belong to the (-2)-curve $\tilde{L}$. Let $g\colon \tilde{S} \to \bar{S}$ be the contraction defined in Remark 4.4. Then $\bar{S}$ is a del Pezzo surface of degree 2 with only one ordinary double point $g(\tilde{L})$. Set $\bar{\Omega} = g(\tilde{\Omega})$, $\bar{E} = g(E)$, $\bar{C} = g(\tilde{C})$ and $\bar{Q} = g(Q)$. Then we have $\pi(\bar{E}) = \pi(\bar{C})$ since $\bar{E} + \bar{C}$ is an anticanonical divisor on $\bar{S}$. The point $\pi(\bar{Q})$ lies on R if and only if Q lies on $\tilde{C}$. The log pair $(\bar{S}, \bar{\Omega} + (m-1)\bar{E})$ is not log canonical at $\bar{Q}$ since g is an isomorphism in a neighborhood of Q. Since the divisor $\bar{\Omega} + (m-1)\bar{E}$ is $\mathbb{Q}$-linearly equivalent to $-K_{\bar{S}}$ by our construction, Lemma 3.2 shows that $Q \in \tilde{C}$.

Note that $\bar{C} + \bar{E}$ is the unique curve in $|-K_{\bar{S}}|$ that is singular at $\bar{Q}$. But $(\bar{S}, \bar{C} + \bar{E})$ is log canonical at $\bar{Q}$. Hence, by Lemma 3.3, so is $(\bar{S}, \bar{\Omega}+(m-1)\bar{E})$. This is a contradiction. Therefore, Q must belong to the (-2)-curve $\tilde{L}$.

Now we can apply [8, Theorem 1.28] to the log pair $(\tilde{S}, a\tilde{L} + (m-1)E + \tilde{\Omega})$ at the point Q to obtain a contradiction immediately. Indeed, it is enough to set $M = 1$, $A = 1$, $N = 0$, $B = 2$, and $\alpha = \beta = 1$ in [8, Theorem 1.28] and check that all the conditions of that theorem are satisfied. However, there is a much simpler way to obtain a contradiction.

There exists another line M on S that intersects L at a point. The line M does not intersect the conic C since $1 = T_P \cdot M = (L + C) \cdot M = L \cdot M$. In particular, P does not lie on M. Let $h\colon \tilde{S} \to \check{S}$ be the contraction of the proper transform of the line M on $\tilde{S}$. Since M is a (-1)-curve and $P \notin M$, the surface $\check{S}$ is a smooth cubic surface in $\mathbb{P}^3$.

Set $\check{\Omega} = h(\tilde{\Omega})$, $\check{E} = h(E)$, $\check{L} = h(\tilde{L})$, $\check{C} = h(\tilde{C})$, $\check{P} = h(Q)$ and $\check{D} = h(\tilde{D})$. Then $(\check{S}, \check{D})$ is not log canonical at $\check{P}$ since h is an isomorphism in a neighborhood of the point Q. On the other hand, $\check{L} + \check{C} + \check{E}$ is an anticanonical divisor of $\check{S}$. Since $\check{P}$ is the intersection point of $\check{L}$ and $\check{E}$ and the divisor $\check{D}$ is $\mathbb{Q}$-linearly equivalent to $-K_{\check{S}}$, Lemma 4.8 implies that $(\check{S}, \check{D})$ is log canonical at $\check{P}$. This is a contradiction. □

As already mentioned, Theorem 1.12 follows from Lemmas 4.3 and 4.5–4.9. Thus Theorem 1.12 has been proved under the assumption that Lemma 4.8 is valid. This assumption will be confirmed in the following section.

5. The proof of Lemma 4.8

To prove Lemma 4.8, we keep the notation of Section 4. We write $T_P = L+M+N$, where L, M, and N are three coplanar lines on S. We may assume that P is the intersection point of L and M, whereas it does not lie on N. We also write $D = a_0 L + b_0 M + c_0 N + \Omega_0$, where a_0, b_0 and c_0 are non-negative rational numbers and Ω_0 is an effective $\mathbb{Q}$-divisor on S whose support contains none of the lines L, M or N. Write $m_0 = \mathrm{mult}_P(\Omega_0)$.

Suppose that (S, D) is not log canonical at P. Let us look for a contradiction.

