J. Eur. Math. Soc. 18, 1983–2017

DOI 10.4171/JEMS/636

Chloé Perin · Rizos Sklinos

Forking and JSJ decomposition in the free group

Received October 8, 2013

Abstract. We give a description of the model-theoretic relation of forking independence in terms of JSJ decompositions in non-abelian free groups.

Keywords. Forking independence, torsion-free hyperbolic groups, curve complex

1. Introduction

In this paper we examine the model-theoretic notion of forking independence in non-abelian free groups.

Forking independence was introduced by Shelah as part of the machinery needed in his classification program (see Section 2). It is an abstract independence relation between two tuples of a structure over a set of parameters.

In the tame context of stable first-order structures, forking independence enjoys certain nice properties (see Fact 2.11). As a matter of fact, the existence of an independence relation having these properties characterizes stable theories and the relation in this case must be exactly forking independence. So it is not surprising that forking independence has grown to have its own ontology and many useful notions have been introduced around it, first within the context of stable theories and then adapted and developed more generally.

In some stable algebraic structures, such as a module or an algebraically closed field, forking independence admits an algebraic interpretation. In the latter case this is easily described: if $\bar{b}$, $\bar{c}$ are finite tuples in an algebraically closed field $\mathcal{K}$ and L is a subfield, then $\bar{b}$ is independent of $\bar{c}$ over L if and only if the transcendence degree of $L(\bar{b}\bar{c})$ over $L(\bar{c})$ is the same as the transcendence degree of $L(\bar{b})$ over L. For a description in the case of modules we refer the reader to [Gar81], [PP83].

C. Perin: IRMA, UMR 7501, 7 rue René-Descartes, 67084 Strasbourg Cedex, France;
e-mail: cperin@math.unistra.fr

R. Sklinos (corresponding author): Institut für Mathematische Logik und Grundlagenforschung, Westfälische Wilhelms-Universität Münster,
Einsteinstrasse 62, 48149 Münster, Germany; e-mail: rizozs@gmail.com

Mathematics Subject Classification (2010): Primary 03C60; Secondary 20F65

Philosophically speaking, in every "natural" stable structure one should be able to understand forking independence in terms of the underlying geometric or algebraic nature. Sela [Sel13] proved that (non-cyclic) torsion-free hyperbolic groups are stable, thus it is natural to ask whether the forking independence relation can be given an algebraic interpretation in these groups. This paper, following this line of thought, gives such an interpretation in free groups and in some torsion-free hyperbolic groups in terms of Grushko and JSJ decompositions.

The first main result of this paper is:

Theorem 1. *Let $\bar{b}, \bar{c}$ be tuples of elements in the free group $\mathbb{F}_n$ and let A be a free factor of $\mathbb{F}_n$. Then $\bar{b}$ and $\bar{c}$ are independent over A if and only if $\mathbb{F}_n$ admits a free decomposition $\mathbb{F}_n = \mathbb{F} * A * \mathbb{F}'$ with $\bar{b} \in \mathbb{F} * A$ and $\bar{c} \in A * \mathbb{F}'$.*

Thus two finite tuples are independent over A if and only if they live in "essentially disjoint" parts of the Grushko decomposition of $\mathbb{F}_n$ relative to A (i.e. the maximal decomposition of $\mathbb{F}_n$ as a free product in which A is contained in one of the factors). The essential ingredients of the proof of our first result is the homogeneity of non-abelian free groups and a result of independent interest concerning the stationarity of types in the theory of non-abelian free groups (see Theorem 3.1).

The relative Grushko decomposition of a group with respect to a set of parameters is a way to see all the splittings of the group as a free product in which the set of parameters is contained in one of the factors. The relative cyclic JSJ decomposition is a generalization of this: it is a graph of group decompositions which encodes all the splittings of the group as an amalgamated product or an HNN extension over a cyclic group, for which the parameter set is contained in one of the factors (see Section 4).

The second result deals with the case where the parameter set is not contained in any proper free factor, so that the relative Grushko decomposition is trivial, and tells us that two tuples are then independent over A if and only if they live in "essentially disjoint" parts of the cyclic JSJ decomposition of $\mathbb{F}_n$ relative to A.

Theorem 2. *Let $\mathbb{F}_n$ be freely indecomposable with respect to A. Let (Λ, v_A) be the pointed cyclic JSJ decomposition of $\mathbb{F}_n$ with respect to A. Let $\bar{b}$ and $\bar{c}$ be tuples in $\mathbb{F}_n$, and denote by $\Lambda_{A\bar{b}}$ (respectively $\Lambda_{A\bar{c}}$) the minimal subgraphs of groups of Λ whose fundamental group contains the subgroups $\langle A, \bar{b}\rangle$ (respectively $\langle A, \bar{c}\rangle$) of $\mathbb{F}_n$. Then $\bar{b}$ and $\bar{c}$ are independent over A if and only if each connected component of $\Lambda_{A\bar{b}} \cap \Lambda_{A\bar{c}}$ contains at most one non-Z-type vertex, and such a vertex is of non-surface type.*

This is a special case of Theorem 8.2, where we prove it for torsion-free hyperbolic groups which are concrete over a set A of parameters. A group is said to be concrete with respect to the set A of parameters if it is freely indecomposable with respect to A, and does not admit an extended hyperbolic floor structure over A—that is, A is not contained in a proper retract of G which satisfies certain properties (see [LPS11]).

In this second result, the middle step between the purely model-theoretic notion of forking independence and the purely geometric one of JSJ decomposition is that of understanding the automorphism group of $\mathbb{F}_n$ relative to A. Indeed, the JSJ decomposition enables us to give in this setting a very good description (up to a finite index) of the group of automorphisms which fix A pointwise. On the other hand, in many cases the model-theoretic definitions bear a strong relation to properties of invariance under automorphisms (as can be seen in Section 2).

It is remarkable that once more the tools of geometric group theory prove so convenient to understand the first-order theory of torsion-free hyperbolic groups. The proof of Theorem 2 makes use of recent deep results of this field, such as Masur and Minsky's results on the curve complex of a surface.

The paper is organised as follows. Section 2 gives a thorough exposition of the model-theoretic notions needed, and the necessary background on forking independence. Section 3 is devoted to the proof of Theorem 1. Section 4 gives relations between the automorphisms and JSJ decomposition. In Section 5 we prove that torsion-free hyperbolic groups are atomic over sets of parameters with respect to which they are concrete. Section 6 recalls some properties of algebraic closures, and a description of algebraic closures in torsion-free hyperbolic groups. Section 7 gives a brief introduction to the curve complex and states the results needed in Section 8, which is devoted to the proof of Theorem 2. In the final section we give some examples and make some further remarks on our results.

2. An introduction to forking independence

In this section we give an almost complete account of the model-theoretic background needed for the rest of the paper.

2.1. Basic notions

We first recall and fix notation for the basic notions of model theory. Let $\mathcal{M}$ be a structure, and $\mathcal{T}h(\mathcal{M})$ its complete first-order theory (the set of all first-order sentences that hold in $\mathcal{M}$).

An *n-type* $p(\bar{x})$ of $\mathcal{T}h(\mathcal{M})$ is a set of formulas without parameters in n variables which is consistent with $\mathcal{T}h(\mathcal{M})$. A type $p(\bar{x})$ is called *complete* if for every $\phi(\bar{x})$ either ϕ or $\neg\phi$ is in $p(\bar{x})$. For example, if $\bar{a}$ is a tuple in $\mathcal{M}$, the set $\mathrm{tp}^{\mathcal{M}}(\bar{a})$ of all formulas satisfied by $\bar{a}$ is a complete type.

If $A \subset \mathcal{M}$ is a set of parameters, we denote by $S_n^{\mathcal{M}}(A)$ the set of all complete n-types of $\mathcal{T}h((\mathcal{M}, \{a\}_{a\in A}))$ (where $(\mathcal{M}, \{a\}_{a\in A})$ is the structure obtained from $\mathcal{M}$ by adding the elements of A as constant symbols). We also note that the set of n-types over the empty set of a first-order theory T is usually denoted by $S_n(T)$.

It is easy to see that $S_n^{\mathcal{M}}(A)$ is a Stone space when equipped with the topology defined by the basis of open sets $[\phi(\bar{x})] = \{p \in S_n^{\mathcal{M}}(A) : \phi \in p\}$, where $\phi(\bar{x})$ is a formula with parameters in A. A type $p \in S_n^{\mathcal{M}}(A)$ is *isolated* if there is a formula $\phi \in p$ such that $[\phi] = \{p\}$.

Definition 2.1. Let A be a subset of a structure $\mathcal{M}$. Then $\mathcal{M}$ is called *atomic* over A if every type in $S_n^{\mathcal{M}}(A)$ which is realized in $\mathcal{M}$ is isolated.

Equivalently, if $\mathcal{M}$ is a countable structure, then $\mathcal{M}$ is atomic if the orbit $\mathrm{Aut}(\mathcal{M}).\bar{b}$ of every finite tuple $\bar{b} \in \mathcal{M}$ is $\emptyset$-definable. If the orbits are merely $\emptyset$-type-definable (i.e. the solution set of a type in $\mathcal{M}$), then we obtain the notion of homogeneity.

Definition 2.2. A countable structure $\mathcal{M}$ is said to be *homogeneous* if for any two finite tuples $\bar{b}, \bar{c} \in \mathcal{M}$ such that $\mathrm{tp}^{\mathcal{M}}(\bar{b}) = \mathrm{tp}^{\mathcal{M}}(\bar{c})$ there is an automorphism of $\mathcal{M}$ sending $\bar{b}$ to $\bar{c}$.

Let $\bar{a}, A \subset \mathcal{M}$. We say $\bar{a}$ is *algebraic* (respectively *definable*) over A if there is a formula $\phi(\bar{x})$ with parameters in A such that $\bar{a} \in \phi(\mathcal{M})$ and $\phi(\mathcal{M})$ is finite (respectively has cardinality one). We denote the set of algebraic (respectively definable) tuples over A by $\mathrm{acl}_{\mathcal{M}}(A)$ (respectively $\mathrm{dcl}_{\mathcal{M}}(A)$). The following lemma is immediate.

Lemma 2.3. *Let $\mathcal{M}$ be countable and atomic over A. Then for any $\bar{a} \in \mathcal{M}$, $\bar{a}$ is algebraic* (*respectively definable*) *over A if and only if $\{f(\bar{a}) : f \in \mathrm{Aut}(\mathcal{M}/A)\}$ is finite* (*respectively has cardinality one*).

Proof. Just note that the orbit of any tuple under $\mathrm{Aut}(\mathcal{M}/A)$ is a definable set over A. $\square$

2.2. *Stability theory*

Stability theory is an important part of modern model theory. The rudiments of stability can be found in the seminal work of Morley proving the Łoś conjecture, namely that a countable theory is categorical in an uncountable cardinal if and only if it is categorical in all uncountable cardinals. A significant aspect of Morley's work is that he assigned an invariant (a dimension for some independence relation) to a model that determined the model up to isomorphism. In full generality most of the results concerning stability are attributed to Shelah [She90]. Shelah has established several dividing lines separating "well behaved" theories from theories which do not have a structure theorem classifying their models. One such dividing line is stable versus unstable, where if a first-order theory T is unstable then it has the maximum number of models, 2^{κ}, for each cardinal $\kappa \geq 2^{|T|}$.

A first-order theory T is *stable* if it prevents the definability of an infinite linear order. To state it more formally, we say that a first-order formula $\phi(\bar{x}, \bar{y})$ in a structure $\mathcal{M}$ has the *order property* if there are sequences $(\bar{a}_n)_{n<\omega}, (\bar{b}_n)_{n<\omega}$ such that $\mathcal{M} \models \phi(\bar{a}_n, \bar{b}_m)$ if and only if $m < n$.

Definition 2.4. A first-order theory T is *stable* if no formula has the order property in a model of T.

By the discussion above it is apparent that the development of an abstract independence relation enabling us to assign a dimension to several sets will be useful. This is what brought Shelah to define forking independence.

The rest of the subsection is devoted to a thorough description of forking independence in stable theories. Unless otherwise stated, all the results in this subsection which are stated without a proof can be found in [Pil96, Chapter 1, Sections 1–2].

We fix a stable first-order theory T and we work in a "big" saturated model $\mathbb{M}$ of T, which is usually called the *monster model* (see [Mar02, p. 218]).

We write $\mathrm{tp}(\bar{a}/A)$ for $\mathrm{tp}^{\mathbb{M}}(\bar{a}/A)$ and $S_n(A)$ for $S_n^{\mathbb{M}}(A)$.

Definition 2.5. A formula $\phi(\bar{x}, \bar{b})$ *forks over* A if there are $n < \omega$ and an infinite sequence $(\bar{b}_i)_{i<\omega}$ such that $\text{tp}(\bar{b}/A) = \text{tp}(\bar{b}_i/A)$ for $i < \omega$, and the set $\{\phi(\bar{x}, \bar{b}_i) : i < \omega\}$ is n-inconsistent.

A tuple $\bar{a}$ is *independent* of B over A (denoted $\bar{a} \underset{A}{\downarrow} B$) if there is no formula in $\text{tp}(\bar{a}/B)$ which forks over A.

In Section 2 of [LPS11] we give an intuitive account of the above definition.

The following observation is immediate.

Remark 2.6. Let $\mathbb{M} \models \phi(\bar{x}) \to \psi(\bar{x})$ and suppose $\psi(\bar{x})$ forks over A. Then $\phi(\bar{x})$ forks over A.

Definition 2.7. If $p \in S_n(A)$ and $A \subseteq B$, then $q := \text{tp}(\bar{a}/B)$ is called a *non-forking extension of* p if $p \subseteq q$ and $\bar{a} \underset{A}{\downarrow} B$.

Definition 2.8. A type $p \in S_n(A)$ is called *stationary* if for any $B \supseteq A$, p has a unique non-forking extension over B.

Definition 2.9. Let $C = \{\bar{c}_i : i \in I\}$ be a set of tuples. We say that C is an *independent set over* A if for every $i \in I$, $\bar{c}_i \underset{A}{\downarrow} \bigcup C \setminus \{\bar{c}_i\}$.

If p is a type over A which is stationary and $(a_i)_{i<\kappa}$, $(b_i)_{i<\kappa}$ are both independent sets over A of realizations of p, then $\text{tp}((a_i)_{i<\kappa}/A) = \text{tp}((b_i)_{i<\kappa}/A)$. This allows us to denote by $p^{(\kappa)}$ the type of κ-independent realizations of p. It is not hard to see that if p is stationary then so is $p^{(\kappa)}$.

We observe the following behavior of forking independence inside a countable atomic model of T.

Lemma 2.10. *Let $\mathcal{M} \models T$. Let $\bar{b}, A \subset \mathcal{M}$, and suppose $\mathcal{M}$ is countable and atomic over A. Suppose $X := \phi(\mathcal{M}, \bar{b})$ contains a non-empty almost A-invariant subset* (*i.e. one with finitely many images under* $\text{Aut}(\mathcal{M}/A)$). *Then $\phi(\bar{x}, \bar{b})$ does not fork over A.*

Proof. Suppose it does; then there is an infinite sequence $(\bar{b}_i)_{i<\omega}$ in $\mathbb{M}$ such that $\text{tp}(\bar{b}_i/A) = \text{tp}(\bar{b}/A)$ and $\{\phi(\bar{x}, \bar{b}_i) : i < \omega\}$ is k-inconsistent for some $k < \omega$. Since $\mathcal{M}$ is atomic over A, we infer that $\text{tp}(\bar{b}/A)$ is isolated, say by $\psi(\bar{y})$. Thus, for arbitrarily large λ the following sentence (over A) is true:

$$\begin{aligned}\mathbb{M} \models \exists \bar{y}_1, \dots, \bar{y}_\lambda\, [(\psi(\bar{y}_1) \wedge \cdots \wedge \psi(\bar{y}_\lambda)) \\ \wedge \text{“any } k\text{-subset of } \{\phi(\bar{x}, \bar{y}_1), \dots, \phi(\bar{x}, \bar{y}_\lambda)\} \text{ is inconsistent”}].\end{aligned}$$

But the above sentence is true in $\mathcal{M}$, and this contradicts the hypothesis that $\phi(\bar{x}, \bar{b})$ contains a non-empty almost A-invariant set. $\square$

We list some properties of forking independence (recall that we have assumed that the theory T is stable).

