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Local profile of fully bubbling solutions to $SU(n + 1)$ Toda systems

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Abstract. We prove that for locally defined singular $SU(n + 1)$ Toda systems in \mathbb{R}^2 , the profile of fully bubbling solutions near the singular source can be accurately approximated by global solutions. The main ingredients of our new approach are the classification theorem of Lin–Wei–Ye [22] and the non-degeneracy of the linearized Toda system [22], which let us overcome the difficulties that come from lack of symmetry and the singular source.

Keywords. $SU(n + 1)$ Toda system, non-degeneracy, a priori estimate, classification theorem, fully bubbling, blowup solutions

1. Introduction

Let (M, g) be a compact Riemann surface, Δ the Beltrami–Laplace operator of the metric g , and K the Gauss curvature. The $SU(n + 1)$ Toda system is the nonlinear PDE

$$\Delta u_i + \sum_{j=1}^n a_{ij} h_j e^{u_j} - K(x) = 4\pi \sum_j \gamma_{ij} \delta_{q_j}, \quad 1 \leq i \leq n, \quad (1.1)$$

where h_i ($i = 1, \dots, n$) are positive smooth functions on M , δ_q stands for the Dirac measure at $q \in M$, and $A = (a_{ij})$ is the Cartan matrix given by

$$A = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ 0 & -1 & 2 & & 0 \\ \vdots & \vdots & & & \vdots \\ 0 & \cdots & -1 & 2 & -1 \\ 0 & \cdots & 0 & -1 & 2 \end{pmatrix}.$$

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The Toda system (1.1) has aroused a lot of attention in recent years because of its close connection to many different fields of mathematics and physics. For $n = 1$, (1.1) reduces to the Gauss curvature equation in two-dimensional surfaces. Without the singular source and for $M = \mathbb{S}^2$, it is the well-known Nirenberg problem. In general it is related to the existence of a metric of positive constant curvature with conic singularities [10, 11, 36, 37]. In the past three decades, equation (1.1) with $n = 1$ has been extensively studied (see [5], [7], [21] for example). For general n and $h_i \equiv 1$ ($i = 1, \dots, n$) equation (1.1) is connected with holomorphic curves in $\mathbb{C}P^n$ via the classical infinitesimal Plücker formulae (see [15]). This geometric connection is important because from it, it has been found out that equation (1.1) with $h_i \equiv 1$ is an integrable system (see [13], [16], for example). Recently, by using this connection, Lin–Wei–Ye [22] have been able to completely classify all the entire solutions of (1.1) in \mathbb{R}^2 with one singular source and finite energy.

In mathematical physics, equation (1.1) has also played an important role in Chern–Simons gauge theory. For example, in the relativistic $SU(n + 1)$ Chern–Simons model proposed by physicists (see [17] for $n = 1$ and [14] for $n > 1$), (1.1) governs the limiting equations as physical parameters tend to 0 and is used to explain the physics of high temperature superconductivity. In the past twenty years, the connections of (1.1) with $n = 1$ and the Chern–Simons–Higgs equation have been explored extensively. See [33] and [26]. However, for $n \geq 2$ only very few works are devoted to this direction of research. See [1], [27] and [34]. For recent developments on equation (1.1) and related subjects, we refer the readers to [3, 4, 18, 19, 22, 23, 24, 28, 29, 30, 31, 32, 40] and the references therein.

One of the fundamental issues concerning (1.1) is the bubbling phenomenon, which could lead to a priori bounds for solutions of (1.1). For $n = 1$, the bubbling phenomenon has been studied thoroughly in the past twenty years. Basically there are two kinds of bubbling behavior of solutions near blowup points. One is called “simple blowup”, which means the bubbling profile could be well controlled locally by entire bubbling solutions in \mathbb{R}^2 . For the case without singular sources, this was proved by Y. Y. Li [21], applying the method of moving planes. If there is a singular source $4\pi\gamma\delta_0$ on the right hand side of the equation, this was proved by Bartolucci–Chen–Lin–Tarantello [2] for $\gamma \notin \mathbb{N}$, and recently by Kuo–Lin [20] if $\gamma \in \mathbb{N}$. On the other hand, the non-simple blowup could occur at $\gamma \in \mathbb{N}$ only. The sharp profile of the non-simple blowup has been proved in [20]. The study of the bubbling phenomenon is important not only to a priori bounds; it also provides a lot of geometric information near blowup points [6, 8, 27].

For $n \geq 2$, (1.1) is an elliptic system. It is expected that the behavior of bubbling solutions is more complicated than for $n = 1$. One major difficulty comes from the partial blowup phenomenon, that is, after a suitable scaling, blowup solutions may converge to a global solution of a smaller system. To understand the partial blowup phenomenon, we have to first study the full blowup behavior, and to obtain an accurate description of this class of bubbling solutions. When $n = 2$ and (1.1) has no singular sources, the bubbling behavior of fully bubbling solutions has been studied by Jost–Lin–Wang [18] and Lin–Wei–Zhao [24]. In [18] it is proved that any sequence of fully bubbling solutions is a simple blowup at any blowup point. The proof in [18] uses a deep application of holonomy theory, which is a very effective generalization of the Pohozaev identity. Unfortunately, the holonomy method cannot be extended to cover the case with singular sources.

The purpose of this article is to extend the results of [18] to any $n \geq 2$ and to include (1.1) with singular sources. Before stating our main results, we set up the problem. Since this is a local problem, for simplicity we consider

$$\Delta u_i^k + \sum_{j=1}^n a_{ij} h_j^k e^{u_j^k} = 4\pi \gamma_i \delta_0, \quad B_1 \subset \mathbb{R}^2, \tag{1.2}$$

where B_1 is the unit ball. We shall use B_r to denote the ball centered at the origin with radius r .

For $u^k = (u_1^k, \dots, u_n^k)$, $h^k = (h_1^k, \dots, h_n^k)$ and γ_i ($i = 1, \dots, n$) we make the usual assumptions:

- (H) (i) $1/C \leq h_i^k \leq C$, $\|h_i^k\|_{C^2(B_1)} \leq C$, $h_i^k(0) = 1$, $i = 1, \dots, n$,
- (ii) $\gamma_i > -1$, $i = 1, \dots, n$,
- (iii) $\int_{B_1} h_i^k e^{u_i^k} \leq C$, $i = 1, \dots, n$, C is independent of k ,
- (iv) $|u_i^k(x) - u_i^k(y)| \leq C$ for all $x, y \in \partial B_1$, $i = 1, \dots, n$,
- (v) $\max_{K \subset \subset B_1 \setminus \{0\}} u_i^k \leq C$, and 0 is the only blowup point.

If (u_1^k, \dots, u_n^k) is a global solution of (1.1) in M , it is easy to see that all assumptions of (H) are satisfied. We also note that assumption (iv) in (H) is necessary for our analysis: Chen [12] proved that without it, even for $n = 1$, blowup solutions can be very complicated near their blowup points. The assumption $h_i^k(0) = 1$ in (i) is just for convenience.

Let

$$-2 \log \epsilon_k = \max_{x \in B_1, i=1, \dots, n} \frac{\tilde{u}_i^k(x)}{1 + \gamma_i}, \quad \text{where } \tilde{u}_i^k(x) = u_i^k(x) - 2\gamma_i \log |x|, \tag{1.3}$$

and

$$\tilde{v}_i^k(y) = \tilde{u}_i^k(\epsilon_k y) + 2(1 + \gamma_i) \log \epsilon_k, \quad i = 1, \dots, n. \tag{1.4}$$

Then clearly \tilde{v}_i^k satisfies

$$\Delta \tilde{v}_i^k(y) + \sum_{j=1}^n a_{ij} |y|^{2\gamma_j} h_j^k(\epsilon_k y) e^{\tilde{v}_j^k} = 0, \quad |y| \leq \epsilon_k^{-1}. \tag{1.5}$$

Our major assumption is that $\tilde{v}^k = (\tilde{v}_1^k, \dots, \tilde{v}_n^k)$ converges to an $SU(n + 1)$ Toda system uniformly over all compact subsets of \mathbb{R}^2 :

Definition 1.1. We say u^k of (1.2) is a *fully bubbling sequence* if \tilde{v}^k converges in $C_{loc}^{1,\alpha}(\mathbb{R}^2)$ to $\tilde{v} = (\tilde{v}_1, \dots, \tilde{v}_n)$ that solves the following $SU(n + 1)$ Toda system in \mathbb{R}^2 :

$$\begin{aligned} \Delta \tilde{v}_i + \sum_{j=1}^n a_{ij} |y|^{2\gamma_j} e^{\tilde{v}_j} &= 0 \quad \text{in } \mathbb{R}^2, \quad i = 1, \dots, n, \\ \int_{\mathbb{R}^2} |y|^{2\gamma_i} e^{\tilde{v}_i} &< \infty, \quad i = 1, \dots, n. \end{aligned} \tag{1.6}$$

