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Complete bounded embedded complex curves in \mathbb{C}^2

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Abstract. We prove that any convex domain of \mathbb{C}^2 carries properly embedded complete complex curves. In particular, we give the first examples of complete bounded embedded complex curves in \mathbb{C}^2 .

Keywords. Riemann surfaces, complex curves, complete holomorphic embeddings

1. Introduction

Let M^k be a k -dimensional connected complex manifold, $k \in \mathbb{N}$. A holomorphic immersion $X: M^k \rightarrow \mathbb{C}^n$, $n \geq k$, is said to be *complete* if the pull back X^*g of the Euclidean metric g on \mathbb{C}^n is a complete Riemannian metric on M^k . This is equivalent $X \circ \gamma$ having infinite Euclidean length for any divergent arc γ in M^k . (Given a non-compact topological space W , an arc $\gamma: [0, 1) \rightarrow W$ is said to be *divergent* if $\gamma(t)$ leaves any compact subset of W when $t \rightarrow 1$.)

An immersion $X: M^k \rightarrow \mathbb{C}^n$ is said to be an *embedding* if $X: M^k \rightarrow X(M^k)$ is a homeomorphism. In this case $X(M^k)$ is said an embedded submanifold of \mathbb{C}^n . If $\Omega \subset \mathbb{C}^n$ is a domain, a map $X: M^k \rightarrow \Omega$ is said to be *proper* if $X^{-1}(K)$ is compact for any compact set $K \subset \Omega$. Proper injective immersions $M^k \rightarrow \Omega$ are embeddings.

In 1977, Yang [28, 29] asked whether there exist complete holomorphic embeddings $M^k \rightarrow \mathbb{C}^n$, $1 \leq k < n$, with bounded image. The first affirmative answer was given two years later by Jones [21] for $k = 1$ and $n \geq 3$. Only recently have Alarcón and Forstnerič [4], as application of Jones' result, provided examples for any $k \in \mathbb{N}$ and $n \geq 3k$. The problem has remained open in the lowest dimensional case: complex curves in \mathbb{C}^2 (see [4, Question 1]). This particular case is especially interesting for topological and analytical reasons that will be more apparent later in this introduction.

The aim of this paper is to fill this gap, proving considerably more:

Theorem 1.1. *Any convex domain $\mathcal{B} \subset \mathbb{C}^2$ carries complete properly embedded complex curves.*

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The topology of the curves in Theorem 1.1 is not controlled (see Question 1.5 below). The conclusion of Theorem 1.1 is obvious when $\mathcal{B} = \Omega \times \mathbb{C}$, where $\Omega \subset \mathbb{C}$ is a convex domain (the flat curve $\{p\} \times \mathbb{C}$, $p \in \Omega$, is complete and properly embedded in $\Omega \times \mathbb{C}$). Further, complete holomorphic graphs over Ω were constructed in [1, 2, 3]. Regarding the case $\mathcal{B} = \mathbb{C}^2$, Bell and Narasimhan [8] conjectured that any open Riemann surface can be properly holomorphically embedded in \mathbb{C}^2 (obviously, this is possible in no other convex domain of \mathbb{C}^2). This classical problem is still open (cf. [13, 14, 10, 7] and references therein). Anyway, all the complex curves in these particular instances are far from being bounded.

Following Yang's results [29], no complete complex hypersurface of \mathbb{C}^n , $n > 1$, has strongly negative holomorphic sectional curvature, and the existence of a complete bounded complex k -dimensional submanifold of \mathbb{C}^n , $n > k$, implies the existence of such a submanifold of \mathbb{C}^{2n} with strongly negative holomorphic sectional curvature. Related existence results can be found in [4]. Theorem 1.1 has nice consequences regarding these questions:

Corollary 1.2. *Let $k \in \mathbb{N}$. There exist*

- (i) *complete bounded embedded complex k -dimensional submanifolds of \mathbb{C}^{2k} , and*
- (ii) *complete bounded embedded complex k -dimensional submanifolds of \mathbb{C}^{4k} with strongly negative holomorphic sectional curvature.*

Proof. Let $X: \mathcal{R} \rightarrow B$ be a complete holomorphic embedding given by Theorem 1.1, where \mathcal{R} is an open Riemann surface and $B \subset \mathbb{C}^2$ is the Euclidean open ball of radius $1/\sqrt{k}$ centered at the origin. Denote by $\mathcal{R}^k = \mathcal{R} \times \cdots \times \mathcal{R}$ the cartesian product of k copies of \mathcal{R} and likewise for B^k . Then the map

$$\varphi: \mathcal{R}^k \rightarrow B^k \subset \mathbb{C}^{2k}, \quad \varphi(p_1, \dots, p_k) = (X(p_1), \dots, X(p_k)),$$

is a complete bounded holomorphic embedding, proving (i) (see [4, Corollary 1]).

To check (ii), notice that $\varphi(\mathcal{R}^k) \subset B_1$, where $B_1 \subset \mathbb{C}^{2k}$ is the Euclidean open ball of radius 1 centered at the origin. Setting $F: B_1 \rightarrow \mathbb{C}^{4k}$, $F(z_1, \dots, z_{2k}) = (z_1, \dots, z_{2k}, e^{z_1}, \dots, e^{z_{2k}})$. Then the map $F \circ \varphi: \mathcal{R}^k \rightarrow \mathbb{C}^{4k}$ proves (ii) (see [29, Sec. 1]). \square

An interesting question is whether, given $k \in \mathbb{N}$, the dimensions $2k$ and $4k$ in the above corollary are optimal.

There are many known examples of complete bounded *immersed* complex curves in \mathbb{C}^2 : Jones [21] constructed a simply-connected one, Martín, Umehara, and Yamada [22] provided examples with some finite topologies, and Alarcón and López [6] gave examples of arbitrary topological type. On the other hand, Alarcón and Forstnerič [4] showed that every bordered Riemann surface is a complete curve in a ball of \mathbb{C}^2 . Furthermore, the curves in [6, 4] have the extra property of being proper in any given convex domain. However, the construction of complete bounded *embedded* complex curves in \mathbb{C}^2 turns out to be a much more involved problem. The main reason is that (in contrast to what happens in \mathbb{C}^n , $n \geq 3$, where the general position of complex curves is embedded) *self-intersections of complex curves in \mathbb{C}^2 are stable under deformations*. Nevertheless, there is a simple self-intersection removal method which consists in replacing every normal

crossing in a complex curve by an embedded annulus. Unfortunately, this surgery does not necessarily preserve the length of divergent arcs (hence completeness); indeed, self-intersection points of immersed complex curves generate *shortcuts* in the arising desingularized curves, so divergent arcs of shorter length.

In order to overcome this difficulty, we consider a stronger notion of completeness (Def. 1.3). Given a holomorphic immersion $X: M^k \rightarrow \mathbb{C}^n$, we denote by $\text{dist}_{X(M^k)}$ the (intrinsic) induced Euclidean distance in $X(M^k)$ given by

$$\text{dist}_{X(M^k)}(p, q) = \inf\{\ell(\gamma) : \gamma \subset X(M^k) \text{ a rectifiable arc connecting } p \text{ and } q\}$$

for any $p, q \in X(M^k)$, where $\ell(\cdot)$ means Euclidean length in \mathbb{C}^n . If X is injective, the function $\text{dist}_{X(M^k)} \circ (X, X): M^k \times M^k \rightarrow \mathbb{R}$ is the intrinsic distance in M^k induced by X ; otherwise it is a pseudo-distance. We call $\text{dist}_{X(M^k)}$ and $(X(M^k), \text{dist}_{X(M^k)})$ the *image distance* and the *image metric space* of $X: M^k \rightarrow \mathbb{C}^n$.

Definition 1.3. A holomorphic immersion $X: M^k \rightarrow \mathbb{C}^n$ is said to be *image complete* if $(X(M^k), \text{dist}_{X(M^k)})$ is a complete metric space (in other words, if every rectifiable divergent arc in $X(M^k)$ has infinite Euclidean length).

Obviously, image completeness implies completeness, and both notions are equivalent for injective immersions. The image distance is very convenient for our purposes since it is preserved by self-intersection removal procedures. As a matter of fact, the proof of Theorem 1.1 is connected with the general existence theorem 1.4 below. As far as the authors' knowledge extends, the following are the first known examples of image complete bounded immersed complex curves in \mathbb{C}^2 .

Theorem 1.4. *Let S be an open orientable smooth surface and let $\mathcal{B} \subset \mathbb{C}^2$ be a convex domain. Then there exist a complex structure \mathcal{J} on S and an image complete proper holomorphic immersion $(S, \mathcal{J}) \rightarrow \mathcal{B}$.*

Let us say a few words about the proof of Theorem 1.1 (see the more general Theorem 3.1 in Sec. 3). The proof relies on a recursive process involving approximation by embedded complex curves in \mathbb{C}^2 (Lemma 3.2), which is the core of the paper. In that lemma we prove that any embedded compact complex curve \mathcal{C} with boundary $b\mathcal{C}$ in the frontier $\text{Fr } \mathcal{D}$ of a regular strictly convex domain \mathcal{D} can be approximated by another *embedded* complex curve \mathcal{C}' with $b\mathcal{C}' \subset \text{Fr } \mathcal{D}'$, where \mathcal{D}' is any given larger convex domain. The curve \mathcal{C}' has possibly higher topological genus than \mathcal{C} and contains a biholomorphic copy of it, roughly speaking $\mathcal{C} \subset \mathcal{C}'$. Furthermore, this procedure can be done so that $\mathcal{C}' \setminus \mathcal{C}$ lies in $\mathcal{D}' \setminus \mathcal{D}$ and the intrinsic Euclidean distance in \mathcal{C}' from \mathcal{C} to $b\mathcal{C}'$ is suitably larger than the distance between \mathcal{D} and $\text{Fr } \mathcal{D}'$ in \mathbb{C}^2 . These facts will be the key to obtaining properness and completeness while preserving boundedness in the proof of Theorem 3.1.

In order to prove Lemma 3.2 (see Sec. 4), we introduce some configurations of slabs in \mathbb{C}^2 that we call *tangent nets* (see Subsec. 4.1). Given a regular strictly convex domain $\mathcal{D} \Subset \mathbb{C}^2$, a tangent net \mathcal{T} for \mathcal{D} is a tubular neighborhood of a finite collection of (affine) hyperplanes tangent to the frontier $\text{Fr } \mathcal{D}$ (see Def. 4.1 and Fig. 4.1). Given another regular

strictly convex domain \mathcal{D}' with $\mathcal{D} \Subset \mathcal{D}' \Subset \mathbb{C}^2$, we show the existence of tangent nets \mathcal{T} for \mathcal{D} with the property that any Jordan arc in \mathcal{T} connecting $\text{Fr } \mathcal{D}$ and $\text{Fr } \mathcal{D}'$ has large length compared to the distance between \mathcal{D} and $\text{Fr } \mathcal{D}'$ in \mathbb{C}^2 (see Lemma 4.2). The second step in the proof of Lemma 3.2 is approximation by immersed complex curves along tangent nets (see Lemma 4.3 in Subsec. 4.2). We show that any immersed compact complex curve Σ in \mathbb{C}^2 with boundary $b\Sigma \subset \text{Fr } \mathcal{D}$ can be approximated by another one $\tilde{\Sigma}$ such that $b\tilde{\Sigma} \subset \text{Fr } \mathcal{D}'$ and $\tilde{\Sigma} \cap (\overline{\mathcal{D}'} \setminus \mathcal{D})$ is contained in a suitable tangent net for \mathcal{D} . This allows us to estimate the growth of the *image diameter* (according to Def. 1.3) of $\tilde{\Sigma}$, and conclude that it is large compared to the distance between \mathcal{D} and $\text{Fr } \mathcal{D}'$. This represents a clear innovation with respect to previous constructions where only the growth of the *intrinsic diameter* could be estimated (cf. [26, 6, 4] and references therein). We conclude the proof of Lemma 3.2 by combining the above two results with a desingularization adapted to our setting (see Lemma 4.5 in Subsec. 4.3). To the best of the authors' knowledge, this is the first such application of the surgery technique in the literature. Since this method increases the topology, the complex curves in Theorem 1.1 could have infinite genus.

On the other hand, Theorem 1.4 follows from a standard recursive application of Lemmas 4.2 and 4.3 (see the more precise Theorem 5.1 in Sec. 5).

Since complex curves in \mathbb{C}^2 are area-minimizing surfaces in \mathbb{R}^4 , our results are connected with the so-called *Calabi–Yau problem for embedded surfaces*. This problem deals with the existence of complete embedded minimal surfaces in bounded domains of \mathbb{R}^3 . Although it still remains open, it is known that solutions must have either infinite genus or uncountably many ends (see Colding and Minicozzi [9] and Meeks, Pérez, and Ros [23]). On the other hand, the construction of embedded complex discs in \mathbb{C}^2 is a subject with vast literature: see for instance [15, 12, 11, 16, 17]. Thus, in view of Theorem 1.1, one is led to ask:

Question 1.5. *Do there exist complete bounded holomorphic embeddings $M \rightarrow \mathbb{C}^2$ with M an open Riemann surface of finite topology? What if M is the complex unit disc?*

Our main tools are the classical Runge and Mergelyan approximation theorems for holomorphic functions and basic convex body theory.

2. Preliminaries

We denote by $\|\cdot\|$, $\langle \cdot, \cdot \rangle$, $\text{dist}(\cdot, \cdot)$, $\ell(\cdot)$, and $\text{diam}(\cdot)$ the Euclidean norm, inner product, distance, length, and diameter in \mathbb{R}^n , $n \in \mathbb{N}$. Given two points p and q in \mathbb{R}^n , we denote by $[p, q]$ (resp., $]p, q[$) the closed (resp., open) straight line segment in \mathbb{R}^n connecting p and q .

In the complex Euclidean space $\mathbb{C}^n \cong \mathbb{R}^{2n}$ we denote by $\langle \cdot, \cdot \rangle: \mathbb{C}^n \times \mathbb{C}^n \rightarrow \mathbb{C}$ the bilinear Hermitian product defined by $\langle (\zeta_1, \dots, \zeta_n), (\xi_1, \dots, \xi_n) \rangle = \sum_{i=1}^n \zeta_i \bar{\xi}_i$, where $\bar{\cdot}$ means complex conjugation. Observe that $\langle \cdot, \cdot \rangle = \Re(\langle \cdot, \cdot \rangle)$. Given $p \in \mathbb{C}^n$, we denote $\langle p \rangle^\perp = \{q \in \mathbb{C}^n: \langle p, q \rangle = 0\}$, $\text{span}_{\mathbb{R}}(p) = \{tp: t \in \mathbb{R}\}$, and $\text{span}_{\mathbb{C}}(p) = \{\zeta p: \zeta \in \mathbb{C}\}$.

Given an n -dimensional topological real manifold M with boundary, we denote by bM the $(n-1)$ -dimensional topological manifold determined by its boundary points.

For any subset $A \subset M$, we denote by A° , \bar{A} , and $\text{Fr } A = \bar{A} \setminus A^\circ$ the interior, the closure, and the frontier of A in M , respectively. Given subsets A and B of M , we write $A \Subset B$ if \bar{A} is compact and $\bar{A} \subset B^\circ$. By a *domain* in M we mean an open connected subset of $M \setminus bM$. By a *region* in M we mean a proper topological subspace of M which is an n -dimensional compact manifold with non-empty boundary.

A topological surface S is said to be *open* if it is non-compact and $bS = \emptyset$. A domain \mathcal{R} in an open connected Riemann surface \mathcal{N} is said to be a *bordered domain* if $\mathcal{R} \Subset \mathcal{N}$ and $\bar{\mathcal{R}}$ is a region with smooth boundary $b\bar{\mathcal{R}} = \text{Fr } \mathcal{R}$. In this case, $b\bar{\mathcal{R}}$ consists of finitely many smooth Jordan curves.

Given a compact topological space K and a continuous map $f: K \rightarrow \mathbb{R}^n$, we denote by

$$\|f\|_{0,K} := \max_{p \in K} \|f(p)\|$$

the maximum norm of f on K . The corresponding space of continuous functions $K \rightarrow \mathbb{R}^n$ will be endowed with the C^0 topology associated to $\|\cdot\|_{0,K}$.

Let \mathcal{N} be an open Riemann surface endowed with a nowhere-vanishing holomorphic 1-form $\vartheta_{\mathcal{N}}$ (such a 1-form exists by the Gunning–Narasimhan theorem [20]). Let K be a compact set in \mathcal{N} . A function $f: K \rightarrow \mathbb{C}^n$, $n \in \mathbb{N}$, is said to be holomorphic if it is the restriction to K of a holomorphic function defined on a domain in \mathcal{N} containing K . In that case, we denote by

$$\|f\|_{1,K;\vartheta_{\mathcal{N}}} := \max_{P \in K} \max \left\{ \|f(P)\|, \left\| \frac{df}{\vartheta_{\mathcal{N}}}(P) \right\| \right\} \quad (2.1)$$

the C^1 maximum norm of f on K (with respect to $\vartheta_{\mathcal{N}}$). If there is no ambiguity, we write $\|f\|_{1,K}$ instead of $\|f\|_{1,K;\vartheta_{\mathcal{N}}}$. The space of holomorphic functions $K \rightarrow \mathbb{C}^n$ will be endowed with the C^1 topology associated to the norm $\|\cdot\|_{1,K;\vartheta_{\mathcal{N}}}$, which does not depend on the choice of $\vartheta_{\mathcal{N}}$.

Given a holomorphic immersion $f: K \rightarrow \mathbb{C}^n$, a point $w \in f(K)$ is said to be a *double point* of f (or of $f(K)$) if $f^{-1}(w)$ contains more than one point. A double point $w \in f(K)$ is said to be a *normal crossing* if $f^{-1}(w)$ consists of precisely two points, P and Q , and $df_P(T_P\mathcal{N})$ and $df_Q(T_Q\mathcal{N})$ are transverse.

Remark 2.1. It is well known that any holomorphic function $K \rightarrow \mathbb{C}^n$, $n \geq 3$, can be approximated in the C^1 topology on K by holomorphic embeddings.