By Lemma 4.1, (S, D) is log canonical outside finitely many points. In particular, $0 \le a_0, b_0, c_0 \le 1$. Also, Lemma 2.1 implies that $m_0 + a_0 + b_0 > 1$ and Lemma 4.2 gives $a_0, b_0 > 0$.

Lemma 5.1. *We have $m_0 + a_0 + b_0 > c_0 + 1$.*

Proof. Since $(S, a_0 L + b_0 M + \Omega_0)$ is not log canonical at P, Lemma 2.4 yields

$$1 + a_0 - c_0 = L \cdot (D - a_0 L - c_0 N) = L \cdot (b_0 M + \Omega_0) > 1,$$

which implies $a_0 > c_0$. Similarly, $b_0 > c_0$.

The log pair $(S, L + M + N)$ is log canonical. Since $(S, a_0 L + b_0 M + c_0 N + \Omega_0)$ is not log canonical at P, it follows from Lemma 2.2 and its proof that

$$\left(S, \frac{1}{1 - c_0} D - \frac{c_0}{1 - c_0} T_P\right)$$

is not log canonical at P. Then Lemma 2.1 shows that

$$\begin{aligned}\mathrm{mult}_P\left(\frac{1}{1 - c_0} D - \frac{c_0}{1 - c_0} T_P\right) &= \mathrm{mult}_P\left(\frac{a_0 - c_0}{1 - c_0} L + \frac{b_0 - c_0}{1 - c_0} M + \frac{1}{1 - c_0} \Omega_0\right)\\ &= \frac{a_0 - c_0}{1 - c_0} + \frac{b_0 - c_0}{1 - c_0} + \frac{m_0}{1 - c_0} > 1.\end{aligned}$$

This yields the conclusion. □

Since $a_0, b_0, c_0 \le 1$ and $(S, L + M + N)$ is log canonical, the effective $\mathbb{Q}$-divisor Ω_0 cannot be the zero-divisor. Let r be the number of irreducible components of the support of Ω_0. Then

$$\Omega_0 = \sum_{i=1}^{r} e_i C_{i0},$$

where e_i's are positive rational numbers and C_{i0}'s are irreducible reduced curves of degrees d_{i0} on S. Consequently,

$$3 = -K_S \cdot \left(a_0 L + b_0 M + c_0 N + \sum_{i=1}^{r} e_i C_{i0}\right) = a_0 + b_0 + c_0 + \sum_{i=1}^{r} e_i d_{i0}. \tag{5.1}$$

We have

$$K_{\tilde{S}} + a_0 \tilde{L} + b_0 \tilde{M} + c_0 \tilde{N} + (a_0 + b_0 + m_0 - 1)E + \sum_{i=1}^{r} e_i \tilde{C}_{i0} = f^*(K_S + D).$$

Recall that $a_0 + b_0 + m_0 = m$.

Lemma 5.2. *We have $m = a_0 + b_0 + m_0 \le 2$.*

Proof. This immediately follows from the three inequalities

$$1 = L \cdot (a_0L + b_0M + c_0N + \Omega_0) = -a_0 + b_0 + c_0 + L \cdot \Omega_0 \geq -a_0 + b_0 + c_0 + m_0,$$
$$1 = M \cdot (a_0L + b_0M + c_0N + \Omega_0) = a_0 - b_0 + c_0 + M \cdot \Omega_0 \geq a_0 - b_0 + c_0 + m_0,$$
$$1 = N \cdot (a_0L + b_0M + c_0N + \Omega_0) = a_0 + b_0 - c_0 + N \cdot \Omega_0 \geq a_0 + b_0 - c_0. \qquad \square$$

The log pair

$$\Big(\tilde{S}, a_0\tilde{L} + b_0\tilde{M} + c_0\tilde{N} + (a_0 + b_0 + m_0 - 1)E + \sum_{i=1}^{r} e_i\tilde{C}_{i0}\Big) \tag{5.2}$$

is not log canonical at some point Q on E. Since $\mathrm{mult}_P(D) = a_0 + b_0 + m_0 \leq 2$, it follows from Remark 2.5 that there is only one such point.

Let $g\colon \tilde{S} \to \bar{S}$ be the contraction defined in Remark 4.4. Then $\bar{S}$ is a del Pezzo surface of degree 2 with two ordinary double points, $g(\tilde{L})$ and $g(\tilde{M})$.

