



The strong profinite genus of a finitely presented group can be infinite

For Fritz Grunewald, in memoriam

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Abstract. We construct the first examples of finitely presented, residually finite groups Γ that contain an infinite sequence of non-isomorphic finitely presented subgroups $P_n \hookrightarrow \Gamma$ such that the inclusion maps induce isomorphisms of profinite completions $\widehat{P}_n \cong \widehat{\Gamma}$.

Keywords. Profinite completion, profinite genus, Grothendieck pairs

1. Introduction

The profinite completion of a group Γ is the inverse limit of the directed system of finite quotients of Γ ; it is denoted $\widehat{\Gamma}$. If Γ is residually finite then the natural map $\Gamma \rightarrow \widehat{\Gamma}$ is injective. In 1970 Alexander Grothendieck [14] posed the following problem: let Γ_1 and Γ_2 be residually finite groups and let $u : \Gamma_1 \rightarrow \Gamma_2$ be a homomorphism such that the induced map of profinite completions $\widehat{u} : \widehat{\Gamma}_1 \rightarrow \widehat{\Gamma}_2$ is an isomorphism; if Γ_1 and Γ_2 are finitely presented, must u be an isomorphism? This problem was settled in 2004 by Bridson and Grunewald [6] who showed that u need not be an isomorphism. (The corresponding result for finitely generated groups had been established earlier by Platonov and Tavgen' [17].) There has since been a considerable amount of work exploring the extent to which Γ_1 can differ from Γ_2 , but the existence of groups of the sort described in the following theorem has remained unknown.

Theorem A. *There exists a finitely presented, residually finite group Γ and a recursive sequence of finitely presented subgroups $u_n : P_n \hookrightarrow \Gamma$ such that each of the maps $\widehat{u}_n : \widehat{P}_n \rightarrow \widehat{\Gamma}$ is an isomorphism, but $P_m \cong P_n$ if and only if $m = n$.*

The analogous result with the P_n finitely generated was proved in [6, Section 9] (cf. [17], [2]). The difficulties that prevented us from proving Theorem A in [6] are overcome here using two new ideas. First, in place of the results from [3] used to establish the finite presentability of fibre products in [6], we use the Effective Asymmetric 1-2-3 Theorem proved in [8]; this breaking of symmetry is compatible with the Platonov–Tavgen'

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criterion for profinite equivalence [17] as distilled in [2] and [6]. The other new idea is inspired by classical results concerning Nielsen equivalence and is described in two forms, the first of which involves the construction of a particular type of non-Hopfian group (Section 4), while the second involves Nielsen equivalence more directly (Section 8). In each case, we construct a finitely presented group Q that admits epimorphisms $p_0 : G \rightarrow Q$ and $p_n : \Lambda \rightarrow Q$ ($n \in \mathbb{N}$) such that the fibre products $P_n < \Gamma := G \times \Lambda$ of (p_0, p_n) satisfy Theorem A.

Using recent work of Agol [1] and Wise [20] (alternatively [15]), one can arrange for $\Gamma = G \times \Lambda$ to be the fundamental group of a special cube complex and hence deduce the following (see Section 7).

Addendum B. *One can assume that the group Γ in Theorem A is residually torsion-free nilpotent, and each $u_n : P_n \rightarrow \Gamma$ induces an isomorphism of pro-nilpotent completions.*

Fritz Grunewald and I tried to prove Theorem A when writing [6] but had insufficient tools at the time. If Fritz were still alive, the present paper would surely have been a joint one. He is sorely missed.

2. Asymmetric fibre products

Given two epimorphisms $p_1 : \Gamma_1 \rightarrow Q$ and $p_2 : \Gamma_2 \rightarrow Q$, one has the *fibre product*

$$P = \{(x, y) \in \Gamma_1 \times \Gamma_2 \mid p_1(x) = p_2(y)\}.$$

The 1-2-3 Theorem of [3] gives conditions under which P is finitely presentable. These are too restrictive for our purposes but the following refinement from [8] will serve us well.

