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Null structure and local well-posedness in the energy class for the Yang–Mills equations in Lorenz gauge

Abstract. We demonstrate null structure in the Yang–Mills equations in Lorenz gauge. Such structure was found in Coulomb gauge by Klainerman and Machedon, who used it to prove global well-posedness for finite-energy data in the temporal gauge by passing to local Coulomb gauges via Uhlenbeck's Lemma. Compared with the Coulomb gauge, the Lorenz gauge has the advantage—shared with the temporal gauge—that it can be imposed globally in space even for large solutions. Using the null structure and bilinear space-time estimates, we also prove local-in-time well-posedness of the Yang–Mills equations in Lorenz gauge for data with finite energy, with a time of existence depending on the initial energy and on the $H^s \times H^{s-1}$-norm of the initial gauge potential, for some choice of $s < 1$ sufficiently close to 1.

Keywords. Yang–Mills equations, well-posedness, Lorenz gauge, null structure

1. Introduction

From the requirement of invariance of the Dirac equation under certain transformations of the spinor field, one arrives at a corresponding transformation rule for the *gauge field*, representing the external forces. If the transformation group is the abelian group U(1), the gauge field satisfies the familiar Maxwell equations of electrodynamics. In 1954, Yang and Mills [40] succeeded in finding a generalisation of the Maxwell equations for non-abelian transformation groups. The Yang–Mills equations play a central role in particle physics, since in their quantized form they provide a unified description of interactions among elementary particles.

In this paper we are interested in the Cauchy problem for the classical Yang–Mills equations on the Minkowski space-time $\mathbb{R}^{1+3}$ with metric diag$(-1, 1, 1, 1)$.

Consider a gauge field $A = (A_\alpha)_{\alpha=0,1,2,3}$, where A_α is a function from the space-time to the Lie algebra $\mathfrak{g}$ of a compact Lie group G. The corresponding Yang–Mills field strength $F = F^A$ is given by, for $\alpha, \beta \in \{0, 1, 2, 3\}$,

$$F_{\alpha\beta} = F^A_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha + [A_\alpha, A_\beta], \tag{1.1}$$

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where $[\,\cdot\,,\,\cdot\,]$ is the Lie bracket in $\mathfrak{g}$. The Yang–Mills equations are then

$$\partial^\alpha F_{\alpha\beta} + [A^\alpha, F_{\alpha\beta}] = 0 \quad (\beta \in \{0, 1, 2, 3\}). \tag{1.2}$$

The metric on $\mathbb{R}^{1+3}$ is used to raise and lower tensorial indices, so $A^0 = -A_0$ and $A^j = A_j$ for $j \in \{1, 2, 3\}$. The coordinates on $\mathbb{R}^{1+3}$ are (x^0, x^1, x^2, x^3), where $t = x^0$ is time and $x = (x^1, x^2, x^3)$ is spatial position; ∂_α is the partial derivative with respect to x^α, and in particular $\partial_0 = \partial_t$ is the time derivative; the spatial gradient $(\partial_1, \partial_2, \partial_3)$ will be denoted ∇, and $\partial = (\partial_t, \nabla)$ is the full space-time gradient. In compliance with Einstein's summation convention, α is implicitly summed over $\{0, 1, 2, 3\}$ in (1.2); in general, repeated upper/lower Greek (resp. Latin) indices are implicitly summed over $\{0, 1, 2, 3\}$ (resp. $\{1, 2, 3\}$).

By representation theory, we may assume (see [32, Theorems 2.15 and 3.28]) that G is a closed Lie subgroup of the group $\mathrm{U}(n)$ of unitary $n \times n$ matrices for some $n \in \mathbb{N}$, hence (see [32, Theorem 4.6]) $\mathfrak{g}$ is a subalgebra of the Lie algebra $\mathfrak{u}(n)$ of skew-hermitian $n \times n$ matrices, and the Lie bracket is the matrix commutator.

As mentioned already, one can think of the Yang–Mills equations as a non-abelian generalisation of Maxwell's equations. Indeed, if G is the abelian group $\mathrm{U}(1)$ (hence $\mathfrak{g} = i\mathbb{R}$), then (1.2) reduces to Maxwell's equations in vacuum. As in the Maxwell theory, it is the field strength F that is the interesting quantity, whereas A is merely a potential representing F. This representation is not unique, in view of the invariance of the Yang–Mills equations under the *gauge transformation*

$$A_\alpha \mapsto A'_\alpha = S^{-1} A_\alpha S + S^{-1} \partial_\alpha S, \tag{1.3}$$

for a given $S \colon \mathbb{R}^{1+3} \to G$. Then $F' = F^{A'}$ is *gauge equivalent* to $F = F^A$ in the sense that

$$F' = S^{-1} F S.$$

If (A, F) satisfies the Yang–Mills equations, then so does (A', F'), as becomes evident if one reformulates the equations in the gauge covariant form

$$D_A^\alpha F_{\alpha\beta} = 0,$$

where $D_A^\alpha = \partial^\alpha + [A^\alpha, \cdot\,]$ is the gauge covariant derivative, with the property that $X' = S^{-1} X S$ implies $D_{A'} X' = S^{-1}(D_A X) S$.

A key feature of the Yang–Mills equations is energy conservation,

$$\mathcal{E}(t) = \frac{1}{2} \int_{\mathbb{R}^3} |F(t, x)|^2 \, dx = \text{const}$$

for sufficiently smooth solutions decaying at spatial infinity. Here $|F|^2 = \sum_{\alpha,\beta} |F_{\alpha\beta}|^2$ with the norm $|\cdot|$ induced by the inner product $\langle A, B \rangle = \text{tr}(AB^*)$ for $A, B \in \mathfrak{g}$, where B^* is the conjugate transpose.

1.1. Gauge conditions

Identifying gauge equivalent solutions, one has *gauge freedom*, that is, the freedom to choose a representative (A, F) from a given equivalence class. Thus one may complement (1.2) by a condition on A, a *gauge condition*. The classical gauge conditions are

- temporal: $A_0 = 0$,
- Coulomb: $\partial^i A_i = 0$,
- Lorenz: $\partial^\alpha A_\alpha = 0$.

Having chosen a gauge condition, one can formulate and try to solve the Cauchy problem for given data at $t = 0$ satisfying appropriate compatibility conditions. But which gauge condition should one choose? Each of the classical gauge conditions has its advantages, so the ideal situation would be to have all three of them satisfied at once, but as this is not possible in general (although it is in the abelian case), one has to settle for one at a time, and the question is then which one.

A useful approach in order to gain some insight is to first assume that we are given a sufficiently regular solution of the Yang–Mills equations, without assuming any gauge condition at all, and then try to gauge transform it into one of the classical gauges by finding the G-valued gauge function S appearing in (1.3). The temporal gauge can always be imposed since it only requires solving a linear ODE for S. The Coulomb gauge leads to a non-linear elliptic equation for S at fixed times, solvable locally in space but in general not globally, so the Coulomb gauge cannot be imposed globally in space even for the initial data. The Lorenz gauge leads to a non-linear wave equation for S, solvable locally in time and globally in space.

In particular, these considerations show that if global regularity[1] holds for Yang–Mills in any gauge at all, then it must hold in temporal gauge but fails in Coulomb gauge (even at the level of the data). In fact, global regularity also fails in Lorenz gauge when $G = \mathrm{SU}(2)$, since then the non-linear wave equation for S is equivalent to the wave maps equation to the 3-sphere, for which self-similar blow up occurs (see [33], [35, p. 75]).

That global regularity does hold in temporal gauge was proved by Eardley and Moncrief [9]. In fact, they proved global well-posedness for data with one degree of regularity more than the energy regularity. Klainerman and Machedon [15] improved this to global well-posedness in the energy class.

From the discussion so far, it may seem that one need look no further than the temporal gauge. However, although this gauge is ideal for the statement of the global results mentioned above, it is not very well suited for *proving* these results. Indeed, both Eardley and Moncrief and Klainerman and Machedon passed temporarily to other gauges to prove their results. In both cases, the authors first prove a local result, and then use energy conservation to show that an appropriate norm cannot blow up in finite time, so that the local result can be continued indefinitely. Eardley and Moncrief passed to the Cronström gauge ($x^\mu A_\mu = 0$) to prove an L^∞ bound on the field strength. Klainerman and Machedon passed to the Coulomb gauge, since in that gauge a crucial null structure appears in the

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[1] By global regularity we mean here that starting from smooth initial data of any size, the Cauchy problem has a smooth solution for all time.

Yang–Mills equations, a structure which is essential in order to obtain closed estimates for the solutions at the energy regularity.

However, the Coulomb gauge can only be imposed locally in space, via Uhlenbeck's Lemma [39], and this leads to considerable technical difficulties since the Yang–Mills system in Coulomb gauge contains a non-linear elliptic equation in addition to non-linear wave equations, so that one has to take great care with boundary conditions.

