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Symplectic fillings of lens spaces as Lefschetz fibrations

Abstract. We construct a positive allowable Lefschetz fibration over the disk on any minimal (weak) symplectic filling of the canonical contact structure on a lens space. Using this construction we prove that any minimal symplectic filling of the canonical contact structure on a lens space is obtained by a sequence of rational blowdowns from the minimal resolution of the corresponding complex two-dimensional cyclic quotient singularity.

Keywords. Symplectic fillings, lens spaces, canonical contact structure, Lefschetz fibrations

1. Introduction

The link of an isolated complex surface singularity carries a canonical—also known as Milnor fillable—contact structure which is unique up to isomorphism [4]. A Milnor fillable contact structure is Stein fillable since a regular neighbourhood of the exceptional divisor in a minimal resolution of the surface singularity provides a holomorphic filling which can be deformed to be Stein without changing the contact structure on the boundary [3]. In particular, a singularity link with its canonical contact structure always admits a symplectic filling given by the minimal resolution of the singularity.

The canonical contact structure on a lens space (the oriented link of a complex two-dimensional cyclic quotient singularity) is well understood as the quotient of the standard tight contact structure on S^3. The finitely many diffeomorphism types of the minimal symplectic fillings of the canonical contact structure on a lens space were classified by Lisca [11] (see also work of the first author and K. Ono [2]).

In this paper, we give an algorithm to present each minimal symplectic filling of the canonical contact structure on a lens space as an explicit genus-zero PALF (positive allowable Lefschetz fibration) over the disk. The existence of such a genus-zero PALF also follows from [19, Theorem 1] although we do not rely on that result in this paper. Using our construction we prove the following.

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Theorem 4.1. *Any minimal symplectic filling of the canonical contact structure on a lens space is obtained by a sequence of rational blowdowns along linear plumbing graphs starting from the minimal resolution of the corresponding cyclic quotient singularity.*

We would like to emphasize that while the various fillings are related to the plumbing by rational blowdown, the curves that are blown down need not be apparent in the canonical plumbing graph. We also obtain the following corollaries of Theorem 4.1.

Corollary 4.7. *The canonical contact structure on a lens space admits a unique minimal symplectic filling—represented by the Stein structure via the PALF we construct on the minimal resolution—up to symplectic rational blowdown and symplectic deformation equivalence.*

Corollary 4.8. *Any Milnor fibre of any smoothing of a complex two-dimensional cyclic quotient singularity can be obtained, up to diffeomorphism, by a sequence of rational blowdowns along linear plumbing graphs from the Milnor fibre diffeomorphic to the minimal resolution of the singularity.*

We refer the reader to [9] and [15] for background material on Lefschetz fibrations, open books and contact structures. We denote a right-handed Dehn twist along a curve γ as γ again and we use functional notation while writing products of Dehn twists.

2. Symplectic fillings as Lefschetz fibrations

For integers $1 \leq q < p$ with $(p,q) = 1$, recall that the *Hirzebruch–Jung continued fraction* is given by

$$\frac{p}{q} = [a_1, \ldots, a_l] = a_1 - \cfrac{1}{a_2 - \cfrac{1}{\ddots - \cfrac{1}{a_l}}}, \quad a_i \geq 2 \quad \text{for all } 1 \leq i \leq l.$$

The lens space $L(p,q)$ is orientation preserving diffeomorphic to the link of the cyclic quotient singularity whose minimal resolution is given by a linear plumbing graph with vertices having weights $-a_1, \ldots, -a_l$, where $p/q = [a_1, \ldots, a_l]$.

It is known that any tight contact structure on $L(p,q)$, in particular the canonical contact structure ξ_{can}, is supported by a planar open book [17]. According to Wendl [19], if a contact 3-manifold (Y, ξ) is supported by a planar open book $\mathcal{OB}_\xi$, then any strong symplectic filling of (Y, ξ) is symplectic deformation equivalent to a blowup of a PALF whose boundary is $\mathcal{OB}_\xi$. On the other hand, it is also known that every weak symplectic filling of a rational homology sphere can be modified into a strong symplectic filling [13]. We conclude that any minimal symplectic filling of $(L(p,q), \xi_{\text{can}})$ admits a genus-zero PALF over D^2. In this section we give an algorithm to describe any minimal symplectic filling of $(L(p,q), \xi_{\text{can}})$ as an *explicit* genus-zero PALF over D^2.

2.1. Lisca's classification of the fillings

We first briefly review Lisca's classification [11] of symplectic fillings of $(L(p, q), \xi_{\mathrm{can}})$, up to diffeomorphism. Let

$$\frac{p}{p-q} = [b_1, \ldots, b_k],$$

where $b_i \geq 2$ for $1 \leq i \leq k$. A k-tuple of nonnegative integers $(n_1, \ldots, n_k)$ is called *admissible* if each of the denominators in the continued fraction $[n_1, \ldots, n_k]$ is positive. It is easy to see that an admissible k-tuple of nonnegative integers is either (0) or consists only of positive integers. Let $\mathcal{Z}_k \subset \mathbb{Z}^k$ denote the set of admissible k-tuples $\mathbf{n} = (n_1, \ldots, n_k)$ of nonnegative integers such that $[n_1, \ldots, n_k] = 0$, and let

$$\mathcal{Z}_k\left(\tfrac{p}{p-q}\right) = \{(n_1, \ldots, n_k) \in \mathcal{Z}_k \mid 0 \leq n_i \leq b_i \text{ for } i = 1, \ldots, k\}.$$

Note that any k-tuple of positive integers in $\mathcal{Z}_k$ can be obtained from $(1, 1)$ by a sequence of strict blowups.

Definition 2.1. *A strict blowup of an r-tuple of integers at the jth term is a map $\psi_j : \mathbb{Z}^r \to \mathbb{Z}^{r+1}$ defined by*

$$(n_1, \ldots, n_j, n_{j+1}, \ldots, n_r) \mapsto (n_1, \ldots, n_{j-1}, n_j + 1, 1, n_{j+1} + 1, n_{j+2}, \ldots, n_r)$$

for any $1 \leq j \leq r - 1$ and by

$$(n_1, \ldots, n_r) \mapsto (n_1, \ldots, n_{r-1}, n_r + 1, 1)$$

when $j = r$. The left inverse of a strict blowup at the jth term is called a strict blowdown at the $(j + 1)$st term.

Consider the chain of k unknots in S^3 with respective framings $n_1, \ldots, n_k$. For any $\mathbf{n} = (n_1, \ldots, n_k) \in \mathcal{Z}_k$, let $N(\mathbf{n})$ denote the result of Dehn surgery on this framed link. It is easy to see that $N(\mathbf{n})$ is diffeomorphic to $S^1 \times S^2$. Let $\mathbf{L} = \bigcup_{i=1}^k L_i$ denote the framed link in $N(\mathbf{n})$, shown in Figure 1 in the complement of the chain of k unknots, where L_i has $b_i - n_i$ components.

The 4-manifold $W_{p,q}(\mathbf{n})$ with boundary $L(p, q)$ is obtained by attaching 2-handles to $S^1 \times D^3$ along the framed link $\varphi(\mathbf{L}) \subset S^1 \times S^2$ for some diffeomorphism $\varphi : N(\mathbf{n}) \to S^1 \times S^2$. Note that this description is a *relative* handlebody decomposition of $W_{p,q}(\mathbf{n})$
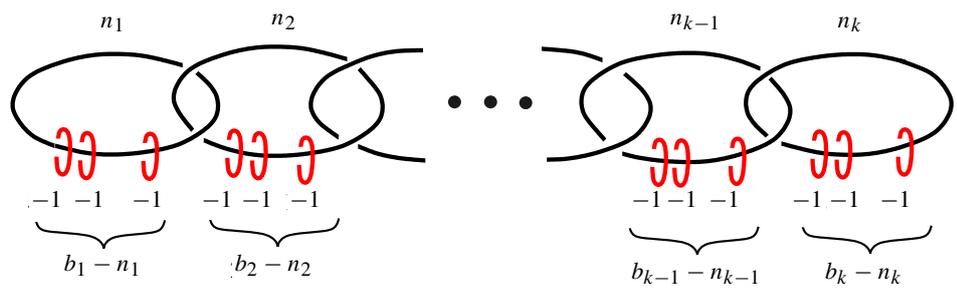

Fig. 1. Lisca's description of the filling $W_{(p,q)}(\mathbf{n})$.

and it is independent of the choice of φ since any self-diffeomorphism of $S^1 \times S^2$ extends to $S^1 \times D^3$. According to Lisca, any symplectic filling of $(L(p, q), \xi_{can})$ is orientation preserving diffeomorphic to a blowup of $W_{p,q}(\mathbf{n})$ for some $\mathbf{n} \in \mathcal{Z}_k\left(\frac{p}{p-q}\right)$.

Remark 2.2. In particular, for $p \neq 4$, $(L(p, 1), \xi_{can})$ has a unique minimal symplectic filling and, for $p \geq 2$, $(L(p^2, p - 1), \xi_{can})$ has two distinct minimal symplectic fillings, up to diffeomorphism.