However, this is no longer true in the lowest dimensional case: double points of an immersed complex curve in \mathbb{C}^2 are stable under deformations. Anyway, any holomorphic function $K \rightarrow \mathbb{C}^2$ can be approximated in the C^1 topology on K by holomorphic immersions all whose double points are normal crossings. We call this property the *general position argument*.

Throughout this paper we will deal with regular convex domains $\mathcal{D} \Subset \mathbb{C}^2$, bordered domains $\mathcal{R} \Subset \mathcal{N}$, and holomorphic immersions $X: \bar{\mathcal{R}} \rightarrow \mathbb{C}^2$ with $X(\bar{\mathcal{R}}) \subset \mathcal{D}$. In this setting, it is interesting to notice the following:

Remark 2.2. If $X(b\bar{\mathcal{R}}) \subset \text{Fr } \mathcal{D}$ then $X(\bar{\mathcal{R}})$ and $\text{Fr } \mathcal{D}$ meet transversally.

Indeed, assume for a moment that $X(\overline{\mathcal{R}})$ and $\text{Fr } \mathcal{D}$ meet tangentially at $p := X(P)$, $P \in b\overline{\mathcal{R}}$. By the basic theory of harmonic functions, there exists a sufficiently small neighborhood U of P in \mathcal{M} such that $\alpha := X^{-1}(p + T_p \text{Fr } \mathcal{D}) \cap U$ consists of a system of at least two analytical arcs intersecting equiangularly at P . Furthermore, contiguous components of $U \setminus \alpha$ lie on opposite sides of $p + T_p \text{Fr } \mathcal{D}$. On the other hand, since $\overline{\mathcal{R}}$ has smooth boundary and $X(b\overline{\mathcal{R}}) \subset \text{Fr } \mathcal{D}$, we have $X(U \cap \overline{\mathcal{R}}) \subset \mathcal{D}$, and so $X(U \cap \overline{\mathcal{R}})$ must lie on one side of $p + T_p \text{Fr } \mathcal{D}$, a contradiction.

A compact (in most cases arcwise-connected) subset K of an open Riemann surface \mathcal{N} is said to be *Runge* if $\mathcal{N} \setminus K$ has no relatively compact connected components in \mathcal{N} ; equivalently, if the inclusion map $i: K \hookrightarrow \mathcal{N}$ induces a group monomorphism $i_*: \mathcal{H}_1(K, \mathbb{Z}) \rightarrow \mathcal{H}_1(\mathcal{N}, \mathbb{Z})$ on homology. In this case we consider $\mathcal{H}_1(K, \mathbb{Z}) \subset \mathcal{H}_1(\mathcal{N}, \mathbb{Z})$ via this monomorphism. Two Runge compact sets K_1 and K_2 of \mathcal{N} are said to be (homeomorphically) *isotopic* if there exists a homeomorphism $\eta: K_1 \rightarrow K_2$ such that the induced homology morphism η_* equals $\text{Id}_{\mathcal{H}_1(K_1, \mathbb{Z})}$. Such an η is said to be an *isotopical homeomorphism*. Two Runge regions K_1 and K_2 of \mathcal{N} are (homeomorphically) isotopic if and only if $\mathcal{H}_1(K_1, \mathbb{Z}) = \mathcal{H}_1(K_2, \mathbb{Z})$.

2.1. Convex domains

A convex domain $\mathcal{D} \subset \mathbb{R}^n$, $\mathcal{D} \neq \mathbb{R}^n$, $n \geq 2$, is said to be *regular* (resp., *analytic*) if its frontier $\text{Fr } \mathcal{D} = \overline{\mathcal{D}} \setminus \mathcal{D}$ is a regular (resp., analytic) hypersurface of \mathbb{R}^n .

Let \mathcal{D} be a regular convex domain of \mathbb{R}^n , $\mathcal{D} \neq \mathbb{R}^n$, $n \geq 2$.

For any $p \in \text{Fr } \mathcal{D}$ we denote by $T_p \text{Fr } \mathcal{D}$ the tangent space to $\text{Fr } \mathcal{D}$ at p . Recall that $\mathcal{D} \cap (p + T_p \text{Fr } \mathcal{D}) = \emptyset$ for all $p \in \text{Fr } \mathcal{D}$.

We denote by $\nu_{\mathcal{D}}: \text{Fr } \mathcal{D} \rightarrow \mathbb{S}^{n-1}$ the outward pointing unit normal to $\text{Fr } \mathcal{D}$. For any $p \in \text{Fr } \mathcal{D}$ and $v \in (T_p \text{Fr } \mathcal{D}) \cap \mathbb{S}^{n-1}$, we denote by $\kappa_{\mathcal{D}}(p, v)$ the normal curvature at p in the direction of v with respect to $-\nu_{\mathcal{D}}$; obviously $\kappa_{\mathcal{D}}(p, v) \geq 0$ since \mathcal{D} is convex. Let $\kappa(p) \geq 0$ be the maximum of the principal curvatures of $\text{Fr } \mathcal{D}$ at p with respect to $-\nu_{\mathcal{D}}$, and set

$$\kappa(\mathcal{D}) := \sup\{\kappa(p) : p \in \text{Fr } \mathcal{D}\} \geq 0. \quad (2.2)$$

The domain \mathcal{D} is said to be *strictly convex* if $\kappa_{\mathcal{D}}(p, v) > 0$ for all $p \in \text{Fr } \mathcal{D}$ and $v \in (T_p \text{Fr } \mathcal{D}) \cap \mathbb{S}^{n-1}$. In this case, $\overline{\mathcal{D}} \cap (p + T_p \text{Fr } \mathcal{D}) = \{p\}$ for all $p \in \text{Fr } \mathcal{D}$. If \mathcal{D} is bounded (i.e., $\mathcal{D} \Subset \mathbb{R}^n$) and strictly convex, then $0 < \kappa(\mathcal{D}) < \infty$.

Assume that \mathcal{D} is bounded and strictly convex. For any $t > -1/\kappa(\mathcal{D})$ we denote by \mathcal{D}_t the bounded regular strictly convex domain in \mathbb{R}^n with frontier $\text{Fr } \mathcal{D}_t = \{p + t\nu_{\mathcal{D}}(p) : p \in \text{Fr } \mathcal{D}\}$, that is, the *parallel convex domain* to \mathcal{D} at (oriented) distance t . Observe that $\mathcal{D} = \mathcal{D}_0$ and $\mathcal{D}_{t_1} \Subset \mathcal{D}_{t_2}$ if $t_1 < t_2$.

For any couple of compact subsets K and O in \mathbb{R}^n , the *Hausdorff distance* between K and O is given by

$$\delta^H(K, O) := \max\left\{\sup_{x \in K} \inf_{y \in O} \|x - y\|, \sup_{y \in K} \inf_{x \in O} \|x - y\|\right\}.$$

A sequence $\{K^j\}_{j \in \mathbb{N}}$ of (possibly unbounded) closed subsets of \mathbb{R}^n is said to converge in the *Hausdorff topology* to a closed subset K^0 of \mathbb{R}^n if $\{K^j \cap B\}_{j \in \mathbb{N}} \rightarrow K^0 \cap B$ in the

Hausdorff distance for any closed Euclidean ball $B \subset \mathbb{R}^n$. If $K^j \Subset K^{j+1} \subset K^0$ for all $j \in \mathbb{N}$ and $\{K^j\}_{j \in \mathbb{N}} \rightarrow K^0$ in the Hausdorff topology, then we write $\{K^j\}_{j \in \mathbb{N}} \nearrow K^0$.

Theorem 2.3 ([25, 24]). *Let $\mathcal{B} \subset \mathbb{R}^n$ be a (possibly neither bounded nor regular) convex domain. Then there exists a sequence $\{\mathcal{D}^j\}_{j \in \mathbb{N}}$ of bounded strictly convex analytic domains in \mathbb{R}^n with $\{\overline{\mathcal{D}^j}\}_{j \in \mathbb{N}} \nearrow \overline{\mathcal{B}}$.*

The following *distance type* function for convex domains will play a fundamental role throughout this paper.

Definition 2.4. Let \mathcal{D} and \mathcal{D}' be bounded regular strictly convex domains in \mathbb{R}^n ($n \geq 2$) with $\mathcal{D} \Subset \mathcal{D}'$. We denote

$$\mathbf{d}(\mathcal{D}, \text{Fr } \mathcal{D}') := \left(\text{dist}(\mathcal{D}, \text{Fr } \mathcal{D}') + \frac{1}{\kappa(\mathcal{D})} \right) \sqrt{\frac{\text{dist}(\mathcal{D}, \text{Fr } \mathcal{D}')}{\text{dist}(\mathcal{D}, \text{Fr } \mathcal{D}') + 2/\kappa(\mathcal{D})}}$$

(see (2.2)).

Remark 2.5. Observe that $\mathbf{d}(\mathcal{D}, \text{Fr } \mathcal{D}') > \text{dist}(\mathcal{D}, \text{Fr } \mathcal{D}')$. Furthermore, $\mathbf{d}(\mathcal{D}, \cdot)$ and $\sqrt{\text{dist}(\mathcal{D}, \cdot)}$ are infinitesimally comparable in the sense that

$$\lim_{n \rightarrow \infty} \frac{\sqrt{\text{dist}(\mathcal{D}, \text{Fr } \mathcal{D}^n)}}{\mathbf{d}(\mathcal{D}, \text{Fr } \mathcal{D}^n)} = \sqrt{2\kappa(\mathcal{D})} > 0$$

for any sequence $\{\mathcal{D}^n\}_{n \in \mathbb{N}}$ of bounded regular strictly convex domains such that $\mathcal{D} \Subset \mathcal{D}^n$ for all $n \in \mathbb{N}$ and $\{\mathcal{D}^n\}_{n \in \mathbb{N}} \rightarrow \overline{\mathcal{D}}$ in the Hausdorff topology.

Lemma 2.7 below will simplify the exposition of the proof of our main results. Its proof relies on Remark 2.5.

Definition 2.6. Let \mathcal{B} be a (possibly neither bounded nor regular) convex domain in \mathbb{R}^n . A sequence $\{\mathcal{D}^k\}_{k \in \mathbb{N}}$ of convex domains in \mathbb{R}^n is said to be *\mathbf{d} -proper in \mathcal{B}* if \mathcal{D}^k is bounded, regular, and strictly convex for all $k \in \mathbb{N}$, $\{\overline{\mathcal{D}^k}\}_{k \in \mathbb{N}} \nearrow \overline{\mathcal{B}}$ in the Hausdorff topology, and

$$\sum_{k \in \mathbb{N}} \mathbf{d}(\mathcal{D}^k, \text{Fr } \mathcal{D}^{k+1}) = \infty.$$

Lemma 2.7. *Any convex domain in \mathbb{R}^n admits a \mathbf{d} -proper sequence of convex domains.*

Proof. Let \mathcal{B} be a convex domain in \mathbb{R}^n . Let $\{C^j\}_{j \in \mathbb{N}}$ be a sequence of bounded strictly convex analytic domains in \mathbb{R}^n with $\{\overline{C^j}\}_{j \in \mathbb{N}} \nearrow \overline{\mathcal{B}}$ (cf. Theorem 2.3). For the sake of simplicity write $d_j := \text{dist}(C^j, \text{Fr } C^{j+1})$ and $\kappa_j := \kappa(C^j)$ for all $j \in \mathbb{N}$.

For each $j \in \mathbb{N}$ choose $m_j \in \mathbb{N}$ large enough that

$$\sum_{a=1}^{m_j} \frac{1}{a} \geq \sqrt{\frac{6d_j\kappa_j^2 + 2\pi^2\kappa_j}{6d_j}}. \quad (2.3)$$

Denote $d_{a,j} = d_j \frac{6}{\pi^2} \sum_{h=1}^a 1/h^2$, and notice that $d_{a,j} < d_j$ (take into account that $\sum_{h=1}^\infty 1/h^2 = \pi^2/6$). Set $C^{0,j} := C^j$ and $C^{a,j} := (C^j)_{d_{a,j}}$, for all $a = 1, \dots, m_j$, the outer parallel convex domain to C^j at distance $d_{a,j}$. Observe that $C^{a,j}$ is analytic and strictly convex,

$$C^j \Subset C^{a,j} \Subset C^{a+1,j} \Subset C^{j+1}, \tag{2.4}$$

$$\text{dist}(C^{a,j}, \text{Fr } C^{a+1,j}) = d_{a+1,j} - d_{a,j} = 6d_j/(\pi(a+1))^2, \tag{2.5}$$

and

$$\kappa(C^{a,j}) = \kappa_j/(1 + d_{a,j}\kappa_j) \leq \kappa_j \quad \text{for all } j \in \mathbb{N} \text{ and } a \in \{0, \dots, m_j - 1\}. \tag{2.6}$$

Set

$$f:]0, \infty[\times]0, \infty[\rightarrow]0, \infty[, \quad f(d, \kappa) = (d + 1/\kappa) \sqrt{\frac{d}{d + 2/\kappa}},$$

and note that $f(d, \cdot)$ is decreasing for all $d > 0$ and $f(6d_j/(\pi(a+1))^2, \kappa(C^{a,j})) = \mathbf{d}(C^{a,j}, \text{Fr } C^{a+1,j})$ for all $j \in \mathbb{N}$ and $a \in \{1, \dots, m_j - 1\}$ (see (2.5)). Therefore,

$$\begin{aligned} \sum_{a=0}^{m_j-1} \mathbf{d}(C^{a,j}, \text{Fr } C^{a+1,j}) &= \sum_{a=0}^{m_j-1} f(6d_j/(\pi(a+1))^2, \kappa(C^{a,j})) \\ &\stackrel{(2.6)}{\geq} \sum_{a=0}^{m_j-1} f(6d_j/(\pi(a+1))^2, \kappa_j) \\ &> \sqrt{\frac{6d_j}{6d_j\kappa_j^2 + 2\pi^2\kappa_j}} \left(\sum_{a=0}^{m_j-1} \frac{1}{a+1} \right) \stackrel{(2.3)}{\geq} 1. \end{aligned}$$

Let $\{\mathcal{D}^k\}_{k \in \mathbb{N}}$ denote the enumeration of $\{C^{a,j} : j \in \mathbb{N}, a \in \{0, \dots, m_j\}\}$ such that $\mathcal{D}^k \Subset \mathcal{D}^{k+1}$ for all $k \in \mathbb{N}$ (see (2.4)). Then

$$\sum_{k \in \mathbb{N}} \mathbf{d}(\mathcal{D}^k, \text{Fr } \mathcal{D}^{k+1}) \geq \sum_{j \in \mathbb{N}} \left(\sum_{a=0}^{m_j-1} \mathbf{d}(C^{a,j}, \text{Fr } C^{a+1,j}) \right) \geq \sum_{j \in \mathbb{N}} 1 = \infty.$$

This property and the fact that $\{\overline{C^j}\}_{j \in \mathbb{N}} \nearrow \overline{B}$ imply that the sequence $\{\mathcal{D}^k\}_{k \in \mathbb{N}}$ is \mathbf{d} -proper in \mathcal{B} . □

3. Complete properly embedded complex curves in convex domains of \mathbb{C}^2

In this section we prove the main result of this paper, Theorem 1.1. It will be a particular instance of the following more precise result.

Theorem 3.1. *Let \mathcal{B} be a (possibly neither bounded nor regular) convex domain in \mathbb{C}^2 . Let $\mathcal{D} \Subset \mathcal{B}$ be a strictly convex bounded regular domain. Let \mathcal{N} be an open Riemann surface equipped with a nowhere-vanishing holomorphic 1-form $\vartheta_{\mathcal{N}}$, and let \mathcal{R} be a bordered domain in \mathcal{N} . Then, for any $\varepsilon \in]0, \min\{\text{dist}(\overline{\mathcal{D}}, \text{Fr } \mathcal{B}), 1/\kappa(\mathcal{D})\}[$ and any holomorphic embedding $X: \overline{\mathcal{R}} \rightarrow \mathbb{C}^2$ such that*

$$X(b\overline{\mathcal{R}}) \subset \text{Fr } \mathcal{D}, \quad (3.1)$$

there exist an open Riemann surface \mathcal{M} (possibly of infinite topological genus) and a complete holomorphic embedding $Y: \mathcal{M} \rightarrow \mathbb{C}^2$ enjoying the following properties:

- (i) $\overline{\mathcal{R}} \subset \mathcal{M}$.
- (ii) $\|Y - X\|_{1, \overline{\mathcal{R}}; \vartheta_{\mathcal{N}}} < \varepsilon$ (see (2.1)).
- (iii) $Y(\mathcal{M}) \subset \mathcal{B}$ and $Y: \mathcal{M} \rightarrow \mathcal{B}$ is a proper map.
- (iv) $Y(\mathcal{M} \setminus \overline{\mathcal{R}}) \subset \mathcal{B} \setminus \mathcal{D}_{-\varepsilon}$.

Theorem 3.1 follows from a recursive process involving the following result on approximation by embedded complex curves.

Lemma 3.2 (Approximation by embedded complex curves). *Let \mathcal{D} and \mathcal{D}' be bounded regular strictly convex domains in \mathbb{C}^2 with $\mathcal{D} \Subset \mathcal{D}'$. Let \mathcal{N} be an open Riemann surface equipped with a nowhere-vanishing holomorphic 1-form $\vartheta_{\mathcal{N}}$ and let \mathcal{U} be a bordered domain in \mathcal{N} . Then, for any $\varepsilon \in]0, \min\{\text{dist}(\overline{\mathcal{D}}, \text{Fr } \mathcal{D}'), 1/\kappa(\mathcal{D})\}[$ and any holomorphic embedding $X: \overline{\mathcal{U}} \rightarrow \mathbb{C}^2$ such that*

$$X(b\overline{\mathcal{U}}) \subset \text{Fr } \mathcal{D}, \quad (3.2)$$

there exist an open Riemann surface \mathcal{N}' , a bordered domain $\mathcal{U}' \Subset \mathcal{N}'$, and a holomorphic embedding $X': \overline{\mathcal{U}'} \rightarrow \mathbb{C}^2$ enjoying the following properties:

- (i) $\overline{\mathcal{U}} \subset \mathcal{U}'$.
- (ii) $\|X' - X\|_{1, \overline{\mathcal{U}}; \vartheta_{\mathcal{N}}} < \varepsilon$.
- (iii) $X'(b\overline{\mathcal{U}'}) \subset \text{Fr } \mathcal{D}'$.
- (iv) $X'(\overline{\mathcal{U}'} \setminus \mathcal{U}) \cap \overline{\mathcal{D}}_{-\varepsilon} = \emptyset$.
- (v) $\ell(X'(\gamma)) > \mathbf{d}(\mathcal{D}, \text{Fr } \mathcal{D}') - \varepsilon$ for any Jordan arc γ in $\overline{\mathcal{U}'}$ connecting $b\overline{\mathcal{U}}$ and $b\overline{\mathcal{U}'}$.