The main purpose of this paper is to show that a fully bubbling sequence u^k can be closely approximated by a sequence of global solutions $U^k = (U_1^k, \dots, U_n^k)$ of

$$\Delta U_i^k + \sum_{j=1}^n a_{ij} e^{U_j^k} = 4\pi \gamma_i \delta_0, \quad \text{in } \mathbb{R}^2, \quad i = 1, \dots, n. \tag{1.7}$$

Theorem 1.2. *Let (H) hold, and let u^k be a fully bubbling sequence as in Definition 1.1 and ϵ_k as defined in (1.3). Then there exists a sequence of global solutions $U^k = (U_1^k, \dots, U_n^k)$ of (1.7) such that for $|y| \leq \epsilon_k^{-1}$ and $i = 1, \dots, n$,*

$$\begin{aligned} &|u_i^k(\epsilon_k y) - U_i^k(\epsilon_k y)| \leq \\ &\begin{cases} C(\sigma) \epsilon_k^\sigma (1 + |y|)^\sigma & \text{if } \min\{\gamma_1, \dots, \gamma_n\} \leq -3/4, \quad \sigma \in (0, \min\{2 + 2\gamma_1, \dots, 2 + 2\gamma_n\}), \\ C \epsilon_k (1 + |y|) & \text{if } \min\{\gamma_1, \dots, \gamma_n\} > -3/4. \end{cases} \end{aligned} \tag{1.8}$$

Moreover, there exists $C > 0$, independent of k , such that

$$|\tilde{U}_i^k(\epsilon_k y) + 2(1 + \gamma_i) \log \epsilon_k + 2(2 + \gamma_i + \gamma_{n+1-i}) \log(1 + |y|)| \leq C \tag{1.9}$$

for $|y| \leq \epsilon_k^{-1}$ and $i = 1, \dots, n$, where $\tilde{U}_i^k(x) = U_i^k(x) - 2\gamma_i \log|x|$ is the regular part of U_i^k .

The global solutions

$$(\tilde{U}_1^k(\epsilon_k y) + 2(1 + \gamma_1) \log \epsilon_k, \dots, \tilde{U}_n^k(\epsilon_k y) + 2(1 + \gamma_n) \log \epsilon_k) \tag{1.10}$$

in Theorem 1.2 are perturbations of $\tilde{v} = (\tilde{v}_1, \dots, \tilde{v}_n)$ in (1.6). In fact, the sequence in (1.10) converges uniformly to \tilde{v} over any fixed compact subset of \mathbb{R}^2 . Thus Theorem 1.2 clearly leads to the following

Corollary 1.3. *Let u^k, ϵ_k be as in Theorem 1.2, and let \tilde{v}^k be defined by (1.4). Then for $i = 1, \dots, n$,*

$$|\tilde{v}_i^k(y) + 2(2 + \gamma_i + \gamma_{n+1-i}) \log(1 + |y|)| \leq C \quad \text{for } |y| \leq \epsilon_k^{-1}. \tag{1.11}$$

Remark 1.4. The estimate in (1.11) holds trivially over any fixed compact subset of \mathbb{R}^2 . So the strength of Corollary 1.3 lies in the fact that the estimate is over $|y| \leq \epsilon_k^{-1}$. Such type of estimate was first established by Li [21] for single Liouville equations.

Estimates similar to (1.8) and (1.11) can be found in [21, 6, 2, 41, 42] for single Liouville equations and in [18, 24] for Toda systems. The proof of Theorem 1.2 is almost entirely different from all the approaches in these works. For example the estimates for single Liouville equations use ODE theory, which is based on the symmetry of global solutions. Lin–Wei–Zhao’s [24] sharp estimates are tailored for regular SU(3) Toda systems because they need to differentiate blowup solutions at blowup points twice (which cannot be expected when a singular source exists), and a lot of algebraic computation to fix the Cauchy

data of blowup solutions. For general singular $SU(n + 1)$ Toda systems, first, the ODE method cannot be used because global solutions may not have any symmetry. Second, fixing the Cauchy data of blowup solutions at a blowup point is impossible, because in addition to the differentiation issue mentioned before, the algebraic computation required to fix the Cauchy data depends on $n^2 + 2n$ parameters and is extremely complicated if n is large. Our approach is purely based on PDE methods and its essential part relies on an important classification theorem of Lin–Wei–Ye [22] for global $SU(n + 1)$ Toda systems and the non-degeneracy property of the corresponding linearized systems. The key point is to choose a sequence of global solutions as approximating solutions. On the one hand these global solutions all tend to a solution of the limit system (1.6), which means all the $n^2 + 2n$ families of parameters corresponding to these global solutions have a limit. On the other hand, one component of the approximating global solutions is very close to the same component of blowup solutions at $n^2 + 2n$ carefully chosen points. The closeness in one component leads to closeness in other components as well.

Theorem 1.2 is an extension of previous work. For example, if $n = 2$ and $\gamma_i = 0$ ($i = 1, 2$), Corollary 1.3 was proved by Jost–Lin–Wang [18]. It is easy to see that Theorem 1.2 is stronger than Corollary 1.3 even for this special case. Lin–Wei–Zhao proved (1.8) for $n = 2$ and $\gamma_i = 0$ ($i = 1, 2$) but Theorem 1.2 also holds when the number of equations is greater than 2 and the singular source at 0 exists.

For some applications such as constructing blowup solutions, more refined estimates than those in Theorem 1.2 are needed. For $SU(3)$ Toda systems with no singularity, Lin–Wei–Zhao [24] obtained more delicate estimates based on Corollary 1.3.

The organization of the article is as follows. In Section 2 we list some facts on $SU(n + 1)$ Toda systems and the non-degeneracy of the linearized systems. The proof of Theorem 1.2 is in Section 3. A key point in that proof is to determine $n^2 + 2n$ points in \mathbb{R}^2 in a specific way. Since this part is somewhat elaborate and elementary, we present it separately in Section 4.

2. Some facts on the linearized $SU(n + 1)$ system

First we list some facts on the entire solutions of $SU(n + 1)$ Toda systems with singularities. For more details see [22]. Let $u = (u_1, \dots, u_n)$ solve

$$\begin{cases} \Delta u_i + \sum_{j=1}^n a_{ij} e^{u_j} = 4\pi \gamma_i \delta_0 & \text{in } \mathbb{R}^2, \quad i = 1, \dots, n, \\ \int_{\mathbb{R}^2} e^{u_i} < \infty, \end{cases} \tag{2.1}$$

where $A = (a_{ij})_{n \times n}$ is the Cartan matrix and $\gamma_i > -1$. Then let

$$u^i = \sum_{j=1}^n a^{ij} u_j, \quad i = 1, \dots, n,$$

where $(a^{ij})_{n \times n} = A^{-1}$. Clearly (u^1, \dots, u^n) satisfies

$$\Delta u^i + e^{\sum_{j=1}^n a_{ij} u^j} = 4\pi \gamma^i \delta_0, \quad \gamma^i = \sum_{j=1}^n a^{ij} \gamma_j, \quad i = 1, \dots, n.$$

The classification theorem of Lin–Wei–Ye [22] asserts that

$$e^{-u^1} = |z|^{-2\gamma^1} \left(\lambda_0 + \sum_{i=1}^n \lambda_i |P_i(z)|^2 \right) \quad (2.2)$$

where for $\mu_i = 1 + \gamma_i$, $i = 1, \dots, n$, we have

$$P_i(z) = z^{\mu_1 + \dots + \mu_i} + \sum_{j=0}^{i-1} c_{ij} z^{\mu_1 + \dots + \mu_j}, \quad i = 1, \dots, n, \quad (2.3)$$

c_{ij} ($j < i$) are complex numbers and $\lambda_i > 0$ ($0 \leq i \leq n$) satisfies

$$\lambda_0 \dots \lambda_n = 2^{-n(n+1)} \prod_{1 \leq i \leq j \leq n} \left(\sum_{k=i}^j \mu_k \right)^{-2}. \quad (2.4)$$

Furthermore if $\mu_{j+1} + \dots + \mu_i \notin \mathbb{N}$ for some $j < i$, then $c_{ij} = 0$. Let

$$\tilde{u}^1 = u^1 - 2\gamma^1 \log |z|.$$

Then

$$\tilde{u}^1 = -\log \left(\lambda_0 + \sum_{i=1}^n \lambda_i |P_i(z)|^2 \right). \quad (2.5)$$

The following lemma classifies the solutions of the linearized system under a mild growth condition at infinity:

Lemma 2.1. *Let Φ_1, \dots, Φ_n solve the linearized $SU(n+1)$ Toda system*

$$\Delta \Phi_i + e^{u^i} \left(\sum_{j=1}^n a_{ij} \Phi_j \right) = 0 \quad \text{in } \mathbb{R}^2, \quad i = 1, \dots, n, \quad (2.6)$$

where u solves (2.1). If

$$|\Phi_i(x)| \leq C(1 + |x|)^\sigma, \quad x \in \mathbb{R}^2, \quad (2.7)$$

for some $\sigma \in (0, \min\{1, 2\mu_1, \dots, 2\mu_n\})$, then

$$e^{-u^1} \Phi_1(z) = \sum_{k=0}^n m_{kk} |z|^{2\beta_k} + 2 \sum_{k=1}^{n-1} |z|^{\beta_k} \sum_{l=k+1}^n |z|^{\beta_l} \operatorname{Re}(m_{kl} e^{-i(\mu_{k+1} + \dots + \mu_l)\theta}) \quad (2.8)$$

where $\theta = \arg(z)$,

$$\beta_0 = -\gamma^1, \quad \beta_i = \gamma^i - \gamma^{i+1} + i, \quad \beta_n = \gamma^n + n, \quad (2.9)$$

$m_{kk} \in \mathbb{R}$ for $k = 0, \dots, n$, and $m_{kl} \in \mathbb{C}$ for $k < l$. Obviously $m_{kl} = 0$ if $\mu_{k+1} + \dots + \mu_l \notin \mathbb{N}$.

