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Proof of the cosmic no-hair conjecture in the $\mathbb{T}^3$-Gowdy symmetric Einstein–Vlasov setting

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Abstract. The currently preferred models of the universe undergo accelerated expansion induced by dark energy. One model for dark energy is a positive cosmological constant. It is consequently of interest to study Einstein's equations with a positive cosmological constant coupled to matter satisfying the ordinary energy conditions: the dominant energy condition etc. Due to the difficulty of analysing the behaviour of solutions to Einstein's equations in general, it is common to either study situations with symmetry, or to prove stability results. In the present paper, we do both. In fact, we analyse, in detail, the future asymptotic behaviour of $\mathbb{T}^3$-Gowdy symmetric solutions to the Einstein–Vlasov equations with a positive cosmological constant. In particular, we prove the cosmic no-hair conjecture in this setting. However, we also prove that the solutions are future stable (in the class of all solutions). Some of the results hold in a more general setting. In fact, we obtain conclusions concerning the causal structure of $\mathbb{T}^2$-symmetric solutions, assuming only the presence of a positive cosmological constant, matter satisfying various energy conditions and future global existence. Adding the assumption of $\mathbb{T}^3$-Gowdy symmetry to this list of requirements, we obtain C^0-estimates for all but one of the metric components. There is consequently reason to expect that many of the results presented in this paper can be generalised to other types of matter.

Keywords. Einstein–Vlasov system, cosmic no-hair conjecture, Gowdy symmetry

1. Introduction

Towards the end of 1998, two research teams studying supernovae of type Ia announced the unexpected conclusion that the universe is expanding at an accelerating rate (cf. [27, 18]). After the observations had been corroborated by other sources, there was a corresponding shift in the class of solutions to Einstein's equations used to model the universe. In particular, physicists attributed the acceleration to a form of matter they referred to as 'dark energy'. However, as the nature of the dark energy remains unclear, there are several models for it. The simplest one is that of a positive cosmological constant (which

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is the one we use in the present paper), but there are several other possibilities (cf., e.g., [24, 25, 26] and references cited therein for some examples). Combining the different observational data, the currently preferred model of the universe is spatially homogeneous and isotropic (i.e., the cosmological principle is assumed to be valid), spatially flat, and has matter of the following forms: ordinary matter (usually modelled by a radiation fluid and dust), dark matter (often modelled by dust), and dark energy (often modelled by a positive cosmological constant).

In the present paper, we are interested in the Einstein–Vlasov system. This corresponds to a different description of the matter than the one usually used. However, this system can also be used in order to obtain models consistent with observations (cf., e.g., [31, Chapter 28]). In fact, Vlasov matter has the property that it naturally behaves as radiation close to the singularity and as dust in the expanding direction, a desirable feature which is usually put in by hand when using perfect fluids to model the matter.

The cosmic no-hair conjecture. The standard starting point in cosmology is the assumption of spatial homogeneity and isotropy. However, it is preferable to prove that solutions generally isotropise and that the spatial variation (as seen by observers) becomes negligible. This is expected to happen in the presence of a positive cosmological constant; in fact, solutions are in that case expected to appear de Sitter like to observers at late times. The latter expectation goes under the name of the *cosmic no-hair conjecture* (see Conjecture 1.11 for a precise formulation). The main objective when studying the expanding direction of solutions to Einstein's equations with a positive cosmological constant is to verify this conjecture.

Spatial homogeneity. Turning to the results that have been obtained so far, it is natural to begin with the spatially homogeneous setting. In 1983, Robert Wald wrote a short, but remarkable, paper [40], in which he proved results concerning the future asymptotic behaviour of spatially homogeneous solutions to Einstein's equations with a positive cosmological constant. In particular, he confirmed the cosmic no-hair conjecture. What is remarkable about the paper is that he was able to obtain conclusions assuming only that certain energy conditions hold and that the solution does not break down in finite time. Concerning the symmetry type, the only issue that comes up in the argument is whether it is compatible with the spatial hypersurfaces of homogeneity having positive scalar curvature or not; positive scalar curvature of these hypersurfaces sometimes leads to recollapse. The results should be contrasted with the case of Einstein's vacuum equation in the spatially homogeneous setting, where the behaviour is strongly dependent on the symmetry type. Since Wald did not prove future global existence, it is necessary to carry out a further analysis in order to confirm the picture obtained in [40] in specific cases. In the case of the Einstein–Vlasov system, this was done in [13]. It is also of interest that it is possible to prove results analogous to those of Wald for more general models for dark energy (see e.g., [24, 25, 26, 14]).

Surface symmetry. Turning to the spatially inhomogeneous setting, there are results in the surface symmetric case with a positive cosmological constant (cf. [39, 38, 37, 15]; see [22] for the definition of surface symmetry). In this case, the isometry group (on a suitable covering space) is 3-dimensional. Nevertheless, the system of equations that result after symmetry reduction is 1+1-dimensional. However, the extra symmetries do eliminate some of the degrees of freedom. Again, the main results are future causal geodesic completeness and a verification of the cosmic no-hair conjecture.

$\mathbb{T}^2$-symmetry. A natural next step to take after surface symmetry is to consider Gowdy or $\mathbb{T}^2$-symmetry. That is the purpose of the present paper. In particular, we prove future causal geodesic completeness of solutions to the $\mathbb{T}^3$-Gowdy symmetric Einstein–Vlasov equations with a positive cosmological constant (note, however, the caveat concerning global existence stated in Subsection 1.1). Moreover, we verify that the cosmic no-hair conjecture holds. It is of interest that most of the arguments go through under the assumption of $\mathbb{T}^2$-symmetry. However, in order to obtain the full picture in this setting, it is necessary to prove one crucial inequality (see Definition 1.1), which we have not yet been able to do in general.

Stability. A fundamental question in the study of cosmological solutions is that of future stability: given initial data corresponding to an expanding solution, do small perturbations thereof yield maximal globally hyperbolic developments which are future causally geodesically complete and globally similar to the future? In the case of a positive cosmological constant, the first result was obtained by Helmut Friedrich [10]; he proved stability of de Sitter space in $3 + 1$ dimensions. Later, he and Michael Anderson [11, 1] generalised the result to higher (even) dimensions and to include various matter fields. Moreover, results concerning radiation fluids were obtained in [16]. However, conformal invariance plays an important role in the arguments presented in these papers. As a consequence, there seems to be a limitation of the types of matter models that can be dealt with using the corresponding methods. The paper [28] was written with the goal of developing methods that are more generally applicable. The papers [29, 36, 32, 34, 35, 12], in which the methods developed in [28] play a central role, indicate that this goal was achieved. In fact, a general future global non-linear stability result for spatially homogeneous solutions to the Einstein–Vlasov equations with a positive cosmological constant was obtained in [31], the ideas developed in [28] being at the core of the argument.

In the present paper, we not only derive detailed future asymptotics of $\mathbb{T}^3$-Gowdy symmetric solutions to the Einstein–Vlasov equations with a positive cosmological constant. We also prove that all the resulting solutions are future stable in the class of all solutions (without symmetry assumptions).

Outlook. As we describe in the next subsection, some of the results concerning $\mathbb{T}^3$-Gowdy symmetric solutions hold irrespective of the matter model (as long as it satisfies the dominant energy condition and the non-negative pressure condition). As a consequence, we expect that it might be possible to derive detailed asymptotics in the case of the Einstein–Maxwell equations (with a positive cosmological constant), and in the case of the Einstein-Euler system (though the issue of shocks may be relevant in the latter case). Due to the stability results demonstrated in [36, 32, 34, 35], it might also be possible to prove stability of the corresponding solutions.

1.1. General results under the assumption of $\mathbb{T}^2$-symmetry

$\mathbb{T}^2$-symmetry. In the present paper, we are interested in $\mathbb{T}^2$-symmetric solutions to Einstein's equations. There are various geometric ways of imposing this type of symmetry (cf., e.g., [7, 33]), but for the purposes of the present paper, we simply assume the topology to be of the form $I \times \mathbb{T}^3$, where I is an open interval contained in $(0, \infty)$. If θ, x and y are 'coordinates' on $\mathbb{T}^3$ and t is the coordinate on I, we also assume the metric to be of the form

$$g = t^{-1/2}e^{\lambda/2}(-dt^2 + \alpha^{-1}d\theta^2) + te^P[dx + Qdy + (G + QH)d\theta]^2 + te^{-P}(dy + Hd\theta)^2, \tag{1.1}$$

where the functions $\alpha > 0$, λ, P, Q, G and H only depend on t and θ (cf., e.g., [33]). Note that translation in the x and y directions defines a smooth action of $\mathbb{T}^2$ on the spacetime (as well as on each constant-t hypersurface). Moreover, the metric is invariant under this action, and the corresponding orbits are referred to as *symmetry orbits*, given by $\{t\} \times \{\theta\} \times \mathbb{T}^2$. Note that the area of a symmetry orbit is proportional to t. For this reason, the foliation of the spacetime corresponding to the metric form (1.1) is referred to as the *constant areal time foliation*. The case of $\mathbb{T}^3$-Gowdy symmetry corresponds to the functions G and H being independent of time; again, there is a more geometric way of formulating this condition: the spacetime is said to be *Gowdy symmetric* if the so-called *twist quantities*, given by

$$J = \epsilon_{\alpha\beta\gamma\delta}X^\alpha Y^\beta \nabla^\gamma X^\delta, \qquad K = \epsilon_{\alpha\beta\gamma\delta}X^\alpha Y^\beta \nabla^\gamma Y^\delta, \tag{1.2}$$

vanish, where $X = \partial_x$ and $Y = \partial_y$ are Killing fields of the above metric and ϵ is the volume form. A basic question to ask concerning $\mathbb{T}^2$-symmetric solutions to Einstein's equations is whether the maximal globally hyperbolic development of initial data admits a constant areal time foliation which is future global. There is a long history of proving such results. The first one was obtained by Vincent Moncrief [17] in the case of vacuum solutions with $\mathbb{T}^3$-Gowdy symmetry. The cases of $\mathbb{T}^2$-symmetric vacuum solutions with and without a positive cosmological constant have also been considered in [8] and [5] respectively. Turning to Vlasov matter, [2] contains an analysis of the existence of foliations in the $\mathbb{T}^3$-Gowdy symmetric Einstein–Vlasov setting. The corresponding results were later extended to the $\mathbb{T}^2$-symmetric case in [4]. However, from our point of view, the most relevant result is that of [33]. By the results of that paper, there is, given $\mathbb{T}^2$-symmetric initial data to the Einstein–Vlasov equations with a positive cosmological constant, a future global foliation of the spacetime of the form (1.1). In other words $I = (t_0, \infty)$. Moreover, if the distribution function is not identically zero, then $t_0 = 0$. Finally, if the initial data have Gowdy symmetry, then the same is true of the development. Strictly speaking, the future global existence result in [33] is based on the observation that the argument should not be significantly different from the proofs in [5, 8, 4]. It would be preferable to have a complete proof of future global existence in the case of interest here, but we shall not provide it in this paper.

Results. Turning to the results, it is of interest to note that some of the conclusions can be obtained without making detailed assumptions concerning the matter content. For that reason, let us assume, for the remainder of this subsection, that we have a solution to Einstein's equations with a positive cosmological constant, where the metric is of the form (1.1), the existence interval I is of the form (t_0, ∞) and the matter satisfies the dominant energy condition and the non-negative pressure condition; recall that the matter is said to satisfy the *dominant energy condition* if $T(u, v) \geq 0$ for all pairs u, v of future directed timelike vectors (where T is the stress energy tensor associated with the matter); and that it is said to satisfy the *non-negative pressure condition* if $T(w, w) \geq 0$ for every spacelike vector w. To begin with, there is a constant $C > 0$ such that $\alpha(t, \theta) \leq Ct^{-3}$ for all $(t, \theta) \in [t_0 + 2, \infty) \times \mathbb{S}^1$ (cf. Proposition 3.3). In fact, this conclusion also holds if we replace the cosmological constant with a non-linear scalar field with a positive lower bound (cf. Remark 3.4). One particular consequence of this estimate for α is that the θ-coordinate of a causal curve converges. Moreover, observers whose θ-coordinates converge to different θ-values are asymptotically unable to communicate. In this sense, there is asymptotic silence. In the case of Gowdy symmetry, more can be deduced. In fact, for every $\epsilon > 0$, there is a $T > t_0$ such that

$$\lambda(t, \theta) \geq -3 \ln t + 2 \ln \frac{3}{4\Lambda} - \epsilon$$

for all $(t, \theta) \in [T, \infty) \times \mathbb{S}^1$ (cf. Proposition 3.5). This estimate turns out to be of crucial importance also in the general $\mathbb{T}^2$-symmetric case. For this reason, we introduce the following terminology.

Definition 1.1. A metric of the form (1.1) which is defined for $t > t_0$ for some $t_0 \geq 0$ is said to have λ-*asymptotics* if for every $\epsilon > 0$ there is a $T > t_0$ such that

$$\lambda(t, \theta) \geq -3 \ln t + 2 \ln \frac{3}{4\Lambda} - \epsilon \quad \text{for all } (t, \theta) \in [T, \infty) \times \mathbb{S}^1.$$

Remark 1.2. All Gowdy solutions have λ-asymptotics under the above assumptions (cf. Proposition 3.5).

Proposition 1.3. *Consider a $\mathbb{T}^2$-symmetric solution to Einstein's equations with a positive cosmological constant. Assume that the matter satisfies the dominant energy condition and the non-negative pressure condition. Assume moreover that the corresponding metric admits a foliation of the form (1.1) on $I \times \mathbb{T}^3$, where $I = (t_0, \infty)$ and $t_0 \geq 0$. Finally, assume that the solution has λ-asymptotics and let $t_1 = t_0 + 2$. Then there is a constant $C > 0$ such that*

$$\left\| \lambda(t, \cdot) + 3 \ln t - 2 \ln \frac{3}{4\Lambda} \right\|_{C^0} \leq Ct^{-1/2},$$
$$t^{-3/2} \langle \alpha^{-1/2}(t, \cdot) \rangle + \| Q(t, \cdot) \|_{C^0} + \| P(t, \cdot) \|_{C^0} \leq C,$$
$$\| H_t(t, \cdot) \|_{L^1} + \| G_t(t, \cdot) \|_{L^1} \leq Ct^{-3/2}$$

for all $(t, \theta) \in [t_1, \infty) \times \mathbb{S}^1$.

Remark 1.4. The choice $t_1 = t_0+2$ may seem unnatural. However, we need to stay away from t_0 (since we do not control the solution close to t_0). Moreover, in some situations we need to know that $\ln t$ is positive and bounded away from zero. Since $t_0 = 0$ for most solutions, it is therefore natural to only consider the interval $t \geq t_0 + 2$ in the study of the future asymptotics.

Remark 1.5. If h is a scalar function on $\mathbb{S}^1$, we use the notation

$$\langle h \rangle = \frac{1}{2\pi} \int_{\mathbb{S}^1} h \, d\theta. \tag{1.3}$$

Sometimes, we shall use the same notation for a scalar function h on $I \times \mathbb{S}^1$. In that case, $\langle h \rangle$ is the function of t defined by $\langle h(t, \cdot) \rangle$. Finally, if $\bar{p} \in \mathbb{R}^3$, we shall also use the notation $\langle \bar{p} \rangle$. However, in that case, $\langle \bar{p} \rangle = (1 + |\bar{p}|^2)^{1/2}$ (cf. Remark 1.19).

Proof of Proposition 1.3. The statement is a consequence of Lemmas 3.7–3.9. □

In particular, in the case of a $\mathbb{T}^3$-Gowdy symmetric solution, there is asymptotic silence in the sense that the θxy-coordinates of a causal curve converge, and causal curves whose asymptotic θxy-coordinates differ are asymptotically unable to communicate (cf. Proposition 3.10).

1.2. Results in the Einstein–Vlasov setting

In order to be able to draw detailed conclusions, we need to restrict our attention to a specific type of matter. In the present paper, we study the Einstein–Vlasov system.

A general description of Vlasov matter. Intuitively, Vlasov matter gives a statistical description of an ensemble of collections of particles. In practice, the matter is described by a distribution function defined on the space of states of particles. The possible states are given by the future directed causal vectors (here and below, we assume that the Lorentz manifolds under consideration are time oriented). Usually, one distinguishes between massive and massless particles. In the latter case, the distribution function is defined on the future light cone, and in the former case, it is defined on the interior.

In the present paper, we are interested in the massive case, and we assume all the particles to have unit mass (for a description of how to reduce the case of varying masses to the case of all particles having unit mass, see [6]). As a consequence, the distribution function is a non-negative function on the *mass shell* $\mathcal{P}$, defined to be the set of future directed unit timelike vectors. In order to connect the matter to Einstein's equations, we need to associate a stress energy tensor with the distribution function. It is given by

$$T^{\mathrm{Vl}}_{\alpha\beta}(r) = \int_{\mathcal{P}_r} f p_\alpha p_\beta \mu_{\mathcal{P}_r}. \tag{1.4}$$

In this expression, $\mathcal{P}_r$ denotes the set of future directed unit timelike vectors based at the spacetime point r. In other words, if $T_{\mathrm{ref}} \in T_r M$ is a future directed timelike vector, then

$$\mathcal{P}_r = \{v \in T_r M : g(v, v) = -1, \ g(T_{\mathrm{ref}}, v) < 0\}.$$

Moreover, the Lorentz metric g induces a Riemannian metric on $\mathcal{P}_r$, and $\mu_{\mathcal{P}_r}$ denotes the corresponding volume form (see (1.18) below for a coordinate representation of $\mu_{\mathcal{P}_r}$). Finally, p_α denotes the components of the one-form obtained by lowering the index of $p \in \mathcal{P}_r$ using the Lorentz metric g. Clearly, it is necessary to demand some degree of fall-off of the distribution function f in order for the integral (1.4) to be well defined. In the present paper, we shall be mainly interested in the case that the distribution function has compact support in the momentum directions (for a fixed spacetime point). However, in Subsections 1.3–1.7 we shall consider a somewhat more general situation. The equation the distribution function has to satisfy is given by

$$\mathcal{L}f = 0. \tag{1.5}$$

Here $\mathcal{L}$ denotes the vector field induced on the mass shell by the geodesic flow (see (1.19) below for a coordinate representation). An alternative way to formulate this equation is to demand that f be constant along $\dot{\gamma}$ for every future directed unit timelike geodesic γ. The intuitive interpretation of the Vlasov equation (1.5) is that collisions between particles are neglected. It is of interest that if f satisfies the Vlasov equation, then the stress energy tensor is divergence free. To conclude, the *Einstein–Vlasov equations with a positive cosmological constant* consist of (1.5) and

$$\text{Ein} + \Lambda g = T^{\text{Vl}}, \tag{1.6}$$

where T^{Vl} is given by the right hand side of (1.4) and Λ is a positive constant. Moreover,

$$\text{Ein} = \text{Ric} - \tfrac{1}{2}Sg$$

is the Einstein tensor, where Ric is the Ricci tensor and S is the scalar curvature of the Lorentz manifold (M, g). The above description is somewhat brief, and the reader interested in more details is referred to, e.g., [9, 23, 3, 31].

Vlasov matter under $\mathbb{T}^2$-symmetry. In the case of $\mathbb{T}^2$-symmetry, it is convenient to use a symmetry reduced version of the distribution function. To this end, introduce the orthonormal frame

$$\begin{aligned} e_0 &= t^{1/4}e^{-\lambda/4}\partial_t, & e_1 &= t^{1/4}e^{-\lambda/4}\alpha^{1/2}(\partial_\theta - G\partial_x - H\partial_y), \\ e_2 &= t^{-1/2}e^{-P/2}\partial_x, & e_3 &= t^{-1/2}e^{P/2}(\partial_y - Q\partial_x). \end{aligned} \tag{1.7}$$

Since the distribution function f is defined on the mass shell, it is convenient to parametrise this set; note that the manifolds we are interested in here are parallelisable (i.e., they have a global frame). An element in $\mathcal{P}$ can be written $v^\alpha e_\alpha$, where

$$v^0 = [1 + (v^1)^2 + (v^2)^2 + (v^3)^2]^{1/2}.$$

As a consequence, we can think of f as depending on v^i, $i = 1, 2, 3$, and the base point. However, due to the symmetry requirements, the distribution function only depends on the $t\theta$-coordinates of the base point. As a consequence, the distribution function can be considered to be a function of (t, θ, v), where $v = (v^1, v^2, v^2)$. In what follows, we shall abuse notation and denote the symmetry reduced function, defined on $I \times \mathbb{S}^1 \times \mathbb{R}^3$, by f. A symmetry reduced version of the equations is found in Section 2.

Remark 1.6. In the $\mathbb{T}^2$-symmetric setting, we always assume the distribution function f has compact support when restricted to constant-t hypersurfaces. Under the assumptions made in the present paper, f has this property, assuming the initial datum for f has compact support.

The first question to ask concerning $\mathbb{T}^2$-symmetric solutions is that of existence of constant areal time foliations for an interval of the form (t_0, ∞). However, due to previous results (cf. [33]), we know that $\mathbb{T}^2$-symmetric solutions to the Einstein–Vlasov equations with a positive cosmological constant are future global in this setting (keeping the caveat stated in Subsection 1.1 in mind). In other words, there is a $t_0 \geq 0$ such that the solution admits a foliation of the form (1.1) on $I \times \mathbb{T}^3$, where $I = (t_0, \infty)$. Consequently, the issue of interest here is the asymptotics. Unfortunately, we are unable to derive detailed asymptotics for all $\mathbb{T}^2$-symmetric solutions. However, we do obtain results for solutions with λ-asymptotics; recall that all $\mathbb{T}^3$-Gowdy symmetric solutions fall into this class.

Theorem 1.7. *Consider a $\mathbb{T}^2$-symmetric solution to the Einstein–Vlasov system with a positive cosmological constant. Choose coordinates so that the corresponding metric takes the form (1.1) on $I \times \mathbb{T}^3$, where $I = (t_0, \infty)$. Assume that the solution has λ-asymptotics and let $t_1 = t_0 + 2$. Then there are smooth functions $\alpha_\infty > 0$, P_∞, Q_∞, G_∞ and H_∞ on $\mathbb{S}^1$ and, for every $0 \leq N \in \mathbb{Z}$, a constant $C_N > 0$ such that*

$$t\|H_t(t,\cdot)\|_{C^N} + t\|G_t(t,\cdot)\|_{C^N} + \|H(t,\cdot) - H_\infty\|_{C^N} + \|G(t,\cdot) - G_\infty\|_{C^N} \leq C_N t^{-3/2}, \tag{1.8}$$

$$t\|P_t(t,\cdot)\|_{C^N} + t\|Q_t(t,\cdot)\|_{C^N} + \|P(t,\cdot) - P_\infty\|_{C^N} + \|Q(t,\cdot) - Q_\infty\|_{C^N} \leq C_N t^{-1}, \tag{1.9}$$

$$\left\|\frac{\alpha_t}{\alpha} + \frac{3}{t}\right\|_{C^N} + \left\|\lambda_t + \frac{3}{t}\right\|_{C^N} \leq C_N t^{-2}, \tag{1.10}$$

$$\|t^3\alpha(t,\cdot) - \alpha_\infty\|_{C^N} + \left\|\lambda(t,\cdot) + 3\ln t - 2\ln\frac{3}{4\Lambda}\right\|_{C^N} \leq C_N t^{-1} \tag{1.11}$$

for all $t \geq t_1$. Define $f_{\mathrm{sc}}(t,\theta,v) = f(t,\theta,t^{-1/2}v)$. Then there is an $R > 0$ such that

$$\mathrm{supp}\, f_{\mathrm{sc}}(t,\cdot) \subseteq \mathbb{S}^1 \times B_R(0)$$

for all $t \geq t_1$, where $B_R(0)$ is the ball of radius R in $\mathbb{R}^3$ centred at 0. Moreover, there is a smooth, non-negative function $f_{\mathrm{sc},\infty}$ on $\mathbb{S}^1 \times \mathbb{R}^3$ with compact support such that

$$t\|\partial_t f_{\mathrm{sc}}(t,\cdot)\|_{C^N(\mathbb{S}^1\times\mathbb{R}^3)} + \|f_{\mathrm{sc}}(t,\cdot) - f_{\mathrm{sc},\infty}\|_{C^N(\mathbb{S}^1\times\mathbb{R}^3)} \leq C_N t^{-1}$$

for all $t \geq t_1$. Turning to the geometry, let $\bar{g}(t,\cdot)$ and $\bar{k}(t,\cdot)$ denote the metric and the second fundamental form induced by g on the hypersurface $\{t\} \times \mathbb{T}^3$, and let $\bar{g}_{ij}(t,\cdot)$ denote the components of $\bar{g}(t,\cdot)$ with respect to the vector fields $\partial_1 = \partial_\theta$, $\partial_2 = \partial_x$ and $\partial_3 = \partial_y$ etc. Then

$$\|t^{-1}\bar{g}_{ij}(t,\cdot) - \bar{g}_{\infty,ij}\|_{C^N} + \|t^{-1}\bar{k}_{ij} - \mathcal{H}\bar{g}_{\infty,ij}\|_{C^N} \leq C_N t^{-1} \tag{1.12}$$

for all $t \geq t_1$, *where* $\mathcal{H} = (\Lambda/3)^{1/2}$ *and*

$$\bar{g}_\infty = \frac{3}{4\Lambda\alpha_\infty} d\theta^2 + e^{P_\infty}[dx + Q_\infty dy + (G_\infty + Q_\infty H_\infty)\, d\theta]^2 + e^{-P_\infty}(dy + H_\infty\, d\theta)^2. \tag{1.13}$$

Moreover, the solution is future causally geodesically complete.

The proof of the above theorem is given in Section 10.

It is of interest to record what the spacetime looks like to an observer. In particular, we wish to prove the cosmic no-hair conjecture in the present setting. The rough statement of this conjecture is that the spacetime appears de Sitter like to late time observers. However, in order to be able to state a theorem, we need a formal definition. Before proceeding to the details, let us provide some intuition. Let

$$g_{\mathrm{dS}} = -dt^2 + e^{2\mathcal{H}t}\bar{g}_E, \tag{1.14}$$

where $\mathcal{H} = (\Lambda/3)^{1/2}$ and $\bar{g}_E$ denotes the standard flat Euclidean metric. Then $(\mathbb{R}^4, g_{\mathrm{dS}})$ corresponds to a part of de Sitter space. It may seem more reasonable to consider de Sitter space itself. However, as far as the asymptotic behaviour of de Sitter space is concerned, (1.14) is as good a model as de Sitter space itself. Consider a future directed and inextendible causal curve in $(\mathbb{R}^4, g_{\mathrm{dS}})$, say $\gamma = (\gamma^0, \bar{\gamma})$, defined on (s_-, s_+). Then $\bar{\gamma}(s)$ converges to some $\bar{x}_0 \in \mathbb{R}^3$ as $s \to s_+-$. Moreover, $\gamma(s) \in C_{\bar{x}_0,\Lambda}$ for all s, where

$$C_{\bar{x}_0,\Lambda} = \{(t, \bar{x}) : |\bar{x} - \bar{x}_0| \leq \mathcal{H}^{-1}e^{-\mathcal{H}t}\}.$$

In practice, it is convenient to introduce a lower bound on the time coordinate and to introduce a margin in the spatial direction. Moreover, it is convenient to work with open sets. We shall therefore be interested in sets of the form

$$C_{\Lambda,K,T} = \{(t, \bar{x}) : t > T,\ |\bar{x}| < K\mathcal{H}^{-1}e^{-\mathcal{H}t}\}; \tag{1.15}$$

note that $\bar{x}_0$ can be translated to zero by an isometry. Since we are interested in the late time behaviour of solutions, it is natural to restrict attention to sets of the form $C_{\Lambda,K,T}$ for some $K \geq 1$ and $T > 0$.

Definition 1.8. Let (M, g) be a time oriented, globally hyperbolic Lorentz manifold which is future causally geodesically complete. Assume moreover that (M, g) is a solution to Einstein's equations with a positive cosmological constant Λ. Then (M, g) is said to be *future asymptotically de Sitter like* if there is a Cauchy hypersurface Σ in (M, g) such that for every future oriented and inextendible causal curve γ in (M, g), the following holds:

- there is an open set D in (M, g) such that $J^-(\gamma)\cap J^+(\Sigma) \subset D$ and D is diffeomorphic to $C_{\Lambda,K,T}$ for a suitable choice of $K \geq 1$ and $T > 0$,
- using $\psi : C_{\Lambda,K,T} \to D$ to denote the diffeomorphism; letting $R(t) = K\mathcal{H}^{-1}e^{-\mathcal{H}t}$; using $\bar{g}_{\mathrm{dS}}(t, \cdot)$ and $\bar{k}_{\mathrm{dS}}(t, \cdot)$ to denote the metric and the second fundamental form induced on $S_t = \{t\} \times B_{R(t)}(0)$ by g_{dS}; using $\bar{g}(t, \cdot)$ and $\bar{k}(t, \cdot)$ to denote the metric and

the second fundamental form induced on S_t by ψ^*g (where ψ^*g denotes the pullback of g by ψ); and letting $N \in \mathbb{N}$, we have

$$\lim_{t\to\infty}\left(\|\bar{g}_{\mathrm{dS}}(t,\cdot)-\bar{g}(t,\cdot)\|_{C^N_{\mathrm{dS}}(S_t)}+\|\bar{k}_{\mathrm{dS}}(t,\cdot)-\bar{k}(t,\cdot)\|_{C^N_{\mathrm{dS}}(S_t)}\right)=0. \tag{1.16}$$

Remark 1.9. In the definition, we use the notation

$$\|h\|_{C^N_{\mathrm{dS}}(S_t)}=\left(\sup_{S_t}\sum_{l=0}^{N}\bar{g}_{\mathrm{dS},i_1j_1}\cdots\bar{g}_{\mathrm{dS},i_lj_l}\bar{g}^{im}_{\mathrm{dS}}\bar{g}^{jn}_{\mathrm{dS}}\overline{\nabla}^{i_1}_{\mathrm{dS}}\cdots\overline{\nabla}^{i_l}_{\mathrm{dS}}h_{ij}\overline{\nabla}^{j_1}_{\mathrm{dS}}\cdots\overline{\nabla}^{j_l}_{\mathrm{dS}}h_{mn}\right)^{1/2}$$

for a covariant 2-tensor field h on S_t, where $\overline{\nabla}_{\mathrm{dS}}$ denotes the Levi-Civita connection associated with $\bar{g}_{\mathrm{dS}}(t,\cdot)$. Note also that, even though $R(t)$ shrinks to zero exponentially, the diameter of S_t, as measured with respect to $\bar{g}_{\mathrm{dS}}(t,\cdot)$, is constant.

Remark 1.10. In some situations it might be more appropriate to adapt the Cauchy hypersurface Σ to the causal curve γ, i.e., to first fix γ and then Σ.

The above definition leads to a formal statement of the cosmic no-hair conjecture.

Conjecture 1.11 (Cosmic no-hair). *Let $\mathcal{A}$ denote the class of initial data such that the corresponding maximal globally hyperbolic developments (MGHD's) are future causally geodesically complete solutions to Einstein's equations with a positive cosmological constant Λ (for some fixed matter model). Then generic elements of $\mathcal{A}$ yield MGHD's that are future asymptotically de Sitter like.*

Remark 1.12. It is probably necessary to exclude certain matter models in order for the statement to be correct. Moreover, the statement, as it stands, is quite vague: there is no precise definition of 'generic'. However, which notion of genericity is most natural might depend on the situation.

Remark 1.13. The Nariai spacetimes, discussed, e.g., in [28, pp. 126-127], are time oriented, globally hyperbolic, causally geodesically complete solutions to Einstein's vacuum equations with a positive cosmological constant that do not exhibit future asymptotically de Sitter like behaviour. They are thus potential counterexamples to the cosmic no-hair conjecture. There is a similar example in the Einstein–Maxwell setting (with a positive cosmological constant) in [28, p. 127]. However, both of these examples are rather special, and it is natural to conjecture them to be unstable. Nevertheless, they constitute the motivation for demanding genericity.

Finally, we are in a position to phrase a result concerning the cosmic no-hair conjecture in the $\mathbb{T}^3$-Gowdy symmetric setting. The proof of the theorem below is given in Section 10.

Theorem 1.14. *Consider a $\mathbb{T}^2$-symmetric solution to the Einstein–Vlasov system with a positive cosmological constant. Choose coordinates so that the corresponding metric takes the form (1.1) on $I \times \mathbb{T}^3$, where $I = (t_0, \infty)$. Assume that the solution has λ-asymptotics. Then the solution is future asymptotically de Sitter like, i.e., the cosmic no-hair conjecture holds.*

Remark 1.15. Recall that all $\mathbb{T}^3$-Gowdy symmetric solutions have λ-asymptotics.

Remark 1.16. In the particular case of interest here, the equality (1.16) can actually be improved to the estimate

$$\|\bar{g}_{\mathrm{dS}}(\tau,\cdot) - \bar{g}(\tau,\cdot)\|_{C^N_{\mathrm{dS}}(S_\tau)} + \|\bar{k}_{\mathrm{dS}}(\tau,\cdot) - \bar{k}(\tau,\cdot)\|_{C^N_{\mathrm{dS}}(S_\tau)} \leq C_N e^{-2\mathcal{H}\tau}$$

for all $\tau > T$ and a suitable constant C_N.

Remark 1.17. The main estimate needed to prove the theorem is (1.12). In situations where such an estimate holds, it is thus to be expected that the solution is future asymptotically de Sitter like.

1.3. Stability, notation and function spaces

Let us now turn to the subject of stability. Combining Theorem 1.7 with the results of [31], it turns out to be possible to prove that the solutions to which Theorem 1.7 applies are also future stable. In the present subsection, we begin by introducing the terminology necessary in order to make a formal statement of this result.

Let (M, g) be a time oriented $n+1$-dimensional Lorentz manifold. We say that (x, U) are *canonical local coordinates* if ∂_{x^0} is future oriented timelike on U and $g(\partial_{\mathsf{x}^i}|_r, \partial_{\mathsf{x}^j}|_r)$, $i, j = 1, \dots, n$, are the components of a positive definite metric for every $r \in U$ (cf. [31, p. 87]). If $p \in \mathcal{P}_r$ for some $r \in U$, we then define

$$\Xi_{\mathsf{x}}(p) = \Xi_{\mathsf{x}}(p^\alpha \partial_{\mathsf{x}^\alpha}|_r) = [\mathsf{x}(r), \bar{p}], \tag{1.17}$$

where $\bar{p} = (p^1, \dots, p^n)$. Note that Ξ_{x} are local coordinates on the mass shell. If f is defined on the mass shell, we shall use the notation $\mathsf{f}_{\mathsf{x}} = f \circ \Xi_{\mathsf{x}}^{-1}$. It is also convenient to introduce the notation $\bar{\mathsf{p}}_{\mathsf{x}}$ according to $\Xi_{\mathsf{x}}(p) = [\mathsf{x}(r), \bar{\mathsf{p}}_{\mathsf{x}}(p)]$, assuming $p \in \mathcal{P}_r$. With this notation, the measure $\mu_{\mathcal{P}_r}$ can be written

$$\mu_{\mathcal{P}_r} = -\frac{|g_{\mathsf{x}}(r)|^{1/2}}{\mathsf{p}_{\mathsf{x},0} \circ \iota_r} \iota_r^* d\bar{\mathsf{p}}_{\mathsf{x}}, \tag{1.18}$$

where $|g_{\mathsf{x}}|$ is the determinant of the metric g expressed in the x-coordinates; $\iota_r : \mathcal{P}_r \to \mathcal{P}$ is the inclusion; $\mathsf{p}^\alpha_{\mathsf{x}}(p)$ are the components of p in the coordinates x; and $\mathsf{p}_{\mathsf{x},\alpha}(p) = g_{\mathsf{x},\alpha\beta}\mathsf{p}^\beta_{\mathsf{x}}(p)$. The reader interested in the derivation of (1.18) is referred to [31, Section 13.3]. Let us also note that the operator $\mathcal{L}$ is given by

$$\mathcal{L} = \mathsf{p}^\alpha_{\mathsf{x}} \frac{\partial}{\partial \mathsf{x}^\alpha} - \Gamma^i_{\alpha\beta} \mathsf{p}^\alpha_{\mathsf{x}} \mathsf{p}^\beta_{\mathsf{x}} \frac{\partial}{\partial \bar{\mathsf{p}}^i_{\mathsf{x}}} \tag{1.19}$$

in the above coordinates.

In order to proceed, we need to introduce function spaces for the distribution functions. To that end, recall [31, Definition 7.1, p. 87]:

Definition 1.18. Let $1 \le n \in \mathbb{Z}$, $\mu \in \mathbb{R}$, (M, g) be a time oriented $n+1$-dimensional Lorentz manifold and $\mathcal{P}$ be the set of future directed unit timelike vectors. The space $\mathfrak{D}^{\infty}_{\mu}(\mathcal{P})$ is defined to consist of the smooth functions $f : \mathcal{P} \to \mathbb{R}$ such that, for every choice of canonical local coordinates (x, U), $n+1$-multiindex α and n-multiindex β, the derivative $\partial_x^{\alpha}\partial_{\bar{p}}^{\beta}\mathsf{f}_{\mathsf{x}}$ (where x symbolises the first $n+1$ and $\bar{p}$ the last n variables), considered as a function from $\mathsf{x}(U)$ to the set of functions from $\mathbb{R}^n$ to $\mathbb{R}$, belongs to

$$C[\mathsf{x}(U), L^2_{\mu+|\beta|}(\mathbb{R}^n)]. \tag{1.20}$$

Remark 1.19. The space $L^2_{\mu}(\mathbb{R}^n)$ is the weighted L^2-space corresponding to the norm

$$\|h\|_{L^2_{\mu}} = \left(\int_{\mathbb{R}^n} \langle\bar{p}\rangle^{2\mu}|h(\bar{p})|^2\, d\bar{p}\right)^{1/2}, \tag{1.21}$$

where $\langle\bar{p}\rangle = (1+|\bar{p}|^2)^{1/2}$; recall the comments in Remark 1.5.

Remarks 1.20. If $f \in \mathfrak{D}^{\infty}_{\mu}(\mathcal{P})$ for some $\mu > n/2+1$, then the stress energy tensor is a well defined smooth function (cf. [31, Proposition 15.37, p. 246]). Moreover, the stress energy tensor is divergence free if f satisfies the Vlasov equation.

It is worth pointing out that it is possible to introduce more general function spaces, corresponding to a finite degree of differentiability (cf. [31, Definition 15.1, p. 234]). However, the above definition is sufficient for our purposes. The above function spaces are suitable when discussing functions on the mass shell. However, we also need to introduce function spaces for the initial datum for the distribution function. If $(\bar{\mathsf{x}}, U)$ are local coordinates on a manifold Σ, we introduce local coordinates on $T\Sigma$ by $\bar{\Xi}_{\bar{\mathsf{x}}}(\bar{p}^i\partial_{\bar{\mathsf{x}}^i}|_{\bar{\xi}}) = (\bar{\mathsf{x}}(\bar{\xi}), \bar{p})$ in analogy with (1.17). Moreover, if $\bar{f}$ is defined on $T\Sigma$, we shall use the notation $\bar{\mathsf{f}}_{\bar{\mathsf{x}}} = \bar{f} \circ \bar{\Xi}_{\bar{\mathsf{x}}}^{-1}$. Let us recall [31, Definition 7.5, p. 89]:

Definition 1.21. Let $1 \le n \in \mathbb{Z}$, $\mu \in \mathbb{R}$ and Σ be an n-dimensional manifold. The space $\bar{\mathfrak{D}}^{\infty}_{\mu}(T\Sigma)$ is defined to consist of the smooth functions $\bar{f} : T\Sigma \to \mathbb{R}$ such that, for every choice of local coordinates $(\bar{\mathsf{x}}, U)$, n-multiindex α and n-multiindex β, the derivative $\partial_{\bar{x}}^{\alpha}\partial_{\bar{p}}^{\beta}\bar{\mathsf{f}}_{\bar{\mathsf{x}}}$ (where $\bar{x}$ symbolises the first n and $\bar{p}$ the last n variables), considered as a function from $\bar{\mathsf{x}}(U)$ to the set of functions from $\mathbb{R}^n$ to $\mathbb{R}$, belongs to

$$C[\bar{\mathsf{x}}(U), L^2_{\mu+|\beta|}(\mathbb{R}^n)].$$

Remark 1.22. According to the criteria appearing in Definitions 1.18 and 1.21, we need to verify continuity conditions for every choice of local coordinates. However, it turns out to be sufficient to consider a fixed collection of local coordinates covering the manifold of interest (cf. [31, Lemma 15.9, p. 235 and Lemma 15.19, p. 237]).

Finally, in order to be able to state a stability result, we need a norm. To this end, recall [31, Definition 7.7, pp. 89–90]:

Definition 1.23. Let $1 \leq n \in \mathbb{Z}$, $0 \leq l \in \mathbb{Z}$, $\mu \in \mathbb{R}$ and Σ be a compact n-dimensional manifold. Let, moreover, $\bar{\chi}_i$, $i = 1, \dots, j$, be a finite partition of unity subordinate to a cover consisting of coordinate neighbourhoods, say $(\bar{\mathrm{x}}_i, U_i)$. Then $\|\cdot\|_{H^l_{\mathrm{Vl},\mu}}$ is defined by

$$\|\bar{f}\|_{H^l_{\mathrm{Vl},\mu}} = \left(\sum_{i=1}^{j} \sum_{|\alpha|+|\beta|\leq l} \int_{\bar{\mathrm{x}}_i(U_i)\times\mathbb{R}^n} \langle \bar{\varrho} \rangle^{2\mu+2|\beta|} \bar{\chi}_i(\bar{\xi}) (\partial^{\alpha}_{\bar{\xi}} \partial^{\beta}_{\bar{\varrho}} \bar{\mathfrak{f}}_{\bar{\mathrm{x}}_i})^2 (\bar{\xi}, \bar{\varrho})\, d\bar{\xi}\, d\bar{\varrho} \right)^{1/2} \tag{1.22}$$

for each $\bar{f} \in \bar{\mathfrak{D}}^{\infty}_{\mu}(T\Sigma)$.