Roughly speaking, this lemma ensures that any embedded compact complex curve $X: \overline{\mathcal{U}} \rightarrow \mathbb{C}^2$ with boundary in the frontier of a regular strictly convex domain $\mathcal{D} \Subset \mathbb{C}^2$ can be approximated by another *embedded* complex curve $X': \overline{\mathcal{U}'} \rightarrow \mathbb{C}^2$ with boundary in the frontier of a larger convex domain \mathcal{D}' . This can be done so that $X'(\overline{\mathcal{U}'} \setminus \mathcal{U})$ lies outside \mathcal{D} and the intrinsic Euclidean diameter of $X'(\overline{\mathcal{U}'})$ exceeds in $\mathbf{d}(\mathcal{D}, \text{Fr } \mathcal{D}')$ the one of $X(\mathcal{U})$ (see Def. 2.4). These facts will be the key to obtaining properness and completeness while preserving boundedness in the proof of Theorem 3.1. We point out that \mathcal{U}' has a possibly higher topological genus than \mathcal{U} .

Lemma 3.2 will be proved later in Sec. 4 (see in particular Subsec. 4.4). We are now ready to prove our main result.

Proof of Theorem 3.1. Denote $\mathcal{D}^0 := \mathcal{D}$ and let $\{\mathcal{D}^n\}_{n \in \mathbb{N}}$ be a \mathbf{d} -proper sequence of convex domains in \mathcal{B} with $\mathcal{D}^0 \Subset \mathcal{D}^1$ (see Def. 2.6 and Lemma 2.7). Set $\mathcal{N}_0 = \mathcal{N}$, $\vartheta_0 = \vartheta_{\mathcal{N}}$, $\mathcal{U}_0 = \mathcal{R}$, and $X_0 = X$. Fix any $\varepsilon_0 \in]0, \varepsilon/2[$.

Let us recursively construct a sequence $\{\Xi_n = (\mathcal{N}_n, \vartheta_n, \mathcal{U}_n, X_n, \epsilon_n)\}_{n \in \mathbb{N}}$, where

- \mathcal{N}_n is an open Riemann surface,
- ϑ_n is a nowhere-vanishing holomorphic 1-form on \mathcal{N}_n ,
- $\mathcal{U}_n \Subset \mathcal{N}_n$ is a bordered domain,
- $X_n : \bar{\mathcal{U}}_n \rightarrow \mathbb{C}^2$ is a holomorphic embedding, and
- $\epsilon_n \in]0, \min\{\text{dist}(\mathcal{D}^{n-1}, \text{Fr } \mathcal{D}^n), 1/\kappa(\mathcal{D}^{n-1})\}[$,

so that the following properties are satisfied for all $n \in \mathbb{N}$:

(A_n) $\bar{\mathcal{U}}_{n-1} \subset \mathcal{U}_n$ (in particular, the closure of \mathcal{U}_{n-1} in \mathcal{N}_{n-1} agrees with the one in \mathcal{N}_n).

(B_n) $\min_{\bar{\mathcal{U}}_{n-1}} |\vartheta_{n-1}/\vartheta_n| > 1$.

(C_n) ϵ_n satisfies

(C.1_n) $\epsilon_n < \epsilon_{n-1}/2 < \epsilon/2^{n+1}$ and

(C.2_n) every holomorphic function $F : \bar{\mathcal{U}}_{n-1} \rightarrow \mathbb{C}^2$ with $\|F - X_{n-1}\|_{1, \bar{\mathcal{U}}_{n-1}; \vartheta_{n-1}} < 2\epsilon_n$ is an embedding on $\bar{\mathcal{U}}_{n-1}$.

(D_n) $\|X_n - X_{n-1}\|_{1, \bar{\mathcal{U}}_{n-1}; \vartheta_{n-1}} < \epsilon_n$.

(E_n) $X_n(b\bar{\mathcal{U}}_n) \subset \text{Fr } \mathcal{D}^n$; hence $X_n(\bar{\mathcal{U}}_n)$ and $\text{Fr } \mathcal{D}^n$ meet transversally (see Remark 2.2).

(F_n) $X_n(\bar{\mathcal{U}}_a \setminus \mathcal{U}_{a-1}) \cap \bar{\mathcal{D}}_{-\epsilon_a}^{a-1} = \emptyset$ for all $a \in \{1, \dots, n\}$.

(G_n) $\ell(X_n(\gamma)) > \mathbf{d}(\mathcal{D}^{a-1}, \text{Fr } \mathcal{D}^a) - \epsilon_a$ for any Jordan arc γ in $\bar{\mathcal{U}}_a$ connecting $b\bar{\mathcal{U}}_{a-1}$ and $b\bar{\mathcal{U}}_a$, for all $a \in \{1, \dots, n\}$.

The basis of the induction is given by setting $\Xi_0 = (\mathcal{N}_0, \vartheta_0, \mathcal{U}_0, X_0, \epsilon_0)$. Remark 2.2 shows that $X_0(\bar{\mathcal{U}}_0)$ and $\text{Fr } \mathcal{D}^0$ meet transversally, proving (E₀). Properties (j₀), $j \neq E$, are empty.

For the inductive step, let $n \in \mathbb{N}$, assume that we have already constructed Ξ_m for all $m \in \{0, \dots, n-1\}$, and let us construct Ξ_n .

Let ϵ_n be a real number in $]0, \min\{\text{dist}(\mathcal{D}^{n-1}, \text{Fr } \mathcal{D}^n), 1/\kappa(\mathcal{D}^{n-1})\}[$ and satisfying (C_n) to be specified later. By (E_{n-1}), Lemma 3.2 applies to the data

$$(\mathcal{D}, \mathcal{D}', \mathcal{N}, \vartheta_{\mathcal{N}}, \mathcal{U}, \epsilon, X) = (\mathcal{D}^{n-1}, \mathcal{D}^n, \mathcal{N}^{n-1}, \vartheta_{n-1}, \mathcal{U}_{n-1}, \epsilon_n, X_{n-1}),$$

furnishing an open Riemann surface \mathcal{N}^n , a bordered domain $\mathcal{U}_n \Subset \mathcal{N}^n$, and a holomorphic embedding $X_n : \bar{\mathcal{U}}_n \rightarrow \mathbb{C}^2$ satisfying (A_n), (D_n), (E_n), and properties (F_n) and (G_n) for $a = n$. Further, (F_n) and (G_n) for $a \in \{1, \dots, n-1\}$ are ensured by (F_{n-1}), (G_{n-1}), and (D_n), provided that ϵ_n is chosen small enough. Up to taking any nowhere-vanishing holomorphic 1-form ϑ_n in \mathcal{N}_n satisfying (B_n), this closes the induction and concludes the construction of the sequence $\{\Xi_n\}_{n \in \mathbb{N}}$.

Denote by \mathcal{M} the open Riemann surface $\bigcup_{n \in \mathbb{N}} \mathcal{U}_n$; observe that properties (A_n), $n \in \mathbb{N}$, imply Theorem 3.1(i). The sequence $\{X_n : \bar{\mathcal{U}}_n \rightarrow \mathbb{C}^2\}_{n \in \mathbb{N}}$ converges uniformly on compact sets of \mathcal{M} to a holomorphic map $Y : \mathcal{M} \rightarrow \mathbb{C}^2$; just observe that properties (B_n), (C.1_n), and (D_n) guarantee that

$$\|X_n - X_{n-1}\|_{1, \bar{\mathcal{U}}_k; \vartheta_k} < \epsilon_n < \epsilon/2^{n+1} \quad \text{for any } k < n. \tag{3.3}$$

Let us show that the map Y satisfies all the requirements in the theorem.

- Y is an injective immersion. Indeed, for every $k \in \mathbb{N}$, (3.3) and (C.1_n), $n > k$, give

$$\|Y - X_k\|_{1, \overline{\mathcal{U}}_k; \vartheta_k} \leq \sum_{n>k} \|X_n - X_{n-1}\|_{1, \overline{\mathcal{U}}_k; \vartheta_k} < \sum_{n>k} \epsilon_n < 2\epsilon_{k+1} < \epsilon_k. \quad (3.4)$$

This and (C.2_n) ensure that $Y|_{\overline{\mathcal{U}}_k} : \overline{\mathcal{U}}_k \rightarrow \mathbb{C}^2$ is an embedding for all $k \in \mathbb{N}$, hence Y is an injective immersion as claimed.

- Y is complete. Indeed, from (G_n), $n \in \mathbb{N}$, and taking limits as $n \rightarrow \infty$, we infer that $\ell(Y(\gamma)) \geq \mathbf{d}(\mathcal{D}^{n-1}, \text{Fr } \mathcal{D}^n) - \epsilon_n$ for any Jordan arc γ in $\overline{\mathcal{U}}_n$ connecting $b\overline{\mathcal{U}}_{n-1}$ and $b\overline{\mathcal{U}}_n$, for all $n \in \mathbb{N}$. Therefore, if $\alpha \in \mathcal{M}$ is a divergent arc in \mathcal{M} with initial point in $\mathcal{R} = \mathcal{U}_0$, one infers that $\ell(Y(\alpha)) \geq \sum_{n \in \mathbb{N}} (\mathbf{d}(\mathcal{D}^{n-1}, \text{Fr } \mathcal{D}^n) - \epsilon_n) = \infty$: take into account that $\{\overline{\mathcal{U}}_n\}_{n \in \mathbb{N}}$ is an exhaustion by compact sets of \mathcal{M} , the series $\sum_{n \in \mathbb{N}} \epsilon_n$ is convergent (see (C.1_n)), and $\sum_{n \in \mathbb{N}} \mathbf{d}(\mathcal{D}^{n-1}, \text{Fr } \mathcal{D}^n)$ is divergent (recall that $\{\mathcal{D}^n\}_{n \in \mathbb{N}}$ is \mathbf{d} -proper in \mathcal{B} ; see Def. 2.6). This ensures the completeness of Y .

- Item (ii) is given by (3.4) for $k = 0$ (recall that $\epsilon_0 < \varepsilon$).

- $Y(\mathcal{M}) \subset \mathcal{B}$ and $Y : \mathcal{M} \rightarrow \mathcal{B}$ is proper. For the first assertion, let $P \in \mathcal{M}$ and take $k \in \mathbb{N}$ such that $P \in \mathcal{U}_k$. From (E_n) and the Convex Hull Property, $X_n(P) \in \mathcal{D}^n$ for all $n \geq k$. Letting $n \rightarrow \infty$, we find that $Y(P) \in \overline{\mathcal{B}}$ and so, by the convexity of \mathcal{B} and the maximum principle for harmonic functions, $Y(P) \in \mathcal{B}$.

Then, properties (F_n), $n \in \mathbb{N}$, and the fact that $\{\overline{\mathcal{D}^{n-1}_{-\epsilon_n}}\}_{n \in \mathbb{N}}$ is an exhaustion of \mathcal{B} by compact sets imply that

$$Y(\mathcal{M} \setminus \overline{\mathcal{U}}_{k-1}) \subset \mathcal{B} \setminus \overline{\mathcal{D}^{k-1}_{-\epsilon_k}} \quad \text{for all } k \in \mathbb{N}. \quad (3.5)$$

This inclusion for $k = 1$ proves (iv). To check that $Y : \mathcal{M} \rightarrow \mathcal{B}$ is proper, let $K \subset \mathcal{B}$ be a compact subset. Since $\{\overline{\mathcal{D}^{n-1}_{-\epsilon_n}}\}_{n \in \mathbb{N}}$ is an exhaustion of \mathcal{B} , there exists $k \in \mathbb{N}$ such that $K \subset \overline{\mathcal{D}^{n-1}_{-\epsilon_n}}$ for all $n \geq k$. Therefore, (3.5) gives $Y^{-1}(K) \subset \overline{\mathcal{U}}_{k-1}$. This shows that $Y^{-1}(K)$ is compact and proves (iii). \square

4. Approximation by embedded complex curves

In this section we prove Lemma 3.2. The proof consists of three main steps. In the first step (Subsec. 4.1), we introduce the notion of *tangent net* for a convex domain, and prove existence of tangent nets with useful geometrical properties. The second step is approximation by complex curves along tangent nets (see Subsec. 4.2). In the final step we prove desingularization for complex curves in \mathbb{C}^2 (see Subsec. 4.3). Lemma 3.2 will follow by combining these results (see Subsec. 4.4).

4.1. Tangent nets

The aim of this section is to introduce the notion of *tangent net* (Def. 4.1) and prove existence of tangent nets with useful properties (see Lemma 4.2).

Definition 4.1. Let \mathcal{D} be a bounded regular strictly convex domain in \mathbb{R}^n , $n \geq 2$. Let $\Delta \subset \text{Fr } \mathcal{D}$ be a finite set and define

$$\Gamma := \bigcup_{p \in \Delta} (p + T_p \text{Fr } \mathcal{D}) \subset \mathbb{R}^n \setminus \mathcal{D}.$$

The set

$$\mathcal{T} := \{q \in \mathbb{R}^n : \text{dist}(q, \Gamma) < \epsilon\}$$

is said to be a *tangent net* of radius $\epsilon > 0$ for \mathcal{D} (see Fig. 4.1). Observe that if $\epsilon < 1/\kappa(\mathcal{D})$ then $\mathcal{T} \subset \mathbb{R}^n \setminus \overline{\mathcal{D}}_{-\epsilon}$.

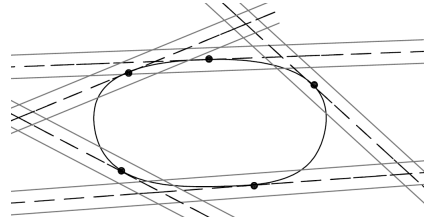


Fig. 4.1. A tangent net.

The sets $\mathcal{T}^0 := \Delta$ and $\mathcal{T}^1 := \Gamma$ are called the *0-skeleton* and the *1-skeleton* of \mathcal{T} , respectively. For any $p \in \mathcal{T}^0$, the set $\mathcal{T}(p) := \{q \in \mathbb{R}^n : \text{dist}(q, p + T_p \text{Fr } \mathcal{D}) < \epsilon\}$ is the *slab of \mathcal{T} based at p* .

The following *Pythagoras' type* result will be crucial.

Lemma 4.2. Let \mathcal{D} and \mathcal{D}' be bounded regular strictly convex domains in \mathbb{R}^n ($n \geq 2$) with $\mathcal{D} \Subset \mathcal{D}'$. Let $A \subset \text{Fr } \mathcal{D}$ consist of a finite collection of smooth immersed compact arcs and closed curves. Then for any $\epsilon > 0$ there exists a tangent net \mathcal{T} of radius $< \epsilon$ for \mathcal{D} such that

- (i) $A \subset \mathcal{T}$ and
- (ii) $\ell(\gamma) > \mathbf{d}(\mathcal{D}, \text{Fr } \mathcal{D}') - \epsilon$ for any Jordan arc $\gamma \subset \mathcal{T}$ connecting $\text{Fr } \mathcal{D}$ and $\text{Fr } \mathcal{D}'$.

Proof. For the sake of simplicity, denote $d_0 := \text{dist}(\mathcal{D}, \text{Fr } \mathcal{D}')$ and $\kappa_0 := \kappa(\mathcal{D})$.

Write $A = \bigcup_{i=1}^{\mu} \alpha_i$, where α_i is either a smooth closed immersed curve or a smooth immersed compact arc in $\text{Fr } \mathcal{D}$ for all $i \in \{1, \dots, \mu\}$, $\mu \in \mathbb{N}$. Denote

$$\mathfrak{L} := 1 + \max\{\ell(\alpha_i) : i = 1, \dots, \mu\} < \infty. \tag{4.1}$$

For any $m \in \mathbb{N}$ set

$$\epsilon_m := \frac{1}{\kappa_0} \left(1 - \cos\left(\frac{\mathfrak{L}\kappa_0}{m}\right) \right). \tag{4.2}$$

Since $\lim_{m \rightarrow \infty} m\epsilon_m = 0$, we have

$$\max \left\{ \epsilon_m, \frac{4(m\mu + 1)\epsilon_m}{\sqrt{(d_0\kappa_0 + 1)^2 - 1}} \right\} < \epsilon \tag{4.3}$$

for large enough m .

Let $m \in \mathbb{N}$ satisfy (4.3) and set $I := \{1, \dots, \mu\} \times \{1, \dots, m\}$.

From (4.1), for any $i \in \{1, \dots, \mu\}$ there exist m points $p_{i,1}, \dots, p_{i,m}$ splitting α_i into m arcs of the same length $< \mathfrak{L}/m$. Denote $\Delta := \{p_{i,j} : (i, j) \in I\}$, let \mathcal{T} be the tangent net of radius ϵ_m for \mathcal{D} with 0-skeleton $\mathcal{T}^0 = \Delta$, and observe that

$$\text{dist}_{\text{Fr } \mathcal{D}}(q, \mathcal{T}^0) < \mathfrak{L}/m \quad \text{for all } q \in A, \tag{4.4}$$

where $\text{dist}_{\text{Fr } \mathcal{D}}$ is the intrinsic distance in $\text{Fr } \mathcal{D}$.