This work grew out of the question whether it would be possible to carry out the argument of Klainerman and Machedon with the Coulomb gauge replaced by the Lorenz gauge, the motivation being that the Lorenz gauge in certain respects is more natural than the Coulomb gauge. For example, the Yang–Mills equations in Lorenz gauge is purely a system of non-linear wave equations, so one has finite speed of propagation. Moreover, the Lorenz gauge can be imposed globally in space. In this paper we make a modest start in this direction by considering the local well-posedness of the Yang–Mills equations in Lorenz gauge for low-regularity initial data. Our main results are:

- The Yang–Mills equations in Lorenz gauge have a null structure similar to the one discovered by Klainerman and Machedon in Coulomb gauge.
- The $3 + 1$-dimensional Yang–Mills equations in Lorenz gauge are locally well posed for finite-energy initial data.

We remark that partial null structure is also present in the temporal gauge, as proved by Tao [37], but the resulting low-regularity theory has so far been limited to small-norm data. Lorenz-gauge null structures were first discovered in [6] for the Maxwell–Dirac equations, then for the Maxwell–Klein–Gordon equations [30] (see also [25] and other gauge field theories (see [2, 31, 11]).

We close this section with a more detailed discussion of our local well-posedness result. The null structure is discussed in Section 2.

1.2. The Cauchy problem

Consider initial data at $t = 0$,

$$A_\alpha(0) = a_\alpha, \qquad \partial_t A_\alpha(0) = \dot{a}_\alpha.$$

Thus,

$$F_{\alpha\beta}(0) = f_{\alpha\beta},$$

where

$$\begin{cases} f_{ij} = \partial_i a_j - \partial_j a_i + [a_i, a_j], \\ f_{0i} = \dot{a}_i - \partial_i a_0 + [a_0, a_i]. \end{cases} \tag{1.4}$$

Note that (1.2) with $\beta = 0$ imposes the (gauge invariant) constraint

$$\partial^i f_{0i} + [a^i, f_{0i}] = 0. \tag{1.5}$$

Additional constraints are imposed by the chosen gauge condition. Regularity will be measured in the Sobolev spaces $H^s = (I - \Delta)^{-s/2} L^2(\mathbb{R}^3)$.

Definition 1.1. A *temporal-gauge H^s data set* consists of $(a_\alpha, \dot{a}_\alpha) \in H^s \times H^{s-1}$ with $a_0 = \dot{a}_0 = 0$ and such that (1.5) is satisfied: $\partial^i \dot{a}_i + [a^i, \dot{a}_i] = 0$. We say that the data have *finite energy* if $f_{\alpha\beta} \in L^2$.

Definition 1.2. A *Lorenz-gauge H^s data set* consists of $(a_\alpha, \dot{a}_\alpha) \in H^s \times H^{s-1}$ satisfying the Lorenz constraint $\dot{a}_0 = \partial^i a_i$ and the Yang–Mills constraint (1.5). We say that the data have *finite energy* if $f_{\alpha\beta} \in L^2$.

Remark 1.3. Any temporal-gauge data set $(a_i', \dot{a}_i')$ trivially induces a Lorenz-gauge data set $(a_\alpha, \dot{a}_\alpha)$ given by

$$a_0 = 0, \quad \dot{a}_0 = \partial^i a_i', \quad (a_i, \dot{a}_i) = (a_i', \dot{a}_i').$$

Moreover, a Lorenz-gauge evolution A of $(a_\alpha, \dot{a}_\alpha)$ is formally gauge equivalent to a temporal-gauge evolution A' of $(a_i', \dot{a}_i')$ by the gauge transformation (1.3) with S the solution of the linear ODE $\partial_t S = -A_0 S$ with initial value $S(0) = I$.

In both the temporal and Lorenz gauges, the Yang–Mills equations are non-linear wave equations, so by classical methods the Cauchy problem is solvable locally in time for sufficiently regular initial data, and finite speed of propagation holds. Using estimates of Strichartz type one can lower the regularity requirement on the data almost down to the energy level $s = 1$, but to actually reach this level one must have null structure and use bilinear space-time estimates.

Let us first recall some results for Yang–Mills in temporal gauge; then we move on to the Lorenz gauge.

In temporal gauge, Segal [28] established local well-posedness for $s \geq 3$. Eardley and Moncrief [8] improved this to $s \geq 2$ for the more general Yang–Mills–Higgs equations (Yang–Mills coupled to a scalar field), and moreover they were able to prove a global result [9] by using conservation of energy. Global well-posedness for finite-energy data ($s = 1$) was proved by Klainerman and Machedon [15] and extended to Yang–Mills–Higgs by Keel [12]. Tao [37] proved local well-posedness for $s > 3/4$, for data with small norm. The scaling-critical regularity is $s = 1/2$.

As mentioned earlier, although the finite-energy result in [15] is formulated in temporal gauge, the Coulomb gauge is used to obtain the main estimates, via a local gauge change based on Uhlenbeck's Lemma [39]. In the abelian case, on the other hand, a global Coulomb gauge can be used without problems, as in the works [14, 5, 21] on the Maxwell–Klein–Gordon system, which is the special case of Yang–Mills–Higgs corresponding to $G = U(1)$. Local and global regularity properties of the Yang–Mills and Maxwell–Klein–Gordon equations have also been studied in higher space dimensions, and in particular in $1 + 4$ dimensions, which is the energy-critical case (see [17, 27, 34, 18, 19]). Structure-preserving numerical schemes for Yang–Mills and Maxwell–Klein–Gordon have been found in [4, 3].

Recently, Oh [22, 23] introduced a new approach based on the Yang–Mills heat gauge, and in particular recovered the global well-posedness result from [15].

Let us now turn our attention to the Lorenz gauge. Then the gauge field A and the field strength F satisfy a system of non-linear wave equations of the form

$$\begin{cases} \Box A = \Pi(A, \partial A) + \Pi(A, F), \\ \Box F = \Pi(A, \partial F) + \Pi(\partial A, \partial A) + \Pi(A, A, F) + \Pi(A, A, A, A), \end{cases}$$

where $\Box = \partial^\alpha \partial_\alpha = -\partial_t^2 + \Delta$ is the D'Alembertian and each instance of $\Pi(\dots)$ denotes a multilinear form in the given arguments. The difficult terms here are the bilinear ones. But as we show in the next section, $\Pi(A, \partial A)$, $\Pi(A, \partial F)$ and $\Pi(\partial A, \partial A)$ are all null forms in Lorenz gauge. The term $\Pi(A, F)$, on the other hand, is not. Nevertheless, this term benefits from the regularising effect of the null structure in the wave equation for F. This structure implies that F is more regular than ∂A, hence $\Pi(A, F)$ is more regular than a generic term $\Pi(A, \partial A)$.

Therefore, although one could expand F using (1.1) and reduce to a non-linear wave equation for A only, of the form

$$\Box A = \Pi(A, \partial A) + \Pi(A, A, A), \tag{1.6}$$

this is not a good idea in Lorenz gauge. In Coulomb gauge, on the other hand, the spatial part (A_i) of A will satisfy an equation of this form where *all* the bilinear terms are null forms, as shown in [15].

We remark that by using a mixed-norm estimate for the homogeneous wave equation due to Pecher [24] (generalising the estimates of Strichartz [36]), one can prove (essentially as in [26]) that generic equations of the form (1.6) are locally well posed for H^s data for all $s > 1$, and this is sharp, in view of the counterexamples of Lindblad [20]. If $\Pi(A, \partial A)$ is a null form, however, one can do better, and in particular $s = 1$ is allowed, as proved by Klainerman and Machedon [13, 14].

Using the null structure and bilinear space-time estimates, we prove the following local well-posedness result for finite-energy data. Observe that every H^1 data set has finite energy.

Theorem 1.4. *Fix* $15/16 < s < 1$. *Given any Lorenz-gauge* H^1 *data set* $(a, \dot{a})$ *(which necessarily has finite energy), there exists a time* $T > 0$, *depending on*

$$\|a\|_{H^s} + \|\dot{a}\|_{H^{s-1}} + \|f\|_{L^2},$$

and there exists a solution $(A, F = F^A)$ *of the Yang–Mills–Lorenz equations on* $(-T, T) \times \mathbb{R}^3$ *with the given initial data. The solution has the regularity*

$$A \in C([-T, T]; H^s) \cap C^1([-T, T]; H^{s-1}), \quad F \in C([-T, T]; L^2),$$

and is unique in a certain subspace containing the solutions. The solution depends continuously on the data in the $H^s \times H^{s-1} \times L^2$-*norm. Higher regularity persists, hence smooth data give a smooth solution. Moreover, the energy is conserved.*

Our focus here was on reaching the energy regularity, but it is clear from the proof of the theorem that the regularity can be lowered, and in particular the lower bound 15/16 for s is far from sharp.

As mentioned above, the motivation for this work was the question whether it is possible to carry out the argument of Klainerman and Machedon with the Coulomb gauge replaced by the Lorenz gauge. To continue this program, one needs a local version of the above theorem. This is work in progress.

In the next section we prove the key null identities. In Section 3 we write out the wave equations for A and F in Lorenz gauge, and use the null identities to show that all the bilinear terms with derivatives are null forms. The proof of local well-posedness is given in Sections 4 and 5.

The proof of null structure that we give is adapted to 3+1 dimensions, hence holds in lower dimensions, but should also generalise to higher dimensions. Thus, the methods developed here should adapt to give a low-regularity theory for the Yang–Mills equations in Lorenz gauge also in other space dimensions. Local well-posedness at high regularity in $1 + 2$ dimensions and global well-posedness in $1 + 1$ dimensions have been proved in [10].