Theorem 2.1. *There exists an algorithm that, given the following data describing group homomorphisms $f_i : \Gamma_i \rightarrow Q$ ($i = 1, 2$), will output a finite presentation of the fibre product P of these maps, together with a map $P \rightarrow \Gamma_1 \times \Gamma_2$ defined on the generators, provided that both the f_i are surjective and at least one of the kernels $\ker f_i$ is finitely generated. (If either of these conditions fails, the procedure will not halt.)*

Input:

- A finite presentation $\mathcal{Q} \equiv \langle X \mid R \rangle$ for Q .
- A finite presentation $\langle \underline{a}^{(i)} \mid \underline{r}^{(i)} \rangle$ for Γ_i ($i = 1, 2$).
- For each $a \in \underline{a}^{(i)}$, a word \tilde{a} in the free group on X such that $\tilde{a} = f_i(a)$ in Q .
- A finite set of identity sequences that generates $\pi_2 \mathcal{Q}$ as a $\mathbb{Z}Q$ -module.

We shall only need this theorem in the case where Q is aspherical, i.e. $\pi_2 \mathcal{Q} = 0$, in which case the algorithm simplifies considerably. The algorithmic nature of the construction is needed to justify the word “recursive” in the statement of Theorem A.

The above theorem allows us to present fibre products. We shall use it in combination with the following criterion for proving that the inclusions of certain fibre products induce isomorphisms of profinite completions. This criterion is essentially due to Platonov and Tavgen’ [17]. They dealt only with the case $G_1 = G_2$ and $p_1 = p_2$, but the distillation of their argument described in [6, Section 5] applies directly to the asymmetric case.

Theorem 2.2. *Let $p_1 : G_1 \rightarrow Q$ and $p_2 : G_2 \rightarrow Q$ be epimorphisms and let $P \subset G_1 \times G_2$ be the associated fibre product. If G_1 and G_2 are finitely generated, Q has no finite quotients, and $H_2(Q, \mathbb{Z}) = 0$, then the inclusion $u : P \hookrightarrow G_1 \times G_2$ induces an isomorphism $\hat{u} : \hat{P} \rightarrow \hat{G}_1 \times \hat{G}_2$.*

3. A Rips construction

We assume that the reader is familiar with the theory of (Gromov) hyperbolic groups. We shall use a version of the well-known Rips construction [18] to construct hyperbolic groups with controlled properties. In the current setting we need to control the automorphisms of the groups constructed, and for this we appeal to the following lemma, the essence of which is taken from [7].

Lemma 3.1. *Let Γ be a torsion-free hyperbolic group and let $N \triangleleft \Gamma$ be a non-trivial subgroup that is finitely generated and normal. If $\text{Out}(\Gamma)$ is infinite, then Γ/N is virtually cyclic.*

Proof. It follows from Rips's theory of group actions on \mathbb{R} -trees and Paulin's Theorem [16], [11] that if $\text{Out}(\Gamma)$ is infinite then Γ acts on a simplicial tree with cyclic arc stabilizers (see [5, Corollary 1.3]). Let A be the stabilizer of an edge in this tree. Proposition 2.2 of [7] implies that either N is contained in A , or else NA has finite index in Γ . The first possibility cannot occur, because it would imply that N was an infinite cyclic normal subgroup of Γ , and the only torsion-free hyperbolic group with such a subgroup is \mathbb{Z} . Thus NA has finite index in Γ and Γ/N is commensurable with a quotient of A , which is cyclic. \square

The original Rips construction was an algorithm that took as input a finite presentation \mathcal{Q} for a group Q and gave as output a small cancellation presentation for a group Γ and an epimorphism $\Gamma \rightarrow Q$ with finitely generated kernel. There have since been many refinements of this construction in which extra properties are imposed on Γ . The most important of these from our point of view is Haglund and Wise's proof [15] that one can require Γ to be *virtually special*. (An alternative proof of this can be obtained by combining Wise's results about cubulating small cancellation groups [20] with Agol's proof [1] that cubulated hyperbolic groups are virtually special.) For us the key properties of virtually special groups are that they are residually finite and (what is more) that each has a subgroup of finite index that is residually torsion-free-nilpotent [13]. We summarize this discussion as follows:

Proposition 3.2. *There exists an algorithm that, given a finite group-presentation $\mathcal{Q} \equiv \langle X \mid R \rangle$, will construct a finite presentation $\mathcal{P} \equiv \langle X \cup A \mid R' \cup V \rangle$ for a group Γ so that*

- (1) Γ is torsion-free, hyperbolic and residually finite.
- (2) $N := \langle A \rangle$ is normal in Γ .
- (3) Γ/N is isomorphic to the group with presentation \mathcal{Q} .
- (4) If \mathcal{Q} is not virtually cyclic, then $\text{Out}(\Gamma)$ is finite.
- (5) If \mathcal{Q} has no finite quotients, then one may assume that Γ is special; in particular it is residually torsion-free-nilpotent.