Remark 1.24. Clearly, the norm depends on the choice of partition of unity and on the choice of coordinates. However, different choices lead to equivalent norms. Here, we are mainly interested in the case $\Sigma = \mathbb{T}^3$, in which case it is not necessary to introduce local coordinates or a partition of unity.

1.4. The Einstein–Vlasov-non-linear scalar field system

In the present paper, we are mainly interested in the Einstein–Vlasov system with a positive cosmological constant. However, in the proof of future stability of $\mathbb{T}^3$-Gowdy symmetric solutions, we use two results. First, we use the fact that solutions that start out close to de Sitter space are future stable. Second, we use Cauchy stability. There are results of this type in the literature. However, they are formulated in the Einstein–Vlasov-non-linear scalar field setting. In order to make it clear that the statements appearing in the literature can be applied in our setting, it is therefore necessary to briefly describe the Einstein–Vlasov-non-linear scalar field system. This is the purpose of the present subsection.

In 3+1-dimensions, the Einstein–Vlasov-non-linear scalar field system can be written

$$R_{\alpha\beta} - T_{\alpha\beta} + \tfrac{1}{2}(\mathrm{tr}\, T) g_{\alpha\beta} = 0, \tag{1.23}$$
$$\nabla^{\alpha}\nabla_{\alpha}\phi - V' \circ \phi = 0, \tag{1.24}$$
$$\mathcal{L}f = 0 \tag{1.25}$$

(cf. [31, (7.13)–(7.15), p. 91]). In these equations, $\phi \in C^{\infty}(M)$ is referred to as the *scalar field*; $V : \mathbb{R} \to \mathbb{R}$ is a smooth function referred to as the *potential*; ∇ is the Levi-Civita connection associated with the metric g; and

$$T_{\alpha\beta} = T^{\mathrm{sf}}_{\alpha\beta} + T^{\mathrm{Vl}}_{\alpha\beta},$$

where T^{Vl} is defined in (1.4) and

$$T^{\mathrm{sf}}_{\alpha\beta} = \nabla_{\alpha}\phi\nabla_{\beta}\phi - \left[\tfrac{1}{2}\nabla^{\gamma}\phi\nabla_{\gamma}\phi + V(\phi)\right] g_{\alpha\beta}.$$

Assuming that $V'(0) = 0$, it is consistent to demand that ϕ be zero in (1.24). Moreover, if $\phi = 0$, then $T^{\mathrm{sf}} = -V(0)g$. Letting $\Lambda = V(0)$, the equations (1.23)–(1.25) then reduce to the Einstein–Vlasov system with a positive cosmological constant Λ, assuming $V(0) > 0$. In order to prove future stability in the Einstein–Vlasov-non-linear scalar field setting, it is not sufficient to demand that $V'(0) = 0$ and $V(0) > 0$. It is also of interest to know that $V''(0) > 0$. We shall therefore make this assumption from now on. Given V such that $V'(0) = 0$, $V(0) > 0$ and $V''(0) > 0$, it is convenient to introduce

$$\mathcal{H} = (V(0)/3)^{1/2}, \tag{1.26}$$

$$\chi = V''(0)/\mathcal{H}^2 \tag{1.27}$$

(cf. [31, (7.9) and (7.10), p. 90]). Note that in the non-linear scalar field setting, we always assume $V(0)$ is positive and we equate it with Λ. In particular, (1.26) is thus consistent with previous definitions of $\mathcal{H}$ (cf., e.g., the statement of Theorem 1.7). If we are interested in the Einstein–Vlasov system with a positive cosmological constant Λ, it is sufficient to choose

$$V(\phi) = \Lambda + \Lambda\phi^2. \tag{1.28}$$

Then $V(0) = \Lambda > 0$, $V'(0) = 0$ and $V''(0) = 2\Lambda > 0$. Moreover, $\mathcal{H} = (\Lambda/3)^{1/2}$ and $\chi = 6$. Clearly, (1.28) is an arbitrary choice; there are many other possibilities.

Let us now recall the definition of initial data given in [31, Definition 7.11, pp. 93–94] (note that the dimension n is here assumed to equal 3):

Definition 1.25. Let $5/2 < \mu \in \mathbb{R}$. *Initial data* for (1.23)–(1.25) consist of an oriented 3-dimensional manifold Σ, a non-negative function $\bar{f} \in \bar{\mathfrak{D}}^{\infty}_{\mu}(T\Sigma)$, a Riemannian metric $\bar{g}$, a symmetric covariant 2-tensor field $\bar{k}$ and two functions $\bar{\phi}_0$ and $\bar{\phi}_1$ on Σ, all assumed to be smooth and to satisfy

$$\bar{r} - \bar{k}_{ij}\bar{k}^{ij} + (\mathrm{tr}\,\bar{k})^2 = \bar{\phi}_1^2 + \overline{\nabla}^i\bar{\phi}_0\overline{\nabla}_i\bar{\phi}_0 + 2V(\bar{\phi}_0) + 2\rho^{\mathrm{Vl}}, \tag{1.29}$$

$$\overline{\nabla}^j\bar{k}_{ji} - \overline{\nabla}_i(\mathrm{tr}\,\bar{k}) = \bar{\phi}_1\overline{\nabla}_i\bar{\phi}_0 - \bar{J}_i^{\mathrm{Vl}}, \tag{1.30}$$

where $\overline{\nabla}$ is the Levi-Civita connection of $\bar{g}$, $\bar{r}$ is the associated scalar curvature, indices are raised and lowered by $\bar{g}$, and ρ^{Vl} and $\bar{J}_i^{\mathrm{Vl}}$ are given by (1.33) and (1.34) below respectively. Given initial data, the *initial value problem* is that of finding a solution (M, g, f, ϕ) to (1.23)–(1.25) (in other words, a 4-dimensional manifold M, a smooth time oriented Lorentz metric g on M, a non-negative function $f \in \mathfrak{D}^{\infty}_{\mu}(\mathcal{P})$ and a $\phi \in C^{\infty}(M)$ such that (1.23)–(1.25) are satisfied), and an embedding $i : \Sigma \to M$ such that

$$i^*g = \bar{g}, \quad \phi \circ i = \bar{\phi}_0, \quad \bar{f} = i^*(f \circ \mathrm{pr}^{-1}_{i(\Sigma)}),$$

and if N is the future directed unit normal and κ is the second fundamental form of $i(\Sigma)$, then $i^*\kappa = \bar{k}$ and $(N\phi) \circ i = \bar{\phi}_1$. Such a quadruple (M, g, f, ϕ) is referred to as a *development* of the initial data, the existence of an embedding i being tacit. If, in addition to the above conditions, $i(\Sigma)$ is a Cauchy hypersurface in (M, g), the quadruple is said to be a *globally hyperbolic development*.

Remark 1.26. The map $\mathrm{pr}_{i(\Sigma)}$ is the diffeomorphism from the mass shell above $i(\Sigma)$ to the tangent space of $i(\Sigma)$ defined by mapping a vector v to its component perpendicular to the normal of $i(\Sigma)$.

Remark 1.27. If $\bar{\phi}_0 = \bar{\phi}_1 = 0$, the equations (1.29) and (1.30) become

$$\bar{r} - \bar{k}_{ij}\bar{k}^{ij} + (\operatorname{tr}\bar{k})^2 = 2\Lambda + 2\rho^{\mathrm{Vl}}, \tag{1.31}$$

$$\overline{\nabla}^j\bar{k}_{ji} - \overline{\nabla}_i(\operatorname{tr}\bar{k}) = -\bar{J}_i^{\mathrm{Vl}}. \tag{1.32}$$

These are the constraint equations for the Einstein–Vlasov system with a positive cosmological constant Λ.

The *energy density* and *current* induced by the initial data are given by

$$\rho^{\mathrm{Vl}}(\bar{\xi}) = \int_{T_{\bar{\xi}}\Sigma} \bar{f}(\bar{p})[1 + \bar{g}(\bar{p}, \bar{p})]^{1/2}\bar{\mu}_{\bar{g},\bar{\xi}}, \tag{1.33}$$

$$\bar{J}^{\mathrm{Vl}}(\bar{X}) = \int_{T_{\bar{\xi}}\Sigma} \bar{f}(\bar{p})\bar{g}(\bar{X}, \bar{p})\bar{\mu}_{\bar{g},\bar{\xi}}. \tag{1.34}$$

In these expressions, $\bar{\xi} \in \Sigma$, $\bar{X} \in T_{\bar{\xi}}\Sigma$, $\bar{\mu}_{\bar{g},\bar{\xi}}$ is the volume form on $T_{\bar{\xi}}\Sigma$ induced by $\bar{g}$, and $\bar{p} \in T_{\bar{\xi}}\Sigma$. It is important to note that under the assumptions of the above definition, the energy density is a smooth function and the current is a smooth one-form field on Σ (cf. [31, Lemma 15.40, p. 246]).

Given initial data, there is a unique maximal globally hyperbolic development thereof (cf. [31, Corollary 23.44, p. 418 and Lemma 23.2, p. 398]). The definition of a maximal globally hyperbolic development is given by [31, Definition 7.14, p. 94]:

Definition 1.28. Given initial data for (1.23)–(1.25), a *maximal globally hyperbolic development* of the data is a globally hyperbolic development (M, g, f, ϕ) with embedding $i : \Sigma \to M$ such that if (M', g', f', ϕ') is any other globally hyperbolic development of the same data with embedding $i' : \Sigma \to M'$, then there is a map $\psi : M' \to M$ which is a diffeomorphism onto its image such that $\psi^*g = g'$, $\psi^*f = f'$, $\psi^*\phi = \phi'$ and $\psi \circ i' = i$.

It is worth noting that the maximal globally hyperbolic development is independent of the parameter μ. The above discussion of the initial value problem for the Einstein–Vlasov-non-linear scalar field system is somewhat brief, and the reader interested in a more detailed discussion is referred to [31, Chapter 7].

1.5. Future stability in the spatially homogeneous and isotropic setting

In the proof of stability of the $\mathbb{T}^3$-Gowdy symmetric solutions, we need to refer to [31, Theorem 7.16, pp. 104–106]. However, the statement of this theorem is based on terminology introduced in [31]. Moreover, in the statement of Theorem 1.35, we refer to the conclusions of [31, Theorem 7.16]. For this reason, we here provide not only the notational background, but also the statement of [31, Theorem 7.16]. However, the reader interested in why the particular formulation of the theorem is natural is referred to [31, Sections 7.6–7.7].

The rough idea is to only make local assumptions concerning the initial data and to derive future global conclusions concerning the solution. Given a 3-manifold Σ, we therefore focus on a local coordinate patch $(\bar{\mathrm{x}}, U)$. Here U is the neighbourhood in which we make assumptions in the statement of the theorem. The conditions on the initial data are phrased in terms of Sobolev norms on U. Given a tensor field $\mathfrak{T}$ on Σ, we therefore define

$$\|\mathfrak{T}\|_{H^l(U)} = \Big(\sum_{i_1,\dots,i_s=1}^{3}\sum_{j_1,\dots,j_r=1}^{3}\sum_{|\alpha|\le l}\int_{\bar{\mathrm{x}}(U)}|\partial^\alpha \mathfrak{T}^{i_1\cdots i_s}_{j_1\cdots j_r}\circ\bar{\mathrm{x}}^{-1}|^2\,d\bar{x}\Big)^{1/2}. \tag{1.35}$$

In this expression, the components of $\mathfrak{T}$ are computed with respect to the coordinates $\bar{\mathrm{x}}$ and the derivatives are taken with respect to $\bar{x}$. In what follows, norms of the type $\|\mathfrak{T}\|_{H^l(U)}$ are always computed using a particular choice of local coordinates. The choice we have in mind should be clear from the context. In Theorem 1.29, we also use the notation

$$\|\partial_m\bar{g}\|_{H^l(U)} = \Big(\sum_{i,j=1}^{3}\sum_{|\alpha|\le l}\int_{\bar{\mathrm{x}}(U)}|\partial^\alpha\partial_m\bar{g}_{ij}\circ\bar{\mathrm{x}}^{-1}|^2\,d\bar{x}\Big)^{1/2}. \tag{1.36}$$

To measure the local size of the distribution function, we need a weighted Sobolev norm. However, it is also necessary to allow the freedom to rescale the momentum variable in the definition of the norm. Since we have already motivated the need for this rescaling freedom in [31, Subsection 7.6.1, pp. 100–102], we shall not do so here. Given a constant w, we simply define the local norm for the distribution function by

$$^{w}\|\bar{f}\|_{H^l_{\mathrm{Vl},\mu}(U)} = \Big(\sum_{|\alpha|+|\beta|\le l}\int_{\mathbb{R}^3}\int_{\bar{\mathrm{x}}(U)}(e^{-w})^{2|\beta|}\langle e^{w}\bar{p}\rangle^{2\mu+2|\beta|}|\partial^\alpha_{\bar{\xi}}\partial^\beta_{\bar{p}}\bar{\mathfrak{f}}_{\bar{\mathrm{x}}}|^2(\bar{\xi},\bar{p})\,d\bar{\xi}\,d\bar{p}\Big)^{1/2}. \tag{1.37}$$

Here $\bar{\Xi}_{\bar{\mathrm{x}}}$ are the coordinates on TU associated with $\bar{\mathrm{x}}$ (cf. Subsection 1.3), and $\bar{\mathfrak{f}}_{\bar{\mathrm{x}}} = \bar{f}\circ\bar{\Xi}^{-1}_{\bar{\mathrm{x}}}$.

Given the above notation, [31, Theorem 7.16, pp. 104–106] takes the following form for $n = 3$.

Theorem 1.29. *Let $5/2 < \mu \in \mathbb{R}$ and $7/2 < k_0 \in \mathbb{Z}$. Let V be a smooth function on $\mathbb{R}$ such that $V(0) = V_0 > 0$, $V'(0) = 0$ and $V''(0) > 0$. Let $\mathcal{H}, \chi > 0$ be defined by (1.26) and (1.27) respectively and let $K_{\mathrm{Vl}} \ge 0$. There is an $\varepsilon > 0$, depending only on μ and V, such that if*

- *$(\Sigma, \bar{g}, \bar{k}, \bar{f}, \bar{\phi}_0, \bar{\phi}_1)$ are initial data for (1.23)–(1.25) with $\dim\Sigma = 3$,*
- *$\bar{\mathrm{x}} : U \to B_1(0)$ are local coordinates with $\bar{\mathrm{x}}(U) = B_1(0)$,*
- *the inequality*
$$|e^{-2K}\bar{g}_{ij} - \delta_{ij}| \le \varepsilon \tag{1.38}$$
holds on U for all $i, j = 1, \dots, n$, where K is defined by $e^K = 4/\mathcal{H}$,
- *with the notation introduced in (1.35) and (1.36), we have*
$$\sum_{j=1}^{3}\mathcal{H}^2\|\partial_j\bar{g}\|_{H^{k_0}(U)} + \mathcal{H}\|\bar{k}-\mathcal{H}\bar{g}\|_{H^{k_0}(U)} + \|\bar{\phi}_0\|_{H^{k_0+1}(U)} + \mathcal{H}^{-1}\|\bar{\phi}_1\|_{H^{k_0}(U)} \le \varepsilon e^{-K_{\mathrm{Vl}}}, \tag{1.39}$$
- *using the notation introduced in (1.37) we have, with $w = K + K_{\mathrm{Vl}}$,*
$$^{w}\|\bar{f}\|_{H^{k_0}_{\mathrm{Vl},\mu}(U)} \le \mathcal{H}^2\varepsilon^{5/2}e^{-3K/2-K_{\mathrm{Vl}}}, \tag{1.40}$$

then the maximal globally hyperbolic development (M, g, f, ϕ) *of the initial data has the property that if* $i : \Sigma \to M$ *is the associated embedding, then all causal geodesics that start in* $i \circ \bar{\mathrm{x}}^{-1}[B_{1/4}(0)]$ *are future complete. Furthermore, there is a* $t_- < 0$ *and a smooth map*

$$\psi : (t_-, \infty) \times B_{5/8}(0) \to M, \tag{1.41}$$

which is a diffeomorphism onto its image, such that all causal curves that start in $i \circ \bar{\mathrm{x}}^{-1}[B_{1/4}(0)]$ *remain in the image of* ψ *to the future, and* g*,* f *and* ϕ *have expansions of the form* (1.42)–(1.55) *in the solid cylinder* $[0, \infty) \times B_{5/8}(0)$ *when pulled back by* ψ*. Finally,* $\psi(0, \bar{\xi}) = i \circ \bar{\mathrm{x}}^{-1}(\bar{\xi})$ *for* $\bar{\xi} \in B_{5/8}(0)$*. In the formulae below, Latin indices refer to the natural Euclidean coordinates on* $B_{5/8}(0)$ *and* t *is the natural time coordinate on the solid cylinder. Let* $\zeta = 4\chi/9$*,*

$$\lambda_{\mathrm{pre}} = \begin{cases} \frac{3}{2}[1 - (1-\zeta)^{1/2}], & \zeta \in (0, 1), \\ \frac{3}{2}, & \zeta \geq 1, \end{cases}$$

and $\lambda_{\mathrm{m}} = \min\{1, \lambda_{\mathrm{pre}}\}$*. There is a smooth Riemannian metric* $\bar{\varrho}$ *on* $B_{5/8}(0)$ *and, for every* $l \geq 0$*, a constant* K_l *such that*

$$\|e^{2\mathcal{H}t+2K} g^{ij}(t, \cdot) - \bar{\varrho}^{ij}\|_{C^l} + \|e^{-2\mathcal{H}t-2K} g_{ij}(t, \cdot) - \bar{\varrho}_{ij}\|_{C^l} \leq K_l e^{-2\lambda_{\mathrm{m}}\mathcal{H}t}, \tag{1.42}$$

$$\|e^{-2\mathcal{H}t-2K} \partial_t g_{ij}(t, \cdot) - 2\mathcal{H}\bar{\varrho}_{ij}\|_{C^l} \leq K_l e^{-2\lambda_{\mathrm{m}}\mathcal{H}t} \tag{1.43}$$

for all $l \geq 0$ *and* $t \geq 0$*. Here* $\bar{\varrho}^{ij}$ *denotes the components of the inverse of* $\bar{\varrho}$*. Furthermore,* C^l *denotes the* C^l*-norm on* $B_{5/8}(0)$*. Turning to* g_{0m}*, there is a* $b > 0$ *and, for every* $l \geq 0$*, a constant* K_l *such that*

$$\|g_{0m}(t, \cdot) - \bar{\mathfrak{v}}_m\|_{C^l} + \|\partial_0 g_{0m}(t, \cdot)\|_{C^l} \leq K_l e^{-b\mathcal{H}t} \tag{1.44}$$

for all $l \geq 0$ *and* $t \geq 0$*, where*

$$\bar{\mathfrak{v}}_m = \frac{1}{\mathcal{H}} \bar{\varrho}^{ij} \gamma_{imj} \tag{1.45}$$

and γ_{imj} *denote the Christoffel symbols of the metric* $\bar{\varrho}$*, given by*

$$\gamma_{imj} = \tfrac{1}{2}(\partial_i \bar{\varrho}_{jm} + \partial_j \bar{\varrho}_{im} - \partial_m \bar{\varrho}_{ij}).$$

Let $\bar{k}_{ij}$ *denote the components of the second fundamental form* (*induced on the constant-t hypersurfaces*) *with respect to the standard coordinates on* $B_{5/8}(0)$*. If* $\lambda_{\mathrm{m}} < 1$*, then for every* $l \geq 0$*, there is a constant* K_l *such that*

$$\|g_{00}(t, \cdot) + 1\|_{C^l} + \|\partial_0 g_{00}(t, \cdot)\|_{C^l} \leq K_l e^{-2\lambda_{\mathrm{m}}\mathcal{H}t},$$
$$\|e^{-2\mathcal{H}t-2K} \bar{k}_{ij}(t, \cdot) - \mathcal{H}\bar{\varrho}_{ij}\|_{C^l} \leq K_l e^{-2\lambda_{\mathrm{m}}\mathcal{H}t}$$

for all $l \geq 0$ *and* $t \geq 0$*. If* $\lambda_{\mathrm{m}} = 1$*, then for every* $l \geq 0$*, there is a constant* K_l *such that*

$$\|[\partial_0 g_{00} + 2\mathcal{H}(g_{00} + 1)](t, \cdot)\|_{C^l} \leq K_l e^{-2\mathcal{H}t},$$
$$\|g_{00}(t, \cdot) + 1\|_{C^l} \leq K_l (1+t^2)^{1/2} e^{-2\mathcal{H}t},$$
$$\|e^{-2\mathcal{H}t-2K} \bar{k}_{ij}(t, \cdot) - \mathcal{H}\bar{\varrho}_{ij}\|_{C^l} \leq K_l (1+t^2)^{1/2} e^{-2\mathcal{H}t}$$

for all $l \geq 0$ and $t \geq 0$. In order to describe the asymptotics concerning ϕ, let $\varphi = e^{\lambda_{\rm pre}\mathcal{H}t}\phi$. If $\zeta < 1$, there is a smooth function φ_0, a constant $b > 0$ and, for every $l \geq 0$, a constant K_l such that

$$\|\varphi(t,\cdot) - \varphi_0\|_{C^l} + \|\partial_0\varphi\|_{C^l} \leq K_l e^{-b\mathcal{H}t} \tag{1.46}$$

for all $l \geq 0$ and $t \geq 0$. If $\zeta = 1$, there are smooth functions φ_0 and φ_1, a constant $b > 0$ and, for every $l \geq 0$, a constant K_l such that

$$\|\partial_0\varphi(t,\cdot) - \varphi_1\|_{C^l} + \|\varphi(t,\cdot) - \varphi_1 t - \varphi_0\|_{C^l} \leq K_l e^{-b\mathcal{H}t} \tag{1.47}$$

for all $l \geq 0$ and $t \geq 0$. Finally, if $\zeta > 1$, there is an antisymmetric matrix A, given by

$$A = \begin{pmatrix} 0 & \delta\mathcal{H} \\ -\delta\mathcal{H} & 0 \end{pmatrix},$$

where $\delta = 3(\zeta - 1)^{1/2}/2$, smooth functions φ_0 and φ_1, a constant $b > 0$ and, for every $l \geq 0$, a constant K_l such that

$$\left\| e^{-At}\begin{pmatrix} \delta\mathcal{H}\varphi \\ \partial_0\varphi \end{pmatrix}(t,\cdot) - \begin{pmatrix} \varphi_0 \\ \varphi_1 \end{pmatrix} \right\|_{C^l} \leq K_l e^{-b\mathcal{H}t} \tag{1.48}$$

for all $l \geq 0$ and $t \geq 0$. In order to describe the asymptotics for the distribution function, let $\mathsf{x} = \psi^{-1}$. Then (x, U) are canonical local coordinates, where

$$U = \psi[(t_-, \infty) \times B_{5/8}(0)].$$

Let $\mathsf{f}_{\mathsf{x}} = f \circ \Xi_{\mathsf{x}}^{-1}$ and

$$\mathsf{h}(t, \bar{x}, \bar{q}) = \mathsf{f}_{\mathsf{x}}(t, \bar{x}, e^{-2\mathcal{H}t - K - K_{\rm VI}}\bar{q}). \tag{1.49}$$

Introduce moreover the notation

$$\|\bar{\mathsf{f}}\|_{H^l_{\rm VI,\mu}[B_{5/8}(0)\times\mathbb{R}^3]} = \left(\sum_{|\alpha|+|\beta|\leq l} \int_{B_{5/8}(0)} \int_{\mathbb{R}^3} \langle\bar{p}\rangle^{2\mu+2|\beta|} |\partial^\alpha_{\bar{x}}\partial^\beta_{\bar{p}}\bar{\mathsf{f}}(\bar{x},\bar{p})|^2 \, d\bar{p}\, d\bar{x}\right)^{1/2}$$

for $\bar{\mathsf{f}} \in C^\infty[B_{5/8}(0) \times \mathbb{R}^3]$. Then there is a constant $b > 0$ and, for every l, a constant K_l such that

$$\|\partial_t \mathsf{h}(t,\cdot)\|_{H^l_{\rm VI,\mu}[B_{5/8}(0)\times\mathbb{R}^3]} \leq K_l e^{-b\mathcal{H}t} \tag{1.50}$$

for all $l \geq 0$ and $t \geq 0$. There is also a function $\bar{\mathsf{h}} \in C^\infty[B_{5/8}(0) \times \mathbb{R}^3]$, a constant $b > 0$ and, for every l, a constant K_l such that

$$\begin{aligned} \|\bar{\mathsf{h}}\|_{H^l_{\rm VI,\mu}[B_{5/8}(0)\times\mathbb{R}^3]} &< \infty, \\ \|\mathsf{h}(t,\cdot) - \bar{\mathsf{h}}\|_{H^l_{\rm VI,\mu}[B_{5/8}(0)\times\mathbb{R}^3]} &\leq K_l e^{-b\mathcal{H}t} \end{aligned} \tag{1.51}$$

for all $l \geq 0$ *and* $t \geq 0$. *Furthermore,* $\bar{\mathfrak{h}} \geq 0$. *Concerning the stress energy tensor associated with the Vlasov matter, there is a* $b > 0$ *and, for every* $l \geq 0$, *a constant* K_l *such that*

$$\left\| e^{3(\mathcal{H}t+K_{\mathrm{Vl}})} T_{00}^{\mathrm{Vl}} - \int_{\mathbb{R}^3} \bar{\mathfrak{h}} |\bar{\varrho}|^{1/2} d\bar{q} \right\|_{C^l} \leq K_l e^{-b\mathcal{H}t}, \tag{1.52}$$

$$\left\| e^{3(\mathcal{H}t+K_{\mathrm{Vl}})} T_{0i}^{\mathrm{Vl}} + \int_{\mathbb{R}^3} \bar{\mathfrak{q}}_i \bar{\mathfrak{h}} |\bar{\varrho}|^{1/2} d\bar{q} \right\|_{C^l} \leq K_l e^{-b\mathcal{H}t}, \tag{1.53}$$

$$\| e^{2\mathcal{H}t+3K_{\mathrm{Vl}}} T_{ij}^{\mathrm{Vl}} \|_{C^l} \leq K_l \tag{1.54}$$

for all $l \geq 0$ *and all* $t \geq 0$, *where* $|\bar{\varrho}|$ *denotes the absolute value of the determinant of* $\bar{\varrho}$,

$$\bar{\mathfrak{q}}_i = \bar{\mathfrak{v}}_i + e^{K-K_{\mathrm{Vl}}} \bar{\varrho}_{ij} \bar{q}^j$$

and $\bar{\mathfrak{v}}_i$ *is defined in* (1.45). *Finally, if* $\mu > 9/2$, *there is a constant* $b > 0$ *and, for every* $l \geq 0$, *a constant* K_l *such that*

$$\left\| e^{3(\mathcal{H}t+K_{\mathrm{Vl}})} T_{ij}^{\mathrm{Vl}} - \int_{\mathbb{R}^3} \bar{\mathfrak{h}} \bar{\mathfrak{q}}_i \bar{\mathfrak{q}}_j |\bar{\varrho}|^{1/2} d\bar{q} \right\|_{C^l} \leq K_l e^{-b\mathcal{H}t} \tag{1.55}$$

for all $l \geq 0$ *and* $t \geq 0$.

Remark 1.30. In case one is only interested in the Einstein–Vlasov setting with a positive cosmological constant, more detailed information can be obtained (cf. [31, Proposition 32.8, pp. 609–611]).

1.6. *Cauchy stability*

In what follows, we also need a Cauchy stability result in the Einstein–Vlasov-non-linear scalar field setting. There are such results in the literature (cf. [31]). However, for the convenience of the reader, we introduce the necessary terminology and quote the relevant result in the present subsection.

First, we need to introduce the notion of a background solution (cf. [31, Definition 24.2, p. 421]). In the 3-dimensional case, this definition takes the following form.

Definition 1.31. Let $5/2 < \mu \in \mathbb{R}$, Σ be a closed 3-dimensional manifold, and let g be a smooth time oriented Lorentz metric on $M = I \times \Sigma$, where I is an open interval. Let ∂_t denote differentiation with respect to the first coordinate and assume that $g(\partial_t, \partial_t) = g_{00} < 0$ and the hypersurfaces $\Sigma_t = \{t\} \times \Sigma$ are spacelike with respect to g for $t \in I$. Finally, assume that $\phi \in C^\infty(M)$ and $f \in \mathfrak{D}_\mu^\infty(\mathcal{P})$, together with g, satisfy (1.23)–(1.25). Then (M, g, f, ϕ) is called a *background solution*.

Remark 1.32. In the case of $\mathbb{T}^2$-symmetric solutions, the metric is of the form (1.1). Moreover, the distribution functions of interest have compact support on constant time hypersurfaces. As a consequence, it is clear that the $\mathbb{T}^2$-symmetric solutions we consider in the present paper are background solutions in the sense of the above definition.

Next, we introduce the notion of induced initial data on constant-t hypersurfaces (cf. [31, Definition 24.3, p. 421]). In the 3-dimensional case, this definition takes the following form.

Definition 1.33. Let $5/2 < \mu \in \mathbb{R}$, Σ be a closed 3-dimensional manifold, and let g be a smooth time oriented Lorentz metric on $M = I \times \Sigma$, where I is an open interval. Let furthermore $\phi \in C^\infty(M)$, $f \in \mathfrak{D}^\infty_\mu(\mathcal{P})$ and assume that (g, f, ϕ) solve (1.23)–(1.25). Let $t \in I$ and assume $\Sigma_t = \{t\} \times \Sigma$ is spacelike with respect to g. Let κ be the second fundamental form and N be the future directed unit normal of Σ_t. Finally, let $\iota_t : \Sigma \to M$ be defined by $\iota_t(\bar{x}) = (t, \bar{x})$ and

$$\bar{g} = \iota_t^* g, \qquad \bar{k} = \iota_t^* \kappa, \qquad \bar{f} = \iota_t^*(f \circ \mathrm{pr}_{\Sigma_t}^{-1}), \qquad \bar{\phi}_0 = \iota_t^* \phi, \qquad \bar{\phi}_1 = \iota_t^*(N\phi).$$

Then $(\bar{g}, \bar{k}, \bar{f}, \bar{\phi}_0, \bar{\phi}_1)$ are referred to as the *initial data induced on* Σ_t by (g, f, ϕ), or simply the initial data induced on Σ_t if the solution is understood from the context.

Finally, we formulate the Cauchy stability result we need here (cf. [31, Corollary 24.10, p. 432]). In the 3-dimensional case, this result takes the following form.

Theorem 1.34. *Let $5/2 < \mu \in \mathbb{R}$ and $5/2 < l \in \mathbb{Z}$. Let $(M_{\mathrm{bg}}, g_{\mathrm{bg}}, f_{\mathrm{bg}}, \phi_{\mathrm{bg}})$ be a background solution with $M_{\mathrm{bg}} = I_{\mathrm{bg}} \times \Sigma$ and recall the notation Σ, Σ_t etc. from Definition 1.31 (the interval denoted by I in Definition 1.31 will here be denoted by I_{bg}). Assume that $0 \in I_{\mathrm{bg}}$ and let $(\bar{g}_{\mathrm{bg}}, \bar{k}_{\mathrm{bg}}, \bar{f}_{\mathrm{bg}}, \bar{\phi}_{\mathrm{bg},0}, \bar{\phi}_{\mathrm{bg},1})$ be the initial data induced on Σ_0 by $(g_{\mathrm{bg}}, f_{\mathrm{bg}}, \phi_{\mathrm{bg}})$. Make a choice of $H^l_{\mathrm{Vl},\mu}(T\Sigma)$-norms and a choice of Sobolev norms $\|\cdot\|_{H^l}$ on tensor fields on Σ. Let $J \subset I_{\mathrm{bg}}$ be a compact interval and let $\epsilon > 0$. Then there is a $\delta > 0$ such that if $(\Sigma, \bar{g}, \bar{k}, \bar{f}, \bar{\phi}_0, \bar{\phi}_1)$ are initial data for the Einstein–Vlasov-non-linear scalar field system satisfying*

$$\begin{aligned} \|\bar{g} - \bar{g}_{\mathrm{bg}}\|_{H^{l+1}} + \|\bar{k} - \bar{k}_{\mathrm{bg}}\|_{H^l} + \|\bar{\phi}_0 - \bar{\phi}_{\mathrm{bg},0}\|_{H^{l+1}} \\ + \|\bar{\phi}_1 - \bar{\phi}_{\mathrm{bg},1}\|_{H^l} + \|\bar{f} - \bar{f}_{\mathrm{bg}}\|_{H^l_{\mathrm{Vl},\mu}(T\Sigma)} \le \delta, \end{aligned}$$

then there is an open interval I containing 0 and a solution (g, f, ϕ) to (1.23)–(1.25) on $M = I \times \Sigma$ such that

- *the initial data induced on Σ_0 by (g, f, ϕ) are given by $(\bar{g}, \bar{k}, \bar{f}, \bar{\phi}_0, \bar{\phi}_1)$,*
- *∂_t is timelike with respect to g and Σ_t is a spacelike Cauchy hypersurface with respect to g for all $t \in I$,*
- *$J \subset I$ and if the initial data induced on Σ_t (for $t \in I_{\mathrm{bg}} \cap I$) by (g, f, ϕ) and $(g_{\mathrm{bg}}, f_{\mathrm{bg}}, \phi_{\mathrm{bg}})$ are $(\bar{g}_t, \bar{k}_t, \bar{f}_t, \bar{\phi}_{t,0}, \bar{\phi}_{t,1})$ and $(\bar{g}_{\mathrm{bg},t}, \bar{k}_{\mathrm{bg},t}, \bar{f}_{\mathrm{bg},t}, \bar{\phi}_{\mathrm{bg},t,0}, \bar{\phi}_{\mathrm{bg},t,1})$ respectively, then*

$$\begin{aligned} \|\bar{g}_t - \bar{g}_{\mathrm{bg},t}\|_{H^{l+1}} + \|\bar{k}_t - \bar{k}_{\mathrm{bg},t}\|_{H^l} + \|\bar{\phi}_{t,0} - \bar{\phi}_{\mathrm{bg},t,0}\|_{H^{l+1}} \\ + \|\bar{\phi}_{t,1} - \bar{\phi}_{\mathrm{bg},t,1}\|_{H^l} + \|\bar{f}_t - \bar{f}_{\mathrm{bg},t}\|_{H^l_{\mathrm{Vl},\mu}(T\Sigma)} \le \epsilon \qquad (1.56) \end{aligned}$$

for all $t \in J$.

1.7. Stability of $\mathbb{T}^3$-Gowdy symmetric solutions

Combining Theorems 1.7, 1.29 and 1.34 yields a future stability result for the $\mathbb{T}^2$-symmetric solutions considered in Theorem 1.7. Moreover, the solutions are stable in the Einstein–Vlasov-non-linear scalar field setting.

Theorem 1.35. *Consider a $\mathbb{T}^2$-symmetric solution to the Einstein–Vlasov system with a positive cosmological constant Λ. Choose coordinates so that the corresponding metric takes the form (1.1) on $I \times \mathbb{T}^3$, where $I = (t_0, \infty)$. Assume that the solution has λ-asymptotics. Choose a $t \in I$ and let $i : \mathbb{T}^3 \to I \times \mathbb{T}^3$ be given by $i(\bar{x}) = (t, \bar{x})$. Let $\bar{g}_{\mathrm{bg}} = i^* g$ and let $\bar{k}_{\mathrm{bg}}$ denote the pullback (under i) of the second fundamental form induced on $i(\mathbb{T}^3)$ by g. Let moreover*

$$\bar{f}_{\mathrm{bg}} = i^*(f \circ \mathrm{pr}^{-1}_{i(\mathbb{T}^3)}).$$

Make a choice of $\mu > 5/2$, a choice of norms as in Definition 1.23 and a choice of Sobolev norms on tensor fields on $\mathbb{T}^3$. Let in addition $V : \mathbb{R} \to \mathbb{R}$ be a smooth function such that $V(0) = \Lambda$, $V'(0) = 0$ and $V''(0) > 0$. Then there is an $\epsilon > 0$ such that if $(\mathbb{T}^3, \bar{g}, \bar{k}, \bar{f}, \bar{\phi}_0, \bar{\phi}_1)$ are initial data for (1.23)–(1.25) with $\bar{f} \in \bar{\mathfrak{D}}^\infty_\mu(T\mathbb{T}^3)$ satisfying

$$\|\bar{g} - \bar{g}_{\mathrm{bg}}\|_{H^5} + \|\bar{k} - \bar{k}_{\mathrm{bg}}\|_{H^4} + \|\bar{f} - \bar{f}_{\mathrm{bg}}\|_{H^4_{\mathrm{Vl},\mu}} + \|\bar{\phi}_0\|_{H^5} + \|\bar{\phi}_1\|_{H^4} \le \epsilon,$$

then the maximal globally hyperbolic development (M, g, f, ϕ) of the initial data is future causally geodesically complete. Moreover, there is a Cauchy hypersurface Σ in (M, g) such that for each point of Σ, there is a neighbourhood $(\bar{\mathrm{x}}, U)$ such that Theorem 1.29 applies. In particular, the asymptotics stated in Theorem 1.29 hold.

Remark 1.36. Up to the point where we appeal to Theorem 1.29, Cauchy stability applies. It should therefore be possible to obtain detailed control over the perturbed solutions for the entire future. The interested reader is encouraged to write down the details.

Remark 1.37. The function $\bar{f}_{\mathrm{bg}}$ has compact support, but $\bar{f}$ need not.

The proof of Theorem 1.35 is given in Section 10.

1.8. Outline

Finally, let us give an outline of the paper. In Section 2, we write down the equations in the case that the metric takes the form (1.1) (the reader interested in the derivation is referred to Appendix A). In Section 3, we collect the conclusions which are not dependent on the particular type of matter model (as long as it satisfies the dominant energy condition and the non-negative pressure condition). The section ends with conclusions concerning the causal structure of $\mathbb{T}^3$-Gowdy symmetric spacetimes. Turning to the more detailed conclusions, we specialise to the case of Vlasov matter. The natural first step is to derive light cone estimates, i.e., to consider the behaviour along characteristics. This is the subject of Section 4. As opposed to the vacuum case, we need to control the characteristics associated with the Vlasov equation at the same time as the first derivatives of the metric components. Fortunately, the e_2 and e_3 components of the momentum are controlled automatically due to symmetry. However, an argument is required in the case of the e_1 component. In order to obtain control of higher order derivatives, we need to take derivatives of the characteristic system (associated with the Vlasov equation, i.e. with the geodesic flow). Naively, this should require control of second order derivatives of the metric functions, something we do not have. Nevertheless, by an appropriate choice of variables, controlling first order derivatives turns out to be sufficient. A similar choice was already suggested in [2, Lemma 3, p. 363] (cf. also [4, Lemma 3, p. 257]). However, in the present setting, it is not sufficient to derive a system involving only first order derivatives of the metric functions. We also need to be able to use the system to derive the desired type of asymptotics for the derivatives of the characteristic system. It turns out to be possible to do so, and we write down the required arguments in Section 6. After obtaining this conclusion, we proceed inductively to derive higher order estimates for the characteristic system and the metric components. The required arguments are written down in Sections 7 and 8. In order to obtain the desired conclusions concerning the distribution function, it is convenient to consider L^2-based energies. This subject is treated in Section 9. Finally, in Section 10, we prove the main theorems of the paper. As an appendix, we include a derivation of Einstein's equations as well as of the Vlasov equation (cf. Appendix A). We also provide a summary of the most important notation in Appendix B.