Let us show that \mathcal{T} is as required.

First, let us check item (i). In view of (4.4), it suffices to check that the slab $\mathcal{T}(p_{i,j})$ contains the intrinsic geodesic ball in $\text{Fr } \mathcal{D}$ with center $p_{i,j}$ and radius \mathfrak{L}/m , for all $(i, j) \in I$. Indeed, let $S_{i,j} \subset \overline{\mathcal{D}}$ denote the Euclidean sphere in \mathbb{R}^n of radius $1/\kappa_0$ tangent to $\text{Fr } \mathcal{D}$ at $p_{i,j}$. Basic trigonometry and (4.2) imply that $\mathcal{T}(p_{i,j})$ contains the intrinsic geodesic ball in $S_{i,j}$ with center $p_{i,j}$ and radius \mathfrak{L}/m . Then the assertion follows from Rauch's theorem and the definition of κ_0 (see (2.2)).

Let us show that \mathcal{T} satisfies item (ii). Let $\gamma \subset \mathcal{T}$ be as in (ii) and denote by $p_0 \in \text{Fr } \mathcal{D}$ and $q_0 \in \text{Fr } \mathcal{D}'$ the endpoints of γ . Without loss of generality, assume that $\gamma \subset \mathcal{T} \cap \overline{\mathcal{D}'}$. Let C be the cone in \mathbb{R}^n given by

$$C := \bigcup_{x \in \Lambda} [x, q_0], \quad \text{where } \Lambda := \{x \in \text{Fr } \mathcal{D} : q_0 \in x + T_x \text{Fr } \mathcal{D}\}.$$

Denote by Ω the compact region in $\mathbb{R}^n \setminus \mathcal{D}$ bounded by $\text{Fr } \mathcal{D}$ and C (see Figure 4.2).

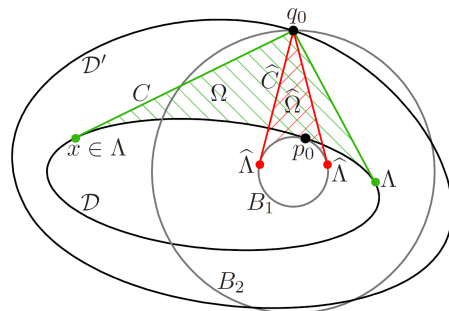


Fig. 4.2. Proof of Lemma 4.2.

Assume first that $p_0 \in \text{Fr } \mathcal{D} \setminus \text{Fr } \Omega$. In this case there exists $x_0 \in \Lambda$ such that $\ell(\gamma) \geq \ell([x_0, q_0])$. Since \mathcal{D} and \mathcal{D}' are strictly convex, the definition of κ_0 and Pythagoras' theorem give

$$\ell([x_0, q_0]) \geq \sqrt{d_0^2 + 2d_0/\kappa_0} > \mathbf{d}(\mathcal{D}, \text{Fr } \mathcal{D}'),$$

and we are done; the latter inequality follows from a straightforward computation.

Assume now that $p_0 \in \text{Fr } \mathcal{D} \cap \text{Fr } \Omega$. Let $B_1 \subset \mathcal{D}$ be the Euclidean open ball in \mathbb{R}^n of radius $1/\kappa_0$ tangent to $\text{Fr } \mathcal{D}$ at p_0 . Let B_2 be the Euclidean open ball in \mathbb{R}^n with the same

center as B_1 and such that $q_0 \in \text{Fr } B_2$. Denote

$$\widehat{\Lambda} := \{x \in \text{Fr } B_1 : q_0 \in x + T_x \text{Fr } B_1\}, \quad \widehat{C} := \bigcup_{x \in \widehat{\Lambda}} [x, q_0], \quad \widehat{\mathcal{T}} := \mathcal{T} \cup \widehat{C}. \quad (4.5)$$

Denote by $\widehat{\Omega}$ the compact region in $\mathbb{R}^n \setminus B_1$ bounded by $\text{Fr } B_1$ and \widehat{C} , and notice that $\widehat{\Omega} \subset \overline{\mathcal{D}'}$ (see Figure 4.2). Since $[p_0, q_0] \cap \overline{\mathcal{D}} = \{p_0\}$ and $p_0 \in \overline{B_1} \subset \overline{\mathcal{D}}$, we have $[p_0, q_0] \cap \overline{B_1} = \{p_0\}$ as well, and so $p_0 \in \text{Fr } \mathcal{D} \cap \text{Fr } \widehat{\Omega}$.

If $\gamma \cap (\mathbb{R}^n \setminus \widehat{\Omega}) \neq \emptyset$, let p_1 be the first point of γ in \widehat{C} and let $\gamma_0 \subset \gamma \cap \widehat{\Omega}$ be the subarc of γ with endpoints p_0 and p_1 . Observe that the arc $\widehat{\gamma}_0 := \gamma_0 \cup [p_1, q_0] \subset \widehat{\mathcal{T}} \cap \widehat{\Omega}$ connects p_0 and q_0 and satisfies $\ell(\widehat{\gamma}_0) \leq \ell(\gamma)$. Therefore, to finish the proof it suffices to show that $\ell(\widehat{\gamma}) > \mathbf{d}(\mathcal{D}, \text{Fr } \mathcal{D}') - \epsilon$ for any compact arc $\widehat{\gamma} \subset \widehat{\mathcal{T}} \cap \widehat{\Omega}$ with endpoints p_0 and q_0 . Let $\widehat{\gamma}$ be such an arc.

Up to a rigid motion, assume that B_1 and B_2 are centered at $\vec{0} \in \mathbb{R}^n$ and $q_0 = (\vec{0}, r_2) \in \mathbb{R}^{n-1} \times \mathbb{R}$, where r_2 is the radius of B_2 . Since the radius of B_1 equals $1/\kappa_0$, $p_0 \in \text{Fr } B_1 \cap \text{Fr } \mathcal{D}$, and $q_0 \in \text{Fr } B_2 \cap \text{Fr } \mathcal{D}'$, it follows that

$$r_2 \geq d_0 + 1/\kappa_0. \quad (4.6)$$

In this setting, the set $\widehat{\Lambda}$ in (4.5) is

$$\widehat{\Lambda} = \left\{ \left(\vec{x}, \frac{1}{r_2 \kappa_0^2} \right) \in \mathbb{R}^{n-1} \times \mathbb{R} : \|\vec{x}\| = \frac{\sqrt{r_2^2 \kappa_0^2 - 1}}{r_2 \kappa_0^2} \right\}. \quad (4.7)$$

Since the endpoint q_0 of $\widehat{\gamma}$ is the vertex of the cone \widehat{C} (see (4.5)), there exist $a \in \mathbb{N}$ satisfying

$$a - 1 \leq \sharp I = m\mu, \quad (4.8)$$

a compact polygonal arc $\beta = \bigcup_{i=1}^a L_i \subset \widehat{\mathcal{T}} \cap \widehat{\Omega}$ with endpoints p_0 and q_0 , and an injective map $\{1, \dots, a - 1\} \ni i \mapsto \sigma_i \in I$, such that:

- $L_i = [(\vec{x}_i, y_i), (\vec{x}_{i+1}, y_{i+1})] \subset \mathbb{R}^{n-1} \times \mathbb{R}$, $i = 1, \dots, a$.
- $(\vec{x}_1, y_1) = p_0$ and $(\vec{x}_{a+1}, y_{a+1}) = q_0 = (\vec{0}, r_2)$ in $\mathbb{R}^{n-1} \times \mathbb{R}$.
- $L_i \subset \mathcal{T}_{\sigma_i}$ for all $i = 1, \dots, a - 1$.
- $L_a \subset \widehat{C}$ (possibly $L_a = \{q_0\}$).
- $\ell(\beta) \leq \ell(\widehat{\gamma})$.

To finish it suffices to show that $\ell(\beta) > \mathbf{d}(\mathcal{D}, \text{Fr } \mathcal{D}') - \epsilon$.

Since \mathcal{T} is a tangent net of radius ϵ_m for \mathcal{D} and the slope of any segment in $\mathcal{T} \cap \widehat{\Omega}$ is at most the one of the cone \widehat{C} (that is, the slope of the segment $[q_1, q_0]$ over $\mathbb{R}^{n-1} \times \{0\}$ for any $q_1 = (\vec{x}_{q_1}, y_{q_1}) \in \widehat{\Lambda}$, which equals $(r_2 - y_{q_1})/\|\vec{x}_{q_1}\| = \sqrt{r_2^2 \kappa_0^2 - 1}$), basic trigonometry gives

$$h_i := |y_{i+1} - y_i| \leq \|\vec{x}_{i+1} - \vec{x}_i\| \sqrt{r_2^2 \kappa_0^2 - 1} + 2r_2 \kappa_0 \epsilon_m \quad \forall i \in \{1, \dots, a\}. \quad (4.9)$$

Since $(\vec{x}_1, y_1) = p_0 \in \text{Fr } B_1 = \mathbb{S}^{n-1}(1/\kappa_0)$, we have $y_1 \leq 1/\kappa_0$; and since $y_{a+1} = r_2$, we see that

$$\sum_{i=1}^a h_i \geq r_2 - 1/\kappa_0. \quad (4.10)$$

From (4.9), we obtain

$$\ell(\beta) = \sum_{i=1}^a \ell(L_i) = \sum_{i=1}^a \sqrt{\|\vec{x}_{i+1} - \vec{x}_i\|^2 + h_i^2} \geq F - G, \quad (4.11)$$

where

$$F = \frac{r_2 \kappa_0}{\sqrt{r_2^2 \kappa_0^2 - 1}} \sum_{i=1}^a h_i,$$

$$G = \frac{r_2 \kappa_0}{\sqrt{r_2^2 \kappa_0^2 - 1}} \sum_{i=1}^a \left(h_i - \frac{1}{r_2 \kappa_0} \sqrt{(h_i - 2\epsilon_m r_2 \kappa_0)^2 + (r_2^2 \kappa_0^2 - 1) h_i^2} \right).$$

On the one hand, since the function

$$f:]1/\kappa_0, \infty[\rightarrow]0, \infty[, \quad f(t) = \frac{t^2 \kappa_0 - t}{\sqrt{t^2 \kappa_0^2 - 1}},$$

is increasing, we infer from (4.10) and (4.6) that

$$F \geq \frac{r_2^2 \kappa_0 - r_2}{\sqrt{r_2^2 \kappa_0^2 - 1}} = f(r_2) \geq f(d_0 + 1/\kappa_0) = \mathbf{d}(\mathcal{D}, \text{Fr } \mathcal{D}'). \quad (4.12)$$

On the other hand,

$$G = \frac{r_2 \kappa_0}{\sqrt{r_2^2 \kappa_0^2 - 1}} \sum_{i=1}^a \frac{-4\epsilon_m^2 + \frac{4\epsilon_m}{r_2 \kappa_0} h_i}{h_i + \frac{1}{r_2 \kappa_0} \sqrt{(h_i - 2\epsilon_m r_2 \kappa_0)^2 + (r_2^2 \kappa_0^2 - 1) h_i^2}}$$

$$< \frac{4\epsilon_m}{\sqrt{r_2^2 \kappa_0^2 - 1}} \sum_{i=1}^a \frac{h_i}{h_i + \frac{1}{r_2 \kappa_0} \sqrt{(h_i - 2\epsilon_m r_2 \kappa_0)^2 + (r_2^2 \kappa_0^2 - 1) h_i^2}} \leq \frac{4\epsilon_m a}{\sqrt{r_2^2 \kappa_0^2 - 1}}.$$

Therefore, taking into account (4.6), (4.8), and (4.3), we get

$$G < \frac{4(m\mu + 1)\epsilon_m}{\sqrt{(d_0 \kappa_0 + 1)^2 - 1}} < \epsilon.$$

This inequality, (4.11), and (4.12) prove the lemma. \square

4.2. Deforming curves along tangent nets

The following approximation result is the second key to the proof of Lemma 3.2 (see Def. 4.1 for notation).

Lemma 4.3. *Let \mathcal{D} and \mathcal{D}' be bounded regular strictly convex domains in \mathbb{C}^2 with $\mathcal{D} \Subset \mathcal{D}'$. Let $\varepsilon \in]0, \min\{\text{dist}(\overline{\mathcal{D}}, \text{Fr } \mathcal{D}'), 1/\kappa(\mathcal{D})\}[$ and let \mathcal{T} be a tangent net of radius ε for \mathcal{D} . Let $\delta \in]0, \varepsilon[$, let \mathcal{N} be an open connected Riemann surface equipped with a nowhere-vanishing holomorphic 1-form $\vartheta_{\mathcal{N}}$, let $\mathcal{R} \Subset \mathcal{N}$ be a bordered domain, and let $X: \overline{\mathcal{R}} \rightarrow \mathbb{C}^2$ be a holomorphic immersion such that*

$$X(b\overline{\mathcal{R}}) \subset \mathcal{T} \cap \mathcal{D}_\delta \quad (\text{hence } X(\overline{\mathcal{R}}) \subset \mathcal{D}_\delta). \quad (4.13)$$

Then there exist a bordered domain $\mathcal{S} \Subset \mathcal{N}$ and a holomorphic immersion $Y: \overline{\mathcal{S}} \rightarrow \mathbb{C}^2$ enjoying the following properties:

- (a) $\mathcal{R} \Subset \mathcal{S}$ and \mathcal{R} and \mathcal{S} are homeomorphically isotopic (i.e., $\overline{\mathcal{S}} \setminus \mathcal{R}$ consists of a finite collection of pairwise disjoint compact annuli).
- (b) $\|Y - X\|_{1, \overline{\mathcal{R}}, \vartheta_{\mathcal{N}}} < \delta$ (see (2.1)).
- (c) $Y(\overline{\mathcal{S}} \setminus \mathcal{R}) \subset \overline{\mathcal{D}'} \setminus \overline{\mathcal{D}}_{-\varepsilon}$.
- (d) $Y(b\overline{\mathcal{S}}) \subset \text{Fr } \mathcal{D}'$, hence $Y(\overline{\mathcal{S}}) \subset \overline{\mathcal{D}'}$.
- (e) $Y(\overline{\mathcal{S}}) \subset \mathcal{D}_\delta \cup \mathcal{T}$.

Before going into the proof of Lemma 4.3, a few words about its geometrical implications. Roughly speaking, the lemma ensures that an immersed compact complex curve $X(\overline{\mathcal{R}}) \subset \mathbb{C}^2$ with boundary $X(b\overline{\mathcal{R}})$ lying close to the frontier of a regular strictly convex domain $\mathcal{D} \Subset \mathbb{C}^2$ can be approximated by another one $Y(\overline{\mathcal{S}}) \subset \mathbb{C}^2$ with boundary $Y(b\overline{\mathcal{S}})$ in the frontier of a larger convex domain \mathcal{D}' . The main point is that this can be done in such a way that the piece of $Y(\overline{\mathcal{S}})$ outside \mathcal{D} lies in a given tangent net \mathcal{T} for \mathcal{D} containing $X(b\overline{\mathcal{R}})$ (see Lemma 4.3(e)).

Notice that the intrinsic Euclidean diameter of the complex curve $Y: \mathcal{S} \rightarrow \mathbb{C}^2$ exceeds in $\text{dist}(\mathcal{D}, \text{Fr } \mathcal{D}')$ the one of $X: \mathcal{R} \rightarrow \mathbb{C}^2$. Combining this lemma with a suitable choice of \mathcal{T} according to Lemma 4.2, one can also guarantee that the *image diameter* of the curve exceeds in $\mathbf{d}(\mathcal{D}, \text{Fr } \mathcal{D}')$ the one of the initial curve X (see Defs. 2.4 and 1.3). This will be the key to obtaining image completeness while preserving boundedness in the proof of Theorem 1.4 (Sec. 5). The main novelty of Lemma 4.3 with respect to previous related constructions (cf. [26, 6, 4] and references therein) is to estimate the image diameter of the curve instead of the intrinsic one.

From the technical point of view, the proof of the lemma relies on approximating $X(\overline{\mathcal{R}})$ by another immersed curve $\overline{\Sigma} \subset \mathbb{C}^2$ with boundary $b\overline{\Sigma}$ in $\mathbb{C}^2 \setminus \overline{\mathcal{D}'}$ such that $\overline{\Sigma} \subset \mathcal{D}_\delta \cup \mathcal{T}$. Lemma 4.3 will then follow up to trimming off the curve $\overline{\Sigma}$ in order to ensure item (d). The construction of $\overline{\Sigma}$ depends on the classical Runge and Mergelyan approximation theorems, and consists of three main steps that we now roughly describe.

First, we split $b\overline{\mathcal{R}}$ into a finite collection of pairwise disjoint Jordan arcs $\alpha_{i,j}$ so that $X(\alpha_{i,j})$ lies in a slab $\mathcal{T}(p_{i,j})$ of \mathcal{T} with $p_{i,j} \in \mathcal{T}^0$ (see items (i)–(iv) below).

In the second step (properties (v)–(vii) below), we attach to $X(\overline{\mathcal{R}})$ a family of Jordan arcs $\lambda_{i,j} \subset \mathbb{C}^2$ with initial point at an endpoint of $X(\alpha_{i,j}) \subset X(b\overline{\mathcal{R}})$ and final point in $\mathbb{C}^2 \setminus \overline{\mathcal{D}'}$. Each $\lambda_{i,j}$ is chosen to be close to a segment inside $\mathcal{T}(p_{i,j})$. We then approximate $X(\overline{\mathcal{R}}) \cup \bigcup_{i,j} \lambda_{i,j}$ by a new curve $F(\overline{\mathcal{M}})$, $\mathcal{R} \Subset \mathcal{M} \Subset \mathcal{N}$ (see properties (viii)–(xiii)

below). The bordered domain \mathcal{M} is chosen so that the final point of $r_{i,j} = F^{-1}(\lambda_{i,j})$ lies in $b\overline{\mathcal{M}}$.