2.2. Another description of the fillings

Here we give another description of $W_{p,q}(\mathbf{n})$ which will lead to a construction of a genus-zero PALF on this 4-manifold with boundary. We refer to Figure 1 in the following discussion. First we slide the unknot with framing n_{k-1} over the unknot with framing n_k and denote by n'_{k-1} the framing of the new unknot. Next we slide the unknot with framing n_{k-2} over the unknot with framing n'_{k-1} and proceed inductively until we slide the unknot with framing n_1 over the one with framing n'_2 and let n'_1 denote its new framing. By setting $n'_k = n_k$, the new framings of the surgery curves are given by $n'_1, \ldots, n'_k$, all of which can be computed inductively by the standard formula for a handle-slide:

$$n'_i = n_i + n'_{i+1} - 2$$

for $1 \leq i \leq k - 1$. Notice that these handle-slides are performed in the complement of the link $\mathbf{L}$ in Figure 1 and the result of Dehn surgery on the new framed link is also diffeomorphic to $S^1 \times S^2$.

Moreover, this new surgery link can be viewed as the closure of a braid in S^3. We order the strands of this braid using the subindices of their associated framings. To visualize this braid, imagine a trivial braid with k strands, wrap the kth strand $n'_k - 1$ times around the first $k - 1$ strands and then wrap the strand indexed by $k - 1$ around the first $k - 2$ strands $n'_{k-1} - 1$ times and proceed inductively. See Figure 2 for an illustration of "wrapping around". To be more precise, this braid is given by

$$\prod_{j=2}^{k}(\sigma_{j-1}^{-1} \cdots \sigma_1^{-1} \sigma_1^{-1} \cdots \sigma_{j-1}^{-1})^{n'_j - 1}$$

where $\sigma_1, \ldots, \sigma_{k-1}$ are the standard generators in the braid group with k strands.
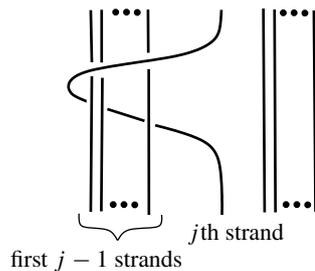

Fig. 2. The jth strand wraps around the first $j - 1$ strands once.

Each component L_i of $\mathbf{L}$ can now be viewed as an unknot linking the first i strands of this braid. As a result we get another relative handlebody description of the 4-manifold $W_{p,q}(\mathbf{n})$, where the chain of unknots with framings $n_1, \ldots, n_k$ in Lisca's description is replaced by unknots with framings $n'_1, \ldots, n'_k$ braided as described above and the link $\mathbf{L}$ plays the same role in both descriptions.

2.3. Open book decompositions of $S^1 \times S^2$

Let ξ_{st} denote the standard contact structure in $S^1 \times S^2$. Our aim in this section is to construct an open book decomposition compatible with $(S^1 \times S^2, \xi_{st})$ corresponding to a strict blowup sequence from (0) to an arbitrary positive k-tuple $\mathbf{n} = (n_1, \ldots, n_k) \in \mathcal{Z}_k$. It is well-known that the open book whose page is an annulus and whose monodromy is the identity is compatible with $(S^1 \times S^2, \xi_{st})$. We say that this open book *corresponds to* $(0) \in \mathcal{Z}_1$. If $k > 1$, we stabilize this open book once so that the new page is a disk with two holes and the new monodromy is a right-handed Dehn twist around one of the holes. This is the open book corresponding to $(1, 1) \in \mathcal{Z}_2$. The holes in the disk are ordered linearly from *left* to *right* and the Dehn twist is around the second hole as shown in Figure 3(a).
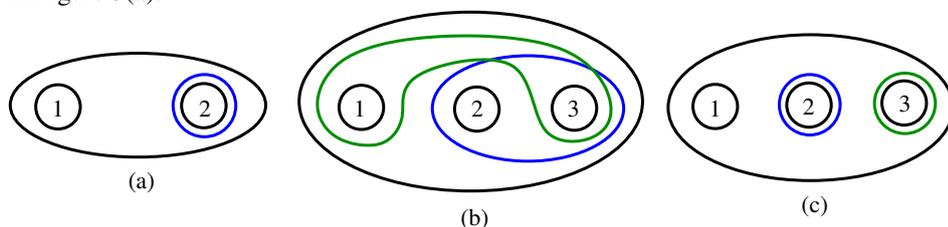

Fig. 3. Positive stabilizations.

Depending on a blowup sequence from $(1, 1)$ to $(n_1, \ldots, n_k)$, we inductively stabilize $k - 2$ times the open book corresponding to $(1, 1)$ as follows: For the initial step corresponding to the blowup $(1, 1) \to (2, 1, 2)$ we just split the second hole in Figure 3(a) into two holes, so that both holes lie in the interior of the Dehn twist. Then we relabel the holes 1, 2, 3 consecutively from left to right and add a stabilizing right-handed Dehn twist which encircles the holes labelled 1 and 3 as depicted in Figure 3(b). This is certainly a positive stabilization, as one can attach a 1-handle in the interior of the second hole in Figure 3(a), and let the stabilizing curve go over this 1-handle.

Corresponding to the alternative blowup $(1, 1) \to (1, 2, 1)$, we just insert a third hole to the *right* of the second hole so that this hole is *not included*—as opposed to the previous case—in the Dehn twist which already exists in the initial open book. Then we add a stabilizing right-handed Dehn twist around this new hole as shown in Figure 3(c).

Suppose that the page of the open book, corresponding to the result of $r - 2$ consecutive blowups starting from $(1, 1)$, is a disk D_r with r holes (for $3 \le r \le k - 1$) so that the monodromy is the product of $r - 1$ right-handed Dehn twists

$$x_1 \cdots x_{r-1}.$$

Assume that the holes are ordered linearly from *left* to *right* on the disk. If the next blowup occurs at the jth term, for $1 \le j \le r - 1$, then we insert a new hole between the jth and $(j + 1)$st holes (imagine splitting the $(j + 1)$st hole into two) and relabel the holes $1, \ldots, r + 1$ consecutively from left to right. Let D_{r+1} denote the new disk with $r + 1$ holes and let $\widetilde{x}_i$ denote the right-handed Dehn twist on D_{r+1} induced from x_i. This means that if x_i encircles the $(j + 1)$st hole in D_r, then $\widetilde{x}_i$ encircles the same holes as x_i plus the new hole inserted to obtain D_{r+1}, otherwise x_i and $\widetilde{x}_i$ encircle the same holes. To complete the stabilization, we add a right-handed Dehn twist along a curve β_j encircling the holes labelled $1, \ldots, j, j + 2$, skipping the *new hole* now labelled $j + 1$ in D_{r+1}. As a result the monodromy of the new open book is given by the product

$$\widetilde{x}_1 \cdots \widetilde{x}_{r-1} \beta_j.$$

If, on the other hand, the next blowup occurs at the rth term, we insert an $(r + 1)$st hole to the right and add a stabilizing right-handed Dehn twist α_{r+1} around this new hole labelled $r + 1$. In this case, it is clear how to lift the Dehn twist x_i in D_r to $\widetilde{x}_i$ in D_{r+1} and the resulting monodromy is

$$\widetilde{x}_1 \cdots \widetilde{x}_{r-1} \alpha_{r+1}.$$

The page of the resulting open book decomposition of $S^1 \times S^2$ corresponding to a strict blowup sequence from $(1, 1)$ to the positive k-tuple $\mathbf{n} = (n_1, \ldots, n_k)$ is a disk D_k with k holes and the monodromy is given as the product of $k - 1$ right-handed Dehn twists (ordered by the induction) along the inserted stabilizing curves at each blowup. Note that if we think of the holes in D_k as being arranged counterclockwise in an annular neighbourhood of the boundary, then each of the Dehn twists we consider is a *convex* Dehn twist.

The open book decomposition we have just constructed leads to yet another surgery description of $S^1 \times S^2$. Take the closure of a trivial braid with k strands each of which has 0-framing and insert (-1)-framed surgery curves (ordered from top to bottom) corresponding to the stabilizing curves linking this braid according to the algorithm given above. By blowing down all the (-1)-surgery curves we get a *framed braid* with k strands whose closure represents $S^1 \times S^2$.

2.4. Equivalence of the two framed braids

We claim that the framed braid with k strands obtained by blowing down all the (-1)-surgery curves in Section 2.3 is exactly the same as the framed braid obtained in Section 2.2 by handle-slides on the given chain of k unknots. Our aim in this section is to prove this claim by induction. Let us use $(n_1, \ldots, n_k)' = (n_1', \ldots n_k')$ to denote the new framings of the surgery curves after performing the handle-slides in Section 2.2.