2. Lorenz-gauge null form identities

Recall the definition of Klainerman's null forms,

$$\begin{cases} Q_0(u, v) = \partial_\alpha u \partial^\alpha v = -\partial_t u \partial_t v + \partial^i u \partial_i v, \\ Q_{\alpha\beta}(u, v) = \partial_\alpha u \partial_\beta v - \partial_\beta u \partial_\alpha v. \end{cases} \tag{2.1}$$

If u, v are $\mathfrak{g}$-valued and the product is the matrix product, we can also define commutator versions:

$$\begin{cases} Q_0[u, v] = [\partial_\alpha u, \partial^\alpha v] = Q_0(u, v) - Q_0(v, u), \\ Q_{\alpha\beta}[u, v] = [\partial_\alpha u, \partial_\beta v] - [\partial_\beta u, \partial_\alpha v] = Q_{\alpha\beta}(u, v) + Q_{\alpha\beta}(v, u). \end{cases} \tag{2.2}$$

Note that $Q_0[\cdot, \cdot]$ is antisymmetric and $Q_{\alpha\beta}[\cdot, \cdot]$ is symmetric. Observe also the identity

$$[\partial_\alpha u, \partial_\beta u] = \tfrac{1}{2}([\partial_\alpha u, \partial_\beta u] - [\partial_\beta u, \partial_\alpha u]) = \tfrac{1}{2} Q_{\alpha\beta}[u, u]. \tag{2.3}$$

We now prove the key identities that enable us to reveal null structures in Lorenz gauge. They involve certain linear combinations of the above null forms. To simplify the notation, we make the following definition.

Definition 2.1. For $\mathfrak{g}$-valued u_α and v, let

$$\mathfrak{Q}[u, v] = -\tfrac{1}{2}\varepsilon^{ijk}\varepsilon_{klm} Q_{ij}[R^l u^m, v] - Q_{0i}[R^i u_0, v],$$

where ε_{ijk} is the antisymmetric symbol with $\varepsilon_{123} = 1$, and

$$R_i = |\nabla|^{-1}\partial_i = (-\Delta)^{-1/2}\partial_i$$

are the Riesz transforms.

Lemma 2.2. *Assume $A_\alpha, \phi \in \mathcal{S}$ with values in $\mathfrak{g}$. Then*

$$[A^\alpha, \partial_\alpha \phi] = \mathfrak{Q}[|\nabla|^{-1} A, \phi] - [|\nabla|^{-1} R^i (\partial^\alpha A_\alpha), \partial_i \phi].$$

Proof. Split A into its temporal part A_0 and its spatial part $\mathbf{A} = (A_1, A_2, A_3)$, and write the latter as the sum of its curl-free and divergence-free parts $\mathbf{A}^{\text{cf}}$ and $\mathbf{A}^{\text{df}}$:

$$\mathbf{A} = -(-\Delta)^{-1} \nabla (\nabla \cdot \mathbf{A}) + (-\Delta)^{-1} \nabla \times (\nabla \times \mathbf{A}) =: \mathbf{A}^{\text{cf}} + \mathbf{A}^{\text{df}}.$$

Now write

$$A^\alpha \partial_\alpha \phi = (-A_0 \partial_t \phi + \mathbf{A}^{\text{cf}} \cdot \nabla \phi) + \mathbf{A}^{\text{df}} \cdot \nabla \phi$$

Since $(\mathbf{A}^{\text{df}})^i = \varepsilon^{ijk} \varepsilon_{klm} R_j R^l A^m$ one has, as observed in [14],

$$\mathbf{A}^{\text{df}} \cdot \nabla \phi = \varepsilon^{ijk} \varepsilon_{klm} (R_j R^l A^m) \partial_i \phi = -\tfrac{1}{2} \varepsilon^{ijk} \varepsilon_{klm} Q_{ij}(|\nabla|^{-1} R^l A_m, \phi).$$

Next, writing

$$\mathbf{A}^{\text{cf}} = -(-\Delta)^{-1} \nabla (\nabla \cdot \mathbf{A}) = -(-\Delta)^{-1} \nabla (\partial_t A_0 + \partial^\alpha A_\alpha),$$

we see that

$$\begin{aligned}
-A_0 \partial_t \phi + \mathbf{A}^{\text{cf}} \cdot \nabla \phi &= -A_0 \partial_t \phi - (-\Delta)^{-1} \partial^i (\partial_t A_0 + \partial^\alpha A_\alpha) \partial_i \phi \\
&= (-\Delta)^{-1} (\partial^i \partial_i A_0) \partial_t \phi - (-\Delta)^{-1} \partial^i (\partial_t A_0) \partial_i \phi - |\nabla|^{-1} R^i (\partial^\alpha A_\alpha) \partial_i \phi \\
&= \partial_i (|\nabla|^{-1} R^i A_0) \partial_t \phi - \partial_t (|\nabla|^{-1} R^i A_0) \partial_i \phi - |\nabla|^{-1} R^i (\partial^\alpha A_\alpha) \partial_i \phi \\
&= -Q_{0i} (|\nabla|^{-1} R^i A_0, \phi) - |\nabla|^{-1} R^i (\partial^\alpha A_\alpha) \partial_i \phi.
\end{aligned}$$

Thus,

$$\begin{aligned}
A^\alpha \partial_\alpha \phi = &-\tfrac{1}{2} \varepsilon^{ijk} \varepsilon_{klm} Q_{ij}(|\nabla|^{-1} R^l A^m, \phi) - Q_{0i}(|\nabla|^{-1} R^i A_0, \phi) \\
&- |\nabla|^{-1} R^i (\partial^\alpha A_\alpha) \partial_i \phi.
\end{aligned}$$

Similarly, we find that

$$\begin{aligned}
(\partial_\alpha \phi) A^\alpha = &\tfrac{1}{2} \varepsilon^{ijk} \varepsilon_{klm} Q_{ij}(\phi, |\nabla|^{-1} R^l A^m) + Q_{0i}(\phi, |\nabla|^{-1} R^i A_0) \\
&- (\partial_i \phi) |\nabla|^{-1} R^i (\partial^\alpha A_\alpha).
\end{aligned}$$

Subtracting the last two identities, we obtain the desired conclusion. $\square$

We also need the identity

$$[\partial_t A^\alpha, \partial_\alpha \phi] = [\partial^\alpha A_\alpha, \partial_t \phi] + Q_{0i}[A^i, \phi]. \tag{2.4}$$

Indeed, we calculate

$$\begin{aligned}
[\partial_t A^\alpha, \partial_\alpha \phi] &= [-\partial_t A_0, \partial_t \phi] + [\partial_t A^i, \partial_i \phi] = [\partial^\alpha A_\alpha - \partial^i A_i, \partial_t \phi] + [\partial_t A^i, \partial_i \phi] \\
&= [\partial^\alpha A_\alpha, \partial_t \phi] - [\partial_i A^i, \partial_t \phi] + [\partial_t A^i, \partial_i \phi] = [\partial^\alpha A_\alpha, \partial_t \phi] + Q_{0i}[A^i, \phi].
\end{aligned}$$

3. The equations in Lorenz gauge

3.1. Wave equation for A

In view of (1.1), the Yang–Mills equations (1.2) can be written as

$$\Box A_\beta - \partial_\beta(\partial^\alpha A_\alpha) + [\partial^\alpha A_\alpha, A_\beta] + [A_\alpha, \partial^\alpha A_\beta] + [A^\alpha, F_{\alpha\beta}] = 0,$$

where $\Box = \partial^\alpha \partial_\alpha = -\partial_t^2 + \Delta$. Imposing the Lorenz gauge condition

$$\partial^\alpha A_\alpha = 0,$$

this simplifies to

$$\Box A_\beta = -[A^\alpha, \partial_\alpha A_\beta] - [A^\alpha, F_{\alpha\beta}], \tag{3.1}$$

and in view of Lemma 2.2, the first term on the right is a null form:

$$\Box A_\beta = -\mathfrak{Q}[|\nabla|^{-1} A, A_\beta] - [A^\alpha, F_{\alpha\beta}]. \tag{3.2}$$

Expanding the last term in (3.1), one could also write

$$\Box A_\beta = -2[A^\alpha, \partial_\alpha A_\beta] + [A^\alpha, \partial_\beta A_\alpha] - [A^\alpha, [A_\alpha, A_\beta]],$$

but this is not a good idea. The cubic term causes no problems, but the new bilinear term $[A^\alpha, \partial_\beta A_\alpha]$ is not a null form, as far as we know,[2] and in fact it is worse than the term $[A^\alpha, F_{\alpha\beta}]$, since F has better regularity than ∂A. The reason is that F itself satisfies a non-linear wave equation with null structure in the bilinear terms, as will soon transpire.