2. Symmetry assumptions and equations

In this paper, we study $\mathbb{T}^2$-symmetric solutions of Einstein's equations. Since it will be convenient to express the equations using the orthonormal frame (1.7), let us introduce the notation

$$\rho = T(e_0, e_0), \qquad J_i = -T(e_0, e_i), \qquad P_i = T(e_i, e_i), \qquad S_{ij} = T(e_i, e_j), \tag{2.1}$$

where we do not sum over any indices; here and below, we tacitly assume that Latin indices range from 1 to 3 and Greek indices range from 0 to 3. It is also convenient to introduce the notation

$$J = -t^{5/2}\alpha^{1/2}e^{P-\lambda/2}(G_t + QH_t), \qquad K = QJ - t^{5/2}\alpha^{1/2}e^{-P-\lambda/2}H_t. \tag{2.2}$$

Note that these objects are the twist quantities introduced in (1.2) (cf. Appendix A.3). In order to derive Einstein's equations, it is useful to calculate the Einstein tensor for a metric of the form (1.1). The corresponding, somewhat lengthy, computations appear in Appendix A. Using the above notation, the calculations yield the conclusion that the 00 and 11 components of Einstein's equations can be written

$$\begin{aligned}\lambda_t - 2\frac{\alpha_t}{\alpha} = {} & t[P_t^2 + \alpha P_\theta^2 + e^{2P}(Q_t^2 + \alpha Q_\theta^2)] + \frac{e^{\lambda/2-P}J^2}{t^{5/2}} \\ & + \frac{e^{\lambda/2+P}(K-QJ)^2}{t^{5/2}} + 4t^{1/2}e^{\lambda/2}(\rho + \Lambda),\end{aligned} \tag{2.3}$$

$$\lambda_t = t[P_t^2 + \alpha P_\theta^2 + e^{2P}(Q_t^2 + \alpha Q_\theta^2)] - \frac{e^{\lambda/2-P}J^2}{t^{5/2}} - \frac{e^{\lambda/2+P}(K-QJ)^2}{t^{5/2}} + 4t^{1/2}e^{\lambda/2}(P_1 - \Lambda), \tag{2.4}$$

respectively. The 22 component minus the 33 component can be written

$$\partial_t(t\alpha^{-1/2}P_t) = \partial_\theta(t\alpha^{1/2}P_\theta) + t\alpha^{-1/2}e^{2P}(Q_t^2 - \alpha Q_\theta^2) + \frac{\alpha^{-1/2}e^{\lambda/2-P}J^2}{2t^{5/2}} - \frac{\alpha^{-1/2}e^{\lambda/2+P}(K-QJ)^2}{2t^{5/2}} + t^{1/2}e^{\lambda/2}\alpha^{-1/2}(P_2 - P_3). \tag{2.5}$$

The 22 component plus the 33 component can be written

$$\partial_t\left[t\alpha^{-1/2}\left(\lambda_t - 2\frac{\alpha_t}{\alpha} - \frac{3}{t}\right)\right] = \partial_\theta(t\alpha^{1/2}\lambda_\theta) - t\alpha^{-1/2}[P_t^2 + e^{2P}Q_t^2 - \alpha(P_\theta^2 + e^{2P}Q_\theta^2)] - 2t\alpha^{-1/2}\left(\frac{e^{\lambda/2-P}J^2}{t^{7/2}} + \frac{e^{\lambda/2+P}(K-QJ)^2}{t^{7/2}}\right) + \alpha^{-1/2}\lambda_t + 2t^{1/2}e^{\lambda/2}\alpha^{-1/2}(2\Lambda - P_2 - P_3). \tag{2.6}$$

The 01, 02, 03, 12 and 13 components are equivalent to

$$\lambda_\theta = 2t(P_tP_\theta + e^{2P}Q_tQ_\theta) - 4t^{1/2}e^{\lambda/2}\alpha^{-1/2}J_1, \tag{2.7}$$

$$J_\theta = 2t^{5/4}\alpha^{-1/2}e^{P/2+\lambda/4}J_2, \tag{2.8}$$

$$K_\theta = 2t^{5/4}\alpha^{-1/2}e^{-P/2+\lambda/4}J_3 + 2t^{5/4}\alpha^{-1/2}e^{P/2+\lambda/4}QJ_2, \tag{2.9}$$

$$J_t = -2t^{5/4}e^{\lambda/4+P/2}S_{12}, \tag{2.10}$$

$$K_t = -2t^{5/4}e^{\lambda/4+P/2}QS_{12} - 2t^{5/4}e^{-P/2+\lambda/4}S_{13}, \tag{2.11}$$

respectively. Finally, the 23 component reads

$$\partial_t(t\alpha^{-1/2}e^{2P}Q_t) - \partial_\theta(t\alpha^{1/2}e^{2P}Q_\theta) = t^{-5/2}\alpha^{-1/2}e^{\lambda/2+P}J(K-QJ) + 2t^{1/2}\alpha^{-1/2}e^{\lambda/2+P}S_{23}. \tag{2.12}$$

For future reference, we also note that

$$\frac{\alpha_t}{\alpha} = -\frac{e^{-P+\lambda/2}J^2}{t^{5/2}} - \frac{e^{P+\lambda/2}(K-QJ)^2}{t^{5/2}} - 4t^{1/2}e^{\lambda/2}\Lambda - 2t^{1/2}e^{\lambda/2}(\rho - P_1), \tag{2.13}$$

$$\lambda_t - \frac{\alpha_t}{\alpha} = t[P_t^2 + \alpha P_\theta^2 + e^{2P}(Q_t^2 + \alpha Q_\theta^2)] + 2t^{1/2}e^{\lambda/2}(\rho + P_1). \tag{2.14}$$

2.1. Preliminary calculations

Since the metric components only depend on two variables, it is natural to derive estimates by integrating along characteristics. In the present subsection, we record a general calculation which is of interest in that context. To begin with, let us define

$$\partial_\pm = \partial_t \pm \alpha^{1/2}\partial_\theta, \quad \mathcal{A}_\pm = (\partial_\pm P)^2 + e^{2P}(\partial_\pm Q)^2. \tag{2.15}$$

One reason for introducing $\mathcal{A}_\pm$ is the equality (2.16) derived below; since the right hand side only contains first derivatives of the metric components, it is possible to integrate along the characteristics to control $\mathcal{A}_\pm$.

Lemma 2.1. *Consider a $\mathbb{T}^2$-symmetric solution to Einstein's equations with a cosmological constant Λ such that the metric takes the form (1.1). Then*

$$\begin{aligned}\partial_\pm\mathcal{A}_\mp = &-\left(\frac{2}{t}-\frac{\alpha_t}{\alpha}\right)\mathcal{A}_\mp \mp \frac{2}{t}\alpha^{1/2}(P_\theta\partial_\mp P + e^{2P}Q_\theta\partial_\mp Q)\\ &+\frac{e^{-P+\lambda/2}J^2}{t^{7/2}}\partial_\mp P - \frac{e^{P+\lambda/2}(K-QJ)^2}{t^{7/2}}\partial_\mp P + 2\frac{e^{\lambda/2}J(K-QJ)}{t^{7/2}}e^P\partial_\mp Q\\ &+2t^{-1/2}e^{\lambda/2}(P_2-P_3)\partial_\mp P + 4t^{-1/2}e^{\lambda/2}S_{23}e^P\partial_\mp Q.\end{aligned} \tag{2.16}$$

Remark 2.2. In this calculation, the cosmological constant need not be positive.

Proof of Lemma 2.1. The statement follows from a lengthy computation. For the benefit of the reader, let us write down some of the intermediate steps. Using (2.5), we obtain

$$\begin{aligned}\partial_\pm\partial_\mp P = &-\frac{1}{t}P_t + \frac{\alpha_t}{2\alpha}\partial_\mp P + e^{2P}(Q_t^2 - \alpha Q_\theta^2)\\ &+\frac{e^{-P+\lambda/2}J^2}{2t^{7/2}} - \frac{e^{P+\lambda/2}(K-QJ)^2}{2t^{7/2}} + t^{-1/2}e^{\lambda/2}(P_2-P_3).\end{aligned} \tag{2.17}$$

Similarly, due to (2.12), we obtain

$$\begin{aligned}\partial_\pm\partial_\mp Q = &-\frac{1}{t}Q_t + \frac{\alpha_t}{2\alpha}\partial_\mp Q - 2(Q_tP_t - \alpha Q_\theta P_\theta) + \frac{e^{\lambda/2-P}J(K-QJ)}{t^{7/2}}\\ &+2t^{-1/2}e^{\lambda/2-P}S_{23}.\end{aligned} \tag{2.18}$$

If we combine (2.17) and (2.18) with the fact that

$$-4(Q_tP_t - \alpha P_\theta Q_\theta)\partial_\mp Q + 2\partial_\pm P(\partial_\mp Q)^2 = -2\partial_\mp P(Q_t^2 - \alpha Q_\theta^2),$$

a calculation yields the conclusion of the lemma. □

2.2. *Vlasov matter*

The equations (2.3)–(2.14) hold in general. However, we are here particularly interested in matter of Vlasov type. In order to derive the relevant form of the Vlasov equation, recall the conventions concerning f introduced in Subsection 1.2. Recall moreover that the Vlasov equation is equivalent to f being constant along future directed unit timelike geodesics. As a consequence, it can be calculated (cf. Appendix A.7) that the Vlasov equation takes the form

$$\begin{aligned}
&\frac{\partial f}{\partial t}+\frac{\alpha^{1/2}v^1}{v^0}\frac{\partial f}{\partial\theta}-\bigg[\frac{1}{4}\alpha^{1/2}\lambda_\theta\,v^0+\frac{1}{4}\bigg(\lambda_t-\frac{2\alpha_t}{\alpha}-\frac{1}{t}\bigg)v^1-\alpha^{1/2}e^P Q_\theta\frac{v^2v^3}{v^0}\\
&\qquad+\frac{1}{2}\alpha^{1/2}P_\theta\frac{(v^3)^2-(v^2)^2}{v^0}-t^{-7/4}e^{\lambda/4}\big(e^{-P/2}Jv^2+e^{P/2}(K-QJ)v^3\big)\bigg]\frac{\partial f}{\partial v^1}\\
&\quad-\bigg[\frac{1}{2}\bigg(P_t+\frac{1}{t}\bigg)v^2+\frac{1}{2}\alpha^{1/2}P_\theta\frac{v^1v^2}{v^0}\bigg]\frac{\partial f}{\partial v^2}\\
&\quad-\bigg[\frac{1}{2}\bigg(\frac{1}{t}-P_t\bigg)v^3-\frac{1}{2}\alpha^{1/2}P_\theta\frac{v^1v^3}{v^0}+e^Pv^2\bigg(Q_t+\alpha^{1/2}Q_\theta\frac{v^1}{v^0}\bigg)\bigg]\frac{\partial f}{\partial v^3}=0. \qquad (2.19)
\end{aligned}$$

Turning to the stress energy tensor, it satisfies

$$T(e_\mu,e_\nu)=\int_{\mathbb{R}^3}v_\mu v_\nu f\frac{1}{-v_0}\,dv, \qquad (2.20)$$

where $v_\alpha=\eta_{\alpha\beta}v^\beta$ and $\eta=\mathrm{diag}\{-1,1,1,1\}$. In particular, in the Vlasov case, we have

$$\rho=\int_{\mathbb{R}^3}v^0f\,dv,\quad P_k=\int_{\mathbb{R}^3}\frac{(v^k)^2}{v^0}f\,dv,\quad J_k=\int_{\mathbb{R}^3}v^kf\,dv,\quad S_{jk}=\int_{\mathbb{R}^3}\frac{v^jv^k}{v^0}f\,dv, \qquad (2.21)$$

where $j,k=1,2,3$.

3. Preliminary conclusions concerning the asymptotics

In the present section, we are interested in $\mathbb{T}^2$-symmetric solutions to Einstein's equations such that the corresponding metric admits a foliation of the form (1.1) on $I\times\mathbb{T}^3$, where $I=(t_0,\infty)$ and $t_0\geq 0$. For the sake of brevity, we shall refer below to solutions of this form as *future global*, and we shall speak of t_0 and $t_1=t_0+2$ without further introduction.

It is useful to begin by recalling the following consequences of the non-negative pressure condition and the dominant energy condition.

Lemma 3.1. *Consider a solution to Einstein's equations with a cosmological constant* Λ *and a metric of the form* (1.1)*. Let* ρ, P_i, J_i *and* S_{ij}, $i,j=1,2,3$, *be defined by* (2.1)*. If the stress energy tensor satisfies the non-negative pressure condition, then, for* $i=1,2,3$,

$$0\leq P_i. \qquad (3.1)$$

If the stress energy tensor satisfies the dominant energy condition, then, for $i,j=1,2,3$,

$$0\leq\rho, \qquad (3.2)$$
$$|P_i|\leq\rho, \qquad (3.3)$$
$$|J_i|\leq\rho, \qquad (3.4)$$
$$|S_{ij}|\leq\rho. \qquad (3.5)$$

Proof. By definition, $P_i = T(e_i, e_i)$. Since e_i is a spacelike vector field, the non-negative pressure condition implies that (3.1) holds. The dominant energy condition states that $T(u, v) \geq 0$ for future directed timelike vectors u and v. By continuity, this also holds for future directed causal vectors. Since e_0 is future directed timelike, $\rho = T(e_0, e_0) \geq 0$, so (3.2) follows. Note that $e_0 \pm e_i$ is a future directed causal vector field. In particular,

$$0 \leq T(e_0 - e_i, e_0 + e_j) = \rho + J_i - J_j - S_{ij}.$$

Since S_{ij} is symmetric, adding this inequality to the one obtained by interchanging i and j yields $S_{ij} \leq \rho$. Similarly,

$$0 \leq T(e_0 \pm e_i, e_0 \pm e_j) = \rho \mp J_i \mp J_j + S_{ij}.$$

Adding the two inequalities yields $-S_{ij} \leq \rho$. Thus (3.5) holds. The proof of (3.3) is similar. Finally,

$$0 \leq T(e_0, e_0 \pm e_i) = \rho \mp J_i,$$

so that (3.4) holds. □

Before deriving estimates describing the asymptotics of solutions, let us make the following remark.

Remark 3.2. In what follows, the constants appearing in the estimates we state are allowed to depend on the solution, unless otherwise indicated.

Proposition 3.3. *Given a future global solution to Einstein's equations with a cosmological constant* $\Lambda > 0$*,* $\mathbb{T}^2$*-symmetry and a stress energy tensor satisfying the dominant energy condition, there is a constant* $C > 0$ *such that*

$$\alpha(t, \theta) \leq Ct^{-3} \tag{3.6}$$

for all $(t, \theta) \in [t_1, \infty) \times \mathbb{S}^1$.

Remark 3.4. The same conclusion holds if we replace the cosmological constant with a non-linear scalar field with a potential having a positive lower bound; in other words, if we set $\Lambda = 0$ and consider stress energy tensors of the form $T = T^{\mathrm{o}} + T^{\mathrm{sf}}$, where T^{o} is the stress energy tensor associated with matter fields satisfying the dominant energy condition, and T^{sf} is the stress energy tensor associated with a non-linear scalar field with a potential V having a positive lower bound.

Proof of Proposition 3.3. Due to (2.14) and the fact that the matter satisfies the dominant energy condition (so that (3.3) holds), we conclude that $\lambda_t - \alpha_t/\alpha \geq 0$. There is thus a $c_0 > 0$ such that

$$(\alpha^{-1/2} e^{\lambda/2})(t, \theta) \geq c_0$$

for all $(t, \theta) \in [t_1, \infty) \times \mathbb{S}^1$. Combining this observation with (2.13) and (3.3), we obtain

$$\partial_t \alpha^{-1/2} = -\frac{\alpha_t}{2\alpha} \alpha^{-1/2} \geq 2t^{1/2} \alpha^{-1/2} e^{\lambda/2} \Lambda \geq c_1 t^{1/2}$$

for some constant $c_1 > 0$ and all $(t, \theta) \in [t_1, \infty) \times \mathbb{S}^1$. Integrating this inequality, we obtain the conclusion of the proposition. □

In the Gowdy case, the second and third terms on the right hand side of (2.4) are zero, so we can extract more information. In fact, we have the following observation.

Proposition 3.5. *Consider a future global solution to Einstein's equations with a cosmological constant $\Lambda > 0$, $\mathbb{T}^3$-Gowdy symmetry and matter satisfying the non-negative pressure condition. Then for every $\epsilon > 0$, there is a $T > t_0$ such that*

$$\lambda(t,\theta) \geq -3\ln t + 2\ln\frac{3}{4\Lambda} - \epsilon$$

for all $(t,\theta) \in [T,\infty)\times\mathbb{S}^1$.

Proof. Let

$$\hat{\lambda} = \lambda + 3\ln t - 2\ln\frac{3}{4\Lambda}. \qquad (3.7)$$

Then (2.4) with $J = K = 0$ yields

$$\partial_t\hat{\lambda} = t[P_t^2 + \alpha P_\theta^2 + e^{2P}(Q_t^2 + \alpha Q_\theta^2)] + 4t^{1/2}e^{\lambda/2}P_1 + \frac{3}{t}(1 - e^{\hat{\lambda}/2}).$$

Since $P_1 \geq 0$ due to the non-negative pressure condition (cf. (3.1)), we conclude that

$$\partial_t\hat{\lambda} \geq \frac{3}{t}(1 - e^{\hat{\lambda}/2}).$$

For every $\epsilon > 0$, there is thus a T such that $\hat{\lambda}(t,\theta) \geq -\epsilon$ for all $(t,\theta) \in [T,\infty)\times\mathbb{S}^1$. □

In order to proceed, it is convenient to introduce an energy:

$$\begin{aligned} E_{\text{bas}} &= \int_{\mathbb{S}^1} t\alpha^{-1/2}\left(\lambda_t - 2\frac{\alpha_t}{\alpha} - 4t^{1/2}e^{\lambda/2}\Lambda\right)d\theta \\ &= \int_{\mathbb{S}^1}\left(t^2\alpha^{-1/2}[P_t^2 + \alpha P_\theta^2 + e^{2P}(Q_t^2 + \alpha Q_\theta^2)] + \frac{\alpha^{-1/2}e^{\lambda/2-P}J^2}{t^{3/2}}\right. \\ &\qquad \left. + \frac{\alpha^{-1/2}e^{\lambda/2+P}(K - QJ)^2}{t^{3/2}} + 4t^{3/2}\alpha^{-1/2}e^{\lambda/2}\rho\right)d\theta. \end{aligned} \qquad (3.8)$$

Let us motivate this particular choice. The energy E_{bas} is quite similar to the energy defined in [4, (42), p. 251]. However, there is one fundamental difference. The integrand in the energy defined in [4] contains a term of the form $\alpha^{-1/2}U_t^2$, where $U = (P + \ln t)/2$. Using U instead of P as a variable is convenient in global existence arguments, since some of the formulae become less involved. However, the variable U is poorly adapted to the actual asymptotics of solutions. The reason is that, in the end, P_t converges to zero as t^{-2}. The dominant term in U is thus $(\ln t)/2$. If one uses U instead of P in the energy, the best estimate of E_{bas} one could hope for would be $E_{\text{bas}}(t) \leq Ct^{3/2}$. Below, we prove that $E_{\text{bas}} \leq Ct^{1/2}$ (cf. Lemma 3.7). In addition, it is possible to derive a good estimate for the time derivative of E_{bas} (cf. the proof of Lemma 3.6 below).

On a more general level, it is natural to ask why it is necessary to use L^2-based energy estimates at all. Since the problem is $1 + 1$-dimensional, should it not be sufficient to consider the behaviour along characteristics? The problem in our setting is that we wish to derive detailed quantitative information for arbitrary initial data. In particular, we are not in a situation where we can use bootstrap arguments. For this reason, we need to proceed step by step. First, it is necessary to derive not only rough, but quite detailed, control of some of the metric components, in particular λ. This leads, for example, to estimates of the form (3.19) and (3.20) below. Only once we have estimates of this form, is it meaningful to turn to the characteristic system: see, e.g., the last two terms on the right hand side of (4.2) and the proof of Lemma 4.3.

If the metric has λ-asymptotics (recall Definition 1.1), we can estimate E_{bas}.

Lemma 3.6. *Consider a future global solution to Einstein's equations with a cosmological constant $\Lambda > 0$, $\mathbb{T}^2$-symmetry, λ-asymptotics, and a stress energy tensor satisfying the dominant energy condition and the non-negative pressure condition. Then for every $a > 1/2$, there is a constant $C_a > 0$ such that*

$$E_{\mathrm{bas}}(t) \le C_a t^a \tag{3.9}$$

for all $t \ge t_1$.

Proof. Due to (2.6), we obtain

$$\begin{aligned}
&\partial_t\left[t\alpha^{-1/2}\left(\lambda_t - 2\frac{\alpha_t}{\alpha} - 4t^{1/2}e^{\lambda/2}\Lambda\right)\right] \\
&\quad = \partial_\theta(t\alpha^{1/2}\lambda_\theta) + 2t\alpha^{1/2}(P_\theta^2 + e^{2P}Q_\theta^2) - \frac{3}{2}\frac{\alpha^{-1/2}e^{\lambda/2-P}J^2}{t^{5/2}} \\
&\qquad - \frac{3}{2}\frac{\alpha^{-1/2}e^{\lambda/2+P}(K-QJ)^2}{t^{5/2}} - 2t^{5/2}\alpha^{-1/2}e^{\lambda/2}\Lambda[P_t^2 + e^{2P}Q_t^2 + \alpha(P_\theta^2 + e^{2P}Q_\theta^2)] \\
&\qquad + t^{1/2}\alpha^{-1/2}e^{\lambda/2}(3\rho + P_1 - 2P_2 - 2P_3) - 4t^2\alpha^{-1/2}e^{\lambda}\Lambda(\rho + P_1).
\end{aligned} \tag{3.10}$$

Since the matter satisfies the non-negative pressure condition, we know that $P_i \ge 0$ (cf. (3.1)), so that

$$\begin{aligned}
\frac{dE_{\mathrm{bas}}}{dt} \le& \int_{\mathbb{S}^1} 2t\alpha^{1/2}(P_\theta^2 + e^{2P}Q_\theta^2)\,d\theta - \int_{\mathbb{S}^1} 2t^{5/2}\alpha^{1/2}e^{\lambda/2}\Lambda(P_\theta^2 + e^{2P}Q_\theta^2)\,d\theta \\
&+ \int_{\mathbb{S}^1} t^{1/2}\alpha^{-1/2}e^{\lambda/2}(3\rho + P_1)\,d\theta - \int_{\mathbb{S}^1} 4t^2\alpha^{-1/2}e^{\lambda}\Lambda(\rho + P_1)\,d\theta.
\end{aligned}$$

Using the consequences of Lemma 3.1 and the fact that the solution has λ-asymptotics, we conclude that for every $a > 1/2$, there is a $T \ge t_1$ such that

$$\frac{dE_{\mathrm{bas}}}{dt} \le \frac{a}{t}E_{\mathrm{bas}}$$

for all $t \ge T$. As a consequence, $E_{\mathrm{bas}}(t) \le Ct^a$ for $t \ge t_1$. □

Using the estimate for E_{bas} derived in Lemma 3.6, it is possible to extract more information concerning the asymptotics.

Lemma 3.7. *Consider a future global solution to Einstein's equations with a cosmological constant $\Lambda > 0$, $\mathbb{T}^2$-symmetry, λ-asymptotics, and a stress energy tensor satisfying the dominant energy condition and the non-negative pressure condition. Then there is a constant $C > 0$ such that*

$$\left\| \lambda(t,\cdot) + 3\ln t - 2\ln\frac{3}{4\Lambda}\right\|_{C^0} \leq Ct^{-1/2}, \tag{3.11}$$

$$E_{\text{bas}}(t) \leq Ct^{1/2} \tag{3.12}$$

for all $t \geq t_1$.

Proof. From the estimate $E_{\text{bas}}(t) \leq C_a t^a$, the fact that $\alpha^{1/2} \leq Ct^{-3/2}$, (2.4) and (3.3), we conclude that

$$\langle\lambda_t\rangle = -4t^{1/2}\langle e^{\lambda/2}\rangle\Lambda + O(t^{a-5/2}) \tag{3.13}$$

(the notation $\langle\lambda_t\rangle$ was introduced in Remark 1.5). Due to (2.7) and (3.4), we also have

$$|\lambda_\theta| \leq t\alpha^{-1/2}[P_t^2 + \alpha P_\theta^2 + e^{2P}(Q_t^2 + \alpha Q_\theta^2)] + 4t^{1/2}\alpha^{-1/2}e^{\lambda/2}\rho.$$

By (3.9), we thus obtain

$$\int_{\mathbb{S}^1} |\lambda_\theta|\, d\theta \leq Ct^{a-1}. \tag{3.14}$$

Recall that $\hat{\lambda}$ is defined in (3.7) and note that, by (3.13),

$$\langle\hat{\lambda}_t\rangle = \frac{3}{t}(1 - \langle e^{\hat{\lambda}/2}\rangle) + O(t^{a-5/2}). \tag{3.15}$$

Let us first prove that $\langle\hat{\lambda}\rangle$ converges to zero. To this end, let $\epsilon > 0$. Since the solution has λ-asymptotics, there is a T such that $\langle\hat{\lambda}\rangle(t) \geq -\epsilon$ for all $t \geq T$. To prove that there is a T such that $\langle\hat{\lambda}\rangle(t) \leq \epsilon$ for all $t \geq T$, let us assume that $\langle\hat{\lambda}\rangle(t) \geq \epsilon$ for some t. From (3.14), we conclude that $\hat{\lambda}(t,\theta) \geq \epsilon/2$ for all $\theta \in \mathbb{S}^1$ (assuming t is large enough). Inserting this into (3.15), we conclude that

$$\langle\hat{\lambda}_t\rangle \leq \frac{2}{t}(1 - e^{\epsilon/4})$$

for t large enough. Since the right hand side is negative and non-integrable, $\langle\hat{\lambda}\rangle$ has to decay until it is smaller than ϵ (assuming the starting time t is large enough). Moreover, $\langle\hat{\lambda}\rangle$ cannot exceed ϵ at a later time. To obtain a quantitative estimate, note that

$$\langle\hat{\lambda}_t\rangle = \frac{3}{t}(1 - e^{\langle\hat{\lambda}\rangle/2}) + O(t^{a-2}),$$

where we have used the fact that $\hat{\lambda}$ is bounded to the future as well as (3.14). Hence

$$\begin{aligned}\partial_t\langle\hat{\lambda}\rangle^2 = 2\langle\hat{\lambda}\rangle\langle\hat{\lambda}_t\rangle &= \frac{6}{t}\langle\hat{\lambda}\rangle\left[1 - \left(1 + \frac{1}{2}\langle\hat{\lambda}\rangle + O(\langle\hat{\lambda}\rangle^2)\right)\right] + O(t^{a-2}\langle\hat{\lambda}\rangle)\\ &= -\frac{3}{t}\langle\hat{\lambda}\rangle^2 + \frac{1}{t}O(\langle\hat{\lambda}\rangle^3) + O(t^{a-2}\langle\hat{\lambda}\rangle).\end{aligned}$$

Let $0 < b < 1 - a$ and define $\mathcal{E} = t^{2b}\langle\hat{\lambda}\rangle^2$. Then

$$\frac{d\mathcal{E}}{dt} = \frac{2b}{t}\mathcal{E} - \frac{3}{t}\mathcal{E} + \frac{1}{t}O(\langle\hat{\lambda}\rangle\mathcal{E}) + t^{-1}O(t^{b+a-1}\mathcal{E}^{1/2}).$$

As a consequence, there is a constant $C > 0$ such that $\partial_t\mathcal{E} \leq 0$ when $\mathcal{E} \geq C$ and t is large enough. In particular, $\mathcal{E}$ is bounded to the future. For every $0 < b < 1/2$, there is thus a constant C_b such that

$$\left\|\lambda(t,\cdot) + 3\ln t - 2\ln\frac{3}{4\Lambda}\right\|_{C^0} \leq C_b t^{-b}$$

for all $t \geq t_1$. Due to this estimate, we can return to the argument presented in the proof of Lemma 3.6 and obtain the improvement $E_{\text{bas}}(t) \leq Ct^{1/2}$ for $t \geq t_1$. As a consequence, we can go through the above arguments with $a = 1/2$ and $b = 1/2$. □

Lemma 3.8. *Consider a future global solution to Einstein's equations with a cosmological constant $\Lambda > 0$, $\mathbb{T}^2$-symmetry, λ-asymptotics, and a stress energy tensor satisfying the dominant energy condition and the non-negative pressure condition. Then there is a constant $C > 0$ such that*

$$t^{-3/2}\langle\alpha^{-1/2}(t,\cdot)\rangle + \|Q(t,\cdot)\|_{C^0} + \|P(t,\cdot)\|_{C^0} \leq C \tag{3.16}$$

for all $t \geq t_1$.

Proof. Using (2.13), (3.7), (3.11), (3.12) and Lemma 3.1, we estimate

$$\begin{aligned}
\partial_t\langle\alpha^{-1/2}\rangle &= -\frac{1}{2}\left\langle\alpha^{-1/2}\frac{\alpha_t}{\alpha}\right\rangle \\
&= \frac{1}{2\pi}\int_{\mathbb{S}^1}\left[\frac{\alpha^{-1/2}e^{-P+\lambda/2}J^2}{2t^{5/2}} + \frac{\alpha^{-1/2}e^{P+\lambda/2}(K-QJ)^2}{2t^{5/2}}\right]d\theta \\
&\quad + \frac{1}{2\pi}\int_{\mathbb{S}^1}[2t^{1/2}\alpha^{-1/2}e^{\lambda/2}\Lambda + t^{1/2}\alpha^{-1/2}e^{\lambda/2}(\rho - P_1)]\,d\theta \\
&\leq Ct^{-1/2} + \frac{1}{2\pi}\int_{\mathbb{S}^1} 2t^{1/2}\alpha^{-1/2}e^{\lambda/2}\Lambda\,d\theta \\
&\leq Ct^{-1/2} + \frac{3}{2t}\langle e^{\hat{\lambda}/2}\alpha^{-1/2}\rangle \leq \frac{3}{2t}\langle\alpha^{-1/2}\rangle + Ct^{-3/2}\langle\alpha^{-1/2}\rangle + Ct^{-1/2}.
\end{aligned}$$

Let $A = \langle\alpha^{-1/2}\rangle + t$. Then

$$\frac{dA}{dt} = \partial_t\langle\alpha^{-1/2}\rangle + 1 \leq \frac{3}{2t}A + Ct^{-3/2}A.$$

Consequently,

$$\ln\frac{A(t)}{A(t_1)} \leq \frac{3}{2}\ln t + C_0,$$

so that $\langle\alpha^{-1/2}\rangle \leq Ct^{3/2}$ for $t \geq t_1$. Combining this estimate with (3.12) yields

$$\int_{\mathbb{S}^1}|P_\theta|\,d\theta \leq \left(\int_{\mathbb{S}^1}\alpha^{1/2}P_\theta^2\,d\theta\right)^{1/2}\left(\int_{\mathbb{S}^1}\alpha^{-1/2}\,d\theta\right)^{1/2} \leq Ct^{-3/4}t^{3/4} \leq C \tag{3.17}$$

for $t \geq t_1$. On the other hand, using (3.6) and (3.12) gives

$$|\partial_t \langle P \rangle| = |\langle P_t \rangle| \leq \frac{1}{2\pi} \int_{\mathbb{S}^1} |P_t|\, d\theta \leq \frac{1}{\sqrt{2\pi}} \left(\int_{\mathbb{S}^1} P_t^2\, d\theta \right)^{1/2} \leq C t^{-3/2}. \tag{3.18}$$

Consequently, $\langle P \rangle$ is bounded to the future. Combining these two observations implies

$$\| P(t, \cdot) \|_{C^0} \leq C$$

for all $t \geq t_1$. Combining this estimate for P with the bound (3.12), one can derive L^1-estimates for Q_θ and Q_t analogous to (3.17) and (3.18). Consequently, Q is also bounded to the future. □

Lemma 3.9. *Consider a future global solution to Einstein's equations with a cosmological constant $\Lambda > 0$, $\mathbb{T}^2$-symmetry, λ-asymptotics, and a stress energy tensor satisfying the dominant energy condition and the non-negative pressure condition. Then there is a constant $C > 0$ such that*

$$\left\| \frac{e^{\lambda/2 - P} J^2}{t^{5/2}} \right\|_{C^0} \leq C t^{-2}, \tag{3.19}$$

$$\left\| \frac{e^{P + \lambda/2} (K - QJ)^2}{t^{5/2}} \right\|_{C^0} \leq C t^{-2} \tag{3.20}$$

for all $t \geq t_1$. Moreover, for $t \geq t_1$,

$$\| H_t \|_{L^1} + \| G_t \|_{L^1} \leq C t^{-3/2}.$$

Proof. Combining (2.8), (3.4), (3.11), (3.12) and (3.16), we conclude that

$$\int_{\mathbb{S}^1} |J_\theta|\, d\theta \leq C t^{5/4} \int_{\mathbb{S}^1} \alpha^{-1/2} e^{\lambda/4} \rho\, d\theta \leq C t^{5/4} t^{-3/2} t^{3/4} \int_{\mathbb{S}^1} t^{3/2} \alpha^{-1/2} e^{\lambda/2} \rho\, d\theta \leq C t. \tag{3.21}$$

Hence the spatial variation of J is not greater than Ct. Combining (2.10), (3.5), (3.6), (3.11), (3.12) and (3.16) yields

$$\int_{\mathbb{S}^1} |J_t|\, d\theta \leq C t^{5/4} \int_{\mathbb{S}^1} e^{\lambda/4} \rho\, d\theta \leq C t^{5/4} t^{-3/2} t^{3/4} t^{-3/2} \int_{\mathbb{S}^1} t^{3/2} \alpha^{-1/2} e^{\lambda/2} \rho\, d\theta \leq C t^{-1/2}. \tag{3.22}$$

As a consequence,

$$|\langle J \rangle| \leq C t^{1/2}. \tag{3.23}$$

Combining (3.21) and (3.23) yields

$$\| J(t, \cdot) \|_{C^0} \leq C t. \tag{3.24}$$

From (3.11), (3.16) and (3.24), we conclude that (3.19) holds.

Let us now turn to $K - QJ$. First, note that the L^1-norm of Q_θ is bounded to the future. The argument is similar to (3.17), keeping in mind that (3.16) holds. Moreover,

$$\int_{\mathbb{S}^1} |K_\theta - Q_\theta J - QJ_\theta|\,d\theta \le \int_{\mathbb{S}^1} |K_\theta - QJ_\theta|\,d\theta + \int_{\mathbb{S}^1} |Q_\theta J|\,d\theta \le \int_{\mathbb{S}^1} |K_\theta - QJ_\theta|\,d\theta + Ct,$$

where we have used (3.24) and the fact that the L^1-norm of Q_θ is bounded. On the other hand, (2.8) and (2.9) yield

$$K_\theta - QJ_\theta = 2t^{5/4}\alpha^{-1/2}e^{-P/2+\lambda/4}J_3.$$

Keeping (3.4) in mind, we can thus argue as in the proof of (3.21) to conclude that

$$\int_{\mathbb{S}^1} |K_\theta - Q_\theta J - QJ_\theta|\,d\theta \le Ct. \tag{3.25}$$

In particular, the spatial variation of $K - QJ$ is bounded by Ct. On the other hand, the L^1-norm of Q_t is bounded by $Ct^{-3/2}$ (cf. (3.18) and (3.16)). Combining this observation with (3.24) yields

$$\begin{aligned}\int_{\mathbb{S}^1} |K_t - Q_t J - QJ_t|\,d\theta &\le \int_{\mathbb{S}^1} |K_t - QJ_t|\,d\theta + \int_{\mathbb{S}^1} |Q_t J|\,d\theta \\ &\le \int_{\mathbb{S}^1} |K_t - QJ_t|\,d\theta + Ct^{-1/2}.\end{aligned} \tag{3.26}$$

Moreover, due to (2.10) and (2.11),

$$K_t - QJ_t = -2t^{5/4}e^{-P/2+\lambda/4}S_{13}.$$

Keeping (3.5) in mind, we can proceed as in (3.22) to obtain

$$\int_{\mathbb{S}^1} |K_t - QJ_t|\,d\theta \le Ct^{-1/2}.$$

Due to (3.26) and this estimate, the mean value of $K - QJ$ cannot grow faster than $Ct^{1/2}$. Combining this observation with (3.25) yields

$$\|K - QJ\|_{C^0} \le Ct. \tag{3.27}$$

Keeping (3.11) and (3.16) in mind, we obtain (3.20). As (2.2) holds, we conclude that

$$\int_{\mathbb{S}^1} |H_t|\,d\theta \le \int_{\mathbb{S}^1} t^{-5/2}\alpha^{-1/2}e^{P+\lambda/2}|K - QJ|\,d\theta.$$

Combining this with (3.11), (3.16) and (3.27) yields the desired L^1-estimate for H_t. A similar argument for G_t gives the remaining conclusion of the lemma. □

3.1. Causal structure of $\mathbb{T}^3$-Gowdy symmetric solutions

It is of interest to note that in the $\mathbb{T}^3$-Gowdy symmetric case, it is sufficient to assume future global existence and energy conditions in order to conclude that there is asymptotic silence. In fact, we have the following result.

Proposition 3.10. *Consider a future global $\mathbb{T}^3$-Gowdy symmetric solution to Einstein's equations with a cosmological constant $\Lambda > 0$ and a stress energy tensor satisfying the dominant energy condition and the non-negative pressure condition. Then there is a constant C, depending only on the solution, such that if*

$$\gamma(s) = [s, \theta(s), x(s), y(s)] = [s, \bar{\gamma}(s)]$$

is a causal curve, then

$$|\dot{\bar{\gamma}}(s)|^2 \leq Cs^{-3} \tag{3.28}$$

for $s \geq t_1$. In particular, there is a point $\bar{x}_0 \in \mathbb{T}^3$ such that

$$d[\bar{\gamma}(s), \bar{x}_0] \leq Cs^{-1/2} \tag{3.29}$$

for all $s \geq t_1$, where d is the standard metric on $\mathbb{T}^3$.

Proof. The causality of the curve is equivalent to the estimate

$$\alpha^{-1}\dot{\theta}^2 + s^{3/2}e^{P-\lambda/2}[\dot{x} + Q\dot{y} + (G + QH)\dot{\theta}]^2 + s^{3/2}e^{-P-\lambda/2}(\dot{y} + H\dot{\theta})^2 \leq 1. \tag{3.30}$$

Note that in the case of Gowdy symmetry, G and H are time-independent. In particular, they are bounded. Due to (3.16) we also know that Q is bounded for $t \geq t_1$. On the other hand, combining (3.30) with (3.6), (3.11) and (3.16) yields

$$|\dot{\theta}| \leq Cs^{-3/2},$$
$$|\dot{y} + H\dot{\theta}| \leq Cs^{-3/2},$$
$$|\dot{x} + Q\dot{y} + (G + QH)\dot{\theta}| \leq Cs^{-3/2}$$

for $s \geq t_1$. Thus (3.28) holds, an estimate which implies (3.29). □

4. Light cone estimates

In the presence of matter of Vlasov type, it is necessary to consider the characteristic system in parallel with the light cone estimates for the metric components. Let us therefore begin by writing down the characteristic system:

$$\frac{d\Theta}{ds} = \alpha^{1/2}\frac{V^1}{V^0}, \tag{4.1}$$

$$\frac{dV^1}{ds} = -\frac{1}{4}\alpha^{1/2}\lambda_\theta V^0 - \frac{1}{4}\left(\lambda_t - 2\frac{\alpha_t}{\alpha} - \frac{1}{s}\right)V^1 + \alpha^{1/2}e^P Q_\theta \frac{V^2V^3}{V^0} - \frac{1}{2}\alpha^{1/2}P_\theta \frac{(V^3)^2 - (V^2)^2}{V^0} + \frac{e^{-P/2+\lambda/4}J}{s^{7/4}}V^2 + \frac{e^{P/2+\lambda/4}(K - QJ)}{s^{7/4}}V^3, \tag{4.2}$$

$$\frac{dV^2}{ds} = -\frac{1}{2}\left(P_t + \frac{1}{s}\right)V^2 - \frac{1}{2}\alpha^{1/2}P_\theta\frac{V^1V^2}{V^0}, \tag{4.3}$$

$$\frac{dV^3}{ds} = -\frac{1}{2}\left(\frac{1}{s} - P_t\right)V^3 + \frac{1}{2}\alpha^{1/2}P_\theta\frac{V^1V^3}{V^0} - e^P Q_t V^2 - \alpha^{1/2}e^P Q_\theta\frac{V^1V^2}{V^0}. \tag{4.4}$$

Note that in this system of equations, functions such as $\alpha^{1/2}$ should be evaluated at $[s, \Theta(s)]$. In view of the Vlasov equation (2.19), it is clear that the distribution function is constant along characteristics. It is important to note that only the case of V^1 requires an analysis; for V^2 and V^3 we automatically obtain the following estimate.

Lemma 4.1. *Consider a $\mathbb{T}^2$-symmetric solution to the Einstein–Vlasov equations with a cosmological constant $\Lambda > 0$ and existence interval (t_0, ∞), where $t_0 \geq 0$. Assume that the solution has λ-asymptotics and let $t_1 = t_0 + 2$. Then there is a constant $C > 0$, depending only on the solution, such that if Θ, V is a solution to (4.1)–(4.4) with initial data $\Theta(t_1)$, $V(t_1)$ such that $[t_1, \Theta(t_1), V(t_1)]$ is in the support of f, then*

$$|V^2(s)| + |V^3(s)| \leq Cs^{-1/2} \quad \textit{for all } s \geq t_1.$$

Remark 4.2. As mentioned in Remark 1.6, we tacitly assume $f(t_1, \cdot)$ to have compact support.

Proof of Lemma 4.1. Due to (4.3) and (4.4), it can be verified that

$$s^{1/2}e^{P/2}V^2, \qquad s^{1/2}Qe^{P/2}V^2 + s^{1/2}e^{-P/2}V^3 \tag{4.5}$$

are conserved along characteristics. Since we know P and Q to be uniformly bounded (cf. Lemma 3.8), we obtain the conclusion of the lemma. □

Let us now turn to V^1. First, we have the following estimate.

Lemma 4.3. *Consider a $\mathbb{T}^2$-symmetric solution to the Einstein–Vlasov equations with a cosmological constant $\Lambda > 0$ and existence interval (t_0, ∞), where $t_0 \geq 0$. Assume that the solution has λ-asymptotics and let $t_1 = t_0 + 2$. Then there is a constant $C > 0$, depending only on the solution, such that if Θ, V is a solution to (4.1)–(4.4) with initial data $\Theta(t_1)$, $V(t_1)$ such that $[t_1, \Theta(t_1), V(t_1)]$ is in the support of f, then*

$$\begin{aligned}\frac{d(V^1)^2}{ds} \leq & -\frac{1}{s}(V^1)^2 + Cs^{-1/2}e^{\lambda/2}(\mathcal{Q}^1)^2\frac{|V^1|}{V^0}\\ & + CsF\frac{|V^1|}{V^0} + Cs^{-1}F^{1/2}\frac{|V^1|}{V^0} + Cs^{-3/2}(V^1)^2 + Cs^{-2}|V^1|\end{aligned} \tag{4.6}$$

for all $s \geq t_1$, where

$$\mathcal{Q}^1(t) := \sup\{|v^1| : (t, \theta, v^1, v^2, v^3) \in \operatorname{supp} f\} \tag{4.7}$$

and

$$F(t) = \sup_{\theta \in \mathbb{S}^1} \mathcal{A}_+(t, \theta) + \sup_{\theta \in \mathbb{S}^1} \mathcal{A}_-(t, \theta), \tag{4.8}$$

where $\mathcal{A}_\pm$ is defined in (2.15).