Proof. This lemma is proved in [22] when all Φ_i are bounded functions. Here we mention the minor modifications when a mild growth condition in (2.7) is assumed. Let

$$w_i(y) = -\frac{1}{2\pi} \int_{\mathbb{R}^2} (\log |y - \eta|) e^{\mu_i(\eta)} \left(\sum_{j=1}^n a_{ij} \Phi_j(\eta) \right) d\eta.$$

From (2.8) and $e^{\mu_i(z)} = O(|z|^{-4-2\nu_{n+1-i}})$ we see that $e^{\mu_i(z)} (\sum_{j=1}^n a_{ij} \Phi_j(z)) = O(|z|^{-2-\delta})$ for some $\delta > 0$ when $|z|$ is large. Thus $w_i(y) = O(\log |y|)$ for $|y|$ large. From $\Delta(\Phi_i - w_i) = 0$ in \mathbb{R}^2 and $|\Phi_i(z) - w_i(z)| \leq O(|z|^{1-\delta})$ for some $\delta > 0$ we have

$$\Phi_i = w_i + C.$$

Then using the integral representation of Φ_i we can further obtain $\nabla^k \Phi_i = O(|z|^{-k})$ as $|z| \rightarrow \infty$. The remaining part of the proof is the same as in [22, proof of Lemma 6.1]. \square

From (2.9) it is easy to verify that

$$\beta_i - \beta_{i-1} = \mu_i, \quad 1 \leq i \leq n. \tag{2.10}$$

Thus β_i is increasing because $\mu_i = 1 + \gamma_i > 0$. Using (2.2) and (2.10) in (2.8), we have

$$\begin{aligned} \Phi_1 &= \frac{1}{\lambda_0 + \sum_i \lambda_i |P_i(z)|^2} \left\{ \sum_{k=0}^n m_{kk} |z|^{2\beta_k + 2\gamma^1} \right. \\ &\quad \left. + 2 \sum_{k=0}^{n-1} |z|^{\beta_k + \gamma^1} \sum_{l=k+1}^n |z|^{\beta_l + \gamma^1} \operatorname{Re}(m_{kl} e^{-i(\mu_{k+1} + \dots + \mu_l)\theta}) \right\} \\ &= \frac{1}{\lambda_0 + \sum_i \lambda_i |P_i(z)|^2} \left\{ \sum_{k=0}^n m_{kk} |z|^{2\mu_1 + \dots + 2\mu_k} + 2 \sum_{k=0}^{n-1} |z|^{\mu_1 + \dots + \mu_k} \right. \\ &\quad \left. \times \left(\sum_{l=k+1}^n |z|^{\mu_1 + \dots + \mu_l} \operatorname{Re}(m_{kl} e^{-i(\mu_{k+1} + \dots + \mu_l)\theta}) \right) \right\}. \tag{2.11} \end{aligned}$$

Lemma 2.2.

$$\frac{m_{00}}{\lambda_0} + \dots + \frac{m_{nn}}{\lambda_n} = 0.$$

Proof. It is proved in [22] that the linearized system is non-degenerate, which means all solutions to (2.6) are obtained by differentiating $n^2 + 2n$ parameters of (u^1, \dots, u^n) . In particular

$$\Phi_1 = c_1 \frac{\partial u^1}{\partial \lambda_1} + \dots + c_n \frac{\partial u^1}{\partial \lambda_n} + c_{n+1} \frac{\partial u^1}{\partial c_{01}^{\mathbb{R}}} + \dots + c_{n^2+2n} \frac{\partial u^1}{\partial c_{n,n-1}^{\mathbb{I}}}, \tag{2.12}$$

where $c_{ij}^{\mathbb{R}}$ is the real part of c_{ij} , and $c_{ij}^{\mathbb{I}}$ is the imaginary part. Direct computation from (2.2) and (2.4) shows

$$\frac{\partial u^1}{\partial \lambda_k} = -\frac{|P_k|^2 + \frac{\partial \lambda_0}{\partial \lambda_k}}{\lambda_0 + \sum_{i=1}^n \lambda_i |P_i|^2} = -\frac{|P_k|^2 - \frac{\lambda_0}{\lambda_k}}{\lambda_0 + \sum_{i=1}^n \lambda_i |P_i|^2}$$

for $k = 1, \dots, n$. Comparing (2.11) and (2.12) we have

$$m_{kk} = -c_k, \quad k = 1, \dots, n, \quad m_{00} = \frac{c_1 \lambda_0}{\lambda_1} + \dots + \frac{c_n \lambda_0}{\lambda_n}.$$

Then it is easy to see that

$$\frac{m_{00}}{\lambda_0} + \frac{m_{11}}{\lambda_1} + \dots + \frac{m_{nn}}{\lambda_n} = 0. \quad \square$$

From Lemma 2.2 we see that there are $n^2 + 2n$ unknowns in Φ_1 . We write Φ_1 as

$$\begin{aligned} \Phi_1 = & \frac{1}{\lambda_0 + \sum_{i=1}^n \lambda_i |P_i(z)|^2} \left\{ \sum_{k=1}^n m_{kk} |z|^{2\mu_1 + \dots + 2\mu_k} - \sum_{k=1}^n \frac{\lambda_0}{\lambda_k} m_{kk} \right. \\ & \left. + 2 \sum_{k=0}^{n-1} |z|^{2\mu_1 + \dots + 2\mu_k} \sum_{l=k+1}^n \operatorname{Re}(\bar{m}_{kl} z^{\mu_{k+1} + \dots + \mu_l}) \right\}. \end{aligned} \quad (2.13)$$

3. The proof of Theorem 1.2

Recall that $\tilde{v}^k = (\tilde{v}_1^k, \dots, \tilde{v}_n^k)$ satisfies (1.5) and \tilde{v}^k converges in $C_{\text{loc}}^{1,\alpha}(\mathbb{R}^2)$ to a solution $\tilde{v} = (\tilde{v}_1, \dots, \tilde{v}_n)$ of (1.6). By the classification theorem of Lin–Wei–Ye [22], there exists $\Lambda = (\lambda_i, c_{ij})$ ($i = 0, \dots, n, j < i$) such that $\tilde{v}^1(z)$ is defined as in (2.5) where λ_i and P_i satisfy (2.4) and (2.3), respectively. To emphasize the dependence on Λ , we denote \tilde{v}_i and \tilde{v}^i as $\tilde{v}_i(z, \Lambda)$ and $\tilde{v}^i(z, \Lambda)$, respectively.

The following matrix plays an important role in the argument below: For $p_1, \dots, p_{n^2+2n} \in \mathbb{R}^2$, set

$$\mathbf{M} = (\Theta(p_1), \dots, \Theta(p_{n^2+2n})), \quad (3.1)$$

where

$$\Theta(p) = \left(\frac{\partial \tilde{v}^1}{\partial \lambda_0}(p), \dots, \frac{\partial \tilde{v}^1}{\partial \lambda_{n-1}}(p), \frac{\partial \tilde{v}^1}{\partial c_{10}^{\mathbb{R}}}(p), \dots, \frac{\partial \tilde{v}^1}{\partial c_{n,n-1}^{\mathbb{I}}}(p) \right)'$$

where $()'$ stands for transpose. In Section 4 we shall show that if we choose p_1, \dots, p_{n^2+2n} appropriately with respect to Λ , the matrix \mathbf{M} is invertible.