Proof. The only item that is not covered by the preceding discussion and Lemma 3.1 is (5), where the phrase “one may assume” needs explaining. Rips’s original construction gives a short exact sequence $1 \rightarrow N \rightarrow \Gamma \rightarrow Q \rightarrow 1$ satisfying items (1) to (4) but even with the work of Agol and Wise in hand one knows only that Γ is *virtually* special. To remedy this, we pass to a subgroup of finite index $\Gamma_0 < \Gamma$ that is special. Since Q has no finite quotients, $\Gamma_0 \rightarrow Q$ is still onto. And the kernel, being of finite index in N , is still finitely generated. Thus we may replace Γ and N by Γ_0 and $\Gamma_0 \cap N$ preserving properties (1) to (4). \square

4. Non-Hopfian groups with no finite quotients

A group H is termed *non-Hopfian* if there is an epimorphism $H \twoheadrightarrow H$ with non-trivial kernel. Let

$$S = \langle a, t \mid ta^2t^{-1} = a^3 \rangle.$$

Famously, Baumslag and Solitar [4] recognised that this group is non-Hopfian.

Lemma 4.1. *The given presentation of S is aspherical, $S/\langle\langle t \rangle\rangle$ is trivial, and $\psi : a \mapsto a^2, t \mapsto t$ defines an epimorphism with non-trivial kernel.*

Proof. To see that ψ is onto, observe that t and $a = a^3a^{-2} = ta^2t^{-1}a^{-2}$ are in the image. Britton’s lemma assures us that $c := [a, tat^{-1}] \in \ker \psi$ is non-trivial. It is obvious that $S/\langle\langle t \rangle\rangle$ is trivial. A standard topological argument shows that the natural presentations of HNN extensions of free groups are aspherical (cf. [6, p. 364]). Alternatively, we can appeal to the fact that 1-relator presentations where the relation is not a proper power are aspherical. \square

The group S belongs to the family of groups considered in section 4.2 of [6], where it is proved that a certain amalgamated free product $B = S_1 *_L S_2$ has no non-trivial finite quotients. Here, S_1 and S_2 are isomorphic copies of S (with subscripts to distinguish them), L is a free group of rank 2, and the amalgamation makes the identification $c_1 = t_2$ and $t_1 = c_2$, where $c = [a, tat^{-1}]$, as above. Thus B admits the following *aspherical* presentation:

$$B = \langle a_1, t_1, a_2, t_2 \mid t_1a_1^2t_1^{-1}a_1^{-3}, t_2a_2^2t_2^{-1}a_2^{-3}, t_2^{-1}[a_1, t_1a_1t_1^{-1}], t_1^{-1}[a_2, t_2a_2t_2^{-1}] \rangle.$$

The features of B that we need in this section are the following, which are established in [6, p. 365]. (Other features will be used in Section 8.) A finite presentation of a group is termed *balanced* if it has the same number of generators as relations.

Lemma 4.2. *B is an infinite group that has a balanced, aspherical presentation and no non-trivial finite quotients. In particular, B is torsion-free and $H_1(B, \mathbb{Z}) = H_2(B, \mathbb{Z}) = 0$.*

4.1. The group we seek

Let B be a group satisfying the hypotheses of Lemma 4.2, fix an element of infinite order $b \in B$ and define

$$Q = S *_Z B$$

where the amalgamation identifies $t \in S$ with $b \in B$.

Proposition 4.3. *Q is a non-Hopfian group that has a balanced, aspherical presentation and no non-trivial finite quotients. In particular, Q is torsion-free and $H_1(Q, \mathbb{Z}) = H_2(Q, \mathbb{Z}) = 0$.*

Proof. Let $B = \langle X \mid R \rangle$ be a balanced aspherical presentation and let β be a word in the generators that equals $b^{-1} \in B$. Then

$$Q \equiv \langle a, t, X \mid ta^2t^{-1}a^{-3}, t\beta, R \rangle$$

is an aspherical balanced presentation for Q . In any finite quotient of Q , the image of B is trivial, hence the image of $t = b$ is trivial. And since S is in the normal closure of t , it too has trivial image.