Fact 2.11. (i) (Existence of non-forking extensions) *Let $p \in S_n(A)$ and $A \subseteq B$. Then there is a non-forking extension of p over B.*
(ii) (Symmetry) $\bar{a} \underset{A}{\perp} \bar{b}$ *if and only if* $\bar{b} \underset{A}{\perp} \bar{a}$.
(iii) (Local character) *For any $\bar{a}$, A, there is $A' \subseteq A$ with $|A'| \leq |T|$ such that $\bar{a} \underset{A'}{\perp} A$.*
(iv) (Transitivity) *Let $A \subseteq B \subseteq C$. Then $\bar{a} \underset{A}{\perp} C$ if and only if $\bar{a} \underset{A}{\perp} B$ and $\bar{a} \underset{B}{\perp} C$.*
(v) (Boundedness) *Every type over a model is stationary.*

In fact the above properties of forking independence characterize stable theories in the sense that if a theory T admits a sufficiently saturated model on which we can define an independence relation on triples of sets satisfying (i)–(v), then T is stable and the relation is exactly forking independence.

The following lemma is useful in practice.

Lemma 2.12. *Let $A \subseteq B$. Then $\bar{a} \underset{A}{\perp} B$ if and only if* $\mathrm{acl}(\bar{a}A) \underset{\mathrm{acl}(A)}{\perp} \mathrm{acl}(B)$.

The following theorem answers the question of how much "information" a type should include in order to be stationary.

Theorem 2.13 (Finite Equivalence Relation Theorem). *Let $p_1, p_2 \in S_n(B)$ be two distinct types, let $A \subseteq B$ and suppose that neither p_1 nor p_2 forks over A. Then there is a finite equivalence relation $E(\bar{x}, \bar{y})$ definable over A such that $p_1(\bar{x}) \cup p_2(\bar{y}) \models \neg E(\bar{x}, \bar{y})$.*

Shelah observed that "seeing" equivalence classes of definable equivalence relations as real elements gives a mild expansion of our theory, which we denote by T^{eq}, with many useful properties (we refer the reader to [Pil96, p. 10] for the construction). In this setting we denote by $\mathrm{acl}^{\mathrm{eq}}$ (respectively $\mathrm{dcl}^{\mathrm{eq}}$) the algebraic closure (respectively definable closure) calculated in $\mathbb{M}^{\mathrm{eq}}$ (the monster model of T^{eq}, which actually is an expansion of $\mathbb{M}$).

The following lemma is an easy application of the finite equivalence relation theorem.

Lemma 2.14. *Let A be a set of parameters in $\mathbb{M}$. Then* $\mathrm{acl}^{\mathrm{eq}}(A) = \mathrm{dcl}^{\mathrm{eq}}(A)$ *if and only if every type $p \in S_n(A)$ is stationary.*

Proof. ($\Rightarrow$) Suppose, for the sake of contradiction, that there is a type q in $S_n(A)$ which is not stationary. Let q_1, q_2 be distinct non-forking extensions of q to some set $B \supset A$. By the finite equivalence relation theorem, we may take B to be $\mathrm{acl}^{\mathrm{eq}}(A)$. Now the hypothesis yields a contradiction as a type over A extends uniquely to $\mathrm{dcl}^{\mathrm{eq}}(A)$.

($\Leftarrow$) Suppose that there is e in $\mathrm{acl}^{\mathrm{eq}}(A) \setminus \mathrm{dcl}^{\mathrm{eq}}(A)$. Then $\mathrm{tp}(e/A)$ is not stationary. Indeed, consider two distinct images of e under automorphisms fixing A. Then these images have different types over $\mathrm{acl}^{\mathrm{eq}}(A)$ and these types do not fork over A. By the construction of T^{eq}, there is a tuple $\bar{b}$ in $\mathbb{M}$ such that $e \in \mathrm{dcl}^{\mathrm{eq}}(\bar{b})$. Now, it is easy to see that $\mathrm{tp}(\bar{b}/A)$ is not stationary. $\square$

2.3. *Stable groups*

A group, $\mathcal{G} := (G, \cdot, \ldots)$, in the sense of model theory is a structure equipped with a group operation, but possibly also with some additional relations and functions. In the case where the additional structure is definable by multiplication alone, we speak of a *pure group*.

We define a *stable group* to be a group whose first-order theory $\mathcal{T}h(G, \cdot, \ldots)$ is stable. Although for the purpose of this paper it would be enough to consider pure groups, there is no harm in developing stable group theory in greater generality. All results in this subsection can be found in [Poi01].

Definition 2.15. Let $\mathcal{G}$ be a group. We say $\mathcal{G}$ is *connected* if there is no definable proper subgroup of finite index.

Definition 2.16. Let $\mathcal{G}$ be a stable group. Let X be a definable subset of G. Then X is *left* [*right*] *generic* if finitely many left [right] translates of X by elements of G cover G.

As in a stable group $\mathcal{G}$ a definable set $X \subseteq G$ is left generic if and only if it is right generic, we simply say generic.

Definition 2.17. Let $\mathcal{G}$ be a stable group and let $A \subseteq G$. A type $p(x) \in S_1(A)$ is *generic* if every formula in $p(x)$ is generic.

Lemma 2.18. *Let $p(x)$ be a generic type of the stable group $\mathcal{G}$. Then any non-forking extension of $p(x)$ is generic.*

It is not hard to see the following:

Fact 2.19. *Let $\mathcal{G}$ be a stable group. Then $\mathcal{G}$ is connected if and only if there is, over any set of parameters, a unique generic type.*

This has an immediate corollary.

Corollary 2.20. *Let $\mathcal{G}$ be a connected stable group. Then every generic type is stationary. In fact, generic types are exactly the non-forking extensions of the generic type over $\emptyset$.*

2.4. *Torsion-free hyperbolic groups*

In this subsection we see torsion-free hyperbolic groups as $\mathcal{L}$-structures in their natural language $\mathcal{L} := \{\cdot, ^{-1}, 1\}$, i.e. the language of groups. We denote by $\mathbb{F}_n := \langle e_1, \ldots, e_n \rangle$ the free group of rank n and we assume that $n > 1$. We start with the deep result of Kharlampovich–Myasnikov [KM06] and Sela [Sel06].

Theorem 2.21. *Let $\mathbb{F}$ be a non-abelian free factor of $\mathbb{F}_n$. Then $\mathbb{F} \prec \mathbb{F}_n$.*

This allows us to denote by T_{fg} the common theory of non-abelian free groups. Since connectedness (in the sense defined in the previous subsection) is a first-order property, we can state a result of Poizat [Poi83] in the following way.

Theorem 2.22. T_{fg} *is connected.*

As a matter of fact, the theory of every (non-cyclic) torsion-free hyperbolic group is connected (see [O11]).

Theorem 2.23. *Let G be a torsion-free hyperbolic group not elementarily equivalent to a free group. Then $\mathcal{T}h(G)$ is connected.*

On the other hand, Sela [Sel13] proved the following:

Theorem 2.24. *Let G be a (non-cyclic) torsion-free hyperbolic group. Then $\mathcal{T}h(G)$ is stable.*

Thus, in every theory of a (non-cyclic) torsion-free hyperbolic group there is a unique generic type over any set of parameters.

We specialize to T_{fg} and we denote by p_0 the generic type over the empty set. By Corollary 2.20 the generic type p_0 is stationary, thus we can define $p_0^{(\kappa)}$ to be the type of κ-independent realizations of p_0. Pillay [Pil09] gave an interpretation of the generic type in terms of its solution set in $\mathbb{F}_n$; in fact, he proved more generally

Theorem 2.25. *An m-tuple $a_1, \ldots, a_m$ realizes $p_0^{(m)}$ in $\mathbb{F}_n$ if and only if $a_1, \ldots, a_m$ is part of a basis of $\mathbb{F}_n$.*

In particular, the above theorem states that $\mathrm{tp}^{\mathbb{F}_n}(e_1)$ is generic.

An immediate consequence is that if $\mathbb{F}_n = \mathbb{F} * \mathbb{F}' * \mathbb{F}''$, then $\mathbb{F} \underset{\mathbb{F}'}{\downarrow} \mathbb{F}''$. We note that the above theorem has been generalized by the authors to finitely generated models of T_{fg} with appropriate modifications (see [PS12]).

The following theorem has been proved by the authors [PS12] and Ould Houcine [O11] independently.

Theorem 2.26. $\mathbb{F}_n$ *is homogeneous.*

As a matter of fact, we will see in Section 5 that the proof of this result can be adapted to give the following

Theorem 2.27. *Let G be a torsion-free hyperbolic group concrete with respect to a subgroup A. Then G is atomic over A.*

3. Forking over free factors

In this section we describe forking independence in non-abelian free groups over (possibly trivial) free factors. We begin with a result of more general interest.

Theorem 3.1. *Let $A \subset \mathbb{F}_n$. Then every type in $S_m(A)$ is stationary if and only if $\mathrm{tp}^{\mathbb{F}_n}(e_1, \ldots, e_n/A)$ is stationary.*

Proof. For the non-trivial direction it is enough, by Lemma 2.14, to prove that $\mathrm{acl}^{\mathrm{eq}}(A) = \mathrm{dcl}^{\mathrm{eq}}(A)$. Let $a \in \mathrm{acl}^{\mathrm{eq}}(A)$. Then $a \in \mathrm{dcl}^{\mathrm{eq}}(e_1, \ldots, e_n)$, and since $\mathrm{tp}^{\mathbb{F}_n}(e_1, \ldots, e_n/A)$ is stationary, we find that $\mathrm{tp}^{\mathbb{F}_n^{\mathrm{eq}}}(a/A)$ is stationary. Thus, $a \in \mathrm{dcl}^{\mathrm{eq}}(A)$ as desired. □

We get the following corollaries.

Corollary 3.2. *Let $p(\bar{x}) \in S_m(T_{\mathrm{fg}})$. Then $p(\bar{x})$ is stationary.*

Proof. Since $p_0(x)$ is stationary, it follows that $p_0^{(2)}(x, y) := \mathrm{tp}^{\mathbb{F}_2}(e_1, e_2)$ is. Now use Theorem 3.1 for $A = \emptyset$. □

Corollary 3.3. *Suppose a realizes p_0 in some model of T_{fg} and $p \in S_m(a)$. Then p is stationary.*

Proof. Suppose b realizes the unique non-forking extension of p_0 over a. Then $\langle a, b\rangle \cong \mathbb{F}_2$, and $\mathrm{tp}^{\langle a,b\rangle}(a, b/a)$ is stationary. Now use Theorem 3.1 for $A = \{a\}$. □

We are now ready to describe forking independence over free factors. For $m < n < \omega$, we will denote the free group of rank $n - m$ generated by $e_{m+1}, \ldots, e_n$ by $\mathbb{F}_{m,n}$.

Theorem 3.4. *Let $\bar{a}, \bar{b} \in \mathbb{F}_n$ and let A be a free factor of $\mathbb{F}_n$. Then $\bar{a} \underset{A}{\perp} \bar{b}$ if and only if $\mathbb{F}_n$ admits a free decomposition $\mathbb{F}_n = \mathbb{F} * A * \mathbb{F}'$ with $\bar{a} \in \mathbb{F} * A$ and $\bar{b} \in A * \mathbb{F}'$.*

Proof. ($\Leftarrow$) This direction is immediate as a basis of $\mathbb{F}_n$ is an independent set over $\emptyset$ by Theorem 2.25.

($\Rightarrow$) We may assume that $A = \mathbb{F}_m$ for some $m < n$ (we also include the case where $\mathbb{F}_m$ is trivial). Let $\bar{a}(x_1, \ldots, x_n)$ be a tuple of words in variables $x_1, \ldots, x_n$ such that $\bar{a}(e_1, \ldots, e_n) = \bar{a}$. We consider the tuple $\bar{a}' := \bar{a}(e_1, \ldots, e_m, e_{n+1}, \ldots, e_{2n-m})$ in $\mathbb{F}_{2n-m}$. As $e_{n+1}, \ldots, e_{2n-m}$ is independent of $e_1, \ldots, e_n$ over $e_1, \ldots, e_m$, we deduce that $\bar{a}'$ is independent of $\mathbb{F}_m\bar{b}$ over $\mathbb{F}_m$. We also note that $p := \mathrm{tp}^{\mathbb{F}_{2n-m}}(\bar{a}/\mathbb{F}_m) = \mathrm{tp}^{\mathbb{F}_{2n-m}}(\bar{a}'/\mathbb{F}_m)$ as there is an automorphism of $\mathbb{F}_{2n-m}$ fixing $\mathbb{F}_m$ taking $\bar{a}$ to $\bar{a}'$. But p is stationary (for $\mathbb{F}_m$ trivial this follows from Corollary 3.2, for $\mathbb{F}_m \cong \mathbb{Z}$ from Corollary 3.3, and in any other case $\mathbb{F}_m$ is a model, so this follows from Fact 2.11(v)), thus $\mathrm{tp}^{\mathbb{F}_{2n-m}}(\bar{a}/\mathbb{F}_m\bar{b}) = \mathrm{tp}^{\mathbb{F}_{2n-m}}(\bar{a}'/\mathbb{F}_m\bar{b})$. By homogeneity of $\mathbb{F}_{2n-m}$ there is an automorphism $f \in \mathrm{Aut}(\mathbb{F}_{2n-m}/\bar{b})$ which sends $\bar{a}'$ to $\bar{a}$. We consider the decomposition $\mathbb{F}_{2n-m} = \mathbb{F}_m * \mathbb{F}_{m+1,n} * \mathbb{F}_{n+1,2n-m}$. Applying f we get $\mathbb{F}_{2n-m} = \mathbb{F}_m * f(\mathbb{F}_{m+1,n}) * f(\mathbb{F}_{n+1,2n-m})$ with $\bar{b} \in \mathbb{F}_m * f(\mathbb{F}_{m+1,n})$ and $\bar{a} \in \mathbb{F}_m * f(\mathbb{F}_{n+1,2n-m})$. But $\mathbb{F}_n$ is a subgroup of $\mathbb{F}_{2n-m}$, thus by the Kurosh subgroup theorem we get a decomposition of $\mathbb{F}_n$ as desired. □

4. JSJ decompositions and modular groups

The main theme of this section is the description of the modular group $\mathrm{Mod}_A(G)$ of automorphisms of a torsion-free hyperbolic group G which is freely indecomposable with respect to a subgroup A. We briefly explain the outline of the section and the tools used.

In the first subsection we are concerned with cyclic JSJ splittings of G relative to A. These are splittings of a group G which in some sense encode all the splittings of G over cyclic subgroups in which A is elliptic.

The next three subsections are devoted to "elementary" automorphisms associated to a splitting of a group. These are automorphisms that can be read off locally from the splitting, namely Dehn twists and vertex automorphisms. Under certain conditions we prove that "elementary" automorphisms almost commute.

In the final subsection we prove that one can read off the modular group of G from its JSJ splitting relative to A, and we moreover give a normal form theorem for modular automorphisms. These results are not new (see [Lev05] and [GL15]), but the hands-on proofs we give in this specific setting hopefully helps to gain low-level intuition. We also describe the normal form of automorphisms which fix pointwise a subgroup of G containing A.