Lemma 5.3. *The point Q on the exceptional curve E belongs to either $\tilde{L}$ or $\tilde{M}$.*

Proof. Suppose Q lies on neither $\tilde{L}$ nor $\tilde{M}$. Set $\bar{E} = g(E)$, $\bar{N} = g(\tilde{N})$ and $\bar{Q} = g(Q)$. In addition, write $\bar{C}_{i0} = g(\tilde{C}_{i0})$ for each i. Then $\pi(\bar{E}) = \pi(\bar{N})$. The point $\pi(\bar{Q})$ lies outside the quartic curve R since $\bar{Q}$ is a smooth point of the anticanonical divisor $\bar{E} + \bar{N}$ on $\bar{S}$.

Since g is an isomorphism in a neighborhood of Q, the log pair

$$\Big(\bar{S}, c_0\bar{N} + (a_0 + b_0 + m_0 - 1)\bar{E} + \sum_{i=1}^{r} e_i\bar{C}_{i0}\Big) \tag{5.3}$$

is not log canonical at $\bar{Q}$. The divisor $c_0\bar{N} + (a_0 + b_0 + m_0 - 1)\bar{E} + \sum_{i=1}^{r} e_i\bar{C}_{i0}$ is an effective anticanonical $\mathbb{Q}$-divisor on the surface $\bar{S}$. Hence, we can apply Lemma 3.2 to the log pair (5.3) to obtain a contradiction. $\square$

From now on we may assume without loss of generality that Q is the intersection point of $\tilde{L}$ and E.

Let $\rho\colon S \dashrightarrow \mathbb{P}^2$ be the linear projection from the point P. Then ρ is a generically 2-to-1 rational map. Thus it induces a birational involution τ_P of the cubic surface S, classically known as the *Geiser involution* associated to P (see [27]).

Remark 5.4. By construction, τ_P is biregular outside $L \cup M \cup N$. In fact, one can show that τ_P is biregular outside P and N. Moreover, $\tau_P(L) = L$ and $\tau_P(M) = M$.

For each i, set $C_{i1} = \tau_P(C_{i0})$ and denote by d_{i1} the degree of the curve C_{i1}. We then employ new effective $\mathbb{Q}$-divisors

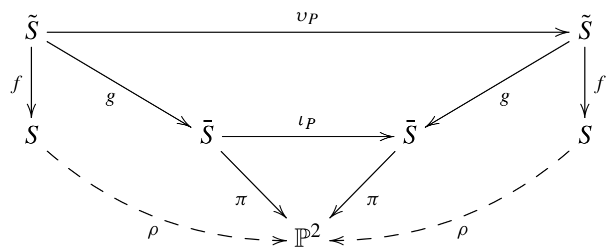
$$\Omega_1 = \sum_{i=1}^{r} e_iC_{i1}, \qquad D_1 = a_1L + b_1M + c_1N + \Omega_1,$$

where $a_1 = a_0$, $b_1 = b_0$ and $c_1 = a_0 + b_0 + m_0 - 1$. Note that $a_0 + b_0 + m_0 - 1 > 0$ by Lemma 2.1 (cf. Lemma 5.1).

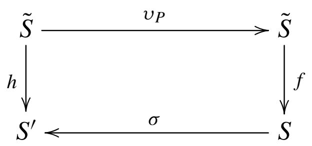
Lemma 5.5. *The divisor D_1 is an effective anticanonical $\mathbb{Q}$-divisor on the surface S. The log pair (S, D_1) is not log canonical at the intersection point of L and N.*

Proof. Let $h\colon \tilde{S} \to S'$ be the contraction of the (-1)-curve $\tilde{N}$. Then S' is a smooth cubic surface in $\mathbb{P}^3$. Set $E' = h(E)$, $L' = h(\tilde{L})$, $M' = h(\tilde{M})$, $Q' = h(Q)$ and $C'_{i0} = h(\tilde{C}_{i0})$ for each i. Then the integral divisor $L' + M' + E'$ is an anticanonical divisor of S'. In particular, L', M' and E' are coplanar lines on S'. Moreover, Q' is the intersection point of L' and E' by the assumption right after Lemma 5.3. It does not lie on M'.