First of all, we claim that the framings of each strand with the same index are equal in both braids. Suppose that our claim holds before we apply a blowup to an r-tuple $(n_1, \ldots, n_r)$. One can verify that the effect of a blowup of $(n_1, \ldots, n_r)$ at the jth term,

for $1 \leq j \leq r - 1$, is given by

$$(n_1, \ldots, n_{j-1}, n_j + 1, 1, n_{j+1} + 1, n_{j+2}, \ldots, n_r)'$$
$$= (n'_1 + 1, \ldots, n'_{j-1} + 1, n'_j + 1, n'_{j+1}, n'_{j+1} + 1, n'_{j+2}, \ldots, n'_r).$$

On the other hand, for the induction step in the framed surgery presentation described in Section 2.3, we insert a zero framed new strand between the jth and the $(j + 1)$st strand and relabel the strands consecutively from left to right so that the new strand has index $j + 1$. We also insert a new (-1)-surgery curve linking the strands $1, \ldots, j, j + 2$ avoiding the new $(j+1)$st strand. The induction hypothesis implies that by blowing down all the (-1)-curves except the new one, the framings of the strands are given by

$$(n'_1, \ldots, n'_j, n'_{j+1}, n'_{j+1}, n'_{j+2}, \ldots, n'_r).$$

We simply observe that blowing down the last inserted (-1)-surgery curve adds 1 to the new framing of the first j strands and the $(j + 2)$nd strand, which is consistent with the blowup formula above.

Next we show that the two braids are in fact equivalent in the complement of **L**. As the induction hypothesis we suppose that the two braids are equivalent for an r-tuple $(n_1, \ldots, n_r)$ and then we apply a blowup to $(n_1, \ldots, n_r)$ at the jth term, for $1 \leq j \leq r - 1$, as the induction step. According to the braid description in Section 2.3, we insert a new strand between the jth and $(j + 1)$st strand, which is a parallel copy of the $(j + 1)$st
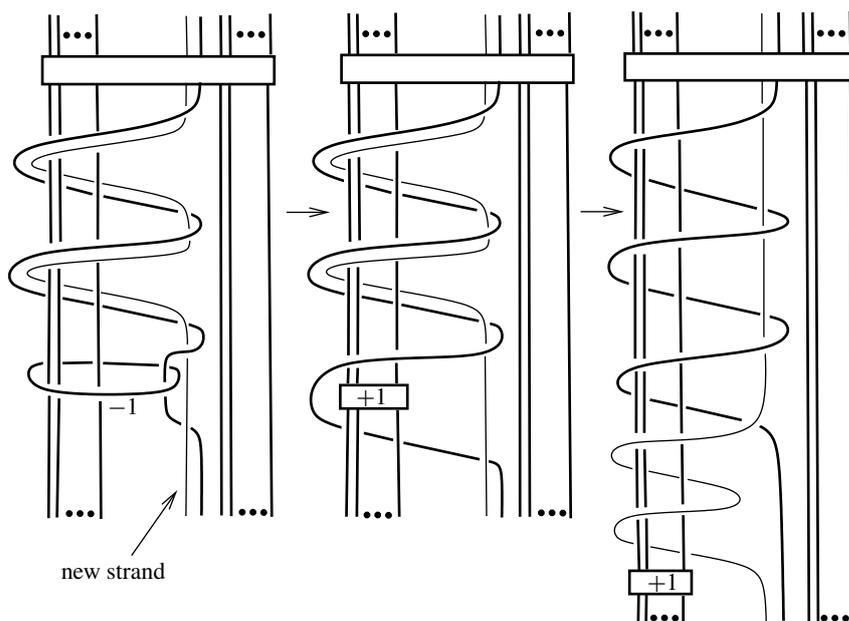

Fig. 4. Blowing down the (-1)-curve.

strand, corresponding to the blowup at the jth term. The induction hypothesis implies that by blowing down all (-1)-curves except the new one, with the new indexing, the $(j+1)$st strand links the $(j+2)$nd strand n'_{j+1} times. This is because the $(j+1)$st strand is nothing but a parallel copy of the $(j+2)$nd strand, and their linking is determined by the framing of the former $(j+1)$st strand. Similarly, they both wrap around the strands to the left of them $n'_{j+1} - 1$ times. The effect of blowing down the last inserted (-1)-curve linking the strands $1, \ldots, j, j+2$ avoiding the new strand (now indexed with $j+1$) is illustrated on the left in Figure 4, where the new strand is represented by the thin curve.

By blowing down the last (-1)-curve, the strands $1, \ldots, j, j+2$ will acquire a full right twist, as shown in the middle of Figure 4. When we pull the "spring" in the thin curve down, it becomes clear how this $(j+1)$st strand wraps around the strands to the left of it $n'_{j+1} - 1$ times as depicted on the right in Figure 4. In this new braid the number of times any strand wraps around the strands to the left of it is consistent with the blowup formula given above. In particular, the $(j+2)$nd strand wraps around the strands to the left of it n'_{j+1} times.

To verify our claim for the case of a blowup of an r-tuple at the jth term for $j = r$ is much easier and it is left to the reader.

2.5. Genus-zero PALF on the fillings

The open book decomposition of $S^1 \times S^2$ described in Section 2.3, corresponding to any sequence of strict blowups from (0) to a k-tuple $\mathbf{n} \in \mathcal{Z}_k\left(\frac{p}{p-q}\right)$, is compatible with the unique tight contact structure on $S^1 \times S^2$. The genus-zero PALF over D^2 whose boundary is given by this open book is diffeomorphic to $S^1 \times D^3$ since the tight contact $S^1 \times S^2$ has a unique Stein filling up to diffeomorphism. A handlebody decomposition of this PALF on $S^1 \times D^3$ can be obtained from the closure of the framed braid in Section 2.3 by converting the 0-framed surgery curves—the strands of the braid—to dotted circles representing 1-handles, where each (-1)-surgery curve linking the strands of this braid represents a vanishing cycle.

Inserting the link $\mathbf{L}$ into this diagram completes the handlebody decomposition of the desired PALF on $W_{p,q}(\mathbf{n})$, since each component of $\mathbf{L}$ also represents a vanishing cycle. This is because each component of $\mathbf{L}$ can be Legendrian realized on the planar page of the open book of $S^1 \times S^2$.

As a consequence, the resulting contact structure on $L(p, q)$ is obtained by Legendrian surgery from the standard tight contact $S^1 \times S^2$. The ordered vanishing cycles of this PALF on $W_{p,q}(\mathbf{n})$ can be explicitly described on a disk with k holes by the algorithm given in Section 2.3, where we add a Dehn twist corresponding to each component of $\mathbf{L}$ at the end. Summarizing we obtain

Theorem 2.3. *There is an algorithm to present any minimal symplectic filling of the canonical contact structure on a lens space as an explicit genus-zero PALF over the disk.*

We would like to point out that the PALF in Theorem 2.3 can be obtained explicitly, which therefore leads to an *absolute* handlebody decomposition of any symplectic filling at hand as opposed to the relative decomposition depicted in Figure 1.

2.6. An example

In the following we illustrate our algorithm to construct a genus-zero PALF on the symplectic filling $W_{(81,47)}(\mathbf{n})$ of the canonical contact structure on $L(81, 47)$, where $\mathbf{n} = (3, 2, 1, 3, 2)$. Note that $\frac{81}{47} = [2, 4, 3, 3, 2]$ and $\frac{81}{81-47} = [3, 2, 3, 3, 3]$.

According to Lisca's classification, $W_{(81,47)}(\mathbf{n})$ represents one of the six distinct diffeomorphism classes of minimal symplectic fillings of the canonical contact structure on $L(81, 47)$. The link $\mathbf{L}$ in Lisca's description of the filling in question has three components in total, two of which are linking the third and one linking the fifth unknot in the chain $\mathbf{n}$ (see Figure 5).
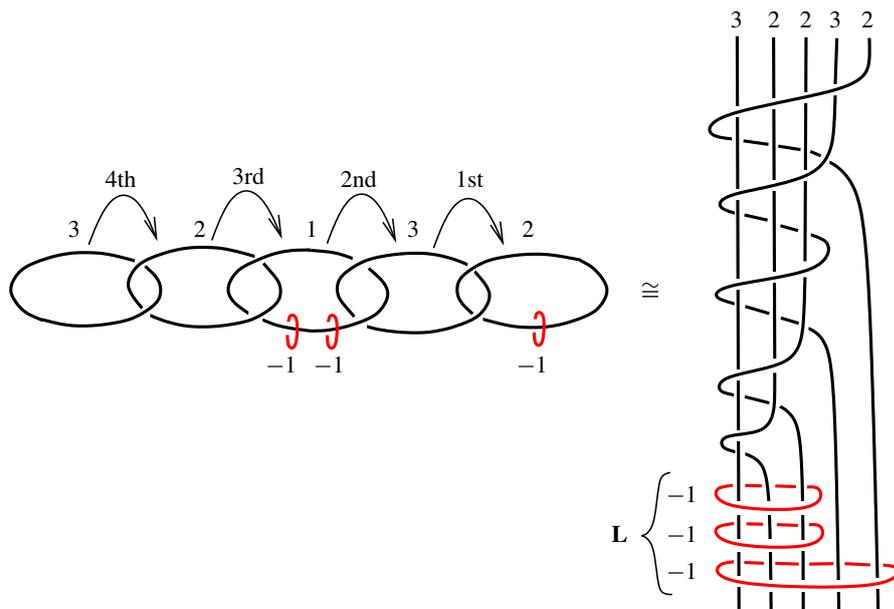

Fig. 5. Handle-slides.