4.1. G-trees and JSJ decompositions

We will use definitions and results about graphs of groups from [Ser83]. Let Γ be a graph; we will denote by $V(\Gamma)$ and $E(\Gamma)$ respectively its vertex and edge sets. The set $E(\Gamma)$ is always assumed to be stable under the involution which to an edge e associates its inverse edge $\bar{e}$.

Let G be a finitely generated group. A *G-tree* is a simplicial tree T endowed with an action of G without inversions of edges. We say T is *minimal* if it admits no proper G-invariant subtree. A *cyclic G-tree* is a G-tree whose edge stabilizers are infinite cyclic. If A is a subset of G, a *(G, A)-tree* is a G-tree in which A fixes a point. Following [GL07], we call a (not necessarily simplicial) surjective equivariant map $d : T_1 \to T_2$ between two (G, A)-trees a *domination map*. A surjective simplicial map $p : T_1 \to T_2$ which consists in collapsing some orbits of edges to points is called a *collapse map*. In this case, we also say that T_1 *refines* T_2.

We also define:

Definition 4.1 (Bass–Serre presentation). Let G be a finitely generated group, and let T be a G-tree. Denote by Λ the corresponding quotient graph of groups and by p the quotient map $T \to \Lambda$. A *Bass–Serre presentation* for Λ is a triple $(T^1, T^0, (t_e)_{e\in E_1})$ consisting of

- a subtree T^1 of T which contains exactly one edge of $p^{-1}(e)$ for each edge e of Λ;
- a subtree T^0 of T^1 which contains exactly one vertex of $p^{-1}(v)$ for each vertex v of Λ;
- for each edge $e \in E_1 := \{e = uv \mid u \in T^0,\ v \in T^1 \setminus T^0\}$, an element t_e of G such that $t_e^{-1} \cdot v$ lies in T^0.

We call t_e the *stable letter* associated to e.

One can give an explicit presentation of the group G whose generating set is the union of the stabilizers of vertices of T^0 together with the stable letters t_e, hence the name.

For JSJ decompositions, we will use the framework described in [GL09a] and [GL09b] (see also the brief summary given in Section 3 of [PS12]). We recall here the main definitions and results we will use. Unless mentioned otherwise, all G-trees are assumed to be minimal.

Deformation space. The *deformation space* of a cyclic (G, A)-tree T is the set of all cyclic (G, A)-trees T' such that T dominates T' and T' dominates T. A cyclic (G, A)-tree is *universally elliptic* if its edge stabilizers are elliptic in every cyclic (G, A)-tree. If T is a universally elliptic cyclic (G, A)-tree, and T' is any cyclic (G, A)-tree, it is easy to see that there is a tree $\hat{T}$ which refines T and dominates T' (see [GL09a, Lemma 3.2]).

JSJ trees. A cyclic relative *JSJ tree* for G with respect to A is a universally elliptic cyclic (G, A)-tree which dominates any other universally elliptic cyclic (G, A)-tree. All these JSJ trees belong to the same deformation space, which we denote $\mathcal{D}_{\mathrm{JSJ}}$. Guirardel and Levitt show that if G is finitely presented and A is finitely generated, the JSJ deformation space always exists (see [GL09a, Theorem 5.1]). It is easily seen to be unique.

Rigid and flexible vertices. A vertex stabilizer in a (relative) JSJ tree is said to be *rigid* if it is elliptic in any cyclic (G, A)-tree, and *flexible* if not. In the case of a torsion-free hyperbolic group G and a finitely generated subgroup A of G with respect to which G is freely indecomposable, the flexible vertices of a cyclic JSJ tree of G with respect to A are *surface type* vertices [GL09a, Theorem 8.20], i.e. their stabilizers are fundamental groups of hyperbolic surfaces with boundary, any adjacent edge group is contained in a maximal boundary subgroup, and any maximal boundary subgroup contains either exactly one adjacent edge group, or exactly one conjugate of A [GL09a, Remark 8.19]. Note that (since the vertices are not rigid) these surfaces cannot be thrice punctured spheres [GL09a, Remark 8.19]. Nor can they be once punctured Klein bottles or twice punctured projective planes. Indeed, otherwise the JSJ tree T can be refined to a tree $\hat{T}$ by the splitting of this surface corresponding to one or two curves bounding Möbius bands. This new (G, A)-tree is still universally elliptic, since there are no incompatible splittings of the surface, but T does not dominate $\hat{T}$; we get a contradiction.

We give a simple example of a JSJ decomposition at the level of graph of groups.

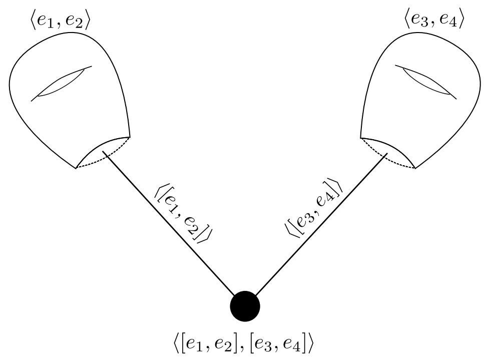


Fig. 1. A JSJ decomposition of the free group $\mathbb{F}_4$ on $e_1, \ldots, e_4$ relative to $A = \langle [e_1, e_2], [e_3, e_4]\rangle$.

The tree of cylinders. In [GL11], *cylinders* in cyclic G-trees are defined as equivalence classes of edges under the equivalence relation given by commensurability of stabilizers, and to any G-tree T is associated its *tree of cylinders*. It can be obtained from T as follows: the vertex set is the union $V_0(T_c) \cup V_1(T_c)$ where $V_0(T_c)$ contains a vertex w'

for each vertex w of T contained in at least two distinct cylinders, and $V_1(T_c)$ contains a vertex v_c for each cylinder c of T. There is an edge between vertices w' and v_c lying in $V_0(T_c)$ and $V_1(T_c)$ respectively if and only if w belongs to the cylinder c.

We get a tree which is bipartite: every edge in the tree of cylinders joins a vertex from $V_0(T_c)$ (which is cyclically stabilized) to a vertex of $V_1(T_c)$. Since the action of G on T sends cylinders to cylinders, the tree of cylinders admits an obvious G-action. Note also that if H stabilizes an edge e of T, its centralizer $C(H)$ preserves the cylinder containing e since the translates of e are also stabilized by H; in particular there is a vertex in T_c whose stabilizer is $C(H)$. It is moreover easy to see that this vertex is unique.

It turns out that the tree of cylinders is in fact an invariant of the deformation space [GL11, Corollary 4.10].

Case of freely indecomposable torsion-free hyperbolic groups. By [GL11, Theorem 2], if G is a torsion-free hyperbolic group freely indecomposable with respect to a finitely generated subgroup A, the tree of cylinders T_c of the cyclic JSJ deformation space of G with respect to A is itself a JSJ tree, and it is moreover *strongly* 2-*acylindrical*: if a non-trivial element stabilizes two distinct edges, they are adjacent to a common cyclically stabilized vertex.

Moreover, in this case the tree of cylinders is not only universally elliptic, but in fact *universally compatible*: given any cyclic (G, A)-tree T, there is a refinement $\hat{T}$ of T_c which *collapses* onto T [GL11, Theorem 6].

The JSJ deformation space being unique, it must be preserved under the action of $\mathrm{Aut}_A(G)$ on (isomorphism classes of) (G, A)-trees defined by twisting the G-actions. Thus the tree of cylinders is a fixed point of this action, that is, for any automorphism $\phi \in \mathrm{Aut}_A(G)$, there is an automorphism $f : T_c \to T_c$ such that for any $x \in T_c$ and $g \in G$ we have $f(g \cdot x) = \phi(g) \cdot f(x)$.

JSJ relative to a non-finitely generated subgroup. Let G be a torsion-free hyperbolic group freely indecomposable with respect to a subgroup A. By [PS12, Proposition 3.7], there is a finitely generated subgroup A_0 of A such that G is freely indecomposable with respect to A_0 and A is elliptic in any cyclic JSJ tree of G with respect to A_0. The tree of cylinders of the cyclic JSJ deformation space with respect to A_0 clearly admits a common refinement with any cyclic (G, A)-tree, and has all the properties described above in the case A was finitely generated. So whenever we refer to the tree of cylinders of the cyclic JSJ deformation space with respect to A (for a possibly non-finitely generated group), we tacitly mean the tree of cylinders of the cyclic JSJ deformation space with respect to A_0.

The pointed cyclic JSJ tree. For our purposes, we need a tree with a basepoint which is a slight variation of the tree of cylinders. Note that this tree is not minimal.

Definition 4.2. Let G be a torsion-free hyperbolic group freely indecomposable with respect to a subgroup A. Let T_c be the tree of cylinders of the cyclic JSJ deformation space of G with respect to A.

We define the pointed cyclic JSJ tree (T, v) of G with respect to A as follows:

- If A is cyclic, let u be either (if it exists) the unique vertex whose stabilizer is exactly the centralizer $C(A)$ of A, or otherwise the unique vertex fixed by $C(A)$. We take T to be the tree T_c to which we add one orbit of vertices $G.v$, one orbit of edges $G.e$ with $e = vu$, and we set $\mathrm{Stab}(v) = \mathrm{Stab}(e) = C(A)$.

- If A is not cyclic, we take $T = T_c$ and we let v be the unique vertex fixed by A.

Definition 4.3. A vertex of the pointed cyclic JSJ tree is said to be a *Z-type vertex* if it is cyclically stabilized and distinct from the basepoint v.

Remark 4.4. It is not hard to see that the pointed cyclic JSJ tree of G with respect to A is strongly 2-acylindrical, universally compatible, and a fixed point of the action of $\mathrm{Aut}_A(G)$ on (G, A)-trees defined by twisting the G-action.

4.2. Dehn twists

Let G be a finitely generated group.

Definition 4.5. Let $e = uv$ be an edge in a G-tree T, and let a be an element in the centralizer in G of $\mathrm{Stab}(e)$. The G-tree T' obtained from T by collapsing all the edges not in the orbit of e induces a splitting of G as an amalgamated product $G = U *_{\mathrm{Stab}(e)} V$ or as an HNN extension $U*_{\mathrm{Stab}(e)}$ with stable letter t, where U is the stabilizer of the image vertex of u in T'.

The *Dehn twist* by a about e is the automorphism of G which restricts to the identity on U and to conjugation by a on V (respectively sends t to at in the HNN case).

The proof of the following lemma is immediate. We first recall that a G-tree is called *non-trivial* if there is no globally fixed point. It is not hard to see that if G is finitely generated and T is a non-trivial G-tree then T contains a unique minimal G-invariant subtree.

Lemma 4.6. *Let G be a finitely generated group, and let T be a cyclic G-tree. Suppose H is a finitely generated subgroup of G whose minimal subtree T_H in T contains no translate of e. Then any Dehn twist about e restricts to a conjugation on H by an element which depends only on the connected component of $T \setminus G.e$ containing T_H.*

The following lemma describes Dehn twists with respect to Bass–Serre presentations.

Lemma 4.7. *Let G be a finitely generated group, and let T be a cyclic G-tree with a Bass–Serre presentation $(T^1, T^0, (t_f)_{f\in E_1})$. Let τ_e be a Dehn twist by an element a about an edge $e = uv$ of T^1. Then*

- *for each vertex x of T^0, the restriction of τ_e to G_x is a conjugation by an element g_x which is 1 if x and u are in the same connected component of $T^1 \setminus \{e\}$, and a otherwise;*
- *for any edge $f = xy'$ of T^1 with $x \in T^0$ and $t_f^{-1} \cdot y' = y \in T^0$, we have*

$$\tau_e(t_f) = \begin{cases} g_x t_f g_y^{-1} & \text{if } f \neq e, \\ a t_f & \text{if } f = e, \\ t_f a^{-1} & \text{if } f = \bar{e}. \end{cases}$$

Proof. The proof is straightforward, it suffices to note that the images of x, y and y' under the map p which collapses all the edges not in the orbit of e all belong to $\{p(u), p(v)\}$, and to consider the various possibilities. $\square$

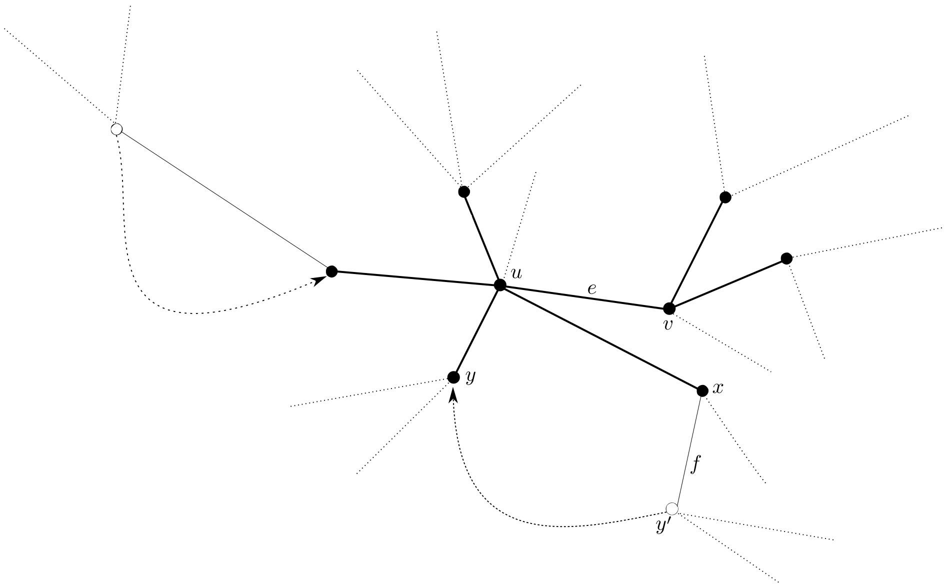


Fig. 2. A choice of T^1 and T^0 (thick subtree).

Remark 4.8. If τ is the Dehn twist by a about e, and τ' the Dehn twist by a^{-1} about $\bar{e}$, we have $\tau = \mathrm{Conj}(a) \circ \tau'$.

In particular, if $(T^1, T^0, (t_f)_f)$ is a Bass–Serre presentation for T such that e is in T^1, and if R is a connected component of $T^0 \setminus \{e\}$, then there exists an element g such that $\mathrm{Conj}(g) \circ \tau$ is a Dehn twist about e or $\bar{e}$ which restricts to the identity on G_x for any vertex x of R.

The next lemma gives a useful relation between Dehn twists about edges adjacent to a common cyclically stabilized vertex:

Lemma 4.9. *Let G be a finitely generated group, and let T be a G-tree. Suppose v is a vertex of T whose stabilizer is cyclic, and let $e_1 = u_1 v, \dots, e_r = u_r v$ be representatives of the orbits of edges adjacent to v. Let z be an element in the centralizer of* $\mathrm{Stab}(v)$*, and denote by τ_i the Dehn twist about e_i by z. Then*

$$\tau_1 \dots \tau_r = \mathrm{Conj}(z^{r-1}).$$

Proof. Choose a Bass–Serre presentation $(T^1, T^0, (t_f)_f)$ such that $v \in T^0$ and all the edges e_i are contained in T^1.

It is easy to see that both $\tau_1 \dots \tau_r$ and $\mathrm{Conj}(z^{r-1})$ restrict to the identity on $\mathrm{Stab}(v)$ and on $\langle z \rangle$. If w is a vertex of T^0 other than v, the Dehn twist τ_{e_i} restricts on $\mathrm{Stab}(w)$ to a conjugation by an element g_w^i which is 1 if w lies in the same connected component of $T^1 \setminus \{e_i\}$ as u_i, and z otherwise. Now the first alternative holds for exactly one value of i, thus $\tau_{e_1} \dots \tau_{e_r}$ restricts to a conjugation by z^{r-1} on $\mathrm{Stab}(w)$.