Let ι_P be the biregular involution of the surface $\bar{S}$ induced by the double cover π. Then ι_P induces a biregular involution υ_P of $\tilde{S}$ since $\tilde{S}$ is the minimal resolution of singularities of $\bar{S}$. Thus, we have a commutative diagram



This shows $\tau_P = f \circ \upsilon_P \circ f^{-1}$. On the other hand, $\upsilon_P(E) = \tilde{N}$ since $\pi \circ g(E) = \pi \circ g(\tilde{N})$. This means that there exists an isomorphism $\sigma\colon S \to S'$ that makes the diagram



commute. By construction, $\sigma(L) = L'$, $\sigma(M) = M'$, $\sigma(N) = E'$, and $\sigma(C_{i1}) = C'_{i0}$ for every i. Recall that Q' is the intersection point of L' and E'.

Since h is an isomorphism locally around Q, the log pair

$$\Big(S', a_0 L' + b_0 M' + (a_0 + b_0 + m_0 - 1)E' + \sum_{i=1}^{r} e_i C'_{i0}\Big)$$

is not log canonical at Q'. Since

$$a_0 \tilde{L} + b_0 \tilde{M} + c_0 \tilde{N} + (a_0 + b_0 + m_0 - 1)E + \sum_{i=1}^{r} e_i \tilde{C}_{i0} \sim_{\mathbb{Q}} -K_{\tilde{S}},$$

we have $a_0 L' + b_0 M' + (a_0 + b_0 + m_0 - 1)E' + \sum_{i=1}^{r} e_i C'_{i0} \sim_{\mathbb{Q}} -K_{S'}$. Therefore,

$$a_0 L + b_0 M + (a_0 + b_0 + m_0 - 1)N + \sum_{i=1}^{r} e_i C_{i1} \sim_{\mathbb{Q}} -K_S,$$

and

$$\Big(S, a_0L + b_0M + (a_0 + b_0 + m_0 - 1)N + \sum_{i=1}^{r} e_iC_{i1}\Big)$$

is not log canonical at the intersection point of L and N. □

Now we are able to replace the original effective $\mathbb{Q}$-divisor D by the new effective $\mathbb{Q}$-divisor D_1. By Lemma 5.5, both have the same properties that we have been using so far. However, the $\mathbb{Q}$-divisor Ω_1 is *slightly better* than Ω_0 in the sense of the following lemma.

Lemma 5.6. *The degree of the* $\mathbb{Q}$*-divisor* Ω_1 *is strictly smaller than the degree of* Ω_0*, i.e.,*

$$\sum_{i=1}^{r} e_i d_{i1} < \sum_{i=1}^{r} e_i d_{i0}.$$

Proof. Since $D_1 \sim_{\mathbb{Q}} -K_S$ by Lemma 5.5, we obtain

$$\begin{aligned} 3 &= -K_S \cdot \Big(a_0L + b_0M + (a_0 + b_0 + m_0 - 1)N + \sum_{i=1}^{r} e_iC_{i1}\Big) \\ &= 2a_0 + 2b_0 + m_0 - 1 + \sum_{i=1}^{r} e_i d_{i1}. \end{aligned}$$

On the other hand, $a_0 + b_0 + c_0 + \sum_{i=1}^{r} e_i d_{i0} = 3$ by (5.1). Thus,

$$\sum_{i=1}^{r} e_i d_{i1} = \sum_{i=1}^{r} e_i d_{i0} - (a_0 + b_0 + m_0 - 1 - c_0) < \sum_{i=1}^{r} e_i d_{i0}$$

because $a_0 + b_0 + m_0 - 1 - c_0 > 0$ by Lemma 5.1. □

Repeating this process, we can obtain a sequence of effective anticanonical $\mathbb{Q}$-divisors

$$D_k = a_kL + b_kM + c_kN + \Omega_k$$

on the surface S such that for each k, the log pair (S, D_k) is not log canonical at one of the three intersection points $L \cap M$, $L \cap N$ and $M \cap N$. Note that

$$\Omega_k = \sum_{i=1}^{r} e_iC_{ik},$$

where C_{ik}'s are irreducible reduced curves of degrees d_{ik}. We then obtain a strictly decreasing sequence of rational numbers

$$\sum_{i=1}^{r} e_i d_{i0} > \sum_{i=1}^{r} e_i d_{i1} > \cdots > \sum_{i=1}^{r} e_i d_{ik} > \cdots$$

by Lemma 5.6. This is a contradiction since the subset

$$\Big\{\sum_{i=1}^{r} e_i n_i \;\Big|\; n_1, \ldots, n_r \in \mathbb{N}\Big\} \subset \mathbb{Q}$$

is discrete and bounded from below. This completes the proof of Lemma 4.8.