In the final step, we first split the boundary $b\overline{\mathcal{M}}$ into finitely many arcs $\beta_{i,j}$ in accordance with the $\alpha_{i,j}$'s and the $r_{i,j}$'s (properties (xiv)–(xvi) below). The arcs $r_{i,j}$'s split $\mathcal{M} \setminus \overline{\mathcal{R}}$ into a finite collection of topological discs $\mathfrak{A}_{i,j}$, where $\alpha_{i,j} \cup \beta_{i,j} \subset \text{Fr}\overline{\mathfrak{A}_{i,j}}$. Then, we stretch $F(\mathfrak{A}_{i,j})$ outside of $\overline{\mathcal{D}}$ along the slab $\mathcal{T}(p_{i,j})$ in a complex direction orthogonal to $\lambda_{i,j}$, hence preserving what was already done in the second step. This gives a curve $\overline{\Sigma}$ as announced above ($\overline{\Sigma}$ corresponds to $Y_n(\overline{\mathcal{M}})$ for $n = \text{IJ}$, see properties (1_n)–(6_n) below).

Proof of Lemma 4.3. Recall that $\langle \cdot, \cdot \rangle$ denotes the bilinear Hermitian product of \mathbb{C}^2 and $\nu_{\mathcal{D}}: \text{Fr}\mathcal{D} \rightarrow \mathbb{S}^3$ the outward pointing unit normal to $\text{Fr}\mathcal{D}$. Denote by $\mathcal{J}: \mathbb{C}^2 \rightarrow \mathbb{C}^2$, $\mathcal{J}(\zeta, \xi) = (i\zeta, i\xi)$, the canonical complex structure of \mathbb{C}^2 .

We begin with the following reduction. Since $\nu_{\mathcal{D}}: \text{Fr}\mathcal{D} \rightarrow \mathbb{S}^3$ is a diffeomorphism, we can assume without loss of generality that

$$(\nu_{\mathcal{D}}(p_1))^\perp \cap (\nu_{\mathcal{D}}(p_2))^\perp = \{0\} \quad \forall \{p_1, p_2\} \subset \mathcal{T}^0, p_1 \neq p_2. \tag{4.14}$$

Indeed, just replace \mathcal{T} by another tangent net $\widehat{\mathcal{T}}$ for \mathcal{D} satisfying $X(b\overline{\mathcal{R}}) \subset \widehat{\mathcal{T}}$, $\widehat{\mathcal{T}} \cap \overline{\mathcal{D}} \subset \mathcal{T} \cap \overline{\mathcal{D}}$, and (4.14). To do so, choose $\widehat{\mathcal{T}}$ with 0-skeleton and radius ($< \varepsilon$) close enough to the ones of \mathcal{T} and use the fact that condition (4.14) determines an open and dense subset in the space of tangent nets for \mathcal{D} .

Since $(\nu_{\mathcal{D}}(p))^\perp = T_p \text{Fr}\mathcal{D} \cap \mathcal{J}(T_p \text{Fr}\mathcal{D})$ for all $p \in \text{Fr}\mathcal{D}$, condition (4.14) implies that $(T_{p_1} \text{Fr}\mathcal{D} \cap T_{p_2} \text{Fr}\mathcal{D}) \setminus ((\nu_{\mathcal{D}}(p_1))^\perp \cup (\nu_{\mathcal{D}}(p_2))^\perp) \neq \emptyset$ for any couple $\{p_1, p_2\} \subset \mathcal{T}^0$, $p_1 \neq p_2$. For every such couple, fix

$$v_{\{p_1, p_2\}} \in ((T_{p_1} \text{Fr}\mathcal{D} \cap T_{p_2} \text{Fr}\mathcal{D}) \setminus ((\nu_{\mathcal{D}}(p_1))^\perp \cup (\nu_{\mathcal{D}}(p_2))^\perp)) \cap \mathbb{S}^3. \tag{4.15}$$

The *first step* of the proof consists in suitably splitting the boundary curves of $\overline{\mathcal{R}}$. Denote by $\alpha_1, \dots, \alpha_{\text{I}}, \text{I} \in \mathbb{N}$, the connected components of $b\overline{\mathcal{R}}$, which are smooth Jordan curves in \mathcal{N} . From (4.13), there exist a natural number $\text{J} \geq 3$, a family $\{\alpha_{i,j} \subset \alpha_i : (i, j) \in \mathfrak{H} := \{1, \dots, \text{I}\} \times \mathbb{Z}_{\text{J}}\}$ of Jordan subarcs (here $\mathbb{Z}_{\text{J}} = \{0, 1, \dots, \text{J} - 1\}$ denotes the additive cyclic group of integers modulo J), and points $\{p_{i,j} : (i, j) \in \mathfrak{H}\} \subset \mathcal{T}^0$, meeting the following requirements:

- (i) $\bigcup_{j=1}^{\text{J}} \alpha_{i,j} = \alpha_i$ for all $i \in \{1, \dots, \text{I}\}$.
- (ii) $\alpha_{i,j} \cap \alpha_{i,k} = \emptyset$ for all $(i, j) \in \mathfrak{H}$ and $k \in \mathbb{Z}_{\text{J}} \setminus \{j - 1, j, j + 1\}$.
- (iii) $\alpha_{i,j}$ and $\alpha_{i,j+1}$ have a common endpoint $Q_{i,j}$ and are otherwise disjoint for all $(i, j) \in \mathfrak{H}$.
- (iv) $X(\alpha_{i,j}) \subset \mathcal{T}(p_{i,j}) \cap \mathcal{D}_\delta$ for all $(i, j) \in \mathfrak{H}$, where $\mathcal{T}(p_{i,j})$ is the slab of \mathcal{T} based at $p_{i,j} \in \mathcal{T}^0$ (see Def. 4.1).

To find such a partition, choose the arcs $\alpha_{i,j}$ so that $X(\alpha_{i,j}) \subset \mathbb{C}^2$ has sufficiently small diameter for all $(i, j) \in \mathfrak{H}$. Take into account (4.13) in order to ensure (iv). Notice that the map $\mathfrak{H} \ni (i, j) \mapsto p_{i,j} \in \mathcal{T}^0$ is not necessarily either injective or surjective.

In the *second step* we attach to $X(\overline{\mathcal{R}})$ a suitable family of Jordan arcs. In the Riemann surface \mathcal{N} , take for every $(i, j) \in \mathfrak{H}$ an analytic Jordan arc $r_{i,j} \subset \mathcal{N} \setminus \mathcal{R}$ attached

transversally to $b\overline{\mathcal{R}}$ at $Q_{i,j}$ and otherwise disjoint from $\overline{\mathcal{R}}$. In addition, choose those arcs to be pairwise disjoint. Denote by $P_{i,j}$ the other endpoint of $r_{i,j}$, $(i, j) \in \mathfrak{H}$.

For every $(i, j) \in \mathfrak{H}$, there exists a smooth regular embedded arc $\lambda_{i,j}$ in \mathbb{C}^2 enjoying the following properties:

- (v) $\lambda_{i,j} \subset \mathcal{T}(p_{i,j}) \cap \mathcal{T}(p_{i,j+1})$. In particular, $\lambda_{i,j} + T_{p_{i,k}} \text{Fr } \mathcal{D} := \bigcup_{q \in \lambda_{i,j}} (q + T_{p_{i,k}} \text{Fr } \mathcal{D}) \subset \mathcal{T}(p_{i,k}) \subset \mathcal{T}$ for $k = j, j + 1$.
- (vi) $\lambda_{i,j}$ is attached transversally to $X(b\overline{\mathcal{R}})$ at $X(Q_{i,j})$ and matches smoothly with $X(\overline{\mathcal{R}})$ at $X(Q_{i,j})$.
- (vii) $|\langle o_{i,j} - X(Q_{i,j}), \mathcal{J}(v_{\mathcal{D}}(p_{i,k})) \rangle| > 1 + \text{diam}(\mathcal{D}')$ for $k = j, j + 1$, where $o_{i,j}$ is the endpoint of $\lambda_{i,j}$, $o_{i,j} \neq X(Q_{i,j})$ (recall that $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product).

Indeed, the arc $\lambda_{i,j}$ can be obtained as a slight deformation of the segment

$$[X(Q_{i,j}), X(Q_{i,j}) + c_{i,j}v_{\{p_{i,j}, p_{i,j+1}\}}] \subset \mathbb{C}^2,$$

where $v_{\{p_{i,j}, p_{i,j+1}\}}$ is given by (4.15) and $c_{i,j} > 0$ is a large enough constant so that the above segment formally meets (vii) (notice that $\langle v_{\{p_{i,j}, p_{i,j+1}\}}, \mathcal{J}(v_{\mathcal{D}}(p_{i,k})) \rangle \neq 0$, $k = j, j + 1$; see (4.15)). For item (v), take into account (iii), (iv), and (4.15). Further, (vi) trivially follows up to a slight deformation of the segment.

Extend X , with the same name, to a smooth function $\overline{\mathcal{R}} \cup \bigcup_{(i,j) \in \mathfrak{H}} r_{i,j} \rightarrow \mathbb{C}^2$ mapping the arc $r_{i,j}$ diffeomorphically onto $\lambda_{i,j}$ for all $(i, j) \in \mathfrak{H}$. In this setting, Mergelyan's theorem furnishes a bordered domain $\mathcal{M} \Subset \mathcal{N}$ and a holomorphic immersion $Y_0: \overline{\mathcal{M}} \rightarrow \mathbb{C}^2$, as close as desired to X in the \mathcal{C}^1 topology on $\overline{\mathcal{R}} \cup \bigcup_{(i,j) \in \mathfrak{H}} r_{i,j}$, such that:

- (viii) $\mathcal{R} \Subset \mathcal{M}$ and $\overline{\mathcal{M}} \setminus \mathcal{R}$ consists of I pairwise disjoint compact annuli $\mathfrak{A}_1, \dots, \mathfrak{A}_I$.
- (ix) $\alpha_i \subset \text{Fr } \mathfrak{A}_i$, $r_{i,j} \subset \mathfrak{A}_i$, and $r_{i,j} \cap \text{Fr } \mathfrak{A}_i = \{Q_{i,j}, P_{i,j}\}$ for all $(i, j) \in \mathfrak{H}$.
- (x) $\|Y_0 - X\|_{1, \overline{\mathcal{R}}; \vartheta_{\mathcal{N}}} < \delta / (1 + IJ)$, where $\delta > 0$ is as in the statement of the lemma.
- (xi) $Y_0(r_{i,j}) \subset \overline{\mathcal{T}}(p_{i,j}) \cap \mathcal{T}(p_{i,j+1})$ for all $(i, j) \in \mathfrak{H}$ (see (v)).
- (xii) $Y_0(\alpha_{i,j}) \subset \mathcal{T}(p_{i,j}) \cap \mathcal{D}_\delta$ for all $(i, j) \in \mathfrak{H}$ (take into account (iv)).
- (xiii) $|\langle Y_0(P_{i,j}) - Y_0(Q_{i,j}), \mathcal{J}(v_{\mathcal{D}}(p_{i,k})) \rangle| > 1 + \text{diam}(\mathcal{D}')$ for all $(i, j) \in \mathfrak{H}$ and $k \in \{j, j + 1\}$ (see (vii)).

Write $\beta_i = (\text{Fr } \mathfrak{A}_i) \setminus \alpha_i$ for the connected component of $\text{Fr } \mathfrak{A}_i$ disjoint from α_i , $i = 1, \dots, I$. For every $(i, j) \in \mathfrak{H}$ denote by $\mathfrak{A}_{i,j}$ the connected component of $\mathfrak{A}_i \setminus (\alpha_i \cup \bigcup_{j \in \mathbb{Z}_J} r_{i,j})$ containing $\alpha_{i,j}$ in its frontier. Observe that $\overline{\mathfrak{A}_{i,j}}$ is a closed disc in \mathfrak{A}_i bounded by $r_{i,j-1}$, $\alpha_{i,j}$, $r_{i,j}$, and a subarc $\beta_{i,j}$ of β_i connecting the points $P_{i,j-1}$ and $P_{i,j}$ (see Fig. 4.3).

In the *final step* of the construction, we *stretch* $F(\mathfrak{A}_{i,j})$ outside of $\overline{\mathcal{D}'}$ along the slab $\mathcal{T}(p_{i,j})$. For every $(i, j) \in \mathfrak{H}$, choose a closed disc $K_{i,j} \subset \mathfrak{A}_{i,j}$ with $\text{Fr } K_{i,j}$ close enough to $\text{Fr } \mathfrak{A}_{i,j}$ so that:

- (xiv) $K_{i,j} \cap \beta_{i,j}$ is a Jordan arc containing neither $P_{i,j-1}$ nor $P_{i,j}$.
- (xv) $Y_0(\overline{\mathfrak{A}_{i,j} \setminus K_{i,j}}) \subset \mathcal{T}(p_{i,j})$ (use (xi), (xii), and a continuity argument).

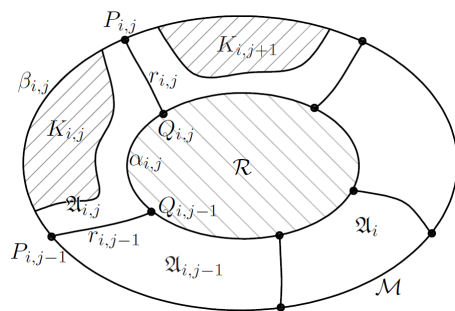


Fig. 4.3. \mathfrak{A}_i .

(xvi) $\pi_{i,j}(Y_0(\overline{\beta_{i,j} \setminus K_{i,j}})) \cap \pi_{i,j}(\overline{\mathcal{D}'}) = \emptyset$, where

$$\pi_{i,j}: \mathbb{C}^2 \rightarrow \text{span}_{\mathbb{R}}(\mathcal{J}(v_{\mathcal{D}}(p_{i,j})))$$

denotes the orthogonal projection (use $\{Y_0(Q_{i,j-1}), Y_0(Q_{i,j})\} \subset \mathcal{D}'$ (see (xii)), property (xiii), and a continuity argument again; see Fig. 4.3).

Let $\sigma: \{1, \dots, \text{IJ}\} \rightarrow \mathfrak{H}$ be a bijective map. To finish, we construct in a recursive process a sequence of holomorphic immersions $Y_n: \mathcal{M} \rightarrow \mathbb{C}^2, n \in \{0, 1, \dots, \text{IJ}\}$, enjoying the following properties:

(1_n) $\|Y_n - Y_{n-1}\|_{1, \overline{\mathcal{M} \setminus \mathfrak{A}_{\sigma(n)}}; \vartheta_{\mathcal{N}}} < \delta / (1 + \text{IJ})$.

(2_n) $\langle Y_n - Y_{n-1}, v_{\mathcal{D}}(p_{\sigma(n)}) \rangle = 0$.

(3_n) $Y_n(\mathfrak{A}_{\sigma(a)} \setminus K_{\sigma(a)}) \subset \mathcal{T}_{\sigma(a)}$ for all $a \in \{1, \dots, \text{IJ}\}$.

(4_n) $\pi_{\sigma(a)}(Y_n(\overline{\beta_{\sigma(a)} \setminus K_{\sigma(a)}})) \cap \pi_{\sigma(a)}(\overline{\mathcal{D}'}) = \emptyset$ for all $a \in \{1, \dots, \text{IJ}\}$.

(5_n) $Y_n(K_{\sigma(a)}) \cap \overline{\mathcal{D}'} = \emptyset$ for all $a \in \{1, \dots, n\}$.

(6_n) $Y_n(\overline{\mathcal{R}}) \subset \mathcal{D}_{\delta}$.

The basis of the induction corresponds to the already given immersion Y_0 . Indeed, notice that (6₀) is implied by (xii) and the Convex Hull Property; (3₀) and (4₀) agree with (xv) and (xvi); and (1₀), (2₀), and (5₀) are empty conditions.

For the inductive step, assume that we have constructed $Y_m: \overline{\mathcal{M}} \rightarrow \mathbb{C}^2$ for all $m \in \{0, \dots, n-1\}$ meeting the above requirements for some $n \in \{1, \dots, \text{IJ}\}$. Let us find an immersion Y_n satisfying (1_n), ..., (6_n).

For simplicity, write $w_n := v_{\mathcal{D}}(p_{\sigma(n)})$, and fix $u_n \in \langle w_n \rangle^{\perp} \cap \mathbb{S}^3 \subset T_{p_{\sigma(n)}} \text{Fr } \mathcal{D}$. Since $\{u_n, w_n\}$ is a $\langle \cdot, \cdot \rangle$ -orthonormal basis of \mathbb{C}^2 , we have

$$Y_{n-1} = \langle Y_{n-1}, u_n \rangle u_n + \langle Y_{n-1}, w_n \rangle w_n. \tag{4.16}$$

Recall that $\overline{\mathcal{M} \setminus \mathfrak{A}_{\sigma(n)}} \cap K_{\sigma(n)} = \emptyset$, and consider the holomorphic function $\phi: \overline{\mathcal{M} \setminus \mathfrak{A}_{\sigma(n)}} \cup K_{\sigma(n)} \rightarrow \mathbb{C}$ given by

$$\phi|_{\overline{\mathcal{M} \setminus \mathfrak{A}_{\sigma(n)}}} = \langle Y_{n-1}, u_n \rangle|_{\overline{\mathcal{M} \setminus \mathfrak{A}_{\sigma(n)}}} \quad \text{and} \quad \phi|_{K_{\sigma(n)}} = \zeta_n, \tag{4.17}$$

where $\zeta_n \in \mathbb{C}$ is a constant with modulus large enough that

$$(\zeta_n u_n + \text{span}_{\mathbb{C}}(w_n)) \cap \overline{\mathcal{D}'} = \emptyset. \tag{4.18}$$

Such a constant exists since $\overline{\mathcal{D}'}$ is compact. Since $\overline{\mathcal{M} \setminus \mathfrak{A}_{\sigma(n)} \cup K_{\sigma(n)}}$ is a Runge subset of a domain in \mathcal{N} containing $\overline{\mathcal{M}}$, Runge's theorem furnishes a holomorphic function $\varphi: \overline{\mathcal{M}} \rightarrow \mathbb{C}$ as close to ϕ as desired in the \mathcal{C}^1 topology on $\overline{\mathcal{M} \setminus \mathfrak{A}_{\sigma(n)} \cup K_{\sigma(n)}}$.