If $f = wx'$ is an edge of $T^1 \setminus T^0$ with w in T^0, note first that f is different from all the edges e_i (though we may have $f = \bar{e}_i$). By Lemma 4.7, if $x = t_f^{-1} \cdot x'$ we have

$$\tau_i(t_f) = \begin{cases} g_w^i t_f (g_x^i)^{-1} & \text{if } f \neq \bar{e}_i, \\ t_f z^{-1} & \text{if } f = \bar{e}_i. \end{cases}$$

If f is different from $\bar{e}_i$ for all values of i, we conclude as before by noting that g_w^i (respectively g_x^i) is z for all but one value of i. If $f = \bar{e}_i$, then $w = v$, $x' = u_i$ and x is not in the same connected component as u_i, so $g_w^j = z$ for all j, and $g_x^j = z$ for all but one value of j, and this value cannot be i. Thus in both cases, we get $\tau_1 \dots \tau_r(t_f) = z^{r-1} t_f z^{1-r}$. □

4.3. *Vertex automorphisms*

We want to extend automorphisms of stabilizers of vertices in a G-tree to automorphisms of G. For this we give

Definition 4.10. Let G be a finitely generated group acting on a tree T, and let v be a vertex in T. Denote by p the map collapsing all the orbits of edges of T except those of the edges adjacent to v.

An automorphism σ of G is called a *vertex automorphism* associated to v if $\sigma(G_v) = G_v$, and if for every edge $e = vw$ of $p(T)$ adjacent to v, it restricts to a conjugation by an element g_e on the stabilizer of e, as well as on the stabilizer of w if w is not in the orbit of v.

Remark 4.11. If v is a vertex in a G-tree T, and if σ_0 is an automorphism of $\mathrm{Stab}_G(v)$ which restricts to a conjugation by an element g_e on the stabilizer of each edge e adjacent to v, we can extend σ_0 to a vertex automorphism σ of G. For this, choose a Bass–Serre presentation $(T^1, T^0, (t_f)_f)$ for G with respect to $p(T)$ such that $p(v) \in T^0$, and such that the orbits of edges adjacent to $p(v)$ are represented in T^1 by edges adjacent to $p(v)$. We then define σ as follows:

- on $G_{p(v)}$, σ restricts to σ_0;
- for any vertex x of T^0 distinct from $p(v)$, σ restricts on G_x to conjugation by g_e where $e = p(v)x$;
- for $f = p(v)x'$ an edge of $T^1 \setminus T^0$ with $x = t_f^{-1} \cdot x'$ in T^0, we set

$$\sigma(t_f) = \begin{cases} g_f t_f g_e^{-1} \text{ where } e = p(v)x & \text{if } x \neq p(v), \\ g_f t_f g_{f'}^{-1} \text{ where } f' = t_f^{-1} f & \text{if } x = p(v). \end{cases}$$

Remark 4.12. If σ' is another vertex automorphism associated to v such that $\sigma|_{G_v} = \sigma'|_{G_v}$, then for any vertex w of $p(T)$ adjacent to $p(v)$ by an edge e, the restriction of σ to G_w is conjugation by an element g'_w such that $g_w^{-1} g'_w$ lies the centralizer of G_e.

It is therefore easy to deduce that $\sigma^{-1} \circ \sigma'$ is a product of Dehn twists about edges of $p(T)$.

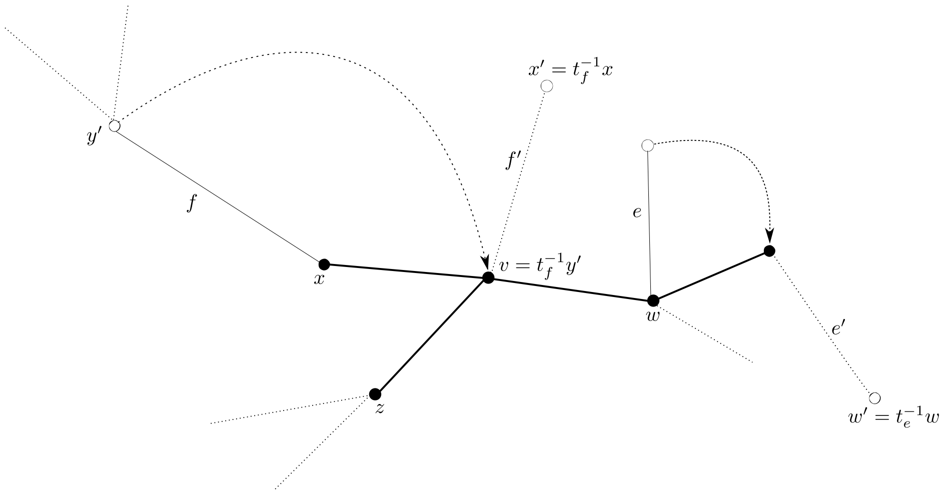


Fig. 3. A Bass–Serre presentation $(T^1, T^0, (t_f)_f)$ for the action of G on T together with the translates of the edges in $T^1 \setminus T^0$ by the stable letters.

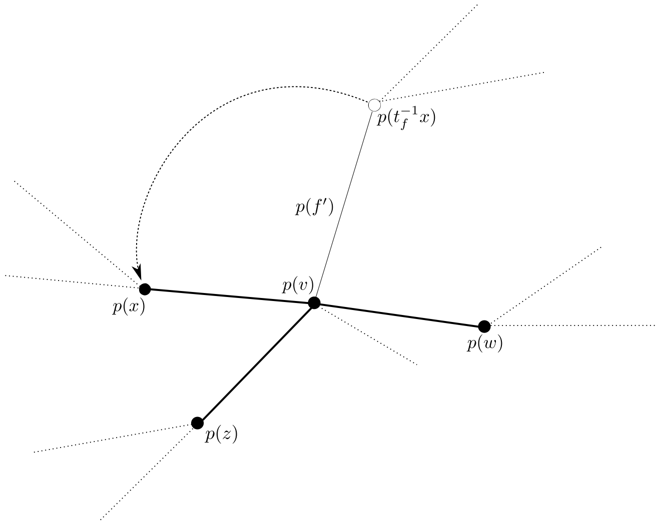


Fig. 4. A Bass–Serre presentation for the corresponding action of G on $p(T)$.

We now give an analogue of Lemma 4.6 for vertex automorphisms.

Lemma 4.13. *Let H be a finitely generated subgroup of G, and denote by T_H the minimal subtree of H in T. If no translate of T_H contains v, then any vertex automorphism σ_v associated to v restricts to a conjugation on H.*

Proof. The image of T_H by p is a vertex x of $p(T)$, thus σ_v restricts to a conjugation on $\mathrm{Stab}(x)$ which contains H. □

The following lemma describes vertex automorphisms with respect to Bass–Serre presentations.

Lemma 4.14. *Let G be a finitely generated group, and let T be a G-tree. Let $(T^1, T^0, (t_f)_{f\in E_1})$ be a Bass–Serre presentation for G with respect to T. Let σ be a vertex automorphism of G associated to a vertex v of T^0. For a vertex u or an edge e of $T \setminus G.v$, denote by $[u]$ (respectively $[e]$) the connected component of $T \setminus G.v$ containing u (respectively e). Then there exists an element g_R of G_v associated to each connected component R of $T \setminus G.v$ adjacent to v such that*

- *for each vertex u of $T^0 \setminus \{v\}$, the restriction of σ to G_u is conjugation by $g_{[u]}$;*
- *for any edge f of $T^1 \setminus T^0$ with $f' = t_f^{-1} \cdot f$, we have $\sigma(t_f) = g_{[f]} z t_f (g_{[f']})^{-1}$ for z in $C(\mathrm{Stab}(f))$.*

Proof. Let R be a connected component of $T \setminus G \cdot v$ adjacent to v. The collapse map p sends all the vertices in R to the same vertex x_R of $p(T)$ adjacent to $p(v)$ and different from $p(v)$, so their stabilizers are contained in $\mathrm{Stab}(x_R)$, on which σ restricts to conjugation by an element g_R of G_v by definition. If R does not contain any vertices, it is reduced to a single edge e adjacent to v and we let g_R be such that σ restricts to a conjugation by g_R on the stabilizer of e.

For the second point, note that σ restricts to conjugation by $g_{[f]}$ on $\mathrm{Stab}(f)$, and to conjugation by $g_{[f']}$ on $\mathrm{Stab}(f') = t_f^{-1}\,\mathrm{Stab}(f) t_f$. Hence for any $h \in \mathrm{Stab}(f)$ we have

$$g_{[f']} t_f^{-1}\, h\, t_f g_{[f']}^{-1} = \sigma(t_f^{-1} h t_f) = \sigma(t_f)^{-1} g_{[f]}\, h\, g_{[f']}^{-1} \sigma(t_f).$$

This implies that $\sigma(t_f) = g_{[f]} z t_f g_{[f']}^{-1}$ for some z in the centralizer of $\mathrm{Stab}(f)$. □

Remark 4.15. Let σ be a vertex automorphism with support $v \in T$, and let R be a connected component of $T \setminus G.v$ adjacent to v; for any vertex x of R, σ restricts on G_x to conjugation by an element g_R of $\mathrm{Stab}(v)$. Then $\mathrm{Conj}(g_R^{-1}) \circ \sigma$ is a vertex automorphism with support v, and it restricts to the identity on G_x for any vertex x of R.

4.4. Elementary automorphisms

Definition 4.16. Let T be a G-tree. If ρ is a Dehn twist about an edge e of T, or a vertex automorphism associated to a vertex v of T, we say it is an *elementary automorphism* associated to T. We call the edge e (respectively the vertex v) the *support* of ρ and denote it $\mathrm{Supp}(\rho)$.

Lemma 4.17. *Suppose ρ is an elementary automorphism associated to a G-tree T. Then for any $g \in G$, $\mathrm{Conj}(g\rho(g^{-1})) \circ \rho$ is an elementary automorphism of $g \cdot \mathrm{Supp}(\rho)$.*

Proof. Denote by ρ' the automorphism $\mathrm{Conj}(g) \circ \rho \circ \mathrm{Conj}(g^{-1}) = \mathrm{Conj}(g\rho(g^{-1})) \circ \rho$. It is easy to check (for example using Lemmas 4.7 and 4.14) that if ρ is the Dehn twist about an edge e by $z \in C(\mathrm{Stab}(e))$, then ρ' is the Dehn twist about $g \cdot e$ by gzg^{-1}, and if ρ is a vertex automorphism associated to v, then ρ' is a vertex automorphism associated to $g \cdot v$ which restricts to $\mathrm{Conj}(g) \circ \rho|_{G_v} \circ \mathrm{Conj}(g^{-1})$ on $G_{g\cdot v}$. □

Under some conditions on G and T, elementary automorphisms commute up to conjugation.

Proposition 4.18. *Let T be a cyclic G-tree whose edge stabilizers have cyclic centralizers. Let ρ and σ be two elementary automorphisms associated to T with supports in distinct orbits. Then there exists $g \in G$ such that*

$$\rho \circ \sigma = \mathrm{Conj}(g) \circ \sigma \circ \rho.$$

Proof. Choose a Bass–Serre presentation $(T^1, T^0, (t_e)_e)$ for G with respect to T. By Lemma 4.17, we may assume that both ρ and σ have support in T^1.

Let $R^0, \dots, R^m$ denote the connected components of $T^1 \setminus \{\mathrm{Supp}(\rho)\}$, and $S^0, \dots, S^n$ the connected components of $T^1 \setminus \{\mathrm{Supp}(\sigma)\}$, and assume without loss of generality that $\mathrm{Supp}(\sigma)$ lies in R^0 and $\mathrm{Supp}(\rho)$ lies in S^0. Note that $S^1, \dots, S^n$ are contained in R^0, and $R^1, \dots, R^m$ are contained in S^0.

By Lemmas 4.7 and 4.14, there are elements g_j (respectively h_k) such that ρ (respectively σ) restricts to conjugation by g_j (respectively h_k) on the stabilizer G_u of each vertex u which lies in R^j (respectively S^k) and is not in the orbit of the support of ρ (respectively σ). Also, if f is an edge of T^1 which lies in R^j (respectively S^k), then ρ (respectively σ) restricts to conjugation by g_j (respectively h_k) on the stabilizer G_f of f.

Moreover, we claim that $\sigma(g_j) = h_0 g_j h_0^{-1}$ and $\rho(h_k) = g_0 h_k g_0^{-1}$. If ρ is a vertex automorphism associated to a vertex v, Lemma 4.14 shows that g_j is an element of G_v on which σ restricts to conjugation by h_0. If ρ is a Dehn twist about $e = \mathrm{Supp}(\rho)$, the element g_j is in $C(G_e)$; since $C(G_e)$ is cyclic by hypothesis, and since σ restricts to conjugation by h_0 on G_e, it must also send g_j to $h_0 g_j h_0^{-1}$.

Let u be a vertex of T^0 which lies in $R^0 \cap S^k$. On G_u, we see that $\rho \circ \sigma$ restricts to conjugation by $\rho(h_k) g_0 = g_0 h_k$. Similarly $\sigma \circ \rho$ restricts to conjugation by $\sigma(g_0) h_k = h_0 g_0 h_0^{-1} h_k$. Thus $\rho \circ \sigma$ restricts to $\mathrm{Conj}(g_0 h_0 g_0^{-1} h_0^{-1}) \circ \sigma \circ \rho$ on G_u. The case of u in $R^j \cap S^0$ is symmetric.

If u is the support of one of the two elementary automorphisms, without loss of generality σ, the restriction of $\sigma \circ \rho$ on G_u is $\mathrm{Conj}(\sigma(g_0)) \circ \sigma|_{G_u}$, while the restriction of $\rho \circ \sigma$ on G_u is $\mathrm{Conj}(g_0) \circ \sigma|_{G_u}$. Thus for any vertex u of T^0, $\rho \circ \sigma$ restricts to $\mathrm{Conj}(g_0 h_0 g_0^{-1} h_0^{-1}) \circ \sigma \circ \rho$ on G_u.

Let now $e = uv'$ be an edge of $T^1 \setminus T^0$ such that v' lies in T^1 but not in T^0, and $v = t_e^{-1} \cdot v'$ is in T^0. Suppose u lies in $R^0 \cap S^k$ and v in $R^j \cap S^0$. By Lemmas 4.7 and 4.14, we know that $\rho(t_e) = g_0 z t_e g_j^{-1}$ and $\sigma(t_e) = h_k w t_e h_0^{-1}$ for some z, w in $C(\mathrm{Stab}(f))$. Note that ρ and σ restrict on $C(\mathrm{Stab}(f))$ to conjugation by g_0 and h_k respectively. From this we see that $\rho \circ \sigma(t_e) = \mathrm{Conj}(g_0 h_0 g_0^{-1} h_0^{-1}) \circ \sigma \circ \rho(t_e)$.

The remaining cases (when both u and v lie in $R^0 \cap S^k$, and where one or both of u, v coincide with the support of ρ or σ) are dealt with in a similar way. □

4.5. Modular groups

Let G be a torsion-free hyperbolic group which is freely indecomposable with respect to a subgroup H. As in [PS12], we define the relative modular group $\mathrm{Mod}_H(G)$ as the subgroup of $\mathrm{Aut}_H(G)$ generated by Dehn twists about one-edge cyclic splittings of G in which H is elliptic.

Recall that we have the following result [RS94, Corollary 4.4]:

Theorem 4.19. *Let G be a torsion-free hyperbolic group freely indecomposable with respect to a (possibly trivial) subgroup H. Then the modular group $\mathrm{Mod}_H(G)$ has finite index in $\mathrm{Aut}_H(G)$.*

We now relate cyclic JSJ decompositions and modular groups. This will enable us to give a "normal form" for modular automorphisms.

First we define a group of automorphisms associated to a G-tree.