3.2. Wave equation for F

Regardless of the choice of gauge, F satisfies

$$\Box F_{\beta\gamma} + [A^\alpha, \partial_\alpha F_{\beta\gamma}] + \partial^\alpha[A_\alpha, F_{\beta\gamma}] + [A^\alpha, [A_\alpha, F_{\beta\gamma}]] + 2[F^\alpha{}_\beta, F_{\gamma\alpha}] = 0. \tag{3.3}$$

We recall the derivation below. The initial conditions are

$$(F, \partial_t F)(0) = (f, \dot{f}),$$

where

$$\begin{cases} f_{ij} = \partial_i a_j - \partial_j a_i + [a_i, a_j], \\ f_{0i} = \dot{a}_i - \partial_i a_0 + [a_0, a_i], \\ \dot{f}_{ij} = \partial_i \dot{a}_j - \partial_j \dot{a}_i + [\dot{a}_i, a_j] + [a_i, \dot{a}_j], \\ \dot{f}_{0i} = \partial^j f_{ji} + [a^\alpha, f_{\alpha i}], \end{cases} \tag{3.4}$$

the expression for $\dot{f}_{0i}$ coming from (1.2) with $\beta = i$.

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[2] This is in contrast to the situation in Coulomb gauge, since there one can apply the projection $\mathbf{P}$ to divergence-free fields on both sides of the wave equation for A_j, and use the fact that $\mathbf{P}[A^i, \partial_j A_i]$ is a null form (see [14, 15]).

Expanding the last term in (3.3) yields

$$\Box F_{\beta\gamma} = -[\partial^\alpha A_\alpha, F_{\beta\gamma}]$$
$$- 2[A^\alpha, \partial_\alpha F_{\beta\gamma}] + 2[\partial_\gamma A^\alpha, \partial_\alpha A_\beta] - 2[\partial_\beta A^\alpha, \partial_\alpha A_\gamma]$$
$$+ 2[\partial^\alpha A_\beta, \partial_\alpha A_\gamma] + 2[\partial_\beta A^\alpha, \partial_\gamma A_\alpha]$$
$$- [A^\alpha, [A_\alpha, F_{\beta\gamma}]] + 2[F_{\alpha\beta}, [A^\alpha, A_\gamma]] - 2[F_{\alpha\gamma}, [A^\alpha, A_\beta]]$$
$$- 2[[A^\alpha, A_\beta], [A_\alpha, A_\gamma]].$$

After imposing the Lorenz gauge condition, the first term on the right-hand side disappears, the second term is a null form by Lemma 2.2, the third and fourth terms are null forms by either Lemma 2.2 or the identity (2.4), the fifth term is identical to $2Q_0[A_\beta, A_\gamma]$, and the sixth term equals $Q_{\beta\gamma}[A^\alpha, A_\alpha]$ by the identity (2.3).

The conclusion is that, in Lorenz gauge,

$$\Box F_{ij} = -2\mathfrak{Q}[|\nabla|^{-1}A, F_{ij}] + 2\mathfrak{Q}[|\nabla|^{-1}\partial_j A, A_i] - 2\mathfrak{Q}[|\nabla|^{-1}\partial_i A, A_j]$$
$$+ 2Q_0[A_i, A_j] + Q_{ij}[A^\alpha, A_\alpha]$$
$$- [A^\alpha, [A_\alpha, F_{ij}]] + 2[F_{\alpha i}, [A^\alpha, A_j]] - 2[F_{\alpha j}, [A^\alpha, A_i]]$$
$$- 2[[A^\alpha, A_i], [A_\alpha, A_j]], \tag{3.5}$$

and

$$\Box F_{0i} = -2\mathfrak{Q}[|\nabla|^{-1}A, F_{0i}] + 2\mathfrak{Q}[|\nabla|^{-1}\partial_i A, A_0] - 2Q_{0j}[A^j, A_i]$$
$$+ 2Q_0[A_0, A_i] + Q_{0i}[A^\alpha, A_\alpha]$$
$$- [A^\alpha, [A_\alpha, F_{0i}]] + 2[F_{\alpha 0}, [A^\alpha, A_i]] - 2[F_{\alpha i}, [A^\alpha, A_0]]$$
$$- 2[[A^\alpha, A_0], [A_\alpha, A_i]]. \tag{3.6}$$

This completes the derivation of the null structure.

To end this section, we recall the derivation of (3.3). Let D_α be the covariant derivative $D_\alpha = \partial_\alpha + [A_\alpha, \cdot]$, and note the commutation identity

$$D_\alpha D_\beta X - D_\beta D_\alpha X = [F_{\alpha\beta}, X],$$

which follows from the Jacobi identity

$$[X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] = 0.$$

In particular, $D^\alpha D_\beta F_{\gamma\alpha} = D_\beta D^\alpha F_{\gamma\alpha} + [F^\alpha{}_\beta, F_{\gamma\alpha}]$ and $D^\alpha D_\gamma F_{\alpha\beta} = D_\gamma D^\alpha F_{\alpha\beta} + [F^\alpha{}_\gamma, F_{\alpha\beta}]$. Therefore, applying D^α to both sides of the Bianchi identity

$$D_\alpha F_{\beta\gamma} + D_\beta F_{\gamma\alpha} + D_\gamma F_{\alpha\beta} = 0$$

yields

$$D^\alpha D_\alpha F_{\beta\gamma} + D_\beta D^\alpha F_{\gamma\alpha} + [F^\alpha{}_\beta, F_{\gamma\alpha}] + D_\gamma D^\alpha F_{\alpha\beta} + [F^\alpha{}_\gamma, F_{\alpha\beta}] = 0.$$

But by the Yang–Mills equation, $D^\alpha F_{\gamma\alpha} = 0$ and $D^\alpha F_{\alpha\beta} = 0$. By antisymmetry, $[F^\alpha{}_\gamma, F_{\alpha\beta}] = [F_{\alpha\gamma}, F^\alpha{}_\beta] = -[F^\alpha{}_\beta, F_{\alpha\gamma}] = [F^\alpha{}_\beta, F_{\gamma\alpha}]$. Thus,

$$D^\alpha D_\alpha F_{\beta\gamma} + 2[F^\alpha{}_\beta, F_{\gamma\alpha}] = 0.$$

Finally, in view of the identity

$$D^\alpha D_\alpha X = \Box X + [A^\alpha, \partial_\alpha X] + \partial^\alpha [A_\alpha, X] + [A^\alpha, [A_\alpha, X]],$$

one obtains (3.3).

4. Local well-posedness

In this section we begin the proof of the main result, Theorem 1.4.

By iteration, we solve simultaneously the non-linear wave equations for A and F, written in terms of null forms as in (3.2), (3.5) and (3.6):

$$\begin{cases} \Box A_\beta = \mathfrak{M}_\beta(A, \partial_t A, F, \partial_t F), \\ \Box F_{\beta\gamma} = \mathfrak{N}_{\beta\gamma}(A, \partial_t A, F, \partial_t F), \end{cases} \tag{4.1}$$

where

$$\mathfrak{M}_\beta(A, \partial_t A, F, \partial_t F) = -\mathfrak{Q}[|\nabla|^{-1}A, A_\beta] - [A^\alpha, F_{\alpha\beta}],$$

$$\begin{aligned} \mathfrak{N}_{ij}(A, \partial_t A, F, \partial_t F) = {}&-2\mathfrak{Q}[|\nabla|^{-1}A, F_{ij}] + 2\mathfrak{Q}[|\nabla|^{-1}\partial_j A, A_i] \\ &- 2\mathfrak{Q}[|\nabla|^{-1}\partial_i A, A_j] + 2Q_0[A_i, A_j] + Q_{ij}[A^\alpha, A_\alpha] \\ &- [A^\alpha, [A_\alpha, F_{ij}]] + 2[F_{\alpha i}, [A^\alpha, A_j]] - 2[F_{\alpha j}, [A^\alpha, A_i]] \\ &- 2[[A^\alpha, A_i], [A_\alpha, A_j]], \end{aligned}$$

$$\begin{aligned} \mathfrak{N}_{0i}(A, \partial_t A, F, \partial_t F) = {}&-2\mathfrak{Q}[|\nabla|^{-1}A, F_{0i}] + 2\mathfrak{Q}[|\nabla|^{-1}\partial_i A, A_0] \\ &- 2Q_{0j}[A^j, A_i] + 2Q_0[A_0, A_i] + Q_{0i}[A^\alpha, A_\alpha] \\ &- [A^\alpha, [A_\alpha, F_{0i}]] + 2[F_{\alpha 0}, [A^\alpha, A_i]] - 2[F_{\alpha i}, [A^\alpha, A_0]] \\ &- 2[[A^\alpha, A_0], [A_\alpha, A_i]]. \end{aligned}$$

The initial conditions are

$$(A, \partial_t A)(0) = (a, \dot{a}), \qquad (F, \partial_t F)(0) = (f, \dot{f}). \tag{4.2}$$

We shall prove the following local well-posedness result.

Theorem 4.1. *Let* $15/16 < s < 1$. *Given any initial data* $(a, \dot{a}) \in H^s \times H^{s-1}$ *and* $(f, \dot{f}) \in L^2 \times H^{-1}$, *there exists a* $T > 0$, *depending continuously on*

$$\|a\|_{H^s} + \|\dot{a}\|_{H^{s-1}} + \|f\|_{L^2} + \|\dot{f}\|_{H^{-1}},$$

and there exist

$$\begin{cases} A \in C([-T, T]; H^s) \cap C^1([-T, T]; H^{s-1}), \\ F \in C([-T, T]; L^2) \cap C^1([-T, T]; H^{-1}), \end{cases} \tag{4.3}$$

solving (4.1) *on* $S_T = (-T, T) \times \mathbb{R}^3$ *in the sense of distributions, and satisfying the initial condition* (4.2).