Let $\tilde{v}^{i,k} = \sum_j a^{ij} \tilde{v}_j^k$. Then $\tilde{v}^{i,k}$ converges uniformly to $\tilde{v}^i(\cdot, \Lambda)$ over any fixed compact subset of \mathbb{R}^2 . Since the difference between $\tilde{v}^{i,k}$ and $\tilde{v}^i(\cdot, \Lambda)$ is only $o(1)$, we need to find a sequence of global solutions that approximates better. Suppose the sequence of global solutions is represented by $\Lambda_k := (\lambda_i^k, c_{ij}^k)$: the regular part of the first component is

$$\tilde{v}^1(z, \Lambda_k) = -\log\left(\lambda_0^k + \sum_{i=1}^n \lambda_i^k |P_i^k(z)|^2\right)$$

with

$$P_i^k(z) = z^{\mu_1^k + \dots + \mu_i^k} + \sum_{j=0}^{i-1} c_{ij}^k z^{\mu_1^k + \dots + \mu_j^k}.$$

Other components $\tilde{v}^i(z, \Lambda_k)$ are determined by the equation

$$\Delta \tilde{v}^i(y, \Lambda_k) + |y|^{2\gamma_i} e^{\sum_j a_{ij} \tilde{v}^j(y, \Lambda_k)} = 0 \quad \text{in } \mathbb{R}^2, \quad i = 1, \dots, n.$$

Finally we set

$$v^i(z, \Lambda_k) = \tilde{v}^i(z, \Lambda_k)(z) + 2\gamma^i \log |z|, \quad \gamma^i = \sum_j a^{ij} \gamma_j, \quad i = 1, \dots, n. \quad (3.2)$$

We claim that if

$$\tilde{v}^1(p_l, \Lambda_k) = \tilde{v}^{1,k}(p_l), \quad l = 1, \dots, n^2 + 2n, \quad (3.3)$$

then

$$\lambda_i^k \rightarrow \lambda_i, \quad c_{ij}^k \rightarrow c_{ij}. \quad (3.4)$$

Indeed, since $\tilde{v}^{1,k}(p_l) = \tilde{v}^1(p_l, \Lambda) + o(1)$ for $l = 1, \dots, n^2 + 2n$, (3.4) clearly follows from the invertibility of \mathbf{M} . In other words there exist $\Lambda_k \rightarrow \Lambda$ such that (3.3) holds.

Let $v_i(\cdot, \Lambda_k) = \sum_j a_{ij} v^j(\cdot, \Lambda_k)$. Here we point out that

$$v_i(\cdot, \Lambda_k) = \tilde{U}_i^k(\epsilon_k \cdot) + 2(1 + \gamma_i) \log \epsilon_k, \quad i = 1, \dots, n,$$

which is the global sequence in (1.10) and in the statement of Theorem 1.2.

In order to obtain estimates (1.8) we write (2.13) as

$$\begin{aligned} & \Phi_1(z) \left(\lambda_0 + \sum_i \lambda_i |P_i(z)|^2 \right) \\ &= \sum_{k=1}^n m_{kk} \left(|z|^{2\mu_1 + \dots + 2\mu_k} - \frac{\lambda_0}{\lambda_k} \right) + 2 \sum_{k=0}^{n-1} \sum_{l=k+1}^n |z|^{2\mu_1 + \dots + 2\mu_k + \mu_{k+1} + \dots + \mu_l} \\ & \quad \times \left(\cos((\mu_{k+1} + \dots + \mu_l)\theta) m_{kl}^1 + \sin((\mu_{k+1} + \dots + \mu_l)\theta) m_{kl}^2 \right) \\ &= \mathbf{X} \hat{\Theta}(z), \end{aligned} \quad (3.5)$$

where

$$\mathbf{X} = (m_{11}, \dots, m_{nn}, m_{01}^1, \dots, m_{n-1,n}^2), \quad m_{kl} = m_{kl}^1 + \sqrt{-1} m_{kl}^2.$$

So $\hat{\Theta}(z)$ is a column vector (as also is $\Theta(p)$). Our choice of p_1, \dots, p_{n^2+2n} (explained in Section 4) also makes

$$\mathbf{M}_1 = (\hat{\Theta}(p_1), \dots, \hat{\Theta}(p_{n^2+2n}))$$

invertible.

Let $\Phi_i^k = \tilde{v}^{i,k} - \tilde{v}^i(\cdot, \Lambda_k)$. By (1.5) and the definition of $\tilde{v}^{i,k}$ we have

$$\Delta \tilde{v}^{i,k} + |y|^{2\gamma_i} h_i^k(\epsilon_k y) e^{\sum_j a_{ij} \tilde{v}^j(y)} = 0, \quad |y| \leq \epsilon_k^{-1}.$$

Hence the equation for $(\Phi_1^k, \dots, \Phi_n^k)$ can be written as

$$\Delta \Phi_i^k(y) + |y|^{2\gamma_i} e^{\xi_i^k(y)} \left(\sum_j a_{ij} \Phi_j^k(y) \right) = O(\epsilon_k |y|) |y|^{2\gamma_i} e^{\sum_j a_{ij} \tilde{v}^{j,k}} \tag{3.6}$$

where, by the mean value theorem,

$$e^{\xi_i^k} = \frac{e^{\sum_j a_{ij} \tilde{v}^{j,k}} - e^{\sum_j a_{ij} \tilde{v}^j(\cdot, \Lambda_k)}}{\sum_j a_{ij} (\tilde{v}^{j,k} - \tilde{v}^j(\cdot, \Lambda_k))} = \int_0^1 e^{\sum_j a_{ij} (t \tilde{v}^{j,k} + (1-t) \tilde{v}^j(\cdot, \Lambda_k))} dt.$$

By [23, Theorems 4.1 and 4.2], $e^{\xi_i^k}$ converges uniformly to $e^{\tilde{v}_i(\cdot, \Lambda)}$ over each compact subset of \mathbb{R}^2 , and moreover,

$$|y|^{2\gamma_i} e^{\xi_i^k(y)} = O(1 + |y|)^{-4-2\gamma_{n+1-i}+o(1)}, \quad |y| \leq \epsilon_k^{-1}. \tag{3.7}$$

Also by [23, Theorems 4.1 and 4.2] we can estimate the right hand side of (3.6). Thus (3.6) can be written as

$$\Delta \Phi_i^k + |y|^{2\gamma_i} e^{\xi_i^k(y)} \left(\sum_{j=1}^n a_{ij} \Phi_j^k \right) = \frac{O(\epsilon_k)}{(1 + |y|)^{3+2\gamma_{n+1-i}}}, \quad |y| \leq \epsilon_k^{-1}. \tag{3.8}$$

It is immediate to observe that the oscillation of Φ_i^k on $\partial B_{\epsilon_k}^{-1}$ is finite. Thus for convenience we use the following functions to eliminate that oscillation:

$$\begin{cases} \Delta \psi_i^k = 0 & \text{in } B_{\epsilon_k}^{-1}, \\ \psi_i^k = \Phi_i^k - \frac{1}{2\pi\epsilon_k^{-1}} \int_{\partial B_{\epsilon_k}^{-1}} \Phi_i^k & \text{on } \partial B_{\epsilon_k}^{-1}. \end{cases}$$

A standard estimate gives

$$|\psi_i^k(y)| \leq C\epsilon_k |y|, \quad |y| \leq \epsilon_k^{-1}. \tag{3.9}$$

Let $\tilde{\Phi}_i^k = \Phi_i^k - \psi_i^k$. Then by (3.8) and (3.9) we have

$$\Delta \tilde{\Phi}_i^k + |y|^{2\gamma_i} e^{\xi_i^k(y)} \left(\sum_{j=1}^n a_{ij} \tilde{\Phi}_j^k \right) = \frac{O(\epsilon_k)}{(1 + |y|)^{3+2\gamma_{n+1-i}}}, \quad |y| \leq \epsilon_k^{-1}, \tag{3.10}$$

and it follows from (3.3) and (3.9) that

$$\tilde{\Phi}_1^k(p_l) = O(\epsilon_k), \quad l = 1, \dots, n^2 + 2n. \tag{3.11}$$

We consider two cases.