6. α-functions of smooth del Pezzo surfaces

In this section, we prove Theorem 1.25. Let S_d be a smooth del Pezzo surface of degree d. We first make a simple but useful observation.

Lemma 6.1. *Let $f : S_d \to S$ be the blow-down of a (-1)-curve E on the del Pezzo surface S_d. Then S is a smooth del Pezzo surface and $\alpha_{S_d}(P) \geq \alpha_S(f(P))$ for every point P of S_d outside E.*

Proof. It is easy to check that $-K_S$ is ample. The second statement immediately follows from the definition of the α-function. □

We have already shown that the α-function $\alpha_{\mathbb{P}^2}$ of the projective plane is constant with value $1/3$ (see Example 1.22) and the α-function $\alpha_{\mathbb{P}^1\times\mathbb{P}^1}$ of the quadric surface is constant with value $1/2$ (see Example 1.23).

Lemma 6.2. *The α-function $\alpha_{\mathbb{F}_1}$ of the blow-up $\mathbb{F}_1$ of $\mathbb{P}^2$ at one point is the constant function with value $1/3$.*

Proof. Let P be a point on $\mathbb{F}_1$. Let $\pi : \mathbb{F}_1 \to \mathbb{P}^1$ be the $\mathbb{P}^1$-bundle morphism onto $\mathbb{P}^1$. Let C be its section with $C^2 = -1$ and let L_P be the fiber of the morphism π over $\pi(P)$. Since $2C + 3L_P \sim -K_{\mathbb{F}_1}$, we have $\alpha_{\mathbb{F}_1}(P) \leq 1/3$. But $\alpha(\mathbb{F}_1) = 1/3$ by Theorem 1.17. Thus, $\alpha_{\mathbb{F}_1}$ is the constant function with value $1/3$ by Lemma 1.21. □

The surface S_7 is the blow-up of $\mathbb{P}^2$ at two distinct points Q_1 and Q_2. Let E be the proper transform of the line passing through Q_1 and Q_2 by the two-point blow-up $f : S_7 \to \mathbb{P}^2$ with exceptional curves E_1 and E_2.

Lemma 6.3. *The α-function of a del Pezzo surface S_7 of degree 7 is*

$$\alpha_{S_7}(P) = \begin{cases} 1/2 & \text{if } P \notin E, \\ 1/3 & \text{if } P \in E. \end{cases}$$

Proof. Let $P \in S$. Then $\alpha_{S_7}(P) \geq \alpha(S) = 1/3$ by Theorem 1.17 and Lemma 1.21.

If $P \in E$, then $\alpha_{S_7}(P) \leq 1/3$ since $2E_1 + 2E_2 + 3E \sim -K_S$. It follows that $\alpha_{S_7}(P) = 1/3$.

Suppose that $P \notin E$. Let L be a line on $\mathbb{P}^2$ whose proper transform by the blow-up f passes through P. Since $f^*(2L) + E$ is an effective anticanonical divisor passing through P, we have $\alpha_{S_7}(P) \leq 1/2$.

Let $g : S \to \mathbb{P}^1 \times \mathbb{P}^1$ be the birational morphism obtained by contracting the (-1)-curve E. Then g is an isomorphism around P. We have $\alpha_{S_7}(P) \geq \alpha_{\mathbb{P}^1\times\mathbb{P}^1}(g(P))$ by Lemma 6.1. Since $\alpha_{\mathbb{P}^1\times\mathbb{P}^1}$ is constant with value $1/2$, we obtain $\alpha_{S_7}(P) = 1/2$. □

Lemma 6.4. *The α-function α_{S_6} of a del Pezzo surface S_6 of degree 6 is constant with value $1/2$.*