The solution has the regularity, for some $b > 1/2$,

$$\left(A \pm \frac{1}{i \langle \nabla \rangle} \partial_t A \right) \in X_{\pm}^{s,b}(S_T), \qquad \left(F \pm \frac{1}{i \langle \nabla \rangle} \partial_t F \right) \in X_{\pm}^{0,b}(S_T),$$

and it is the unique solution with this property. (See Definition 4.3 below for the definition of the spaces used here.)

The solution depends continuously on the data. Moreover, higher regularity persists, in the sense that if, for some $k \in \mathbb{N}$, we know that

$$\nabla^\alpha(a, \dot{a}) \in H^s \times H^{s-1}, \qquad \nabla^\alpha(f, \dot{f}) \in L^2 \times H^{-1},$$

for all multi-indices $\alpha \in \mathbb{N}_0^3$ with $|\alpha| \leq k$, then $\partial^\alpha(A, F)$ belongs to (4.3) for all $\alpha \in \mathbb{N}_0^{1+3}$ with $|\alpha| \leq k$. In particular, if the data are smooth and compactly supported, then the solution is smooth.

Remark 4.2. In this theorem, we do not assume any compatibility conditions on the data, hence $F = F^A$ and the Lorenz gauge condition $\partial^\alpha A_\alpha = 0$ will not necessarily hold. They will hold, however, if we assume the constraints (3.4), (1.5) and $\dot{a}_0 = \partial^i a_i$. Indeed, setting

$$u = \partial^\alpha A_\alpha, \qquad V = F - F^A,$$

we have

$$u(0) = \partial_t u(0) = 0, \qquad V(0) = \partial_t V(0) = 0,$$

and (u, V) satisfies a system of wave equations of the form

$$\begin{cases} \Box u = \Pi(A, \partial u) + \Pi(\partial A, V) + \Pi(A, \partial V) + \Pi(\partial \nabla A, |\nabla|^{-1} Ru) \\ \qquad + \Pi(\nabla A, \partial |\nabla|^{-1} Ru) + \Pi(A, A, u) + \Pi(A, A, V) + \Pi(A, \partial A, |\nabla|^{-1} Ru), \\ \Box V = \Pi(\partial A, V) + \Pi(A, \partial V) + \Pi(\partial F, |\nabla|^{-1} Ru) + \Pi(\partial \nabla A, |\nabla|^{-1} Ru) \\ \qquad + \Pi(\nabla A, \partial |\nabla|^{-1} Ru) + \Pi(A, A, V) + \Pi(A, \partial A, |\nabla|^{-1} Ru), \end{cases}$$

where $R = |\nabla|^{-1} \nabla$. By regularisation of the data (as in [15, Proposition 1.2]), persistence of higher regularity, and continuous dependence on the data, we may assume smoothness. Therefore, the unique solution is $(u, V) = (0, 0)$. Thus $(A, F = F^A)$ is a solution of the actual Yang–Mills equations (1.2), and the Lorenz gauge condition holds. The regularisation argument shows that our solutions are limits of smooth solutions, and since conservation of energy holds for the latter, it also holds for our solutions.

Let us now turn to the proof of Theorem 4.1. It is convenient to recast the system in first-order form, since this will allow us to treat the null forms in a unified way (see Section 5.2). To avoid certain singularities at low frequency, we first rewrite our system so that we have the Klein–Gordon operator $\Box - 1$ on the left-hand side:

$$\begin{cases} (\Box - 1) A_\beta = -A_\beta + \mathfrak{M}_\beta(A, \partial_t A, F, \partial_t F), \\ (\Box - 1) F_{\beta\gamma} = -F_{\beta\gamma} + \mathfrak{N}_{\beta\gamma}(A, \partial_t A, F, \partial_t F). \end{cases}$$

Now apply the change of variables $(A, \partial_t A, F, \partial_t F) \mapsto (A_+, A_-, F_+, F_-)$ given by

$$A_\pm = \frac{1}{2}\left(A \pm \frac{1}{i\langle\nabla\rangle}\partial_t A\right), \quad F_\pm = \frac{1}{2}\left(F \pm \frac{1}{i\langle\nabla\rangle}\partial_t F\right).$$

Equivalently,

$$(A, \partial_t A, F, \partial_t F) = \big(A_+ + A_-, i\langle\nabla\rangle(A_+ - A_-), F_+ + F_-, i\langle\nabla\rangle(F_+ - F_-)\big). \quad (4.4)$$

Then our system transforms to

$$\begin{cases} (i\partial_t + \langle\nabla\rangle)A_+ = -\dfrac{1}{2\langle\nabla\rangle}\mathfrak{M}'(A_+, A_-, F_+, F_-), \\[2mm] (i\partial_t - \langle\nabla\rangle)A_- = +\dfrac{1}{2\langle\nabla\rangle}\mathfrak{M}'(A_+, A_-, F_+, F_-), \\[2mm] (i\partial_t + \langle\nabla\rangle)F_+ = -\dfrac{1}{2\langle\nabla\rangle}\mathfrak{N}'(A_+, A_-, F_+, F_-), \\[2mm] (i\partial_t - \langle\nabla\rangle)F_- = +\dfrac{1}{2\langle\nabla\rangle}\mathfrak{N}'(A_+, A_-, F_+, F_-), \end{cases} \quad (4.5)$$

where

$$\mathfrak{M}'(A_+, A_-, F_+, F_-) = -(A_+ + A_-) + \mathfrak{M}(A, \partial_t A, F, \partial_t F),$$
$$\mathfrak{N}'(A_+, A_-, F_+, F_-) = -(F_+ + F_-) + \mathfrak{N}(A, \partial_t A, F, \partial_t F),$$

and on the right-hand side it is understood that we use the substitution (4.4) on the arguments of $\mathfrak{M}$ and $\mathfrak{N}$.

The transformed initial data are

$$\begin{cases} A_\pm(0) = a_\pm := \dfrac{1}{2}\left(a \pm \dfrac{1}{i\langle\nabla\rangle}\dot{a}\right) \in H^s, \\[3mm] F_\pm(0) = f_\pm := \dfrac{1}{2}\left(f \pm \dfrac{1}{i\langle\nabla\rangle}\dot{f}\right) \in L^2. \end{cases} \quad (4.6)$$

We solve the transformed system by iterating in $X^{s,b}$-spaces associated to the dispersive operators $i\partial_t \pm \langle\nabla\rangle$. Spaces of this type have become an indispensable tool in the study of non-linear dispersive PDEs since the seminal work of Bourgain [1]. For an exposition of the theory, see [38, Section 2.6].

Definition 4.3. For $(s, b) \in \mathbb{R}^2$, define $X_\pm^{s,b}$ to be the completion of $\mathcal{S}(\mathbb{R}^{1+3})$ with respect to the norm

$$\|u\|_{X_\pm^{s,b}} = \left\|\langle\xi\rangle^s\langle-\tau \pm \langle\xi\rangle\rangle^b \widetilde{u}(\tau, \xi)\right\|_{L^2_{\tau,\xi}},$$

where $\widetilde{u}(\tau, \xi) = \mathcal{F}_{t,x}u(\tau, \xi)$ is the space-time Fourier transform of $u(t, x)$. The restriction to a time slab $S_T = (-T, T) \times \mathbb{R}^3$, denoted $X_\pm^{s,b}(S_T)$, is defined as the quotient space $X_\pm^{s,b}/\mathcal{M}$, where $\mathcal{M}$ is the closed subspace consisting of $u \in X_\pm^{s,b}$ such that $u = 0$ on S_T.

We recall some facts about $X^{s,b}$-spaces (see, e.g., [38, Section 2.6]).

Lemma 4.4. *Let $s \in \mathbb{R}$, $b > 1/2$ and $T > 0$. Then $X_\pm^{s,b}(S_T) \hookrightarrow C([-T, T]; H^s)$.*

Lemma 4.5. *Let $s \in \mathbb{R}$, $1/2 < b \leq b' < 1$ and $0 < T < 1$. Then for any $g \in H^s$ and $G \in X_\pm^{s,b'-1}(S_T)$, the Cauchy problem*

$$(i\partial_t \pm \langle\nabla\rangle)u = G \quad on \ S_T, \quad u(0) = g,$$

has a unique solution $u \in X_\pm^{s,b}(S_T)$. Moreover,

$$\|u\|_{X_\pm^{s,b}(S_T)} \leq C(\|g\|_{H^s} + T^{b'-b}\|G\|_{X_\pm^{s,b'-1}(S_T)}),$$

where C depends only on b and b'.

Remark 4.6. For sufficiently regular G (say $G \in C([-T, T]; H^s)$), the solution in the last lemma is $u(t) = e^{\pm it\langle\nabla\rangle}g - i \int_0^t e^{\pm i(t-t')\langle\nabla\rangle}G(t') \, dt'$ for $-T \leq t \leq T$.

By using Lemma 4.5 and a standard iteration argument (which we omit), local well-posedness can be deduced from the following non-linear estimates.