Definition 4.20. Let G be a finitely generated group, H a subgroup of G, and T a cyclic (G, H)-tree with a distinguished set of orbits of vertices which are of surface type.

The *group of elementary automorphisms of G with respect to T*, $\mathrm{Aut}_H^T(G)$, is the subgroup of $\mathrm{Aut}_H(G)$ generated by the Dehn twists about edges of T, the vertex automorphisms associated to surface type vertices, and the inner automorphisms.

Lemma 4.21 (Normal Form Lemma). *Let T be a cyclic (G, H)-tree whose edge stabilizers have cyclic centralizers. Let $(T^1, T^0, (t_f)_f)$ be a Bass–Serre presentation for G with respect to T. Then any element θ of $\mathrm{Aut}_H^T(G)$ can be written as a product of the form*

$$\mathrm{Conj}(z) \circ \rho_1 \circ \cdots \circ \rho_r$$

where the ρ_j are Dehn twists about distinct edges of T^1 or vertex automorphisms associated to distinct surface type vertices of T^0. Moreover, we can permute the list of supports of the ρ_j. Finally, if H fixes a non-surface type vertex x of T^0, we can in fact choose the ρ_j to fix $\mathrm{Stab}_G(x)$ pointwise, and thus z to lie in the centralizer of H.

Proof. This follows easily from Lemma 4.17 and Proposition 4.18. The last statement follows from Remarks 4.8 and 4.15. ◻

The universal properties of the JSJ imply the following result, which can be seen as a special case of [GL15, Theorem 5.4].

Proposition 4.22. *Let G be a torsion-free hyperbolic group and let H be a subgroup of G with respect to which G is freely indecomposable. Let T be the pointed cyclic JSJ tree of G with respect to H. Then $\mathrm{Aut}_H^T(G) = \mathrm{Mod}_H(G)$.*

To prove it, we will use the following lemmas, which relate elementary automorphisms associated to G-trees $\hat{T}$ and T when $\hat{T}$ is a refinement of T. The proof of the first of these results is immediate.

Lemma 4.23. *Let $\hat{T}$ and T be two G-trees and suppose $p : \hat{T} \to T$ is a collapse map. Let τ be a Dehn twist by an element a about an edge e of T. Then τ is the Dehn twist by a about the unique edge $\hat{e}$ such that $p(\hat{e}) = e$.*

Lemma 4.24. *Let $\hat{T}$ and T be two G-trees and suppose $p : \hat{T} \to T$ is a collapse map. Let $\hat{\tau}$ be a Dehn twist by an element a about an edge $\hat{e}$ of $\hat{T}$. If $p(\hat{e})$ is an edge, then $\hat{\tau}$ is a Dehn twist by a about $p(\hat{e})$. If $p(\hat{e})$ is a vertex v and $a \in \mathrm{Stab}(\hat{e})$, then $\hat{\tau}$ is a vertex automorphism associated to v. Its restriction to G_v is the Dehn twist by a about the edge $\hat{e}$ of the minimal subtree $\hat{T}_v$ of $\mathrm{Stab}(v)$ in $\hat{T}$.*

Proof. If $p(\hat{e})$ is an edge e, it is easy to see that the one-edge splittings induced by $\hat{e}$ and by $p(\hat{e})$ are the same, which proves the claim. Suppose now that $p(\hat{e})$ is a vertex v.

We choose a Bass–Serre presentation $(\hat{T}^1, \hat{T}^0, (t_f)_f)$ for G with respect to $\hat{T}$ such that

- $\hat{e}$ is in $\hat{T}^1$;
- if $T^i = p(\hat{T}^i)$ for $i = 0, 1$, the triple $(T^1, T^0, (t_f)_f)$ is a Bass–Serre presentation for G with respect to T

(this can be done by taking for T^0 the lift of a maximal subtree of $\hat{T}/G$ which contains a maximal subtree of each of the maximal subgraphs collapsed under p).

The minimal subtree $\hat{T}_v$ of G_v in $\hat{T}$ is contained in the preimage of v by p. In particular, any translate of $\hat{T}_v$ by an element of $G \setminus G_v$ is disjoint from $\hat{T}_v$, so two vertices (respectively two edges) of $\hat{T}_v$ are in the same orbit under G_v if and only if they are in the same orbit under the action of G. By our choice of Bass–Serre presentation, $(\hat{T}^1 \cap \hat{T}_v, \hat{T}^0 \cap \hat{T}_v, (t_f)_{f \in E((\hat{T}^1 \setminus \hat{T}^0) \cap \hat{T}_v)})$ is a Bass–Serre presentation for G_v with respect to $\hat{T}_v$.

From this it is easy to check that the restriction of $\hat{\tau}$ to G_v is the Dehn twist by a about $\hat{e}$ with respect to the action of G_v on $\hat{T}_v$.

Consider now the map $\pi : T \to \pi(T)$ which collapses all the orbits of edges of T which are not adjacent to v. Let $f = \pi(v)w$ be an edge adjacent to $\pi(v)$. There is a unique edge $\hat{f}$ of T such that $\pi \circ p(\hat{f}) = f$. Then $\hat{\tau}$ restricts on $\mathrm{Stab}(\hat{f})$ to a conjugation by an element which is either 1 or a; both fix $\pi(v)$. If w is not in the orbit of $\pi(v)$, the subtree $\pi^{-1}(w)$ of T is stabilized by G_w and does not meet any translates of e. By Lemma 4.6, $\hat{\tau}$ restricts on G_w to a conjugation by 1 or by a, which both fix $\pi(v)$. Thus $\hat{\tau}$ is a vertex automorphism with respect to $\pi(v)$. □

Lemma 4.25. *Let G be a torsion-free hyperbolic group which is freely indecomposable with respect to a subgroup H. Suppose T is a cyclic (G, H)-tree with a distinguished set of orbits of vertices which are of surface type. Then $\mathrm{Aut}_H^T(G)$ is a subgroup of $\mathrm{Mod}_H(G)$.*

Proof. Suppose τ is a Dehn twist about an edge e of T which fixes H pointwise. By definition, τ is the Dehn twist by a associated to the one-edge splitting obtained from T by collapsing all the edges which are not in the orbit of e, thus it lies in $\mathrm{Mod}_H(G)$.

Now let σ be a vertex type automorphism associated to a surface type vertex v of T, with corresponding surface Σ. It is a classical result that the group of automorphisms of the fundamental group of a surface is generated by the Dehn twists δ_c by elements c corresponding to simple closed curves γ on the surface. Thus it is enough to show the result for a vertex type automorphism σ which restrict to a Dehn twist δ_c on G_v.

Denote by T^+ the refinement of T obtained by refining v by the G_v-tree dual to the curve γ on Σ, and let e^+ be the edge of T^+ stabilized by c. By Lemma 4.24, the Dehn twist τ^+ by c about e^+ is an elementary automorphism associated to T with support v, whose restriction to G_v is exactly δ_c. By Remark 4.12, σ and τ^+ differ by a product of Dehn twists about edges of T. These Dehn twists as well as τ^+ are all elements of $\mathrm{Mod}_H(G)$ by the first part of the proof, thus so is σ. □

Proof of Proposition 4.22. By Lemma 4.25, we have $\mathrm{Aut}_H^T(G) \leq \mathrm{Mod}_H(G)$.

Conversely, let T' be a cyclic G-tree with a unique orbit of edges in which H is elliptic. Let a be an element in the centralizer of the stabilizer of some edge e, and denote by τ the Dehn twist about e by a.

As noted in Remark 4.4, T is universally compatible, so it admits a refinement $\hat{T}$ which collapses onto T' via some map $p : \hat{T} \to T'$. By Lemma 4.23, τ is a Dehn twist about an edge $\hat{e}$ of $\hat{T}$. Note that if $p(\hat{e})$ is a vertex, then it must be a surface type vertex, so in particular the stabilizer of $\hat{e}$ is maximal cyclic. By Lemma 4.24, this Dehn twist is an elementary automorphism associated to T. Hence τ is in $\mathrm{Aut}_H^T(G)$. □

Lemma 4.26. *Let $A \leq H$ be subgroups of a torsion-free hyperbolic group G which is freely indecomposable with respect to A. Let T be a cyclic (G, A)-tree with a distinguished set of orbits of vertices which are of surface type. Suppose ρ is an elementary automorphism associated to T whose support does not lie in any translate of the minimal subtree T_H of H in T. Then there exist $g \in G$ and $\sigma \in \mathrm{Mod}_H(G)$ such that $\rho = \mathrm{Conj}(g) \circ \sigma$.*

Proof. Consider the tree T' obtained from T by collapsing each subtree in the orbit of T_H. It is a (G, H)-tree with a distinguished set of orbits of vertices which are of surface type (inherited from T) and ρ is an elementary automorphism associated to T'. By Remarks 4.8 and 4.15 there is $g \in G$ and an elementary automorphism σ associated to T' such that $\rho = \mathrm{Conj}(g) \circ \sigma$ and σ fixes H pointwise. By Lemma 4.25, $\sigma \in \mathrm{Mod}_H(G)$. □

The following result can be seen as a generalization of Lemma 4.21.

Proposition 4.27. *Let G be a torsion-free hyperbolic group, freely indecomposable with respect to a subgroup A. Let T be the pointed JSJ tree of G with respect to A. Let H be a finitely generated subgroup of G containing A, and denote by T_H the minimal subtree of H in T. If θ is an element of $\mathrm{Mod}_H(G)$, it can be written as*

$$\theta = \mathrm{Conj}(z) \circ \tau_{e_1} \circ \cdots \circ \tau_{e_p} \circ \sigma_{v_1} \circ \cdots \circ \sigma_{v_q}$$

where each τ_{e_i} is a Dehn twist about an edge e_i of T and each σ_{v_j} is a vertex automorphism σ_{v_j} associated with a flexible vertex v_j of T such that:

- *the edge e_i does not lie in any translate of T_H for any i,*
- *if v_j lies in some translate of T_H then the restriction of σ_{v_j} to the corresponding surface group fixes an element representing a non-boundary parallel simple closed curve.*

Proof. By definition of the modular group, and by Proposition 4.18, it is enough to prove the result when $\theta = \tau$ is the Dehn twist by some element a about an edge e of a cyclic (G, H)-tree T' which has a unique orbit of edges. Since G is torsion-free hyperbolic, we may further assume that the stabilizer of e is maximal cyclic so that the tree T' is 1-acylindrical.

Note that T' is in particular a cyclic (G, A)-tree; by universal compatibility of the pointed JSJ tree, T admits a refinement $\hat{T}$ which collapses onto T'. We thus have collapse maps $p : \hat{T} \to T$ and $p' : \hat{T} \to T'$. The tree $\hat{T}$ is obtained by refining each surface type vertex v by the minimal subtree T_v in T' of its stabilizer G_v.

Let x be a vertex of T' stabilized by H. Let $\hat{x}$ be a vertex of $\hat{T}$ such that $p'(\hat{x}) = x$. Denote by $\hat{T}_H$ the minimal subtree of H in $\hat{T}$; it is covered by translates of paths $[\hat{x}, h \cdot \hat{x}]$. We have $p'(h \cdot \hat{x}) = x$ for all $h \in H$, so $p'(\hat{T}_H) = \{x\}$.

By Lemma 4.23, τ is a Dehn twist by a about an edge $\hat{e}$ such that $p'(\hat{e}) = e$. Note that $\hat{e}$ does not lie in $\hat{T}_H$ since the image of $\hat{T}_H$ by p is a single vertex.

By Lemma 4.24, if $p(\hat{e})$ is an edge, τ is a Dehn twist about $p(\hat{e})$. Also, if $p(\hat{e})$ is an edge, it lies outside of $p(\hat{T}_H)$ which contains T_H, so it lies outside of T_H.

If $p(\hat{e})$ is a vertex, it must be a surface type vertex, and $\mathrm{Stab}(e)$ is generated by an element corresponding to a simple closed curve on the corresponding surface; in particular it is maximal cyclic and a is in $\mathrm{Stab}(e)$. Thus by Lemma 4.24, τ is a vertex automorphism associated to the vertex $p(\hat{e})$ of T. Moreover, the restriction of τ to $\mathrm{Stab}(v)$ is a Dehn twist about an edge of the minimal tree of $\mathrm{Stab}(v)$ in $\hat{T}$ which is dual to a set of non-boundary parallel simple closed curves on the corresponding surface; this finishes the proof. □

5. Isolating types

In this section, we show that if G is a torsion-free hyperbolic group which is freely indecomposable with respect to A and does not admit the structure of an extended hyperbolic tower over A, then the orbits of tuples of elements of G under $\mathrm{Aut}_A(G)$ are definable over A (equivalently, G is atomic over A).

For the notion of an extended hyperbolic floor we refer the reader to [LPS11]. We give the following definition:

Definition 5.1. Let G be a torsion-free hyperbolic group and let $A \subset G$. Then G is *concrete* with respect to A if:

(i) G is freely indecomposable with respect to A;
(ii) G does not admit the structure of an extended hyperbolic floor over A.

Lemma 5.19 of [Per11] shows that if $G = \mathbb{F}$ is a free group and A is not contained in a free factor, then $\mathbb{F}$ is concrete with respect to A.

Here are some further examples where G is concrete with respect to A:

Example 5.2. (i) G is the fundamental group of the connected sum of four projective planes, and A is any set of parameters that contains a non-cyclic subgroup (see [LPS11, Lemma 3.12]).

(ii) $G := \langle a, b, c, d \mid [a, b] = [c, d] \rangle$ is the fundamental group of the connected sum of two tori, and A is any set of parameters that contains a subgroup of the form $\langle a, b, g \rangle$ with $g \notin \langle a, b \rangle$ (see the proof of [LPS11, Lemma 6.1]).

The proof of the result below is essentially contained in [PS12], but we include it here for reference. For the special case of free groups see also [O11, Proposition 5.9].

Theorem 5.3. *Let G be a torsion-free hyperbolic group. Suppose G is concrete with respect to A. Then for any $\bar{b} \in G$, the orbit of $\bar{b}$ under* $\mathrm{Aut}_A(G)$ *is definable over A, so in particular* $\mathrm{tp}^G(\bar{b}/A)$ *is isolated.*

Proof. By Lemma 3.7 in [PS12], there is a finite subset A_0 of A such that G is freely indecomposable with respect to A_0, A is elliptic in any G-tree in which A_0 is elliptic, and $\mathrm{Mod}_{A_0}(G) = \mathrm{Mod}_A(G)$. By [PS12, Corollary 4.5], we can assume moreover that any embedding $j : G \to G$ which restricts to the identity on A_0 restricts to the identity on A.

By [PS12, Theorem 4.4], there is a finite set $\{\eta_j : G \to Q_j\}_{j=1}^m\}$ of quotient maps such that any non-injective endomorphism $\theta : G \to G$ which restricts to the identity on A_0 factors through one of the maps η_j after precomposition with an element σ of $\mathrm{Mod}_A(G)$. For each j, choose a non-trivial element u_j in $\mathrm{Ker}(\eta_j)$.

Let $\gamma_1, \dots, \gamma_r$ be a generating set for G. Write each element a of A_0, each element u_j, and the tuple $\bar{b}$ as a word $w_a(\gamma_1, \dots, \gamma_r)$ (respectively $w_{u_j}(\gamma_1, \dots, \gamma_r)$ and a tuple $\bar{w}_{\bar{b}}(\gamma_1, \dots, \gamma_r)$).

Let Λ be a JSJ decomposition of G with respect to A. Two endomorphisms h and h' of $\mathbb{F}$ are said to be Λ-*related* if h and h' coincide up to conjugation on rigid vertex groups of Λ, and for any flexible vertex group S of Λ, $h(S)$ is non-abelian if and only if $h'(S)$ is non-abelian. It is easy to see that there is a formula $\mathrm{Rel}(\bar{x}, \bar{y})$ such that for any pair of endomorphisms h and h' of G, the morphism h' is Λ-related to h if and only if $G \models \mathrm{Rel}(h(\gamma_1, \dots, \gamma_r), h'(\gamma_1, \dots, \gamma_r))$ (see [Per08, Lemma 5.18]).