First we slide 2-handles in the chain over each other and obtain a new surgery diagram as shown on the right in Figure 5. The new unknots can be drawn as the closure of a braid and their framings are given by $(n'_1, \ldots, n'_5) = (3, 2, 2, 3, 2)$. In addition, two components of $\mathbf{L}$ link the first three strands, and one component links all the strands of this braid.

On the other hand, positive stabilizations of the standard open book of $S^1 \times S^2$ corresponding to the blowup sequence

$$(1, 1) \to (2, 1, 2) \to (3, 1, 2, 2) \to (3, 2, 1, 3, 2) = \mathbf{n}$$

are depicted in Figure 6. The monodromy of our PALF on $W_{(81,47)}((3, 2, 1, 3, 2))$ is given as the product

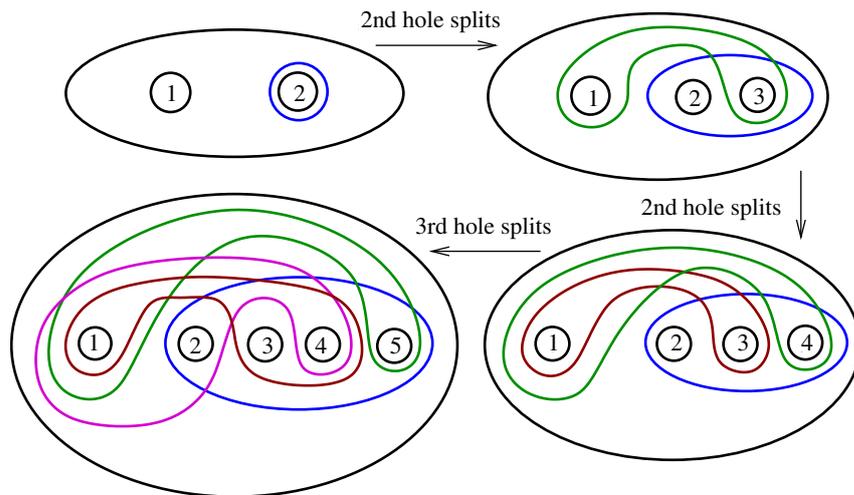$$x_1 x_2 x_3 \beta_2 \gamma_3^2 \gamma_5$$

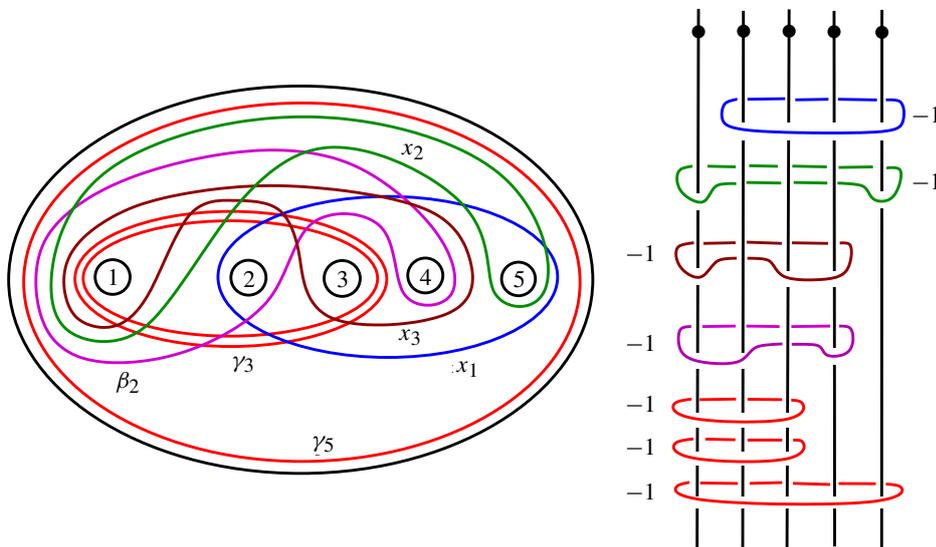**Fig. 6.** Positive stabilizations of the standard open book.

Fig. 7. Monodromy $x_1x_2x_3\beta_2\gamma_3^2\gamma_5$ of the PALF on $W_{(81,47)}((3, 2, 1, 3, 2))$ and its handlebody diagram.

of right-handed Dehn twists along the four stabilizing curves x_1, x_2, x_3, β_2 in the order they appear and three more right-handed Dehn twists corresponding to the link **L** (see Figure 7). Two of these latter ones are along two disjoint copies of a convex curve γ_3 encircling the first three holes and one is along a convex curve γ_5 encircling all the holes. Moreover, a handle decomposition of $W_{(81,47)}((3, 2, 1, 3, 2))$ including five 1-handles, where one can explicitly see the PALF, is shown in Figure 7.

3. Monodromy substitutions and rational blowdowns

The lantern relation in the mapping class group of a sphere with four holes was discovered by Dehn although Johnson named it the lantern relation after rediscovering it in [10]. This relation and its generalizations have been effectively used recently in solving some interesting problems in low-dimensional topology. The key point is that the lantern relation (cf. Figure 8) holds in any subsurface of another surface which is homeomorphic to a sphere with four holes.
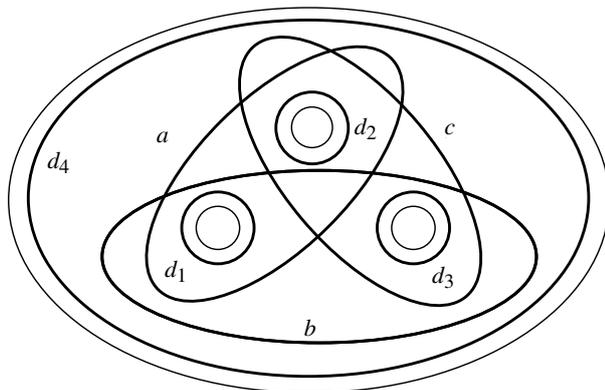

Fig. 8. The lantern relation $d_1 d_2 d_3 d_4 = abc$ on a four-holed sphere

Suppose that there is a "piece" in the monodromy factorization of a (not necessarily positive or allowable) Lefschetz fibration which appears as the left-hand side of the lantern relation. Deleting that piece from the monodromy word and inserting the right-hand side is called a *lantern substitution*. It was shown in [5] that the effect of this substitution in the total space of the fibration is a rational blowdown operation, which can be easily seen as follows: The PALF with monodromy $d_1 d_2 d_3 d_4$ is diffeomorphic to the D^2 bundle over S^2 with Euler number -4, while the PALF with monodromy abc is diffeomorphic to a rational 4-ball with boundary $L(4, 1)$. Cutting a submanifold diffeomorphic to the D^2-bundle over S^2 with Euler number -4 from a 4-manifold and gluing in a rational 4-ball was named a *rational blowdown operation* by Fintushel and Stern [7].

We would like to point out that the genus-zero PALF with monodromy $d_1 d_2 d_3 d_4$ and the genus-zero PALF with monodromy abc represent the two distinct diffeomorphism classes of the minimal symplectic fillings of $(L(4, 1), \xi_{\text{can}})$.

Since the linear plumbing of $p-1$ disk bundles over S^2 with Euler numbers $-(p+2)$, $-2, \ldots, -2$ has boundary $L(p^2, p-1)$, which also bounds a rational 4-ball, the cut-and-paste operation described above is defined similarly for this case [7]. The corresponding monodromy substitution was discovered and named the *daisy relation* in [6], which is essentially obtained by repeated applications of the lantern substitution. In fact, the PALFs given by the products of right-handed Dehn twists appearing on the two sides of the daisy relation represent the two distinct diffeomorphism classes of the minimal symplectic fillings of $(L(p^2, p-1), \xi_{\text{can}})$ for any $p \geq 2$.

A generalization of Fintushel and Stern's rational blowdown operation was introduced in [16] involving the lens space $L(p^2, pq - 1)$ as the boundary. The corresponding monodromy substitution for this rational blowdown can be computed by the technique introduced in [6].

A *rational blowdown along a linear plumbing graph* is the replacement of a neighbourhood of a configuration of spheres in a smooth 4-manifold which intersect according to a linear plumbing graph whose boundary is $L(p^2, pq - 1)$ by a rational 4-ball with the same oriented boundary.

4. Symplectic fillings and rational blowdowns

Our goal in this section is to prove our main result. We point out that our proof does not rely on the results in [6].

Theorem 4.1. *Any minimal symplectic filling of the canonical contact structure on a lens space is obtained, up to diffeomorphism, by a sequence of rational blowdowns along linear plumbing graphs from the minimal resolution of the corresponding complex two-dimensional cyclic quotient singularity.*

Remark 4.2. According to [18] (see also [8]), the rational blowdowns in Theorem 4.1 can be realized as *symplectic* rational blowdowns.

It will be convenient to make the following definitions for the proof of Theorem 4.1.