To see that Q is non-Hopfian we consider the homomorphism $\Psi : Q \rightarrow Q$ whose restriction to S is the epimorphism ψ of Lemma 4.1 and whose restriction to B is the identity: Ψ is well-defined because $\psi(t) = t$ and $S \cap B = \langle t \rangle$; it is onto because S and B lie in the image; and it has non-trivial kernel because ψ does. \square

5. Isomorphisms between fibre products

A subgroup $H < G_1 \times G_2$ of a direct product is termed a *subdirect product* if the coordinate projections map it *onto* G_1 and G_2 , and it is said to be *full* if both of the intersections $H \cap G_i$ are non-trivial. The fibre product of any pair of epimorphisms $G_1 \rightarrow Q$ and $G_2 \rightarrow Q$ is a subdirect product, and it is full provided both maps have non-trivial kernel. (All subdirect products of $G_1 \times G_2$ arise in this way: see [9, p. 632].)

Lemma 5.1. *Let Γ_1 and Γ_2 be torsion-free, non-elementary hyperbolic groups, let $P, P' < \Gamma_1 \times \Gamma_2$ be full subdirect products, let $N_i = P \cap \Gamma_i$ and let $N'_i = P' \cap \Gamma_i$. Then every isomorphism $\phi : P \rightarrow P'$ maps $N_1 \times N_2$ isomorphically onto $N'_1 \times N'_2$ (sending the direct summands to direct summands) and extends uniquely to an isomorphism $\Phi : \Gamma_1 \times \Gamma_2 \rightarrow \Gamma_1 \times \Gamma_2$.*

Proof. Let $N_i = P \cap \Gamma_i$ and $N'_i = P' \cap \Gamma_i$ and note that these are the kernels of the coordinate projections restricted to P and P' . Note too that N_i and N'_i are normal in Γ_i because, for example, N_1 is normal in P and the projection of P onto Γ_1 fixes N_1 . A non-trivial normal subgroup of a torsion-free, non-elementary hyperbolic group contains a non-abelian free group, so the centraliser in P of any $n \in N_1$ contains such a free group (since it contains N_2). On the other hand, elements of P that do not lie in $N_1 \cup N_2$ are of the form (γ_1, γ_2) with $\gamma_i \in \Gamma_i \setminus \{1\}$, and non-trivial elements of torsion-free hyperbolic groups have cyclic centralisers. Thus $N_1 \cup N_2$ consists of precisely those $x \in P$

with non-abelian centraliser. And $N'_1 \cup N'_2 \subset P'$ can be characterised similarly. It follows that every isomorphism $\phi : P \rightarrow P'$ sends $N_1 \cup N_2$ bijectively to $N'_1 \cup N'_2$, and therefore maps $N_1 \times N_2$ isomorphically onto $N'_1 \times N'_2$. Further consideration of centralisers shows that ϕ sends direct factors to direct factors: either $\phi(N_1) = N'_1$ and $\phi(N_2) = N'_2$ or else $\phi(N_1) = N'_2$ and $\phi(N_2) = N'_1$.

If $\phi(N_1) = N'_1$ and $\phi(N_2) = N'_2$, then the coordinate projections give us natural identifications

$$P/N_2 = \Gamma_1 = P'/N'_2 \quad \text{and} \quad P/N_1 = \Gamma_2 = P'/N'_1,$$

via which the maps $pN_2 \mapsto \phi(p)N'_2$ and $pN_1 \mapsto \phi(p)N'_1$ define the unique isomorphisms $\phi_1 : \Gamma_1 \rightarrow \Gamma_1$ and $\phi_2 : \Gamma_2 \rightarrow \Gamma_2$ such that $\Phi := (\phi_1, \phi_2) \in \text{Aut}(\Gamma_1) \times \text{Aut}(\Gamma_2)$ restricts to $\phi : P \rightarrow P'$.

If $\phi(N_1) = N'_2$ and $\phi(N_2) = N'_1$ then instead we obtain isomorphisms $\tilde{\phi}_1 : \Gamma_1 \rightarrow \Gamma_2$ and $\tilde{\phi}_2 : \Gamma_2 \rightarrow \Gamma_1$ such that $\Phi := \tilde{\phi}_1 \times \tilde{\phi}_2 \in \text{Aut}(\Gamma_1 \times \Gamma_2)$ restricts to $\phi : P \rightarrow P'$. \square

6. The first proof of Theorem A

We shall need the following elementary lemma.