Lemma 4.7. *Let $0 < T < 1$, $b = 1/2 + \varepsilon$, $b' = 1/2 + 2\varepsilon$ and $1 - \varepsilon < s < 1$, where $0 < \varepsilon \leq 1/16$. Then*

$$\|\mathfrak{M}'(A_+, A_-, F_+, F_-)\|_{X_\pm^{s-1,b'-1}(S_T)} \lesssim N(1 + N^3),$$

$$\|\mathfrak{N}'(A_+, A_-, F_+, F_-)\|_{X_\pm^{-1,b'-1}(S_T)} \lesssim N(1 + N^3),$$

where

$$N = \|A_+\|_{X_+^{s,b}(S_T)} + \|A_-\|_{X_-^{s,b}(S_T)} + \|F_+\|_{X_+^{0,b}(S_T)} + \|F_-\|_{X_-^{0,b}(S_T)}.$$

Moreover, we have the difference estimates

$$\|\mathfrak{M}'(A_+, A_-, F_+, F_-) - \mathfrak{M}'(A'_+, A'_-, F'_+, F'_-)\|_{X_\pm^{s-1,b'-1}(S_T)} \lesssim \delta(1 + N^3),$$

$$\|\mathfrak{M}'(A_+, A_-, F_+, F_-) - \mathfrak{M}'(A'_+, A'_-, F'_+, F'_-)\|_{X_\pm^{-1,b'-1}(S_T)} \lesssim \delta(1 + N^3),$$

where

$$\delta = \sum_\pm (\|A_\pm - A'_\pm\|_{X_\pm^{s,b}(S_T)} + \|F_\pm - F'_\pm\|_{X_\pm^{0,b}(S_T)}).$$

Thus, by iteration we obtain a solution (A_+, A_-, F_+, F_-) of the transformed system on S_T for $T > 0$ sufficiently small (the relevant condition is $T^\varepsilon(1 + N^3) \ll 1$). The solution has the regularity

$$A_\pm \in X_\pm^{s,b}(S_T), \quad F_\pm \in X_\pm^{0,b}(S_T),$$

and is unique in this space. By Lemma 4.4,

$$A_\pm \in C([-T, T]; H^s), \quad F_\pm \in C([-T, T]; L^2), \tag{4.7}$$

Standard arguments also give continuous dependence on the data and persistence of higher regularity. We omit the details.

Finally, we can transform back to the original formulation of the system by defining $A = A_+ + A_-$ and $F = F_+ + F_-$. Pairwise addition of the equations in (4.5) reveals that $\partial_t A = i\langle\nabla\rangle(A_+ - A_-)$ and $\partial_t F = i\langle\nabla\rangle(F_+ - F_-)$. Thus, $\mathfrak{M}'(A_+, A_-, F_+, F_-) = -A + \mathfrak{M}(A, \partial_t A, F, \partial_t F)$ and similarly for $\mathfrak{N}'$ and $\mathfrak{N}$. Since $\square - 1$ can be factored as either $(i\partial_t + \langle\nabla\rangle)(i\partial_t - \langle\nabla\rangle)$ or $(i\partial_t - \langle\nabla\rangle)(i\partial_t + \langle\nabla\rangle)$, we now see that (4.5) implies (4.1). Moreover, the map $(A_+, A_-, F_+, F_-) \mapsto (A, \partial_t A, F, \partial_t F)$ is continuous from the space (4.7) to the space (4.3).

It remains to prove the estimates in Lemma 4.7. Note that $\mathfrak{M}'$ and $\mathfrak{N}'$ contain linear terms for which the estimates are trivial. We now turn to the estimates for the non-linear parts, $\mathfrak{M}$ and $\mathfrak{N}$.

5. Estimates for the non-linear terms

Here we prove Lemma 4.7. In addition to the $X_\pm^{s,b}$-norms defined in the last section, we will need the wave-Sobolev norms

$$\|u\|_{H^{s,b}} = \|\langle\xi\rangle^s\langle|\tau| - |\xi|\rangle^b \widetilde{u}(\tau, \xi)\|_{L_{\tau,\xi}^2}.$$

Norms of this type were first applied to the study of regularity questions for non-linear wave equations with null forms by Klainerman and Machedon [13, 16].

Note the relations

$$\begin{cases} \|u\|_{H^{s,b}} \leq \|u\|_{X_\pm^{s,b}} & \text{if } b \geq 0, \\ \|u\|_{X_\pm^{s,b}} \leq \|u\|_{H^{s,b}} & \text{if } b \leq 0, \end{cases} \tag{5.1}$$

between the two types of norms.

Since the operators $\mathfrak{M}$ and $\mathfrak{N}$ are local in time, it suffices to prove the estimates in Lemma 4.7 without the restriction to S_T. Since $b' - 1 < 0$, we may replace the X-norms on the left-hand sides of the estimates by the corresponding H-norms, in view of (5.1). We will only prove the first two estimates in the lemma; the difference estimates follow from the same arguments, in view of the multilinearity of the terms constituting $\mathfrak{M}$ and $\mathfrak{N}$.

Thus, we have reduced our task to proving

$$\|\mathfrak{M}(A, \partial_t A, F, \partial_t F)\|_{H^{s-1,b'-1}} \lesssim N(1 + N^3), \tag{5.2}$$

$$\|\mathfrak{N}(A, \partial_t A, F, \partial_t F)\|_{H^{-1,b'-1}} \lesssim N(1 + N^3), \tag{5.3}$$

where

$$N = \|A_+\|_{X_+^{s,b}} + \|A_-\|_{X_-^{s,b}} + \|F_+\|_{X_+^{0,b}} + \|F_-\|_{X_-^{0,b}},$$

and

$$b = 1/2 + \varepsilon, \quad b' = 1/2 + 2\varepsilon, \quad 1 - \varepsilon < s < 1, \tag{5.4}$$

for $\varepsilon > 0$ sufficiently small; in fact, $\varepsilon \leq 1/16$ suffices, as we will see in the course of the proof.

On the left-hand sides of the above estimates, as well as the ones that follow, it is understood that the substitution (4.4) is used on the arguments of $\mathfrak{M}$ and $\mathfrak{N}$.

To simplify the notation, we write

$$\|A\|_{X^{s,b}} = \|A_+\|_{X_+^{s,b}} + \|A_-\|_{X_-^{s,b}}, \qquad \|F\|_{X^{0,b}} = \|F_+\|_{X_+^{0,b}} + \|F_-\|_{X_-^{0,b}}.$$

Looking at the types of terms that appear in $\mathfrak{M}$ and $\mathfrak{N}$ (defined after (4.1)), and recalling also the definition of $\mathfrak{Q}$ from Section 2 and noting that the Riesz transforms R_i are bounded on all the spaces involved, we reduce the argument to proving the following estimates, where Q denotes any of the null forms Q_0, Q_{0i} or Q_{ij}, and $\Pi(\dots)$ denotes a multilinear form in its arguments:

$$\|Q[|\nabla|^{-1}A, A]\|_{H^{s-1,b'-1}} \lesssim \|A\|_{X^{s,b}} \|A\|_{X^{s,b}}, \tag{5.5}$$

$$\|Q[|\nabla|^{-1}A, F]\|_{H^{-1,b'-1}} \lesssim \|A\|_{X^{s,b}} \|F\|_{X^{0,b}}, \tag{5.6}$$

$$\|Q[A, A]\|_{H^{-1,b'-1}} \lesssim \|A\|_{X^{s,b}} \|A\|_{X^{s,b}}, \tag{5.7}$$

$$\|\Pi(A, F)\|_{H^{s-1,b'-1}} \lesssim \|A\|_{X^{s,b}} \|F\|_{X^{0,b}}, \tag{5.8}$$

$$\|\Pi(A, A, F)\|_{H^{-1,b'-1}} \lesssim \|A\|_{X^{s,b}} \|A\|_{X^{s,b}} \|F\|_{X^{0,b}}, \tag{5.9}$$

$$\|\Pi(A, A, A, A)\|_{H^{-1,b'-1}} \lesssim \|A\|_{X^{s,b}} \|A\|_{X^{s,b}} \|A\|_{X^{s,b}} \|A\|_{X^{s,b}}. \tag{5.10}$$

The difficult estimates are the first four, where the regularity is sharp, except in the third estimate, where there is a little room. We shall in fact reduce all of them to product estimates of the form (after extracting regularity gains due to the null structure)

$$\|uv\|_{H^{-s_0,b_0}} \lesssim \|u\|_{H^{s_1,b_1}} \|v\|_{H^{s_2,b_2}}. \tag{5.11}$$

Definition 5.1. Let $s_0, s_1, s_2, b_0, b_1, b_2 \in \mathbb{R}$. If (5.11) holds for all $u, v \in \mathcal{S}(\mathbb{R}^{1+3})$, we say that the matrix $\left(\begin{smallmatrix} s_0 & s_1 & s_2 \\ b_0 & b_1 & b_2 \end{smallmatrix}\right)$ is a *product*.

Estimates of this type were first studied by Klainerman and Machedon [13]. The full range of products up to certain borderline cases was determined in [7]. For our present purposes, the following simplified version of the product law will suffice. Note, however, that it includes some borderline cases (corresponding to equality in one and only one of the last two conditions below), which will be of crucial importance for us.