Definition 4.3. For a positive k-tuple $\mathbf{n} = (n_1, \ldots, n_k) \in \mathcal{Z}_k$, we say that $\mathbf{n}$ has *height s*, and write $\mathrm{ht}(\mathbf{n}) = s$, if s is the minimal number of strict blowups required to obtain $\mathbf{n}$ from an l-tuple of the form $(1, 2, \ldots, 2, 1) \in \mathbb{Z}^l$, which we will denote by $\mathbf{u}_l$, for $l \geq 2$. We set $\mathbf{u}_1 = (0)$ and define $\mathrm{ht}(\mathbf{u}_1) = 0$.

It is easy to check that
$$\mathrm{ht}(\mathbf{n}) = |\mathbf{n}| - 2(k - 1),$$
for any $\mathbf{n} = (n_1, \ldots, n_k) \in \mathcal{Z}_k$, where $|\mathbf{n}| = n_1 + \cdots + n_k$.

In addition, we slightly generalize the definition of the 4-manifold $W_{p,q}(\mathbf{n})$ as follows:

Definition 4.4. For a pair of k-tuples $\mathbf{n} = (n_1, \ldots, n_k), \mathbf{m} = (m_1, \ldots, m_k) \in \mathbb{Z}^k$, with $\mathbf{n} \in \mathcal{Z}_k$, we will denote by $W(\mathbf{n}, \mathbf{m})$ the 4-manifold constructed as in Section 2 from the 3-manifold $N(\mathbf{n}) \cong S^1 \times S^2$ and the framed link $\mathbf{L} = \bigcup_{i=1}^k L_i$ associated to $\mathbf{m}$, where L_i consists of $|m_i|$ components as in Figure 1 with the components having framings -1 if $m_i > 0$ and framings $+1$ if $m_i < 0$.

Note that if each $m_i \geq 0$ and $b_i := n_i + m_i \geq 2$ for all i, then there are unique integers $1 \leq q < p$ with $(p, q) = 1$ such that
$$\frac{p}{p - q} = [b_1, \ldots, b_k].$$

In this case $W(\mathbf{n}, \mathbf{m})$ is just the minimal symplectic filling $W_{p,q}(\mathbf{n})$ of $L(p, q)$ given by Lisca. Also note that if $\mathbf{m}$ has precisely one component m_j which is different from 0 with $m_j = \pm 1$ and $n_j = 1$, then $W(\mathbf{n},\mathbf{m})$ is a rational 4-ball. To see this, note that $H_1(W(\mathbf{n}, \mathbf{m}), \mathbb{Q})$ and $H_2(W(\mathbf{n}, \mathbf{m}), \mathbb{Q})$ are trivial precisely when the matrix describing the linking of the attaching circles of the 2-handles with the dotted circles representing the 1-handles is nondegenerate, and it is easy to check that the latter holds when one imposes the above conditions on $\mathbf{m}$ and $\mathbf{n}$.

By the algorithm in Section 2, the 4-manifold $W(\mathbf{n}, \mathbf{m})$ with boundary admits a genus-zero ALF (*achiral* Lefschetz fibration) over D^2. In other words, the monodromy of the Lefschetz fibration will include left-handed Dehn twists if $m_i < 0$ for some i. In the following by the *monodromy factorization* of $W(\mathbf{n}, \mathbf{m})$ we mean the monodromy factorization of this Lefschetz fibration over D^2 (which may include some left-handed Dehn twists). Moreover, by a *cancelling pair of Dehn twists* we mean the composition of a right-handed and a left-handed Dehn twist along two parallel copies of some curve on a surface. Our proof of Theorem 4.1 is based on the following preliminary result.

Lemma 4.5. *Given a pair of k-tuples $\mathbf{n} = (n_1, \ldots, n_k), \mathbf{m} = (m_1, \ldots, m_k) \in \mathbb{Z}^k$ with $\mathbf{n} \in \mathcal{Z}_k$ and $s = \mathrm{ht}(\mathbf{n}) \geq 1$, there exists a sequence of k-tuples $\mathbf{n}_0, \ldots, \mathbf{n}_s \in \mathcal{Z}_k$ with $\mathbf{n}_0 = \mathbf{u}_k$ and $\mathbf{n}_s = \mathbf{n}$ such that, setting $\mathbf{m}_i = \mathbf{n} + \mathbf{m} - \mathbf{n}_i$, the monodromy factorization of $W(\mathbf{n}_i, \mathbf{m}_i)$ can be obtained from the monodromy factorization of $W(\mathbf{n}_{i-1}, \mathbf{m}_{i-1})$ by a lantern substitution together with, possibly, the introduction or removal of some cancelling pairs of Dehn twists for $1 \leq i \leq s$.*

Proof. The proof will be by induction on s. Suppose that $s = \mathrm{ht}(\mathbf{n}) = 1$. This means that $\mathbf{n} = \psi_j(\mathbf{u}_{k-1})$ for some $1 \leq j \leq k - 2$, where $\psi_j \colon \mathbb{Z}^{k-1} \to \mathbb{Z}^k$ denotes the strict blowup at the jth term. Letting $\mathbf{m}' = (m'_1, \ldots, m'_k) = \mathbf{n} + \mathbf{m} - \mathbf{u}_k$, we find that

$$m_i = \begin{cases} m'_i - 1 & \text{if } i = j, \\ m'_i + 1 & \text{if } i = j + 1, \\ m'_i - 1 & \text{if } i = j + 2, \\ m'_i & \text{otherwise} \end{cases}$$

for any $\mathbf{m} = (m_1, \ldots, m_k) \in \mathbb{Z}^k$. We compute the monodromy factorizations ϕ and ϕ' of $W(\mathbf{n}, \mathbf{m})$ and $W(\mathbf{u}_k, \mathbf{m}')$, respectively. For this, consider a disk D_k with k holes ordered linearly from left to right and label the boundary of the ith hole α_i, for $2 \leq i \leq k$. Also, label the convex curve containing the first i holes γ_i, for $1 \leq i \leq k$, and label the convex curve containing the $(j + 1)$st and the $(j + 2)$nd holes δ_j. Finally label the convex curve containing the first j holes plus the $(j + 2)$nd hole β_j. Here "convex" is used as in the sense of Section 2.3. Following the algorithm given in the same section, we find that

$$\phi' = \alpha_2 \cdots \alpha_k \gamma_1^{m'_1} \cdots \gamma_k^{m'_k}$$

and

$$\phi = \alpha_2 \cdots \alpha_j \delta_j \alpha_{j+3} \cdots \alpha_k \beta_j \gamma_1^{m'_1} \cdots \gamma_{j-1}^{m'_{j-1}} \gamma_j^{m'_j - 1} \gamma_{j+1}^{m'_{j+1} + 1} \gamma_{j+2}^{m'_{j+2} - 1} \gamma_{j+3}^{m'_{j+3}} \cdots \gamma_k^{m'_k}.$$

We see that ϕ can be obtained from ϕ' by the single lantern substitution

$$\alpha_{j+1}\alpha_{j+2}\gamma_j\gamma_{j+2} = \delta_j\beta_j\gamma_{j+1}.$$

Note, however, that if either $m'_j \le 0$ or $m'_{j+2} \le 0$, then we will need to introduce a cancelling pair of Dehn twists into the monodromy factorization ϕ' before we can apply the lantern substitution. Also, if $m'_{j+1} \le -1$, then after applying the lantern substitution we will remove a cancelling pair of Dehn twists which appears in the monodromy. This finishes the proof for $s = 1$ by setting $\mathbf{m}_0 = \mathbf{m}'$.