Claim 4.4. *If φ is chosen close enough to ϕ in the \mathcal{C}^1 topology on $\overline{\mathcal{M} \setminus \mathfrak{A}_{\sigma(n)} \cup K_{\sigma(n)}}$, then the function $Y_n: \overline{\mathcal{M}} \rightarrow \mathbb{C}^2$ given by*

$$Y_n := \varphi u_n + \langle Y_{n-1}, w_n \rangle w_n \tag{4.19}$$

satisfies properties $(1_n), \dots, (6_n)$.

Indeed, first of all observe that, up to slightly modifying φ , Y_n can be assumed to be an immersion by a general position argument. Since $\varphi \approx \phi = \langle Y_{n-1}, u_n \rangle$ on $\overline{\mathcal{M} \setminus \mathfrak{A}_{\sigma(n)}}$, we have $Y_n \approx Y_{n-1}$ on $\overline{\mathcal{M} \setminus \mathfrak{A}_{\sigma(n)}}$, and (1_n) and (6_n) hold (take into account (4.17), (4.19), (4.16), and (6_{n-1})). Property (2_n) directly follows from (4.19), (4.16), and the definition of u_n and w_n .

To check (3_n) we distinguish two cases. If $a \neq n$, then $Y_n \approx Y_{n-1}$ on $\overline{\mathcal{M} \setminus \mathfrak{A}_{\sigma(n)}} \supset \overline{\mathfrak{A}_{\sigma(a)} \setminus K_{\sigma(a)}}$; hence (3_{n-1}) implies that $Y_n(\overline{\mathfrak{A}_{\sigma(a)} \setminus K_{\sigma(a)}}) \subset \mathcal{T}_{\sigma(a)}$. If $a = n$ then the inclusion $Y_n(\overline{\mathfrak{A}_{\sigma(n)} \setminus K_{\sigma(n)}}) \subset \mathcal{T}_{\sigma(n)}$ is ensured by (2_n) , (3_{n-1}) , and the fact that $\mathcal{T}_{\sigma(n)}$ is foliated by affine hyperplanes $\langle \cdot, \cdot \rangle$ -orthogonal to $\nu_{\mathcal{D}}(p_{\sigma(n)})$.

For (4_n) we distinguish two cases again. If $a \neq n$, then (4_{n-1}) and the fact that $Y_n \approx Y_{n-1}$ on $\overline{\mathcal{M} \setminus \mathfrak{A}_{\sigma(n)}} \supset \overline{\beta_{\sigma(a)} \setminus K_{\sigma(a)}}$ give $\pi_{\sigma(a)}(Y_n(\overline{\beta_{\sigma(a)} \setminus K_{\sigma(a)}})) \cap \pi_{\sigma(a)}(\overline{\mathcal{D}'}) = \emptyset$ as well. If $a = n$ then the assertion follows from (2_n) , (4_{n-1}) , and the definition of $\pi_{\sigma(n)}$.

Finally, property (5_n) for $a < n$ is guaranteed by (5_{n-1}) and the fact that $Y_n \approx Y_{n-1}$ on $K_{\sigma(a)}$; whereas for $a = n$ it is ensured by (4.18) and $\varphi \approx \phi$ on $K_{\sigma(n)}$.

This proves the claim, closes the induction, and concludes the construction of the immersions $Y_n: \overline{\mathcal{M}} \rightarrow \mathbb{C}^2$, $n \in \{1, \dots, \text{IJ}\}$.

Let \mathcal{S} denote the connected component of $Y_{\text{IJ}}^{-1}(\mathcal{D}') \subset \mathcal{M} \Subset \mathcal{N}$ containing $\overline{\mathcal{R}}$ (see (6_{IJ})). Up to a slight deformation of Y_{IJ} , assume that $\mathcal{S} \Subset \mathcal{N}$ is a bordered domain. Define $Y := Y_{\text{IJ}}|_{\overline{\mathcal{S}}}: \overline{\mathcal{S}} \rightarrow \mathbb{C}^2$ and let us check that Y meets all the requirements in the statement of the lemma.

Indeed, properties (x) and (1_n) , $n \in \{1, \dots, \text{IJ}\}$, give

$$\|Y_{\text{IJ}} - X\|_{1, \overline{\mathcal{R}}; \vartheta_{\mathcal{N}}} < \delta, \tag{4.20}$$

proving Lemma 4.3(b).

Properties (4_{IJ}) and (5_{IJ}) imply that $Y_{\text{IJ}}(b\overline{\mathcal{M}}) \cap \overline{\mathcal{D}'} = \emptyset$; observe that $b\overline{\mathcal{M}} = \bigcup_{a=1}^{\text{IJ}} \beta_{\sigma(a)}$. This property and the definition of $\overline{\mathcal{S}}$ ensure item (d) in the lemma.

From (6_{IJ}) it follows that

$$Y(\overline{\mathcal{R}}) \subset \mathcal{D}_{\delta} \Subset \mathcal{D}', \tag{4.21}$$

hence $\mathcal{R} \Subset \mathcal{S}$ and Lemma 4.3(a) holds by the maximum principle. Furthermore, (4.21) and (5_{IJ}) show that $b\overline{\mathcal{S}} \subset \overline{\mathcal{M} \setminus (\overline{\mathcal{R}} \cup \bigcup_{a=1}^{\text{IJ}} K_{\sigma(a)})} = \bigcup_{a=1}^{\text{IJ}} \overline{\mathfrak{A}_{\sigma(a)} \setminus K_{\sigma(a)}}$, and so $\overline{\mathcal{S}} \setminus \mathcal{R} \subset \bigcup_{a=1}^{\text{IJ}} \overline{\mathfrak{A}_{\sigma(a)} \setminus K_{\sigma(a)}}$ as well. Then (3_{IJ}) gives

$$Y(\overline{\mathcal{S}} \setminus \mathcal{R}) \subset \mathcal{T} \cap \overline{\mathcal{D}'} \subset \overline{\mathcal{D}'} \setminus \overline{\mathcal{D}}_{-\varepsilon} \tag{4.22}$$

(for the latter inclusion, take into account that \mathcal{T} has radius ε), proving Lemma 4.3(c). Finally, (4.21) and (4.22) guarantee item (e). \square

4.3. The desingularization lemma

In this subsection we prove the following desingularization for complex curves in \mathbb{C}^2 ; it is the third key to the proof of Lemma 3.2.

Lemma 4.5. *Let $\mathcal{D} \subset \mathbb{C}^2$ be a strictly convex bounded regular domain. Let \mathcal{N} be an open Riemann surface, let $\vartheta_{\mathcal{N}}$ be a nowhere-vanishing holomorphic 1-form on \mathcal{N} , and let \mathcal{R} and \mathcal{M} be bordered domains in \mathcal{N} with $\mathcal{R} \Subset \mathcal{M}$. Let $X: \mathcal{N} \rightarrow \mathbb{C}^2$ be a holomorphic immersion satisfying*

- (I) $X(b\overline{\mathcal{M}}) \subset \text{Fr } \mathcal{D}$ (hence $X(\overline{\mathcal{R}}) \subset \mathcal{D}$) and
- (II) *there are no double points of $X(\overline{\mathcal{M}})$ in $X(\overline{\mathcal{R}})$; in particular, $X|_{\overline{\mathcal{R}}}$ is an embedding.*

Then for any $\epsilon > 0$ there exist an open Riemann surface \mathcal{W} , a bordered domain $\mathcal{S} \Subset \mathcal{W}$, and a holomorphic embedding $F: \mathcal{W} \rightarrow \mathbb{C}^2$ such that:

- (A) $\overline{\mathcal{R}} \subset \mathcal{S}$.
- (B) $\|F - X\|_{1, \overline{\mathcal{R}}; \vartheta_{\mathcal{N}}} < \epsilon$ and the Hausdorff distance $\mathfrak{d}^H(X(\overline{\mathcal{M}} \setminus \mathcal{R}), F(\overline{\mathcal{S}} \setminus \mathcal{R})) < \epsilon$. In particular, $\mathfrak{d}^H(X(\overline{\mathcal{M}}), F(\overline{\mathcal{S}})) < \epsilon$.
- (C) $F(b\overline{\mathcal{S}}) \subset \text{Fr } \mathcal{D}$.

The proof of the lemma consists in replacing every normal crossing in $X(\overline{\mathcal{M}})$ by an embedded annulus. It is important to point out that although this surgery increases the topology, the arising embedded complex curve $F(\overline{\mathcal{S}})$ contains a biholomorphic copy of $\overline{\mathcal{R}}$, which is C^1 close to $X(\overline{\mathcal{R}})$.

Roughly speaking, we take a holomorphic defining function $\mathcal{P}_0: \overline{\mathcal{D}} \rightarrow \mathbb{C}$ of $X(\overline{\mathcal{M}})$ so that $X(\overline{\mathcal{M}}) \equiv \{(\zeta, \xi) \in \overline{\mathcal{D}}: \mathcal{P}_0(\zeta, \xi) = 0\}$. Then we take a nearby smooth level set $\mathcal{C}_\lambda := \{(\zeta, \xi) \in \overline{\mathcal{D}}: \mathcal{P}_0(\zeta, \xi) = \lambda\}$, with λ close to 0. If λ is close enough to 0, \mathcal{C}_λ is an embedded complex curve containing a biholomorphic copy of $\overline{\mathcal{R}}$, and the surface $F(\overline{\mathcal{S}}) := \mathcal{C}_\lambda$ is as desired.

Proof of Lemma 4.5. Let $\mathcal{M}' \Subset \mathcal{N}$ be a bordered domain such that $\mathcal{M} \Subset \mathcal{M}'$,

$$X(b\overline{\mathcal{M}'}) \cap \overline{\mathcal{D}} = \emptyset, \text{ and there are no double points of } X(\overline{\mathcal{M}'}) \text{ in } X(\overline{\mathcal{R}}) \quad (4.23)$$

(take into account properties (I) and (II)).

Let $F_0: \overline{\mathcal{M}'} \rightarrow \mathbb{C}^2$ be a slight deformation of $X: \overline{\mathcal{M}'} \rightarrow \mathbb{C}^2$ such that:

- (i) $F_0: \overline{\mathcal{M}'} \rightarrow \mathbb{C}^2$ is a holomorphic immersion.
- (ii) $F_0(\overline{\mathcal{R}}) \subset \mathcal{D}$, $F_0(b\overline{\mathcal{M}'}) \cap \overline{\mathcal{D}} = \emptyset$ (see (4.23)), and $F_0(\mathcal{M}')$ and $\text{Fr } \mathcal{D}$ meet transversally.
- (iii) F_0 is as close to X as desired in the C^1 topology on $\overline{\mathcal{M}'}$; in particular
 - $\|F_0 - X\|_{1, \overline{\mathcal{R}}; \vartheta_{\mathcal{N}}} < \epsilon/2$,
 - there are no double points of $F_0(\overline{\mathcal{M}'})$ in $F_0(\overline{\mathcal{R}})$ (in particular, $F_0|_{\overline{\mathcal{R}}}: \overline{\mathcal{R}} \rightarrow \mathbb{C}^2$ is an embedding), and
 - $\mathfrak{d}^H(X(\overline{\mathcal{M}} \setminus \mathcal{R}), F_0(\overline{\mathcal{S}_0} \setminus \mathcal{R})) < \epsilon/2$, where $\mathcal{S}_0 \Subset \mathcal{M}'$ is the connected component of $F_0^{-1}(\mathcal{D})$ containing $\overline{\mathcal{R}}$.

(iv) All the double points of $F_0(\overline{S}_0)$ are normal crossings and lie in \mathcal{D} (take into account Remark 2.1).

Denote by $\mathcal{A} := \{\{P, P^*\} \subset \overline{S}_0 : P \neq P^* \text{ and } F_0(P) = F_0(P^*)\}$ the (finite) *double points set* of $F_0|_{\overline{S}_0}$, and write $F_0(\mathcal{A}) := \{F_0(P) : \{P, P^*\} \in \mathcal{A}\} \subset \mathbb{C}^2$. Notice from (ii) and (iii) that

$$F_0(\overline{S}_0) \text{ and } \text{Fr } \mathcal{D} \text{ meet transversally and } F_0(\mathcal{A}) \cap (F_0(\overline{\mathcal{R}}) \cup \text{Fr } \overline{\mathcal{D}}) = \emptyset. \tag{4.24}$$

Without loss of generality, \overline{S}_0 can be assumed to be homeomorphic to $\overline{\mathcal{M}}$, but not biholomorphic.

The domain \mathcal{D} is a Stein manifold whose second cohomology group $\mathcal{H}^2(\overline{\mathcal{D}}, \mathbb{Z})$ vanishes. This implies that any divisor in $\overline{\mathcal{D}}$ is principal (see for instance [27, p. 98]), hence there exists a holomorphic function $\mathcal{P}_0 : \overline{\mathcal{D}} \rightarrow \mathbb{C}$ such that

$$F_0(\overline{S}_0) = \{(\zeta, \xi) \in \overline{\mathcal{D}} : \mathcal{P}_0(\zeta, \xi) = 0\}.$$

From (iv) and the fact that F_0 is an immersion, it is not hard to check that $q \in F_0(\mathcal{A}_0)$ if and only if

$$\frac{\partial \mathcal{P}_0}{\partial \zeta}(q) = \frac{\partial \mathcal{P}_0}{\partial \xi}(q) = \mathcal{P}_0(q) = 0 \quad \text{and} \quad \text{H}(\mathcal{P}_0)_q \neq 0, \tag{4.25}$$

where $\text{H}(\mathcal{P}_0)_q$ denotes the Hessian of \mathcal{P}_0 at q .

The next step of the proof consists in removing from $F_0(\overline{S}_0)$ all the normal crossings. To do so, we deform this curve in an appropriate way. For each $\lambda \in \mathbb{C} \setminus \{0\}$ consider the holomorphic function

$$\mathcal{P}_\lambda : \overline{\mathcal{D}} \rightarrow \mathbb{C}, \quad \mathcal{P}_\lambda(\zeta, \xi) := \mathcal{P}_0(\zeta, \xi) - \lambda,$$

and set

$$\mathcal{S}_\lambda := \{(\zeta, \xi) \in \mathcal{D} : \mathcal{P}_\lambda(\zeta, \xi) = 0\}.$$

Obviously,

$$\lim_{\lambda \rightarrow 0} \mathcal{P}_\lambda = \mathcal{P}_0 \quad \text{uniformly on } \mathbb{C}^2. \tag{4.26}$$

Claim 4.6. *If $|\lambda| > 0$ is small enough, then there exists an open embedded complex curve \mathcal{C}_λ in \mathbb{C}^2 such that \mathcal{C}_λ and $\text{Fr } \mathcal{D}$ meet transversally and $\mathcal{C}_\lambda \cap \overline{\mathcal{D}} = \mathcal{S}_\lambda$.*

Proof. To prove the claim, it suffices to check that 0 is a regular value for $\mathcal{P}_\lambda|_{\overline{\mathcal{D}}}$.

Consider the holomorphic function $f : \overline{\mathcal{D}} \times \mathbb{C} \rightarrow \mathbb{C}^3$ given by

$$f(p, \lambda) = \left(\frac{\partial \mathcal{P}_0}{\partial \zeta}, \frac{\partial \mathcal{P}_0}{\partial \xi}, \mathcal{P}_\lambda \right)(p).$$

Obviously, 0 is a regular value for $\mathcal{P}_\lambda|_{\overline{\mathcal{D}}}$ if and only if $f^{-1}(0, 0, 0) \subset S_0$ (take into account that $\mathcal{S}_\lambda \cap F_0(S_0) = \emptyset$ for $\lambda \neq 0$). Since any double point p of \mathcal{S}_λ satisfies $\frac{\partial \mathcal{P}_0}{\partial \zeta}(p) = \frac{\partial \mathcal{P}_0}{\partial \xi}(p) = 0$, conditions (4.24)–(4.26) imply that the double points set of \mathcal{S}_λ

converges, as λ goes to 0, to $F_0(\mathcal{A})$. On the other hand, the Jacobian of f satisfies

$$\text{Jac } f_{(q,0)} = -\text{H}(\mathcal{P}_0)_q \neq 0 \quad \text{for any } q \in F_0(\mathcal{A})$$

(see (iv) and (4.25)). Therefore, f is a local biholomorphism around all points $(q, 0)$, $q \in F_0(\mathcal{A})$, and we are done.

The claim follows from (4.24), (4.26), and the fact that \mathcal{S}_λ is a submanifold of $\overline{\mathcal{D}}$. \square

As a consequence of Claim 4.6, the embedded complex curve \mathcal{S}_λ is a (connected) bordered domain in \mathcal{C}_λ with $b\overline{\mathcal{S}}_\lambda \subset \text{Fr } \mathcal{D}$.