Theorem 5.2 (D'Ancona, Foschi, Selberg [7]). *Let $s_0, s_1, s_2 \in \mathbb{R}$ and $b_0, b_1, b_2 \geq 0$. Assume that*

$$\sum b_i > 1/2, \tag{5.12}$$

$$\sum s_i > 2 - \sum b_i, \tag{5.13}$$

$$\sum s_i > 3/2 - \min_{i \neq j}(b_i + b_j), \tag{5.14}$$

$$\sum s_i > 3/2 - \min(b_0 + s_1 + s_2, s_0 + b_1 + s_2, s_0 + s_1 + b_2), \tag{5.15}$$

$$\sum s_i \geq 1, \tag{5.16}$$

$$\min_{i \neq j}(s_i + s_j) \geq 0, \tag{5.17}$$

*and that the last two inequalities are **not both equalities**. Then*

$$P = \begin{pmatrix} s_0 & s_1 & s_2 \\ b_0 & b_1 & b_2 \end{pmatrix}$$

is a product. Conversely, if P is a product, then the conditions (5.12)–(5.17), *with >*
replaced by ≥ throughout, must hold.

Remark 5.3. If $b_0 \geq 0$ and $b_1, b_2 > 1/2$, the conditions in the theorem reduce to

$$\sum s_i > 3/2 - (b_0 + s_1 + s_2), \tag{5.18}$$

$$\sum s_i \geq 1, \tag{5.19}$$

$$\min_{i \neq j}(s_i + s_j) \geq 0, \tag{5.20}$$

where the last two inequalities are not both allowed to be equalities.

The key tools needed to prove (5.5)–(5.10) are now at hand. We start with (5.8), then
we prove the null form estimates, and finally the much easier trilinear and quadrilinear
estimates.

5.1. Estimate for $\Pi(A, F)$

In view of (5.1), we can reduce (5.8) to

$$\|uv\|_{H^{s-1,b'-1}} \lesssim \|u\|_{H^{s,b}} \|v\|_{H^{0,b}}.$$

This holds by Theorem 5.2; in fact, we can use the simplification in Remark 5.3. Since
$\sum s_i = 1$, the estimate is sharp, and we require $\min_{i \neq j}(s_i + s_j)$ to be strictly positive,
which translates to the condition $0 < s < 1$. Finally, (5.18) yields the condition $s >
b' - 1/2$. These conditions are satisfied with our choice (5.4) of exponents.

5.2. Null form estimates I: Preliminary reductions

Since the matrix commutator structure plays no role in the estimates under consideration,
we reduce the reasoning to the ordinary null forms for $\mathbb{C}$-valued functions u and v,

$$\begin{cases} Q_0(u, v) = -\partial_t u \partial_t v + \partial^i u \partial_i v, \\ Q_{0i}(u, v) = \partial_t u \partial_i v - \partial_i u \partial_t v, \\ Q_{ij}(u, v) = \partial_i u \partial_j v - \partial_j u \partial_i v. \end{cases}$$

This reduction is justified in view of (2.2), which shows that the commutator null forms
are linear combinations of the ordinary ones.

Substituting $u = u_+ + u_-$, $\partial_t u = i\langle\nabla\rangle(u_+ - u_-)$, and similarly for v and $\partial_t v$, one obtains

$$Q_0(u, v) = \sum_{\pm,\pm'}(\pm 1)(\pm'1)[\langle D\rangle u_\pm\langle D\rangle v_{\pm'} - (\pm D^i)u_\pm(\pm'D_i)v_{\pm'}],$$

$$Q_{0i}(u, v) = \sum_{\pm,\pm'}(\pm 1)(\pm'1)[-\langle D\rangle u_\pm(\pm'D_i)v_{\pm'} + (\pm D_i)u_\pm\langle D\rangle v_{\pm'}],$$

$$Q_{ij}(u, v) = \sum_{\pm,\pm'}(\pm 1)(\pm'1)[-(\pm D_i)u_\pm(\pm'D_j)v_{\pm'} + (\pm D_j)u_\pm(\pm'D_i)v_{\pm'}],$$

where $D = (D_1, D_2, D_3) = -i\nabla$ has Fourier symbol ξ. In terms of the symbols

$$q_0(\xi, \eta) = \langle\xi\rangle\langle\eta\rangle - \xi\cdot\eta,$$
$$q_{0i}(\xi, \eta) = -\langle\xi\rangle\eta_i + \xi_i\langle\eta\rangle,$$
$$q_{ij}(\xi, \eta) = -\xi_i\eta_j + \xi_j\eta_i,$$

we have, more conveniently,

$$Q_0(u, v) = \sum_{\pm,\pm'}(\pm 1)(\pm'1)B_{q_0(\pm\xi,\pm'\eta)}(u_\pm, v_{\pm'}),$$

$$Q_{0i}(u, v) = \sum_{\pm,\pm'}(\pm 1)(\pm'1)B_{q_{0i}(\pm\xi,\pm'\eta)}(u_\pm, v_{\pm'}),$$

$$Q_{ij}(u, v) = \sum_{\pm,\pm'}(\pm 1)(\pm'1)B_{q_{ij}(\pm\xi,\pm'\eta)}(u_\pm, v_{\pm'}),$$

where for a given symbol $\sigma(\xi, \eta)$ we denote by $B_{\sigma(\xi,\eta)}(\cdot, \cdot)$ the operator defined by

$$\mathcal{F}_{t,x}\{B_{\sigma(\xi,\eta)}(u, v)\}(\tau, \xi) = \int \sigma(\xi - \eta, \eta)\widetilde{u}(\tau - \lambda, \xi - \eta)\widetilde{v}(\lambda, \eta)\,d\lambda\,d\eta.$$

We now estimate the symbols appearing above.

Lemma 5.4. *For all non-zero $\xi, \eta \in \mathbb{R}^3$,*

$$|q_0(\xi, \eta)| \lesssim |\xi||\eta|\theta(\xi, \eta)^2 + \frac{1}{\min(\langle\xi\rangle, \langle\eta\rangle)},$$

$$|q_{0j}(\xi, \eta)| \lesssim |\xi||\eta|\theta(\xi, \eta) + \frac{|\xi|}{\langle\eta\rangle} + \frac{|\eta|}{\langle\xi\rangle},$$

$$|q_{ij}(\xi, \eta)| \leq |\xi||\eta|\theta(\xi, \eta),$$

where $\theta(\xi, \eta) = \arccos\left(\frac{\xi\cdot\eta}{|\xi||\eta|}\right) \in [0, \pi]$ is the angle between ξ and η.

Proof. This is easily seen by writing the symbols in the following way:

$$q_0(\xi, \eta) = |\xi|\,|\eta|\left(1 - \frac{\xi \cdot \eta}{|\xi|\,|\eta|}\right) + \langle\xi\rangle\langle\eta\rangle - |\xi|\,|\eta|,$$

$$q_{0i}(\xi, \eta) = |\xi|\,|\eta|\left(\frac{\xi_i}{|\xi|} - \frac{\eta_i}{|\eta|}\right) + \xi_i(\langle\eta\rangle - |\eta|) - (\langle\xi\rangle - |\xi|)\eta_i,$$

$$|q_{ij}(\xi, \eta)| \leq |\xi \times \eta|. \qquad \square$$

In view of this, and since the norms we use only depend on the absolute value of the space-time Fourier transform, we can reduce any estimate for $Q(u, v)$ to a corresponding estimate for the three expressions

$$B_{\theta(\pm\xi, \pm'\eta)}(|\nabla|u, |\nabla|v), \quad \langle\nabla\rangle u \langle\nabla\rangle^{-1} v \quad \text{and} \quad \langle\nabla\rangle^{-1} u \langle\nabla\rangle v.$$

Thus, (5.5)–(5.7) can be reduced to the following:

$$\|B_{\theta(\pm\xi, \pm'\eta)}(u, v)\|_{H^{s-1, b'-1}} \lesssim \|u\|_{X_\pm^{s,b}} \|v\|_{X_{\pm'}^{s-1,b}}, \tag{5.21}$$

$$\|B_{\theta(\pm\xi, \pm'\eta)}(u, v)\|_{H^{-1, b'-1}} \lesssim \|u\|_{X_\pm^{s,b}} \|v\|_{X_{\pm'}^{-1,b}}, \tag{5.22}$$

$$\|B_{\theta(\pm\xi, \pm'\eta)}(u, v)\|_{H^{-1, b'-1}} \lesssim \|u\|_{X_\pm^{s-1,b}} \|v\|_{X_{\pm'}^{s-1,b}}, \tag{5.23}$$

$$\|uv\|_{H^{s-1,0}} \lesssim \||\nabla|u\|_{H^{s-1,b}} \|v\|_{H^{s+1,b}}, \tag{5.24}$$

$$\|uv\|_{H^{s-1,0}} \lesssim \||\nabla|u\|_{H^{s+1,b}} \|v\|_{H^{s-1,b}}, \tag{5.25}$$

$$\|uv\|_{H^{-1,0}} \lesssim \||\nabla|u\|_{H^{s-1,b}} \|v\|_{H^{1,b}}, \tag{5.26}$$

$$\|uv\|_{H^{-1,0}} \lesssim \||\nabla|u\|_{H^{s+1,b}} \|v\|_{H^{-1,b}}, \tag{5.27}$$

$$\|uv\|_{H^{-1,0}} \lesssim \|u\|_{H^{s-1,b}} \|v\|_{H^{s+1,b}}, \tag{5.28}$$

$$\|uv\|_{H^{-1,0}} \lesssim \|u\|_{H^{s+1,b}} \|v\|_{H^{s-1,b}}, \tag{5.29}$$

where we have also used (5.1) to change X-norms to H-norms on the right-hand sides of the last six estimates.