Proof. By (4.2), we have

$$\frac{d(V^1)^2}{ds} = -\frac{1}{2}\alpha^{1/2}\lambda_\theta V^0 V^1 - \frac{1}{2}\left(\lambda_t - 2\frac{\alpha_t}{\alpha} - \frac{1}{s}\right)(V^1)^2 + 2\alpha^{1/2}e^P Q_\theta \frac{V^1 V^2 V^3}{V^0}$$
$$- \alpha^{1/2} P_\theta V^1 \frac{(V^3)^2 - (V^2)^2}{V^0} + 2\frac{e^{-P/2+\lambda/4} J}{s^{7/4}} V^1 V^2$$
$$+ 2\frac{e^{P/2+\lambda/4}(K - QJ)}{s^{7/4}} V^1 V^3.$$

However, using Lemmas 3.9 and 4.1, we can estimate the last two terms by $Cs^{-2}|V^1|$. We thus have

$$\frac{d(V^1)^2}{ds} \le -\frac{1}{2}\alpha^{1/2}\lambda_\theta V^0 V^1 - \frac{1}{2}\left(\lambda_t - 2\frac{\alpha_t}{\alpha} - \frac{1}{s}\right)(V^1)^2 + Cs^{-1}F^{1/2}\frac{|V^1|}{V^0} + Cs^{-2}|V^1|,$$

where we have used Lemma 4.1. By (2.3) and (2.7), the sum of the first and the second terms on the right hand side can be written

$$2s^{1/2}e^{\lambda/2}(J_1 V^0 - \rho V^1)V^1 - 2s^{1/2}e^{\lambda/2}\Lambda(V^1)^2 - \frac{e^{-P+\lambda/2}J^2}{2s^{5/2}}(V^1)^2$$
$$- \frac{e^{P+\lambda/2}(K - QJ)^2}{2s^{5/2}}(V^1)^2 - s\alpha^{1/2}(P_t P_\theta + e^{2P} Q_t Q_\theta)V^0 V^1$$
$$- \frac{1}{2}s[P_t^2 + \alpha P_\theta^2 + e^{2P}(Q_t^2 + \alpha Q_\theta^2)](V^1)^2 + \frac{1}{2s}(V^1)^2. \qquad (4.9)$$

Note that

$$-s\alpha^{1/2}P_t P_\theta V^0 V^1 - \frac{1}{2}s[P_t^2 + \alpha P_\theta^2](V^1)^2 \le s\alpha^{1/2}|P_t P_\theta| V^0 |V^1| - \frac{1}{2}s[P_t^2 + \alpha P_\theta^2](V^1)^2$$
$$\le s\alpha^{1/2}|P_t P_\theta|\,|V^1|(V^0 - |V^1|) - \frac{1}{2}s(|P_t| - \alpha^{1/2}|P_\theta|)^2(V^1)^2.$$

Combining this estimate with a similar estimate for Q, we conclude that the second last and third last terms in (4.9) can be estimated by

$$s\alpha^{1/2}(|P_t P_\theta| + e^{2P}|Q_t Q_\theta|)|V^1|\frac{1 + (V^2)^2 + (V^3)^2}{V^0 + |V^1|} \le CsF\frac{|V^1|}{V^0}.$$

By Lemma 3.7, we have

$$-2s^{1/2}e^{\lambda/2}\Lambda + \frac{1}{2s} = -\frac{1}{s} + O(s^{-3/2}).$$

Combining the above observations with Lemma 3.9, we conclude that

$$\frac{d(V^1)^2}{ds} \le -\frac{1}{s}(V^1)^2 + 2s^{1/2}e^{\lambda/2}(J_1 V^0 - \rho V^1)V^1$$
$$+ CsF\frac{|V^1|}{V^0} + Cs^{-1}F^{1/2}\frac{|V^1|}{V^0} + Cs^{-3/2}(V^1)^2 + Cs^{-2}|V^1|. \qquad (4.10)$$

Let us estimate the term

$$T_1 := 2s^{1/2}e^{\lambda/2}(J_1V^0 - \rho V^1)V^1 = 2s^{1/2}e^{\lambda/2}(I^- + I^+)V^1, \tag{4.11}$$

where

$$I^-(s) = \int_{\mathbb{R}^2}\int_{-\infty}^{0}[v^1V^0(s) - v^0V^1(s)]f(s,\Theta(s),v)\,dv^1\,dv^2\,dv^3,$$
$$I^+(s) = \int_{\mathbb{R}^2}\int_{0}^{\infty}[v^1V^0(s) - v^0V^1(s)]f(s,\Theta(s),v)\,dv^1\,dv^2\,dv^3.$$

There are two cases to distinguish: $V^1(s) \geq 0$ and $V^1(s) < 0$. When $V^1(s) \geq 0$, I^- is non-positive and can be dropped. Furthermore, for $v^1 \geq 0$,

$$\begin{aligned} v^1V^0 - v^0V^1 &= \frac{(v^1)^2(V^0)^2 - (v^0)^2(V^1)^2}{v^1V^0 + v^0V^1}\\ &= \frac{(v^1)^2(1+(V^2)^2+(V^3)^2)}{v^1V^0 + v^0V^1} - \frac{(V^1)^2(1+(v^2)^2+(v^3)^2)}{v^1V^0 + v^0V^1}\\ &\leq \frac{v^1(1+(V^2)^2+(V^3)^2)}{V^0}. \end{aligned}$$

Letting $E_{\mathrm{vel}} = \{(v^2, v^3) : |v^2| \leq Cs^{-1/2}, |v^3| \leq Cs^{-1/2}\}$, where C is the constant appearing in Lemma 4.1, we obtain

$$\begin{aligned} T_1 &\leq 2\|f(t_1,\cdot)\|_\infty s^{1/2}e^{\lambda/2}V^1\int_{E_{\mathrm{vel}}}\int_0^{\mathcal{Q}^1}\frac{v^1(1+(V^2)^2+(V^3)^2)}{V^0}dv^1\,dv^2\,dv^3\\ &\leq Cs^{-1/2}e^{\lambda/2}\int_0^{\mathcal{Q}^1}v^1\,dv^1\frac{V^1}{V^0} \leq Cs^{-1/2}e^{\lambda/2}(\mathcal{Q}^1)^2\frac{V^1}{V^0}, \end{aligned} \tag{4.12}$$

where $\mathcal{Q}^1(t)$ is defined in the statement of the lemma. When $V^1 < 0$, an analogous argument can be given and it follows that in both cases

$$T_1 \leq Cs^{-1/2}e^{\lambda/2}(\mathcal{Q}^1)^2\frac{|V^1|}{V^0}. \tag{4.13}$$

Combining (4.10), (4.11) and (4.13) yields the conclusion of the lemma. □

Lemma 4.4. *Consider a $\mathbb{T}^2$-symmetric solution to the Einstein–Vlasov equations with a cosmological constant $\Lambda > 0$ and existence interval (t_0, ∞), where $t_0 \geq 0$. Assume that the solution has λ-asymptotics and let $t_1 = t_0 + 2$. Then there is a constant $C > 0$, depending only on the solution, such that*

$$t^3\|P_t^2 + \alpha P_\theta^2 + e^{2P}(Q_t^2 + \alpha Q_\theta^2)\|_{C^0} + t[\mathcal{Q}^1(t)]^2 \leq C \quad \textit{for all } t \geq t_1.$$

Proof. Let us use (2.16) to derive an estimate for F. Note first that

$$-\left(\frac{2}{t} - \frac{\alpha_t}{\alpha}\right) \leq -\left(\frac{2}{t} + 4t^{1/2}e^{\lambda/2}\Lambda\right) \leq -\frac{5}{t} + O(t^{-3/2}),$$

where we have used (2.13), (3.3) and (3.11). Note also that the second term on the right hand side of (2.16) can be written

$$\frac{1}{2t}(\mathcal{A}_{\mp} - \mathcal{A}_{\pm}) + \frac{2}{t}\alpha(P_\theta^2 + e^{2P}Q_\theta^2) \le \frac{1}{t}\mathcal{A}_{\mp} + \frac{1}{2t}(\mathcal{A}_+ + \mathcal{A}_-).$$

Combining these estimates with Lemma 3.9, we conclude that

$$\begin{aligned}\partial_{\pm}\mathcal{A}_{\mp} \le & -\frac{4}{t}\mathcal{A}_{\mp} + Ct^{-3/2}\mathcal{A}_{\mp} + \frac{1}{2t}(\mathcal{A}_+ + \mathcal{A}_-) + Ct^{-3}\mathcal{A}_{\mp}^{1/2} \\ & + 2t^{-1/2}e^{\lambda/2}(P_2 - P_3)\partial_{\mp}P + 4t^{-1/2}e^{\lambda/2}S_{23}e^{P}\partial_{\mp}Q.\end{aligned}$$

Moreover, due to (2.21) and Lemma 4.1,

$$|P_k| \le Ct^{-2}\ln(1+\mathcal{Q}^1), \qquad |S_{23}| \le Ct^{-2}\ln(1+\mathcal{Q}^1)$$

for $k = 2, 3$. As a consequence,

$$\partial_{\pm}\mathcal{A}_{\mp} \le -\frac{4}{t}\mathcal{A}_{\mp} + Ct^{-3/2}\mathcal{A}_{\mp} + \frac{1}{2t}(\mathcal{A}_+ + \mathcal{A}_-) + Ct^{-3}\mathcal{A}_{\mp}^{1/2} + Ct^{-4}\mathcal{A}_{\mp}^{1/2}\ln(1+\mathcal{Q}^1).$$

Defining

$$\hat{\mathcal{A}}_{\pm} = t^4\mathcal{A}_{\pm} + t, \tag{4.14}$$

we obtain

$$\partial_{\pm}\hat{\mathcal{A}}_{\mp} \le \frac{1}{2t}(\hat{\mathcal{A}}_+ + \hat{\mathcal{A}}_-) + Ct^{-3/2}\hat{\mathcal{A}}_{\mp} + Ct^{-2}\hat{\mathcal{A}}_{\mp}^{1/2}\ln(1+\mathcal{Q}^1).$$

Introducing

$$\hat{F}(t) = \sup_{\theta\in\mathbb{S}^1}\hat{\mathcal{A}}_+(t,\theta) + \sup_{\theta\in\mathbb{S}^1}\hat{\mathcal{A}}_-(t,\theta), \tag{4.15}$$

we obtain

$$\hat{F}(t) \le \hat{F}(t_1) + \int_{t_1}^{t}\left(\frac{1}{s}\hat{F}(s) + Cs^{-3/2}\hat{F}(s) + Cs^{-2}\hat{F}^{1/2}(s)\ln(1+\mathcal{Q}^1)\right)ds. \tag{4.16}$$

Introduce

$$R^1(s) = [s(V^1(s))^2 + 1]^{1/2}, \qquad \hat{\mathcal{Q}}^1(s) = [s(\mathcal{Q}^1(s))^2 + 1]^{1/2}. \tag{4.17}$$

Then (4.6) implies that

$$\frac{d(R^1)^2}{ds} \le Cs^{-2}(\hat{\mathcal{Q}}^1)^2 + Cs^{-2}\hat{F} + Cs^{-5/2}\hat{F} + Cs^{-3/2}(\hat{\mathcal{Q}}^1)^2.$$

Integrating this inequality from t_1 to t and taking the supremum over initial data belonging to the support of f, we obtain

$$[\hat{\mathcal{Q}}^1(t)]^2 \le [\hat{\mathcal{Q}}^1(t_1)]^2 + \int_{t_1}^{t}\left(Cs^{-2}\hat{F} + Cs^{-3/2}(\hat{\mathcal{Q}}^1)^2\right)ds. \tag{4.18}$$

Adding (4.16) and (4.18) and introducing

$$\mathcal{G} = \hat{F} + (\hat{\mathcal{Q}}^1)^2, \tag{4.19}$$

we obtain

$$\mathcal{G}(t) \leq \mathcal{G}(t_1) + \int_{t_1}^{t} \left(\frac{1}{s}\mathcal{G}(s) + Cs^{-3/2}\mathcal{G}(s) \right) ds.$$

In particular, $\mathcal{G}(t) \leq Ct$, so that $\hat{F}(t) \leq Ct$ and $\mathcal{Q}^1$ is bounded. Returning to (4.6) with this information in mind, we conclude that

$$\frac{d(R^1)^2}{ds} \leq Cs^{-3/2}(\hat{\mathcal{Q}}^1)^2.$$

By arguments similar to ones given above, we conclude that $\hat{\mathcal{Q}}^1$ is bounded. □

5. Intermediate estimates

Before proceeding, it is useful to collect the estimates that follow from the above arguments.

Lemma 5.1. *Consider a $\mathbb{T}^2$-symmetric solution to the Einstein–Vlasov equations with a cosmological constant $\Lambda > 0$ and existence interval (t_0, ∞), where $t_0 \geq 0$. Assume that the solution has λ-asymptotics and let $t_1 = t_0 + 2$. Then there is a constant $C > 0$, depending only on the solution, such that*

$$\left\| \lambda(t,\cdot) + 3\ln t - 2\ln\frac{3}{4\Lambda} \right\|_{C^0} \leq Ct^{-1}, \tag{5.1}$$

$$\left\| \frac{\alpha_t}{\alpha} + \frac{3}{t} \right\|_{C^0} + \left\| \lambda_t + \frac{3}{t} \right\|_{C^0} \leq Ct^{-2}, \tag{5.2}$$

$$\|\alpha^{1/2}\lambda_\theta\|_{C^0} \leq Ct^{-2} \tag{5.3}$$

for all $t \geq t_1$. Moreover,

$$\|J_t\|_{C^0} + \|K_t\|_{C^0} \leq Ct^{-2}, \tag{5.4}$$

$$\left\| \frac{e^{P+\lambda/2}(K-QJ)^2}{t^{5/2}} \right\|_{C^0} + \left\| \frac{e^{-P+\lambda/2}J^2}{t^{5/2}} \right\|_{C^0} \leq Ct^{-4}, \tag{5.5}$$

$$\|J_\theta\|_{C^0} + \|K_\theta\|_{C^0} \leq C, \tag{5.6}$$

$$\left\| \partial_\theta\left(\frac{e^{P+\lambda/2}(K-QJ)^2}{t^{5/2}} \right) \right\|_{C^0} + \left\| \partial_\theta\left(\frac{e^{-P+\lambda/2}J^2}{t^{5/2}} \right) \right\|_{C^0} \leq Ct^{-4} \tag{5.7}$$

for all $t \geq t_1$. Finally,

$$\|\rho\|_{C^0} + t^{1/2}\|J_i\|_{C^0} + t\|P_i\|_{C^0} + t\|S_{im}\|_{C^0} \leq Ct^{-3/2} \tag{5.8}$$

for all $t \geq t_1$.

Remark 5.2. Note that as a consequence of (5.2),

$$\|\partial_t(\alpha^{-1/2}e^{\lambda/2})\|_{C^0} \leq Ct^{-2} \quad \text{for all } t \geq t_1.$$

Proof of Lemma 5.1. Due to (2.21) and Lemmas 4.1 and 4.4, the estimate (5.8) holds. Combining Lemmas 3.9 and 4.4 with (2.4) and (5.8), we conclude that

$$\partial_t \hat{\lambda} = \frac{3}{t} - \frac{3}{t} e^{\hat{\lambda}/2} + O(t^{-2}),$$

where we have used the notation (3.7). Combining this observation with Lemma 3.7, we conclude that there is a constant C such that

$$\partial_t \hat{\lambda}^2 \leq -\frac{3}{t} \hat{\lambda}^2 + C t^{-2} |\hat{\lambda}|.$$

Introducing $L = t^2 \hat{\lambda}^2$, we obtain

$$\partial_t L \leq -\frac{1}{t} L + \frac{C}{t} L^{1/2}.$$

In particular, L decreases once it exceeds a certain value. As a consequence, L is bounded, and we obtain (5.1). Combining (2.4), (2.13), (5.1) and (5.8) and Lemmas 3.9 and 4.4, we then obtain (5.2). As a consequence of this estimate, $t^3 \alpha$ converges to a strictly positive function. In particular, there are constants $C_i > 0$, $i = 1, 2$, such that

$$C_1 \leq t^3 \alpha(t, \theta) \leq C_2 \tag{5.9}$$

for all $t \geq t_1$.

From (2.7), (3.11), (5.8) and Lemma 4.4, we obtain (5.3). Returning to (2.10) and (2.11), keeping (3.11), (3.16) and (5.8) in mind, we conclude that (5.4) holds. As a consequence, J and K are bounded. Combining this observation with (3.11) and (3.16) yields (5.5). From (2.8), (2.9), (3.11), (3.16), (5.8) and (5.9), we also obtain (5.6). By (3.16), (5.9) and Lemma 4.4, we know that P_θ and Q_θ are bounded for $t \geq t_1$. Moreover, λ_θ is bounded for $t \geq t_1$ due to (5.3) and (5.9). Combining these observations with (5.1), (5.6) and the fact that J, K, Q and P are bounded, we obtain (5.7). □

6. Derivatives of the characteristic system

Solutions to the Vlasov equation can be expressed in terms of the initial datum for the distribution function and appropriate solutions to the characteristic system (4.1)–(4.4). To see this, let us begin by introducing the notation Θ, V for the solution to (4.1)–(4.4) corresponding to the initial data

$$\Theta(t; t, \theta, v) = \theta, \qquad V(t; t, \theta, v) = v. \tag{6.1}$$

Here we write $\Theta(s; t, \theta, v)$ and $V(s; t, \theta, v)$ in order to clarify the dependence on the initial data. In particular, $\Theta(s; t, \theta, v)$ and $V(s; t, \theta, v)$, considered as functions of s, constitute a solution to (4.1)–(4.4). The purpose of the variables (t, θ, v) appearing after the semi-colon is simply to indicate that the relations (6.1) hold. We shall write $d\Theta/ds$ and dV/ds to indicate differentiation with respect to the first variable. Moreover, $\partial_t \Theta$, $\partial_\theta \Theta$, $\partial_{v^i} \Theta$ etc. will denote differentiation with respect to the variables appearing after the semi-colon.

Given a fixed $\tau \in (t_0, \infty)$, where (t_0, ∞) is the existence interval of the solution to the Einstein–Vlasov system under consideration, we know that

$$f(t, \theta, v) = f[\tau, \Theta(\tau; t, \theta, v), V(\tau; t, \theta, v)]. \tag{6.2}$$

Since $f(\tau, \cdot)$ is a smooth function with compact support, it is sufficient to estimate the derivatives of solutions to the characteristic system in order to estimate the derivatives of f. Unfortunately, differentiating the characteristic system leads to second order derivatives of P, Q etc., quantities over which we have no control. However, using the ideas introduced in [2], this problem can be circumvented. In fact, let ∂ be a shorthand for ∂_t, ∂_θ or ∂_{v^i} and let

$$\Psi = \alpha^{-1/2} e^{\lambda/2} \partial\Theta, \tag{6.3}$$

$$\begin{aligned} Z^1 = \partial V^1 + \Bigg[&\frac{1}{4}\alpha^{-1/2}\left(\lambda_t - 2\frac{\alpha_t}{\alpha} - 4s^{1/2}e^{\lambda/2}\Lambda\right)V^0 - \frac{1}{2}\alpha^{-1/2}P_t V^0 \frac{(V^2)^2 - (V^3)^2}{(V^0)^2 - (V^1)^2} \\ &+ \frac{1}{2}P_\theta V^1 \frac{(V^2)^2 - (V^3)^2}{(V^0)^2 - (V^1)^2} - \alpha^{-1/2} e^P Q_t \frac{V^0 V^2 V^3}{(V^0)^2 - (V^1)^2} \\ &+ e^P Q_\theta \frac{V^1 V^2 V^3}{(V^0)^2 - (V^1)^2}\Bigg]\partial\Theta, \end{aligned} \tag{6.4}$$

$$Z^2 = \partial V^2 + \frac{1}{2} P_\theta V^2 \partial\Theta, \tag{6.5}$$

$$Z^3 = \partial V^3 - \left(\frac{1}{2}P_\theta V^3 - e^P Q_\theta V^2\right)\partial\Theta. \tag{6.6}$$

It is then possible to derive an ODE for (Ψ, Z^1, Z^2, Z^3) such that the coefficients are controlled due to previous arguments. The definitions (6.3)–(6.6) differ slightly from those of [2]. The reason is that in the present context, it is not sufficient to know that no second order derivatives of P, Q etc. occur; we need to analyse, in detail, all the terms that appear, and to use the resulting system in order to derive specific estimates for $\partial\Theta$ and ∂V^i. The relevant result is the following.

Lemma 6.1. *Consider a $\mathbb{T}^2$-symmetric solution to the Einstein–Vlasov equations with a cosmological constant $\Lambda > 0$ and existence interval (t_0, ∞), where $t_0 \geq 0$. Assume that the solution has λ-asymptotics and let $t_1 = t_0 + 2$. Then there is a constant $C > 0$, depending only on the solution, such that*

$$\frac{dZ^1}{ds} = -\frac{1}{2s}Z^1 + c_{1,\theta}\Psi + c_{1,j}Z^j, \tag{6.7}$$

$$\frac{dZ^2}{ds} = -\frac{1}{2s}Z^2 + c_{2,2}Z^2, \tag{6.8}$$

$$\frac{dZ^3}{ds} = -\frac{1}{2s}Z^3 + c_{3,2}Z^2 + c_{3,3}Z^3, \tag{6.9}$$

$$\frac{d\Psi}{ds} = c_{\theta,\theta}\Psi + c_{\theta,i}Z^i, \tag{6.10}$$

where Einstein's summation convention applies to i and j, and

$$|c_{i,j}(s;t,\theta,v)| + |c_{\theta,i}(s;t,\theta,v)| + |c_{i,\theta}(s;t,\theta,v)| + s^{1/2}|c_{\theta,\theta}(s;t,\theta,v)| \leq Cs^{-3/2}$$

for all $(t,\theta,v) \in [t_1,\infty) \times \mathbb{S}^1 \times \mathbb{R}^3$ in the support of f and for all $s \in [t_1,t]$.

Proof. Let us begin by noting that

$$e^{P/2}s^{1/2}Z^2 = \partial(s^{1/2}e^{P/2}V^2),$$
$$s^{1/2}e^{-P/2}Z^3 + e^{P/2}Qs^{1/2}Z^2 = \partial(s^{1/2}Qe^{P/2}V^2 + s^{1/2}e^{-P/2}V^3).$$

Since the quantities appearing in (4.5) are preserved along characteristics, we obtain

$$\frac{d}{ds}(e^{P/2}s^{1/2}Z^2) = 0, \tag{6.11}$$

$$\frac{d}{ds}(s^{1/2}e^{-P/2}Z^3 + e^{P/2}Qs^{1/2}Z^2) = 0. \tag{6.12}$$

We also have

$$\frac{d\Psi}{ds} = \frac{1}{2}\left(\lambda_t - \frac{\alpha_t}{\alpha}\right)\Psi + \frac{1}{2}\alpha^{1/2}\lambda_\theta\frac{V^1}{V^0}\Psi + e^{\lambda/2}\partial\left(\frac{V^1}{V^0}\right). \tag{6.13}$$

In the end, we shall express $\partial(V^1/V^0)$ in terms of Z^i and Ψ. However, there is no immediate gain in doing so here. The most cumbersome part of the argument is to compute the derivative of Z^1. This calculation can be divided into several parts. We first consider

$$\frac{d}{ds}(\partial V^1) = \partial\left(\frac{dV^1}{ds}\right).$$

When calculating the right hand side, it is convenient to divide the result into terms which include a ∂V^i factor, $i = 1,2,3$, and terms which do not. Combining Lemmas 4.1, 4.4 and 5.1, we see that the terms which include such a factor can be written

$$-\frac{1}{2s}\partial V^1 + c_i\partial V^i,$$

where $c_i(s) = O(s^{-2})$; note that the coefficient of V^1 in (4.2) is

$$-\frac{1}{2s} + O(s^{-2}).$$

It is straightforward to calculate the remaining terms, and we conclude that

$$\begin{aligned}\frac{d}{ds}(\partial V^1) =& -\frac{1}{4}\partial_\theta(\alpha^{1/2}\lambda_\theta)V^0\partial\Theta - \frac{1}{4}\partial_\theta\left(\lambda_t - 2\frac{\alpha_t}{\alpha}\right)V^1\partial\Theta + \partial_\theta(\alpha^{1/2}e^P Q_\theta)\frac{V^2V^3}{V^0}\partial\Theta \\ &- \frac{1}{2}\partial_\theta(\alpha^{1/2}P_\theta)\frac{(V^3)^2-(V^2)^2}{V^0}\partial\Theta + s^{-7/4}\partial_\theta(e^{\lambda/4}e^{-P/2}J)V^2\partial\Theta \\ &+ s^{-7/4}\partial_\theta[e^{\lambda/4}e^{P/2}(K-QJ)]V^3\partial\Theta - \frac{1}{2s}\partial V^1 + c_i\partial V^i,\end{aligned}$$

where $c_i(s) = O(s^{-2})$. Combining an argument which is identical to the proof of (5.7) with Lemma 4.1, we can estimate the coefficients of $\partial\Theta$ in the third last and fourth last terms on the right hand side. In fact, we obtain

$$\begin{aligned}\frac{d}{ds}(\partial V^1) = &-\frac{1}{2s}\partial V^1 - \frac{1}{4}\partial_\theta(\alpha^{1/2}\lambda_\theta)V^0\partial\Theta - \frac{1}{4}\partial_\theta\left(\lambda_t - 2\frac{\alpha_t}{\alpha}\right)V^1\partial\Theta\\ &+ \partial_\theta(\alpha^{1/2}e^P Q_\theta)\frac{V^2V^3}{V^0}\partial\Theta - \frac{1}{2}\partial_\theta(\alpha^{1/2}P_\theta)\frac{(V^3)^2-(V^2)^2}{V^0}\partial\Theta\\ &+ c_\theta\partial\Theta + c_i\partial V^i,\end{aligned}$$

where $c_i(s) = O(s^{-2})$ and $c_\theta(s) = O(s^{-3})$. As a next step, it is of interest to consider the terms that arise when d/ds hits a V^α in the second term in the definition of Z^1. Before writing down the result, let us note that, due to Lemmas 4.1, 4.4 and 5.1,

$$\frac{dV^i}{ds} = -\frac{1}{2s}V^i + O(s^{-2}), \qquad \frac{dV^0}{ds} = O(s^{-2}). \tag{6.14}$$

From these observations, Lemmas 4.1, 4.4 and 5.1, as well as (5.9) and the definition of Z^1, we conclude that when d/ds hits a V^α in the second term in Z^1, the resulting expression can be written $c_\theta\partial\Theta$, where $c_\theta(s) = O(s^{-2})$.

Note that every term appearing in the second term in the definition of Z^1 can be written in the form

$$h(\cdot,\Theta)\psi(V)\alpha^{-1/2}(\cdot,\Theta)\partial\Theta. \tag{6.15}$$

We have already estimated the terms that arise when d/ds hits ψ. Let us therefore consider the terms that arise when the derivative hits the remaining factors. Omitting the arguments, we need to consider

$$\begin{aligned}&\left(h_t + \alpha^{1/2}h_\theta\frac{V^1}{V^0}\right)\psi\alpha^{-1/2}\partial\Theta - \frac{\alpha_t}{2\alpha^{3/2}}h\psi\partial\Theta\\ &\qquad + h\psi\left[-\frac{\alpha_\theta}{2\alpha^{3/2}}\alpha^{1/2}\frac{V^1}{V^0}\partial\Theta + \alpha^{-1/2}\frac{\alpha_\theta}{2\alpha^{1/2}}\frac{V^1}{V^0}\partial\Theta + \partial\left(\frac{V^1}{V^0}\right)\right]\\ &\qquad = \left(\partial_t(\alpha^{-1/2}h) + h_\theta\frac{V^1}{V^0}\right)\psi\partial\Theta + h\psi\partial\left(\frac{V^1}{V^0}\right).\end{aligned} \tag{6.16}$$

In all the terms of interest, $h\psi = O(s^{-2})$, where we have used Lemmas 4.1, 4.4 and 5.1. As a consequence, this expression can be written

$$\left(\partial_t(\alpha^{-1/2}h) + h_\theta\frac{V^1}{V^0}\right)\psi\partial\Theta + c_i\partial V^i,$$

where $c_i(s) = O(s^{-2})$. Combining the above observations, we conclude that

$$\begin{aligned}\frac{dZ^1}{ds} &= -\frac{1}{2s}\partial V^1 - \frac{1}{4}\partial_\theta(\alpha^{1/2}\lambda_\theta)V^0\partial\Theta - \frac{1}{4}\partial_\theta\left(\lambda_t - 2\frac{\alpha_t}{\alpha}\right)V^1\partial\Theta \\ &+ \partial_\theta(\alpha^{1/2}e^P Q_\theta)\frac{V^2V^3}{V^0}\partial\Theta - \frac{1}{2}\partial_\theta(\alpha^{1/2}P_\theta)\frac{(V^3)^2-(V^2)^2}{V^0}\partial\Theta \\ &+ \frac{1}{4}\partial_t\left[\alpha^{-1/2}\left(\lambda_t - 2\frac{\alpha_t}{\alpha} - 4s^{1/2}e^{\lambda/2}\Lambda\right)\right]V^0\partial\Theta \\ &+ \frac{1}{4}\partial_\theta\left(\lambda_t - 2\frac{\alpha_t}{\alpha} - 4s^{1/2}e^{\lambda/2}\Lambda\right)V^1\partial\Theta - \frac{1}{2}\partial_t(\alpha^{-1/2}P_t)V^0\frac{(V^2)^2-(V^3)^2}{(V^0)^2-(V^1)^2}\partial\Theta \\ &- \frac{1}{2}P_{t\theta}V^1\frac{(V^2)^2-(V^3)^2}{(V^0)^2-(V^1)^2}\partial\Theta + \frac{1}{2}P_{t\theta}V^1\frac{(V^2)^2-(V^3)^2}{(V^0)^2-(V^1)^2}\partial\Theta \\ &+ \frac{1}{2}\partial_\theta(\alpha^{1/2}P_\theta)\frac{(V^1)^2}{V^0}\frac{(V^2)^2-(V^3)^2}{(V^0)^2-(V^1)^2}\partial\Theta - \partial_t(\alpha^{-1/2}e^P Q_t)\frac{V^0V^2V^3}{(V^0)^2-(V^1)^2}\partial\Theta \\ &- \partial_\theta(e^P Q_t)\frac{V^1V^2V^3}{(V^0)^2-(V^1)^2}\partial\Theta + \partial_t(e^P Q_\theta)\frac{V^1V^2V^3}{(V^0)^2-(V^1)^2}\partial\Theta \\ &+ \partial_\theta(\alpha^{1/2}e^P Q_\theta)\frac{V^1}{V^0}\frac{V^1V^2V^3}{(V^0)^2-(V^1)^2}\partial\Theta + c_\theta\partial\Theta + c_i\partial V^i, \end{aligned} \tag{6.17}$$

where $c_\theta(s) = O(s^{-2})$ and $c_i(s) = O(s^{-2})$. How to interpret the terms in this equation should be clear from (6.15) and (6.16). However, there is one term which is slightly ambiguous, namely the sixth one on the right hand side of (6.17). For clarity, let us point out that the coefficient of $V^0\partial\Theta$ in this term should be interpreted as the time derivative of

$$\frac{1}{4}\left[\alpha^{-1/2}\left(\lambda_t - 2\frac{\alpha_t}{\alpha} - 4t^{1/2}e^{\lambda/2}\Lambda\right)\right],$$

evaluated at $[s, \Theta(s)]$. The expression (6.17) can be simplified somewhat. First, the terms involving $P_{t\theta}$ cancel. Moreover,

$$-\partial_\theta(e^P Q_t) + \partial_t(e^P Q_\theta) = e^P(P_t Q_\theta - P_\theta Q_t).$$

Using Lemma 4.4 and (5.9), we can estimate this expression in order to conclude that the sum of the fourth last and fifth last terms is of the form $c_\theta\partial\Theta$, where $c_\theta = O(s^{-3})$ (to obtain this conclusion, we also use Lemma 4.1). By Lemma 4.4, (5.1), (5.3) and (5.9), the sum of the third and the seventh terms is $c_\theta\partial\Theta$, where $c_\theta = O(s^{-2})$. Since

$$\begin{aligned}-\frac{(V^3)^2-(V^2)^2}{V^0} + \frac{(V^1)^2}{V^0}\frac{(V^2)^2-(V^3)^2}{(V^0)^2-(V^1)^2} &= V^0\frac{(V^2)^2-(V^3)^2}{(V^0)^2-(V^1)^2}, \\ \frac{V^2V^3}{V^0} + \frac{V^1}{V^0}\frac{V^1V^2V^3}{(V^0)^2-(V^1)^2} &= \frac{V^0V^2V^3}{(V^0)^2-(V^1)^2},\end{aligned}$$

the terms involving a factor of $\partial_\theta(\alpha^{1/2}P_\theta)$ can be written

$$\frac{1}{2}\partial_\theta(\alpha^{1/2}P_\theta)V^0\frac{(V^2)^2-(V^3)^2}{(V^0)^2-(V^1)^2}\partial\Theta,$$

and the terms involving a factor of $\partial_\theta(\alpha^{1/2}e^P Q_\theta)$ can be written

$$\partial_\theta(\alpha^{1/2}e^P Q_\theta)\frac{V^0V^2V^3}{(V^0)^2-(V^1)^2}\partial\Theta.$$

Combining these observations yields

$$\begin{aligned}\frac{dZ^1}{ds} = &-\frac{1}{2s}\partial V^1+\frac{1}{4}\partial_t\left[\alpha^{-1/2}\left(\lambda_t-2\frac{\alpha_t}{\alpha}-4s^{1/2}e^{\lambda/2}\Lambda\right)\right]V^0\partial\Theta-\frac{1}{4}\partial_\theta(\alpha^{1/2}\lambda_\theta)V^0\partial\Theta\\ &-\frac{1}{2}[\partial_t(\alpha^{-1/2}P_t)-\partial_\theta(\alpha^{1/2}P_\theta)]V^0\frac{(V^2)^2-(V^3)^2}{(V^0)^2-(V^1)^2}\partial\Theta\\ &-[\partial_t(\alpha^{-1/2}e^PQ_t)-\partial_\theta(\alpha^{1/2}e^PQ_\theta)]\frac{V^0V^2V^3}{(V^0)^2-(V^1)^2}\partial\Theta+c_\theta\partial\Theta+c_i\partial V^i,\end{aligned} \tag{6.18}$$

where $c_\theta(s)=O(s^{-2})$ and $c_i(s)=O(s^{-2})$. Combining (2.5), (5.9) and Lemmas 4.4 and 5.1 yields

$$\partial_t(\alpha^{-1/2}P_t)-\partial_\theta(\alpha^{1/2}P_\theta)=O(t^{-1}).$$

Due to this estimate and Lemmas 4.4 and 4.1, the fourth term on the right hand side of (6.18) is $c_\theta\partial\Theta$, where $c_\theta(s)=O(s^{-2})$. Keeping (2.12) in mind, a similar argument yields the same conclusion concerning the fifth term on the right hand side of (6.18). Finally, keeping (3.10) in mind, a similar argument yields the conclusion that the combination of the second and third terms on the right hand side of (6.18) is $c_\theta\partial\Theta$, where $c_\theta(s)=O(s^{-3/2})$. To conclude,

$$\frac{dZ^1}{ds}=-\frac{1}{2s}\partial V^1+c_\theta\partial\Theta+c_i\partial V^i, \tag{6.19}$$

where $c_\theta(s)=O(s^{-3/2})$ and $c_i(s)=O(s^{-2})$. On the other hand, due to (5.1) and (5.9), the function $\alpha^{-1/2}e^{\lambda/2}$ can be bounded from above and below by positive constants (for $t\geq t_1$). In other words, $\partial\Theta$ and Ψ are interchangeable when deriving equations of the form (6.7)–(6.10). Moreover, due to (5.9) and Lemmas 4.1, 4.4 and 5.1,

$$Z^i=\partial V^i+c_{i,\theta}\Psi, \tag{6.20}$$

where $c_{i,\theta}(s)=O(s^{-1/2})$. We can now prove the lemma. First, combining (6.11), (6.12) and Lemma 4.4 yields (6.8) and (6.9). Combining (6.13) and (6.20) with Lemmas 4.1, 4.4 and 5.1 gives (6.10). Finally, combining (6.19) and (6.20) yields (6.7). □

Lemma 6.2. *Consider a $\mathbb{T}^2$-symmetric solution to the Einstein–Vlasov equations with a cosmological constant $\Lambda>0$ and existence interval (t_0,∞), where $t_0\geq 0$. Assume that the solution has λ-asymptotics and let $t_1=t_0+2$. Then there is a constant $C>0$, depending only on the solution, such that*

$$(\ln s)^2|\partial_\theta\Theta(s;t,\theta,v)|+s^{1/2}|\partial_\theta V^i(s;t,\theta,v)|\leq C(\ln t)^2 \tag{6.21}$$

for all $s\in[t_1,t]$ and $(t,\theta,v)\in[t_1,\infty)\times\mathbb{S}^1\times\mathbb{R}^3$ in the support of f.

Remark 6.3. Arguments similar to the ones below yield estimates for $\partial_t\Theta$, $\partial_{v^i}\Theta$ etc.

Proof of Lemma 6.2. Let

$$\hat{Z}^i(s;t,\theta,v) = s^{1/2}Z^i(s;t,\theta,v), \qquad \hat{\Psi}(s;t,\theta,v) = (\ln s)^2\Psi(s;t,\theta,v). \tag{6.22}$$

Then, due to Lemma 6.1,

$$\begin{aligned}
\frac{d\hat{Z}^1}{ds} &= c_{1,\theta}s^{1/2}(\ln s)^{-2}\hat{\Psi} + c_{1,j}\hat{Z}^j,\\
\frac{d\hat{Z}^2}{ds} &= c_{2,2}\hat{Z}^2,\\
\frac{d\hat{Z}^3}{ds} &= c_{3,2}\hat{Z}^2 + c_{3,3}\hat{Z}^3,\\
\frac{d\hat{\Psi}}{ds} &= \frac{2}{s\ln s}\hat{\Psi} + c_{\theta,\theta}\hat{\Psi} + c_{\theta,i}s^{-1/2}(\ln s)^2\hat{Z}^i
\end{aligned}$$

for $s \in [t_1, t]$, with coefficients as in Lemma 6.1. Introducing

$$\hat{E} = \sum_{i=1}^{3}(\hat{Z}^i)^2 + (\hat{\Psi})^2, \tag{6.23}$$

we conclude that there is a constant $C > 0$, depending only on the solution, such that

$$\frac{d\hat{E}}{ds} \geq -C\frac{1}{s(\ln s)^2}\hat{E}$$

for $s \in [t_1, t]$. As a consequence,

$$\hat{E}(s;t,\theta,v) \leq C\hat{E}(t;t,\theta,v) \tag{6.24}$$

for $s \in [t_1, t]$. Let us now assume $\partial = \partial_\theta$. Then

$$\hat{\Psi}(t;t,\theta,v) = O[(\ln t)^2].$$

Moreover,

$$\hat{Z}^i(t;t,\theta,v) = [t^{1/2}\partial_\theta V^i + O(1)\Psi](t;t,\theta,v) = O(1).$$

As a consequence, $\hat{E}(t;t,\theta,v) = O[(\ln t)^4]$. Thus (6.24) implies (6.21); note that the estimate for $\partial_\theta\Theta$ is immediate and

$$|\partial_\theta V^i| \leq |Z^i| + Cs^{-1/2}|\partial_\theta\Theta|.$$

□

7. Higher order light cone estimates

Before proceeding to higher order light cone estimates, let us record some consequences of the estimates obtained in Lemma 6.2.

Lemma 7.1. *Consider a $\mathbb{T}^2$-symmetric solution to the Einstein–Vlasov equations with a cosmological constant $\Lambda > 0$ and existence interval (t_0, ∞), where $t_0 \geq 0$. Assume that the solution has λ-asymptotics and let $t_1 = t_0 + 2$. Then there is a constant $C > 0$, depending only on the solution, such that*

$$\|\partial_\theta \rho\|_{C^0} + t^{1/2}\|\partial_\theta J_i\|_{C^0} + t\|\partial_\theta S_{ij}\|_{C^0} + t\|\partial_\theta P_k\|_{C^0} \leq Ct^{-3/2}(\ln t)^2, \tag{7.1}$$

$$\left\|\partial_\theta\left(\frac{\alpha_t}{\alpha}\right)\right\|_{C^0} \leq Ct^{-3/2}, \tag{7.2}$$

$$\left\|\frac{\alpha_\theta}{\alpha}\right\|_{C^0} \leq C \tag{7.3}$$

for all $t \geq t_1$.

Remark 7.2. Note that $\partial_\theta(\alpha_t/\alpha) = \partial_t(\alpha_\theta/\alpha) = \partial_t\partial_\theta \ln\alpha$.

Proof of Lemma 7.1. The estimate (7.1) follows from (6.2), (6.21) and the fact that $|v^i| \leq Ct^{-1/2}$ in the support of $f(t, \cdot)$. Consider (2.13). Since Lemmas 4.4 and 5.1 together with (5.9) imply that J, K, Q and P are bounded in C^1 and that λ_θ is $O(t^{-1/2})$, the first two terms on the right hand side of (2.13) are $O(t^{-4})$ in C^1. Since λ_θ is $O(t^{-1/2})$, the θ-derivative of the third term on the right hand side of (2.13) is $O(t^{-3/2})$. Due to (7.1), the θ-derivative of the last term is better. Thus (7.2) holds, so that

$$\left\|\partial_t\left(\frac{\alpha_\theta}{\alpha}\right)\right\|_{C^0} \leq \left\|\partial_\theta\left(\frac{\alpha_t}{\alpha}\right)\right\|_{C^0} \leq Ct^{-3/2}.$$

Integrating this estimate yields (7.3). □

In what follows, we shall proceed inductively in order to derive estimates for higher order derivatives. Let us therefore assume that we have a $\mathbb{T}^2$-symmetric solution to the Einstein–Vlasov equations with a cosmological constant $\Lambda > 0$ and existence interval (t_0, ∞), where $t_0 \geq 0$. Assume moreover that the solution has λ-asymptotics and let $t_1 = t_0 + 2$. Let us make the following inductive assumption.