Case 1: $\min\{\gamma_1, \dots, \gamma_n\} \leq -3/4$. In this case we set

$$H_k = \max_i \max_{|y| \leq \epsilon_k^{-1}} \frac{|\tilde{\Phi}_i^k(y)|}{(1 + |y|)^\sigma \epsilon_k^\sigma}$$

for any fixed $\sigma \in (0, \min\{1, 2\mu_1, \dots, 2\mu_n\})$. Our goal is to show that H_k is bounded. Indeed, suppose $H_k \rightarrow \infty$ and let the maximum be attained at y_k . Let

$$\hat{\Phi}_i^k(y) = \frac{\tilde{\Phi}_i^k(y)}{H_k(1 + |y_k|)^\sigma \epsilon_k^\sigma}.$$

This definition immediately implies

$$|\hat{\Phi}_i^k(y)| = \frac{|\tilde{\Phi}_i^k(y)|}{H_k \epsilon_k^\sigma (1 + |y|)^\sigma} \frac{(1 + |y|)^\sigma}{(1 + |y_k|)^\sigma} \leq \frac{(1 + |y|)^\sigma}{(1 + |y_k|)^\sigma}. \tag{3.12}$$

Next we write the equation for $(\hat{\Phi}_1^k, \dots, \hat{\Phi}_n^k)$ as

$$\Delta \hat{\Phi}_i^k + |y|^{2\gamma_i} e^{\xi_i^k} \left(\sum_j a_{ij} \hat{\Phi}_j^k \right) = \frac{O(\epsilon_k^{1-\sigma})(1 + |y|)^{-3-2\gamma_{n+1-i}}}{H_k(1 + |y_k|)^\sigma},$$

and we observe that $\hat{\Phi}_i^k$ has no oscillation on $\partial B_{\epsilon_k^{-1}}$.

We first consider the case that along a subsequence, $y_k \rightarrow y^*$. In this case, $(\hat{\Phi}_1^k, \dots, \hat{\Phi}_n^k)$ converges to (Φ_1, \dots, Φ_n) that satisfies

$$\begin{cases} \Delta \Phi_i + e^{v_i} \sum_j a_{ij} \Phi_j = 0 & \text{in } \mathbb{R}^2, \quad i = 1, \dots, n, \\ |\Phi_i(y)| \leq C(1 + |y|)^\sigma, \quad i = 1, \dots, n, \quad \sigma \in (0, \min\{1, 2\mu_1, \dots, 2\mu_n\}), \\ \Phi_1(p_l) = 0, \quad l = 1, \dots, n^2 + 2n, \end{cases} \tag{3.13}$$

where $v_i(y) = \tilde{v}_i(y) + 2\gamma_i \log |y|$. Note that the last equation in (3.13) holds because of (3.11). From the first two conditions of (3.13) and Lemma 2.1 we obtain (2.8). Then by (3.5),

$$\mathbf{M}\hat{\Theta}(p_l) = 0, \quad l = 1, \dots, n^2 + 2n.$$

Since \mathbf{M} is invertible, we have

$$m_{11} = \dots = m_{n,n} = m_{1,0}^1 = \dots = m_{n,n-1}^2 = 0.$$

Thus $\Phi_1 \equiv 0$, which means $\Phi_i \equiv 0$ for all i . This contradicts $|\Phi_i(y^*)| = 1$ for some i .

The only remaining case we need to consider is when $y_k \rightarrow \infty$. To get a contradiction we evaluate

$$\begin{aligned} \hat{\Phi}_i^k(y_k) - \hat{\Phi}_i^k(0) &= \int_{B_{\epsilon_k^{-1}}} (G_k(y_k, \eta) - G_k(0, \eta)) \left(|\eta|^{2\gamma_i} e^{\xi_i^k(\eta)} \left(\sum_j a_{ij} \tilde{\Phi}_j^k(\eta) \right) \right. \\ &\quad \left. + \frac{O(\epsilon_k^{1-\sigma})(1 + |\eta|)^{-3-2\gamma_{n+1-i}}}{H_k(1 + |y_k|)^\sigma} \right) d\eta \end{aligned} \tag{3.14}$$

where G_k is the Green function on $B_{\epsilon_k^{-1}}$ with the Dirichlet boundary condition. To evaluate the right hand side above we use (3.12), (3.7) and the following estimate of the Green function (see [30] for the proof):

For $y \in \Omega_k := B_{1/\epsilon_k}$, let

$$\begin{aligned} \Sigma_1 &= \{\eta \in \Omega_k; |\eta| < |y|/2\}, \\ \Sigma_2 &= \{\eta \in \Omega_k; |y - \eta| < |y|/2\}, \\ \Sigma_3 &= \Omega_k \setminus (\Sigma_1 \cup \Sigma_2). \end{aligned}$$

Then for $|y| > 2$,

$$|G_k(y, \eta) - G_k(0, \eta)| \leq \begin{cases} C(\log |y| + |\log |\eta||), & \eta \in \Sigma_1, \\ C(\log |y| + |\log |y - \eta||), & \eta \in \Sigma_2, \\ C|y|/|\eta|, & \eta \in \Sigma_3. \end{cases} \quad (3.15)$$

Using (3.15) to estimate the right hand side of (3.14) is standard. Here we just point out that we use (3.12) to estimate $\hat{\Phi}_j^k(\eta)$ in the first term and it is essential to use $\epsilon_k^{1-\sigma}$ for the second term, as $\min\{2\mu_1, \dots, 2\mu_n\} + \sigma$ may be less than or equal to 1 in this case. After these standard estimates we see that the right hand side of (3.14) is $o(1)$. However we know $|\hat{\Phi}_i^k(y_k)| = 1$ for some i and it is easy to prove $|\hat{\Phi}_i^k(0)| \rightarrow 0$ by exactly the same argument used in the proof of $y_k \rightarrow \infty$. Thus we obtain a contradiction, proving

$$|\tilde{\Phi}_i^k(y)| \leq C\epsilon_k^\sigma (1 + |y|)^\sigma.$$

Case 2: $\min\{\gamma_1, \dots, \gamma_n\} > -3/4$. In this case we set

$$H_k = \max_i \max_{|y| \leq \epsilon_k^{-1}} \frac{|\tilde{\Phi}_i^k(y)|}{(1 + |y|)^\sigma \epsilon_k} \quad \text{and} \quad \hat{\Phi}_i^k(y) = \frac{\tilde{\Phi}_i^k(y)}{H_k(1 + |y_k|)^\sigma}.$$

Here we choose σ not only in $(0, \min\{1, 2\mu_1, \dots, 2\mu_n\})$, but also satisfying

$$\min\{2\mu_1, \dots, 2\mu_n\} + \sigma > 1. \quad (3.16)$$

Since $\min\{2\mu_1, \dots, 2\mu_n\} > 1/2$, such a σ can be found. By the definition of H_k , the estimate (3.12) still holds. The equation for $\hat{\Phi}_i^k$ becomes

$$\Delta \hat{\Phi}_i^k + |y|^{2\gamma_i} e^{\xi_i^k} \left(\sum_j a_{ij} \hat{\Phi}_j^k \right) = \frac{O((1 + |y|)^{-3-2\gamma_{n+1-i}})}{H_k(1 + |y_k|)^\sigma}.$$

Assume H_k is attained at y_k . Then by the same argument as in Case 1, $|y_k| \rightarrow \infty$. In order to get a contradiction, we observe that (3.14) becomes

$$\begin{aligned} \hat{\Phi}_i^k(y_k) - \hat{\Phi}_i^k(0) &= \int_{B_{\epsilon_k^{-1}}} (G_k(y_k, \eta) - G_k(0, \eta)) \left(|\eta|^{2\gamma_i} e^{\xi_i^k(\eta)} \left(\sum_j a_{ij} \tilde{\Phi}_j^k(\eta) \right) \right. \\ &\quad \left. + \frac{O((1 + |\eta|)^{-3-2\gamma_{n+1-i}})}{H_k(1 + |y_k|)^\sigma} \right) d\eta. \end{aligned} \quad (3.17)$$

Using the same estimate on G_k and (3.16) we see that the right hand side of (3.17) is $o(1)$, thus we get a contradiction as in Case 1, proving

$$|\tilde{\Phi}_i^k(y)| \leq C\epsilon_k(1 + |y|)^\sigma \quad \text{in Case 2.}$$

Note that the main reason that the power of ϵ_k can be 1 is that (3.16) holds. Theorem 1.2 follows from the estimates of $\tilde{\Phi}_i^k$ and (3.9). \square

4. The determination of p_1, \dots, p_{n^2+2n}

In this section we explain how p_1, \dots, p_{n^2+2n} are chosen to make the matrices \mathbf{M} and \mathbf{M}_1 both invertible.