Now suppose that t is a positive integer and it is known that for every pair of k-tuples $\mathbf{n}, \mathbf{m} \in \mathbb{Z}^k$ with $\mathbf{n} \in \mathcal{Z}_k$ and $s = \text{ht}(\mathbf{n}) \le t$ there exists a sequence of k-tuples $\mathbf{n}_0, \ldots, \mathbf{n}_s \in \mathcal{Z}_k$ with $\mathbf{n}_0 = \mathbf{u}_k$ and $\mathbf{n}_s = \mathbf{n}$ such that, setting $\mathbf{m}_i = \mathbf{n} + \mathbf{m} - \mathbf{n}_i$, the monodromy factorization of $W(\mathbf{n}_i, \mathbf{m}_i)$ can be obtained from the monodromy factorization of $W(\mathbf{n}_{i-1}, \mathbf{m}_{i-1})$ by a lantern substitution together with, possibly, the introduction or removal of some cancelling pairs of Dehn twists for $1 \le i \le s$. Let $\mathbf{n}, \mathbf{m} \in \mathbb{Z}^k$ be a pair of k-tuples with $\mathbf{n} \in \mathcal{Z}_k$ and $s = \text{ht}(\mathbf{n}) = t + 1$. Then there is a $(k - 1)$-tuple $\mathbf{n}' \in \mathcal{Z}_{k-1}$ such that $\mathbf{n} = \psi_j(\mathbf{n}')$ and $\text{ht}(\mathbf{n}') = t$. Let $\rho_{j+1}: \mathbb{Z}^k \to \mathbb{Z}^{k-1}$ denote the map $(l_1, \ldots, l_k) \mapsto (l_1, \ldots, \widehat{l_{j+1}}, \ldots, l_k)$ given by omitting the $(j + 1)$st entry. By the induction hypothesis, there is a sequence of $(k - 1)$-tuples $\mathbf{n}'_0, \ldots, \mathbf{n}'_t \in \mathcal{Z}_{k-1}$ with $\mathbf{n}'_0 = \mathbf{u}_{k-1}$ and $\mathbf{n}'_t = \mathbf{n}'$ such that, setting $\mathbf{m}'_i = \mathbf{n}' + \rho_{j+1}(\mathbf{m}) - \mathbf{n}'_i$, the monodromy factorization of $W(\mathbf{n}'_i, \mathbf{m}'_i)$ can be obtained from the monodromy factorization of $W(\mathbf{n}'_{i-1}, \mathbf{m}'_{i-1})$ by a lantern substitution together with, possibly, the introduction or removal of some cancelling pairs of Dehn twists for $1 \le i \le t$. Consider the sequence $\mathbf{n}_i = \psi_j(\mathbf{n}'_{i-1})$ for $1 \le i \le s = t + 1$ of k-tuples in $\mathcal{Z}_k$ obtained by taking strict blowups at the jth term of the $(k - 1)$-tuples in the sequence $\mathbf{n}'_0, \ldots, \mathbf{n}'_t$. Let $\mathbf{n}_0 = \mathbf{u}_k$ and set $\mathbf{m}_i = \mathbf{n} + \mathbf{m} - \mathbf{n}_i$ for $0 \le i \le s$. We claim that the monodromy factorization of $W(\mathbf{n}_i, \mathbf{m}_i)$ can be obtained from the monodromy factorization of $W(\mathbf{n}_{i-1}, \mathbf{m}_{i-1})$ by a lantern substitution together with, possibly, the introduction or removal of some cancelling pairs of Dehn twists for $1 \le i \le s$.

For $i = 1$, the proof follows from the above since $\text{ht}(\mathbf{n}_1) = 1$. Suppose that $i > 1$. Then the monodromy factorization ϕ'_{i-2} of $W(\mathbf{n}'_{i-2}, \mathbf{m}'_{i-2})$ has the form

$$\phi'_{i-2} = c_1 \cdots c_l,$$

where c_r denotes a convex Dehn twist of D_{k-1} for $1 \le r \le l$. It follows that the monodromy factorization ϕ_{i-1} of $W(\mathbf{n}_{i-1}, \mathbf{m}_{i-1})$ has the form

$$\phi_{i-1} = \widetilde{c}_1 \cdots \widetilde{c}_l \beta_j \gamma_{j+1}^{m_{i-1,j+1}},$$

where β_j and γ_{j+1} are convex Dehn twists of D_k as before and $m_{i-1,j+1}$ denotes the $(j + 1)$st component of $\mathbf{m}_{i-1}$. Here we have used the convention that if σ is a convex Dehn twist of D_{k-1} around a collection H of holes, then $\widetilde{\sigma}$ denotes the convex Dehn twist of D_k around the collection $\widetilde{H}$ of holes given by

$$\widetilde{H} = \{r \mid 1 \le r \le j + 1 \text{ and } r \in H\} \cup \{r + 1 \mid j + 1 \le r \le k - 1 \text{ and } r \in H\}.$$

By the induction hypothesis, the monodromy factorization ϕ'_{i-1} of $W(\mathbf{n}'_{i-1}, \mathbf{m}'_{i-1})$ is obtained from the monodromy factorization ϕ'_{i-2} of $W(\mathbf{n}'_{i-2}, \mathbf{m}'_{i-2})$ via a lantern relation of the form

$$c_{i_1} c_{i_2} c_{i_3} c_{i_4} = c_{i_5} c_{i_6} c_{i_7},$$

where, for each r, c_{i_r} is a convex Dehn twist of D_{k-1} which may or may not be included in the set $\{c_1, \ldots, c_l\}$ of convex Dehn twists, together with, possibly, the introduction or removal of some cancelling pairs of Dehn twists. It follows easily that the monodromy factorization ϕ_i of $W(\mathbf{n}_i, \mathbf{m}_i)$ is obtained from the monodromy factorization ϕ_{i-1} of $W(\mathbf{n}_{i-1}, \mathbf{m}_{i-1})$ via a lantern relation of the form

$$\widetilde{c}_{i_1} \widetilde{c}_{i_2} \widetilde{c}_{i_3} \widetilde{c}_{i_4} = \widetilde{c}_{i_5} \widetilde{c}_{i_6} \widetilde{c}_{i_7},$$

together with, possibly, the introduction or removal of some cancelling pairs of Dehn twists, completing the proof of the induction step and the lemma. □

Proof of Theorem 4.1. Let $\mathbf{n} = (n_1, \ldots, n_k)$, $\mathbf{m} = (m_1, \ldots, m_k)$ be k-tuples in $\mathbb{Z}^k$ with $\mathbf{n} \in \mathcal{Z}_k$ and $\mathbf{m}$ nonnegative. Assume that $n_j + m_j > 1$ for all $1 < j < k$. We prove that $W(\mathbf{n}, \mathbf{m})$ is obtained from $W(\mathbf{u}_k, \mathbf{m}_0)$ by a sequence of rational blowdowns, where $\mathbf{m}_0 = \mathbf{n} + \mathbf{m} - \mathbf{u}_k$. The statement of the theorem follows from this.

Let $s = \mathrm{ht}(\mathbf{n})$. If $s = 0$, then $W(\mathbf{n}, \mathbf{m})$ corresponds to the filling of a lens space by the canonical plumbing and there is nothing to check. Suppose that $s \geq 1$ and consider the sequence

$$\mathbf{n} = \mathbf{n}^0 \to \mathbf{n}^1 \to \cdots \to \mathbf{n}^s \tag{4.1}$$

given by taking the strict blowdown at the leftmost possible 1. (Note, in particular, that if the first term of $\mathbf{n}$ is 1, then, according to the definition of a strict blowdown, $\mathbf{n}$ cannot be strictly blown down at this term.) Here $\mathbf{n}^i \in \mathbb{Z}^{k-i}$ for $0 \leq i \leq s$, and $\mathbf{n}^s = \mathbf{u}_{k-s}$. From the proof of Lemma 4.5, there is an associated sequence $\mathbf{u}_k = \mathbf{n}_0, \ldots, \mathbf{n}_s = \mathbf{n}$ such that, setting $\mathbf{m}_i = \mathbf{n} + \mathbf{m} - \mathbf{n}_i$, the monodromy factorization of $W(\mathbf{n}_i, \mathbf{m}_i)$ is obtained from the monodromy factorization of $W(\mathbf{n}_{i-1}, \mathbf{m}_{i-1})$ by a lantern substitution together with, possibly, the introduction or removal of some cancelling pairs of Dehn twists. Let $0 = i_0 < i_1 < \cdots < i_r = s$ be the sequence of indices such that $\mathbf{m}_i$ has all components nonnegative if and only $i = i_j$ for some j. We claim that $W(\mathbf{n}_{i_j}, \mathbf{m}_{i_j})$ is obtained from $W(\mathbf{n}_{i_{j-1}}, \mathbf{m}_{i_{j-1}})$ by a rational blowdown for $1 \leq j \leq r$. The proof is by induction on r.

Suppose that $r = 1$, that is, $i_1 = s$. We first show that $\mathbf{n} = \mathbf{n}_s$ contains exactly one component n_j equal to 1 with $1 < j < k$. On the contrary, suppose that $\mathbf{n}$ contains at least two such components. Consider the strict blowdown sequence in (4.1) and let $\mathbf{n}^t$ be the first tuple which has fewer interior components equal to 1 than $\mathbf{n}$. It follows from the assumption that $t < s$. Let $\mathbf{m} = \mathbf{m}^0, \ldots, \mathbf{m}^s$ denote the associated sequence constructed as follows: if $\mathbf{n}^i$ is obtained from $\mathbf{n}^{i-1}$ by a strict blowdown at the jth term, let $\mathbf{m}^i = \rho_j(\mathbf{m}^{i-1})$, where, as before, $\rho_j: \mathbb{Z}^{k-i+1} \to \mathbb{Z}^{k-i}$ is the map given by omitting the jth entry. For each pair $(\mathbf{n}^i, \mathbf{m}^i)$, consider the sequence $(\mathbf{n}_0^i = \mathbf{u}_{k-i}, \mathbf{m}_0^i), \ldots, (\mathbf{n}_{s-i}^i = \mathbf{n}^i, \mathbf{m}_{s-i}^i = \mathbf{m}^i)$ constructed as in the proof of Lemma 4.5 from the portion of the blowdown sequence (4.1) beginning at $\mathbf{n}^i$. Now consider the following diagram:

$$(\mathbf{n} = \mathbf{n}_s^0, \mathbf{m} = \mathbf{m}_s^0) \dashrightarrow (\mathbf{n}^t = \mathbf{n}_{s-t}^t, \mathbf{m}_{s-t}^t) \dashrightarrow (\mathbf{n}_0^0 = \mathbf{u}_{k-s}, \mathbf{m}_0^s)$$