Inductive Assumption 7.3. For some $1 \leq N \in \mathbb{Z}$, there are constants $0 \leq m_j \in \mathbb{Z}$ and C_j, $j = 1, \ldots, N$, (depending only on N and the solution) such that

$$s^{1/2}\left|\frac{\partial^j V}{\partial\theta^j}(s; t, \theta, v)\right| + \left|\frac{\partial^j \Theta}{\partial\theta^j}(s; t, \theta, v)\right| \leq C_j(\ln t)^{m_j}, \tag{7.4}$$

$$\|P_\theta\|_{C^{N-1}} + \|Q_\theta\|_{C^{N-1}} + t^{3/2}\|P_t\|_{C^{N-1}} + t^{3/2}\|Q_t\|_{C^{N-1}} \leq C_{N-1} \tag{7.5}$$

for all $j = 1, \ldots, N$, $(t, \theta, v) \in [t_1, \infty) \times \mathbb{S}^1 \times \mathbb{R}^3$ in the support of f and $s \in [t_1, t]$.

Remarks 7.4. The induction hypothesis holds for $N = 1$. In what follows, C_j and m_j will change from line to line. However, they are only allowed to depend on N and the solution.

In this section we prove that if Inductive Assumption 7.3 holds, then (7.5) holds with N replaced by $N + 1$. In the next section, we close the induction argument by proving that (7.4) holds with j replaced by $N + 1$.

We shall need the following consequences of the inductive assumption.

Lemma 7.5. *Consider a $\mathbb{T}^2$-symmetric solution to the Einstein–Vlasov equations with a cosmological constant $\Lambda > 0$ and existence interval (t_0, ∞), where $t_0 \geq 0$. Assume that the solution has λ-asymptotics and let $t_1 = t_0 + 2$. Assume moreover that Inductive Assumption 7.3 holds for some $1 \leq N \in \mathbb{Z}$. Then there are constants C_j, $j = 0, \dots, N$, and m_N, depending only on N and the solution, such that*

$$\|\rho\|_{C^N} + t^{1/2}\|J_i\|_{C^N} + t\|P_i\|_{C^N} + t\|S_{im}\|_{C^N} \leq C_N t^{-3/2}(\ln t)^{m_N}, \tag{7.6}$$

$$t^{1/2}\|\partial_\theta^{l+1}\lambda\|_{C^0} \leq C_l, \tag{7.7}$$

$$\|\alpha^{-1}\partial_\theta^j\alpha\|_{C^0} \leq C_j, \tag{7.8}$$

$$\|J_\theta\|_{C^N} + \|K_\theta\|_{C^N} \leq C_N(\ln t)^{m_N}, \tag{7.9}$$

$$\left\|\partial_\theta^{l+1}\left(\frac{\alpha_t}{\alpha}\right)\right\|_{C^0} \leq C_l t^{-3/2} \tag{7.10}$$

for $t \geq t_1$, $0 \leq j \leq N$, $0 \leq l \leq N-1$ and $i, m = 1, 2, 3$.

Proof. For $N = 1$, the conclusions follow from Lemmas 4.4, 5.1 and 7.1, (5.9) and the equations (2.8) and (2.9). We may thus assume that $N \geq 2$. An immediate consequence of the inductive assumption is that, for $0 \leq j \leq N$ and $t \geq t_1$,

$$\left|\frac{\partial^j f}{\partial\theta^j}\right| \leq C_j(\ln t)^{m_j}$$

(cf. (6.2) and (7.4)). This yields (7.6). To obtain control of the θ-derivatives of α and λ, we proceed inductively. Let us make the inductive assumption that

$$\|\alpha^{-1}\partial_\theta^j\alpha\|_{C^0} \leq C_j, \tag{7.11}$$

$$\|\partial_\theta^j\lambda\|_{C^0} \leq C_j t^{-1/2} \tag{7.12}$$

for $1 \leq j \leq l < N$. Note that we know this is true for $l = 1$. Differentiating (2.7) l times with respect to θ and appealing to (7.5), (7.6), (7.11) and (7.12), we conclude that (7.12) holds with j replaced by $l+1$. In order to improve our knowledge concerning α, let us begin by improving our estimates for the θ-derivatives for J and K. Differentiating (2.8) and (2.9) $0 \leq j \leq l$ times and using (7.5), (7.6), (7.11) and (7.12), we conclude that

$$\|\partial_\theta^{j+1}J\|_{C^0} + \|\partial_\theta^{j+1}K\|_{C^0} \leq C_j(\ln t)^{m_j} \tag{7.13}$$

for $t \geq t_1$ and $0 \leq j \leq l$. Differentiating (2.13) $l+1$ times with respect to θ, using (7.5), (7.6), (7.13) as well as the fact that (7.12) holds for $1 \leq j \leq l+1$, we obtain

$$\left\|\partial_\theta^{l+1}\left(\frac{\alpha_t}{\alpha}\right)\right\|_{C^0} = \left\|\partial_t\partial_\theta^{l}\left(\frac{\alpha_\theta}{\alpha}\right)\right\|_{C^0} \leq C_l t^{-3/2} \tag{7.14}$$

for $t \geq t_1$. Thus

$$\left\|\partial_\theta^{l}\left(\frac{\alpha_\theta}{\alpha}\right)\right\|_{C^0} \leq C_l$$

for $t \geq t_1$. Combining this estimate with the inductive hypothesis, we conclude that (7.11) holds with j replaced by $l+1$. Thus (7.12) and (7.11) hold for $1 \leq j \leq N$. Consequently, (7.7)–(7.9) hold. In addition, (7.14) implies (7.10). □

We are now in a position to derive higher order light cone estimates.

Lemma 7.6. *Consider a $\mathbb{T}^2$-symmetric solution to the Einstein–Vlasov equations with a cosmological constant $\Lambda > 0$ and existence interval (t_0, ∞), where $t_0 \geq 0$. Assume that the solution has λ-asymptotics and let $t_1 = t_0 + 2$. Assume moreover that Inductive Assumption 7.3 holds for some $1 \leq N \in \mathbb{Z}$. Then there is a constant $C_N > 0$, depending only on N and the solution, such that*

$$t^{3/2}\|\partial_\theta^N P_t\|_{C^0} + \|\partial_\theta^N P_\theta\|_{C^0} + t^{3/2}\|e^P \partial_\theta^N Q_t\|_{C^0} + \|e^P \partial_\theta^N Q_\theta\|_{C^0} \leq C_N \tag{7.15}$$

for all $t \geq t_1$. As a consequence, (7.5) holds with N replaced by $N+1$.

Proof. If $N = 1$, then in some of the sums below, the lower summation limit is larger than the upper limit; such sums are meant to be zero. Moreover, terms which are bounded by Ct^{-3} for $t \geq t_1$ will sometimes be written $O(t^{-3})$. Let us compute

$$\begin{aligned}
&\partial_\pm[\partial_\theta^N P_t \mp \partial_\theta^N(\alpha^{1/2}P_\theta)]\\
&\quad= \partial_\theta^N P_{tt} \mp \partial_\theta^N\left(\frac{\alpha_t}{2\alpha}\alpha^{1/2}P_\theta + \alpha^{1/2}P_{t\theta}\right) \pm \alpha^{1/2}\partial_\theta^{N+1}P_t - \alpha^{1/2}\partial_\theta^{N+1}(\alpha^{1/2}P_\theta)\\
&\quad= \partial_\theta^N P_{tt} \mp \frac{1}{2}\sum_{j=0}^{N-1}\beta_j\partial_\theta^{N-j}\left(\frac{\alpha_t}{\alpha}\right)\partial_\theta^j(\alpha^{1/2}P_\theta) \mp \frac{1}{2}\frac{\alpha_t}{\alpha}\partial_\theta^N(\alpha^{1/2}P_\theta)\\
&\qquad\mp \frac{N\alpha_\theta}{2\alpha^{1/2}}\partial_\theta^N P_t \mp \sum_{j=0}^{N-2}\beta_j\partial_\theta^{N-j}(\alpha^{1/2})\partial_\theta^{j+1}(P_t) - \partial_\theta^N[\alpha^{1/2}\partial_\theta(\alpha^{1/2}P_\theta)]\\
&\qquad+ \frac{N\alpha_\theta}{2\alpha^{1/2}}\partial_\theta^N(\alpha^{1/2}P_\theta) + \sum_{j=0}^{N-2}\beta_j\partial_\theta^{N-j}(\alpha^{1/2})\partial_\theta^{j+1}(\alpha^{1/2}P_\theta),
\end{aligned}$$

where the β_j are binomial coefficients. Note that all the sums are $O(t^{-3})$ due to Inductive Assumption 7.3, Lemma 7.5 and (5.9). Let us use (2.5) in order to compute

$$\begin{aligned}
&\partial_\theta^N[P_{tt} - \alpha^{1/2}\partial_\theta(\alpha^{1/2}P_\theta)] = \partial_\theta^N\left(P_{tt} - \alpha P_{\theta\theta} - \frac{\alpha_\theta}{2}P_\theta\right)\\
&= -\frac{1}{t}\partial_\theta^N P_t + \frac{\alpha_t}{2\alpha}\partial_\theta^N P_t + \sum_{j=0}^{N-1}\beta_j\partial_\theta^{N-j}\left(\frac{\alpha_t}{2\alpha}\right)\partial_\theta^j P_t + \sum_{j=0}^{N-1}\beta_j\partial_\theta^{N-j}(e^{2P})\partial_\theta^j(Q_t^2 - \alpha Q_\theta^2)\\
&\quad+ e^{2P}\sum_{j=1}^{N-1}\beta_j(\partial_\theta^{N-j}(Q_t - \alpha^{1/2}Q_\theta))(\partial_\theta^j(Q_t + \alpha^{1/2}Q_\theta))\\
&\quad+ 2e^{2P}[Q_t\partial_\theta^N Q_t - \alpha^{1/2}Q_\theta\partial_\theta^N(\alpha^{1/2}Q_\theta)] - \partial_\theta^N\left(\frac{e^{P+\lambda/2}(K-QJ)^2}{2t^{7/2}}\right)\\
&\quad+ \partial_\theta^N\left(\frac{e^{-P+\lambda/2}J^2}{2t^{7/2}}\right) + t^{-1/2}\partial_\theta^N[e^{\lambda/2}(P_2 - P_3)].
\end{aligned}$$

By Inductive Assumption 7.3, Lemmas 7.5 and 5.1, and (5.9), the sums are $O(t^{-3})$, as also are the last three terms on the right hand side. We thus obtain

$$\begin{aligned}&\partial_\pm[\partial_\theta^N P_t \mp \partial_\theta^N(\alpha^{1/2}P_\theta)]\\&\quad= -\frac{1}{t}\partial_\theta^N P_t + \frac{\alpha_t}{2\alpha}[\partial_\theta^N P_t \mp \partial_\theta^N(\alpha^{1/2}P_\theta)] \mp \frac{N\alpha_\theta}{2\alpha^{1/2}}[\partial_\theta^N P_t \mp \partial_\theta^N(\alpha^{1/2}P_\theta)]\\&\qquad+2e^{2P}[Q_t\partial_\theta^N Q_t - \alpha^{1/2}Q_\theta\partial_\theta^N(\alpha^{1/2}Q_\theta)] + O(t^{-3}).\end{aligned}$$

Introducing

$$\mathcal{A}_{N+1,\pm} = [\partial_\theta^N P_t \pm \partial_\theta^N(\alpha^{1/2}P_\theta)]^2 + e^{2P}[\partial_\theta^N Q_t \pm \partial_\theta^N(\alpha^{1/2}Q_\theta)]^2, \tag{7.16}$$

we conclude that

$$\begin{aligned}&\partial_\pm[\partial_\theta^N P_t \mp \partial_\theta^N(\alpha^{1/2}P_\theta)]^2\\&\quad\le -\frac{5}{t}[\partial_\theta^N P_t \mp \partial_\theta^N(\alpha^{1/2}P_\theta)]^2 \mp \frac{2}{t}\partial_\theta^N(\alpha^{1/2}P_\theta)[\partial_\theta^N P_t \mp \partial_\theta^N(\alpha^{1/2}P_\theta)]\\&\qquad+C_N t^{-3}\mathcal{A}_{N+1,\mp}^{1/2} + C_N t^{-3/2}(\mathcal{A}_{N+1,+} + \mathcal{A}_{N+1,-}),\end{aligned}$$

where we have used Inductive Assumption 7.3, Lemma 7.5, (5.2) and (5.9). Now consider

$$\begin{aligned}&\partial_\pm[\partial_\theta^N Q_t \mp \partial_\theta^N(\alpha^{1/2}Q_\theta)]\\&\quad= \partial_\theta^N Q_{tt} \mp \frac{1}{2}\sum_{j=0}^{N-1}\beta_j\partial_\theta^{N-j}\left(\frac{\alpha_t}{\alpha}\right)\partial_\theta^j(\alpha^{1/2}Q_\theta) \mp \frac{1}{2}\frac{\alpha_t}{\alpha}\partial_\theta^N(\alpha^{1/2}Q_\theta)\\&\qquad\mp\frac{N\alpha_\theta}{2\alpha^{1/2}}\partial_\theta^N Q_t \mp \sum_{j=0}^{N-2}\beta_j\partial_\theta^{N-j}(\alpha^{1/2})\partial_\theta^j Q_{t\theta} - \partial_\theta^N[\alpha^{1/2}\partial_\theta(\alpha^{1/2}Q_\theta)]\\&\qquad+\frac{N\alpha_\theta}{2\alpha^{1/2}}\partial_\theta^N(\alpha^{1/2}Q_\theta) + \sum_{j=0}^{N-2}\beta_j\partial_\theta^{N-j}(\alpha^{1/2})\partial_\theta^{j+1}(\alpha^{1/2}Q_\theta).\end{aligned}$$

As above, all the sums are $O(t^{-3})$ due to Inductive Assumption 7.3, Lemma 7.5 and (5.9). Using (2.12), we compute

$$\begin{aligned}&\partial_\theta^N[Q_{tt} - \alpha^{1/2}\partial_\theta(\alpha^{1/2}Q_\theta)]\\&\quad= -\frac{1}{t}\partial_\theta^N Q_t + \frac{\alpha_t}{2\alpha}\partial_\theta^N Q_t + \sum_{j=0}^{N-1}\beta_j\partial_\theta^{N-j}\left(\frac{\alpha_t}{2\alpha}\right)\partial_\theta^j Q_t - 2(\partial_\theta^N P_t)Q_t\\&\qquad-2P_t(\partial_\theta^N Q_t) + 2\partial_\theta^N(\alpha^{1/2}P_\theta)\alpha^{1/2}Q_\theta + 2\alpha^{1/2}P_\theta\partial_\theta^N(\alpha^{1/2}Q_\theta)\\&\qquad-2\sum_{j=1}^{N-1}\beta_j(\partial_\theta^{N-j}P_t)(\partial_\theta^j Q_t) + 2\sum_{j=1}^{N-1}\beta_j(\partial_\theta^{N-j}(\alpha^{1/2}P_\theta))(\partial_\theta^j(\alpha^{1/2}Q_\theta))\\&\qquad+\partial_\theta^N\left(\frac{e^{\lambda/2-P}J(K-QJ)}{t^{7/2}}\right) + 2t^{-1/2}\partial_\theta^N(e^{\lambda/2-P}S_{23}).\end{aligned}$$

By Inductive Assumption 7.3, Lemmas 7.5 and 5.1, and (5.9), the sums are $O(t^{-3})$, as are the last two terms on the right hand side. Thus

$$\begin{aligned}&\partial_\pm[\partial_\theta^N Q_t \mp \partial_\theta^N(\alpha^{1/2}Q_\theta)]\\&= -\frac{1}{t}\partial_\theta^N Q_t + \frac{\alpha_t}{2\alpha}[\partial_\theta^N Q_t \mp \partial_\theta^N(\alpha^{1/2}Q_\theta)] \mp \frac{N\alpha_\theta}{2\alpha^{1/2}}[\partial_\theta^N Q_t \mp \partial_\theta^N(\alpha^{1/2}Q_\theta)]\\&\quad -2(\partial_\theta^N P_t)Q_t - 2P_t(\partial_\theta^N Q_t) + 2\partial_\theta^N(\alpha^{1/2}P_\theta)\alpha^{1/2}Q_\theta + 2\alpha^{1/2}P_\theta\partial_\theta^N(\alpha^{1/2}Q_\theta) + O(t^{-3}).\end{aligned}$$

Consequently,

$$\begin{aligned}&\partial_\pm[\partial_\theta^N Q_t \mp \partial_\theta^N(\alpha^{1/2}Q_\theta)]^2\\&\qquad \le -\frac{5}{t}[\partial_\theta^N Q_t \mp \partial_\theta^N(\alpha^{1/2}Q_\theta)]^2 \mp \frac{2}{t}\partial_\theta^N(\alpha^{1/2}Q_\theta)[\partial_\theta^N Q_t \mp \partial_\theta^N(\alpha^{1/2}Q_\theta)]\\&\qquad\quad + C_N t^{-3}\mathcal{A}_{N+1,\mp}^{1/2} + C_N t^{-3/2}(\mathcal{A}_{N+1,+} + \mathcal{A}_{N+1,-}),\end{aligned}$$

where we have used Inductive Assumption 7.3, Lemma 7.5, (5.2) and (5.9). Adding up the above estimates, we conclude that

$$\begin{aligned}\partial_\pm\mathcal{A}_{N+1,\mp} \le& -\frac{5}{t}\mathcal{A}_{N+1,\mp} + \frac{1}{2t}(\mathcal{A}_{N+1,\mp} - \mathcal{A}_{N+1,\pm}) + \frac{1}{t}(\mathcal{A}_{N+1,+} + \mathcal{A}_{N+1,-})\\&+ C_N t^{-3/2}(\mathcal{A}_{N+1,+} + \mathcal{A}_{N+1,-}) + C_N t^{-3}\mathcal{A}_{N+1,\mp}^{1/2}.\end{aligned}$$

Let us introduce

$$\hat{\mathcal{A}}_{N+1,\pm} = t^{7/2}\mathcal{A}_{N+1,\pm} + t^{1/2}, \quad \hat{F}_{N+1,\pm} = \sup_{\theta\in\mathbb{S}^1}\hat{\mathcal{A}}_{N+1,\pm}, \quad \hat{F}_{N+1} = \hat{F}_{N+1,+} + \hat{F}_{N+1,-}. \tag{7.17}$$

Then

$$\partial_\pm\hat{\mathcal{A}}_{N+1,\mp} \le \frac{1}{2t}\hat{\mathcal{A}}_{N+1,\pm} + C_N t^{-3/2}(\hat{\mathcal{A}}_{N+1,+} + \hat{\mathcal{A}}_{N+1,-}).$$

Integrating this differential inequality, taking the supremum etc., we obtain

$$\hat{F}_{N+1}(t) \le \hat{F}_{N+1}(t_1) + \int_{t_1}^t \left(\frac{1}{2s}\hat{F}_{N+1}(s) + C_N s^{-3/2}\hat{F}_{N+1}(s)\right) ds.$$

As a consequence, $\hat{F}_{N+1}(t) \le C_N t^{1/2}$. Combining this estimate with Inductive Assumption 7.3, Lemma 7.5 and (5.9), we obtain (7.15). □

8. Higher order derivatives of the characteristic system

In the previous section we showed that (7.5) holds with N replaced by $N+1$, that is,

$$\|P_\theta\|_{C^N} + \|Q_\theta\|_{C^N} + t^{3/2}\|P_t\|_{C^N} + t^{3/2}\|Q_t\|_{C^N} \le C_N \tag{8.1}$$

for all $t \ge t_1$. We also need to prove that (7.4) holds with j replaced with $N+1$. Before stating the relevant result, let us make the following preliminary observation.

Lemma 8.1. *Consider a $\mathbb{T}^2$-symmetric solution to the Einstein–Vlasov equations with a cosmological constant $\Lambda > 0$ and existence interval (t_0, ∞), where $t_0 \geq 0$. Assume that the solution has λ-asymptotics and let $t_1 = t_0 + 2$. Assume moreover that Inductive Assumption 7.3 holds for some $1 \leq N \in \mathbb{Z}$. Then there are constants C_j and m_j, $j = 0, \dots, N$, depending only on N and the solution, such that*

$$\|\rho\|_{C^N} + t^{1/2}\|J_i\|_{C^N} + t\|P_i\|_{C^N} + t\|S_{im}\|_{C^N} \leq C_N t^{-3/2}(\ln t)^{m_N}, \tag{8.2}$$

$$\|\alpha^{-1}\partial_\theta^j \alpha\|_{C^0} + t^{1/2}\|\partial_\theta^{j+1}\lambda\|_{C^0} \leq C_j, \tag{8.3}$$

$$\|J_\theta\|_{C^N} + \|K_\theta\|_{C^N} \leq C_N (\ln t)^{m_N}, \tag{8.4}$$

$$\left\|\partial_\theta^{l+1}\left(\frac{\alpha_t}{\alpha}\right)\right\|_{C^0} + \|\partial_\theta^{l+1}\lambda_t\|_{C^0} \leq C_l t^{-3/2}, \tag{8.5}$$

$$\left\|\lambda_t - 2\frac{\alpha_t}{\alpha} - 4t^{1/2}e^{\lambda/2}\Lambda\right\|_{C^N} \leq C_N t^{-2}, \tag{8.6}$$

$$\left\|\lambda_t - \frac{\alpha_t}{\alpha}\right\|_{C^N} \leq C_N t^{-2} \tag{8.7}$$

for $t \geq t_1$, $0 \leq j \leq N$, $0 \leq l \leq N-1$ and $i, m = 1, 2, 3$. Moreover, using the notation

$$\Psi_j = \partial_\theta^j \Psi, \quad Z_j^i = \partial_\theta^j Z^i, \quad V_j^i = \partial_\theta^j V^i, \quad \Theta_j = \partial_\theta^j \Theta \tag{8.8}$$

(where the ∂-operator used to define Z and Ψ is given by ∂_θ), there are functions $c_{i,\theta}$, $i = 1, 2, 3$, such that

$$|\Psi_l(s; t, \theta, v)| + s^{1/2}|Z_l(s; t, \theta, v)| \leq C_l (\ln t)^{m_l}, \tag{8.9}$$

$$|\Psi_j(s; t, \theta, v) - (\alpha^{-1/2}e^{\lambda/2})[s, \Theta(s; t, \theta, v)]\Theta_{j+1}(s; t, \theta, v)| \leq C_j (\ln t)^{m_j}, \tag{8.10}$$

$$|Z_j^i(s; t, \theta, v) - V_{j+1}^i(s; t, \theta, v) - (c_{i,\theta}\Psi_j)(s; t, \theta, v)| \leq C_j s^{-1/2}(\ln t)^{m_j}, \tag{8.11}$$

$$|c_{i,\theta}(s; t, \theta, v)| \leq C_0 s^{-1/2} \tag{8.12}$$

for all $(t, \theta, v) \in [t_1, \infty) \times \mathbb{S}^1 \times \mathbb{R}^3$ in the support of f, $0 \leq j \leq N$, $0 \leq l \leq N-1$ and $s \in [t_1, t]$.

Remark 8.2. Due to (8.3), we have

$$|\partial_\theta^j \alpha^p| \leq C_{p,j}\alpha^p$$

for all $(t, \theta) \in [t_1, \infty) \times \mathbb{S}^1$, $p \in \mathbb{R}$ and $0 \leq j \leq N$. In particular, spatial derivatives of powers of α can effectively be ignored. In the derivation of the estimates below, it is useful to keep this observation in mind.

Proof of Lemma 8.1. Combining Lemma 7.5 with (8.1), (2.4) and (2.7), we obtain (8.2)–(8.5); recall that P, Q, J and K are bounded to the future. The estimate (8.6) is a consequence of Lemma 7.5, (8.1) and the fact that

$$\lambda_t - 2\frac{\alpha_t}{\alpha} - 4t^{1/2}e^{\lambda/2}\Lambda = t[P_t^2 + \alpha P_\theta^2 + e^{2P}(Q_t^2 + \alpha Q_\theta^2)] + \frac{e^{-P+\lambda/2}J^2}{t^{5/2}} + \frac{e^{P+\lambda/2}(K - QJ)^2}{t^{5/2}} + 4t^{1/2}e^{\lambda/2}\rho$$

(cf. (2.13) and (2.14)). For similar reasons, (8.7) holds (cf. (2.14)). Turning to (8.9)–(8.12), note that, up to numerical factors, $\partial_\theta^j \Psi(s; t, \theta, v)$ can be written as a sum of terms of the form

$$\partial_\theta^k (\alpha^{-1/2} e^{\lambda/2})[s, \Theta(s; t, \theta, v)] \Theta_{i_1}(s; t, \theta, v) \cdots \Theta_{i_{k+1}}(s; t, \theta, v),$$

where $i_1 + \cdots + i_{k+1} = j + 1$. Since $k \leq j \leq N$, the first factor is bounded due to (8.3). By Inductive Assumption 7.3, the factors Θ_{i_j} can be estimated by $C(\ln t)^{m_j}$ if $i_j \leq N$. The only way a factor Θ_{N+1} could occur is if $k = 0$ and all the derivatives hit Θ_1 in the definition of Ψ. These observations yield (8.10) and the estimate

$$|\Psi_l(s; t, \theta, v)| \leq C_l (\ln t)^{m_l}$$

for $0 \leq l \leq N-1$. The proof of the remaining estimates is similar in nature, but somewhat more involved. □

We now finish the induction argument by proving that (7.4) holds with j replaced by $N + 1$.

Lemma 8.3. *Consider a $\mathbb{T}^2$-symmetric solution to the Einstein–Vlasov equations with a cosmological constant $\Lambda > 0$ and existence interval (t_0, ∞), where $t_0 \geq 0$. Assume that the solution has λ-asymptotics and let $t_1 = t_0 + 2$. Assume, moreover, that Inductive Assumption 7.3 holds for some $1 \leq N \in \mathbb{Z}$. Then (7.4) holds with j replaced by $N + 1$.*

Proof. The strategy of the proof is very similar to that of the proof of Lemma 6.1. The idea is to derive a system of ODE's for Z_N^i and Ψ_N analogous to (6.7)–(6.10), and then to use arguments similar to those in the proof of Lemma 6.2. Deriving appropriate equations for Z_N^i, $i = 2, 3$, turns out to be relatively easy, due to (6.11) and (6.12). In fact, we obtain the desired conclusions concerning Z_N^i, $i = 2, 3$, without much effort (cf. (8.15) and (8.16) below). Deriving an equation for Ψ_N also turns out to be quite easy (cf. (8.14)). Similarly to the proof of Lemma 6.1, the main difficulty lies in deriving an equation for Z_N^1. Once the desired equation has been obtained, we rescale Z_N^1 and Ψ_N according to (8.23) and introduce an energy according to (8.24); note that these definitions are analogous to the ones in the proof of Lemma 6.2. Finally, the equations imply a differential inequality for the energy $\hat{E}_N$ which can be integrated to yield the desired estimate.

Before proceeding to the proof, we introduce some notation. Let b be a C^1 function on $M = (t_0, \infty) \times \mathbb{S}^1$. Evaluating it along a characteristic, we obtain

$$B(s; t, \theta, v) = b[s, \Theta(s; t, \theta, v)].$$

Differentiating B with respect to θ yields

$$\frac{\partial B}{\partial \theta}(s; t, \theta, v) = \frac{\partial b}{\partial \theta}[s, \Theta(s; t, \theta, v)] \frac{\partial \Theta}{\partial \theta}(s; t, \theta, v). \tag{8.13}$$

On the other hand, distinguishing between B and b is quite cumbersome in the arguments that we are about to carry out. As a consequence, we shall write b when we mean B.

Moreover, we shall use $\partial_\vartheta b$ as a shorthand for $\partial_\theta B$, whereas $\partial_\theta b$ should be interpreted as the first factor on the right hand side of (8.13), and $\partial_t b$ as mapping $(s; t, \theta, v)$ to

$$\frac{\partial b}{\partial t}[s, \Theta(s; t, \theta, v)].$$

In particular, $b_\vartheta = b_\theta \Theta_1$. Finally, if in some expression a ϑ-derivative hits a V or a Θ, it is to be interpreted as an ordinary θ-derivative.

Note first that

$$\frac{d\Psi}{ds} = \frac{1}{2}\left(\lambda_t - \frac{\alpha_t}{\alpha}\right)\Psi + \frac{1}{2}\alpha^{1/2}\lambda_\theta \frac{V^1}{V^0}\Psi + e^{\lambda/2}\frac{\partial V^1}{V^0} - e^{\lambda/2}\frac{V^1}{(V^0)^3}\sum_{i=1}^{3} V^i \partial V^i$$

due to (6.13). Differentiating this equality N times with respect to ϑ, we obtain

$$\frac{d\Psi_N}{ds} = \frac{1}{2}\left(\lambda_t - \frac{\alpha_t}{\alpha}\right)\Psi_N + \frac{1}{2}\alpha^{1/2}\lambda_\theta \frac{V^1}{V^0}\Psi_N + e^{\lambda/2}\frac{V^1_{N+1}}{V^0} - e^{\lambda/2}\frac{V^1}{(V^0)^3}\sum_{i=1}^{3} V^i V^i_{N+1} + O[s^{-2}(\ln t)^{m_N}],$$

where we have used (5.9), Inductive Assumption 7.3 and Lemma 8.1. Due to (8.11), this equation can be written

$$\frac{d\Psi_N}{ds} = c^N_{\theta,\theta}\Psi_N + c^N_{\theta,i} Z^i_N + O[s^{-2}(\ln t)^{m_N}], \tag{8.14}$$

where $c^N_{\theta,\theta} = O(s^{-2})$, $c^N_{\theta,i} = O(s^{-3/2})$ and we sum over i but not N. Turning to Z^2, we have

$$\frac{dZ^2}{ds} = -\frac{1}{2s}Z^2 - \frac{1}{2}\left(P_t + \alpha^{1/2}P_\theta \frac{V^1}{V^0}\right)Z^2$$

(cf. (6.11)). Differentiating this equality N times with respect to ϑ, we obtain

$$\frac{dZ^2_N}{ds} = -\frac{1}{2s}Z^2_N - \frac{1}{2}\left(P_t + \alpha^{1/2}P_\theta \frac{V^1}{V^0}\right)Z^2_N + O[s^{-2}(\ln t)^{m_N}],$$

where we have used (5.9), (8.1), Inductive Assumption 7.3 and Lemma 8.1. Hence

$$\frac{d}{ds}(s^{1/2}e^{P/2}Z^2_N) = O[s^{-3/2}(\ln t)^{m_N}].$$

Integrating this equality from s to t, we obtain (assuming $N \geq 1$)

$$-(s^{1/2}e^{P/2}Z^2_N)(s; t, \theta, v) = O[(\ln t)^{m_N}];$$

note that

$$(t^{1/2}e^{P/2}Z^2_N)(t; t, \theta, v) = t^{1/2}e^{P/2}\frac{1}{2}(\partial^{N+1}_\theta P)v^2 = O(1)$$

due to (8.1). In particular,

$$|Z^2_N(s; t, \theta, v)| \leq C_N s^{-1/2}(\ln t)^{m_N} \tag{8.15}$$

for $s \in [t_1, t]$. Turning to Z^3, we have

$$\frac{dZ^3}{ds} = -\frac{1}{2s}Z^3 + \frac{1}{2}\left(P_t + \alpha^{1/2}P_\theta\frac{V^1}{V^0}\right)Z^3 - e^P\left(Q_t + \alpha^{1/2}Q_\theta\frac{V^1}{V^0}\right)Z^2$$

(cf. (6.12) and (6.11)). Differentiating N times with respect to ϑ, we obtain

$$\frac{dZ^3_N}{ds} = -\frac{1}{2s}Z^3_N + \frac{1}{2}\left(P_t + \alpha^{1/2}P_\theta\frac{V^1}{V^0}\right)Z^3_N - e^P\left(Q_t + \alpha^{1/2}Q_\theta\frac{V^1}{V^0}\right)Z^2_N + O[s^{-2}(\ln t)^{m_N}].$$

Due to (8.15), the third term on the right hand side is $O[s^{-2}(\ln t)^{m_N}]$. We can thus proceed as in the proof of (8.15) to obtain

$$|Z^3_N(s; t, \theta, v)| \le C_N s^{-1/2}(\ln t)^{m_N} \tag{8.16}$$

for $s \in [t_1, t]$. Finally, we need to derive an equation for Z^1_N. Just as in the derivation of the equation for Z^1, it is natural to divide the analysis into several steps. Consider first

$$\partial_\vartheta^{j+1}\left(\frac{dV^1}{ds}\right)$$

for $0 \le j \le N$. All the terms appearing in dV^1/ds can be written $h\psi \circ V$. When differentiating an expression of this form, the terms that arise are (up to numerical factors) of the form $\partial_\vartheta^k h \partial_\theta^l \psi \circ V$. If both k and l are ≥ 1, the resulting term is $O[s^{-2}(\ln t)^{m_j}]$. If all the derivatives hit ψ, we obtain (after summing over all the terms appearing in dV^1/ds)

$$-\frac{1}{2s}V^1_{j+1} + c_i^j V^i_{j+1} + O[s^{-2}(\ln t)^{m_j}]$$

where $c_i^j = O(s^{-2})$ and we sum over i but not j. If all the derivatives hit h, we obtain (after summing over all the terms appearing in dV^1/ds)

$$-\frac{1}{4}\partial_\vartheta^{j+1}(\alpha^{1/2}\lambda_\theta)V^0 - \frac{1}{4}\partial_\vartheta^{j+1}\left(\lambda_t - 2\frac{\alpha_t}{\alpha}\right)V^1 + \partial_\vartheta^{j+1}(\alpha^{1/2}e^P Q_\theta)\frac{V^2V^3}{V^0} - \frac{1}{2}\partial_\vartheta^{j+1}(\alpha^{1/2}P_\theta)\frac{(V^3)^2 - (V^2)^2}{V^0} + c_\theta^j\Theta_{j+1} + O[s^{-3}(\ln t)^{m_j}],$$

where $c_\theta^j = O(s^{-3})$ and we have used (5.6); note that, due to (8.1) and Lemma 8.1, we control $N+1$ θ-derivatives of the first factor in each of the last two terms appearing on the right hand side of (4.2). Adding up, we conclude that

$$\begin{aligned}\partial_\vartheta^{j+1}\left(\frac{dV^1}{ds}\right) = &-\frac{1}{2s}V^1_{j+1} + c_i^j V^i_{j+1} + c_\theta^j\Theta_{j+1} - \frac{1}{4}\partial_\vartheta^{j+1}(\alpha^{1/2}\lambda_\theta)V^0 \\ &- \frac{1}{4}\partial_\vartheta^{j+1}\left(\lambda_t - 2\frac{\alpha_t}{\alpha}\right)V^1 + \partial_\vartheta^{j+1}(\alpha^{1/2}e^P Q_\theta)\frac{V^2V^3}{V^0} \\ &- \frac{1}{2}\partial_\vartheta^{j+1}(\alpha^{1/2}P_\theta)\frac{(V^3)^2-(V^2)^2}{V^0} + O[s^{-2}(\ln t)^{m_j}],\end{aligned} \tag{8.17}$$

where $c_i^j = O(s^{-2})$ and $c_\theta^j = O(s^{-3})$ and we sum over i but not j.

The second term in the definition of Z^1 is a sum of terms of the form

$$h\,\psi \circ V\,\alpha^{-1/2}\,\Theta_1. \tag{8.18}$$

The relevant h's are

$$h_1 = \frac{1}{4}\left(\lambda_t - 2\frac{\alpha_t}{\alpha} - 4s^{1/2}e^{\lambda/2}\Lambda\right), \quad h_2 = -\frac{1}{2}P_t, \quad h_3 = \frac{1}{2}\alpha^{1/2}P_\theta,$$
$$h_4 = -e^P Q_t, \quad h_5 = \alpha^{1/2}e^P Q_\theta,$$

and the relevant ψ's are

$$\psi_1 = V^0, \quad \psi_2 = V^0\frac{(V^2)^2 - (V^3)^2}{(V^0)^2 - (V^1)^2}, \quad \psi_3 = V^1\frac{(V^2)^2 - (V^3)^2}{(V^0)^2 - (V^1)^2},$$
$$\psi_4 = \frac{V^0V^2V^3}{(V^0)^2 - (V^1)^2}, \quad \psi_5 = \frac{V^1V^2V^3}{(V^0)^2 - (V^1)^2}.$$

We want to differentiate (8.18) with respect to s and then N times with respect to ϑ. Before going into the details, let us record the following estimate:

$$s^{1/2}|(\partial_\vartheta^j h_1)(s;t,\theta,v)| + \sum_{i=2}^{5}|(\partial_\vartheta^j h_i)(s;t,\theta,v)| \le C_N s^{-3/2}(\ln t)^{m_N} \tag{8.19}$$

for $0 \le j \le N$, $(t, v, \theta) \in [t_1, \infty) \times \mathbb{S}^1 \times \mathbb{R}^3$ in the support of f and $s \in [t_1, t]$. In the case of h_i, $i = 2, \ldots, 5$, (8.19) is an immediate consequence of the inductive hypothesis, (5.9), (8.1) and Lemma 8.1, and in the case of h_1, it is a consequence of (8.6). We also have

$$\begin{aligned} |\psi_1(s;t,\theta,v)| + s\sum_{i=2}^{5}|\psi_i(s;t,\theta,v)| &\le C,\\ \sum_{i=1}^{5}|(\partial_\theta^{j+1}\psi_i \circ V)(s;t,\theta,v)| &\le C_j s^{-1}(\ln t)^{m_j} \end{aligned} \tag{8.20}$$

for $0 \le j \le N-1$, $(t, v, \theta) \in [t_1, \infty) \times \mathbb{S}^1 \times \mathbb{R}^3$ in the support of f and $s \in [t_1, t]$; this is an immediate consequence of the inductive hypothesis.