Consider now the following formula $\phi(\bar{z}, A_0)$:

$$\exists x_1, \dots, x_r \left\{ \bar{z} = \bar{w}_{\bar{b}}(x_1, \dots, x_r) \wedge \bigwedge_{a \in A_0} a = w_a(x_1, \dots, x_r) \right\}$$
$$\wedge \forall y_1, \dots, y_r \left\{ \mathrm{Rel}(\bar{x}, \bar{y}) \to \bigvee_j w_{u_j}(y_1, \dots, y_r) \neq 1 \right\}.$$

Suppose $G \models \phi(\bar{c}, A_0)$. Then the endomorphism $h : G \to G$ given by $\gamma_j \mapsto x_j$ sends $\bar{b}$ to $\bar{c}$ and fixes A_0, moreover no endomorphism h' which is Λ-related to h factors through one of the maps η_i. This implies that h is injective; but by the relative co-Hopf property for torsion-free hyperbolic groups (see [PS12, Corollary 4.2]), this in turn implies that h is an automorphism fixing A_0. By our choice of A_0, in fact $h \in \mathrm{Aut}_A(G)$. Thus the set defined by $\phi(\bar{z}, A_0)$ is contained in the orbit of $\bar{b}$ under $\mathrm{Aut}_A(G)$.

To finish the proof it is enough to show that $G \models \phi(\bar{b}, A_0)$.

It is obvious that the first part of the sentence $\phi(\bar{b})$ is satisfied by G (just take x_j to be γ_j). If the second part is not satisfied, this means that there exists an endomorphism $h' : G \to G$ which is Λ-related to the identity, and which kills one of the elements u_j. Thus h' restricts to conjugation on the rigid vertex groups of Λ, sends surface type flexible vertex groups to non-abelian images, and is non-injective; it is a non-injective preretraction $G \to G$ with respect to Λ. By [Per11, Proposition 5.11], this implies that G admits a structure of a hyperbolic floor over A_0, thus over A, a contradiction. □

We further remark that if $A \subset G$ is not contained in any proper retract of G, then the isolating formula can be taken to be Diophantine, i.e. $\exists \bar{y}\ (\Sigma(\bar{x}, \bar{y}, \bar{a}) = 1)$. This follows easily from a recent result of Groves [Gro12]:

Theorem 5.4. *Let G be a torsion-free hyperbolic group. Suppose A is not contained in any proper retract of G. Then any endomorphism of G that fixes A is an automorphism.*

6. Algebraic closures

As Lemma 2.12 shows, the notion of forking independence is preserved under taking algebraic closures of the triple under consideration. For instance, in order to prove that two tuples $\bar{b}$, $\bar{c}$ fork over a set A of parameters, it is enough (by transitivity of forking and the above mentioned lemma) to show that some elements b' and c' in the respective algebraic closures $\mathrm{acl}(A\bar{b})$ and $\mathrm{acl}(A\bar{c})$ fork over A. Thus, it will be useful to understand $\mathrm{acl}(A)$ for A a subset of a torsion-free hyperbolic group G.

It is not hard to see that if the subgroup generated by A is cyclic, the algebraic closure of A is the maximal cyclic subgroup containing A (see [OV11, Lemma 3.1]).

If G is concrete with respect to A, we can use the results of the previous section to get

Proposition 6.1. *Let G be a torsion-free hyperbolic group which is concrete with respect to a subgroup A. Let (T, v_A) be the pointed cyclic JSJ tree of G with respect to A. Then* $\mathrm{Stab}(v_A)$ *is contained in the algebraic closure of A in G.*

Proof. Let $\bar{b}$ be a tuple in $\mathrm{Stab}(v_A)$. By Proposition 4.22, $\mathrm{Mod}_A(G)$ is generated by elementary automorphisms associated to (T, v_A) which fix the vertex group $\mathrm{Stab}(v_A)$ pointwise, thus $\bar{b}$ is fixed by $\mathrm{Mod}_A(G)$.

Since $\mathrm{Mod}_A(G)$ has finite index in $\mathrm{Aut}_A(G)$, the orbit of $\bar{b}$ under $\mathrm{Aut}_A(G)$ is finite. But by Theorem 5.3, this orbit is definable over A, thus $\bar{b}$ is in $\mathrm{acl}_A(G)$ □

The converse to this result does not hold: there could be some roots of elements of $\mathrm{Stab}(v_A)$ which are not in $\mathrm{Stab}(v_A)$, yet in torsion-free hyperbolic groups, algebraic closures are closed under taking roots. But this is the only obstruction: this was proved by Ould Houcine and Vallino in the case of free groups [OV11], and extends easily to the case considered here.

We continue with an easy corollary of the above proposition.

Corollary 6.2. *Let G be a torsion-free hyperbolic group which is concrete with respect to a subgroup A. Then there is a finitely generated subgroup A_0 of A such that $A \subseteq \mathrm{acl}_G(A_0)$. In particular $\mathrm{acl}_G(A) = \mathrm{acl}_G(A_0)$.*

Proof. Take A_0 to be the finitely generated subgroup of A given by [PS12, Proposition 3.7]. It is not hard to see that G is concrete with respect to A_0. Let (T, v_{A_0}) be the pointed cyclic JSJ tree of G with respect to A_0. By Proposition 6.1 we have $\mathrm{Stab}(v_{A_0}) \subseteq \mathrm{acl}_G(A_0)$; but A fixes v_{A_0}, thus we get the result. □

Proposition 6.3. *Let G be a torsion-free hyperbolic group concrete with respect to a subgroup A, and let H be a finitely generated non-abelian subgroup of G which contains A. Let (T, v_A) and (T', v_H) be the pointed cyclic JSJ trees of G relative to A and H respectively. Denote by T_H the minimal subtree of H in T.*

- *If U is the non-cyclic stabilizer of a rigid vertex of T_H, then $U \subseteq \mathrm{Stab}(v_H)$.*
- *If S is the stabilizer of a flexible vertex of T_H with corresponding surface Σ, then there is an element γ of S corresponding to a non-boundary parallel simple closed curve on Σ which is contained in $\mathrm{Stab}(v_H)$.*

In particular U and γ are fixed by $\mathrm{Mod}_H(G)$, and contained in $\mathrm{acl}_G(H)$.

Proof. The tree T' is a cyclic tree in which A is elliptic, thus the JSJ tree T admits a refinement $\hat{T}$ which collapses onto T'. We have collapse maps $p : \hat{T} \to T$ and $p' : \hat{T} \to T'$.

Denote by $\hat{T}_H$ the minimal subtree of H in $\hat{T}$; we have $p(\hat{T}_H) = T_H$ and $p'(\hat{T}_H) = \{v_H\}$. Let now x be a vertex of T_H.

If x is a rigid vertex of T with non-cyclic stabilizer U, then $p^{-1}(x)$ is reduced to a point $\hat{x}$ which must lie in $\hat{T}_H$, and also has stabilizer U. Now $p'(\hat{x}) = v_H$ so U lies in $\mathrm{Stab}(v_H) \subseteq \mathrm{acl}_G(H)$.

If x is a surface type vertex of T, the action of $\mathrm{Stab}(x)$ on $p^{-1}(x)$ is dual to a set of disjoint simple closed curves on Σ, so the stabilizer of any vertex $\hat{x}$ in $p^{-1}(x) \cap \hat{T}_H$ corresponds to a subsurface of Σ which is not an annulus parallel to the boundary. In particular, $\mathrm{Stab}(\hat{x})$ contains an element γ corresponding to a non-boundary parallel simple closed curve on Σ. Again $p'(\hat{x}) = v_H$, so $\mathrm{Stab}(\hat{x})$ lies in $\mathrm{acl}_G(H)$. In particular γ is in $\mathrm{Stab}(v_H) \subseteq \mathrm{acl}_G(H)$. □

We finish this section with a result which generalizes Proposition 6.3:

Proposition 6.4. *Let G be a torsion-free hyperbolic group concrete with respect to a subgroup A, and let H be a finitely generated non-abelian subgroup of G which contains A. Let (T, v_A) be the pointed cyclic JSJ tree of G relative to A. Denote by T_H the minimal subtree of H in T. Let v be a vertex of T such that the path from T_H to v consists of edges which all lie in translates of T_H, and does not contain any surface type vertices. Then* $\mathrm{Stab}(v) \subseteq \mathrm{acl}_G(H)$.

Proof. Let (T', v_H) be the pointed cyclic JSJ tree of G relative to H.

The JSJ tree T admits a refinement $\hat{T}$ which collapses onto T'. We have collapse maps $p : \hat{T} \to T$ and $p' : \hat{T} \to T'$. Any non-surface type vertex y of T (respectively any edge e of T) has as preimage by p a single vertex (respectively a unique edge), which we denote by $\hat{y}$ (respectively $\hat{e}$). Moreover, $\mathrm{Stab}(\hat{y}) = \mathrm{Stab}(y)$.

The hypotheses on v imply that the path between $\hat{T}_H$ and $\hat{v}$ consists exactly of the lifts $\hat{e}$ of the edges e of the path $[u, v]$ between T_H and v.

Now each $\hat{e}$ lies in a translate of the path $[\hat{v}_A, h \cdot \hat{v}_A]$ for some $h \in H$, and this path is collapsed under the map p'. Thus all the edges $\hat{e}$ are collapsed under p' so $p'(\hat{v}) = p'(\hat{u})$.

Finally, as in the proof of Proposition 6.3, we can see that for any vertex y of T_H which is of non-surface type we have $p'(\hat{y}) = v_H$. Thus $p'(\hat{u}) = v_H$, and $\mathrm{Stab}(v) = \mathrm{Stab}(\hat{v})$ is contained in $\mathrm{Stab}(v_H) \subseteq \mathrm{acl}_G(H)$. □

7. The curve complex

In this section we give some basic definitions and results about the curve complex assigned to a surface, introduced by Harvey [Har81]. These will be useful for the proof of Theorem 8.2.

Definition 7.1. Let Σ be a surface with (possibly empty) boundary. Then the *curve complex* $\mathcal{C}(\Sigma)$ is the simplicial complex defined as follows:

(i) 0-simplices are simple closed curves (up to free homotopy) on Σ which do not bound a disk, an annulus, or a Möbius band,
(ii) A subset $\{\gamma_0, \dots, \gamma_k\}$ of the set of 0-simplices forms a k-simplex if the curves in the subset can be realized disjointly.

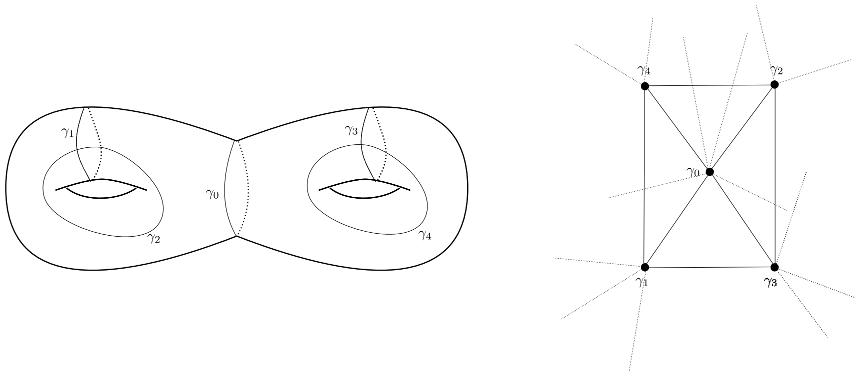


Fig. 5. Part of the curve complex of the orientable surface of genus 2.

Remark 7.2. Let $\Sigma_{g,n}$ be the orientable surface of genus g with n boundary components. In the following sporadic cases Definition 7.1 gives a degenerate discrete set or even the empty set:

- $\Sigma_{0,n}$ for $n \leq 4$;
- $\Sigma_{1,n}$ for $n \leq 1$;
- the n-punctured projective plane for $n \leq 2$;
- the n-punctured Klein bottle with $n \leq 1$.

If Σ is the once-punctured torus $\Sigma_{1,1}$ or the four-punctured sphere $\Sigma_{0,4}$ we modify the second part of the definition: a subset of the set of 0-simplices forms a simplex if the corresponding curves can be realized with intersection number at most 1 (respectively 2 in the case of $\Sigma_{0,4}$). Note that in both cases the resulting simplicial complex is the well known Farey graph.

Combining [MM99, Theorem 1.1] and the results in [BF07, appendix] we get

Theorem 7.3. *Let Σ be a surface which either is a punctured torus or has Euler characteristic at most -2. Then $\mathcal{C}(\Sigma)$ has infinite diameter.*

The mapping class group $\mathcal{MCG}(\Sigma)$ of the surface Σ, that is, the group of isotopy classes of self-homeomorphisms of Σ (fixing each boundary component pointwise), acts on the curve complex of Σ in the obvious way. We observe the following:

Lemma 7.4. *Let Σ be a surface which either is a punctured torus or has Euler characteristic at most -2. Let $R \geq 0$, and let x be a vertex of $\mathcal{C}(\Sigma)$. Then there exists a sequence $(h_n)_{n<\omega}$ of elements of $\mathcal{MCG}(\Sigma)$ such that the translates $h_n(B_R(x))$ of the ball of radius R around x are pairwise disjoint.*

Proof. It is immediate that there are only a finite number of orbits of vertices in $\mathcal{C}(\Sigma)$ under the action of $\mathcal{MCG}(\Sigma)$. Let M be such that any ball of radius M in $\mathcal{C}(\Sigma)$ meets each of these orbits.

Since $\mathcal{C}(\Sigma)$ has infinite diameter, we can find a sequence y_n of vertices such that $d(y_n, y_m) > 2(M+R)$ for $m \neq n$. By our choice of M, each of the balls $B_M(y_n)$ contains a vertex $x_n = h_n(x)$ in the orbit of x. Thus the balls $B_R(x_n)$ are pairwise disjoint. □

Before proving our next lemma we recall the correspondence between the geometric notions mentioned above and their algebraic counterparts.

We fix a surface with a basepoint, $(\Sigma, *)$. We note that the free homotopy class of a simple closed curve α on Σ corresponds to the conjugacy class $[a]$ of an element a representing α in $S := \pi_1(\Sigma, *)$.

Moreover, a mapping class h in $\mathcal{MCG}(\Sigma)$ gives rise to an outer automorphism of the fundamental group S as a surface group with boundary (that is, an outer automorphism that fixes the conjugacy classes corresponding to the boundary components). It is a classical result that this induces an isomorphism between $\mathcal{MCG}(\Sigma)$ and $\mathrm{Out}(S)$. So, we have:

Lemma 7.5. *Let Σ be a surface which either is a punctured torus or has Euler characteristic at most -2. Let $[a], [b]$ be conjugacy classes representing simple closed curves α, β in Σ. Then there is a sequence $(\rho_n)_{n<\omega} \subset \mathrm{Out}(S)$ such that $\rho_i \circ f_1([a]) \neq \rho_j \circ f_2([a])$ for any $f_1, f_2 \in \mathrm{Out}_{[b]}(S)$ (i.e. outer automorphisms fixing the conjugacy class of b) and $i \neq j$.*

Proof. We apply Lemma 7.4 for $R = d_{\mathcal{C}(\Sigma)}(\alpha, \beta)$ and $x = \beta$. It is a straightforward exercise to see that the sequence $(\rho_n)_{n<\omega}$ of outer automorphisms corresponding to the sequence $(h_n)_{n<\omega}$ of mapping classes given by Lemma 7.4 satisfies the conclusion. □

8. Forking over big sets

In this section we bring results from previous sections together in order to prove Theorem 2.