Lemma 6.1. *Let G be a group, let $K \triangleleft G$ be a normal subgroup and let $\phi : G \rightarrow G$ be an automorphism. If $K \subsetneq \phi(K)$, then ϕ has infinite order in $\text{Out}(G)$.*

Proof. If ϕ^m were an inner automorphism for some $m > 0$ then, since K is normal, we would have $\phi^m(K) = K$, whereas $K \subsetneq \phi(K)$ implies $K \subsetneq \phi^m(K)$. \square

We turn to the main argument. Let \mathcal{Q} be the aspherical presentation described in Proposition 4.3 and let

$$1 \rightarrow N \rightarrow \Gamma \xrightarrow{\pi_0} \mathcal{Q} \rightarrow 1$$

be the short exact sequence obtained by applying the Rips construction to it. Let $\Psi : \mathcal{Q} \rightarrow \mathcal{Q}$ be the epimorphism described in the proof of Proposition 4.3 and define $\pi_n = \pi_0 \circ \Psi^n$. Let $P_n < \Gamma \times \Gamma$ be the fibre product of the maps $\pi_0 : \Gamma \times \{1\} \rightarrow \mathcal{Q}$ and $\pi_n : \{1\} \times \Gamma \rightarrow \mathcal{Q}$. The kernel of π_0 is finitely generated, so we have all of the data required to apply Theorem 2.1. Thus we obtain, in a recursive manner, finite presentations for the fibre products P_n . Theorem 2.2 assures us that the inclusion $P_n \hookrightarrow \Gamma \times \Gamma$ induces an isomorphism of profinite completions. Thus the following claim completes the proof of Theorem A.

Claim. $P_n \cong P_m$ if and only if $m = n$.

The intersection of P_n with $\Gamma \times \{1\}$ is $\ker \pi_0$ while its intersection with $\{1\} \times \Gamma$ is $\ker \pi_n$. Thus P_n contains the subgroup $K_n := \ker \pi_0 \times \ker \pi_n$, which is normal in $\Gamma \times \Gamma$. Note that $K_n \subsetneq K_m$ if $m > n$.

Lemma 5.1 tells us that any isomorphism $\phi : P_n \rightarrow P_m$ is the restriction to P_n of an automorphism Φ of $\Gamma \times \Gamma$. The automorphism group of $\Gamma \times \Gamma$ contains $\text{Aut}(\Gamma) \times \text{Aut}(\Gamma)$ as a subgroup of index 2, and Lemma 3.1 tells us that the group of inner automorphisms

has finite index in this. In particular, Φ has finite order in the outer automorphism group of $\Gamma \times \Gamma$. But then $\Phi(K_n) = K_m$, by Lemma 5.1, which contradicts Lemma 6.1 unless $m = n$. This completes the proof of Theorem A. \square

7. Pro-nilpotent equivalences

The *pro-nilpotent* completion of a group G is the inverse limit of its system of nilpotent quotients; equivalently, it is the inverse limit of the sequence $G/G_c \rightarrow G/G_{c-1}$ where G_c is the c -th term of the lower central series of G . If a homomorphism of finitely generated groups induces an isomorphism of profinite completions, then it induces an isomorphism of pro-nilpotent completions [10, Proposition 3.2]. Thus Addendum B will be proved if we can arrange for the group Γ of the previous section to be residually torsion-free-nilpotent. Proposition 3.2(5) assures us that we can do so.

8. Nielsen equivalence and T -equivalence

The proof of Theorem A presented in Section 6 is an implementation of the following naive idea: if one has a group Q of type F_3 and an infinite family of epimorphisms from finitely presented groups $\pi_0 : \Gamma \rightarrow Q$ and $\pi_n : G \rightarrow Q$, where π_0 has a non-trivial finitely generated kernel and the π_n are “truly inequivalent”, then one expects the fibre products $P_n < \Gamma \times G$ of pairs (π_0, π_n) to be non-isomorphic. The most direct way in which one might try to implement this strategy is to let G be a free group and take the π_n to be Nielsen-inequivalent choices of generating sets, but this approach is fraught with technical difficulties. In this section we consider an alternative implementation of the naive strategy that takes up the idea of Nielsen equivalence more directly than our first proof of Theorem A, providing us with different examples.