Proof. Let $P \in S_6$. One can easily check that $\alpha_{S_6}(P) \leq 1/2$. One the other hand, we have a birational morphism $h : S_6 \to S_7$, where S_7 is a del Pezzo surface of degree 7, such that h is an isomorphism around P and the point $h(P)$ is not on the (-1)-curve of S_7 intersecting two different (-1)-curves. Then $\alpha_{S_6}(P) \geq 1/2$ by Lemmas 6.1 and 6.3. □

Lemma 6.5. *The α-function of a del Pezzo surface S_5 of degree 5 is*

$$\alpha_{S_5}(P) = \begin{cases} 1/2 & \text{if there is a } (-1)\text{-curve passing through } P, \\ 2/3 & \text{otherwise.} \end{cases}$$

Proof. Let $P \in S_5$. Suppose that P lies on a (-1)-curve. Then there exists an effective anticanonical divisor not reduced at P. Thus, we have $\alpha_{S_5}(P) \le 1/2$. Meanwhile, $1/2 = \alpha(S_5) \le \alpha_{S_5}(P)$ by Lemma 1.21 and Theorem 1.17. Therefore, $\alpha_{S_5}(P) = 1/2$.

Suppose that P is not on any (-1)-curve. Then there exist exactly five irreducible smooth rational curves $C_1, \dots, C_5$ passing through P, with $-K_S \cdot C_i = 2$ for each i (cf. [7, proof of Lemma 5.8]). Moreover, for every C_i, there are four irreducible smooth rational curves E^i_1, E^i_2, E^i_3 and E^i_4 such that $3C_i + E^i_1 + E^i_2 + E^i_3 + E^i_4$ belongs to the bi-anticanonical linear system $|-2K_{S_5}|$ (cf. Remark 1.14). Therefore, $\alpha_{S_5}(P) \le 2/3$.

Suppose that $\alpha_{S_5}(P) < 2/3$. Then there is an effective anticanonical $\mathbb{Q}$-divisor D such that $(S, \lambda D)$ is not log canonical at P for some positive rational $\lambda < 2/3$. Then $\mathrm{mult}_P(D) > 1/\lambda$ by Lemma 2.1. Let $f\colon S_4 \to S_5$ be the blow-up of S_5 at P with exceptional curve E and let $\tilde{D}$ be the proper transform of D on S_4. Then S_4 is a smooth del Pezzo surface of degree 4. We have

$$K_{S_4} + \lambda\tilde{D} + (\lambda\mathrm{mult}_P(D) - 1)E = f^*(K_{S_5} + \lambda D),$$

which implies that $(S_4, \lambda\tilde{D} + (\lambda\mathrm{mult}_P(D) - 1)E)$ is not log canonical. On the other hand, $(S_4, \lambda\tilde{D} + \lambda(\mathrm{mult}_P(D) - 1)E)$ is log canonical because $\tilde{D} + (\mathrm{mult}_P(D) - 1)E$ is an effective anticanonical $\mathbb{Q}$-divisor of S_4 and $\alpha(S_4) = 2/3$ by Theorem 1.17. However, this is absurd because $\lambda(\mathrm{mult}_P(D) - 1) > \lambda\mathrm{mult}_P(D) - 1$. □

Lemma 6.6. *The α-function on a del Pezzo surface S_4 of degree 4 is*

$$\alpha_{S_4}(P) = \begin{cases} 2/3 & \text{if } P \text{ is on a } (-1)\text{-curve,} \\ 3/4 & \text{if there is an effective anticanonical divisor that consists of} \\ & \text{two } 0\text{-curves meeting tangentially at } P; \\ 5/6 & \text{otherwise.} \end{cases}$$

Proof. Let $P \in S_4$. If P lies on a (-1)-curve L, then there are five mutually disjoint (-1)-curves $E_1, \dots, E_5$ that intersect L. Let $h : S_4 \to \mathbb{P}^2$ be the contraction of all E_i's. Since $h(L)$ is a conic in $\mathbb{P}^2$, we see that $3L + \sum_{1 \le i \le 5} E_i$ is a member of the linear system $|-2K_{S_4}|$ (cf. Remark 1.14). This means that $\alpha_{S_4}(P) \le 2/3$. Therefore, $\alpha_{S_4}(P) = 2/3$ since $\alpha(S_4) \le \alpha_{S_4}(P)$ by Lemma 1.21 and $\alpha(S_4) = 2/3$ by Theorem 1.17.