We start with a lemma that connects forking independence with the modular group of a torsion-free hyperbolic group concrete with respect to a set of parameters.

Lemma 8.1. *Let G be a torsion-free hyperbolic group, and let A be a subset of G with respect to which G is concrete. Let $\bar{b}, \bar{c}$ be tuples in G. Suppose that the orbit $\mathrm{Mod}_{A\bar{c}}(G).\bar{b}$ is preserved by $\mathrm{Mod}_A(G)$. Then $\bar{b} \underset{A}{\downarrow} \bar{c}$.*

Proof. Let $X := \mathrm{Aut}_{A\bar{c}}(G).\bar{b}$. By Proposition 5.3, X is definable over $A\bar{c}$. By Remark 2.6, since X implies any other formula in $\mathrm{tp}(\bar{b}/A\bar{c})$, it is enough to prove that X does not fork over A.

Now $\mathrm{Mod}_{A\bar{c}}(G).\bar{b}$ is a non-empty subset of X preserved by $\mathrm{Mod}_A(G)$; since $\mathrm{Mod}_A(G)$ has finite index in $\mathrm{Aut}_A(G)$, this subset is almost A-invariant; by Lemma 2.10, we get the result. □

We can now state and prove the second main result of the paper.

Theorem 8.2. *Let G be a torsion-free hyperbolic group, and let A be a subset of G with respect to which G is concrete. Let (Λ, v_A) be the pointed cyclic JSJ decomposition of G with respect to A. Let $\bar{b}$ and $\bar{c}$ be tuples of G, and denote by $\Lambda_{A\bar{b}}$ (respectively $\Lambda_{A\bar{c}}$) the minimal subgraphs of groups of Λ whose fundamental group contains the subgroups $\langle A, \bar{b}\rangle$ (respectively $\langle A, \bar{c}\rangle$) of G. Then $\bar{b}$ and $\bar{c}$ are independent over A if and only if each connected component of $\Lambda_{A\bar{b}} \cap \Lambda_{A\bar{c}}$ contains at most one non-Z-type vertex, and such a vertex is of non-surface type.*

Note that since free groups are concrete over any set of parameters with respect to which they are freely indecomposable, Theorem 2 is a corollary of this result.

Remark 8.3. We note that by Corollary 6.2 coupled with Lemma 2.12, there exists a finitely generated subgroup A_0 of A such that tuples $\bar{b}$ and $\bar{c}$ fork over A if and only if they fork over A_0. This easily extends to any finitely generated subgroup of A which is "sufficiently large" (i.e. which contains A_0) .

On the other hand, by [PS12, Proposition 3.6], there exists a finitely generated subgroup A_0 of A such that the minimal subgraph of groups of Λ whose fundamental group contains $\langle A, \bar{b}\rangle$ (respectively $\langle A, \bar{c}\rangle$) is the same as the the minimal subgraph of groups whose fundamental group contains $\langle A_0, \bar{b}\rangle$ (respectively $\langle A_0, \bar{c}\rangle$). Again this extends to any "sufficiently large" finitely generated subgroup of A.

Thus, in proving Theorem 8.2, we can always assume that the set of parameters is a finitely generated group.

We first prove the "if" direction.

Lemma 8.4. *In the setting of Theorem 8.2, suppose that $\Lambda_{A\bar{b}} \cap \Lambda_{A\bar{c}}$ contains at most one non-Z-type vertex, and such a vertex is of non-surface type. Then $\bar{b} \mathop{\downarrow}\limits_{A} \bar{c}$.*

Proof. Let (T, v_A) be the pointed cyclic JSJ tree of G with respect to A. Let $T_{A\bar{b}}$ (respectively $T_{A\bar{c}}$) denote the minimal subtree of the subgroup of G generated by A and $\bar{b}$ (respectively A and $\bar{c}$) in T. Note that by definition, v_A lies in both $T_{A\bar{b}}$ and $T_{A\bar{c}}$.

Choose a Bass–Serre presentation $(T^1, T^0, (t_f)_f)$ for G with respect to Λ such that $v_A \in T^0$.

By Lemma 8.1, it is enough to show that $\mathrm{Mod}_{A\bar{c}}(G).\bar{b}$ is preserved by $\mathrm{Mod}_A(G)$. For this, it is enough to show that for any θ in $\mathrm{Mod}_A(G)$, we can find α in $\mathrm{Mod}_{A\bar{c}}(G)$ such that $\theta(\bar{b}) = \alpha(\bar{b})$.

By Lemma 4.21, θ can be written as a product of the form

$$\mathrm{Conj}(z) \circ \rho_1 \circ \cdots \circ \rho_t$$

where the ρ_j fix A pointwise, are Dehn twists about distinct edges of T^1 or vertex automorphisms associated to distinct surface type vertices of T^0.

The hypothesis on $\Lambda_{A\bar{b}} \cap \Lambda_{A\bar{c}}$ implies that the intersection of $\bigcup_{g\in G} g \cdot T_{A\bar{b}}$ with $\bigcup_{h\in G} h \cdot T_{A\bar{c}}$ contains no surface type vertex. Also, it implies that this intersection meets at most one orbit of an edge of each cylinder of T.

In this light, we may assume that the supports of the ρ_j lie outside of $\bigcup_{g\in G} g \cdot T_{A\bar{b}} \cap \bigcup_{h\in G} h \cdot T_{A\bar{c}}$. Indeed, suppose that $\mathrm{Supp}(\rho_j)$ is in $g \cdot T_{A\bar{b}} \cap h \cdot T_{A\bar{c}}$; it must be an edge by the remark above. By Lemma 4.9 it can be replaced by a product of a conjugation and Dehn twists whose supports are edges in the same cylinder which are not in the orbit of e; they must lie outside of $\bigcup_{g\in G} g \cdot T_{A\bar{b}} \cap \bigcup_{h\in G} h \cdot T_{A\bar{c}}$.

Since for each j, either ρ_j does not belong to any translate of $T_{A\bar{b}}$, or it does not belong to any translate of $T_{A\bar{c}}$, we may assume (using Lemma 4.26 and Proposition 4.18) that there exists r such that $\rho_i \in \mathrm{Mod}_{A\bar{c}}(G)$ for any $i \leq r$ and $\rho_j \in \mathrm{Mod}_{A\bar{b}}(G)$ for any $j > r$.

Also observe that since θ and each ρ_j fix A, either z is trivial, or A is cyclic and $z \in C(A)$. In the first case we can take α to be $\rho_1 \circ \cdots \circ \rho_r$.

In the second case we let τ be the product of the Dehn twists by z about the edges of T^1 which are in the unique cylinder containing v_A, but do not lie in $T_{A\bar{c}}$. Then τ satisfies $\tau(\bar{b}) = \mathrm{Conj}(z)(\bar{b})$, and lies in $\mathrm{Mod}_{A\bar{c}}(G)$. Thus we can take α to be $\rho_1 \circ \cdots \circ \rho_r \circ \tau$. □

To prove the second direction of Theorem 8.2, it is enough to consider the following three cases: (i) for some g, $T_{A\bar{b}} \cap g \cdot T_{A\bar{c}}$ contains a surface type vertex, (ii) for some g, h, h', there are edges from distinct orbits $e = xz$ and $e' = yz$ contained in $T_{A\bar{b}} \cap h \cdot T_{A\bar{c}}$ and $g \cdot T_{A\bar{b}} \cap h' \cdot T_{A\bar{c}}$ respectively, where each of x and y is either the basepoint, or a non-cyclically stabilized vertex of rigid type, and (iii) for some g, $T_{A\bar{b}} \cap g \cdot T_{A\bar{c}}$ contains an edge $e = v_A x$ where v_A is the basepoint and x is non-cyclically stabilized of rigid type.

The following lemma deals with the latter case. Note that in this case, A is cyclic.

Lemma 8.5. *In the setting of Theorem 8.2, let $T_{A\bar{b}}$ (respectively $T_{A\bar{c}}$) denote the minimal subtree of $\langle A, \bar{b}\rangle$ (respectively $\langle A, \bar{c}\rangle$) in T. Suppose that there exists g such that $T_{A\bar{b}} \cap g \cdot T_{A\bar{c}}$ contains an edge $e = v_A x$ where v_A is the basepoint and x is a non-cyclically stabilized vertex of rigid type. Then $\bar{b}$ forks with $\bar{c}$ over A.*

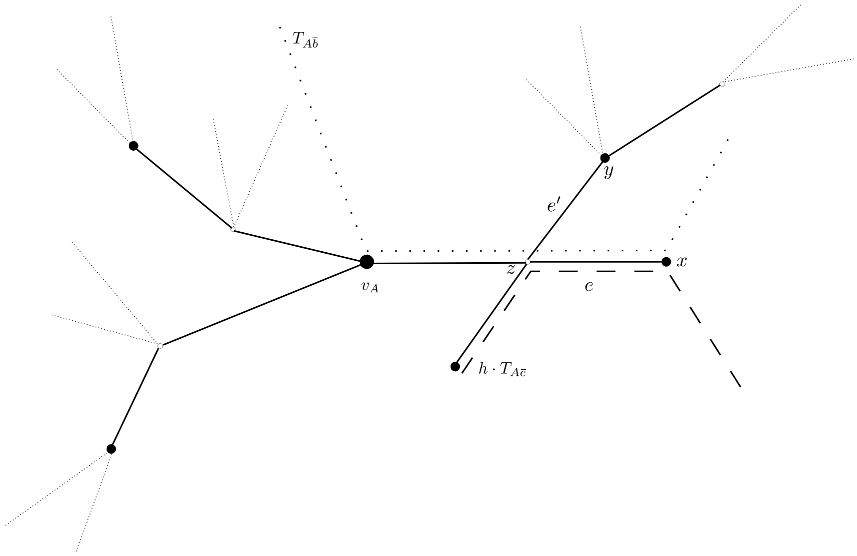
Proof. By definition of the pointed cyclic JSJ tree, since v_A is at distance 1 of a non-cyclically stabilized vertex, it follows that A is cyclic and e is the unique edge adjacent to v_A. Now v_A and e are stabilized by the centralizer $C(A)$ of A, and the stabilizer of x is not cyclic.

By Proposition 6.3, $\mathrm{Stab}(x) \subseteq \mathrm{acl}_G(A\bar{b})$, and by Proposition 6.4, $\mathrm{Stab}(x) \subseteq \mathrm{acl}_G(A\bar{c})$, but $\mathrm{Stab}(x) \nsubseteq \mathrm{acl}_G(A)$. This implies that $\bar{b}$ forks with $\bar{c}$ over A. □

We can now show that if we are in case (ii), the tuples $\bar{b}$ and $\bar{c}$ fork over A.

Lemma 8.6. *In the setting of Theorem 8.2, let $T_{A\bar{b}}$ (respectively $T_{A\bar{c}}$) denote the minimal subtree of $\langle A, \bar{b}\rangle$ (respectively $\langle A, \bar{c}\rangle$) in T. Suppose that there exist g, h, h' and edges from distinct orbits $e = xz$ and $e' = yz$ contained in $T_{A\bar{b}} \cap h \cdot T_{A\bar{c}}$ and $g \cdot T_{A\bar{b}} \cap h' \cdot T_{A\bar{c}}$ respectively, where x and y are non-cyclically stabilized vertices which are either the basepoint, or of rigid type. Then $\bar{b}$ and $\bar{c}$ fork over A.*

Proof. Choose a Bass–Serre presentation $(T^1, T^0, (t_\alpha)_\alpha)$ for G with respect to Λ such that e and e' are in T^1.



Denote the stabilizers of x and y by U and V respectively, and let $\bar{u}$, $\bar{v}$ denote generating tuples of U and V respectively. We assume without loss of generality that $x \neq v_A$ so that U is not cyclic.

Let X denote the orbit of the pair $(\bar{u}, \bar{v})$ under $\mathrm{Aut}_{A\bar{c}}(G)$. We will show that X admits infinitely many pairwise disjoint translates by a sequence of automorphisms in $\mathrm{Aut}_A(G)$; since X is definable over $A\bar{c}$ and contains $(\bar{u}, \bar{v})$, this will imply that $(\bar{u}, \bar{v})$ forks with $\bar{c}$ over A. Now by Propositions 6.3 and 6.4 respectively, both $\bar{u}$ and $\bar{v}$ are in $\mathrm{acl}(A\bar{b})$, so this implies that $\bar{b}$ forks with $\bar{c}$ over A.

Let τ_e be a Dehn twist about e by some element ϵ of $\mathrm{Stab}(e)$.

By uniqueness of the tree T, for any element ϕ of $\mathrm{Aut}_A(G)$ there is an automorphism f of T such that for any $w \in T$ and $g \in G$ we have $f(g \cdot w) = \phi(g) \cdot f(w)$. Recall now that $\mathrm{Mod}_{A\bar{c}}(G)$ has finite index in $\mathrm{Aut}_{A\bar{c}}(G)$; pick $\phi_0, \dots, \phi_l$ such that the classes $\mathrm{Mod}_{A\bar{c}}(G)\phi_j$ cover $\mathrm{Aut}_{A\bar{c}}(G)$, and denote by $f_1, \dots, f_l$ the corresponding automorphisms of T. Since ϕ_j fixes $A\bar{c}$, the automorphism f_j must preserve $T_{A\bar{c}}$. Now e lies in $g \cdot T_{A\bar{c}}$, so the edges $f_j(e)$ also lie in a translate of $T_{A\bar{c}}$.

Denote by $\tau_{f_k(e)}$ the automorphism $\phi_k \tau_e \phi_k^{-1}$; it is a Dehn twist about $f_k(e)$. Choose $j_1, \dots, j_{l'}$ minimal such that $\{f_{j_1}(e), \dots, f_{j_{l'}}(e)\} = \{f_1(e), \dots, f_l(e)\}$ and define

$$\tau = \tau_{f_{j_1}(e)} \circ \cdots \circ \tau_{f_{j_{l'}}(e)}.$$

We will show that for r large enough the sequence $\{\tau^{rn}(X)\}_{n\in\mathbb{N}}$ of translates consists of pairwise disjoint sets. For this it is enough to show that for m large enough, $X \cap \tau^m(X)$ is empty. Suppose that there exist j, k and α, β in $\mathrm{Mod}_{A\bar{c}}(G)$ such that

$$\alpha(\phi_j(u, v)) = \tau^m \beta(\phi_k(u, v)).$$

By Proposition 4.27, any element of $\mathrm{Mod}_{A\bar{c}}(G)$ can be written as a product of (a conjugation and) elementary automorphisms whose supports, if they are edges, are not in any translate of $T_{A\bar{c}}$, hence not in the orbit of the edges $f_j(e)$. In particular β commutes with τ (up to conjugation). Thus there exists $\theta = \beta^{-1}\alpha$ in $\mathrm{Mod}_{A\bar{c}}(G)$ such that $\theta(\phi_j(\bar{u}, \bar{v}))$ is conjugate to $\tau^m(\phi_k(\bar{u}, \bar{v}))$.