All these estimates will be handled using the product law, Theorem 5.2. For those estimates that involve the null form $B_{\theta(\pm\xi, \pm'\eta)}$, however, we must first apply the following angle estimate, which quantifies the fact that in a null form, we can trade in hyperbolic regularity (decay with respect to the distance from the cone in Fourier space) and gain a corresponding amount of elliptic regularity.

Lemma 5.5. *Let $\alpha, \beta, \gamma \in [0, 1/2]$. Then for all pairs of signs $(\pm, \pm')$, all $\tau, \lambda \in \mathbb{R}$ and all non-zero $\xi, \eta \in \mathbb{R}^3$,*

$$\theta(\pm\xi, \pm'\eta) \lesssim \left(\frac{\langle|\tau+\lambda| - |\xi+\eta|\rangle}{\min(\langle\xi\rangle, \langle\eta\rangle)}\right)^\alpha + \left(\frac{\langle-\tau \pm |\xi|\rangle}{\min(\langle\xi\rangle, \langle\eta\rangle)}\right)^\beta + \left(\frac{\langle-\lambda \pm' |\eta|\rangle}{\min(\langle\xi\rangle, \langle\eta\rangle)}\right)^\gamma.$$

For a proof, see for example [29, Lemma 2.1].

Combining this angle estimate with the product law, we deduce the following.

Theorem 5.6. *Let* $\sigma_0, \sigma_1, \sigma_2, \beta_0, \beta_1, \beta_2 \in \mathbb{R}$. *Assume that*

$$0 \le \beta_0 < 1/2 < \beta_1, \beta_2 < 1, \tag{5.30}$$

$$\sum \sigma_i + \beta_0 > 3/2 - (\beta_0 + \sigma_1 + \sigma_2), \tag{5.31}$$

$$\sum \sigma_i > 3/2 - (\sigma_0 + \beta_1 + \sigma_2), \tag{5.32}$$

$$\sum \sigma_i > 3/2 - (\sigma_0 + \sigma_1 + \beta_2), \tag{5.33}$$

$$\sum \sigma_i + \beta_0 \ge 1, \tag{5.34}$$

$$\min(\sigma_0 + \sigma_1, \sigma_0 + \sigma_2, \beta_0 + \sigma_1 + \sigma_2) \ge 0, \tag{5.35}$$

*and that the last two inequalities are **not both equalities**. Then we have the null form estimate, for all combinations of signs* $(\pm, \pm')$,

$$\|B_{\theta(\pm\xi, \pm'\eta)}(u, v)\|_{H^{-\sigma_0, -\beta_0}} \lesssim \|u\|_{X_{\pm}^{\sigma_1, \beta_1}} \|v\|_{X_{\pm'}^{\sigma_2, \beta_2}}.$$

Proof. Applying Lemma 5.5, and using also (5.1) and the symmetric nature of the conditions on (σ_1, β_1) and (σ_2, β_2), we reduce the argument to checking that the following are products:

$$P_1 = \begin{pmatrix} \sigma_0 & \beta_0 + \sigma_1 & \sigma_2 \\ 0 & \beta_1 & \beta_2 \end{pmatrix},$$

$$P_2 = \begin{pmatrix} \sigma_0 & \sigma_1 + 1/2 & \sigma_2 \\ \beta_0 & \beta_1 - 1/2 & \beta_2 \end{pmatrix},$$

$$P_3 = \begin{pmatrix} \sigma_0 & \sigma_1 & \sigma_2 + 1/2 \\ \beta_0 & \beta_1 - 1/2 & \beta_2 \end{pmatrix}.$$

But this follows from Theorem 5.2, as one can readily check. Note that for P_1 the simplification in Remark 5.3 applies. □

5.3. Null form estimates II: Conclusion

Using Theorem 5.6 it is now a routine matter to check that the estimates (5.21)–(5.23) hold. In fact, these estimates correspond to the following matrices $\begin{pmatrix} \sigma_0 & \sigma_1 & \sigma_2 \\ \beta_0 & \beta_1 & \beta_2 \end{pmatrix}$:

$$N_1 = \begin{pmatrix} 1 - s & s & s - 1 \\ 1 - b' & b & b \end{pmatrix},$$

$$N_2 = \begin{pmatrix} 1 & s & -1 \\ 1 - b' & b & b \end{pmatrix},$$

$$N_3 = \begin{pmatrix} 1 & s - 1 & s - 1 \\ 1 - b' & b & b \end{pmatrix},$$

and checking against the conditions in Theorem 5.6 gives the following conditions:

$$s > \max(3/2 - b, b', 1/4 + b', 5/8 + b'/2, 5/6 - b/3),$$

which is consistent with our assumption (5.4), provided $0 < \varepsilon \le 1/16$.

We remark that for N_1 and N_2, equality holds in (5.35), so these estimates are sharp.

It remains to check the estimates (5.24)–(5.29). Ignoring low-frequency issues for the moment, so that we can replace $|\nabla|u$ by $\langle\nabla\rangle u$, we reduce the task to checking

$$\|uv\|_{H^{s-1,0}} \lesssim \|u\|_{H^{s,b}}\|v\|_{H^{s+1,b}},$$
$$\|uv\|_{H^{s-1,0}} \lesssim \|u\|_{H^{s+2,b}}\|v\|_{H^{s-1,b}},$$
$$\|uv\|_{H^{-1,0}} \lesssim \|u\|_{H^{s,b}}\|v\|_{H^{1,b}},$$
$$\|uv\|_{H^{-1,0}} \lesssim \|u\|_{H^{s+2,b}}\|v\|_{H^{-1,b}},$$
$$\|uv\|_{H^{-1,0}} \lesssim \|u\|_{H^{s-1,b}}\|v\|_{H^{s+1,b}},$$
$$\|uv\|_{H^{-1,0}} \lesssim \|u\|_{H^{s+1,b}}\|v\|_{H^{s-1,b}}.$$

These estimates hold by Theorem 5.2. In fact, the reduction in Remark 5.3 applies. There is a lot of room in all the conditions except $\min_{i\neq j}(s_i + s_j) \geq 0$, which holds with equality for the second and fourth estimates.

It remains to prove (5.24)–(5.27) in the case where u is at low frequency, that is, $|\xi| \leq 1$ on the Fourier support of u. But then the frequency η of v is comparable to the output frequency $\xi + \eta$, in the sense that $\langle\xi + \eta\rangle \sim \langle\eta\rangle$, hence it remains to show

$$\|uv\|_{L^2} \lesssim \||\nabla|u\|_{L^2}\|v\|_{H^{0,b}},$$

which follows by writing

$$\|uv\|_{L^2} \leq \|u\|_{L^2_t(L^\infty_x)}\|v\|_{L^\infty_t(L^2_x)}$$

and applying the low-frequency estimate

$$\|u(t)\|_{L^\infty} \lesssim \int_{|\xi|\leq 1} |\widehat{u}(t,\xi)|\,d\xi \leq \left(\int_{|\xi|\leq 1} |\xi|^{-2}\,d\xi\right)^{1/2}\left(\int |\xi|^2|\widehat{u}(t,\xi)|^2\,d\xi\right)^{1/2}$$

and

$$\|v\|_{L^q_t(L^2_x)} \lesssim \|\|\widetilde{v}(\tau,\xi)\|_{L^{q'}_\tau}\|_{L^2_\xi} \leq \|\langle\tau\rangle^{-b}\|_{L^{2+4/(q-2)}_\tau}\|v\|_{H^{0,b}}, \tag{5.36}$$

which is valid for $2 \leq q \leq \infty$.

5.4. Estimate for $\Pi(A, A, F)$

Here we prove (5.9):

$$\|uvw\|_{H^{-1,b'-1}} \lesssim \|u\|_{H^{s,b}}\|v\|_{H^{s,b}}\|w\|_{H^{0,b}}.$$

This holds for $s \geq 1/2 + 2\varepsilon$ by the estimates

$$\|uw\|_{H^{-1,b'-1}} \lesssim \|u\|_{H^{3\varepsilon,0}}\|w\|_{H^{0,b}},$$
$$\|uv\|_{H^{3\varepsilon,0}} \lesssim \|u\|_{H^{1/2+2\varepsilon,b}}\|v\|_{H^{1/2+2\varepsilon,b}},$$

both of which hold by Theorem 5.2.

5.5. *Estimate for* $\Pi(A, A, A, A)$

We prove (5.10):

$$\|u_1 u_2 u_3 u_4\|_{H^{-1,0}} \lesssim \|u_1 u_2 u_3 u_4\|_{L_t^2(L_x^{6/5})}$$

$$\leq \|u_1\|_{L_t^8(L_x^{24/5})} \|u_2\|_{L_t^8(L_x^{24/5})} \|u_3\|_{L_t^8(L_x^{24/5})} \|u_4\|_{L_t^8(L_x^{24/5})}$$

$$\lesssim \|u_1\|_{H^{7/8,b}} \|u_2\|_{H^{7/8,b}} \|u_3\|_{H^{7/8,b}} \|u_4\|_{H^{7/8,b}}$$

where we have applied Sobolev embedding and (5.36).

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