Let us consider the term that arises when d/ds hits the ψ-factor in (8.18). Note that

$$\left|\partial_\vartheta^j\left(\frac{dV^i}{ds}\right)\right| \le C_N s^{-3/2}(\ln t)^{m_j}$$

for all $i = 1, 2, 3$ and all $0 \le j \le N$; for $j = 0$, the estimate is a consequence of (6.14); for $j \ge 1$ and $i = 1$, it is a consequence of (8.17); and in the case of $i = 2, 3$, it follows immediately from (4.3), (4.4) and the induction hypothesis. Hence

$$\partial_\vartheta^N\left(h\frac{d\psi \circ V}{ds}\alpha^{-1/2}\Theta_1\right) = c_\theta^N\Theta_{N+1} + O[s^{-2}(\ln t)^{m_j}],$$

where $c_\theta^N = O(s^{-2})$ and h is one of $h_1, \ldots, h_5$. When the s-derivative hits the remaining terms in (8.18) (not ψ), we obtain

$$\partial_\vartheta^N\left[\left(\partial_t(\alpha^{-1/2}h)+h_\theta\frac{V^1}{V^0}\right)\psi\Theta_1+h\psi\frac{\partial V^1}{V^0}-h\psi\frac{V^1}{(V^0)^3}\sum_{i=1}^3 V^i\partial V^i\right]$$

(cf. (6.16)). Due to (8.19) and (8.20), this expression can be written

$$\partial_\vartheta^N\left[\left(\partial_t(\alpha^{-1/2}h)+h_\theta\frac{V^1}{V^0}\right)\psi\Theta_1\right]+\sum_{i=1}^3 c_i^N V_{N+1}^i+O[s^{-2}(\ln t)^{m_N}],$$

where $c_i^N=O(s^{-2})$. Differentiating the second term in the definition of Z^1 once with respect to s and N times with respect to ϑ, we obtain (by adding up the above)

$$\sum_{i=1}^5\partial_\vartheta^N\left[\left(\partial_t(\alpha^{-1/2}h_i)+(\partial_\theta h_i)\frac{V^1}{V^0}\right)\psi_i\Theta_1\right]+\sum_{i=1}^3 c_i^N V_{N+1}^i+c_\theta^N\Theta_{N+1}+O[s^{-2}(\ln t)^{m_N}], \tag{8.21}$$

where $c_\theta^N=O(s^{-2})$ and $c_i^N=O(s^{-2})$. In order to obtain the desired equation we need to add this expression to (8.17) with $j=N$. However, before doing so, note that

$$-\tfrac{1}{4}\partial_\vartheta^{N+1}(\alpha^{1/2}\lambda_\theta)V^0=-\tfrac{1}{4}\partial_\vartheta^N[\partial_\theta(\alpha^{1/2}\lambda_\theta)V^0\Theta_1]+O[s^{-2}(\ln t)^{m_N}]$$

etc. Due to this observation, we can argue as in the proof of Lemma 6.1. In particular, we obtain a formula analogous to (6.17): the difference is that ∂V^1 should be replaced by V_{N+1}^1 in the first term on the right hand side of (6.17); that ∂_ϑ^N should be applied to all but the first and last two terms on the right hand side of (6.17); and that the last two terms should be replaced by ones analogous to the last three terms on the right hand side of (8.21). Proceeding as in the proof of Lemma 6.1, the corresponding expression can be simplified (cf. the derivation of (6.18)). Most of the steps involved in the derivation of (6.18) consist of algebraic manipulations. However, there are two exceptions. The combination of the fourth last and fifth last terms on the right hand side of (6.17) can be written

$$e^P(P_tQ_\theta-P_\theta Q_t)\frac{V^1V^2V^3}{(V^0)^2-(V^1)^2}\Theta_1.$$

The analogous expression in the present setting is

$$\partial_\vartheta^N\left(e^P(P_tQ_\theta-P_\theta Q_t)\frac{V^1V^2V^3}{(V^0)^2-(V^1)^2}\Theta_1\right)=c_\theta^N\Theta_{N+1}+O[s^{-3}(\ln t)^{m_N}],$$

where $c_\theta^N=O(s^{-3})$ and we have used (8.1), Inductive Assumption 7.3 and Lemma 8.1. The combination of the third and seventh terms on the right hand side of (6.17) can be written

$$-\partial_\theta(s^{1/2}e^{\lambda/2}\Lambda)V^1\Theta_1.$$

In the present setting, the analogous term is

$$-\partial_\vartheta^N[\partial_\theta(s^{1/2}e^{\lambda/2}\Lambda)V^1\Theta_1]=c_\theta^N\Theta_{N+1}+O[s^{-2}(\ln t)^{m_N}],$$

where $c_\theta^N=O(s^{-2})$ and we have used Inductive Assumption 7.3 and Lemma 8.1. Summing up, we obtain

$$\begin{aligned}\frac{dZ_N^1}{ds} = &-\frac{1}{2s}V_{N+1}^1 + \partial_\vartheta^N\Bigg[\frac{1}{4}\partial_t\Big[\alpha^{-1/2}\Big(\lambda_t - 2\frac{\alpha_t}{\alpha} - 4s^{1/2}e^{\lambda/2}\Lambda\Big)\Big]V^0\Theta_1\\ &-\frac{1}{4}\partial_\theta(\alpha^{1/2}\lambda_\theta)V^0\Theta_1 - \frac{1}{2}[\partial_t(\alpha^{-1/2}P_t) - \partial_\theta(\alpha^{1/2}P_\theta)]V^0\frac{(V^2)^2-(V^3)^2}{(V^0)^2-(V^1)^2}\Theta_1\\ &-[\partial_t(\alpha^{-1/2}e^PQ_t) - \partial_\theta(\alpha^{1/2}e^PQ_\theta)]\frac{V^0V^2V^3}{(V^0)^2-(V^1)^2}\Theta_1\Bigg]\\ &+c_\theta^N\Theta_{N+1} + c_i^NV_{N+1}^i + O[s^{-2}(\ln t)^{m_N}],\end{aligned}\tag{8.22}$$

where $c_\theta^N = O(s^{-2})$, $c_i^N = O(s^{-2})$ and we sum over i but not over N. The term of importance is the second one on the right hand side. If all the ϑ-derivatives hit Θ_1, the resulting term can be dealt with as in the proof of Lemma 6.1, and we obtain a $c_\theta^N\Theta_{N+1}$-term, where $c_\theta^N = O(s^{-3/2})$. For all the remaining terms, it is possible to use the equations (as in the proof of Lemma 6.1) to obtain terms of the form $O[s^{-3/2}(\ln t)^{m_N}]$. Let us go through the argument in detail for

$$\partial_\vartheta^N\Big[-\frac{1}{2}[\partial_t(\alpha^{-1/2}P_t) - \partial_\theta(\alpha^{1/2}P_\theta)]V^0\frac{(V^2)^2-(V^3)^2}{(V^0)^2-(V^1)^2}\Theta_1\Big].$$

By using (2.5), this expression can be written

$$\begin{aligned}&-\frac{1}{2}\partial_\vartheta^N\Bigg[\Bigg(-\frac{1}{s}\alpha^{-1/2}P_t + \alpha^{-1/2}e^{2P}(Q_t^2 - \alpha Q_\theta^2) + \frac{\alpha^{-1/2}e^{\lambda/2-P}J^2}{2s^{7/2}}\\ &\quad - \frac{\alpha^{-1/2}e^{\lambda/2+P}(K-QJ)^2}{2s^{7/2}} + s^{-1/2}e^{\lambda/2}\alpha^{-1/2}(P_2-P_3)\Bigg)V^0\frac{(V^2)^2-(V^3)^2}{(V^0)^2-(V^1)^2}\Theta_1\Bigg]\\ &= c_\theta^N\Theta_{N+1} + O[s^{-2}(\ln t)^{m_N}],\end{aligned}$$

where $c_\theta = O(s^{-2})$ and we have used (5.9), (8.1), Inductive Assumption 7.3 and Lemma 8.1. From this argument, and similar ones for the remaining terms in (8.22), we obtain

$$\frac{dZ_N^1}{ds} = -\frac{1}{2s}V_{N+1}^1 + c_\theta^N\Theta_{N+1} + c_i^NV_{N+1}^i + O[s^{-3/2}(\ln t)^{m_N}],$$

where $c_\theta^N = O(s^{-3/2})$, $c_i^N = O(s^{-3/2})$ and we sum over i but not over N. Due to (8.10) and (8.11), we conclude that

$$\frac{dZ_N^1}{ds} = -\frac{1}{2s}Z_N^1 + c_\theta^N\Psi_N + c_i^NZ_N^i + O[s^{-3/2}(\ln t)^{m_N}],$$

where $c_\theta^N = O(s^{-3/2})$, $c_i^N = O(s^{-3/2})$ and we sum over i but not over N. Combining this equation with (8.14)–(8.16) yields

$$\begin{aligned}\frac{d\hat{\Psi}_N}{ds} &= \frac{2}{s\ln s}\hat{\Psi}_N + c_{\theta,\theta}^N\hat{\Psi}_N + c_{\theta,1}^Ns^{-1/2}(\ln s)^2\hat{Z}_N^1 + O[s^{-2}(\ln s)^2(\ln t)^{m_N}],\\ \frac{d\hat{Z}_N^1}{ds} &= c_{1,\theta}^Ns^{1/2}(\ln s)^{-2}\hat{\Psi}_N + c_{1,1}^N\hat{Z}_N^1 + O[s^{-1}(\ln t)^{m_N}],\end{aligned}$$

where $c^N_{\theta,\theta} = O(s^{-2})$, $c^N_{\theta,1} = O(s^{-3/2})$, $c^N_{1,\theta} = O(s^{-3/2})$, $c^N_{1,1} = O(s^{-3/2})$, there is no summation over N and we have used the notation

$$\hat{Z}^1_N(s;t,\theta,v) = s^{1/2} Z^1_N(s;t,\theta,v), \qquad \hat{\Psi}_N(s;t,\theta,v) = (\ln s)^2 \Psi_N(s;t,\theta,v). \quad (8.23)$$

Introducing the energy

$$\hat{E}_N = (\hat{\Psi}_N)^2 + (\hat{Z}^1_N)^2, \quad (8.24)$$

we conclude that

$$\frac{d\hat{E}_N}{ds} \geq -\frac{C_N}{s(\ln s)^2}\hat{E}_N - C_N s^{-1}(\ln t)^{m_N} \hat{E}_N^{1/2}.$$

Letting r_N be such that $r_N(t_1) = 0$ and its derivative is the first factor in the first term on the right hand side, we obtain

$$\frac{d\mathcal{E}_N}{ds} \geq -C_N s^{-1}(\ln t)^{m_N} \mathcal{E}_N^{1/2},$$

where $\mathcal{E}_N = \exp(-r_N)\hat{E}_N$. Dividing by $\mathcal{E}_N^{1/2}$ and integrating from s to t, we obtain

$$\mathcal{E}_N^{1/2}(s;t,\theta,v) \leq \mathcal{E}_N^{1/2}(t;t,\theta,v) + C_N(\ln t)^{m_N}.$$

However, the first term on the right hand side can be estimated by $C_N(\ln t)^2$. Combining the resulting estimate with (8.10), (8.11), (8.15) and (8.16), we conclude that (7.4) holds with $j = N+1$. □

Corollary 8.4. *Consider a $\mathbb{T}^2$-symmetric solution to the Einstein–Vlasov equations with a cosmological constant $\Lambda > 0$ and existence interval (t_0, ∞), where $t_0 \geq 0$. Assume that the solution has λ-asymptotics and let $t_1 = t_0 + 2$. Let $0 \leq k \in \mathbb{Z}$. Then there is a constant C_k, depending only on k and the solution, such that*

$$\|P_t\|_{C^k} + \|Q_t\|_{C^k} + t^{1/2}\|\alpha^{1/2}\lambda_\theta\|_{C^k} \leq C_k t^{-2} \quad (8.25)$$

for all $t \geq t_1$. Moreover,

$$\left\|\frac{\alpha_t}{\alpha} + \frac{3}{t}\right\|_{C^k} + \left\|\lambda_t + \frac{3}{t}\right\|_{C^k} \leq C_k t^{-2} \quad \text{for all } t \geq t_1.$$

Proof. By combining Lemmas 7.6 and 8.3, we know that Inductive Assumption 7.3 holds for all N. In particular, the conclusions of Lemma 8.1 hold for all N. Combining this information with (2.5) and (2.12) yields

$$\|\partial_t(t\alpha^{-1/2}P_t)\|_{C^k} + \|\partial_t(t\alpha^{-1/2}Q_t)\|_{C^k} \leq C_k t^{-1/2}.$$

As a consequence,

$$\|t\alpha^{-1/2}P_t\|_{C^k} + \|t\alpha^{-1/2}Q_t\|_{C^k} \leq C_k t^{1/2}.$$

Due to this estimate, as well as (5.9) and (8.3), we can proceed inductively to get

$$\|P_t\|_{C^k} + \|Q_t\|_{C^k} \leq C_k t^{-2}.$$

Combining this estimate with (2.7), (5.9), (8.1) and Lemma 8.1, we obtain (8.25). Combining (8.25) with (2.4), (2.13) and (5.2) yields the final conclusion. □

9. Energy estimates for the distribution function

In the proof of the existence of $f_{\mathrm{sc},\infty}$ (cf. Theorem 1.7), a natural first step is to estimate L^2-based energies for f. In the process of deriving such estimates, it is useful to consider equations for the derivatives of the distribution function. Such equations take the following general form:

$$\frac{\partial h}{\partial t} + \frac{\alpha^{1/2} v^1}{v^0}\frac{\partial h}{\partial \theta} - \frac{1}{2t} v^i \frac{\partial h}{\partial v^i} = R. \tag{9.1}$$

In case $h = f$, R is given by

$$R = L^i \frac{\partial f}{\partial v^i}, \tag{9.2}$$

where

$$L^1 = \frac{1}{4}\alpha^{1/2}\lambda_\theta v^0 + \frac{1}{4}\left(\lambda_t - \frac{2\alpha_t}{\alpha} - \frac{3}{t}\right)v^1 - \alpha^{1/2}e^P Q_\theta \frac{v^2 v^3}{v^0} + \frac{1}{2}\alpha^{1/2}P_\theta \frac{(v^3)^2 - (v^2)^2}{v^0}$$
$$- t^{-7/4}e^{\lambda/4}\left(e^{-P/2}J v^2 + e^{P/2}(K - QJ)v^3\right), \tag{9.3}$$

$$L^2 = \frac{1}{2}P_t v^2 + \frac{1}{2}\alpha^{1/2}P_\theta \frac{v^1 v^2}{v^0}, \tag{9.4}$$

$$L^3 = -\frac{1}{2}P_t v^3 - \frac{1}{2}\alpha^{1/2}P_\theta \frac{v^1 v^3}{v^0} + e^P v^2\left(Q_t + \alpha^{1/2}Q_\theta \frac{v^1}{v^0}\right). \tag{9.5}$$

The energies we shall consider are

$$E_k[h](t) = \sum_{l+|\beta|\le k} \int_{\mathbb{S}^1}\int_{\mathbb{R}^3} t^{-|\beta|}|\partial_\theta^l \partial_v^\beta h(t,\theta,v)|^2 \alpha^{-1/2}t^{-3/2}\, dv\, d\theta. \tag{9.6}$$

We shall also use the notation $E = E_0$.

Remarks 9.1. The purpose of the factor $\alpha^{-1/2}t^{-3/2}$ is to simplify some of the terms that result upon carrying out partial integrations. We could equally well consider energies of the form

$$H_k[f](t) = \sum_{l+|\beta|\le k} \int_{\mathbb{S}^1}\int_{\mathbb{R}^3} t^{-|\beta|}\langle t^{1/2}v\rangle^{2\mu+2|\beta|}|\partial_\theta^l \partial_v^\beta f(t,\theta,v)|^2\, dv\, d\theta$$

for $\mu \ge 0$ (cf. [31]). However, there is a constant $C > 1$, depending only on the solution, μ and β, such that

$$C^{-1} \le \langle t^{1/2}v\rangle^{2\mu+2|\beta|} \le C$$

for $t \ge t_1$ (where t_1 is as in the statement of the previous lemmas) and (t, θ, v) in the support of f. As a consequence, the corresponding weight is of no practical importance.

Lemma 9.2. *Consider a $\mathbb{T}^2$-symmetric solution to the Einstein–Vlasov equations with a cosmological constant $\Lambda > 0$ and existence interval (t_0, ∞), where $t_0 \geq 0$. Assume that the solution has λ-asymptotics and let $t_1 = t_0 + 2$. Let h be a smooth solution to (9.1) (where the function α appears in the Einstein–Vlasov equations and R is some function) which has compact support when restricted to compact time intervals. Then there is a constant $C > 0$, depending only on the solution to the Einstein–Vlasov equations, such that*

$$\frac{dE[h]}{dt} \leq -\frac{3}{2t}E[h] + 2\int_{\mathbb{S}^1}\int_{\mathbb{R}^3} hR\alpha^{-1/2}t^{-3/2}\,dv\,d\theta + Ct^{-2}E[h]$$

for all $t \geq t_1$.

Remark 9.3. It is important to note that the constant C does not depend on h. Moreover, R should be thought of as being defined by (9.1). In particular, due to the assumptions concerning h, the function R is smooth and has compact support when restricted to compact time intervals.

Proof of Lemma 9.2. Differentiating E with respect to time, we obtain

$$\frac{dE}{dt} = 2\int_{\mathbb{S}^1}\int_{\mathbb{R}^3} h\partial_t h\alpha^{-1/2}t^{-3/2}\,dv\,d\theta + \int_{\mathbb{S}^1}\int_{\mathbb{R}^3} h^2\left(-\frac{3}{2t} - \frac{\alpha_t}{2\alpha}\right)\alpha^{-1/2}t^{-3/2}\,dv\,d\theta. \tag{9.7}$$

Due to (5.2), we can estimate the second term on the right hand side. Consider the first term; using (9.1), it can be written

$$2\int_{\mathbb{S}^1}\int_{\mathbb{R}^3} h\left(-\frac{\alpha^{1/2}v^1}{v^0}\frac{\partial h}{\partial\theta} + \frac{1}{2t}v^i\frac{\partial h}{\partial v^i} + R\right)\alpha^{-1/2}t^{-3/2}\,dv\,d\theta.$$

The term involving $\partial_\theta h$ can be integrated to zero. The term involving R we leave as it is. What remains is to estimate the term

$$\frac{1}{2t}\int_{\mathbb{S}^1}\int_{\mathbb{R}^3} v^i\frac{\partial h^2}{\partial v^i}\alpha^{-1/2}t^{-3/2}\,dv\,d\theta = -\frac{3}{2t}\int_{\mathbb{S}^1}\int_{\mathbb{R}^3} h^2\alpha^{-1/2}t^{-3/2}\,dv\,d\theta.$$

The lemma follows. □

Let us turn to higher order derivatives of the distribution function.

Lemma 9.4. *Consider a $\mathbb{T}^2$-symmetric solution to the Einstein–Vlasov equations with a cosmological constant $\Lambda > 0$ and existence interval (t_0, ∞), where $t_0 \geq 0$. Assume that the solution has λ-asymptotics and let $t_1 = t_0 + 2$. Fix $0 \leq k \in \mathbb{Z}$. Then there is a constant $C_k > 0$, depending only on k and the solution to the Einstein–Vlasov equations, such that*

$$\frac{dE_k[f]}{dt} \leq -\frac{3}{2t}E_k[f] + C_k t^{-3/2}E_k[f]$$

for all $t \geq t_1$. In particular, $t^{3/2}E_k[f]$ is bounded to the future.

Proof. Differentiating (9.1) with $h = f$, we obtain

$$\frac{\partial f_{\beta,l}}{\partial t} + \frac{\alpha^{1/2}v^1}{v^0}\frac{\partial f_{\beta,l}}{\partial \theta} - \frac{1}{2t}v^i\frac{\partial f_{\beta,l}}{\partial v^i} = \partial_v^\beta \partial_\theta^l R + \left[\frac{\alpha^{1/2}v^1}{v^0}\partial_\theta, \partial_v^\beta\partial_\theta^l\right]f - \frac{1}{2t}[v^i\partial_{v^i}, \partial_v^\beta\partial_\theta^l]f, \tag{9.8}$$

where we use the notation $f_{\beta,l} = \partial_v^\beta\partial_\theta^l f$ and assume that $|\beta| + l \leq k$. Let us denote the right hand side of (9.8) by $\mathsf{R}_{\beta,l}$. Due to Lemma 9.2, it is of interest to estimate

$$2\int_{\mathbb{S}^1}\int_{\mathbb{R}^3} t^{-|\beta|} f_{\beta,l}\mathsf{R}_{\beta,l}\alpha^{-1/2}t^{-3/2}\,dv\,d\theta. \tag{9.9}$$

By an inductive argument, it can be proven that the third term on the right hand side of (9.8) is given by $|\beta| f_{\beta,l}/2t$. The corresponding contribution to (9.9) is thus

$$\frac{|\beta|}{t}t^{-|\beta|}E[f_{\beta,l}].$$

Turning to the second term on the right hand side of (9.8), it can (up to numerical factors) be written as a sum of terms of the form

$$\partial_v^{\beta_1}\partial_\theta^{l_1}\left(\frac{\alpha^{1/2}v^1}{v^0}\right)\partial_v^{\beta_2}\partial_\theta^{l_2+1}f,$$

where $\beta_1 + \beta_2 = \beta$, $l_1 + l_2 = l$ and $|\beta_1| + l_1 \geq 1$. Note that the first factor can always be estimated by $C_k t^{-3/2}$. In case $\beta_1 = 0$, it can be estimated by $C_k t^{-2}$ (on the support of f). Due to these observations, we have

$$t^{-|\beta|/2}\left|\left[\frac{\alpha^{1/2}v^1}{v^0}\partial_\theta, \partial_v^\beta\partial_\theta^l\right]f\right| \leq C_k t^{-2}\sum_{l_1+|\beta_1|\leq k} t^{-|\beta_1|/2}|f_{\beta_1,l_1}|.$$

The corresponding contribution to (9.9) can thus be estimated by

$$C_k t^{-2}E_k[f].$$

Finally, let us consider the first term on the right hand side of (9.8). Since R is given by (9.2), the expression $\partial_v^\beta\partial_\theta^l R$ is given by the sum of

$$L^i\partial_{v^i}\partial_v^\beta\partial_\theta^l f \tag{9.10}$$

and terms which (up to numerical factors) can be written

$$(\partial_v^{\beta_1}\partial_\theta^{l_1}L^i)\partial_{v^i}\partial_v^{\beta_2}\partial_\theta^{l_2}f, \tag{9.11}$$

where $|\beta_1| + l_1 \geq 1$. The contribution to (9.9) from (9.10) can be written

$$\int_{\mathbb{S}^1}\int_{\mathbb{R}^3} t^{-|\beta|}L^i(\partial_{v^i}f_{\beta,l}^2)\alpha^{-1/2}t^{-3/2}\,dv\,d\theta.$$

Integrating by parts with respect to v^i and keeping in mind that the L^i are given by (9.3)–(9.5), we conclude that this expression can be estimated by

$$C_k t^{-3/2} E_k[f], \tag{9.12}$$

where we have used Lemma 8.1 and Corollary 8.4. Let us now consider the contribution from terms of the form (9.11). It is natural to divide them into two different categories: either $\beta_1 = 0$, or $\beta_1 \neq 0$. In case $\beta_1 = 0$, the expression (9.11) can be estimated by

$$C_k t^{-3/2} t^{-1/2} |\partial_{v^i} \partial_v^{\beta} \partial_\theta^{l_2} f|.$$

In case $\beta_1 \neq 0$, the expression (9.11) can be estimated by

$$C_k t^{-3/2} |\partial_{v^i} \partial_v^{\beta_2} \partial_\theta^{l_2} f|.$$

In order to obtain these estimates, we have appealed to Lemma 8.1 and Corollary 8.4. As a consequence, the contribution to (9.9) from terms of the form (9.11) can be estimated by (9.12). Adding up the above observations, we conclude that

$$\begin{aligned}\frac{d}{dt}(t^{-|\beta|} E[f_{\beta,l}]) &= -\frac{|\beta|}{t} t^{-|\beta|} E[f_{\beta,l}] + t^{-|\beta|} \frac{dE[f_{\beta,l}]}{dt} \\ &\leq -\frac{|\beta|}{t} t^{-|\beta|} E[f_{\beta,l}] - \frac{3}{2t} t^{-|\beta|} E[f_{\beta,l}] \\ &\quad + 2 \int_{\mathbb{S}^1} \int_{\mathbb{R}^3} t^{-|\beta|} \mathsf{R}_{\beta,l} f_{\beta,l} \alpha^{-1/2} t^{-3/2} \, dv \, d\theta + C t^{-2} t^{-|\beta|} E[f_{\beta,l}] \\ &\leq -\frac{3}{2t} t^{-|\beta|} E[f_{\beta,l}] + C_k t^{-3/2} E_k[f].\end{aligned}$$

Summing over β and l, we obtain

$$\frac{dE_k[f]}{dt} \leq -\frac{3}{2t} E_k[f] + C_k t^{-3/2} E_k[f]. \qquad \square$$

In order to obtain a better understanding of the asymptotics, it is convenient to rescale the distribution function according to

$$f_{\mathrm{sc}}(t, \theta, v) = f(t, \theta, t^{-1/2} v).$$

We have the following conclusions concerning f_{sc}.

Lemma 9.5. *Consider a $\mathbb{T}^2$-symmetric solution to the Einstein–Vlasov equations with a cosmological constant $\Lambda > 0$ and existence interval (t_0, ∞), where $t_0 \geq 0$. Assume that the solution has λ-asymptotics and let $t_1 = t_0 + 2$. Fix $0 \leq k \in \mathbb{Z}$. Then there is a constant C, depending only on the solution, such that in order for $(t, \theta, v) \in [t_1, \infty) \times \mathbb{S}^1 \times \mathbb{R}^3$ to be in the support of f_{sc}, v has to satisfy $|v| \leq C$. Moreover, there is a constant $C_k > 0$, depending only on k and the solution to the Einstein–Vlasov equations, such that*

$$\|\partial_t f_{\mathrm{sc}}(t, \cdot)\|_{C^k(\mathbb{S}^1 \times \mathbb{R}^3)} \leq C_k t^{-2} \quad \text{for all } t \geq t_1.$$

In particular, there is a smooth, non-negative function with compact support, say $f_{\mathrm{sc},\infty}$, on $\mathbb{S}^1 \times \mathbb{R}^3$ such that

$$\|f_{\mathrm{sc}}(t, \cdot) - f_{\mathrm{sc},\infty}\|_{C^k(\mathbb{S}^1 \times \mathbb{R}^3)} \leq C_k t^{-1} \quad \text{for all } t \geq t_1.$$

Proof. The statement concerning the support is an immediate consequence of Lemmas 4.1 and 4.4. In order to derive the desired estimates, let us compute

$$(\partial_t f_{\mathrm{sc}})(t,\theta,v) = (\partial_t f)(t,\theta,t^{-1/2}v) - \frac{1}{2t}t^{-1/2}v^i(\partial_{v^i} f)(t,\theta,t^{-1/2}v).$$

Using the Vlasov equation, we conclude that

$$\partial_t f_{\mathrm{sc}} = -\frac{\alpha^{1/2}t^{-1/2}v^1}{\langle t^{-1/2}v\rangle}\partial_\theta f_{\mathrm{sc}} + R_{\mathrm{sc}},$$

where

$$\begin{aligned} R_{\mathrm{sc}}(t,\theta,v) &= L^i(t,\theta,t^{-1/2}v)(\partial_{v^i} f)(t,\theta,t^{-1/2}v) \\ &= t^{1/2}L^i(t,\theta,t^{-1/2}v)(\partial_{v^i} f_{\mathrm{sc}})(t,\theta,v) \end{aligned}$$

and the L^i are defined in (9.3)–(9.5). Introducing

$$L^i_{\mathrm{sc}}(t,\theta,v) = t^{1/2}L^i(t,\theta,t^{-1/2}v),$$

we thus have

$$\partial_t f_{\mathrm{sc}} = -\frac{\alpha^{1/2}t^{-1/2}v^1}{\langle t^{-1/2}v\rangle}\partial_\theta f_{\mathrm{sc}} + L^i_{\mathrm{sc}}\partial_{v^i} f_{\mathrm{sc}}. \tag{9.13}$$

Due to the properties of the support of f_{sc}, Lemma 8.1 and Corollary 8.4, for each $0 \le k \in \mathbb{Z}$ there is a constant C_k such that

$$\sum_{l+|\beta|\le k} |(\partial_v^\beta \partial_\theta^l L^i_{\mathrm{sc}})(t,\theta,v)| \le C_k t^{-2} \tag{9.14}$$

for all $i = 1,2,3$ and $(t,\theta,v) \in [t_1,\infty)\times\mathbb{S}^1\times\mathbb{R}^3$ in the support of f_{sc}. To estimate the derivatives of f_{sc} in C^k, it is convenient to translate the estimate $E_k \le C_k t^{-3/2}$ into an estimate for f_{sc}. However,

$$\begin{aligned} \int_{\mathbb{S}^1}\int_{\mathbb{R}^3} |(\partial_\theta^l\partial_v^\beta f_{\mathrm{sc}})(t,\theta,v)|^2\,d\theta\,dv &= \int_{\mathbb{S}^1}\int_{\mathbb{R}^3} t^{-|\beta|}|(\partial_\theta^l\partial_v^\beta f)(t,\theta,t^{-1/2}v)|^2\,d\theta\,dv \\ &= t^{3/2}\int_{\mathbb{S}^1}\int_{\mathbb{R}^3} t^{-|\beta|}|(\partial_\theta^l\partial_v^\beta f)(t,\theta,v)|^2\,d\theta\,dv \\ &\le Ct^{3/2}E_k(t) \le C_k, \end{aligned}$$

assuming $l + |\beta| \le k$. From this estimate and Sobolev embedding, we conclude that all derivatives of f_{sc} are bounded for $t \ge t_1$. Combining this observation with (9.13) and (9.14), we conclude that

$$\sum_{l+|\beta|\le k} |\partial_v^\beta\partial_\theta^l\partial_t f_{\mathrm{sc}}| \le C_k t^{-2} \quad \text{for } t \ge t_1. \qquad \square$$

10. Proofs of the main theorems

Finally, we are in a position to prove the main theorems. Let us begin with Theorem 1.7.

Proof of Theorem 1.7. The conclusions concerning the distribution function are direct consequences of Lemma 9.5. Turning to H and G, we have

$$H_t = -t^{-5/2}\alpha^{-1/2}e^{P+\lambda/2}(K - QJ), \quad G_t = -QH_t - t^{-5/2}\alpha^{-1/2}e^{-P+\lambda/2}J \tag{10.1}$$

(cf. (2.2)). In order to estimate H_t and G_t, it is useful to note that P and Q are bounded in every C^k-norm for $t \geq t_0$; this follows by integrating (8.25). Combining this observation with (2.10), (2.11), (3.11) and Lemma 8.1 yields

$$\|J_t\|_{C^N} + \|K_t\|_{C^N} \leq C_N t^{-2}(\ln t)^{m_N}.$$

Thus J and K are uniformly bounded in C^N. Combining this observation with (10.1), (5.9), (3.11), the bound on P and Q in every C^N-norm and Lemma 8.1, we deduce (1.8). From Corollary 8.4, we know that (1.9) and (1.10) hold. Combining (1.10) with (3.11) and (5.9) yields (1.11). Let us turn to the second fundamental form. By definition,

$$\begin{aligned}\bar{k}_{ij} &= \bar{k}(\partial_i, \partial_j) = \langle \nabla_{\partial_i} e_0, \partial_j\rangle = \langle \nabla_{\partial_i}(t^{1/4}e^{-\lambda/4}\partial_t), \partial_j\rangle \\ &= t^{1/4}e^{-\lambda/4}\langle \nabla_{\partial_i}\partial_t, \partial_j\rangle = \tfrac{1}{2}t^{1/4}e^{-\lambda/4}\partial_t g_{ij},\end{aligned}$$

where we have used the fact that ∂_t and ∂_i are perpendicular. In what follows, we would like to prove that

$$\|\bar{k}_{ij} - \mathcal{H}\bar{g}_{ij}\|_{C^N} \leq C_N, \tag{10.2}$$

where $\mathcal{H} = (\Lambda/3)^{1/2}$. Consider the spatial components of the metric (1.1). If a time derivative hits one of P, Q, G or H in such a component, then the resulting expression is bounded in C^N after it has been multiplied by $t^{1/4}e^{-\lambda/4}/2$: this is due to (1.8), (1.9) and (1.11). As a consequence, what we need to consider are the components of the tensor field

$$\begin{aligned}&\frac{1}{2}t^{1/4}e^{-\lambda/4}\left[\left(-\frac{1}{2t} + \frac{1}{2}\lambda_t - \frac{\alpha_t}{\alpha}\right)t^{-1/2}e^{\lambda/2}\alpha^{-1}\,d\theta^2\right. \\ &\qquad\qquad \left. + e^P[dx + Q dy + (G + QH)\,d\theta]^2 + e^{-P}(dy + H\,d\theta)^2\right] \\ &\qquad = \frac{1}{2}t^{-3/4}e^{-\lambda/4}\bar{g} + \frac{1}{2}t^{-1/4}\alpha^{-1}e^{\lambda/4}\left[\frac{1}{2}\left(\lambda_t + \frac{3}{t}\right) - \left(\frac{\alpha_t}{\alpha} + \frac{3}{t}\right)\right]d\theta^2,\end{aligned}$$

where $\bar{g}$ is the spatial part of the metric. Note that the components of the second term on the right hand side are bounded in C^N: this is a consequence of (1.10) and (1.11). Moreover,

$$\tfrac{1}{2}t^{-3/4}e^{-\lambda/4} = \mathcal{H}e^{-\hat{\lambda}/4},$$

where $\hat{\lambda}$ is defined in (3.7). To prove (10.2), it is thus sufficient to demonstrate that

$$\|(e^{-\hat{\lambda}/4} - 1)\bar{g}_{ij}\|_{C^N} \leq C_N.$$

However, $t(e^{-\hat{\lambda}/4}-1)$ is bounded in C^N in view of (1.11), and $t^{-1}\bar{g}_{ij}$ is bounded in C^N due to (1.8), (1.9) and (1.11). Thus (10.2) holds. Let us define $\bar{g}_\infty$ by (1.13); note that this is a smooth Riemannian metric on $\mathbb{T}^3$. Moreover,

$$\|t^{-1}\bar{g}_{ij}(t,\cdot)-\bar{g}_{\infty,ij}\|_{C^N}\leq C_N t^{-1} \tag{10.3}$$

by (1.8), (1.9) and (1.11). Combining this estimate with (10.2), we obtain (1.12). The proof of future causal geodesic completeness is not very complicated, given the above estimates. One can, e.g., proceed as in [28, proof of Propositions 3 and 4, pp. 189–191]. However, we shall not write down the details, since the result follows from the proof of Theorem 1.35. □

Let us now turn to the proof of the cosmic no-hair conjecture.

Proof of Theorem 1.14. We need to verify that the conditions stated in Definition 1.8 are fulfilled. First, note that $\Sigma_t=\{t\}\times\mathbb{T}^3$ is a Cauchy hypersurface for each $t\in(t_0,\infty)$. An argument is required in order to justify this statement, but since the details are quite standard (cf., e.g., [30, proof of Proposition 20.3, p. 215], in particular [30, p. 217]), we omit the details. Let $\gamma=(\gamma^0,\bar{\gamma})$ be a future directed and inextendible causal curve, defined on $I_\gamma=(s_-,s_+)$. Reparametrising the curve if necessary, we can assume that $\gamma^0(s)=s$ and $I_\gamma=(t_0,\infty)$. By the causality of the curve, we know that

$$\bar{g}_{ij}[\gamma(t)]\dot{\bar{\gamma}}^i(t)\dot{\bar{\gamma}}^j(t)\leq -g_{00}[\gamma(t)]\leq Ct^{-2}$$

for $t\geq t_1$, where $t_1=t_0+2$ and we have used (1.11). Combining this estimate with (1.12), we conclude that there is a constant $K_0>1$ (independent of the curve γ, as long as $\gamma^0(t)=t$) such that

$$\bar{g}_{\infty,ij}[\bar{\gamma}(t)]\dot{\bar{\gamma}}^i(t)\dot{\bar{\gamma}}^j(t)\leq \tfrac{1}{4}K_0^2\mathcal{H}^{-2}t^{-3}$$

for all $t\geq t_1$. In particular, there is an $\bar{x}_0\in\mathbb{T}^3$ such that $d_\infty[\bar{\gamma}(t),\bar{x}_0]\leq K_0\mathcal{H}^{-1}t^{-1/2}$ for all $t\geq t_1$, where d_∞ is the topological metric on $\mathbb{T}^3$ induced by $\bar{g}_\infty$. Let $\epsilon_{\text{inj}}>0$ denote the injectivity radius of $(\mathbb{T}^3,\bar{g}_\infty)$. The injectivity radius of a point p of a Riemannian manifold, denoted $\text{inj}(p)$, is defined in [20, Definition 9.2, p. 142], and the injectivity radius of a Riemannian manifold is the infimum of the injectivity radii of the points of the manifold; that $\epsilon_{\text{inj}}>0$ follows from the continuity of inj (cf. [20, p. 178]); readers interested in a more quantitative bound on the injectivity radius are referred to [20, Lemma 51, p. 319]. Then, given $\bar{x}\in\mathbb{T}^3$, there are normal coordinates on $B_{\epsilon_{\text{inj}}}(\bar{x})$, where distances are computed using d_∞ (cf. [19, pp. 72–73] for the definition of normal coordinates). Fix $t_->K_0^2\mathcal{H}^{-2}\epsilon_{\text{inj}}^{-2}+1$ (note that t_- is independent of the curve). By the above arguments and definitions,

$$J^-(\gamma)\cap J^+(\Sigma_{t_-})\subseteq\{(t,\bar{x})\in I\times\mathbb{T}^3: t\geq t_-,\ d_\infty(\bar{x},\bar{x}_0)\leq K_0\mathcal{H}^{-1}t^{-1/2}\} \tag{10.4}$$

and the closed ball of radius $K_0\mathcal{H}^{-1}t_-^{-1/2}$ (with respect to d_∞) and centre $\bar{x}_0$ is contained in the domain of definition of normal coordinates $\bar{\mathsf{x}}$ with centre at $\bar{x}_0$. Denote the set on the right hand side of (10.4) by $D_{t_-,K_0,\bar{x}_0}$. Define

$$\psi(\tau,\bar{\xi})=[e^{2\mathcal{H}\tau},\bar{\mathsf{x}}^{-1}(\bar{\xi})].$$

Then

$$\psi^{-1}(D_{t_-,K_0,\bar{x}_0}) = \{(\tau,\bar{\xi}) \in J \times \mathbb{R}^3 : \tau \geq T_0,\ |\bar{\xi}| \leq K_0\mathcal{H}^{-1}e^{-\mathcal{H}\tau}\},$$

where $T_0 = \mathcal{H}^{-1}(\ln t_-)/2$, and $J = (\tau_0, \infty)$, where $\tau_0 = \mathcal{H}^{-1}(\ln t_0)/2$; if $t_0 = 0$, then $\tau_0 = -\infty$. If we let T be slightly smaller than T_0, and K be slightly larger than K_0, the map ψ is still defined on $C_{\Lambda,K,T}$ (cf. (1.15)). In analogy with Definition 1.8, let $D = \psi(C_{\Lambda,K,T})$ and $R(\tau) = K\mathcal{H}^{-1}e^{-\mathcal{H}\tau}$. We have already verified all of the requirements of Definition 1.8 (with $\Sigma = \Sigma_{t_-}$ etc.) but the last one, i.e., (1.16).

In order to proceed, let $\bar{\mathrm{g}}_{\infty,ij}$ denote the components of $\bar{g}_\infty$ with respect to the coordinates $\bar{\mathrm{x}}$. Let $\bar{\mathrm{g}}_{ij}(\tau,\cdot)$ and $\bar{\mathrm{k}}_{ij}(\tau,\cdot)$ denote the components of $\bar{g}(e^{2\mathcal{H}\tau},\cdot)$ and $\bar{k}(e^{2\mathcal{H}\tau},\cdot)$, respectively, in the coordinates $\bar{\mathrm{x}}$. Moreover, consider $\bar{\mathrm{g}}_{\infty,ij}$, $\bar{\mathrm{g}}_{ij}(\tau,\cdot)$ and $\bar{\mathrm{k}}_{ij}(\tau,\cdot)$ to be functions on the image of $\bar{\mathrm{x}}$, i.e., on $B_{\epsilon_{\mathrm{inj}}}(0)$, with the origin corresponding to $\bar{x}_0$. Note that the estimates (10.2) and (10.3) hold with $\bar{g}_{ij}$ replaced by $\bar{\mathrm{g}}_{ij}$ etc., assuming the domain on which the C^N-norm is computed is suitably restricted. In particular, letting S_τ be as in Definition 1.8, we have

$$\|e^{-2\mathcal{H}\tau}\bar{\mathrm{k}}_{ij}(\tau,\cdot) - \mathcal{H}\bar{\mathrm{g}}_{\infty,ij}\|_{C^N(S_\tau)} + \|e^{-2\mathcal{H}\tau}\bar{\mathrm{g}}_{ij}(\tau,\cdot) - \bar{\mathrm{g}}_{\infty,ij}\|_{C^N(S_\tau)} \leq C_N e^{-2\mathcal{H}\tau}$$

for all $\tau \geq T$. Note that

$$\bar{\mathrm{g}}_{\infty,ij}(0) = \delta_{ij}, \qquad (\partial_l\bar{\mathrm{g}}_{\infty,ij})(0) = 0$$

by the definition of the coordinates $\bar{\mathrm{x}}$. As a consequence, if $\bar{\xi} \in S_\tau$, then

$$|(\partial_l\bar{\mathrm{g}}_{\infty,ij})(\bar{\xi})| = \left|\int_0^1 \frac{d}{ds}[(\partial_l\bar{\mathrm{g}}_{\infty,ij})(s\bar{\xi})]\,ds\right| \leq Ce^{-\mathcal{H}\tau}.$$

Moreover,

$$|\bar{\mathrm{g}}_{\infty,ij}(\bar{\xi}) - \delta_{ij}| \leq Ce^{-2\mathcal{H}\tau}$$

for $\tau \geq T$ and $\bar{\xi} \in S_\tau$. In particular,

$$\|e^{-2\mathcal{H}\tau}\bar{\mathrm{k}}_{ij}(t,\cdot) - \mathcal{H}\delta_{ij}\|_{C^0(S_\tau)} + \|e^{-2\mathcal{H}\tau}\bar{\mathrm{g}}_{ij}(t,\cdot) - \delta_{ij}\|_{C^0(S_\tau)} \leq Ce^{-2\mathcal{H}\tau}$$

for all $\tau \geq T$. Letting $\bar{g}_{\mathrm{dS}}(\tau,\cdot)$ and $\bar{k}_{\mathrm{dS}}(\tau,\cdot)$ be defined as in Definition 1.8, we conclude in particular that

$$\|\bar{g}_{\mathrm{dS}}(\tau,\cdot) - \bar{g}(\tau,\cdot)\|_{C^0_{\mathrm{dS}}(S_\tau)} + \|\bar{k}_{\mathrm{dS}}(\tau,\cdot) - \bar{k}(\tau,\cdot)\|_{C^0_{\mathrm{dS}}(S_\tau)} \leq Ce^{-2\mathcal{H}\tau}$$

for all $\tau \geq T$. In fact, due to the above estimates, we have

$$\|\bar{g}_{\mathrm{dS}}(\tau,\cdot) - \bar{g}(\tau,\cdot)\|_{C^N_{\mathrm{dS}}(S_\tau)} + \|\bar{k}_{\mathrm{dS}}(\tau,\cdot) - \bar{k}(\tau,\cdot)\|_{C^N_{\mathrm{dS}}(S_\tau)} \leq C_N e^{-2\mathcal{H}\tau}$$

for all $\tau \geq T$. □

Finally, we are in a position to prove Theorem 1.35.

Proof Theorem 1.35. The idea is to demonstrate that for late enough t, there is a neighbourhood of each point in $\{t\} \times \mathbb{T}^3$ such that Theorem 1.29 applies in the neighbourhood; combining this with Cauchy stability (cf. Theorem 1.34) then yields the desired result.

Fixing N, there is an $\epsilon > 0$ and a constant C_N such that for every $\bar{x} \in \mathbb{T}^3$, there are normal coordinates $\bar{\mathsf{x}}$ on $W = B_\epsilon(\bar{x})$ with respect to $\bar{g}_\infty$, where distances on $\mathbb{T}^3$ are measured using the topological metric induced by $\bar{g}_\infty$. Moreover, if $\bar{g}_{\infty,ij}$ are the components of $\bar{g}_\infty$ with respect to $\bar{\mathsf{x}}$, and $\bar{g}_\infty^{ij}$ are the components of the inverse, then the derivatives of $\bar{g}_{\infty,ij}$ and $\bar{g}_\infty^{ij}$ up to order N with respect to the coordinates $\bar{\mathsf{x}}$ on W are bounded by C_N. Moreover, the derivatives of $\bar{\mathsf{x}}$, considered as functions of (θ, x, y), up to order $N+1$ are bounded by C_N. Similarly, the derivatives of $\bar{\mathsf{x}}^{-1}$ up to order $N+1$ are bounded by C_N. The arguments required to prove the above statements are similar to those in [31, proof of Lemma 34.9, p. 650]. The important point is that we obtain uniform bounds which hold regardless of the base point.