Since e and e' are not in the same orbit, we can choose a Bass–Serre presentation for G with respect to T such that $f_j(e)$ and $f_j(e')$ are in T^1. The automorphism θ can be written as a product of a conjugation and elementary automorphisms whose supports are not in the orbits of the edges $f_j(e)$ and $f_j(e')$. Thus Lemmas 4.7 and 4.14 imply that $\theta(\phi_j(\bar{u}, \bar{v}))$ is conjugate to the tuple $\phi_j(\bar{u}, \bar{v})$. On the other hand, by definition of τ we know that $\tau^m(\phi_k(\bar{u}, \bar{v}))$ is conjugate to $\phi_k(\bar{u}, \epsilon^m \bar{v} \epsilon^{-m})$, so finally there exists an element γ such that

$$\gamma \phi_j(\bar{u}, \bar{v}) \gamma^{-1} = \phi_k(\bar{u}, \epsilon^m \bar{v} \epsilon^{-m}).$$

For j and k fixed, this holds for at most one value of γ, since $\gamma \phi_j(\bar{u}) \gamma^{-1} = \phi_k(\bar{u})$, and $\bar{u}$ generates a non-abelian subgroup. But then $\gamma \phi_j(\bar{v}) \gamma^{-1} = \phi_k(\epsilon^m \bar{v} \epsilon^{-m})$ can only be true for a single value of m (for each j, k). Thus for m large enough, $X \cap \tau^m(X)$ is empty. □

To finish the proof of Theorem 8.2, we need to deal with the case where translates of the minimal subtrees intersect in a surface type vertex. For this, we will use the results about the curve complex given in Section 7.

Lemma 8.7. *In the setting of Theorem 8.2, let $T_{A\bar{b}}$ (respectively $T_{A\bar{c}}$) denote the minimal subtree of $\langle A, \bar{b} \rangle$ (respectively $\langle A, \bar{c} \rangle$) in T. Suppose that there exists $g \in G$ such that $(g^{-1} \cdot T_{A\bar{b}}) \cap T_{A\bar{c}}$ contains a surface type vertex. Then $\bar{b}$ and $\bar{c}$ fork over A.*

Proof. Denote by v the surface type vertex, by S its stabilizer and by Σ the corresponding surface with boundary. Fix a Bass–Serre presentation $(T^1, T^0, (t_f)_f)$ such that v lies in T^0. Denote by $v_{A\bar{b}}$ and $v_{A\bar{c}}$ the basepoints of the pointed cyclic JSJ trees of G with respect to $\langle A, \bar{b} \rangle$ and $\langle A, \bar{c} \rangle$ respectively.

By Proposition 6.3, there exist $b_0, c_0 \in S$ which correspond to non-boundary parallel simple closed curves on Σ such that b_0^g is contained in $\mathrm{Stab}(v_{A\bar{b}}) \subseteq \mathrm{acl}_G(A\bar{b})$, and c_0 is contained in $\mathrm{Stab}(v_{A\bar{c}}) \subseteq \mathrm{acl}_G(A\bar{c})$.

Denote by X the orbit of b_0^g under $\mathrm{Aut}_{A\bar{c}}(G)$. We will show that X admits infinitely many pairwise disjoint translates by a sequence of automorphisms in $\mathrm{Aut}_A(G)$; since X is definable over $A\bar{c}$, this implies that $\bar{b}$ forks with $\bar{c}$ over A.

As in the proof of Lemma 8.6, pick $\phi_1, \ldots, \phi_l$ such that the classes $\mathrm{Mod}_{A\bar{c}}(G)\phi_i$ cover $\mathrm{Aut}_{A\bar{c}}(G)$ and denote by $f_1, \ldots, f_l$ the corresponding automorphisms of T. Since ϕ_i fixes $A\bar{c}$, the automorphism f_i must preserve $T_{A\bar{c}}$, hence $f_i(v)$ is a surface type vertex in $T_{A\bar{c}}$.

Let $\{v_1, \ldots, v_s\}$ be the vertices of T^0 which lie in the orbit of one of the vertices $f_i(v)$. Up to reindexing we may assume v_j is in the orbit of $f_j(v)$. Denote by S_j the stabilizer of v_j, and by b_j, c_j the images of b_0, c_0 by ϕ_j.

Note that ϕ_j fixes $A\bar{c}$, so $c_j = \phi_j(c_0)$ is also in $\mathrm{Stab}(v_{A\bar{c}})$. Any element of $\mathrm{Mod}_{A\bar{c}}(G)$ fixes $\mathrm{Stab}(v_{A\bar{c}})$ pointwise, so it preserves the conjugacy class of c_j.

By applying Lemma 7.5 to Σ for the conjugacy classes of b_0 and c_0, we get a sequence of automorphisms ρ_n^v in $\mathrm{Aut}(S)$ such that for any two automorphisms σ, σ' of S which preserve the conjugacy class of c_0, if $m \neq n$ the elements $\rho_n^v \sigma(b_0)$ and $\rho_m^v \sigma'(b_0)$ are not conjugate in S (and thus in G).

Let $\rho_n^{v_j}$ be a vertex automorphism associated to v_j whose restriction to S_j is $\phi_j \circ \rho_n^v \circ \phi_j^{-1}$ and define

$$\rho_n = \rho_n^{v_1} \dots \rho_n^{v_s}.$$

We will show that $\rho_n(X) \cap \rho_m(X)$ is empty. Suppose not; then there exist j, k and θ, θ' in $\mathrm{Mod}_{A\bar{c}}(G)$ such that

$$\rho_n \theta(\phi_j(b_0^g)) = \rho_m \theta'(\phi_k(b_0^g)). \tag{8.1}$$

By Remark 4.21, the automorphism θ can be written as a product of a conjugation and elementary automorphisms $\tau_{e_1} \dots \tau_{e_p} \sigma_{u_1} \dots \sigma_{u_q}$ associated to T where σ_{u_j} is a vertex automorphism supported on v_j, and τ_{e_i} is a Dehn twist about the edge e_i. All the elementary automorphisms with support distinct from v_j restrict to conjugations on S_j, so $\theta(\phi_j(b_0^g))$ and $\theta(c_j)$ are conjugates of $\sigma_{v_j}(b_j)$ and $\sigma_{v_j}(c_j)$ respectively.

Now $\theta \in \mathrm{Mod}_{A\bar{c}}(G)$, so as noted above, it preserves the conjugacy class of c_j; hence so does σ_{v_j}. Let $\sigma = \phi_j^{-1} \circ \sigma_{v_j} \circ \phi_j$; the automorphism σ preserves the conjugacy class of c_0.

Similarly, θ' fixes c_k, so in its normal form $\theta' = \mathrm{Conj}(g) \tau'_{e'_1} \dots \tau'_{e'_{p'}} \sigma'_{u'_1} \dots \sigma'_{u'_{q'}}$, the factor σ'_{v_k} is such that $\sigma' = \phi_k^{-1} \circ \sigma'_{v_k} \circ \phi_k$ preserves the conjugacy class of c_0.

Now $\rho_n \theta(\phi_j(b_0^g))$ is conjugate to

$$\rho_n \theta(b_j) = \rho_n \sigma_{v_j}(b_j) = \rho_n \sigma_{v_j} \phi_j(b_0) = \rho_n \phi_j \sigma(b_0),$$

which is itself equal to $\rho_n^{v_j} \phi_j \sigma(b_0) = \phi_j \rho_n^v \sigma(b_0)$. Similarly $\rho_m \theta'(\phi_k(b_0^g))$ is conjugate to $\phi_k \rho_m^v \sigma'(b_0)$.

Thus (8.1) implies that $\phi_j \rho_n^v \sigma(b_0)$ is conjugate to $\phi_k \rho_m^v \sigma'(b_0)$.

Now these are elements representing non-boundary parallel simple closed curves in S_j and in S_k respectively. Thus S_j and S_k are conjugate, so we must have $j = k$. Hence $\phi_j \rho_m^v \sigma(b_0)$ is conjugate to $\phi_j \rho_m^v \sigma'(b_0)$ in S_j, which contradicts our choice of ρ_n^v. □

9. Examples and further remarks

We start by giving some simple examples of forking independence between tuples in non-abelian free groups.

Example 9.1. (i) Let $\bar{\gamma}_1 \in \langle e_1, e_2 \rangle$ and $\bar{\gamma}_2 \in \langle e_3, e_4 \rangle$. Then $\bar{\gamma}_1$ is independent of $\bar{\gamma}_2$ over $\langle [e_1, e_2], [e_3, e_4] \rangle$ in $\mathbb{F}_4$ (see Figure 1).

(ii) Let $\bar{\gamma}_1, \bar{\gamma}_2 \in \mathbb{F}_2 \setminus \langle [e_1, e_2] \rangle$. Then $\bar{\gamma}_1$ forks with $\bar{\gamma}_2$ over $[e_1, e_2]$ (see Figure 6).

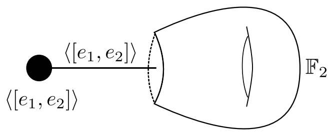


Fig. 6. A graph of groups decomposition of the pointed tree of cylinders of $\mathbb{F}_2$ relative to $H_k = \langle [e_1, e_2]^k \rangle$ for any $k \neq 0$.

We moreover note that our results give a complete description of forking independence (for any two tuples in $\mathbb{F}_2$) over any set of parameters in $\mathbb{F}_2$. The reason is that for any set $A \subseteq \mathbb{F}_2$ of parameters, either $\mathbb{F}_2$ is freely indecomposable with respect to A, or $\mathrm{acl}(A)$ is a free factor of $\mathbb{F}_2$. We would like to connect this observation with the following question we heard from K. Tent.

Question 1. *Is it possible to prove that T_{fg} is stable using the geometric/algebraic description of forking independence?*

Of course, following our discussion after Fact 2.11 one needs to find an independence relation (satisfying the properties of Fact 2.11) in a "saturated enough" model of T_{fg}; still the intuition coming from $\mathbb{F}_2$ might be useful.

One of the difficulties in characterizing forking (between tuples of elements) in a given torsion-free hyperbolic group G, or indeed in any structure which is not saturated, is that the sequence $(\bar{c}_i)_{i<\omega}$ of tuples witnessing the forking of a formula $\phi(x, \bar{c})$ with parameters in G does not have to belong to G, but in general lies in a saturated elementary extension.

Thus, it is possible that one needs to move to and understand automorphisms of saturated elementary extensions (which are known to be non-finitely generated). But in the case of torsion-free hyperbolic groups, we are far from understanding non-finitely generated models, let alone their automorphism groups.

Acknowledgments. We wish to thank Anand Pillay for suggesting the problem. Part of this work was conducted while the second named author was a PhD student under the supervision of A. Pillay and he would like to thank him for teaching him how to fork.

We are grateful to Zlil Sela for a number of helpful suggestions over the course of this work.

We thank Udi Hrushovski for the idea of Theorem 3.1.

Also many thanks to Javier de la Nuez González for a careful reading and spotting that, in an earlier version, Lemma 7.4 was stated too generally.

Finally, this work would not have been made possible without the support of A. Pillay and Z. Sela that allowed the authors to visit each other and collaborate.

References

[BF07] Bestvina, M., Fujiwara, K.: Quasi-homomorphisms on mapping class groups. Glas. Mat. Ser. III **42 (62)**, 213–236 (2007) Zbl 1155.57301 MR 2332668

[Gar81] Garavaglia, S.: Forking in modules. Notre Dame J. Formal Logic **22**, 155–162 (1981) Zbl 0438.03037 MR 0611483

[Gro12] Groves, D.: Test elements in torsion-free hyperbolic groups. New York J. Math. **18**, 651–656 (2012) Zbl 1262.20047 MR 2991417

[GL07] Guirardel, V., Levitt, G.: Deformation spaces of trees. Groups Geom. Dynam. **1**, 135–181 (2007) Zbl 1134.20026 MR 2319455

[GL09a] Guirardel, V., Levitt, G.: JSJ decompositions: definitions, existence, uniqueness. I: The JSJ deformation space. arXiv:0911.3173 (2009)

[GL09b] Guirardel, V., Levitt, G.: JSJ decompositions: definitions, existence, uniqueness. II: Compatibility and acylindricity. arXiv:1002.4564 (2009)

[GL11] Guirardel, V., Levitt, G.: Trees of cylinders and canonical splittings. Geom. Topol. **15**, 977–1012 (2011) Zbl 1272.20026 MR 2821568

[GL15] Guirardel, V., Levitt, G.: Splittings and automorphisms of relatively hyperbolic groups. Groups Geom. Dynam. **9**, 599–663 (2015) Zbl 06466525 MR 3356977

[Har81] Harvey, W. J.: Boundary structure of the modular group. In: Riemann Surfaces and Related Topics (Stony Brook, NY, 1978), Ann. of Math. Stud. 97, Princeton Univ. Press, Princeton, NJ, 245–251 (1981) Zbl 0461.30036 MR 0624817

[KM06] Kharlampovich, O., Myasnikov, A.: Elementary theory of free nonabelian groups. J. Algebra **302**, 451–552 (2006) Zbl 1110.03020 MR 2293770

[Lev05] Levitt, G.: Automorphisms of hyperbolic groups and graphs of groups. Geom. Dedicata **114**, 49–70 (2005) Zbl 1107.20030 MR 2174093

[LPS11] Louder, L., Perin, C., Sklinos, R.: Hyperbolic towers and independent generic sets in the theory of free groups. Notre Dame J. Formal Logic **54**, 521–539 (2013) Zbl 1288.20030 MR 3091669

[Mar02] Marker, D.: Model Theory: An Introduction. Grad. Texts in Math. 217, Springer (2002) Zbl 1003.03034 MR 1924282

[MM99] Masur, H. A., Minsky, Y. N.: Geometry of the complex of curves. I. Hyperbolicity. Invent. Math. **138**, 103–149 (1999) Zbl 0941.32012 MR 1714338

[O11] Ould Houcine, A.: Homogeneity and prime models in torsion-free hyperbolic groups. Confluentes Math. **3**, 121–155 (2011) Zbl 1229.20020 MR 2794551

[OV11] Ould Houcine, A., Vallino, D.: Algebraic & definable closure in free groups. Ann. Inst. Fourier (Grenoble), to appear

[Per08] Perin, C.: Elementary embeddings in torsion-free hyperbolic groups. Ph.D. thesis, Univ. de Caen Basse-Normandie (2008)

[Per11] Perin, C.: Elementary embeddings in torsion free hyperbolic groups. Ann. Sci. École Norm. Sup. (4) **44**, 631–681 (2011) Zbl 1245.20052 MR 2919979

[PS12] Perin, C., Sklinos, R.: Homogeneity in the free group. Duke Math. J. **161**, 2635–2668 (2012) Zbl 1270.20028 MR 2988905

[Pil96] Pillay, A.: Geometric Stability Theory. Oxford Univ. Press (1996) Zbl 0871.03023 MR 1429864

[Pil09] Pillay, A.: On genericity and weight in the free group. Proc. Amer. Math. Soc. **137**, 3911–3917 (2009) Zbl 1184.03023 MR 2529900

[PP83] Pillay, A., Prest, M.: Forking and pushouts in modules. Proc. London Math. Soc. **46**, 365–384 (1983) Zbl 0509.03018 MR 0693046

[Poi83] Poizat, B.: Groupes stables, avec types génériques réguliers. J. Symbolic Logic **48**, 339–355 (1983) Zbl 0525.03024 MR 0704088

[Poi01] Poizat, B.: Stable Groups. Math. Surveys Monogr. 87, Amer. Math. Soc. (2001) Zbl 0969.03047 MR 1827833

[RS94] Rips, E., Sela, Z.: Structure and rigidity in hyperbolic groups I. Geom. Funct. Anal. **4**, 337–371 (1994) Zbl 0818.20042 MR 1274119

[Sel06] Sela, Z.: Diophantine geometry over groups VI: The elementary theory of free groups. Geom. Funct. Anal. **16**, 707–730 (2006) Zbl 1118.20035 MR 2238945

[Sel13] Sela, Z.: Diophantine geometry over groups VIII: Stability. Ann. of Math. (2) **177**, 787–868 (2013) Zbl 1285.20042 MR 3034289

[Ser83] Serre, J.-P.: Arbres, Amalgames, SL_2. Astérisque **46** (1983) Zbl 0369.20013 MR 0476875

[She90] Shelah, S.: Classification Theory and the Number of Nonisomorphic Models. 2nd ed., North-Holland (1990) Zbl 0713.03013 MR 1083551