Let F be a free group with ordered basis $\{x_1, \dots, x_n\}$ and let G be a group. Ordered generating sets $\Sigma = \{s_1, \dots, s_n\} \subset G$ of cardinality n correspond to epimorphisms $q_\Sigma : F \twoheadrightarrow G$; one defines $q_\Sigma(x_i) = s_i$. The automorphism groups $\text{Aut}(F)$ and $\text{Aut}(G)$ act on the set of such epimorphisms by pre-composition and post-composition, respectively. These actions commute. By definition, $\Sigma = \{s_1, \dots, s_n\}$ and $\Sigma' = \{s'_1, \dots, s'_n\}$ (or q_Σ and $q_{\Sigma'}$) are *Nielsen equivalent* if they lie in the same $\text{Aut}(F)$ -orbit, and *T -equivalent* if they lie in the same orbit under the action of $\text{Aut}(F) \times \text{Aut}(G)$. In other words, they are T -equivalent if there are automorphisms $\phi : F \rightarrow F$ and $\psi : G \rightarrow G$ making the following diagram commute:

$$\begin{array}{ccc} F & \xrightarrow{\phi} & F \\ q_\Sigma \downarrow & & \downarrow q_{\Sigma'} \\ G & \xrightarrow{\psi} & G \end{array}$$

There is a considerable literature on Nielsen equivalence but it is a notoriously difficult invariant to compute and little is known when $n > 2$. For 2-generator, 1-relator groups, the situation is better understood.

Example 8.1. Let $S = \langle a, t \mid ta^2t^{-1} = a^3 \rangle$. Because $a = a^3a^{-2} = ta^2t^{-1}a^{-2}$, for every positive integer $n > 0$ the 2-element set $\Sigma_n = \{t, a^{2^n}\}$ generates G . Brunner [12] proves that Σ_n is not T -equivalent to Σ_m if $n \neq m$.

In Section 4 we considered the group

$$B = \langle a_1, t_1, a_2, t_2 \mid t_1a_1^2t_1^{-1}a_1^{-3}, t_2a_2^2t_2^{-1}a_2^{-3}, t_2^{-1}[a_1, t_1a_1t_1^{-1}], t_1^{-1}[a_2, t_2a_2t_2^{-1}] \rangle,$$

which is an amalgam of the form $S *_L S$ with L free of rank 2. Although Nielsen equivalence behaves well with respect to free products [19], it does not behave well with respect to amalgamated free products, so there is no obvious way of adapting the generating sets in Example 8.1 so as to produce an infinite sequence of T -inequivalent generating sets for B . To circumvent this problem, we pass from consideration of maps from the free group of rank 4 to B to consideration of maps $\Lambda \rightarrow B$, where Λ is obtained from the free group of rank 4 taking a trivial HNN extension that distinguishes a free factor F of rank 2.

Let $\Lambda = \langle \alpha_1, \tau_1, \alpha_2, \tau_2, \zeta \mid [\alpha_1, \zeta] = [\tau_1, \zeta] = 1 \rangle$ and let $F = \langle \alpha_1, \tau_1 \rangle < \Lambda$.

Lemma 8.2. *For every automorphism $\phi : \Lambda \rightarrow \Lambda$ there exists $l \in \Lambda$ such that $\text{ad}_l \circ \phi$ sends ζ to $\zeta^{\pm 1}$ and restricts to an automorphism of $C_\Lambda(\zeta) = F \times \langle \zeta \rangle$.*

Proof. A simple calculation with HNN normal forms shows that the only elements $\lambda \in \Lambda$ whose centraliser $C_\Lambda(\lambda)$ contains a non-abelian free group are the conjugates of powers of ζ . And $\langle \zeta \rangle$ is maximal among cyclic subgroups of Λ (as one can see by abelianising, for example). So $\phi(\zeta) = l\zeta^{\pm 1}l^{-1}$ for some $l \in \Lambda$, which implies that $\text{ad}_l \circ \phi$ preserves $C_\Lambda(\zeta) = F \times \langle \zeta \rangle$. □

Lemma 8.3. *Let Q be a group, let $G < Q$ be a subgroup with trivial centraliser, let $q, q' : \Lambda \rightarrow Q$ be epimorphisms with kernels N and N' , and suppose that $q(F) = q'(F) = G$. If $q|_F$ and $q'|_F$ are not T -equivalent, then there is no automorphism $\phi : \Lambda \rightarrow \Lambda$ with $\phi(N) = N'$.*

Proof. By definition, ζ commutes with F and we are assuming that $q(F) = q'(F) = G$ has trivial centraliser, so $q(\zeta) = q'(\zeta) = 1$ and both q and q' factor through the retraction $\rho : F \times \langle \zeta \rangle \rightarrow F$.