On the other hand,

$$\lim_{\lambda \rightarrow 0} \text{d}^{\text{H}}(\overline{\mathcal{S}}_\lambda \cap \mathcal{K}, F_0(\overline{\mathcal{S}}_0) \cap \mathcal{K}) = 0 \quad \text{for any compact } \mathcal{K} \subset \overline{\mathcal{D}}. \quad (4.27)$$

It is interesting to notice that the convergence of $\overline{\mathcal{S}}_\lambda$ to $F_0(\overline{\mathcal{S}}_0)$ as $\lambda \rightarrow 0$ is nice outside the double points set $F_0(\mathcal{A})$, as the following claim shows:

Claim 4.7. *Let $\Omega \Subset S_0$ be a bordered domain such that $F_0(\overline{\Omega}) \cap F_0(\mathcal{A}) = \emptyset$ (in particular, $F_0|_{\overline{\Omega}}: \overline{\Omega} \rightarrow \mathbb{C}^2$ is an embedding). Then, if $|\lambda| > 0$ is small enough, there exist a bordered domain $\Omega_\lambda \Subset \mathcal{S}_\lambda$ and a biholomorphism $\sigma_\lambda: \overline{\Omega} \rightarrow \overline{\Omega}_\lambda$ such that*

$$\lim_{\lambda \rightarrow 0} \|\sigma_\lambda - F_0\|_{1, \overline{\Omega}; \vartheta_{\mathcal{N}}} = 0.$$

Proof. Write $F_0 = (z_0, w_0)$ and choose any holomorphic $G := (f_1, f_2): \overline{\Omega} \rightarrow \mathbb{C}^2$ such that

$$f_2 dz_0 - f_1 dw_0 \text{ vanishes nowhere on } \overline{\Omega}; \quad (4.28)$$

existence of such a G follows from the fact that F_0 is an immersion on $\overline{\Omega}$ and from Riemann–Roch’s theorem. For any $\delta > 0$, set $\mathbb{D}_\delta = \{t \in \mathbb{C}: |t| < \delta\}$ and consider the holomorphic function

$$\Phi: \overline{\Omega} \times \mathbb{D}_1 \rightarrow \mathbb{C}^2, \quad \Phi(P, t) = F_0(P) + tG(P).$$

Notice that Φ is a local biholomorphism around $(P, 0)$, $P \in \Omega$ (see (4.28)). Denote $V_\delta = \Phi(\overline{\Omega} \times \mathbb{D}_\delta)$, $\delta \in]0, 1[$, and choose δ small enough that $\overline{V}_\delta \subset \mathcal{D}$, $V_\delta \cap F_0(\mathcal{A}) = \emptyset$, and

$$\Psi: \overline{\Omega} \times \mathbb{D}_\delta \rightarrow V_\delta, \quad \Psi(P, t) = \Phi(P, t),$$

is a biholomorphism; take into account that $F_0|_{\overline{\Omega}}: \overline{\Omega} \rightarrow \mathbb{C}^2$ is an embedding and $F_0(\overline{\Omega}) \cap F_0(\mathcal{A}) = \emptyset$. Write $\pi: \overline{\Omega} \times \mathbb{D}_1 \rightarrow \overline{\Omega}$ for the natural holomorphic projection.

If δ is small enough, then 0 is a regular value for $\mathcal{P}_\lambda|_{V_\delta}$ for any λ (take into account (4.25) and the fact that $F_0(\overline{\Omega}) \cap F_0(\mathcal{A}) = \emptyset$). Therefore, $\Gamma := \{\mathcal{S}_\lambda \cap V_\delta: \lambda \in \mathbb{C}\}$ is a regular holomorphic foliation of V_δ transverse to the field $G \circ \pi \circ \Psi^{-1}$ (see (4.28)), and so π is one-to-one on sheets of Γ . To finish, it suffices to set $\Omega_\lambda := V_\delta \cap \mathcal{S}_\lambda$ and observe that for $|\lambda| > 0$ small enough:

- $\Omega_\lambda \Subset \mathcal{S}_\lambda$ and $\rho_\lambda := (\pi \circ \Psi^{-1})|_{\overline{\Omega}_\lambda}: \overline{\Omega}_\lambda \rightarrow \overline{\Omega}$ is a biholomorphism, and
 - $\lim_{\lambda \rightarrow 0} \|\sigma_\lambda - F_0\|_{1, \overline{\Omega}; \vartheta_{\mathcal{N}}} = 0$, where $\sigma_\lambda := \rho_\lambda^{-1}$
- (see (4.26)). \square

In view of Claim 4.6, to finish it suffices to find a bordered domain $\mathcal{R}_\lambda \in \mathcal{S}_\lambda \in \mathcal{C}_\lambda$ biholomorphic to \mathcal{R} such that $\overline{\mathcal{R}}_\lambda$ converges to $F_0(\overline{\mathcal{R}})$ as $\lambda \rightarrow 0$ (see (4.29) below).

Indeed, Claim 4.7 applies to \mathcal{R} , furnishing a bordered domain $\mathcal{R}_\lambda \in \mathcal{S}_\lambda$ and a biholomorphism $\sigma_\lambda: \overline{\mathcal{R}} \rightarrow \overline{\mathcal{R}}_\lambda$ for $|\lambda| > 0$ small enough. Furthermore, if $\lambda_0 \in \mathbb{C} \setminus \{0\}$ is sufficiently close to 0, the following conditions are satisfied:

- $\sigma_{\lambda_0}: \overline{\mathcal{R}} \rightarrow \overline{\mathcal{R}}_{\lambda_0}$ is a biholomorphism.
- $\|\sigma_{\lambda_0} - F_0\|_{1, \overline{\mathcal{R}}; \vartheta_{\mathcal{N}}} < \epsilon/2$.
- $\mathfrak{d}^H(F_0(\overline{\mathcal{S}}_0 \setminus \mathcal{R}), \overline{\mathcal{S}}_{\lambda_0} \setminus \mathcal{R}_{\lambda_0}) < \epsilon/2$.

For the last item, take into account that $F_0(\mathcal{A}) \cap F_0(\overline{\mathcal{R}}) = \emptyset$ (see (4.24)), (4.27), and

$$\lim_{\lambda \rightarrow 0} \|\sigma_\lambda - F_0\|_{1, \overline{\mathcal{R}}; \vartheta_{\mathcal{N}}} = 0. \tag{4.29}$$

Set $\mathcal{S} := \mathcal{S}_{\lambda_0}$ and $\mathcal{W} = \mathcal{C}_{\lambda_0}$. Up to identifying $\overline{\mathcal{R}}$ with $\overline{\mathcal{R}}_{\lambda_0}$ via σ_{λ_0} (hence $\overline{\mathcal{R}} \subset \mathcal{S}$) and taking into account (iii) and Claim 4.6, the open Riemann surface \mathcal{W} , the bordered domain $\mathcal{S} \Subset \mathcal{W}$, and the holomorphic embedding $F := \text{Id}: \mathcal{W} \rightarrow \mathcal{W} \hookrightarrow \mathbb{C}^2$ satisfy all the requirements of the lemma. \square

4.4. Proof of Lemma 3.2

By (3.2), $X(\overline{\mathcal{U}})$ and $\text{Fr } \mathcal{D}$ meet transversally (see Remark 2.2). Thus, we can find a small $\rho \in]0, \epsilon/2[$ and a bordered domain $\mathcal{V} \Subset \mathcal{N}$ such that $\mathcal{D}_\rho \Subset \mathcal{D}'$, $\mathcal{U} \Subset \mathcal{V}$, X extends as a holomorphic embedding $X: \overline{\mathcal{V}} \rightarrow \mathbb{C}^2$, $X(b\overline{\mathcal{V}}) \subset \text{Fr } \mathcal{D}_\rho$, $X(\overline{\mathcal{V}} \setminus \mathcal{U}) \subset \overline{\mathcal{D}}_\rho \setminus \mathcal{D}$, and

$$|\mathbf{d}(\mathcal{D}_\rho, \text{Fr } \mathcal{D}') - \mathbf{d}(\mathcal{D}, \text{Fr } \mathcal{D}')| < \epsilon/2. \tag{4.30}$$

Take $\epsilon_0 \in]0, \rho/2[$, and notice that

$$X(\overline{\mathcal{U}}) \subset \overline{\mathcal{D}} \subset \mathcal{D}_{\rho-\epsilon_0} \tag{4.31}$$

(see (3.2) and use the maximum principle). Since $X(b\overline{\mathcal{V}}) \subset \text{Fr } \mathcal{D}_\rho$, Lemma 4.2 furnishes a tangent net \mathcal{T} of radius $\mu \in]0, \min\{\epsilon_0, \text{dist}(\overline{\mathcal{D}}_\rho, \text{Fr } \mathcal{D}'), 1/\kappa(\mathcal{D}_\rho)\}[$ for \mathcal{D}_ρ such that:

- (A1) $X(b\overline{\mathcal{V}}) \subset \mathcal{T}$, and
- (A2) $\ell(\alpha) > \mathbf{d}(\mathcal{D}_\rho, \text{Fr } \mathcal{D}') - \mu$ for any Jordan arc α in \mathcal{T} connecting $\text{Fr } \mathcal{D}_\rho$ and $\text{Fr } \mathcal{D}'$.

Take $\varsigma \in]0, \mu[$ small enough that $\mathcal{D}_{\rho+\varsigma} \Subset \mathcal{D}'$ and

- (B1) $\ell(\alpha) > \mathbf{d}(\mathcal{D}_\rho, \text{Fr } \mathcal{D}') - \mu$ for any Jordan arc α in \mathcal{T} connecting $\text{Fr } \mathcal{D}_{\rho+\varsigma}$ and $\text{Fr } \mathcal{D}'$ (see (A2)), and
- (B2) any holomorphic map $G: \overline{\mathcal{V}} \rightarrow \mathbb{C}^2$ with $\|G - X\|_{1, \overline{\mathcal{V}}; \vartheta_{\mathcal{N}}} < \varsigma$ satisfies

- (B2.1) G is an embedding in $\overline{\mathcal{V}}$ (recall that $X: \overline{\mathcal{V}} \rightarrow \mathbb{C}^2$ is an embedding and use the Cauchy estimates),
- (B2.2) $G(\overline{\mathcal{U}}) \subset \mathcal{D}_{\rho-\epsilon_0}$, and $G(\overline{\mathcal{V}} \setminus \mathcal{U}) \cap \overline{\mathcal{D}}_{-\epsilon} = \emptyset$ (see (4.31) and use the fact $X(\overline{\mathcal{V}} \setminus \mathcal{U}) \subset \overline{\mathcal{D}}_\rho \setminus \mathcal{D}$ is disjoint from $\overline{\mathcal{D}}_{-\epsilon}$).

From (A1) and (3.2), Lemma 4.3 applies to the data

$$(\mathcal{D}, \mathcal{D}', \varepsilon, \mathcal{T}, \delta, \mathcal{N}, \vartheta_{\mathcal{N}}, \mathcal{R}, X) = (\mathcal{D}_{\rho}, \mathcal{D}', \mu, \mathcal{T}, \zeta, \mathcal{N}, \vartheta_{\mathcal{N}}, \mathcal{V}, X),$$

providing a bordered domain $\mathcal{W} \Subset \mathcal{N}$ and a holomorphic immersion $Y: \overline{\mathcal{W}} \rightarrow \mathbb{C}^2$ such that:

- (C1) $\mathcal{V} \Subset \mathcal{W}$ and \mathcal{V} and \mathcal{W} are homeomorphically isotopic.
- (C2) $\|Y - X\|_{1, \overline{\mathcal{V}}; \vartheta_{\mathcal{N}}} < \zeta$; in particular, $Y|_{\overline{\mathcal{V}}}$ is an embedding (see (B2)).
- (C3) $Y(\overline{\mathcal{W}} \setminus \mathcal{V}) \subset \overline{\mathcal{D}'} \setminus \overline{\mathcal{D}}_{\rho-\mu}$.
- (C4) $Y(b\overline{\mathcal{W}}) \subset \text{Fr } \mathcal{D}'$.
- (C5) $Y(\overline{\mathcal{W}}) \subset \mathcal{D}_{\rho+\zeta} \cup \mathcal{T}$.

Notice that

$$Y(\overline{\mathcal{U}}) \subset \mathcal{D}_{\rho-\epsilon_0} \quad \text{and} \quad Y(\overline{\mathcal{V}} \setminus \mathcal{U}) \cap \overline{\mathcal{D}}_{-\epsilon} = \emptyset \quad (4.32)$$

(take into account (C2) and (B2.2)). Since $\mu < \epsilon_0 < \rho$, (C3) and the latter assertion in (4.32) give

$$Y(\overline{\mathcal{W}} \setminus \mathcal{U}) \cap \overline{\mathcal{D}}_{-\epsilon} = \emptyset. \quad (4.33)$$

The fact that $Y|_{\overline{\mathcal{V}}}$ is an embedding (see (C2)), property (C3), the first assertion in (4.32), and the fact $\mu < \epsilon_0$ ensure that there are no double points of $Y(\overline{\mathcal{W}})$ in $Y(\overline{\mathcal{U}})$. From this fact and (C4), Lemma 4.5 applies to the data

$$(\mathcal{D}, \mathcal{N}, \vartheta_{\mathcal{N}}, \mathcal{R}, \mathcal{M}, X, \epsilon) = (\mathcal{D}', \mathcal{N}, \vartheta_{\mathcal{N}}, \mathcal{U}, \mathcal{W}, Y, \eta),$$

where $\eta \in]0, \epsilon - \zeta[$ will be specified later, furnishing an open Riemann surface \mathcal{N}' , a bordered domain \mathcal{U}' , and a holomorphic embedding $F: \mathcal{N}' \rightarrow \mathbb{C}^2$ satisfying:

- (D1) $\overline{\mathcal{U}} \Subset \mathcal{U}'$.
- (D2) $\|F - Y\|_{1, \overline{\mathcal{U}}; \vartheta_{\mathcal{N}}} < \eta$ and $\mathfrak{d}^H(Y(\overline{\mathcal{W}} \setminus \mathcal{U}), F(\overline{\mathcal{U}'} \setminus \mathcal{U})) < \eta$.
- (D3) $F(b\overline{\mathcal{U}'}) \subset \text{Fr } \mathcal{D}'$.

Let us check that the embedding $X' := F|_{\overline{\mathcal{U}'}}: \overline{\mathcal{U}'} \rightarrow \mathbb{C}^2$ is as desired. (D1) and (D3) agree with Lemma 3.2(i) and (iii), respectively. Property (ii) follows from (C2) and (D2); recall that $\eta < \epsilon - \zeta$. Property (iv) is given by (4.33) and (D2) provided that η is chosen small enough.

Finally, let us check (v). Let γ be a Jordan arc in $\overline{\mathcal{U}'}$ connecting $b\overline{\mathcal{U}}$ and $b\overline{\mathcal{U}'}$. From (C5), (D2), and the first assertion in (4.32), it follows that $X'(\overline{\mathcal{U}'}) \subset \mathcal{D}_{\rho+\zeta} \cup \mathcal{T}$ and $X'(\overline{\mathcal{U}}) \subset \mathcal{D}_{\rho-\epsilon_0} \Subset \mathcal{D}_{\rho+\zeta}$, provided that η is small enough. Taking also (D3) into account, we deduce that γ contains a subarc γ' such that $X'(\gamma')$ is contained in \mathcal{T} and connects $\text{Fr } \mathcal{D}_{\rho+\zeta}$ and $\text{Fr } \mathcal{D}'$. By (B1), $\ell(X'(\gamma)) \geq \ell(X'(\gamma')) > \mathbf{d}(\mathcal{D}_{\rho}, \text{Fr } \mathcal{D}') - \mu > \mathbf{d}(\mathcal{D}, \text{Fr } \mathcal{D}') - \epsilon$. For the last inequality, take into account that $\mu < \epsilon/2$ and $\mathbf{d}(\mathcal{D}_{\rho}, \text{Fr } \mathcal{D}') > \mathbf{d}(\mathcal{D}, \text{Fr } \mathcal{D}') - \epsilon/2$ (see (4.30)). This concludes the proof.

5. Image complete complex curves in convex domains

In this section we make use of Lemmas 4.2 and 4.3 in order to prove Theorem 5.1 below. Observe that Theorem 1.4 in the introduction is a particular instance of it.

Let \mathcal{N} be an open Riemann surface. A domain $U \subset \mathcal{N}$ is said to be *homeomorphically isotopic* to \mathcal{N} if there exists a homeomorphism $\mu: U \rightarrow \mathcal{N}$ satisfying $\mu_* = i_*$, where $i: U \hookrightarrow \mathcal{N}$ is the inclusion map and $\mu_*, i_*: \mathcal{H}_1(U, \mathbb{Z}) \rightarrow \mathcal{H}_1(\mathcal{N}, \mathbb{Z})$ are the induced group morphisms. In this case, $\mathcal{H}_1(U, \mathbb{Z})$ and $\mathcal{H}_1(\mathcal{N}, \mathbb{Z})$ will be identified via μ_* .

Theorem 5.1. *Let \mathcal{B} be a (possibly neither bounded nor regular) convex domain in \mathbb{C}^2 and let $\mathcal{D} \Subset \mathcal{B}$ be a bounded regular strictly convex domain. Let \mathcal{N} be an open Riemann surface equipped with a nowhere-vanishing holomorphic 1-form $\vartheta_{\mathcal{N}}$, let $\mathcal{M} \Subset \mathcal{N}$ be a Runge bordered domain, and let $X: \overline{\mathcal{M}} \rightarrow \mathbb{C}^2$ be a holomorphic immersion such that*

$$X(b\overline{\mathcal{M}}) \subset \text{Fr } \mathcal{D}. \tag{5.1}$$

Then, for any $\epsilon \in]0, \min\{\text{dist}(\overline{\mathcal{D}}, \text{Fr } \mathcal{B}), 1/\kappa(\mathcal{D})\}[$ there exist a domain $U \subset \mathcal{N}$ and a holomorphic immersion $Y: U \rightarrow \mathbb{C}^2$ with the following properties:

- (A) $\mathcal{M} \Subset U$ and U is homeomorphically isotopic to \mathcal{N} .
- (B) $\|Y - X\|_{1, \overline{\mathcal{M}}; \vartheta_{\mathcal{N}}} < \epsilon$ (see (2.1)).
- (C) $Y(U) \subset \mathcal{B}$ and $Y: U \rightarrow \mathcal{B}$ is a proper map.
- (D) $Y(U \setminus \overline{\mathcal{M}}) \subset \mathcal{B} \setminus \mathcal{D}_{-\epsilon}$.
- (E) Y is image complete (see Def. 1.3).

Proof. Denote $\mathcal{D}^0 := \mathcal{D}$ and let $\{\mathcal{D}^n\}_{n \in \mathbb{N}}$ be a **d**-proper sequence of convex domains in \mathcal{B} with $\mathcal{D}^0 \Subset \mathcal{D}^1$ (see Def. 2.6 and Lemma 2.7).