Suppose that P does not lie on a (-1)-curve. Set $\omega = 3/4$ when there is an effective anticanonical divisor that consists of two 0-curves intersecting tangentially at P, and $\omega = 5/6$ otherwise.

One can easily find an effective anticanonical divisor F on S_4 such that $(S_4, \lambda F)$ is not log canonical at P for every rational $\lambda > \omega$ (see [31, Proposition 3.2]). This shows that $\alpha_{S_4}(P) \le \omega$. Moreover, it is easy to check that $(S_4, \omega C)$ is log canonical at P for each $C \in |-K_{S_4}|$.

Suppose $\alpha_{S_4}(P) < \omega$. Then there is an effective anticanonical $\mathbb{Q}$-divisor D such that $(S_4, \omega D)$ is not log canonical at P. Note that there are only finitely many effective anticanonical divisors $C_1, \ldots, C_k$ such that (S_4, C_i) is not log canonical at P. Applying Lemma 2.2, we may assume that for each i at least one irreducible component of $\mathrm{Supp}(C_i)$ is not contained in the support of D.

Let $f: S_3 \to S_4$ be the blow-up of S_4 at P with exceptional curve E and let $\tilde{D}$ be the proper transform of the divisor D on S_3. Then S_3 is a smooth cubic surface in $\mathbb{P}^3$ and E is a line in S_3. Moreover, $(S_3, \tilde{D} + (\mathrm{mult}_P(D) - 1)E)$ is not log canonical at some point Q on E because (S_4, D) is not log canonical at P.

Let T_Q be the tangent hyperplane section of S_3 at Q. Note that T_Q contains the line E. Since $\tilde{D} + (\mathrm{mult}_P(D) - 1)E$ is an effective anticanonical $\mathbb{Q}$-divisor on S_3, it follows from Corollary 1.13 that (S_3, T_Q) is not log canonical at Q and the support of $\tilde{D}$ contains all the irreducible components of T_Q. In fact, T_Q is either a union of three lines meeting at Q or a union of a line and a conic intersecting tangentially at Q. The divisor $f(T_Q)$ is an effective anticanonical divisor on S_4 such that the log pair $(S_4, f(T_Q))$ is not log canonical at P. This contradicts our assumption since the support of D contains all the irreducible components of $f(T_Q)$. ◻

Consequently, Theorem 1.25 follows from Examples 1.22 and 1.23, and Lemmas 6.2–6.6.

Appendix

This appendix is devoted to the proof of Lemma 1.10. The proof originates from [19] and [22], where it is dispersed. For the readers' convenience, we give a detailed and streamlined proof here.

Let S be a smooth del Pezzo surface of degree at most 4. Suppose that S contains a $(-K_S)$-polar cylinder, i.e., there is an open affine subset $U \subset S$ and an effective anticanonical $\mathbb{Q}$-divisor D such that $U = S \setminus \mathrm{Supp}(D)$ and $U \cong Z \times \mathbb{A}^1$ for some smooth rational affine curve Z. Set $D = \sum_{i=1}^{r} a_i D_i$, where each D_i is an irreducible reduced curve and each a_i is a positive rational number.

Lemma A.1 ([22, Lemma 4.4]). *The number of irreducible components of the divisor D is not smaller than the rank of the Picard group of S, i.e., $r \geq \mathrm{rk}\,\mathrm{Pic}(S) = 10 - K_S^2 \geq 6$.*

To prove Lemma 1.10, we must show that there exists a point $P \in S$ such that

- the log pair (S, D) is not log canonical at P;
- if there exists a *unique* divisor T in the anticanonical linear system $|-K_S|$ such that (S, T) is not log canonical at P, then there is an effective anticanonical $\mathbb{Q}$-divisor D' on S such that
  - (S, D') is not log canonical at P;
  - the support of T is not contained in the support of D'.

The natural projection $U \cong Z \times \mathbb{A}^1 \to Z$ induces a rational map $\pi: S \dashrightarrow \mathbb{P}^1$ given by a pencil $\mathcal{L}$ on the surface S. Then either $\mathcal{L}$ is base-point-free or its base locus consists of a single point.