Towards a contradiction, suppose that there is an automorphism $\phi : \Lambda \rightarrow \Lambda$ such that $\phi(N) = N'$. Then $q(x) \mapsto q'(\phi(x))$ defines an automorphism $\bar{\phi} : Q \rightarrow Q$. Lemma 8.2 tells us that there is an element $l \in \Lambda$ such that the inner automorphism ad_l conjugates $\phi(F \times \langle \zeta \rangle)$ to $F \times \langle \zeta \rangle$. Thus, we obtain the following commutative diagram:

$$\begin{array}{ccccccc} F & \hookrightarrow & F \times \langle \zeta \rangle & \xrightarrow{\phi} & \phi(F \times \langle \zeta \rangle) & \xrightarrow{\text{ad}_l} & F \times \langle \zeta \rangle \xrightarrow{\rho} F \\ \downarrow q & & \downarrow q & & \downarrow q' & & \downarrow q' \\ G & \xrightarrow{\text{id}} & G & \xrightarrow{\bar{\phi}} & \bar{\phi}(G) & \xrightarrow{\text{ad}_{q'(l)}} & G \xrightarrow{\text{id}} G \\ & & & & & & \downarrow q' \end{array}$$

This diagram shows that $q|_F$ and $q'|_F$ are T -equivalent, contrary to hypothesis. □

The vertex groups in the decomposition $B = S *_L S$ are centreless, and each is a maximal elliptic subgroup (in the sense of Bass–Serre theory), therefore each has trivial centraliser in B . Thus we may apply the preceding lemma to maps $\Lambda \rightarrow B$ with the first factor S in the role of G .

Working with the presentations of Λ and B displayed above, we define $q_n : \Lambda \rightarrow B$ by setting

$$q_n(\alpha_i) = a_i^{2^n}, \quad q_n(\tau_i) = t_i, \quad q_n(\zeta) = 1.$$

Corollary 8.4. *If $n \neq m$, there is no automorphism $\phi : \Lambda \rightarrow \Lambda$ such that $\phi(\ker q_n) = \ker q_m$.*

Proof. The discussion in Example 8.1 shows that each q_n is surjective and that the restriction of q_n to $F = \langle \alpha_1, \tau_1 \rangle$ is T-equivalent to the restriction of q_m if and only if $m = n$. \square

9. A second proof of Theorem A

We apply the Rips construction to the finite presentation of B given above to obtain a short exact sequence $1 \rightarrow N \rightarrow \Gamma \xrightarrow{\pi_0} B \rightarrow 1$ satisfying the conditions of Proposition 3.2. As Γ is special, it is a subgroup of a right-angled Artin group (RAAG). Λ itself is a RAAG, and hence $\Gamma \times \Lambda$ is special; in particular it is residually finite and residually torsion-free-nilpotent. B is given by a finite aspherical presentation and it has no non-trivial finite quotients, so, as in Section 6, we will be done if we can prove that the fibre products

$$\Pi_n = \{(x, y) \mid \pi_0(x) = q_n(y)\} < \Gamma \times \Lambda$$

associated to the epimorphisms $q_n : \Lambda \rightarrow B$ from Corollary 8.4 have the property that $\Pi_n \not\cong \Pi_m$ if $n \neq m$.

By arguing with centralisers, as in the proof of Lemma 5.1, one sees that every isomorphism $\phi : \Pi_n \rightarrow \Pi_m$ is the restriction of an ambient automorphism $(\phi_1, \phi_2) \in \text{Aut}(\Gamma) \times \text{Aut}(\Lambda)$ with

$$\phi_2(\Pi_n \cap \Lambda) = \Pi_m \cap \Lambda.$$

(Here we have taken account of the fact that Γ and Λ are not isomorphic.) But $\Pi_i \cap \Lambda = \ker q_i$, so this contradicts Corollary 8.4 unless $n = m$. \square

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