First we list some facts that can be verified easily by direct computation: Using (2.4) (recall that $\tilde{v}^1 = -\log(\lambda_0 + \sum_{i=1}^n \lambda_i |P_i(z)|^2)$) we have

$$\begin{aligned} \frac{\partial \tilde{v}^1}{\partial \lambda_0} &= \frac{\frac{\lambda_n}{\lambda_0} |P_n(z)|^2 - 1}{\lambda_0 + \sum_i \lambda_i |P_i(z)|^2}, & (4.1) \\ \frac{\partial \tilde{v}^1}{\partial \lambda_i} &= \frac{\frac{\lambda_n}{\lambda_i} |P_n(z)|^2 - |P_i(z)|^2}{\lambda_0 + \sum_i \lambda_i |P_i(z)|^2}, \quad i = 1, \dots, n-1, \\ \frac{\partial \tilde{v}^1}{\partial c_{ij}^{\mathbb{R}}} &= -\frac{2\lambda_i \operatorname{Re}(z^{\mu_1+\dots+\mu_j} \bar{P}_i)}{\lambda_0 + \sum_i \lambda_i |P_i(z)|^2}, \quad j < i, i = 1, \dots, n, \\ \frac{\partial \tilde{v}^1}{\partial c_{ij}^{\mathbb{I}}} &= \frac{2\lambda_i \operatorname{Im}(z^{\mu_1+\dots+\mu_j} \bar{P}_i)}{\lambda_0 + \sum_i \lambda_i |P_i(z)|^2}, \quad j < i, i = 1, \dots, n. \end{aligned}$$

It is easy to verify that for $|z|$ large,

$$z^{\mu_1+\dots+\mu_j} \bar{P}_i = |z|^{2\mu_1+\dots+2\mu_j+\mu_{j+1}+\dots+\mu_i} (e^{-\sqrt{-1}(\mu_{j+1}+\dots+\mu_i)\theta} + O(|z|^{-\delta}))$$

for some $\delta > 0$ that depends only on μ_1, \dots, μ_n . Thus for $|z|$ large,

$$\begin{aligned} \frac{\partial \tilde{v}^1}{\partial c_{ij}^{\mathbb{R}}}(z) &\left(\lambda_0 + \sum_k \lambda_k |P_k(z)|^2 \right) \\ &= -2\lambda_i |z|^{2\mu_1+\dots+2\mu_j+\mu_{j+1}+\dots+\mu_i} (\cos((\mu_{j+1} + \dots + \mu_i)\theta) + O(|z|^{-\delta})), \end{aligned} \quad (4.2)$$

$$\begin{aligned} \frac{\partial \tilde{v}^1}{\partial c_{ij}^{\mathbb{I}}}(z) &\left(\lambda_0 + \sum_k \lambda_k |P_k(z)|^2 \right) \\ &= -2\lambda_i |z|^{2\mu_1+\dots+2\mu_j+\mu_{j+1}+\dots+\mu_i} (\sin((\mu_{j+1} + \dots + \mu_i)\theta) + O(|z|^{-\delta})). \end{aligned} \quad (4.3)$$

By the definition of $P_i(z)$ in (2.3),

$$|P_i(z)|^2 = |z|^{2\mu_1+\dots+2\mu_i} (1 + O(|z|^{-\delta})). \quad (4.4)$$

We also note that

$$\frac{\partial \tilde{v}^1}{\partial \lambda_i} = \frac{\lambda_0}{\lambda_i} \frac{\partial \tilde{v}^1}{\partial \lambda_0} + \frac{\frac{\lambda_0}{\lambda_i} - |P_i(z)|^2}{\lambda_0 + \sum_i \lambda_i |P_i(z)|^2}, \quad i = 1, \dots, n-1.$$

The idea of choosing $n^2 + 2n$ points is to make \mathbf{M} (defined in (3.1)) similar to a Vandermonde type matrix. We shall use different parameters in the definition of p_l , which are either large or small, in order to make the leading terms dominate the other terms.

Now we look at \mathbf{M} . Clearly the factor $\lambda_0 + \sum_k \lambda_k |P_k(p_l)|^2$ can be taken out from the l th column, thus for $|p_l| \gg 1$, \mathbf{M} is invertible if and only if,

$$\mathbf{M}_2 := (\Theta_1(p_1), \dots, \Theta_1(p_{n^2+2n}))$$

is invertible, where, according to (4.2)–(4.4),

$$\Theta_1(p_l) = (|p_l|^{2a_n} (1 + O(1/|p_l|^\delta)), |p_l|^{2a_{n-1}+a_{n,n-1}} \cos(a_{n,n-1}\theta_l)(1 + O(1/|p_l|^\delta)), \\ |p_l|^{2a_{n-1}+a_{n,n-1}} \sin(a_{n,n-1}\theta_l)(1 + O(1/|p_l|^\delta)), \dots)'$$

where

$$a_0 = 0, \quad a_i = \mu_1 + \dots + \mu_i \quad (i = 1, \dots, n), \\ a_{ij} = \mu_{j+1} + \dots + \mu_i \quad (i = 1, \dots, n, \quad j < i),$$

$\theta_l = \arg(p_l)$, and $\delta > 0$ only depends on μ_1, \dots, μ_n . Note that $a_{ij} = a_i - a_j$ and $2a_j + a_{ij} = a_i + a_j$. The powers of $|p_l|$ are arranged in non-decreasing order (so the largest power is $2a_n$, the second largest power is $2a_{n-1} + a_{n,n-1}$, etc.). The powers of $|p_l|$ are either $2a_i$ or $a_i + a_j$. Here we note that some powers appear only once (for example $2a_n$), some powers appear only twice (for example $2a_{n-1} + a_{n,n-1}$), and it is possible that some powers appear more than twice.

Let

$$p_l = s^{1+\epsilon l} N e^{\sqrt{-1}\theta_l}, \quad l = 1, \dots, n^2 + 2n,$$

where $N \gg s \gg 1 \gg \epsilon > 0$ are constants only depending on μ_1, \dots, μ_n, n . The angles θ_l also only depend on these parameters. We shall determine these constants and angles below.

From each row a power of N can be taken out, therefore \mathbf{M}_2 is invertible iff

$$(\Theta_2(p_1), \dots, \Theta_2(p_{n^2+2n}))$$

is invertible, where

$$\Theta_2(p_l) = (s^{2a_n(1+\epsilon l)} (1 + O(1/|p_l|^\delta)), \\ s^{(2a_{n-1}+a_{n,n-1})(1+\epsilon l)} \cos(a_{n,n-1}\theta_l)(1 + O(1/|p_l|^\delta)), \\ s^{(2a_{n-1}+a_{n,n-1})(1+\epsilon l)} \sin(a_{n,n-1}\theta_l)(1 + O(1/|p_l|^\delta)), \dots)'$$

Hence for fixed s , if N is sufficiently large and $O(1/|p_l|^\delta)$ is very small, then \mathbf{M}_2 is invertible iff the following matrix is invertible:

$$\mathbf{M}_3 = (\Theta_3(p_1), \dots, \Theta_3(p_{n^2+2n}))$$

where

$$\Theta_3(p_l) = \left(s^{2a_n(1+\epsilon l)}, s^{(2a_{n-1}+a_{n,n-1})(1+\epsilon l)} \cos(a_{n,n-1}\theta_l), s^{(2a_{n-1}+a_{n,n-1})(1+\epsilon l)} \sin(a_{n,n-1}\theta_l), \dots \right)'$$

We start with the largest entry in \mathbf{M}_3 : $s^{2a_n(1+\epsilon(n^2+2n))}$, which is in row one and column $n^2 + 2n$. We divide row 1 by $s^{2a_n(1+\epsilon(n^2+2n))}$ (we call this *Operation 1*); then the entries in row 1 become

$$s^{2a_n\epsilon(l-n^2-2n)} \quad \text{for } l = 1, \dots, n^2 + 2n.$$

Next we subtract a multiple of row 1 from the other rows to eliminate the last entry in each row (we call this *Operation 2*). For any entry in the cofactor matrix of 1, if before Operation 2 it is of the form $s^a A$, it becomes $s^a(A + O(s^{-\delta}))$ after Operation 2. Indeed, for example, let $s^{2a_{i_0}(1+\epsilon l)}$ be an entry before Operation 2. The last entry of the same row is $s^{2a_{i_0}(1+\epsilon(n^2+2n))}$. In Operation 2 we subtract the $s^{2a_{i_0}(1+\epsilon(n^2+2n))}$ multiple of the first row. The entry in row 1 and the same column of $s^{2a_{i_0}(1+\epsilon l)}$ is $s^{2a_n\epsilon(l-n^2-2n)}$. Thus after Operation 2, $s^{2a_{i_0}(1+\epsilon l)}$ becomes

$$\begin{aligned} & s^{2a_{i_0}(1+\epsilon l)} - s^{2a_{i_0}(1+\epsilon(n^2+2n))} s^{2a_n\epsilon(l-n^2-2n)} \\ &= s^{2a_{i_0}(1+\epsilon l)} (1 - s^{(2a_{i_0}-2a_n)\epsilon(n^2+2n-l)}) = s^{2a_{i_0}(1+\epsilon l)} (1 + O(s^{-\delta})) \end{aligned}$$

where we have used $a_{i_0} < a_n$.

Similarly if an entry before Operation 2 is

$$s^{(2a_j+a_{ij})(1+\epsilon l)} \cos(a_{ij}\theta_l),$$

then after Operation 2 it becomes

$$s^{(2a_j+a_{ij})(1+\epsilon l)} (\cos(a_{ij}\theta_l) + O(s^{-\delta}))$$

for some $\delta > 0$. Eventually s will be chosen large to eliminate the influence of all the perturbations.