Set $N_0 := \mathcal{M}$ and let $\{N_n\}_{n \in \mathbb{N}}$ be an exhaustion of \mathcal{N} by bordered domains such that $\overline{N}_n \subset \mathcal{N}$ is Runge, $N_{n-1} \Subset N_n$ and the Euler characteristic $\chi(\overline{N}_n \setminus N_{n-1})$ is in $\{-1, 0\}$ for all $n \in \mathbb{N}$ (cf. [7, Lemma 4.2]).

Write $U_0 := N_0$, $X_0 := X$, and $\eta_0 := \text{Id}_{U_0}: U_0 \rightarrow U_0$, let $\epsilon_0 \in]0, \epsilon/2[$, and let us construct a sequence $\{\Upsilon_n = (U_n, \eta_n, X_n, \epsilon_n)\}_{n \in \mathbb{N}}$, where

- $U_n \Subset \mathcal{N}$ is a bordered domain and \overline{U}_n is Runge in \mathcal{N} ,
- $\eta_n: \overline{U}_n \rightarrow \overline{N}_n$ is an isotopical homeomorphism,
- $X_n: \overline{U}_n \rightarrow \mathbb{C}^2$ is a holomorphic immersion, and
- $\epsilon_n > 0$,

such that the following properties hold for all $n \in \mathbb{N}$:

- (1_n) $U_{n-1} \Subset U_n$.
- (2_n) $\eta_n|_{\overline{U}_{n-1}} = \eta_{n-1}$.
- (3_n) ϵ_n is a positive real number such that
 - $\epsilon_n < \min\{\epsilon_{n-1}/2, 1/\kappa(\mathcal{D}^{n-1}), \text{dist}(\mathcal{D}^{n-1}, \text{Fr } \mathcal{D}^n)\}$ ($< \epsilon/2^{n+1}$) and
 - any holomorphic function $G: \overline{U}_{n-1} \rightarrow \mathbb{C}^2$ with $\|G - X_{n-1}\|_{1, \overline{U}_{n-1}; \vartheta_{\mathcal{N}}} < 2\epsilon_n$ is an immersion.
- (4_n) $\|X_n - X_{n-1}\|_{1, \overline{U}_{n-1}; \vartheta_{\mathcal{N}}} < \epsilon_n$.

- (5_n) $X_n(\overline{U}_a \setminus U_{a-1}) \subset \mathcal{D}^{a+1} \setminus \overline{\mathcal{D}_{-\epsilon_a}^{a-1}}$ for all $a \in \{1, \dots, n\}$.
- (6_n) $X_n(b\overline{U}_n) \subset \text{Fr } \mathcal{D}^n$; hence $X_n(\overline{U}_n \setminus U_{n-1}) \subset \overline{\mathcal{D}^n} \setminus \overline{\mathcal{D}_{-\epsilon_n}^{n-1}}$.
- (7_n) $\ell(\gamma) > \mathbf{d}(\mathcal{D}^{a-1}, \text{Fr } \mathcal{D}^a) - \epsilon_a$ for any Jordan arc $\gamma \subset X_n(\overline{U}_n) \subset \mathbb{C}^2$ connecting $\text{Fr } \mathcal{D}^{a-1}$ and $\text{Fr } \mathcal{D}^a$, for all $a \in \{1, \dots, n\}$.

The sequence will be constructed in a recursive way. For the basis of the induction take $\Upsilon_0 = (U_0, \eta_0, X_0, \epsilon_0)$. Notice that (6₀) agrees with (5.1), and the remaining properties (j₀), $j \neq 6$, are empty.

For the inductive step, fix $n \in \mathbb{N}$ and assume that we have already constructed Υ_m satisfying the above properties for all $m \in \{0, \dots, n-1\}$. Let us construct Υ_n .

Choose any $\epsilon_n > 0$ satisfying (3_n) and

- (i) $\ell(\gamma) > \mathbf{d}(\mathcal{D}^{n-2}, \text{Fr } \mathcal{D}^{n-1}) - \epsilon_{n-1}$ for any Jordan arc γ in $X_{n-1}(\overline{U}_{n-1})$ connecting $\text{Fr } \mathcal{D}^{n-2}$ and $\text{Fr } \mathcal{D}_{-\epsilon_n}^{n-1}$; take into account (7_{n-1}). When $n = 1$, this condition is empty.

Such an ϵ_n exists since $X_{n-1}: \overline{U}_{n-1} \rightarrow \mathbb{C}^2$ is an immersion.

We distinguish two cases.

- Assume that $\chi(\overline{N}_n \setminus N_{n-1}) = 0$. From (6_{n-1}) and Lemma 4.2, there exists a tangent net \mathcal{T}_n of radius $< \epsilon_n$ for \mathcal{D}^{n-1} such that

- (ii) $X_{n-1}(b\overline{U}_{n-1}) \Subset \mathcal{T}_n$ and
- (iii) $\ell(\gamma) > \mathbf{d}(\mathcal{D}^{n-1}, \text{Fr } \mathcal{D}^n) - \epsilon_n$ for any Jordan arc $\gamma \subset \mathcal{T}_n$ connecting $\text{Fr } \mathcal{D}^{n-1}$ and $\text{Fr } \mathcal{D}^n$.

Let $\delta_n \in]0, \epsilon_n[$ to be specified later, and small enough that

- (iv) $\ell(\gamma) > \mathbf{d}(\mathcal{D}^{n-1}, \text{Fr } \mathcal{D}^n) - \epsilon_n$ for any Jordan arc $\gamma \subset \mathcal{T}_n$ connecting $\text{Fr } \mathcal{D}_{\delta_n}^{n-1}$ and $\text{Fr } \mathcal{D}^n$ (see (iii)).

By properties (ii) and (6_{n-1}), one can apply Lemma 4.3 to the data

$$\mathcal{D} = \mathcal{D}^{n-1}, \quad \mathcal{D}' = \mathcal{D}^n, \quad \varepsilon = \epsilon_n, \quad \mathcal{T} = \mathcal{T}_n, \quad \delta = \delta_n, \quad \mathcal{R} = U_{n-1}, \quad \text{and} \quad X = X_{n-1}.$$

The bordered domain U_n (which is Runge since \overline{U}_{n-1} is) and the holomorphic immersion $X_n: \overline{U}_n \rightarrow \mathbb{C}^2$ furnished by Lemma 4.3 enjoy properties (1_n) and (4_n)–(7_n). Indeed, properties (1_n), (4_n), and (6_n) follow straightforwardly.

Property (5_n) for $a = n$ is given by Lemma 4.3(c), whereas for $a < n$ it is ensured by (5_{n-1}) and Lemma 4.3(b) provided that δ_n is small enough.

Property (7_n) for $a = n$ follows from Lemma 4.3(e) and (iv); for $a = n - 1$ it is guaranteed by (i) and Lemma 4.3(b),(c) provided that δ_n is chosen small enough; and for $a < n - 1$ by (7_{n-1}) and Lemma 4.3(b) provided that δ_n is small enough.

Finally we choose any isotopical homeomorphism $\eta_n: \overline{U}_n \rightarrow \overline{N}_n$ satisfying (2_n); such exists since $\chi(\overline{N}_n \setminus N_{n-1}) = 0 = \chi(\overline{U}_n \setminus U_{n-1})$.

- Assume that $\chi(\overline{N}_n \setminus N_{n-1}) = -1$. Consider a smooth Jordan curve $\widehat{\alpha} \in \mathcal{H}_1(\overline{N}_n, \mathbb{Z}) \setminus \mathcal{H}_1(\overline{N}_{n-1}, \mathbb{Z})$ contained in N_n and intersecting $N_n \setminus N_{n-1}$ in a Jordan arc α with endpoints a, b in $b\overline{N}_{n-1}$ and otherwise disjoint from \overline{N}_{n-1} . Notice that since \overline{N}_{n-1} and \overline{N}_n

are Runge subsets of \mathcal{N} and $\chi(\overline{N}_n \setminus N_{n-1}) = -1$, we have $\mathcal{H}_1(\overline{N}_n, \mathbb{Z}) = \mathcal{H}_1(\overline{N}_{n-1} \cup \alpha, \mathbb{Z})$ and $\overline{N}_{n-1} \cup \alpha \subset \mathcal{N}$ is Runge as well.

Likewise, we choose a smooth Jordan arc $\gamma \subset \mathcal{N} \setminus U_{n-1}$ attached transversally to $b\overline{U}_{n-1}$ at the points $\eta_{n-1}^{-1}(a)$ and $\eta_{n-1}^{-1}(b)$ and otherwise disjoint from \overline{U}_{n-1} . We take γ such that there exists an isotopical homeomorphism $\tau : \overline{U}_{n-1} \cup \gamma \rightarrow \overline{N}_{n-1} \cup \alpha$ such that $\tau|_{\overline{U}_{n-1}} = \eta_{n-1}$ and $\tau(\gamma) = \alpha$.

In \mathbb{C}^2 , choose a smooth regular Jordan arc $\lambda \subset \text{Fr } \mathcal{D}$ attached transversally to $X_{n-1}(b\overline{U}_{n-1})$ at the points $X_{n-1}(\eta_{n-1}^{-1}(a))$ and $X_{n-1}(\eta_{n-1}^{-1}(b))$ and otherwise disjoint from $X_{n-1}(\overline{U}_{n-1})$.

From (6_{n-1}) and the fact that $\lambda \subset \text{Fr } \mathcal{D}$, there exist a tangent net $\widehat{\mathcal{T}}_n$ of radius $< \epsilon_n$ for \mathcal{D}^{n-1} and a positive $\widehat{\delta}_n < \epsilon_n$ such that

- (ii') $X_{n-1}(b\overline{U}_{n-1}) \cup \lambda \Subset \widehat{\mathcal{T}}_n$ and
- (iv') $\ell(\gamma) > \mathbf{d}(\mathcal{D}^{n-1}, \text{Fr } \mathcal{D}^n) - \epsilon_n$ for any Jordan arc $\gamma \subset \widehat{\mathcal{T}}_n$ connecting $\text{Fr } \mathcal{D}_{\widehat{\delta}_n}^{n-1}$ and $\text{Fr } \mathcal{D}^n$.

Extend X_{n-1} , with the same name, to a smooth function $\overline{U}_{n-1} \cup \gamma \rightarrow \mathbb{C}^2$ mapping γ diffeomorphically to λ . In this setting, Mergelyan's theorem furnishes a bordered domain $V_{n-1} \subset \mathcal{N}$ with $U_{n-1} \cup \gamma \Subset V_{n-1} \Subset U_n$, $\chi(\overline{U}_n \setminus V_{n-1}) = 0$, and a holomorphic immersion $\widehat{X}_{n-1} : \overline{V}_{n-1} \rightarrow \mathbb{C}^2$, as close as desired to X_{n-1} in the \mathcal{C}^0 topology on $\overline{U}_{n-1} \cup \gamma$ and in the \mathcal{C}^1 topology on \overline{U}_{n-1} , such that $\widehat{X}_{n-1}(b\overline{V}_{n-1}) \subset \widehat{\mathcal{T}}_n \cap \mathcal{D}_{\widehat{\delta}_n}^{n-1}$. We finish by using Lemma 4.3 as in the previous case for small enough $\widehat{\delta}_n$.

This concludes the construction of the sequence $\{\Upsilon_n\}_{n \in \mathbb{N}}$.

Set $U := \bigcup_{n \in \mathbb{N}} U_n$. For Theorem 5.1(A), use (2_n) , $n \in \mathbb{N}$, and the fact that $\{N_n\}_{n \in \mathbb{N}}$ is an exhaustion of \mathcal{N} ; take into account that $\mathcal{M} = U_0$.

From (4_n) and (3_n) , $n \in \mathbb{N}$, the sequence $\{X_n\}_{n \in \mathbb{N}}$ converges uniformly on compact subsets of U to a holomorphic function $Y : U \rightarrow \mathbb{C}^2$ satisfying item (B).

Let us check that Y meets all the requirements in the theorem.

- Y is an immersion. Indeed, for any $k \in \mathbb{N}$, properties (3_n) and (4_n) , $n > k$, give

$$\|Y - X_k\|_{1, \overline{U}_k; \vartheta_{\mathcal{N}}} \leq \sum_{n>k} \|X_n - X_{n-1}\|_{1, \overline{U}_k; \vartheta_{\mathcal{N}}} < \sum_{n>k} \epsilon_n < 2\epsilon_{k+1} < \epsilon_k; \tag{5.2}$$

hence the latter assertion in (3_n) implies that $Y|_{\overline{U}_k}$ is an immersion for all $k \in \mathbb{N}$, and so is Y .

- $Y(U) \subset \mathcal{B}$ and $Y : U \rightarrow \mathcal{B}$ is proper. To see this, we proceed as in the proof of Theorem 3.1. Up to taking limit as $n \rightarrow \infty$, the inclusion $Y(U) \subset \mathcal{B}$ follows from (6_n) and the Convex Hull Property. Likewise, properties (5_n) , $n \in \mathbb{N}$, and the fact that $\{\mathcal{D}_{-\epsilon_n}^{n-1}\}_{n \in \mathbb{N}}$ is an exhaustion of \mathcal{B} by compact sets imply that

$$Y(U \setminus \overline{U}_{k-1}) \subset \mathcal{B} \setminus \mathcal{D}_{-\epsilon_k}^{k-1} \quad \text{for all } k \in \mathbb{N}. \tag{5.3}$$

This inclusion for $k = 1$ proves (D). The properness of $Y : U \rightarrow \mathcal{B}$ follows from the fact that $\{\mathcal{D}_{-\epsilon_n}^{n-1}\}_{n \in \mathbb{N}}$ is an exhaustion of \mathcal{B} and (5.3). This yields (C).

• Y is image complete. Indeed, let α be a locally rectifiable divergent arc in $Y(U)$, and let us check that $\ell(\alpha) = \infty$. Since $Y: U \rightarrow \mathcal{B}$ is proper, α is a divergent arc in \mathcal{B} as well. Let $n_0 \in \mathbb{N}$ be large enough that the initial point of α lies in \mathcal{D}^{n_0} . For every $a \in \mathbb{N}$ with $a > n_0$, let α_a denote a compact subarc of α in $\overline{\mathcal{D}^a} \setminus \mathcal{D}^{a-1}$ connecting $\text{Fr } \mathcal{D}^{a-1}$ and $\text{Fr } \mathcal{D}^a$. Since $\{\mathcal{D}^n\}_{n \in \mathbb{N}}$ is \mathbf{d} -proper in \mathcal{B} (see Def. 2.6) and $\sum_{n \in \mathbb{N}} \epsilon_n$ converges, it suffices to show that $\ell(\alpha_a) \geq \mathbf{d}(\mathcal{D}^{a-1}, \text{Fr } \mathcal{D}^a) - \epsilon_a$ for all $a > n_0$.

Indeed, fix $a > n_0$. Let $n_1 \in \mathbb{N}$, $n_1 \geq a$, be large enough that $\alpha_a \subset Y(U_{n_1})$; recall that $Y: U \rightarrow \mathcal{B}$ is proper. Let $\beta_a = \bigcup_{j=1}^k \beta_{a,j} \subset U_{n_1}$ be a finite union of compact arcs with $Y(\beta_a) = \alpha_a$. Without loss of generality, we can suppose that the arcs $\{\alpha_{a,j} := Y(\beta_{a,j}) : j = 1, \dots, k\}$ are laid end to end and the endpoints of $\alpha_{a,j}$, $j = 2, \dots, k-1$, are double points of $Y(\overline{U}_{n_1})$.

Since the double points of Y are isolated and stable under deformations and since $\{\|Y - X_n\|_{1, \overline{U}_{n_1}; \partial_N}\}_{n \geq n_1} \rightarrow 0$ (see (5.2)), for any sufficiently large $n \geq n_1$ we can find compact arcs $\beta_{a,j}^n$, $j = 1, \dots, k$, in U_{n_1} such that

- $\alpha_a^n := X_n(\beta_a^n)$ is a Jordan arc in $\overline{\mathcal{D}^a} \setminus \mathcal{D}^{a-1}$ connecting $\text{Fr } \mathcal{D}^{a-1}$ and $\text{Fr } \mathcal{D}^a$, where $\beta_a^n = \bigcup_{j=1}^k \beta_{a,j}^n$, and
- $\{\ell(\alpha_a^n)\}_{n > n_1} \rightarrow \ell(\alpha_a)$.

To see this, just observe that the double points of $X_n|_{U_{n_1}}$ converge to the ones of $Y|_{U_{n_1}}$ as $n \rightarrow \infty$, and choose $\beta_{a,j}^n$ as a sufficiently slight deformation of $\beta_{a,j}$ in U_{n_1} so that $\{\alpha_{a,j}^n := X_n(\beta_{a,j}^n) : j = 1, \dots, k\}$ are laid end to end, the endpoints of $\alpha_{a,j}^n$, $j = 2, \dots, k-1$, are double points of $X_n(U)$, and $\{\ell(X_n(\beta_{a,j}^n)) - \ell(X_n(\beta_{a,j}))\}_{n > n_1} \rightarrow 0$.

By property (7_n), $\ell(\alpha_a^n) > \mathbf{d}(\mathcal{D}^{a-1}, \text{Fr } \mathcal{D}^a) - \epsilon_a$ for any large enough $n \geq n_1$. Letting $n \rightarrow \infty$ yields $\ell(\alpha_a) = \ell(Y(\beta_a)) \geq \mathbf{d}(\mathcal{D}^{a-1}, \text{Fr } \mathcal{D}^a) - \epsilon_a$ as claimed.

This shows item (E) and concludes the proof of the theorem. \square

Added in proof. After this paper was written, Globevnik [18, 19], with a different method, proved that every pseudoconvex domain in \mathbb{C}^n , for any $n \geq 2$, contains a complete closed complex hypersurface; in particular, this answers in the optimal way the question just below Corollary 1.2 as regards assertion (i). More recently, Alarcón, Globevnik, and López [5], also with a new different method, constructed complete closed complex hypersurfaces in the unit ball of \mathbb{C}^n , for any $n \geq 2$, with certain control on the topology; in particular, they affirmatively answered Question 1.5 by giving examples with any finite topology.

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