$$(\mathbf{n}_t^0, \mathbf{m}_t^0) \dashrightarrow (\mathbf{n}_0^t = \mathbf{u}_{k-t}, \mathbf{m}_0^t)$$

$$(\mathbf{n}_0^0 = \mathbf{u}_k, \mathbf{m}_0^0)$$

Note that, by construction, every interior component of $\mathbf{n}_{s-t}^t + \mathbf{m}_{s-t}^t$ is greater than 1. (Here the condition that the strict blowdowns are chosen in a consistent fashion, such as taking each strict blowdown at the leftmost possible 1, as we have done, is essential.) Hence every interior component of $\mathbf{n}_0^t + \mathbf{m}_0^t$, being equal to the corresponding component of $\mathbf{n}_{s-t}^t + \mathbf{m}_{s-t}^t$, is greater than 1. Since $\mathbf{n}_0^t = \mathbf{u}_{k-t} = (1, 2, \ldots, 2, 1)$, it follows that every component of $\mathbf{m}_0^t$ is nonnegative (that this is true for the first and last components of $\mathbf{m}_0^t$ follows from the fact that each component of $\mathbf{m}_{s-t}^t$ is nonnegative). Now note that $\mathbf{m}_{l+1}^{i-1}$ can be obtained from $\mathbf{m}_l^i$ as follows: Suppose that $\mathbf{n}_{s-i}^i$ is obtained from $\mathbf{n}_{s-i+1}^{i-1}$ by strictly blowing down at the jth term (and hence that $\mathbf{n}_l^i$ is obtained from $\mathbf{n}_{l+1}^{i-1}$ also by strictly blowing down at the jth term for $0 \le l \le s-i$); then $\mathbf{m}_{l+1}^{i-1} = \chi_j(\mathbf{m}_l^i)$, where $\chi_j : \mathbb{Z}^{k-i} \to \mathbb{Z}^{k-i+1}$ is the map $(z_1, \ldots, z_{k-i}) \mapsto (z_1, \ldots, z_{j-1}, m_j^{i-1}, z_j, \ldots, z_{k-i})$ given by splicing into the jth position the jth component of $\mathbf{m}^{i-1}$, which, being an entry of $\mathbf{m}$, is nonnegative. It follows that every component of $\mathbf{m}_t^0$ is nonnegative, contradicting the fact that $r = 1$. This proves that $\mathbf{n}$ contains exactly one interior component n_j equal to 1.

We now proceed as follows: Given $\mathbf{n}$, suppose that n_j is the only component that is equal to 1, with $1 < j < k$. Let $\mathbf{m}' = (0, \ldots, 0, 1, 0, \ldots, 0)$, where the 1 is in the jth position. Then $\mathbf{m}' \le \mathbf{m}$, since every interior component of $\mathbf{n} + \mathbf{m}$ is greater than 1 and $\mathbf{m}$ is nonnegative. Now note that $W(\mathbf{u}_k, \mathbf{m}_0')$ can be rationally blown down to $W(\mathbf{n}, \mathbf{m}')$, where $\mathbf{m}_0' = \mathbf{n} + \mathbf{m}' - \mathbf{u}_k$ (since $W(\mathbf{u}_k, \mathbf{m}_0')$ corresponds to the filling of a lens space by the canonical plumbing and $W(\mathbf{n}, \mathbf{m}')$ is a rational 4-ball). By replacing the "piece" of the monodromy factorization of $W(\mathbf{u}_k, \mathbf{m}_0)$ that corresponds to the monodromy factorization of $W(\mathbf{u}_k, \mathbf{m}_0')$, where $\mathbf{m}_0 = \mathbf{n} + \mathbf{m} - \mathbf{u}_k$, by the monodromy factorization of $W(\mathbf{n}, \mathbf{m}')$, we see that $W(\mathbf{n}, \mathbf{m})$ is obtained from $W(\mathbf{u}_k, \mathbf{m}_0)$ by a rational blowdown.

Now assume that $l \ge 1$ and the claim is known to hold whenever $r \le l$. Suppose that $r = l + 1$ and consider the following diagram:

$$(\mathbf{n} = \mathbf{n}_{i_r}^0, \mathbf{m} = \mathbf{m}_{i_r}^0) \dashrightarrow (\mathbf{n}_{i_r-i_1}^{i_1}, \mathbf{m}_{i_r-i_1}^{i_1}) \dashrightarrow (\mathbf{n}_0^{i_r} = \mathbf{u}_{k-i_r}, \mathbf{m}_0^{i_r})$$

$$(\mathbf{n}_{i_1}^0, \mathbf{m}_{i_1}^0) \dashrightarrow (\mathbf{n}_0^{i_1} = \mathbf{u}_{k-i_1}, \mathbf{m}_0^{i_1})$$

$$(\mathbf{n}_0^0 = \mathbf{u}_k, \mathbf{m}_0^0)$$

By the previous step, we know that there is exactly one j with $1 < j < k$ such that the jth component of $\mathbf{n}^0_{i_1}$ is 1. It follows that $W(\mathbf{n}^0_{i_1}, \mathbf{m}^0_{i_1})$ is obtained from $W(\mathbf{n}^0_0, \mathbf{m}^0_0)$ by a rational blowdown. Thus it is sufficient to show that $W(\mathbf{n}^0_{i_r}, \mathbf{m}^0_{i_r})$ is obtained from $W(\mathbf{n}^0_{i_1}, \mathbf{m}^0_{i_1})$ by a sequence of rational blowdowns. For this, consider the pair $(\mathbf{n}^{i_1}_{i_r - i_1}, \mathbf{m}^{i_1}_{i_r - i_1})$. Since in the sequence $\mathbf{m}^{i_1}_0, \mathbf{m}^{i_1}_1, \dots, \mathbf{m}^{i_1}_{s - i_1}$ the only tuples with all components nonnegative are precisely the ones with subscripts $0 < i_2 - i_1 < \cdots < i_r - i_1 = s - i_1$, it follows from the induction hypothesis that $W(\mathbf{n}^{i_1}_{i_r - i_1}, \mathbf{m}^{i_1}_{i_r - i_1})$ is obtained from $W(\mathbf{u}_{k - i_1}, \mathbf{m}^{i_1}_0)$ by a sequence of rational blowdowns. Now, arguing as before we find that $W(\mathbf{n}^0_{i_r}, \mathbf{m}^0_{i_r})$ is obtained from $W(\mathbf{n}^0_{i_1}, \mathbf{m}^0_{i_1})$ by a sequence of rational blowdowns, completing the induction step and the proof of the theorem. $\qquad\square$

The content of [11, Corollary 5.2 and Theorem 6.1] can be recovered as a corollary:

Corollary 4.6. *Any minimal symplectic filling of the canonical contact structure on a lens space can be realized as a Stein filling, i.e. the underlying smooth 4-manifold with boundary admits a Stein structure whose induced contact structure on the boundary agrees with the canonical one.*

Proof. Any minimal symplectic filling of the canonical contact structure on a lens space admits a PALF over D^2 by Theorem 2.3 (also by [19, Theorem 1]). This implies that the underlying smooth 4-manifold with boundary admits a Stein structure whose induced contact structure on the boundary is compatible with the open book induced from the PALF [1]. By the proof of Theorem 4.1, the induced open book on the boundary is fixed for all distinct PALFs constructed for a given lens space. The desired result follows since we know that the induced open book on the boundary of the canonical PALF on the minimal resolution is compatible with the canonical contact structure [14]. $\qquad\square$

Corollary 4.7. *The canonical contact structure on a lens space admits a unique minimal symplectic filling—represented by the Stein structure via the PALF we constructed on the minimal resolution—up to symplectic rational blowdown and symplectic deformation equivalence.*

Proof. This follows from the combination of Theorem 4.1, Remark 4.2, Corollary 4.6 and the fact that each diffeomorphism type of a minimal symplectic filling of the canonical contact structure on a lens space carries a unique symplectic structure up to symplectic deformation equivalence which fills the contact structure in question [2]. $\qquad\square$

Corollary 4.8. *Any Milnor fibre of any smoothing of a complex two-dimensional cyclic quotient singularity can be obtained, up to diffeomorphism, by a sequence of rational blowdowns along linear plumbing graphs from the Milnor fibre diffeomorphic to the minimal resolution of the singularity.*

Proof. This immediately follows from Theorem 4.1 coupled with the results in [12], in which Némethi and Popescu-Pampu prove that the classification of Milnor fibres for a cyclic quotient singularity agrees with Lisca's classification of symplectic fillings for the canonical contact singularity link, up to diffeomorphism. $\qquad\square$

4.1. An example

We would like to describe how one can obtain the symplectic filling $W_{(81,47)}((3, 2, 1, 3, 2))$ from the minimal resolution $W_{(81,47)}((1, 2, 2, 2, 1))$ by a single rational blowdown.