Define K by the condition $e^K = 4/\mathcal{H}$ and define the coordinates $\bar{\mathsf{y}} = e^{-K}t^{1/2}\bar{\mathsf{x}}$ on W. Note that the range of $\bar{\mathsf{y}}$ is $B_{e^{-K}t^{1/2}\epsilon}(0)$. For t large enough (the bound being independent of the base point $\bar{x}$), we then have $e^{-K}t^{1/2}\epsilon > 1$. From now on, we assume that t is large enough for this to be the case. Moreover, we assume that the coordinates $\bar{\mathsf{y}}$ are defined on the image of $B_1(0)$ under $\bar{\mathsf{y}}^{-1}$. Let $\bar{\mathsf{g}}_{ij}$ denote the components of $\bar{g}(t,\cdot)$ with respect to the coordinates $\bar{\mathsf{y}}$. Moreover, let $\bar{\mathsf{g}}_{\infty,ij}$ denote the components of $\bar{g}_\infty$ with respect to the coordinates $\bar{\mathsf{x}}$. Due to (10.3), we have

$$|\partial_{\bar{\xi}}^{\alpha}[(e^{-2K}\bar{\mathsf{g}}_{ij} - \bar{\mathsf{g}}_{\infty,ij}) \circ \bar{\mathsf{y}}^{-1}](\bar{\xi})| \leq C_N t^{-1-|\alpha|/2}$$

for $\bar{\xi} \in B_1(0)$ and $|\alpha| \leq N$; note that

$$\bar{\mathsf{y}}^{-1}(\bar{\xi}) = \bar{\mathsf{x}}^{-1}(e^K t^{-1/2}\bar{\xi}).$$

Since $\bar{\mathsf{g}}_{\infty,ij} \circ \bar{\mathsf{y}}^{-1}(0) = \delta_{ij}$,

$$|e^{-2K}\bar{\mathsf{g}}_{ij} \circ \bar{\mathsf{y}}^{-1} - \delta_{ij}| \leq C_N t^{-1/2} \tag{10.5}$$

on $B_1(0)$; in particular, (1.38) holds with a margin for t large enough. Similarly,

$$|\partial_m(\bar{\mathsf{g}}_{\infty,ij} \circ \bar{\mathsf{y}}^{-1})(u)| = \left|\sum_{l=1}^{n}\int_0^1 \partial_l\partial_m(\bar{\mathsf{g}}_{\infty,ij} \circ \bar{\mathsf{y}}^{-1})(su)u^l\,ds\right| \leq Ct^{-1}$$

on $B_1(0)$. Thus

$$\|e^{-2K}(\partial_k\bar{\mathsf{g}}_{ij} \circ \bar{\mathsf{y}}^{-1})\|_{C^{N-1}[B_1(0)]} \leq C_N t^{-1}. \tag{10.6}$$

Due to (10.2), we also have

$$\|(\bar{\mathsf{k}}_{ij} - \mathcal{H}\bar{\mathsf{g}}_{ij}) \circ \bar{\mathsf{y}}^{-1}\|_{C^N[B_1(0)]} \leq C_N t^{-1}, \tag{10.7}$$

where $\bar{\mathsf{k}}_{ij}$ denotes the components of the second fundamental form $\bar{k}$ calculated using the coordinates $\bar{\mathsf{y}}$. In the end, we shall choose $K_{\text{Vl}} = (\ln t)/2$. As a consequence, (10.6) and (10.7) imply that (1.39) holds with a margin (note that the $\mathbb{T}^2$-symmetric background solution is such that $\bar{\phi}_i$, $i = 0, 1$, vanish), assuming $N \geq 5$; note that in order to prove Theorem 1.35, it is sufficient to apply Theorem 1.29 with $k_0 = 4$.

Let us turn to the distribution function. First of all, recall that if Σ is a spacelike hypersurface in a Lorentz manifold, and $\mathfrak{f}$ is a distribution function defined on the mass shell, then the initial datum for the distribution function (denoted $\bar{\mathfrak{f}}$ and defined on $T\Sigma$) induced by $\mathfrak{f}$ on Σ is given by

$$\bar{\mathfrak{f}} = \mathfrak{f} \circ \mathrm{pr}_\Sigma^{-1},$$

where pr_Σ is the projection from the mass shell over Σ to $T\Sigma$; in other words, if $p \in \mathcal{P}_r$ for some $r \in \Sigma$ and N_r is the future directed unit normal to Σ at r, then $\mathrm{pr}_\Sigma(p)$ is the element of $T_r\Sigma$ corresponding to $p + g(p, N_r)N_r$. In our case, we are interested in the hypersurface $\Sigma = \{t\} \times \mathbb{T}^3$. If $z = (t, \theta, x, y)$, then

$$\bar{\mathfrak{f}}(\bar{p}^i e_i|_z) = \mathfrak{f}(p^\alpha e_\alpha|_z) = f(t, \theta, \bar{p}),$$

where $\bar{p} = (\bar{p}^1, \bar{p}^2, \bar{p}^3)$,

$$p^0 = [1 + (\bar{p}^1)^2 + (\bar{p}^2)^2 + (\bar{p}^3)^2]^{1/2}$$

and $p^i = \bar{p}^i$. However, in the application of Theorem 1.29, we need to express $\bar{\mathfrak{f}}$ in the coordinates $\bar{\mathrm{y}}$. Consequently, we are interested in

$$\bar{f}(\bar{z}, \bar{p}) = \bar{\mathfrak{f}}(\bar{p}^i \partial_{\bar{\mathrm{y}}^i}|_z) = \bar{\mathfrak{f}}(\bar{p}^i A_i{}^j(\bar{z}) e_j|_z) = f[t, \theta, v(\bar{z}, \bar{p})],$$

where $z = (t, \bar{z})$,

$$v(\bar{z}, \bar{p}) = (\bar{p}^i A_i{}^1(\bar{z}), \bar{p}^i A_i{}^2(\bar{z}), \bar{p}^i A_i{}^3(\bar{z}))$$

and $A_i{}^j$ is defined by the requirement that

$$\partial_{\bar{\mathrm{y}}^i}|_z = A_i{}^j(\bar{z}) e_j|_z.$$

Thus

$$A_i{}^j(\bar{z}) = \langle \partial_{\bar{\mathrm{y}}^i}|_z, e_j|_z \rangle = e^K t^{-1/2} \langle \partial_{\bar{\mathrm{x}}^i}|_z, e_j|_z \rangle = e^K t^{-1/2} \frac{\partial \bar{\mathrm{z}}^l}{\partial \bar{\mathrm{x}}^i}(z) \langle \partial_{\bar{\mathrm{z}}^l}|_z, e_j|_z \rangle,$$

where $\bar{\mathrm{z}}$ correspond to the standard coordinates on the torus (which are locally well defined). In particular, $\partial_{\bar{\mathrm{z}}^1} = \partial_\theta$, $\partial_{\bar{\mathrm{z}}^2} = \partial_x$ and $\partial_{\bar{\mathrm{z}}^3} = \partial_y$. By the observations at the beginning of the proof, (1.7) and Theorem 1.7, it is clear that all derivatives of $A_i{}^j$ up to order N are uniformly bounded on the domain of $\bar{\mathrm{y}}$, the bound being independent of the base point $\bar{x}$ and time t (assuming t is sufficiently large). What we need to estimate is

$$\sum_{|\alpha|+|\beta| \le k_0} \int_{\mathbb{R}^3} \int_{\bar{\mathrm{y}}(U)} (e^{-w})^{2|\beta|} \langle e^w \bar{p} \rangle^{2\mu + 2|\beta|} |\partial_{\bar{\xi}}^\alpha \partial_{\bar{p}}^\beta \bar{\mathfrak{f}}_{\bar{\mathrm{y}}}|^2 (\bar{\xi}, \bar{p})\, d\bar{\xi}\, d\bar{p}$$

(cf. (1.37) and (1.40)), where

$$\begin{aligned} \bar{\mathfrak{f}}_{\bar{\mathrm{y}}}(\bar{\xi}, \bar{p}) &= \bar{f}[\bar{\mathrm{y}}^{-1}(\bar{\xi}), \bar{p}] = \bar{f}[\bar{\mathrm{x}}^{-1}(e^K t^{-1/2} \bar{\xi}), \bar{p}] \\ &= f\big[t, \bar{\mathrm{z}}^1 \circ \bar{\mathrm{x}}^{-1}(e^K t^{-1/2} \bar{\xi}), v(\bar{\mathrm{x}}^{-1}(e^K t^{-1/2} \bar{\xi}), \bar{p})\big] \end{aligned}$$

and the constant w remains to be specified. Note that all derivatives of $\bar{\mathsf{x}}^{-1}$ and $\bar{\mathsf{z}}^1 \circ \bar{\mathsf{x}}^{-1}$ up to order N are uniformly bounded. As a consequence,

$$\partial_{\bar{p}}^{\beta}\bar{\mathsf{f}}_{\bar{\mathsf{y}}}(\bar{\xi},\bar{p}) = \sum_{|\gamma|=|\beta|} \partial_v^{\gamma} f\big[t, \bar{\mathsf{z}}^1 \circ \bar{\mathsf{x}}^{-1}(e^K t^{-1/2}\bar{\xi}), v(\bar{\mathsf{x}}^{-1}(e^K t^{-1/2}\bar{\xi}), \bar{p})\big]\psi_\gamma(t^{-1/2}\bar{\xi})$$

for functions ψ_γ with bounded derivatives; note that

$$\frac{\partial v^i}{\partial \bar{p}^j}(\bar{\mathsf{x}}^{-1}(e^K t^{-1/2}\bar{\xi}), \bar{p}) = A_j{}^i \circ \bar{\mathsf{x}}^{-1}(e^K t^{-1/2}\bar{\xi}).$$

Hence, $\partial_{\bar{\xi}}^{\alpha}\partial_{\bar{p}}^{\beta}\bar{\mathsf{f}}_{\bar{\mathsf{y}}}(\bar{\xi},\bar{p})$ consists of sums of terms of the form

$$t^{-|\alpha|/2}\partial_\theta^l \partial_v^{\gamma+\delta} f\big[t, \bar{\mathsf{z}}^1 \circ \bar{\mathsf{x}}^{-1}(e^K t^{-1/2}\bar{\xi}), v(\bar{\mathsf{x}}^{-1}(e^K t^{-1/2}\bar{\xi}), \bar{p})\big]\phi_{\gamma,\delta,l}(t^{-1/2}\bar{\xi})\bar{p}^{\lambda},$$

where $|\lambda| = |\delta|$, $|\gamma| = |\beta|$, $l + |\delta| \leq |\alpha|$, $\phi_{\gamma,\delta,l}$ are bounded functions and $\bar{p}^\lambda = (\bar{p}^1)^{\lambda_1}(\bar{p}^2)^{\lambda_2}(\bar{p}^3)^{\lambda_3}$. On the other hand,

$$f(t,\theta,v) = f_{\mathrm{sc}}(t,\theta,t^{1/2}v),$$

where f_{sc} converges to a smooth function with compact support in every C^k-norm. Moreover, f_{sc} has uniformly compact support. Note also that

$$\partial_\theta^l \partial_v^{\gamma+\delta} f(t,\theta,v) = t^{(|\gamma|+|\delta|)/2}\partial_\theta^l \partial_v^{\gamma+\delta} f_{\mathrm{sc}}(t,\theta,t^{1/2}v).$$

Since there is a uniform constant $C > 1$ (independent of t (large enough) and the base point $\bar{x}$) such that

$$C^{-1}|\bar{p}| \leq |v(\bar{\mathsf{x}}^{-1}(e^K t^{-1/2}\bar{\xi}), \bar{p})| \leq C|\bar{p}|,$$

we conclude that

$$\begin{aligned} t^{-|\alpha|/2}\big|\partial_\theta^l \partial_v^{\gamma+\delta} f\big[t, \bar{\mathsf{z}}^1 \circ \bar{\mathsf{x}}^{-1}(e^K t^{-1/2}\bar{\xi}), v(\bar{\mathsf{x}}^{-1}(e^K t^{-1/2}\bar{\xi}), \bar{p})\big]\phi_{\gamma,\delta,l}(t^{-1/2}\bar{\xi})\bar{p}^{\lambda}\big| \\ \leq C_{\alpha,\beta} t^{(|\gamma|-|\alpha|)/2}\chi(t^{1/2}\bar{p}), \end{aligned}$$

where χ is a smooth function with compact support. As a consequence,

$$|\partial_{\bar{\xi}}^{\alpha}\partial_{\bar{p}}^{\beta}\bar{\mathsf{f}}_{\bar{\mathsf{y}}}(\bar{\xi},\bar{p})| \leq C_{\alpha,\beta} t^{(|\beta|-|\alpha|)/2}\chi(t^{1/2}\bar{p})$$

for $\bar{\xi} \in B_1(0)$. Let us now define $w = K + K_{\mathrm{Vl}}$, where $K_{\mathrm{Vl}} = (\ln t)/2$. Then

$$\begin{aligned} &\int_{\mathbb{R}^3}\int_{\bar{\mathsf{y}}(U)} (e^{-w})^{2|\beta|}\langle e^w \bar{p}\rangle^{2\mu+2|\beta|}|\partial_{\bar{\xi}}^{\alpha}\partial_{\bar{p}}^{\beta}\bar{\mathsf{f}}_{\bar{\mathsf{y}}}|^2(\bar{\xi},\bar{p})\,d\bar{\xi}\,d\bar{p} \\ &\quad \leq C_{\alpha,\beta}^2 \int_{\mathbb{R}^3}\int_{B_1(0)} e^{-2|\beta|K} t^{-|\beta|}\langle e^K t^{1/2}\bar{p}\rangle^{2\mu+2|\beta|} t^{|\beta|-|\alpha|}\chi^2(t^{1/2}\bar{p})\,d\bar{\xi}\,d\bar{p} \\ &\quad \leq C_{\mu,\alpha,\beta} t^{-|\alpha|-3/2}. \end{aligned}$$

The square root of the right hand side of this expression should be compared with the right hand side of (1.40):

$$\mathcal{H}^2\varepsilon^{5/2}e^{-3K/2-K_{\mathrm{VI}}} = \mathcal{H}^2\varepsilon^{5/2}e^{-3K/2}t^{-1/2}.$$

Clearly, we have a margin. As a consequence, for t large enough, every $\bar{x} \in \mathbb{T}^3$ has a neighbourhood such that (1.40) holds with a margin. From (10.5)–(10.7), we also know that (1.38) and (1.39) hold with a margin. We can thus apply Theorem 1.29 with the given $\mu > 5/2$ and $k_0 = 4$. In addition, the covering of $\mathbb{T}^3$ obtained by taking the neighbourhoods $\bar{y}^{-1}[B_{1/4}(0)]$ corresponding to varying base points $\bar{x}$ has a finite subcovering. Appealing to Cauchy stability, Theorem 1.34, we conclude that there is an $\epsilon > 0$ with the properties stated in the theorem. □

Appendix A. Derivation of the equations

The purpose of this appendix is to compute the Einstein tensor associated with the metric (1.1), and to derive an expression for the Vlasov equation. Let us begin by expressing g in terms of suitable one-form fields. Let

$$\begin{aligned}
\xi^0 &= t^{-1/4}e^{\lambda/4}dt,\\
\xi^1 &= t^{-1/4}e^{\lambda/4}\alpha^{-1/2}\,d\theta,\\
\xi^2 &= t^{1/2}e^{P/2}(dx + Qdy + (G + QH)\,d\theta),\\
\xi^3 &= t^{1/2}e^{-P/2}(dy + H\,d\theta).
\end{aligned}$$

With these one-form fields, the metric can be written

$$g = -\xi^0 \otimes \xi^0 + \sum_{i=1}^{3} \xi^i \otimes \xi^i.$$

Using the orthonormal frame $\{e_\alpha\}$ introduced in (1.7), it can be verified that $\xi^\alpha(e_\beta) = \delta^\alpha_\beta$.

A.1. *Commutators*

Let us compute the commutators, in other words the functions $\gamma^\alpha_{\beta\zeta}$ such that

$$[e_\beta, e_\zeta] = \gamma^\alpha_{\beta\zeta}e_\alpha.$$

Clearly, $\gamma^\alpha_{\beta\zeta} = -\gamma^\alpha_{\zeta\beta}$. Consequently, it is sufficient to compute $\gamma^\alpha_{\beta\zeta}$ for $\beta < \zeta$. By a straightforward computation,

$$\gamma^0_{01} = \tfrac{1}{4}t^{1/4}e^{-\lambda/4}\alpha^{1/2}\lambda_\theta, \tag{A.1}$$

$$\gamma^1_{01} = -\tfrac{1}{4}t^{1/4}e^{-\lambda/4}(\lambda_t - 2\alpha_t/\alpha - t^{-1}), \tag{A.2}$$

$$\gamma^2_{01} = t^{-3/2}e^{-P/2}J, \tag{A.3}$$

$$\gamma^3_{01} = t^{-3/2}e^{P/2}(K - QJ), \tag{A.4}$$

where we have used the notation (2.2). Turning to γ_{02}^{α}, the only α for which this object is non-zero is $\alpha = 2$, and we have

$$\gamma_{02}^{2} = -\tfrac{1}{2} t^{1/4} e^{-\lambda/4} (t^{-1} + P_t).$$

The only α's for which γ_{03}^{α} is non-zero are $\alpha = 2$ and $\alpha = 3$, and

$$\gamma_{03}^{2} = -t^{1/4} e^{-\lambda/4} e^{P} Q_t, \qquad \gamma_{03}^{3} = -\tfrac{1}{2} t^{1/4} e^{-\lambda/4} (t^{-1} - P_t).$$

The only α for which γ_{12}^{α} is non-zero is $\alpha = 2$, and

$$\gamma_{12}^{2} = -\tfrac{1}{2} t^{1/4} e^{-\lambda/4} \alpha^{1/2} P_\theta.$$

The only α's for which γ_{13}^{α} is non-zero are $\alpha = 2$ and $\alpha = 3$, and

$$\gamma_{13}^{2} = -t^{1/4} e^{-\lambda/4} \alpha^{1/2} e^{P} Q_\theta, \qquad \gamma_{13}^{3} = \tfrac{1}{2} t^{1/4} e^{-\lambda/4} \alpha^{1/2} P_\theta.$$

Finally, $\gamma_{23}^{\alpha} = 0$. For future reference, let us record the following observations:

$$\begin{aligned} \gamma_{0A}^{A} &= -t^{-3/4} e^{-\lambda/4}, & \gamma_{1A}^{A} &= 0, \\ \gamma_{0i}^{i} &= \gamma_{01}^{1} - t^{-3/4} e^{-\lambda/4}, & \gamma_{1i}^{i} &= 0, \\ \gamma_{0\alpha}^{\alpha} &= \gamma_{01}^{1} - t^{-3/4} e^{-\lambda/4}, & \gamma_{1\alpha}^{\alpha} &= -\gamma_{01}^{0}, \end{aligned}$$

where Greek indices range from 0 to 3, lower case Latin indices range from 1 to 3 and capital Latin indices range from 2 to 3; moreover, Einstein's summation convention is in force.

Non-zero components. Note that for $\gamma_{\beta\zeta}^{\alpha}$ to be non-zero, one of the following conditions has to be satisfied:

- $\{\beta, \zeta\} = \{0, 1\}$,
- one of β and ζ is in $\{0, 1\}$, the other is in $\{2, 3\}$, and α is in $\{2, 3\}$.

A.2. Connection coefficients

Define the connection coefficients $\Gamma_{\beta\zeta}^{\alpha}$ by the relation

$$\nabla_{e_\beta} e_\zeta = \Gamma_{\beta\zeta}^{\alpha} e_\alpha,$$

where ∇ is the Levi-Civita connection associated with the metric g. Note that

$$\Gamma_{\beta\zeta}^{0} = -\langle \nabla_{e_\beta} e_\zeta, e_0 \rangle, \qquad \Gamma_{\beta\zeta}^{i} = \langle \nabla_{e_\beta} e_\zeta, e_i \rangle,$$

since the frame is orthonormal. Let us record some symmetries of these objects.

Symmetries of connection coefficients. Since the connection is metric and the basis is orthonormal, we have

$$\Gamma_{\beta\alpha}^{\alpha} = 0 \quad (\text{no summation over } \alpha).$$

For similar reasons, $\Gamma_{\alpha j}^{i}$ is antisymmetric in i and j. Moreover, since $[e_i, e_j]$ is perpendicular to e_0, we have

$$\Gamma_{j0}^{i} = \langle \nabla_{e_j} e_0, e_i \rangle = -\langle e_0, \nabla_{e_j} e_i \rangle = -\langle e_0, \nabla_{e_i} e_j \rangle = \langle \nabla_{e_i} e_0, e_j \rangle = \Gamma_{i0}^{j}.$$

Thus, Γ^i_{j0} is symmetric in i and j. Note that the computation also shows that $\Gamma^i_{j0} = \Gamma^0_{ij}$. Similarly, since $[e_A, e_B] = 0$, we have

$$\Gamma^A_{B1} = \langle \nabla_{e_B} e_1, e_A \rangle = -\langle e_1, \nabla_{e_B} e_A \rangle = -\langle e_1, \nabla_{e_A} e_B \rangle = \langle \nabla_{e_A} e_1, e_B \rangle = \Gamma^B_{A1}.$$

In particular, Γ^A_{B1} is symmetric in A and B, and $\Gamma^A_{B1} = -\Gamma^1_{AB}$.

Connection coefficients including two or more zero indices. That $\Gamma^0_{\alpha 0} = 0$ follows from the above. Moreover, using the Koszul formula, it can be computed that

$$\Gamma^0_{01} = \Gamma^1_{00} = \gamma^0_{01}$$

and the remaining components satisfy $\Gamma^0_{0A} = \Gamma^A_{00} = 0$.

Connection coefficients including exactly one zero index. As already mentioned, Γ^i_{0j} is antisymmetric and $\Gamma^i_{j0} = \Gamma^0_{ij}$ is symmetric. It is thus sufficient to compute Γ^i_{0j} for $i < j$ and Γ^i_{j0} for $i \le j$. We have

$$\Gamma^1_{0A} = -\tfrac{1}{2}\gamma^A_{01}, \qquad \Gamma^2_{03} = \tfrac{1}{2}\gamma^2_{03}.$$

Moreover,

$$\Gamma^i_{i0} = -\gamma^i_{0i} \quad (\text{no summation over } i), \qquad \Gamma^1_{A0} = -\tfrac{1}{2}\gamma^A_{01}, \qquad \Gamma^2_{30} = -\tfrac{1}{2}\gamma^2_{03}.$$

Connection coefficients including no zero index. Note that, due to the Koszul formula and the properties of the commutators, the only Γ^i_{jk}'s which are non-zero are the ones that have one index equalling 1 and two indices in $\{2, 3\}$. Moreover, since $\Gamma^B_{A1} = -\Gamma^1_{AB}$ is symmetric and Γ^A_{1B} is antisymmetric, it is sufficient to calculate that

$$\Gamma^2_{13} = \tfrac{1}{2}\gamma^2_{13}, \qquad \Gamma^A_{A1} = -\gamma^A_{1A} \quad (\text{no summation over } A), \qquad \Gamma^2_{31} = -\tfrac{1}{2}\gamma^2_{13}.$$

For future reference, note that

$$\Gamma^\alpha_{\alpha 1} = \gamma^0_{01}, \tag{A.5}$$

$$\Gamma^i_{i1} = 0, \tag{A.6}$$

$$\Gamma^\alpha_{\alpha 0} = -\gamma^1_{01} + t^{-3/4}e^{-\lambda/4}, \tag{A.7}$$

$$\Gamma^\alpha_{\alpha A} = 0. \tag{A.8}$$

A.3. *Twist quantities*

The quantities J and K have been defined in two different ways: in (1.2) and in (2.2). In this subsection, we verify that these two definitions yield the same result. In the proof, it is useful to introduce different notation for the different definitions. Let us therefore denote the J and K defined in (1.2) by J_{tw} and K_{tw} respectively, while the quantities defined in (2.2) will be still referred to as J and K. As we have calculated the connection coefficients using the orthonormal frame $\{e_\alpha\}$, it is convenient to carry out the computations relative to this frame. Note that

$$X = t^{1/2}e^{P/2}e_2, \qquad Y = t^{1/2}e^{-P/2}e_3 + t^{1/2}e^{P/2}Qe_2,$$

where $X = \partial_x$ and $Y = \partial_y$. In particular, if X^α and Y^α denote the components of X and Y with respect to the frame $\{e_\alpha\}$, then

- $X^2 = t^{1/2}e^{P/2}$ and $X^\alpha = 0$ for $\alpha \neq 2$,
- $Y^2 = t^{1/2}e^{P/2}Q$, $Y^3 = t^{1/2}e^{-P/2}$ and $Y^0 = Y^1 = 0$.

Consequently,

$$\begin{aligned} J_{\mathrm{tw}} &= \epsilon_{\alpha\beta\zeta\delta}X^\alpha Y^\beta \nabla^\zeta X^\delta = \epsilon_{23\zeta\delta}X^2 Y^3 \nabla^\zeta X^\delta = t(\nabla^0 X^1 - \nabla^1 X^0) \\ &= -t(\nabla_0 X^1 + \nabla_1 X^0), \end{aligned} \tag{A.9}$$

where the indices are frame indices, and we assume that the orientation of M is such that $\epsilon_{0123} = 1$. Similarly,

$$K_{\mathrm{tw}} = -t(\nabla_0 Y^1 + \nabla_1 Y^0). \tag{A.10}$$

It remains to calculate $\nabla_0 X^1$ etc. However,

$$\nabla_\alpha X^\beta = \xi^\beta(\nabla_{e_\alpha}X) = \xi^\beta[e_\alpha(X^\zeta)e_\zeta + X^\zeta \nabla_{e_\alpha}e_\zeta] = e_\alpha(X^\beta) + X^\zeta \Gamma^\beta_{\alpha\zeta}.$$

The calculation for Y is the same. In particular,

$$\nabla_0 X^1 + \nabla_1 X^0 = X^\zeta(\Gamma^1_{0\zeta} + \Gamma^0_{1\zeta}) = X^2(\Gamma^1_{02} + \Gamma^0_{12}) = -X^2\gamma^2_{01}.$$

Combining this with (A.3) and (A.9) yields

$$J_{\mathrm{tw}} = tX^2\gamma^2_{01} = t^{3/2}e^{P/2}t^{-3/2}e^{-P/2}J = J.$$

Next, let us calculate

$$\nabla_0 Y^1 + \nabla_1 Y^0 = Y^\zeta(\Gamma^1_{0\zeta} + \Gamma^0_{1\zeta}) = Y^2(\Gamma^1_{02} + \Gamma^0_{12}) + Y^3(\Gamma^1_{03} + \Gamma^0_{13}) = -Y^2\gamma^2_{01} - Y^3\gamma^3_{01}.$$

Combining this with (A.3), (A.4) and (A.10) yields

$$K_{\mathrm{tw}} = tY^2\gamma^2_{01} + tY^3\gamma^3_{01} = t^{3/2}e^{P/2}Qt^{-3/2}e^{-P/2}J + t^{3/2}e^{-P/2}t^{-3/2}e^{P/2}(K - QJ) = K.$$

A.4. Auxiliary computations

To simplify future calculations, let us make some observations concerning the derivatives of the connection coefficients. Note first that

$$\gamma^i_{0i} = -t^{1/4}e^{-\lambda/4}f_i$$

for suitably chosen functions f_i. Thus, it can be computed that

$$\begin{aligned} -e_0(\gamma^i_{0i}) &= -t^{1/4}e^{-\lambda/4}\partial_t\left(-\exp\left(-\frac{1}{4}\lambda + \frac{1}{2}\ln\alpha - \frac{3}{4}\ln t\right)t\alpha^{-1/2}f_i\right) \\ &= \frac{1}{4}t^{1/4}e^{-\lambda/4}\left(\lambda_t - 2\frac{\alpha_t}{\alpha} - \frac{1}{t}\right)\gamma^i_{0i} + t^{1/4}e^{-\lambda/4}t^{-1}\gamma^i_{0i} + t^{-1/2}e^{-\lambda/2}\alpha^{1/2}\partial_t(t\alpha^{-1/2}f_i), \end{aligned}$$

where we do not sum over i. However, the coefficient of γ_{0i}^i in the first term is $-\gamma_{01}^1$ and the coefficient of γ_{0i}^i in the second term is $-\gamma_{02}^2 - \gamma_{03}^3$. Consequently,

$$e_0(\Gamma_{ii}^0) + \Gamma_{\alpha 0}^\alpha \Gamma_{ii}^0 = t^{-1/2} e^{-\lambda/2} \alpha^{1/2} \partial_t (t\alpha^{-1/2} f_i),$$

where we sum over α but not over i. In particular,

$$e_0(\Gamma_{11}^0) + \Gamma_{\alpha 0}^\alpha \Gamma_{11}^0 = \frac{1}{4} t^{-1/2} e^{-\lambda/2} \alpha^{1/2} \partial_t [t\alpha^{-1/2}(\lambda_t - 2\alpha^{-1}\alpha_t - t^{-1})], \quad \text{(A.11)}$$

$$e_0(\Gamma_{22}^0) + \Gamma_{\alpha 0}^\alpha \Gamma_{22}^0 = \frac{1}{2} t^{-1/2} e^{-\lambda/2} \alpha^{1/2} \partial_t [t\alpha^{-1/2}(t^{-1} + P_t)], \quad \text{(A.12)}$$

$$e_0(\Gamma_{33}^0) + \Gamma_{\alpha 0}^\alpha \Gamma_{33}^0 = \frac{1}{2} t^{-1/2} e^{-\lambda/2} \alpha^{1/2} \partial_t [t\alpha^{-1/2}(t^{-1} - P_t)]. \quad \text{(A.13)}$$

Next, note that γ_{01}^0, γ_{12}^2 and γ_{13}^3 can all be written as $h_i = t^{1/4} e^{-\lambda/4} f_i$ for suitably chosen functions f_i. Moreover,

$$e_1(h_i) = t^{1/4} e^{-\lambda/4} \alpha^{1/2} \partial_\theta (t^{1/4} e^{-\lambda/4} f_i) = t^{1/2} e^{-\lambda/2} \alpha^{1/2} \partial_\theta f_i - \gamma_{01}^0 h_i.$$

Since $\Gamma_{\alpha 1}^\alpha = \gamma_{01}^0$, we conclude that

$$e_1(\Gamma_{00}^1) + \Gamma_{\alpha 1}^\alpha \Gamma_{00}^1 = \frac{1}{4} t^{-1/2} e^{-\lambda/2} \alpha^{1/2} \partial_\theta (t\alpha^{1/2} \lambda_\theta), \quad \text{(A.14)}$$

$$e_1(\Gamma_{22}^1) + \Gamma_{\alpha 1}^\alpha \Gamma_{22}^1 = -\frac{1}{2} t^{-1/2} e^{-\lambda/2} \alpha^{1/2} \partial_\theta (t\alpha^{1/2} P_\theta), \quad \text{(A.15)}$$

$$e_1(\Gamma_{33}^1) + \Gamma_{\alpha 1}^\alpha \Gamma_{33}^1 = \frac{1}{2} t^{-1/2} e^{-\lambda/2} \alpha^{1/2} \partial_\theta (t\alpha^{1/2} P_\theta). \quad \text{(A.16)}$$

The expressions γ_{03}^2 and γ_{13}^2 require a somewhat different treatment. However, similar arguments yield

$$e_1(\Gamma_{23}^1) + \Gamma_{23}^1 \Gamma_{\alpha 1}^\alpha = -\frac{1}{2} t^{-1/2} e^{-\lambda/2 - P} \alpha^{1/2} \partial_\theta (t\alpha^{1/2} e^{2P} Q_\theta) - \frac{1}{2}(\gamma_{13}^3 - \gamma_{12}^2)\gamma_{13}^2, \quad (A.17)$$

$$e_0(\Gamma_{23}^0) + \Gamma_{23}^0 \Gamma_{\alpha 0}^\alpha = \frac{1}{2} t^{-1/2} e^{-\lambda/2 - P} \alpha^{1/2} \partial_t (t\alpha^{-1/2} e^{2P} Q_t) + \frac{1}{2}(\gamma_{03}^3 - \gamma_{02}^2)\gamma_{03}^2. \quad (A.18)$$

A.5. Ricci curvature

The Ricci curvature is given by

$$\begin{aligned}
&\mathrm{Ric}(e_\beta, e_\zeta) \\
&\quad = \sum_\alpha \epsilon_\alpha \langle R_{e_\alpha e_\beta} e_\zeta, e_\alpha \rangle = \sum_\alpha \epsilon_\alpha \langle \nabla_{e_\alpha} \nabla_{e_\beta} e_\zeta - \nabla_{e_\beta} \nabla_{e_\alpha} e_\zeta - \nabla_{[e_\alpha, e_\beta]} e_\zeta, e_\alpha \rangle \\
&\quad = \sum_\alpha \epsilon_\alpha \langle \nabla_{e_\alpha} (\Gamma_{\beta\zeta}^\delta e_\delta) - \nabla_{e_\beta} (\Gamma_{\alpha\zeta}^\delta e_\delta) - \gamma_{\alpha\beta}^\delta \Gamma_{\delta\zeta}^\lambda e_\lambda, e_\alpha \rangle \\
&\quad = \sum_\alpha \epsilon_\alpha \langle e_\alpha(\Gamma_{\beta\zeta}^\delta) e_\delta + \Gamma_{\beta\zeta}^\delta \Gamma_{\alpha\delta}^\lambda e_\lambda - e_\beta(\Gamma_{\alpha\zeta}^\delta) e_\delta - \Gamma_{\alpha\zeta}^\delta \Gamma_{\beta\delta}^\lambda e_\lambda - \gamma_{\alpha\beta}^\delta \Gamma_{\delta\zeta}^\lambda e_\lambda, e_\alpha \rangle \\
&\quad = e_\alpha(\Gamma_{\beta\zeta}^\alpha) + \Gamma_{\beta\zeta}^\delta \Gamma_{\alpha\delta}^\alpha - e_\beta(\Gamma_{\alpha\zeta}^\alpha) - \Gamma_{\alpha\zeta}^\delta \Gamma_{\beta\delta}^\alpha - \gamma_{\alpha\beta}^\delta \Gamma_{\delta\zeta}^\alpha,
\end{aligned}$$

where $\epsilon_0 = -1$ and $\epsilon_i = 1$. Let us begin by computing the 00 component:

$$\begin{aligned}
\mathrm{Ric}(e_0, e_0) &= e_\alpha(\Gamma_{00}^\alpha) + \Gamma_{00}^\delta \Gamma_{\alpha\delta}^\alpha - e_0(\Gamma_{\alpha 0}^\alpha) - \Gamma_{\alpha 0}^\delta \Gamma_{0\delta}^\alpha - \gamma_{\alpha 0}^\delta \Gamma_{\delta 0}^\alpha \\
&= e_1(\Gamma_{00}^1) + \Gamma_{00}^1 \Gamma_{\alpha 1}^\alpha - e_0(\Gamma_{\alpha 0}^\alpha) - \Gamma_{\alpha 0}^\beta \Gamma_{0\beta}^\alpha - \gamma_{\alpha 0}^\beta \Gamma_{\beta 0}^\alpha.
\end{aligned}$$

In order to simplify this expression, note that

$$e_1(\Gamma^1_{00}) + \Gamma^1_{00}\Gamma^\alpha_{\alpha 1} = \tfrac{1}{4}t^{-1/2}e^{-\lambda/2}\alpha^{1/2}\partial_\theta(t\alpha^{1/2}\lambda_\theta),$$

where we have used (A.14). Moreover, due to (A.7),

$$-e_0(\Gamma^\alpha_{\alpha 0}) = -e_0(-\gamma^1_{01} + t^{-3/4}e^{-\lambda/4}) = e_0(\gamma^1_{01}) + \tfrac{1}{4}t^{-1/2}e^{-\lambda/2}(\lambda_t + 3t^{-1}).$$

By the symmetries of the connection coefficients,

$$-\Gamma^\beta_{\alpha 0}\Gamma^\alpha_{0\beta} = -\Gamma^0_{i0}\Gamma^i_{00} - \Gamma^i_{00}\Gamma^0_{0i} - \Gamma^j_{i0}\Gamma^i_{0j} = -\Gamma^1_{00}\Gamma^0_{01} = -(\gamma^0_{01})^2.$$

Finally,

$$\begin{aligned}
-\gamma^\beta_{\alpha 0}\Gamma^\alpha_{\beta 0} &= -\gamma^\beta_{i0}\Gamma^i_{\beta 0} = \gamma^0_{0i}\Gamma^i_{00} + \gamma^j_{0i}\Gamma^i_{j0} = (\gamma^0_{01})^2 + \gamma^1_{01}\Gamma^1_{10} + \gamma^A_{01}\Gamma^1_{A0} + \gamma^B_{0A}\Gamma^A_{B0} \\
&= (\gamma^0_{01})^2 - (\gamma^1_{01})^2 + \gamma^A_{01}\Gamma^1_{A0} + \gamma^B_{0A}\Gamma^A_{B0} \\
&= (\gamma^0_{01})^2 - (\gamma^1_{01})^2 - \tfrac{1}{2}(\gamma^2_{01})^2 - \tfrac{1}{2}(\gamma^3_{01})^2 - (\gamma^2_{02})^2 - (\gamma^3_{03})^2 - \tfrac{1}{2}(\gamma^2_{03})^2 \\
&\quad - \tfrac{1}{2}\gamma^2_{03}\gamma^3_{02}.
\end{aligned}$$

Since $\gamma^3_{02} = 0$, we obtain

$$\begin{aligned}
\mathrm{Ric}(e_0, e_0) = {}& \tfrac{1}{4}t^{-1/2}e^{-\lambda/2}\alpha^{1/2}\partial_\theta(t\alpha^{1/2}\lambda_\theta) + \tfrac{1}{4}t^{-1/2}e^{-\lambda/2}(\lambda_t + 3t^{-1}) + e_0(\gamma^1_{01}) \\
& - (\gamma^1_{01})^2 - \tfrac{1}{2}(\gamma^2_{01})^2 - \tfrac{1}{2}(\gamma^3_{01})^2 - (\gamma^2_{02})^2 - (\gamma^3_{03})^2 - \tfrac{1}{2}(\gamma^2_{03})^2.
\end{aligned}$$

To simplify this expression, we combine (A.7) and (A.11) to conclude

$$\begin{aligned}
e_0(\gamma^1_{01}) - (\gamma^1_{01})^2 = {}& -\tfrac{1}{4}t^{-1/2}e^{-\lambda/2}\alpha^{1/2}\partial_t[t\alpha^{-1/2}(\lambda_t - 2\alpha^{-1}\alpha_t - t^{-1})] \\
& + \tfrac{1}{4}t^{-1/2}e^{-\lambda/2}(\lambda_t - 2\alpha^{-1}\alpha_t - t^{-1}).
\end{aligned}$$

Let us now compute

$$\begin{aligned}
&-\frac{1}{2}(\gamma^2_{01})^2 - \frac{1}{2}(\gamma^3_{01})^2 - (\gamma^2_{02})^2 - (\gamma^3_{03})^2 - \frac{1}{2}(\gamma^2_{03})^2 \\
&\quad = -\frac{1}{2}t^{1/2}e^{-\lambda/2}\left(P_t^2 + e^{2P}Q_t^2 + \frac{e^{\lambda/2-P}J^2}{t^{7/2}} + \frac{e^{\lambda/2+P}(K - QJ)^2}{t^{7/2}}\right) - \frac{1}{2}t^{-3/2}e^{-\lambda/2}.
\end{aligned}$$

Thus

$$\begin{aligned}
\mathrm{Ric}(e_0, e_0) = {}& \frac{1}{4}t^{-1/2}e^{-\lambda/2}\alpha^{1/2}(\partial_\theta(t\alpha^{1/2}\lambda_\theta) - \partial_t[t\alpha^{-1/2}(\lambda_t - 2\alpha^{-1}\alpha_t - t^{-1})]) \\
& - \frac{1}{2}t^{1/2}e^{-\lambda/2}\left(P_t^2 + e^{2P}Q_t^2 + \frac{e^{\lambda/2-P}J^2}{t^{7/2}} + \frac{e^{\lambda/2+P}(K - QJ)^2}{t^{7/2}}\right) \\
& + \frac{1}{2}t^{-1/2}e^{-\lambda/2}\left(\lambda_t - \frac{\alpha_t}{\alpha}\right).
\end{aligned}$$

Using (A.5), (A.8) and the antisymmetry of $\Gamma^i_{\alpha j}$ in i and j, we get

$$\begin{aligned}\mathrm{Ric}(e_0,e_1) &= e_\alpha(\Gamma^\alpha_{01}) + \Gamma^\beta_{01}\Gamma^\alpha_{\alpha\beta} - e_0(\Gamma^\alpha_{\alpha 1}) - \Gamma^\beta_{\alpha 1}\Gamma^\alpha_{0\beta} - \gamma^\beta_{\alpha 0}\Gamma^\alpha_{\beta 1}\\ &= e_0(\Gamma^0_{01}) + \Gamma^0_{01}\Gamma^\alpha_{\alpha 0} - e_0(\gamma^0_{01}) - \Gamma^\beta_{\alpha 1}\Gamma^\alpha_{0\beta} - \gamma^\beta_{\alpha 0}\Gamma^\alpha_{\beta 1}\\ &= \Gamma^0_{01}\Gamma^\alpha_{\alpha 0} - \Gamma^\beta_{\alpha 1}\Gamma^\alpha_{0\beta} - \gamma^\beta_{\alpha 0}\Gamma^\alpha_{\beta 1}.\end{aligned}$$

By (A.7),

$$\Gamma^0_{01}\Gamma^\alpha_{\alpha 0} = \gamma^0_{01}(-\gamma^1_{01} + t^{-3/4}e^{-\lambda/4}) = -\gamma^0_{01}\gamma^1_{01} + \gamma^0_{01}t^{-3/4}e^{-\lambda/4}.$$

Moreover, using the symmetries of the connection coefficients, we obtain

$$-\Gamma^\beta_{\alpha 1}\Gamma^\alpha_{0\beta} = -\Gamma^0_{11}\Gamma^1_{00} - \Gamma^i_{j1}\Gamma^j_{0i} = \gamma^0_{01}\gamma^1_{01}.$$

Thus

$$\Gamma^0_{01}\Gamma^\alpha_{\alpha 0} - \Gamma^\beta_{\alpha 1}\Gamma^\alpha_{0\beta} = \gamma^0_{01}t^{-3/4}e^{-\lambda/4} = \tfrac{1}{4}t^{-1/2}e^{-\lambda/2}\alpha^{1/2}\lambda_\theta.$$