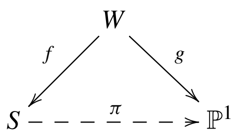
Lemma A.2 ([22, Lemma 4.2]). *The pencil $\mathcal{L}$ is not base-point-free.*

Proof. Suppose that $\mathcal{L}$ is base-point-free. Then π is a morphism, which implies that there exists exactly one irreducible component of $\mathrm{Supp}(D)$ that does not lie in a fiber of π. Moreover, this component is a section. Without loss of generality, we may assume that the component is D_r. Let L be a sufficiently general curve in $\mathcal{L}$. Then

$$2 = -K_S \cdot L = D \cdot L = \sum_{i=1}^{r} a_i D_i \cdot L = a_r D_r \cdot L,$$

and hence $a_r = 2$. This implies that $\alpha(S) \leq 1/2$. However, this contradicts Theorem 1.17 since the degree of S is at most 4. □

Denote the unique base point of the pencil $\mathcal{L}$ by P. Let us show that P is the point we are looking for. Resolving the base locus of $\mathcal{L}$, we obtain a commutative diagram



where f is a composition of blow-ups at smooth points over P and g is a morphism whose general fiber is a smooth rational curve. Denote by $E_1, \dots, E_n$ the exceptional curves of the birational morphism f. Exactly one of them does not lie in the fibers of g. Without loss of generality, we may assume that this is E_n. Then E_n is a section of g.

For every D_i, denote by $\tilde{D}_i$ its proper transform on the surface W. Then every curve $\tilde{D}_i$ lies in a fiber of g.

The following lemma is a slightly stronger version of [22, Lemma 4.6]; its proof is almost the same.

Lemma A.3 (cf. [22, Lemma 4.6]). *For every effective anticanonical $\mathbb{Q}$-divisor H with* $\mathrm{Supp}(H) \subseteq \mathrm{Supp}(D)$*, the log pair (S, H) is not log canonical at the point P.*

Applying Lemma A.3 to (S, D), we see that (S, D) is not log canonical at P. Thus, if there exists no anticanonical divisor T such that (S, T) is not log canonical at P, then we are done. Hence, to complete the proof of Lemma 1.10, we assume that there exists a *unique* divisor $T \in |-K_S|$ such that (S, T) is not log canonical at P. Then Lemma 1.10 follows from the lemma below.

Lemma A.4. *There exists an effective anticanonical $\mathbb{Q}$-divisor D' on S such that the log pair (S, D') is not log canonical at P and* $\mathrm{Supp}(D')$ *does not contain at least one irreducible component of* $\mathrm{Supp}(T)$.

Proof. If $\mathrm{Supp}(D)$ does not contain at least one irreducible component of $\mathrm{Supp}(T)$, then we can simply set $D' = D$. Suppose otherwise, i.e., $\mathrm{Supp}(T) \subseteq \mathrm{Supp}(D)$. Then $T \neq D$. Indeed, the number of irreducible components of $\mathrm{Supp}(D)$ is at least 6 by Lemma A.1. On the other hand, the number of irreducible components of $\mathrm{Supp}(T)$ is at most 4 because $-K_S \cdot T = K_S^2$ and $-K_S$ is ample.

Since $T \neq D$, there exists a positive rational μ such that the $\mathbb{Q}$-divisor $(1+\mu)D-\mu T$ is effective and its support does not contain at least one irreducible component of $\mathrm{Supp}(T)$. Set $D' = (1+\mu)D - \mu T$. Note that D' is also an effective anticanonical $\mathbb{Q}$-divisor on S. By our construction, $\mathrm{Supp}(D') \subseteq \mathrm{Supp}(D)$. Thus, (S, D') is not log canonical at P by Lemma A.3. □

Remark A.5. Note that $U \neq S \setminus \mathrm{Supp}(D')$, which implies that the number of irreducible components of $\mathrm{Supp}(D')$ may be less than $\mathrm{rk}\,\mathrm{Pic}(S)$. Therefore, we can apply Lemma 2.2 only once here. This shows that in the proof of Lemma A.4 we really need to use the *uniqueness* of the divisor T in $|-K_S|$ such that (S, T) is not log canonical at P. Indeed, if there is another divisor T' in $|-K_S|$ such that (S, T') is not log canonical at P, then we would not be able to apply Lemma 2.2 since we might have $D' = T'$.

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