Our strategy is to use high powers of s to simplify the matrix. After the aforementioned row operations it is clear that we only need to consider the cofactor matrix of 1, which we denote A_1 . The highest power of s in A_1 is shared by two entries:

$$s^{(2a_{n-1}+a_{n,n-1})(1+\epsilon(n^2+2n-1))} (\cos(a_{n,n-1}\theta_{n^2+2n-1}) + O(s^{-\delta}))$$

and

$$s^{(2a_{n-1}+a_{n,n-1})(1+\epsilon(n^2+2n-1))} (\sin(a_{n,n-1}\theta_{n^2+2n-1}) + O(s^{-\delta})).$$

We recall that the former is in row 1 of A_1 . We choose $\theta_{n^2+2n-1} = 0$. In A_1 we divide the first row by $s^{(2a_{n-1}+a_{n,n-1})(1+\epsilon(n^2+2n-1))}$; then the largest entry in row 1 of A_1 becomes $1 + O(s^{-\delta})$. We then subtract from other rows a multiple of the first row to eliminate the last entry of each row. For the same reason as before, after these row operations the

invertibility of A_1 is equivalent to the invertibility of the cofactor matrix A_2 of $1 + O(s^{-\delta})$, an $(n^2 + 2n - 2) \times (n^2 + 2n - 2)$ matrix which is barely changed after these transformations. In fact, each entry in A_2 is only multiplied by a factor $1 + O(s^{-\delta})$ in these transformations.

As we continue this process, we face three situations. If the highest power of s without the ϵ part is not repeated, we just apply the same type of row operations as in Operation 1 and Operation 2. If the highest power of s without the ϵ part is shared by only two entries (one is a cosine term, one is a sine term), we just take the corresponding angle to be 0, so the cosine term will dominate all other terms and this case is similar to the previous case. Finally we may run into the following situation: A power of s without the ϵ part is shared by more than two indices:

$$\begin{aligned} &\exists i_0, j_0, i_1, j_1 \text{ such that } 2a_{j_0} + a_{i_0, j_0} = 2a_{j_1} + a_{i_1, j_1}, j_0 \neq j_1. \\ &\exists i_0, j_0, j_1 \text{ such that } 2a_{j_0} + a_{i_0, j_0} = 2a_{j_1}. \end{aligned}$$

In this case we first prove the following simple but important lemma.

Lemma 4.1. *There exists $\epsilon_0 > 0$ that depends only on μ_1, \dots, μ_n and n such that for $\epsilon \in (0, \epsilon_0)$,*

$$|\tilde{p}_a|^{l_1} / |\tilde{p}_b|^{l_2} \rightarrow \infty \quad \text{as } s \rightarrow \infty, \forall a, b \in \{1, \dots, n^2 + 2n\}, \tag{4.5}$$

where l_1, l_2 are any two numbers in the set $\{2a_1, \dots, 2a_n, \dots, 2a_j + a_{ij}, \dots\}$ that satisfy $l_1 > l_2$ and where $|\tilde{p}_a|^{l_1} = s^{(1+\epsilon a)l_1}$, $|\tilde{p}_b|^{l_2} = s^{(1+\epsilon b)l_2}$.

Proof. It is easy to see that for all $a, b \in \{1, \dots, n^2 + 2n\}$, we have $(1 + \epsilon a)l_1 > (1 + \epsilon b)l_2$ if $l_1 > l_2$ and ϵ is sufficiently small. The smallness of ϵ is clearly determined by the set

$$\{2a_1, \dots, 2a_n, \dots, 2a_j + a_{ij}, \dots\}. \quad \square$$

Next we prove two more calculus lemmas.

Lemma 4.2. *Let $N_1 < \dots < N_k$ be positive numbers. Then there exist $\theta_1, \dots, \theta_{2k+1}$ such that the matrix*

$$M_{N_k} = \begin{pmatrix} 1 & \dots & 1 \\ \sin(N_1\theta_1) & \dots & \sin(N_1\theta_{2k+1}) \\ \cos(N_1\theta_1) & \dots & \cos(N_1\theta_{2k+1}) \\ \dots & \dots & \dots \\ \sin(N_k\theta_1) & \dots & \sin(N_k\theta_{2k+1}) \\ \cos(N_k\theta_1) & \dots & \cos(N_k\theta_{2k+1}) \end{pmatrix}$$

satisfies

$$0 < c_1(N_1, \dots, N_k) < |\det(M_{N_k})| < c_2(N_1, \dots, N_k)$$

for positive constants c_1 and c_2 that only depend on N_1, \dots, N_k .

Proof. We use the Taylor expansion of $\sin(N\theta)$ and $\cos(N\theta)$:

$$\begin{aligned} \sin(N_i\theta_j) &= \sum_{l=1}^k (-1)^{l+1} \frac{(N_i\theta_j)^{2l-1}}{(2l-1)!} + O((N_i\theta_j)^{2k+1}), \\ \cos(N_i\theta_j) &= \sum_{l=0}^k (-1)^l \frac{(N_i\theta_j)^{2l}}{(2l)!} + O((N_i\theta_j)^{2k+2}). \end{aligned}$$

We apply the following elementary operations on M_{Nk} : First we subtract a multiple of the first row from other odd-numbered rows to eliminate the first order terms of θ_i ($i = 1, \dots, 2k + 1$). After the cancelation it is easy to see that the entry in row $2j - 1$ ($j > 1$) and column r ($r > 1$) is of the form

$$\sum_{l=2}^k (-1)^{l+1} (a_{l,j}\theta_r)^{2l-1} + O(\theta_r^{2k+1})$$

for some positive constant $a_{l,j}$, which satisfies $a_{l,j} < a_{l,j+1}$. In the second step we use row 3 to eliminate all the $O(\theta^3)$ terms of other odd-numbered rows starting from row 5. After the second step, the entry in row $2j - 1$ ($j > 2$) and column r ($r > 2$) is of the form

$$\sum_{l=3}^k (-1)^{l+1} (\tilde{a}_{l,j}\theta_r)^{2l-1} + O(\theta_r^{2k+1}),$$

with $\tilde{a}_{l,j} > 0$ satisfying $\tilde{a}_{l,j} < \tilde{a}_{l,j+1}$.

After $k - 1$ such operations we see that the entry in row $2j - 1$ and column r is a multiple of θ_r^{2j-1} plus lower order terms. Clearly we can use the terms in row $2k - 1$ to eliminate all the $O(\theta^{2k-1})$ terms in other odd-numbered rows. Then we can use row $2k - 3$ to remove the $O(\theta^{2k-3})$ terms in other odd number rows. After such operations the entry in row $2j - 1$ and column r is $C\theta_r^{2j-1} + O(\theta_r^{2k+1})$. Similar operations can be applied to even-numbered rows. Thus after a finite number of elementary row operations (including multiplying a constant on each row) the matrix M_{Nk} is transformed to

$$\tilde{M}_{Nk} = \begin{pmatrix} 1 & 1 & \dots & 1 \\ \theta_1 & \theta_2 & \dots & \theta_{2k+1} \\ \theta_1^2 & \theta_2^2 & \dots & \theta_{2k+1}^2 \\ \dots & \dots & \dots & \dots \\ \theta_1^{2k-1} & \theta_2^{2k-1} & \dots & \theta_{2k+1}^{2k-1} \\ \theta_1^{2k} & \theta_2^{2k} & \dots & \theta_{2k+1}^{2k} \end{pmatrix} + \text{a matrix of error terms.}$$

The (i, j) entry of the second matrix is $O(\theta_i^{2k+1})$. Now we choose $\theta_i = i\epsilon$ for some $\epsilon > 0$ that depends only on N_1, \dots, N_k . For ϵ sufficiently small, \tilde{M}_{Nk} is invertible if and only if the first matrix is invertible. Finally, we observe that the first matrix of \tilde{M}_{Nk} is a Vandermonde matrix. Lemma 4.2 is established. \square

The proof of the following lemma is very similar and is omitted.

Lemma 4.3. *Let $N_1 < \dots < N_k$ be positive numbers. Then there exist $\theta_1, \dots, \theta_{2k}$ such that the matrix*

$$M_{2N_k} = \begin{pmatrix} \sin(N_1\theta_1) & \dots & \sin(N_1\theta_{2k}) \\ \cos(N_1\theta_1) & \dots & \cos(N_1\theta_{2k}) \\ \dots & \dots & \dots \\ \sin(N_k\theta_1) & \dots & \sin(N_k\theta_{2k}) \\ \cos(N_k\theta_1) & \dots & \cos(N_k\theta_{2k}) \end{pmatrix}$$

satisfies

$$0 < c_1(N_1, \dots, N_k) < |\det(M_{2N_k})| < c_2(N_1, \dots, N_k)$$

for positive constants c_1 and c_2 that only depend on N_1, \dots, N_k .