Finally,

$$\begin{aligned}-\gamma^\beta_{\alpha 0}\Gamma^\alpha_{\beta 1} &= \gamma^\beta_{0i}\Gamma^i_{\beta 1} = \gamma^0_{01}\Gamma^1_{01} + \gamma^j_{0i}\Gamma^i_{j1} = \gamma^2_{02}\Gamma^2_{21} + \gamma^2_{03}\Gamma^3_{21} + \gamma^3_{02}\Gamma^2_{31} + \gamma^3_{03}\Gamma^3_{31}\\ &= -\gamma^2_{02}\gamma^2_{12} - \tfrac{1}{2}\gamma^2_{03}\gamma^2_{13} - \gamma^3_{03}\gamma^3_{13} = -\tfrac{1}{2}t^{1/2}e^{-\lambda/2}\alpha^{1/2}(P_tP_\theta + e^{2P}Q_tQ_\theta).\end{aligned}$$

Thus

$$\mathrm{Ric}(e_0,e_1) = \tfrac{1}{4}t^{-1/2}e^{-\lambda/2}\alpha^{1/2}[\lambda_\theta - 2t(P_tP_\theta + e^{2P}Q_tQ_\theta)].$$

Using (A.8), we get

$$\begin{aligned}\mathrm{Ric}(e_0,e_A) &= e_\alpha(\Gamma^\alpha_{0A}) + \Gamma^\beta_{0A}\Gamma^\alpha_{\alpha\beta} - e_0(\Gamma^\alpha_{\alpha A}) - \Gamma^\beta_{\alpha A}\Gamma^\alpha_{0\beta} - \gamma^\beta_{\alpha 0}\Gamma^\alpha_{\beta A}\\ &= -\tfrac{1}{2}e_1(\gamma^A_{01}) + \Gamma^1_{0A}\Gamma^\alpha_{\alpha 1} - \Gamma^0_{1A}\Gamma^1_{00} - \Gamma^1_{0A}\Gamma^0_{01} - \Gamma^i_{jA}\Gamma^j_{0i} + \gamma^0_{01}\Gamma^1_{0A} + \gamma^i_{0j}\Gamma^j_{iA}\\ &= -\tfrac{1}{2}e_1(\gamma^A_{01}) + (\Gamma^A_{B1} + \Gamma^B_{1A})\Gamma^B_{01} + \gamma^B_{01}\Gamma^1_{BA}.\end{aligned}$$

Let us begin by considering the case $A = 2$:

$$\begin{aligned}-\tfrac{1}{2}e_1(\gamma^2_{01}) &= -\tfrac{1}{2}t^{-5/4}e^{-P/2-\lambda/4}\alpha^{1/2}J_\theta - \tfrac{1}{2}\gamma^2_{12}\gamma^2_{01},\\ (\Gamma^2_{B1} + \Gamma^B_{12})\Gamma^B_{01} &= -\tfrac{1}{2}\gamma^2_{12}\gamma^2_{01} - \tfrac{1}{2}\gamma^2_{13}\gamma^3_{01},\\ \gamma^B_{01}\Gamma^1_{B2} &= \gamma^2_{01}\gamma^2_{12} + \tfrac{1}{2}\gamma^3_{01}\gamma^2_{13}.\end{aligned}$$

Thus

$$\mathrm{Ric}(e_0,e_2) = -\tfrac{1}{2}t^{-5/4}e^{-P/2-\lambda/4}\alpha^{1/2}J_\theta.$$

Moreover

$$\begin{aligned}-\tfrac{1}{2}e_1(\gamma^3_{01}) &= -\tfrac{1}{2}t^{-5/4}e^{P/2-\lambda/4}\alpha^{1/2}(K_\theta - QJ_\theta) + \tfrac{1}{2}\gamma^2_{12}\gamma^3_{01} - \tfrac{1}{2}\gamma^2_{13}\gamma^2_{01},\\ (\Gamma^3_{B1} + \Gamma^B_{13})\Gamma^B_{01} &= -\tfrac{1}{2}\gamma^3_{13}\gamma^3_{01},\\ \gamma^B_{01}\Gamma^1_{B3} &= \tfrac{1}{2}\gamma^2_{01}\gamma^2_{13} + \gamma^3_{01}\gamma^3_{13}.\end{aligned}$$

Keeping in mind that $\gamma_{12}^2 = -\gamma_{13}^3$, we conclude that

$$\mathrm{Ric}(e_0, e_3) = -\tfrac{1}{2} t^{-5/4} e^{P/2-\lambda/4} \alpha^{1/2} (K_\theta - Q J_\theta).$$

Using the symmetries of the connection coefficients, we obtain

$$\begin{aligned}
\mathrm{Ric}(e_1, e_1) &= e_\alpha(\Gamma_{11}^\alpha) + \Gamma_{11}^\delta \Gamma_{\alpha\delta}^\alpha - e_1(\Gamma_{\alpha 1}^\alpha) - \Gamma_{\alpha 1}^\delta \Gamma_{1\delta}^\alpha - \gamma_{\alpha 1}^\delta \Gamma_{\delta 1}^\alpha \\
&= e_0(\Gamma_{11}^0) + \Gamma_{11}^0 \Gamma_{\alpha 0}^\alpha - e_1(\gamma_{01}^0) - \Gamma_{01}^A \Gamma_{1A}^0 - \Gamma_{i1}^0 \Gamma_{10}^i - \Gamma_{i1}^j \Gamma_{1j}^i - \gamma_{01}^0 \Gamma_{01}^0 \\
&\quad - \gamma_{01}^1 \Gamma_{11}^0 - \gamma_{01}^A \Gamma_{A1}^0 - \gamma_{A1}^B \Gamma_{B1}^A \\
&= e_0(\Gamma_{11}^0) + \Gamma_{11}^0 \Gamma_{\alpha 0}^\alpha - e_1(\gamma_{01}^0) + \tfrac{1}{4} \gamma_{01}^A \gamma_{01}^A - \tfrac{1}{4} \gamma_{01}^A \gamma_{01}^A + \gamma_{01}^1 \Gamma_{11}^0 - (\gamma_{01}^0)^2 \\
&\quad - \gamma_{01}^1 \Gamma_{11}^0 - \gamma_{01}^A \Gamma_{A1}^0 - \gamma_{A1}^B \Gamma_{B1}^A \\
&= e_0(\Gamma_{11}^0) + \Gamma_{11}^0 \Gamma_{\alpha 0}^\alpha - e_1(\gamma_{01}^0) - (\gamma_{01}^0)^2 - \gamma_{01}^A \Gamma_{A1}^0 - \gamma_{A1}^B \Gamma_{B1}^A .
\end{aligned}$$

Due to (A.11) and (A.14), the sum of the first four terms is

$$\tfrac{1}{4} t^{-1/2} e^{-\lambda/2} \alpha^{1/2} \big(\partial_t [t \alpha^{-1/2} (\lambda_t - 2\alpha^{-1} \alpha_t - t^{-1})] - \partial_\theta (t \alpha^{1/2} \lambda_\theta) \big).$$

It can also be computed that

$$\begin{aligned}
-\gamma_{01}^A \Gamma_{A1}^0 &= \tfrac{1}{2} t^{-3} [e^{-P} J^2 + e^P (K - QJ)^2], \\
-\gamma_{A1}^B \Gamma_{B1}^A &= -\tfrac{1}{2} t^{1/2} e^{-\lambda/2} \alpha (P_\theta^2 + e^{2P} Q_\theta^2).
\end{aligned}$$

Thus

$$\begin{aligned}
\mathrm{Ric}(e_1, e_1) &= \frac{1}{4} t^{-1/2} e^{-\lambda/2} \alpha^{1/2} \big(\partial_t [t \alpha^{-1/2} (\lambda_t - 2\alpha^{-1} \alpha_t - t^{-1})] - \partial_\theta (t \alpha^{1/2} \lambda_\theta) \big) \\
&\quad + \frac{1}{2} t^{1/2} e^{-\lambda/2} \left(\frac{e^{\lambda/2-P} J^2}{t^{7/2}} + \frac{e^{\lambda/2+P} (K - QJ)^2}{t^{7/2}} \right) - \frac{1}{2} t^{1/2} e^{-\lambda/2} \alpha (P_\theta^2 + e^{2P} Q_\theta^2).
\end{aligned}$$

Let us turn to

$$\begin{aligned}
&\mathrm{Ric}(e_1, e_A) = e_\alpha(\Gamma_{1A}^\alpha) + \Gamma_{1A}^\delta \Gamma_{\alpha\delta}^\alpha - e_1(\Gamma_{\alpha A}^\alpha) - \Gamma_{\alpha A}^\delta \Gamma_{1\delta}^\alpha - \gamma_{\alpha 1}^\delta \Gamma_{\delta A}^\alpha \\
&= e_0(\Gamma_{1A}^0) + \Gamma_{1A}^0 \Gamma_{\alpha 0}^\alpha + \Gamma_{1A}^1 \Gamma_{\alpha 1}^\alpha - \Gamma_{iA}^0 \Gamma_{10}^i - \Gamma_{0A}^i \Gamma_{1i}^0 - \Gamma_{jA}^i \Gamma_{1i}^j - \gamma_{i1}^0 \Gamma_{0A}^i - \gamma_{01}^i \Gamma_{iA}^0 - \gamma_{j1}^i \Gamma_{iA}^j \\
&= -\tfrac{1}{2} e_0(\gamma_{01}^A) + \Gamma_{A0}^1 \Gamma_{B0}^B - \Gamma_{A0}^B \Gamma_{B0}^1 - \Gamma_{0A}^1 \Gamma_{10}^1 - \Gamma_{0A}^B \Gamma_{B0}^1 - \gamma_{01}^1 \Gamma_{1A}^0 - \gamma_{01}^B \Gamma_{BA}^0 .
\end{aligned}$$

We compute

$$\begin{aligned}
-\tfrac{1}{2} e_0(\gamma_{01}^2) &= -\tfrac{1}{2} t^{-5/4} e^{-\lambda/4-P/2} J_t - \tfrac{1}{2} \gamma_{03}^3 \gamma_{01}^2 - \gamma_{02}^2 \gamma_{01}^2, \\
-\Gamma_{02}^1 \Gamma_{10}^1 - \gamma_{01}^1 \Gamma_{12}^0 &= 0, \\
-\Gamma_{20}^B \Gamma_{B0}^1 - \Gamma_{02}^B \Gamma_{B0}^1 &= -\tfrac{1}{2} \gamma_{02}^2 \gamma_{01}^2 - \tfrac{1}{2} \gamma_{03}^2 \gamma_{01}^3, \\
\Gamma_{20}^1 \Gamma_{B0}^B - \gamma_{01}^B \Gamma_{B2}^0 &= \tfrac{3}{2} \gamma_{01}^2 \gamma_{02}^2 + \tfrac{1}{2} \gamma_{01}^2 \gamma_{03}^3 + \tfrac{1}{2} \gamma_{01}^3 \gamma_{03}^2 .
\end{aligned}$$

Adding up, we obtain

$$\mathrm{Ric}(e_1, e_2) = -\tfrac{1}{2}t^{-5/4}e^{-\lambda/4-P/2}J_t.$$

Furthermore,

$$\begin{aligned}
-\tfrac{1}{2}e_0(\gamma_{01}^3) &= -\tfrac{1}{2}t^{-5/4}e^{P/2-\lambda/4}(K_t - QJ_t) - \tfrac{1}{2}\gamma_{03}^2\gamma_{01}^2 - \tfrac{1}{2}\gamma_{02}^2\gamma_{01}^3 - \gamma_{03}^3\gamma_{01}^3,\\
-\Gamma_{03}^1\Gamma_{10}^1 - \gamma_{01}^1\Gamma_{13}^0 &= 0,\\
-\Gamma_{30}^B\Gamma_{B0}^1 - \Gamma_{03}^B\Gamma_{B0}^1 &= -\tfrac{1}{2}\gamma_{03}^3\gamma_{01}^3,\\
\Gamma_{30}^1\Gamma_{B0}^B - \gamma_{01}^B\Gamma_{B3}^0 &= \tfrac{3}{2}\gamma_{01}^3\gamma_{03}^3 + \tfrac{1}{2}\gamma_{01}^3\gamma_{02}^2 + \tfrac{1}{2}\gamma_{01}^2\gamma_{03}^2.
\end{aligned}$$

Adding up, we obtain

$$\mathrm{Ric}(e_1, e_3) = -\tfrac{1}{2}t^{-5/4}e^{P/2-\lambda/4}(K_t - QJ_t).$$

Next,

$$\begin{aligned}
\mathrm{Ric}(e_A, e_B) &= e_\alpha(\Gamma_{AB}^\alpha) + \Gamma_{AB}^\delta\Gamma_{\alpha\delta}^\alpha - \Gamma_{\alpha B}^\delta\Gamma_{A\delta}^\alpha - \gamma_{\alpha A}^\delta\Gamma_{\delta B}^\alpha\\
&= e_0(\Gamma_{AB}^0) + e_1(\Gamma_{AB}^1) + \Gamma_{AB}^0\Gamma_{\alpha 0}^\alpha + \Gamma_{AB}^1\Gamma_{\alpha 1}^\alpha - \Gamma_{\alpha B}^\delta\Gamma_{A\delta}^\alpha - \gamma_{\alpha A}^\delta\Gamma_{\delta B}^\alpha.
\end{aligned}$$

Note that if $A = B = 2$, the first four terms can be written

$$\tfrac{1}{2}t^{-1/2}e^{-\lambda/2}\alpha^{1/2}\big(\partial_t[t\alpha^{-1/2}(t^{-1} + P_t)] - \partial_\theta(t\alpha^{1/2}P_\theta)\big)$$

(cf. (A.12) and (A.15)). Let us therefore turn to

$$\begin{aligned}
-\Gamma_{\alpha 2}^\delta\Gamma_{2\delta}^\alpha - \gamma_{\alpha 2}^\delta\Gamma_{\delta 2}^\alpha &= -\Gamma_{i2}^0\Gamma_{20}^i - \Gamma_{02}^i\Gamma_{2i}^0 - \Gamma_{j2}^i\Gamma_{2i}^j - \gamma_{02}^2\Gamma_{22}^0 - \gamma_{12}^2\Gamma_{22}^1\\
&= -\Gamma_{20}^i(\Gamma_{20}^i + \Gamma_{02}^i) - \Gamma_{A2}^1\Gamma_{21}^A - \Gamma_{12}^A\Gamma_{2A}^1 - \gamma_{02}^2\Gamma_{22}^0 - \gamma_{12}^2\Gamma_{22}^1.
\end{aligned}$$

However, it can be computed that

$$\begin{aligned}
-\Gamma_{20}^i(\Gamma_{20}^i + \Gamma_{02}^i) &= -\tfrac{1}{2}t^{-3}e^{-P}J^2 - \tfrac{1}{2}t^{1/2}e^{-\lambda/2}e^{2P}Q_t^2 - (\gamma_{02}^2)^2,\\
-\Gamma_{A2}^1\Gamma_{21}^A - \Gamma_{12}^A\Gamma_{2A}^1 &= \tfrac{1}{2}t^{1/2}e^{-\lambda/2}\alpha e^{2P}Q_\theta^2 + (\gamma_{12}^2)^2,\\
-\gamma_{02}^2\Gamma_{22}^0 - \gamma_{12}^2\Gamma_{22}^1 &= (\gamma_{02}^2)^2 - (\gamma_{12}^2)^2.
\end{aligned}$$

Adding up gives

$$\begin{aligned}
\mathrm{Ric}(e_2, e_2) &= \frac{1}{2}t^{-1/2}e^{-\lambda/2}\alpha^{1/2}\big(\partial_t[t\alpha^{-1/2}(t^{-1} + P_t)] - \partial_\theta(t\alpha^{1/2}P_\theta)\big)\\
&\quad - \frac{1}{2}t^{1/2}e^{-\lambda/2}\frac{e^{\lambda/2-P}J^2}{t^{7/2}} - \frac{1}{2}t^{1/2}e^{-\lambda/2}e^{2P}(Q_t^2 - \alpha Q_\theta^2).
\end{aligned}$$

Next, consider

$$\mathrm{Ric}(e_2, e_3) = e_0(\Gamma_{23}^0) + e_1(\Gamma_{23}^1) + \Gamma_{23}^0\Gamma_{\alpha 0}^\alpha + \Gamma_{23}^1\Gamma_{\alpha 1}^\alpha - \Gamma_{\alpha 3}^\delta\Gamma_{2\delta}^\alpha - \gamma_{\alpha 2}^\delta\Gamma_{\delta 3}^\alpha.$$

Due to (A.17) and (A.18), the sum of the first four terms can be written

$$\frac{1}{2}t^{-1/2}e^{-\lambda/2-P}\alpha^{1/2}[\partial_t(t\alpha^{-1/2}e^{2P}Q_t)-\partial_\theta(t\alpha^{1/2}e^{2P}Q_\theta)] \\ -\frac{1}{2}(\gamma_{13}^3-\gamma_{12}^2)\gamma_{13}^2+\frac{1}{2}(\gamma_{03}^3-\gamma_{02}^2)\gamma_{03}^2.$$

We compute

$$\begin{aligned}-\Gamma_{\alpha 3}^{\delta}\Gamma_{2\delta}^{\alpha}-\gamma_{\alpha 2}^{\delta}\Gamma_{\delta 3}^{\alpha} &= -\Gamma_{i3}^{0}\Gamma_{20}^{i}-\Gamma_{03}^{i}\Gamma_{2i}^{0}-\Gamma_{j3}^{i}\Gamma_{2i}^{j}-\gamma_{02}^{2}\Gamma_{23}^{0}-\gamma_{12}^{2}\Gamma_{23}^{1}\\ &= -(\Gamma_{30}^{i}+\Gamma_{03}^{i})\Gamma_{20}^{i}-\Gamma_{A3}^{1}\Gamma_{21}^{A}-\Gamma_{13}^{A}\Gamma_{2A}^{1}-\gamma_{02}^{2}\Gamma_{23}^{0}-\gamma_{12}^{2}\Gamma_{23}^{1}.\end{aligned}$$

However,

$$\begin{aligned}-(\Gamma_{30}^{i}+\Gamma_{03}^{i})\Gamma_{20}^{i} &= -\frac{1}{2}\gamma_{01}^{3}\gamma_{01}^{2}-\frac{1}{2}\gamma_{03}^{3}\gamma_{03}^{2},\\ -\Gamma_{A3}^{1}\Gamma_{21}^{A}-\Gamma_{13}^{A}\Gamma_{2A}^{1} &= \frac{1}{2}\gamma_{13}^{3}\gamma_{13}^{2},\\ -\gamma_{02}^{2}\Gamma_{23}^{0}-\gamma_{12}^{2}\Gamma_{23}^{1} &= \frac{1}{2}\gamma_{02}^{2}\gamma_{03}^{2}-\frac{1}{2}\gamma_{12}^{2}\gamma_{13}^{2}.\end{aligned}$$

Adding up, we obtain

$$\begin{aligned}\mathrm{Ric}(e_2,e_3) &= \frac{1}{2}t^{-1/2}e^{-\lambda/2-P}\alpha^{1/2}[\partial_t(t\alpha^{-1/2}e^{2P}Q_t)-\partial_\theta(t\alpha^{1/2}e^{2P}Q_\theta)]\\ &\quad -\frac{1}{2}t^{-3}J(K-QJ).\end{aligned}$$

Finally, let us consider

$$\mathrm{Ric}(e_3,e_3)=e_0(\Gamma_{33}^{0})+e_1(\Gamma_{33}^{1})+\Gamma_{33}^{0}\Gamma_{\alpha 0}^{\alpha}+\Gamma_{33}^{1}\Gamma_{\alpha 1}^{\alpha}-\Gamma_{\alpha 3}^{\delta}\Gamma_{3\delta}^{\alpha}-\gamma_{\alpha 3}^{\delta}\Gamma_{\delta 3}^{\alpha}.$$

Due to (A.13) and (A.16), the sum of the first four terms is

$$\frac{1}{2}t^{-1/2}e^{-\lambda/2}\alpha^{1/2}\big(\partial_t[t\alpha^{-1/2}(t^{-1}-P_t)]+\partial_\theta(t\alpha^{1/2}P_\theta)\big).$$

Let us therefore compute

$$\begin{aligned}-\Gamma_{\alpha 3}^{\delta}\Gamma_{3\delta}^{\alpha}-\gamma_{\alpha 3}^{\delta}\Gamma_{\delta 3}^{\alpha} &= -\Gamma_{i3}^{0}\Gamma_{30}^{i}-\Gamma_{03}^{i}\Gamma_{3i}^{0}-\Gamma_{j3}^{i}\Gamma_{3i}^{j}-\gamma_{03}^{A}\Gamma_{A3}^{0}-\gamma_{13}^{A}\Gamma_{A3}^{1}\\ &= -(\Gamma_{30}^{i}+\Gamma_{03}^{i})\Gamma_{30}^{i}-\Gamma_{A3}^{1}\Gamma_{31}^{A}-\Gamma_{13}^{A}\Gamma_{3A}^{1}-\gamma_{03}^{A}\Gamma_{A3}^{0}-\gamma_{13}^{A}\Gamma_{A3}^{1}.\end{aligned}$$

On the other hand,

$$\begin{aligned}-(\Gamma_{30}^{i}+\Gamma_{03}^{i})\Gamma_{30}^{i} &= -\frac{1}{2}(\gamma_{01}^{3})^2-(\gamma_{03}^{3})^2,\\ -\Gamma_{A3}^{1}\Gamma_{31}^{A}-\Gamma_{13}^{A}\Gamma_{3A}^{1} &= (\gamma_{13}^{3})^2,\\ -\gamma_{03}^{A}\Gamma_{A3}^{0}-\gamma_{13}^{A}\Gamma_{A3}^{1} &= \frac{1}{2}(\gamma_{03}^{2})^2+(\gamma_{03}^{3})^2-\frac{1}{2}(\gamma_{13}^{2})^2-(\gamma_{13}^{3})^2.\end{aligned}$$

Adding up, we obtain

$$-\Gamma_{\alpha 3}^{\delta}\Gamma_{3\delta}^{\alpha}-\gamma_{\alpha 3}^{\delta}\Gamma_{\delta 3}^{\alpha}=-\frac{1}{2}t^{-3}e^{P}(K-QJ)^2+\frac{1}{2}t^{1/2}e^{-\lambda/2}e^{2P}(Q_t^2-\alpha Q_\theta^2).$$

Thus

$$\begin{aligned}\mathrm{Ric}(e_3,e_3) &= \frac{1}{2}t^{-1/2}e^{-\lambda/2}\alpha^{1/2}\big(\partial_t[t\alpha^{-1/2}(t^{-1}-P_t)]+\partial_\theta(t\alpha^{1/2}P_\theta)\big)\\ &\quad -\frac{1}{2}t^{1/2}e^{-\lambda/2}\frac{e^{\lambda/2+P}(K-QJ)^2}{t^{7/2}}+\frac{1}{2}t^{1/2}e^{-\lambda/2}e^{2P}(Q_t^2-\alpha Q_\theta^2).\end{aligned}$$

A.6. The Einstein tensor

Adding up the above, we conclude that the scalar curvature S is given by

$$\begin{aligned}S = {} & \frac{1}{2}t^{-1/2}e^{-\lambda/2}\alpha^{1/2}\big(\partial_t[t\alpha^{-1/2}(\lambda_t - 2\alpha^{-1}\alpha_t - t^{-1})] - \partial_\theta(t\alpha^{1/2}\lambda_\theta)\big) \\ & + \frac{1}{2}t^{1/2}e^{-\lambda/2}[P_t^2 + e^{2P}Q_t^2 - \alpha(P_\theta^2 + e^{2P}Q_\theta^2)] \\ & + \frac{1}{2}t^{1/2}e^{-\lambda/2}\left(\frac{e^{\lambda/2-P}J^2}{t^{7/2}} + \frac{e^{\lambda/2+P}(K-QJ)^2}{t^{7/2}}\right) - \frac{1}{2}t^{-1/2}e^{-\lambda/2}\lambda_t.\end{aligned}$$

As a consequence, if $\text{Ein}_{\alpha\beta} = \text{Ein}(e_\alpha, e_\beta)$, then

$$\begin{aligned}\text{Ein}_{00} = {} & -\frac{1}{4}t^{1/2}e^{-\lambda/2} \\ & \cdot\left[P_t^2 + \alpha P_\theta^2 + e^{2P}(Q_t^2 + \alpha Q_\theta^2) + \frac{e^{\lambda/2-P}J^2}{t^{7/2}} + \frac{e^{\lambda/2+P}(K-QJ)^2}{t^{7/2}}\right] \\ & + \frac{1}{4}t^{-1/2}e^{-\lambda/2}\left(\lambda_t - 2\frac{\alpha_t}{\alpha}\right), \\ \text{Ein}_{11} = {} & -\frac{1}{4}t^{1/2}e^{-\lambda/2} \\ & \cdot\left[P_t^2 + \alpha P_\theta^2 + e^{2P}(Q_t^2 + \alpha Q_\theta^2) - \frac{e^{\lambda/2-P}J^2}{t^{7/2}} - \frac{e^{\lambda/2+P}(K-QJ)^2}{t^{7/2}}\right] \\ & + \frac{1}{4}t^{-1/2}e^{-\lambda/2}\lambda_t, \\ \text{Ein}_{22} = {} & \frac{1}{2}t^{-1/2}e^{-\lambda/2}\alpha^{1/2} \\ & \cdot\left(\partial_t\left[t\alpha^{-1/2}\left(P_t - \frac{1}{2}\lambda_t + \frac{\alpha_t}{\alpha} + \frac{3}{2t}\right)\right] - \partial_\theta\left[t\alpha^{1/2}\left(P_\theta - \frac{1}{2}\lambda_\theta\right)\right]\right) \\ & -\frac{1}{4}t^{1/2}e^{-\lambda/2}[P_t^2 + 3e^{2P}Q_t^2 - \alpha(P_\theta^2 + 3e^{2P}Q_\theta^2)] \\ & -\frac{1}{4}t^{1/2}e^{-\lambda/2}\left(3\frac{e^{\lambda/2-P}J^2}{t^{7/2}} + \frac{e^{\lambda/2+P}(K-QJ)^2}{t^{7/2}}\right) + \frac{1}{4}t^{-1/2}e^{-\lambda/2}\lambda_t.\end{aligned}$$

The last diagonal component is given by

$$\begin{aligned}\text{Ein}_{33} = {} & -\frac{1}{2}t^{-1/2}e^{-\lambda/2}\alpha^{1/2} \\ & \cdot\left(\partial_t\left[t\alpha^{-1/2}\left(P_t + \frac{1}{2}\lambda_t - \frac{\alpha_t}{\alpha} - \frac{3}{2t}\right)\right] - \partial_\theta\left[t\alpha^{1/2}\left(P_\theta + \frac{1}{2}\lambda_\theta\right)\right]\right) \\ & -\frac{1}{4}t^{1/2}e^{-\lambda/2}[P_t^2 - e^{2P}Q_t^2 - \alpha(P_\theta^2 - e^{2P}Q_\theta^2)] \\ & -\frac{1}{4}t^{1/2}e^{-\lambda/2}\left(\frac{e^{\lambda/2-P}J^2}{t^{7/2}} + 3\frac{e^{\lambda/2+P}(K-QJ)^2}{t^{7/2}}\right) + \frac{1}{4}t^{-1/2}e^{-\lambda/2}\lambda_t.\end{aligned}$$

Note that

$$\begin{aligned}\mathrm{Ein}_{22} - \mathrm{Ein}_{33} = {} & t^{-1/2}e^{-\lambda/2}\alpha^{1/2}\big(\partial_t(t\alpha^{-1/2}P_t) - \partial_\theta(t\alpha^{1/2}P_\theta)\big)\\ & - t^{1/2}e^{-\lambda/2}e^{2P}(Q_t^2 - \alpha Q_\theta^2)\\ & - \frac{1}{2}t^{1/2}e^{-\lambda/2}\left(\frac{e^{\lambda/2-P}J^2}{t^{7/2}} - \frac{e^{\lambda/2+P}(K-QJ)^2}{t^{7/2}}\right).\end{aligned}$$

We also have

$$\begin{aligned}\mathrm{Ein}_{22} + \mathrm{Ein}_{33} = {} & -\frac{1}{2}t^{-1/2}e^{-\lambda/2}\alpha^{1/2}\left(\partial_t\left[t\alpha^{-1/2}\left(\lambda_t - 2\frac{\alpha_t}{\alpha} - \frac{3}{t}\right)\right] - \partial_\theta(t\alpha^{1/2}\lambda_\theta)\right)\\ & - \frac{1}{2}t^{1/2}e^{-\lambda/2}[P_t^2 + e^{2P}Q_t^2 - \alpha(P_\theta^2 + e^{2P}Q_\theta^2)]\\ & - t^{1/2}e^{-\lambda/2}\left(\frac{e^{\lambda/2-P}J^2}{t^{7/2}} + \frac{e^{\lambda/2+P}(K-QJ)^2}{t^{7/2}}\right) + \frac{1}{2}t^{-1/2}e^{-\lambda/2}\lambda_t.\end{aligned}$$

The remaining components of the Einstein tensor equal the corresponding components of the Ricci tensor, and so have already been computed. The above calculations yield the expressions (2.3)–(2.12) for Einstein's equations, $\mathrm{Ein} + \Lambda g = T$.

A.7. The Vlasov equation

The distribution function f characterising the Vlasov matter is defined on the mass shell. The mass shell, in its turn, is given by the future directed unit timelike vectors. Since a tangent vector in this set can be written $v^\alpha e_\alpha$, where

$$v^0 = [1 + (v^1)^2 + (v^2)^2 + (v^3)^2]^{1/2},$$

we can think of f as depending on v^i, $i = 1, 2, 3$, and the base point. However, due to the symmetry requirements, the distribution function only depends on the $t\theta$-coordinates of the base point. As a consequence, the distribution function can be considered to be a function of (t, θ, v), where $v = (v^1, v^2, v^3)$. In order to derive an equation for f, recall that the Vlasov equation is equivalent to f being constant along future directed unit timelike geodesics. Consider, therefore, a future directed unit timelike geodesic

$$\gamma(s) = [t(s), \theta(s), x(s), y(s)]$$

in a $\mathbb{T}^2$-symmetric spacetime. Define the functions $v^\alpha(s)$ by the equality

$$\dot{\gamma}(s) = v^\alpha(s)e_\alpha|_{\gamma(s)}.$$

Note that

$$\frac{dt}{ds}(s) = t^{1/4}(s)(e^{-\lambda/4}) \circ \gamma(s)\, v^0(s), \tag{A.19}$$

$$\frac{d\theta}{ds}(s) = t^{1/4}(s)(e^{-\lambda/4}\alpha^{1/2}) \circ \gamma(s)\, v^1(s). \tag{A.20}$$

Let

$$\nu(s) = [t(s), \theta(s), v(s)], \qquad h = f \circ \nu,$$

where $v(s) = [v^1(s), v^2(s), v^3(s)]$. The requirement that f be constant along geodesics is equivalent to $dh/ds = 0$ regardless of the choice of future directed unit timelike geodesic γ. On the other hand,

$$\frac{dh}{ds} = \frac{\partial f}{\partial t} \circ \nu \frac{dt}{ds} + \frac{\partial f}{\partial \theta} \circ \nu \frac{d\theta}{ds} + \sum_{i=1}^{3} \frac{\partial f}{\partial v^i} \circ \nu \frac{dv^i}{ds}.$$

Keeping (A.19) and (A.20) in mind, the requirement that $dh/ds = 0$ is equivalent to

$$\frac{\partial f}{\partial t} \circ \nu + \alpha^{1/2} \circ \gamma \frac{v^1}{v^0} \frac{\partial f}{\partial \theta} \circ \nu + \sum_{i=1}^{3} \frac{\dot{v}^i}{t^{1/4} e^{-\lambda \circ \gamma/4} v^0} \frac{\partial f}{\partial v^i} \circ \nu = 0.$$

In order to derive an expression for $\dot{v}$, note that

$$0 = \ddot{\gamma} = \frac{d}{ds}(v^\alpha e_\alpha) = \dot{v}^\alpha e_\alpha + v^\beta \nabla_{\dot{\gamma}} e_\beta = \dot{v}^\alpha e_\alpha + v^\beta v^\mu \nabla_{e_\mu} e_\beta = (\dot{v}^\alpha + v^\beta v^\mu \Gamma^\alpha_{\mu\beta}) e_\alpha.$$

The geodesic equation can thus be written

$$\dot{v}^\alpha = -v^\beta v^\mu \Gamma^\alpha_{\beta\mu}.$$

Using this formula, it can be calculated that

$$\dot{v}^1 = -\gamma^0_{01} v^0 v^0 + \gamma^1_{01} v^0 v^1 + \gamma^2_{01} v^0 v^2 + \gamma^3_{01} v^0 v^3 - \gamma^2_{12} v^2 v^2 - \gamma^3_{13} v^3 v^3 - \gamma^2_{13} v^2 v^3.$$

Using the formulae for $\gamma^\alpha_{\lambda\mu}$, we conclude that

$$\begin{aligned}\frac{\dot{v}^1}{t^{1/4} e^{-\lambda/4} v^0} = & -\frac{1}{4}\alpha^{1/2}\lambda_\theta\, v^0 - \frac{1}{4}\left(\lambda_t - 2\frac{\alpha_t}{\alpha} - \frac{1}{t}\right) v^1 + \alpha^{1/2} e^P Q_\theta \frac{v^2 v^3}{v^0} \\ & - \frac{1}{2}\alpha^{1/2} P_\theta \frac{(v^3)^2 - (v^2)^2}{v^0} + t^{-7/4} e^{\lambda/4}\big(e^{-P/2} J v^2 + e^{P/2}(K - QJ) v^3\big),\end{aligned}$$

where we have omitted composition with γ for brevity. We also have

$$\dot{v}^2 = \gamma^2_{02} v^0 v^2 + \gamma^2_{12} v^1 v^2,$$

so that

$$\frac{\dot{v}^2}{t^{1/4} e^{-\lambda/4} v^0} = -\frac{1}{2}\left(P_t + \frac{1}{t}\right) v^2 - \frac{1}{2}\alpha^{1/2} P_\theta \frac{v^1 v^2}{v^0}.$$

Finally,

$$\dot{v}^3 = \gamma^2_{03} v^0 v^2 + \gamma^3_{03} v^0 v^3 + \gamma^2_{13} v^1 v^2 + \gamma^3_{13} v^1 v^3,$$

so that

$$\frac{\dot{v}^3}{t^{1/4} e^{-\lambda/4} v^0} = -\frac{1}{2}\left(\frac{1}{t} - P_t\right) v^3 + \frac{1}{2}\alpha^{1/2} P_\theta \frac{v^1 v^3}{v^0} - e^P v^2 \left(Q_t + \alpha^{1/2} Q_\theta \frac{v^1}{v^0}\right).$$

Adding up the above computations, we conclude that the Vlasov equation is equivalent to the requirement that (2.19) holds.

Appendix B. Notation

Metric variables, twist quantities, cosmological constant, frame, manifold

- $\alpha > 0$, λ, P, Q, G and H. These are the functions characterising the metric (cf. (1.1)).
- $\hat{\lambda}$ is defined in (3.7).
- J and K. These are the twist quantities defined in (1.2). They also satisfy (2.2).
- Λ and $\mathcal{H}$. Λ is the positive cosmological constant and $\mathcal{H} = (\Lambda/3)^{1/2}$.
- $\{e_\alpha\}$ is the orthonormal frame defined in (1.7).
- t_0, t_1. In the situations of interest in this paper, the metric (1.1) is defined on $(t_0, \infty) \times \mathbb{T}^3$, where $t_0 \geq 0$. When speaking of a $\mathbb{T}^2$-symmetric solution, we take it for granted that t_0 is defined in this way. Moreover, $t_1 = t_0 + 2$.

Variables for the characteristic system

- Θ, V^1, V^2, V^3 are the basic variables of the characteristic system (4.1)–(4.4). The symbols $\Theta(s; t, \theta, v)$, $V(s; t, \theta, v)$ denote a solution to the characteristic system corresponding to initial data (t, θ, v). In other words, $\Theta(s; t, \theta, v)$, $V(s; t, \theta, v)$, considered as functions of s, are solutions to (4.1)–(4.4). Moreover, $\Theta(t; t, \theta, v) = \theta$, $V(t; t, \theta, v) = v$.
- Ψ and Z^i, $i = 1, 2, 3$. Given a choice of derivative (∂_t, ∂_θ or ∂_{v^i}), say ∂, the variables Ψ and $Z = (Z^1, Z^2, Z^3)$ are defined by (6.3)–(6.6).
- $\hat{\Psi}$ and $\hat{Z}$ are the rescaled versions of Ψ and Z, and they are defined in (6.22).
- Ψ_j, Z^i_j, V^i_j and Θ_j are the higher order derivatives of Ψ, Z, V and Θ. They are defined in (8.8).
- $\hat{Z}^1_N$ and $\hat{\Psi}_N$ are the rescaled versions of Z^1_N and Ψ_N, and they are defined in (8.23).

Matter quantities

- ρ, J_i, P_i and S_{ij}. The quantities ρ, J_i, P_i and S_{ij} are defined in general in (2.1). In the case of Vlasov matter, they are defined in (2.21).

Vlasov matter

- $\mathcal{P}$ denotes the mass shell (the set of future directed unit timelike vectors).
- f denotes the distribution function. For $\mathbb{T}^2$-symmetric solutions, f, however, denotes the symmetry reduced version of the distribution function. In other words, f is considered to be a function of t, θ, v^1, v^2 and v^3, where, if p is an element of the mass shell, (t, θ, x, y) is the base point of p, and v^α are the components of p relative to the orthonormal frame $\{e_\alpha\}$.
- f_{sc} is the rescaled distribution function. It is given by $f_{\text{sc}}(t, \theta, v) = f(t, \theta, t^{-1/2}v)$.
- $T^{\text{Vl}}_{\alpha\beta}$ is the stress energy tensor associated with the Vlasov matter. It is given by (1.4). The components of the stress energy tensor with respect to the frame $\{e_\alpha\}$ are given by (2.1) and (2.21); see also (2.20).
- L^i. The functions L^i are given by (9.3)–(9.5).

The initial value formulation

- pr_Σ is the projection defined in Remark 1.26.
- ρ^{Vl} and $\bar{J}^{\text{Vl}}$ are defined in (1.33) and (1.34).

Auxiliary notation

- $\langle h \rangle$. If h is a scalar function, its mean value is denoted $\langle h \rangle$ (cf. (1.3)).
- $\langle \bar{p} \rangle$. If $\bar{p}$ is a vector, $\langle \bar{p} \rangle = (1 + |\bar{p}|^2)^{1/2}$.

Energies controlling the metric variables, the distribution function and solutions to the characteristic system

- The $H^l_{\mathrm{Vl},\mu}$-norm is defined in (1.22).
- $\partial_\pm$ and $\mathcal{A}_\pm$ are defined in (2.15). They are given by

$$\partial_\pm = \partial_t \pm \alpha^{1/2}\partial_\theta, \qquad \mathcal{A}_\pm = (\partial_\pm P)^2 + e^{2P}(\partial_\pm Q)^2.$$

- E_{bas} is the L^2-based energy introduced in (3.8).
- $\mathcal{Q}^1$ controls the size of the support of f in the v^1-direction (cf. (4.7)). It is given by

$$\mathcal{Q}^1(t) := \sup\{|v^1| : (t, \theta, v^1, v^2, v^3) \in \operatorname{supp} f\}.$$

- F. The sup-norm energy F is introduced in (4.8). It is given by

$$F(t) = \sup_{\theta \in \mathbb{S}^1} \mathcal{A}_+(t, \theta) + \sup_{\theta \in \mathbb{S}^1} \mathcal{A}_-(t, \theta).$$

- $\hat{\mathcal{A}}_\pm$ and $\hat{F}$ are introduced in (4.14) and (4.15) respectively. They are given by

$$\hat{\mathcal{A}}_\pm = t^4 \mathcal{A}_\pm + t, \qquad \hat{F}(t) = \sup_{\theta \in \mathbb{S}^1} \hat{\mathcal{A}}_+(t, \theta) + \sup_{\theta \in \mathbb{S}^1} \hat{\mathcal{A}}_-(t, \theta).$$

- R^1 and $\hat{\mathcal{Q}}^1$ are defined in (4.17). They are given by

$$R^1(s) = [s(V^1(s))^2 + 1]^{1/2}, \qquad \hat{\mathcal{Q}}^1(s) = [s(\mathcal{Q}^1(s))^2 + 1]^{1/2}.$$

- $\mathcal{G}$ is defined in (4.19).
- $\hat{E}$ is introduced in (6.23). It is given by $\hat{E} = \sum_{i=1}^3 (\hat{Z}^i)^2 + (\hat{\Psi})^2$.
- $\mathcal{A}_{N+1,\pm}$ is introduced in (7.16). It is given by

$$\mathcal{A}_{N+1,\pm} = [\partial_\theta^N P_t \pm \partial_\theta^N(\alpha^{1/2} P_\theta)]^2 + e^{2P}[\partial_\theta^N Q_t \pm \partial_\theta^N(\alpha^{1/2} Q_\theta)]^2.$$

- $\hat{\mathcal{A}}_{N+1,\pm}$ and $\hat{F}_{N+1}$ are introduced in (7.17). They are given by

$$\hat{\mathcal{A}}_{N+1,\pm} = t^{7/2}\mathcal{A}_{N+1,\pm} + t^{1/2}, \qquad \hat{F}_{N+1} = \sup_{\theta \in \mathbb{S}^1} \hat{\mathcal{A}}_{N+1,+} + \sup_{\theta \in \mathbb{S}^1} \hat{\mathcal{A}}_{N+1,-}.$$

- $\hat{E}_N$ is defined in (8.24). It is given by $\hat{E}_N = (\hat{\Psi}_N)^2 + (\hat{Z}^1_N)^2$.
- E_k and E. These energies control suitably weighted Sobolev norms of the distribution function. E_k is defined in (9.6). Moreover, $E = E_0$.

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