The monodromy of the the canonical PALF on $W_{(81,47)}((1, 2, 2, 2, 1))$, which is illustrated in Figure 9(a), can be expressed as

$$\alpha_2\alpha_3\alpha_4 a_5\gamma_1^2\gamma_3\gamma_4\gamma_5^2$$

by our algorithm using the blowup sequence

$$(1, 1) \rightarrow (1, 2, 1) \rightarrow (1, 2, 2, 1) \rightarrow (1, 2, 2, 2, 1).$$

In the following we describe a sequence of lantern substitutions, together with introduction or removal of some cancelling pairs of Dehn twists, to obtain the PALF (see Figure 7) we constructed on the symplectic filling $W_{(81,47)}((3, 2, 1, 3, 2))$ from the canonical PALF (see Figure 9(a)) on the minimal resolution $W_{(81,47)}((1, 2, 2, 2, 1))$.

We first insert a cancelling pair of Dehn twists along two parallel copies of a curve encircling the first two holes to obtain the ALF in Figure 9(b) with monodromy

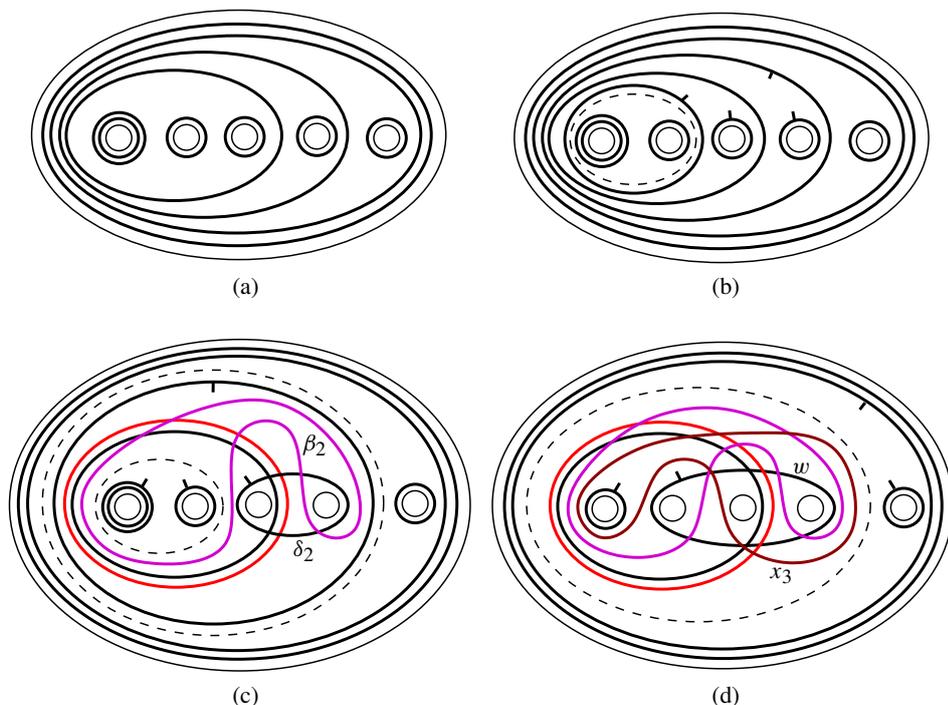$$\alpha_2\alpha_3\alpha_4 a_5\gamma_1^2(\gamma_2^{-1}\gamma_2)\gamma_3\gamma_4\gamma_5^2.$$

Fig. 9. Thicker curves indicate right-handed Dehn twists on a 5-holed disk, whereas left-handed Dehn twists are drawn as dashed curves.

We apply a lantern substitution $\gamma_2\alpha_3\alpha_4\gamma_4 = \delta_2\beta_2\gamma_3$ as indicated in Figure 9(b), to obtain the new ALF depicted in Figure 9(c) with monodromy

$$\alpha_2\alpha_5\gamma_1^2\gamma_2^{-1}(\gamma_4\gamma_4^{-1})\delta_2\beta_2\gamma_3\gamma_3\gamma_5^2,$$

where we also inserted a pair of cancelling Dehn twists along two parallel copies of a curve encircling the first four holes.

Next we apply a second lantern substitution $\gamma_1\alpha_2\delta_2\gamma_4 = \gamma_2wx_3$ indicated in Figure 9(c), to obtain the new ALF depicted in Figure 9(d) with monodromy

$$\alpha_5\gamma_4^{-1}\gamma_1(\gamma_2^{-1}\gamma_2)wx_3\beta_2\gamma_3^2\gamma_5^2 = \alpha_5\gamma_4^{-1}\gamma_1wx_3\beta_2\gamma_3^2\gamma_5^2,$$

where we removed a pair of cancelling Dehn twists encircling the first two holes. A final lantern substitution $\gamma_1w\alpha_5\gamma_5 = \gamma_4x_1x_2$ is applied as indicated in Figure 9(d), together with the removal of a pair of cancelling Dehn twists encircling the first four holes, to obtain a PALF whose monodromy is

$$(\gamma_4^{-1}\gamma_4)x_1x_2x_3\beta_2\gamma_3^2\gamma_5 = x_1x_2x_3\beta_2\gamma_3^2\gamma_5.$$

It is clear that this monodromy is equivalent to the monodromy of the PALF on the symplectic filling $W_{(81,47)}((3, 2, 1, 3, 2))$ depicted in Figure 7.

Using the notation in Lemma 4.5, the above sequence of three lantern substitutions can be expressed as

$$\begin{aligned}
W_{(81,47)}((1, 2, 2, 2, 1)) = {}& W((1, 2, 2, 2, 1), (2, 0, 0, 1, 1, 2)) \\
\rightarrow {}& W((1, 3, 1, 3, 1), (2, -1, 2, 0, 2)) \\
\rightarrow {}& W((2, 2, 1, 4, 1), (1, 0, 2, -1, 2)) \\
\rightarrow {}& W((3, 2, 1, 3, 2), (0, 0, 2, 0, 1)) = W_{(81,47)}((3, 2, 1, 3, 2)).
\end{aligned}$$

We show that the filling $W_{(81,47)}((3, 2, 1, 3, 2))$ is in fact obtained from the minimal resolution $W_{(81,47)}((1, 2, 2, 2, 1))$ by a single rational blowdown as follows: The monodromy of the PALF on $W_{(81,47)}((3, 2, 1, 3, 2))$ can be obtained from the monodromy of the PALF on $W_{(81,47)}((1, 2, 2, 2, 1))$ by a single monodromy substitution (see Figure 10) as

$$\underline{\alpha_2\alpha_3\alpha_4\alpha_5\gamma_1^2\gamma_4\gamma_5}\gamma_3\gamma_5 = \underline{x_1x_2x_3\beta_2\gamma_3}\gamma_3\gamma_5,$$

which is the combination of the three lantern substitutions together with the introduction or removal of cancelling pairs of Dehn twists.

The PALF represented on the left in Figure 10 is diffeomorphic to the linear plumbing of disk bundles over S^2 with Euler numbers $-2, -5, -3$, which can be directly checked by drawing the handlebody diagram of this PALF and applying some handle-slides and cancellations. On the other hand, the PALF on the right is a rational homology 4-ball since the curves in the monodromy spans the rational homology of the genus-zero fibre. We conclude that this monodromy substitution corresponds to a rational blowdown since

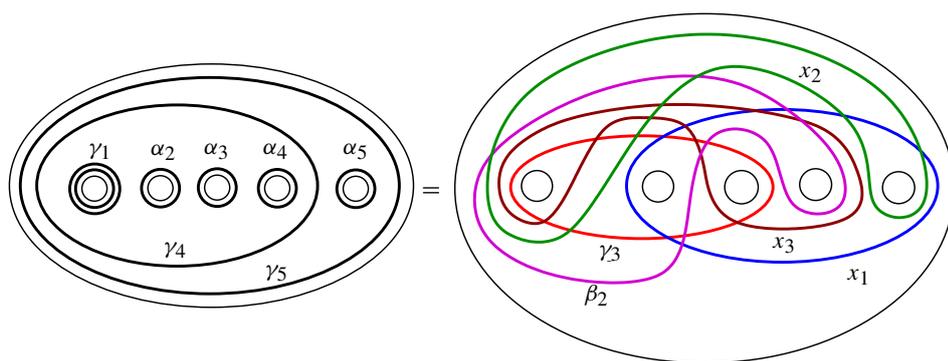$$[-2, -5, -3] = -\frac{5^2}{5 \cdot 3 - 1}.$$

Fig. 10. A monodromy substitution: $\alpha_2\alpha_3\alpha_4\alpha_5\gamma_1^2\gamma_4\gamma_5 = x_1x_2x_3\beta_2\gamma_3$.

Remark 4.9. When we run our algorithm for the two distinct minimal symplectic fillings of $(L(p^2, p-1), \xi_{\mathrm{can}})$, for any $p \geq 2$, we obtain another proof of the daisy relation [6]. Our method would yield many more interesting "positive" relations in the mapping class groups of planar surfaces.

We would like to finish with the following question: Does Theorem 4.1 hold true for minimal symplectic fillings of any Milnor fillable contact 3-manifold supported by a *planar* open book?

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