Now we go back to the case that after finite steps of reduction, the highest power of s without the ϵ part is M and is shared by more than two indices. Our goal is to make the following matrix invertible:

$$A_2 = \begin{pmatrix} B & C \\ D & F \end{pmatrix} \cdot (1 + O(s^{-d}))$$

where $1 + O(s^{-d})$ means each entry in $\begin{pmatrix} B & C \\ D & F \end{pmatrix}$ is multiplied by a quantity of magnitude $1 + O(s^{-d})$, even though these quantities are different from one another. C is either of the form

$$\begin{pmatrix} s^{M(1+\epsilon(l+1))} \sin(N_1\theta_{l+1}) & \dots & s^{M(1+\epsilon(l+2T))} \sin(N_1\theta_{l+2T}) \\ s^{M(1+\epsilon(l+1))} \cos(N_1\theta_{l+1}) & \dots & s^{M(1+\epsilon(l+2T))} \cos(N_1\theta_{l+2T}) \\ \dots & \dots & \dots \\ s^{M(1+\epsilon(l+1))} \sin(N_T\theta_{l+1}) & \dots & s^{M(1+\epsilon(l+2T))} \sin(N_T\theta_{l+2T}) \\ s^{M(1+\epsilon(l+1))} \cos(N_T\theta_{l+1}) & \dots & s^{M(1+\epsilon(l+2T))} \cos(N_T\theta_{l+2T}) \end{pmatrix}$$

or

$$\begin{pmatrix} 1 & \dots & 1 \\ s^{M(1+\epsilon(l+1))} \sin(N_1\theta_{l+1}) & \dots & s^{M(1+\epsilon(l+2T+1))} \sin(N_1\theta_{l+2T+1}) \\ s^{M(1+\epsilon(l+1))} \cos(N_1\theta_{l+1}) & \dots & s^{M(1+\epsilon(l+2T+1))} \cos(N_1\theta_{l+2T+1}) \\ \dots & \dots & \dots \\ s^{M(1+\epsilon(l+1))} \sin(N_T\theta_{l+1}) & \dots & s^{M(1+\epsilon(l+2T+1))} \sin(N_T\theta_{l+2T+1}) \\ s^{M(1+\epsilon(l+1))} \cos(N_T\theta_{l+1}) & \dots & s^{M(1+\epsilon(l+2T+1))} \cos(N_T\theta_{l+2T+1}) \end{pmatrix}$$

We take the first case as an example. B is of the form

$$B = \begin{pmatrix} s^{M(1+\epsilon)} \sin(N_1\theta_1) & \dots & s^{M(1+\epsilon l)} \sin(N_1\theta_l) \\ s^{M(1+\epsilon)} \cos(N_1\theta_1) & \dots & s^{M(1+\epsilon l)} \cos(N_1\theta_l) \\ \dots & \dots & \dots \\ s^{M(1+\epsilon)} \sin(N_T\theta_1) & \dots & s^{M(1+\epsilon l)} \sin(N_T\theta_l) \\ s^{M(1+\epsilon)} \cos(N_T\theta_1) & \dots & s^{M(1+\epsilon l)} \cos(N_T\theta_l) \end{pmatrix}.$$

The importance of Lemma 4.1 is that it makes F minor. For matrices D and F , we just write one row vector of (D, F) as a representative:

$$(s^{H(1+\epsilon)}, \dots, s^{H(1+\epsilon l)}, s^{H(1+\epsilon(l+1))}, \dots, s^{H(1+\epsilon(l+2T))})$$

where

$$(s^{H(1+\epsilon)}, \dots, s^{H(1+\epsilon l)})$$

is a row vector of D , and

$$(s^{H(1+\epsilon(l+1))}, \dots, s^{H(1+\epsilon(l+2T))})$$

is a row vector of F . Here we note that $H < M$, and other rows of \mathbf{A}_2 may have sine or cosine terms.

Now we take $s^{M(1+\epsilon(l+1))}$ out of the $2k$ rows of (B, C) ; after this operation B and C become \tilde{B} and \tilde{C} , where

$$\tilde{B} = \begin{pmatrix} s^{-M\epsilon l} \sin(N_1\theta_l) & s^{-M\epsilon(l-1)} \sin(N_1\theta_2) & \dots & s^{-M\epsilon} \sin(N_1\theta_l) \\ s^{-M\epsilon l} \cos(N_1\theta_l) & s^{-M\epsilon(l-1)} \cos(N_1\theta_2) & \dots & s^{-M\epsilon} \cos(N_1\theta_l) \\ \dots & \dots & \dots & \dots \\ s^{-M\epsilon l} \sin(N_T\theta_l) & s^{-M\epsilon(l-1)} \sin(N_T\theta_2) & \dots & s^{-M\epsilon} \sin(N_T\theta_l) \\ s^{-M\epsilon l} \cos(N_T\theta_l) & s^{-M\epsilon(l-1)} \cos(N_T\theta_2) & \dots & s^{-M\epsilon} \cos(N_T\theta_l) \end{pmatrix}$$

$$\tilde{C} = \begin{pmatrix} \sin(N_1\theta_{l+1}) & s^{M\epsilon} \sin(N_1\theta_{l+2}) & \dots & s^{M(2T-1)\epsilon} \sin(N_1\theta_{l+2T}) \\ \cos(N_1\theta_{l+1}) & s^{M\epsilon} \cos(N_1\theta_{l+2}) & \dots & s^{M(2T-1)\epsilon} \cos(N_1\theta_{l+2T}) \\ \dots & \dots & \dots & \dots \\ \sin(N_T\theta_{l+1}) & s^{M\epsilon} \sin(N_T\theta_{l+2}) & \dots & s^{M(2T-1)\epsilon} \sin(N_T\theta_{l+2T}) \\ \cos(N_T\theta_{l+1}) & s^{M\epsilon} \cos(N_T\theta_{l+2}) & \dots & s^{M(2T-1)\epsilon} \cos(N_T\theta_{l+2T}) \end{pmatrix}$$

After these row operations the major part of \mathbf{A}_2 becomes

$$\mathbf{A}_3 = (A_{31}, A_{32}) = \begin{pmatrix} \tilde{B} & \tilde{C} \\ D & F \end{pmatrix}.$$

Starting from the second column of A_{32} we take away the power of s . For example we divide the second column of A_{32} by $s^{M\epsilon}$, the third column by $s^{2M\epsilon}$ and the $2T$ th column by $s^{M(2T-1)\epsilon}$. Now we see the influence of the representative row vector in F . Before this set of column operations it is

$$(s^{H(1+\epsilon(l+1))}, \dots, s^{H(1+\epsilon(l+2T))})$$

After these column operations it becomes (using $H < M$)

$$s^{H(1+\epsilon(l+1))} (1, O(s^{-d}), \dots, O(s^{-d})).$$

Note that this computation is very similar to those in the proof of Lemma 4.1. We use \tilde{F} to represent the new matrix after the column operations on F .

After these column operations, \tilde{C} becomes

$$\tilde{C}_1 = \begin{pmatrix} \sin(N_1\theta_{l+1}) & \sin(N_1\theta_{l+2}) & \dots & \sin(N_1\theta_{l+2T}) \\ \cos(N_1\theta_{l+1}) & \cos(N_1\theta_{l+2}) & \dots & \cos(N_1\theta_{l+2T}) \\ \dots & \dots & \dots & \dots \\ \sin(N_T\theta_{l+1}) & \sin(N_T\theta_{l+2}) & \dots & \sin(N_T\theta_{l+2T}) \\ \cos(N_T\theta_{l+1}) & \cos(N_T\theta_{l+2}) & \dots & \cos(N_T\theta_{l+2T}) \end{pmatrix}.$$

By Lemma 4.3, \tilde{C}_1 is invertible, which means its row vectors are linearly independent. Thus there is a combination of its row vectors to cancel the representative vector in \tilde{F} (just the major part):

$$s^{H(1+\epsilon(l+1))}(1, 0, \dots, 0).$$

When this same row operation is applied to A_{31} , the representative vector in D ,

$$(s^{H(1+\epsilon)}, \dots, s^{H(1+\epsilon l)}),$$

after the row transformation becomes

$$(s^{H(1+\epsilon)}(1 + O(s^{-d})), \dots, s^{H(1+\epsilon l)}(1 + O(s^{-d})))$$

where we have used $H < M$ again. After these elementary operations, B and F turn into matrices of error terms. Thus the invertibility of A_2 is reduced to the invertibility of the transformation of D , which is of the same nature of D . This method of reduction can be continued and the construction of p_1, \dots, p_{n^2+2n} is complete for the matrix M .

The matrix M_1 is very similar to M and we only require N, s to be large and ϵ to be small in M_1 . Moreover the angles in M_1 are the same as in M . Thus p_1, \dots, p_{n^2+2n} that make M invertible also make M_1 invertible. The construction of p_1, \dots, p_{n^2+2n} is complete.

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